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Some classical views on the parameters of the Grothendieck-Teichmüeller group

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SOME CLASSICAL VIEWS ON THE PARAMETERS OF THE GROTHENDIECK-TEICHMÜLLER GROUP

To John Thompson on the occasion of his 70-th birthday

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Abstract We present two new formulas concerning behaviors of the standard parameters of the Grothendieck-Teichmüller group $\widehat{GT}$, and discuss their relationships with classical mathematics. First, considering a non-Galois etale cover of $\mathbf{P}^1 - \{0,1,\infty\}$ of degree 4, we present a newtype equation satisfied by the Galois image in $\widehat{GT}$. Second, a certain equation in $\mathrm{GL}_2(\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]])$ satisfied by every element of $\widehat{GT}$ is derived as an application of (profinite) free differential calculus.

0. Introduction

The structure of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is one of the most important subjects to study in number theory and arithmetic geometry. One attractive approach has occurred since the fundamental work of G.V.Belyi [Be] published in 1979, which shows that $G_{\mathbb{Q}}$ has a faithful representation in the profinite fundamental group of the projective line minus 3 points. In [Gr], A.Grothendieck predicted that a certain profinite group approximating $G_{\mathbb{Q}}$ can be formulated from the tower of profinite Teichmüller modular groupoids starting from the initial stage $\pi_1(\mathbf{P}^1 - \{0,1,\infty\})$. Based on this significant philosophy, V.G.Drinfeld [Dr] and Y.Ihara [I1] introduced the Grothendieck-Teichmüller group $\widehat{GT}$ in which $G_{\mathbb{Q}}$ sits in a standard way. Unfortunately the fundamental question of whether $G_{\mathbb{Q}} = \widehat{GT}$ has remained unsettled yet; however, it is possible to look at various behaviors of the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ from certain geometric, arithmetic viewpoints (cf. [I2], [LS], [NT] etc.).

In my Florida talk, I reported two equations of different nature in the profinite Grothendieck-Teichmüller group $\widehat{GT}$. The first one (Prop. 1.1 below) is derived from the classical (non-Galois) cover of modular curves $X_0(3) \to X(1)$ of degree 4. This holds on the image of $G_{\mathbb{Q}}$ in $\widehat{GT}$, and is unknown whether to hold on the total $\widehat{GT}$. The second formula (Prop. 3.1, below) is derived from application of the classical Magnus-Gassner formalism in combinatorial group theory. This gives an equation of two-by-two matrices of two formal variables (hence produces infinitely many equations by specializations of variables) which holds on the total $\widehat{GT}$.

In this note, we present proofs of these equations and related results with attempting to show several background materials from different contexts of classical mathematics. Still, concerning ultimate estimation of a (possible) gap between $G_{\mathbb{Q}}$ and $\widehat{GT}$, perspectives have remained obscure from these investigations.

The organization of the sections is as follows: In Sect.1, we summarize a simple typical method (initiated in [NS]-[NT]) to abstract an equation satisfied by $G_{\mathbb{Q}}$ in $\widehat{GT}$ from a certain "doubly 3 point ramified" cover of projective lines, and present Prop. 1.1 as an application. In Sect.2, the same method is examined to apply to the non-compact cases of the list of Singerman's table [Si]. In Sect.3, changed is our focus to the method of classical Magnus-Gassner representations which yields Prop. 3.1. Finally, in Sect.4, we discuss specializations of Prop. 3.1 and discuss several complementary facts.

We refer to [I1], [Sc], [HS] for basic facts on $\widehat{GT}$, and write the standard parameter of $\sigma \in \widehat{GT}$ as $(f_\sigma, \lambda_\sigma) \in \hat{F}_2 \times \hat{\mathbb{Z}}^\times$, where $\hat{F}_2$ is the profinite free group of rank 2 generated by two non-commutative symbols.

1. Hauptmodul and Thompson series

It is now well known that a certain special type of 3 point ramified cover of the projective line $\mathbf{P}^1$ affords newtype equations satisfied by the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. The first example was given in [NS] Theorem 2.2 using a certain combination of two double covers of $\mathbf{P}^1$. Then, in [NT], we investigated several other examples appearing in the Lengendre-Jacobi covers with Galois group S_3 (and its subcovers). This cover is essentially the same as the cover $X(2) \to X(1) = \mathbf{P}^1_J$ of the elliptic modular curve of level 2 over that of level 1 — the J-line. In loc.cit., we also introduced the intermediate covers by the harmonic line $\mathbf{P}^1_u = X_0(2)$ and the equianharmonic line $\mathbf{P}^1_v$ of degrees 3, 2 over X(1) respectively, and studied the Galois covers $X(2) \to \mathbf{P}^1_u$, $X(2) \to \mathbf{P}^1_v$ of degrees 2, 3 respectively.

One finds that the common geometric features of these (Galois) covers $Y(\mathbb{C}) \to X(\mathbb{C})$ are :

(1) Ramification occurs only over the three points $0, 1, \infty$ of $X(\mathbb{C}) = \mathbf{P}^1$.

(2) Exactly three cusps on $Y(\mathbb{C}) \cong \mathbf{P}^1$ do not achieve the least common multiples of ramification indices of those cusps lying over the same image in $\{0, 1, \infty\} \subset X(\mathbb{C})$.

Since the argument in [NT] works also for non-Galois covers satisfying (1),(2), it is natural to seek other covers sharing these two properties. In the fall of 1999, M.Koike delivered a series of lectures at Tokyo Metropolitan University on modular forms for triangle groups (cf. [Ko]), where, among other crucial results, listed are the relationships between the Thompson series for the nine non-compact arithmetic triangle groups classified by K.Takeuchi. For example, the uniformizer (Hauptmodul) of the harmonic line $\mathbf{P}^1_u$ is given by the Thompson series $T_{2B} = 24 + \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$ associated to the conjugacy class $2B$ of the Monster simple group, and satisfies an explicit cubic equation

$$\frac{1728}{J} = \varphi\left(\frac{64}{T_{2B}+40}\right), \quad \varphi(X) := 27\frac{X(1-X)^2}{(1+3X)^2}.$$

None of the conjugacy classes of the Monster corresponds to the equianharmonic line $\mathbf{P}^1_v$, but its Hauptmodul was given by Ford-McKay-Norton as the Thompson series "T_{2a}" satisfying $J - 1728 = T_{2a}^2$. One finds also that the Thompson series T_{3B} generates the function field of $X_0(3)$ and is related to the usual J-function by the quartic equation:

$$\frac{1728}{J} = \varphi\left(\frac{27}{T_{3B}+15}\right), \quad \varphi(X) := 64\frac{X(1-X)^3}{(1+8X)^3}.$$

Once recognizing that this cover has the above properties (1), (2), we immediately obtain the following

Proposition 1.1. *Let $\hat{B}_3$ be the profinite braid group generated by the symbols τ_1, τ_2 with the defining relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. Then, each σ in the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ satisfies the following equation:*

$$f_\sigma(\tau_1^3, \tau_1\tau_2) = (\tau_1\tau_2)^{3\rho_3(\sigma)-3\rho_2(\sigma)} f_\sigma(\tau_1, \tau_1\tau_2)\tau_1^{6\rho_2(\sigma)-6\rho_3(\sigma)}.$$

Here and after, for any positive integer $a > 1$, let $\rho_a : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ be the Kummer 1-cocycle defined by $(\sqrt[n]{a})^{\sigma-1} = \zeta_n^{\rho_a(\sigma)}$ $(\sigma \in G_{\mathbb{Q}},\ n \geq 1,\ \zeta_n = \exp(2\pi i/n))$. These are naturally extended to $\rho_a : \widehat{GT} \to \hat{\mathbb{Z}}$ ([I2]. See also [NT] for $a = 2, 3$).

Proof. We change the coordinates X, J of $X_0(3)$, $X(1)$ to t, r respectively so that the above cover f can be transformed into the form $h : X_0(3) \to X(1)$ given by

$$r = h(t) = \frac{t(t+8)^3}{(t^2-20t-8)^2}.$$

This is ramified only over $r = 0, 1, \infty$, and the ramification indices over $r = 1$ are easily seen from

$$1 - r = \frac{64(1-t)^3}{(t^2-20t-8)^2}.$$

The (0,1) segment of t-line is bijectively mapped to that of u-line. The rest of our argument is almost similar to those given in [NT]. Moreover, observing the principal coefficients of Taylor expansions, we see the "toroidal equivalences" of tangential basepoints: $h(\overrightarrow{01}_t) \sim \frac{1}{2^3}\overrightarrow{01}_r$, $h(\overrightarrow{10}_t) \sim \frac{3^6}{2^6}\overrightarrow{10}_r$ (cf. [N2], [N3, Part II] 5.9). Fixing the canonical connection between $\overrightarrow{01}_u$ and $h(\overrightarrow{01}_t)$ on the real line, we find that the standard loops x_t, y_t of $\mathbf{P}^1_t - \{0,1,\infty\}$ (running around 0,1 respectively) are mapped to x_r, y_r^3 of $\mathbf{P}^1_r - \{0,1,\infty\}$ respectively. Thus,

$$f_\sigma(x_r, y_r^3) = y_r^{-6\rho_2(\sigma)+6\rho_3(\sigma)} f_\sigma(x_r, y_r) x_r^{3\rho_2(\sigma)}$$

in $\pi_1(\mathbf{P}^1_r; e_0|3, e_1|\infty, e_\infty|2)$ which is isomorphic to the profinite completion of the triangle group

$$\widehat{\Delta}(3,\infty,2) := \langle x, y, z \mid xyz = x^3 = z^2 = 1 \rangle.$$

Now, pulling back the above equation by the surjection $\hat{B}_3 \to \widehat{\Delta}(3,\infty,2)$ ($\tau_1\tau_2 \mapsto x_r$, $\tau_1 \mapsto y_r$), we get

$$f_\sigma(\tau_1\tau_2, \tau_1^3) = \tau_1^{-6\rho_2(\sigma)+6\rho_3(\sigma)} f_\sigma(\tau_1\tau_2, \tau_1)(\tau_1\tau_2)^{3\rho_2(\sigma)+3c}$$

for some constant $c = c_\sigma \in \hat{\mathbb{Z}}$. Considering the image on the abelianization of $\hat{B}_3$, we see $c = -\rho_3(\sigma)$. The result is equivalent to the claimed formula. □

2. Singerman pairs of triangle groups

The inclusion pairs $\Delta \subset \Delta_0$ of triangle groups were classified in the paper of D.Singerman [Si] published in 1972. The Galois cases (i.e., the cases where Δ are normal subgroups of Δ_0) are only those three cases corresponding to the Legendre-Jacobi cover together with its subcovers by the harmonic and equianharmonic lines. This is not difficult to see, if one knows (say, from [Kl]) that the Galois covers of genus 0 curves are only limited to those cyclic, dihedral, and platonic covers.

Meanwhile, as for non-Galois covers satisfying (1)-(2) of the previous section, the Fuchsian types of non-normal inclusions of triangle groups are classified into eleven cases in D. Singerman's list ([Si], Theorem 2), all of which are named from A to K:

	non-normal inclusion of triangle groups	index
A	$\Delta(7,7,7) \subset \Delta(2,3,7)$	24
B	$\Delta(2,7,7) \subset \Delta(2,3,7)$	9
C	$\Delta(3,3,7) \subset \Delta(2,3,7)$	8
D	$\Delta(4,8,8) \subset \Delta(2,3,8)$	12
E	$\Delta(3,8,8) \subset \Delta(2,3,8)$	10
F	$\Delta(9,9,9) \subset \Delta(2,3,9)$	12
G	$\Delta(4,4,5) \subset \Delta(2,4,5)$	6
H	$\Delta(n,4n,4n) \subset \Delta(2,3,4n)$	6
I	$\Delta(n,2n,2n) \subset \Delta(2,4,2n)$	4
J	$\Delta(3,n,3n) \subset \Delta(2,3,3n)$	4
K	$\Delta(2,n,2n) \subset \Delta(2,3,2n)$	3

Among them, the non-compact ones occur in the last four cases from H to K with $n = \infty$. We shall check each of the non-compact cases here.

Concerning the problem of finding newtype equations in $\widehat{GT}$ from such a cover $Y \to X$, each lift of one of the basic (real) segments $(0,1),(1,\infty),(\infty,0)$ could produce different equations. For example, the equations (IV) and (GF1) are coming from the same double cover of $\mathbf{P}^1$ but from different segment lifts; there seems no reasoning to deduce one from the other. Still, (especially) in Galois case, if two lifts of a segment can be exchanged under the covering symmetry, then they yield essentially the same equation. In non-Galois case, such symmetry chances become rare so that a lot more different equations could arise from various lifts of basic segments.

The case (H) is the type $\Delta(\infty,4\infty,4\infty) \subset \Delta(2,3,4\infty)$ of index 6. This is essentially the cover $\varphi : X_0(4) \to X(1)$ which is the composition of $X_0(4) \to X_0(2)$ with $X_0(2) \to X(1)$. The corresponding Thompson series is T_{4C}. With suitable coordinates, it is given by

$$s = \varphi(X) = \frac{(X^2-16X+16)^3}{108X^4(X-1)}, \quad 1-s = \frac{(X-2)^2(X^2+32X-32)^2}{108X^4(X-1)}.$$

In this case, all cusps on $X_0(4)$ lie on $\mathbb{R} \cup \{\infty\}$. Unfortunately, the basic segments $(0,1),(1,\infty),(\infty,0)$ on $X_0(4)$ are divided by other cusps with irrational coordinates. So, a simple equation for $\widehat{GT}$ can not be expected here.

The case (I) is the type $\Delta(\infty,2\infty,2\infty)\subset\Delta(2,4,2\infty)$ of index 4. This is the composition of the covers: $X_0(4)\to X_0(2)\to X_0^*(2)$, where $X_0^*(2)$ is the quotient of $X_0(2)$ by the Fricke involution $\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)\notin \mathrm{SL}_2(\mathbb{Z})$. The Hauptmodul of $X_0^*(2)$ is known as T_{2A}. Under suitable change of coordinates, the cover $\varphi: X_0(4)\to X_0^*(2)$ is given by

$$w=\varphi(Z)=\frac{16Z(1-Z)}{(4Z^2-4Z-1)^2},\quad 1-w=\frac{(2Z-1)^4}{(4Z^2-4Z-1)^2}.$$

In this case, the segment $(0,\frac{1}{2})$ on $X_0(4)$ is bijectively mapped onto $(0,1)$ on $X_0^*(2)$. From this, in the similar way to [NT], one can deduce

Proposition 2.1. *The following equation is satisfied by every σ in the image of $G_{\mathbb{Q}}\hookrightarrow\widehat{GT}$:*

$$g_\sigma(x,y^2xy^{-2})=y^{-2\rho_2(\sigma)}f_\sigma(x,y)x^{4\rho_2(\sigma)}\qquad (xyz=y^4=z^2=1).\qquad \text{(GF2)}$$

Here, $g_\sigma(S,T)$ is the unique element of the free profinite group $\hat{F}_2$ of non-commutative symbols S,T characterized by $g_\sigma(T,S)^{-1}g_\sigma(S,T)=f_\sigma(S,T)$ (see [LS]). The above equation is to be understood in the profinite completion of the triangle group $\Delta(\infty,4,2)$. The other basic segments $(1,\infty)$, $(\infty,0)$ on $X_0(4)$ are "irrationally divided". It is not difficult to deduce Prop. 2.1 from combination of [NS] (IV′) and [NT] (GF1).

The case (J) is the type $\Delta(3,\infty,3\infty)\subset\Delta(2,3,3\infty)$ of index 4. This corresponds to the cover $X_0(3)\to X(1)$ discussed in the previous section. Again, the other basic segments $(1,\infty)$, $(\infty,0)$ on $X_0(3)$ are "irrationally divided'

The case (K) is the type $\Delta(2,\infty,2\infty)\subset\Delta(2,3,2\infty)$ of index 3. This is $X_0(2)\to X(1)$. There is one basic segment on $X_0(2)$ bijectively mapped to that on $X(1)$. But examination of it only re-proves the equation (V) of [NT].

We leave discussions on the (co-)compact cases (A)∼(G) of Singerman's list ([S], Theorem 2) for future study. One advantage of non-compact cases above is that one side of the obtained equation can often be expressed by free variables in the issued triangle group, so that the equation could determine the information of f_σ (or its associate, say, g_σ) on this side fully by another side expression. In compact cases, the variables inside f_σ's are necessarily torsion of the triangle group in both sides, so the given information on f_σ would look rather in "reduced" shape. But this would not terribly decrease our interests in fully checking the compact cases. Even in non-compact cases, the above mentioned equations are only those that look relatively simpler than others to be obtained in similar fashions: More thorough investigations (and their combinations) might still yield new nice-looking equations.

3. Magnus-Gassner representation

The standard $\widehat{GT}$-action on the profinite Artin braid group $\hat{B}_3$ of three strands forms a semi-direct product $\widehat{GT} \ltimes \hat{B}_3$. The free profinite group $\hat{F}_2 = \pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01})$ sits naturally in it, and one obtains a conjugate action: $\widehat{GT} \ltimes \hat{B}_3 \to \mathrm{Aut}(\hat{F}_2)$. Here remains a room to apply the classical Magnus-Gassner formalism which enables one to look closely at the action on the meta-abelian quotient $\hat{F}_2/\hat{F}_2''$.

Proposition 3.1. *For each* $\sigma \in \widehat{GT}$*, we have*

$$f_\sigma\left(\left(\begin{smallmatrix}1 & 1-\mathbf{y}\\ 0 & \mathbf{x}\end{smallmatrix}\right), \left(\begin{smallmatrix}\mathbf{y} & 0\\ 1-\mathbf{x} & 1\end{smallmatrix}\right)\right) = \begin{pmatrix} \mathbb{B}_\sigma(\mathbf{x},\mathbf{y}) & \frac{\mathbf{y}-1}{\mathbf{x}-1}(\mathbb{B}_\sigma(\mathbf{x},\mathbf{y})-1) \\ -\frac{\mathbf{x}-1}{\mathbf{y}-1}(\mathbb{B}_\sigma(\mathbf{x},\mathbf{y})-1) & 2-\mathbb{B}_\sigma(\mathbf{x},\mathbf{y}) \end{pmatrix}$$

in the matrix group $\mathrm{GL}_2(\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]])$*, where* $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ *is the commutative ring of the projective limit* $\varprojlim \hat{\mathbb{Z}}[\mathbf{x},\mathbf{y}]/(\mathbf{x}^n-1, \mathbf{y}^n-1)$ $(n \to \infty$ *multiplicatively*$)$*, and* $\mathbb{B}_\sigma(\mathbf{x},\mathbf{y})$ *is the Anderson-Ihara beta function associated to* $\sigma \in \widehat{GT}$*.*

We refer to Ihara's papers [I1-2] for the basic definitions and properties of $\widehat{GT}$ and $\mathbb{B}_\sigma$ (cf. also [Fu]). Let $\widehat{GT} \ltimes \hat{B}_3$ be the above semidirect product of the profinite Artin braid group $\hat{B}_3 = \langle a_1, a_2 \mid a_1a_2a_1 = a_2a_1a_2 \rangle$ by $\widehat{GT}$ in which the structure action is given by

$$\begin{cases} \sigma a_1 \sigma^{-1} = a_1^{\lambda_\sigma}, \\ \sigma a_2 \sigma^{-1} = f_\sigma(a_1^2, a_2^2)^{-1} a_2^{\lambda_\sigma} f_\sigma(a_1^2, a_2^2) \end{cases} \qquad (\sigma \in \widehat{GT}).$$

Lemma 3.2. *Notations being as above, we have the following formula:*

$$f_\sigma(a_1^2, a_2^2) = \eta^{\lambda_\sigma} \sigma \eta^{-1} \sigma^{-1}$$

for $\sigma \in \widehat{GT}$*, where* $\eta = a_1a_2a_1$

Proof. The proof idea goes back to [N3, Part I] Prop.4.12 where only the case $\sigma \in G_{\mathbb{Q}}$ was considered. First we claim that the centralizer of $\hat{B}_3$ in the semidirect product $\widehat{GT} \ltimes \hat{B}_3$ is $\langle \eta^2 \rangle$. In fact, the center of $\hat{B}_3$ is $\langle \eta^2 \rangle$, and the normalizer of $\langle a_1 \rangle$ is $\widehat{GT} \ltimes \langle a_1, \eta^2 \rangle$. So, each element of the centralizer of $\hat{B}_3$ is of the form $a_1^a \eta^{2b} \sigma$ $(a, b \in \hat{\mathbb{Z}}$, $\sigma \in \widehat{GT})$. Its commutativity with a_1 implies $\lambda_\sigma = 1$. Then, writing down its commutativity condition with a_2, we realize $a_1^a f_\sigma(a_1^2, a_2^2)^{-1}$ lies in $\langle a_2, \eta^2 \rangle$, the centralizer of a_2 in $\hat{B}_3$. Taking into consideration that $\langle a_1^2, a_2^2 \rangle$ forms a free profinite subgroup $\hat{F}_2$ and that $f_\sigma \in \hat{F}_2'$, we conclude $a = 0$ and $f_\sigma = 1$. This insures the above claim. Next, applying the conjugation by η (which interchanges a_1 and a_2) to the above structure action of $\widehat{GT} \ltimes \hat{B}_3$, we see that $f_\sigma(a_2^2, a_1^2)\eta\sigma\eta^{-1}$ has the same conjugate action on $\hat{B}_3$ with σ. This means that $f_\sigma(a_2^2, a_1^2)\eta\sigma\eta^{-1}\sigma^{-1}$ lies in the centralizer of $\hat{B}_3$. By the above claim, we obtain

$$f_\sigma(a_2^2, a_1^2)\eta^{2c+1} = \sigma\eta\sigma^{-1}$$

for some constant $c \in \hat{\mathbb{Z}}$. Then, multiplying 2 copies of the both sides respectively yields $\eta^{4c+2} = \eta^{2\lambda_\sigma}$, hence $2c+1 = \lambda_\sigma$. This completes the proof of Lemma 3.2. □

Proof of Proposition 3.1. Now let us consider the inner action of the both sides of Lemma 3.2 on the free subgroup $\hat{F}_2(\subset \hat{B}_3)$ generated by $x := a_1^2$, $y := a_2^2$. For each $\alpha \in \mathrm{Aut}(\hat{F}_2)$, define the transposed Gassner-Magnus matrix ${}^t\overline{\mathfrak{A}}_\alpha$ by

$${}^t\overline{\mathfrak{A}}_\alpha := \begin{pmatrix} (\frac{\partial\alpha(x)}{\partial x})^{\mathrm{ab}} & (\frac{\partial\alpha(y)}{\partial x})^{\mathrm{ab}} \\ (\frac{\partial\alpha(x)}{\partial y})^{\mathrm{ab}} & (\frac{\partial\alpha(y)}{\partial y})^{\mathrm{ab}} \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}[[\hat{F}_2^{\mathrm{ab}}]]).$$

Then, we know that the cocycle property ${}^t\overline{\mathfrak{A}}_{\alpha\beta} = {}^t\overline{\mathfrak{A}}_\alpha \cdot \alpha({}^t\overline{\mathfrak{A}}_\beta)$ holds and that ${}^t\overline{\mathfrak{A}}$ gives a homomorphism on the subgroup $\mathrm{Aut}_1(\hat{F}_2)(\subset \mathrm{Aut}(\hat{F}_2))$ of the "identity on $\hat{F}_2^{\mathrm{ab}}$" automorphisms (cf., e.g., [N1,2]). Since the inner automorphisms $\mathrm{Int}(a_1^2)$, $\mathrm{Int}(a_2^2)$ are in $\mathrm{Aut}_1(\hat{F}_2)$, we may calculate ${}^t\overline{\mathfrak{A}}$ of the left hand side of Lemma 3.2 as $f_\sigma({}^t\overline{\mathfrak{A}}_{\mathrm{Int}(x)}, {}^t\overline{\mathfrak{A}}_{\mathrm{Int}(y)})$. Although $\mathrm{Int}(\sigma), \mathrm{Int}(\eta) \in \mathrm{Aut}(\hat{F}_2)$ are not in Aut_1, using the above action formula and the cocycle property, one can calculate the ${}^t\overline{\mathfrak{A}}$ of RHS of Lemma 3.2 independently. The rest of the proof is rather direct. □

K.Hashimoto and H.Tsunogai remarked to the author that the matrices $A := \left(\begin{smallmatrix} 1 & 1-\mathbf{y} \\ 0 & \mathbf{x} \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} \mathbf{y} & 0 \\ 1-\mathbf{x} & 1 \end{smallmatrix}\right)$ appearing in Prop. 3.1 satisfy

$$ABA^{-1}B^{-1}B^{-1}A^{-1}BABAB^{-1}A^{-1}A^{-1}B^{-1}AB = I.$$

This suggests that the mapping $x \mapsto A$, $y \mapsto B$ from $\hat{F}_2$ to $\mathrm{GL}_2(\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]])$ would annihilate the double commutator subgroup $\hat{F}_2''$. This is in fact the case. More precisely, let $i(\gamma)$ denote the inner automorphism of $\hat{F}_2$ by $\gamma \in \hat{F}_2$. Then, $A = {}^t\overline{\mathfrak{A}}_{i(x)}$, $B = {}^t\overline{\mathfrak{A}}_{i(y)}$. Noticing this, we claim that the Magnus-Gassner mapping $\gamma \mapsto {}^t\overline{\mathfrak{A}}_{i(\gamma)}$ gives the faithful representation

$$\hat{F}_2/\hat{F}_2'' \hookrightarrow \mathrm{GL}_2(\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]), \quad ([\gamma] \mapsto {}^t\overline{\mathfrak{A}}_{i(\gamma)}). \tag{$\natural$}$$

Proof. The Magnus-Gassner matrix for $i(\gamma) \in \mathrm{Aut}_1(\hat{F}_2)$ is

$${}^t\overline{\mathfrak{A}}_{i(\gamma)} = \begin{pmatrix} (\gamma + (1-x)\frac{\partial\gamma}{\partial x})^{\mathrm{ab}} & ((1-y)\frac{\partial\gamma}{\partial x})^{\mathrm{ab}} \\ ((1-x)\frac{\partial\gamma}{\partial y})^{\mathrm{ab}} & (\gamma + (1-y)\frac{\partial\gamma}{\partial y})^{\mathrm{ab}} \end{pmatrix}$$

Therefore ${}^t\overline{\mathfrak{A}}_{i(\gamma)} = I$ if and only if $(\frac{\partial\gamma}{\partial x})^{\rm ab} = (\frac{\partial\gamma}{\partial y})^{\rm ab} = 0$ and $\gamma \in \hat{F}_2{}'$. The Blanchfield-Lyndon sequence (the profinite case is due to Ihara [I2]) tells us that $\hat{F}_2{}'/\hat{F}_2{}''$ is embedded into $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]$ by $\gamma \mapsto ((\frac{\partial\gamma}{\partial x})^{\rm ab}, (\frac{\partial\gamma}{\partial y})^{\rm ab})$. Therefore we see ${}^t\overline{\mathfrak{A}}_{i(\gamma)} = I \Leftrightarrow \gamma \in \hat{F}_2{}''$. □

4. Representations of f_σ in $\mathrm{GL}_2(\hat{\mathbb{Z}})$

In this last section, we shall discuss matrix representations of the parameter(s) of $\widehat{GT}$ following the above results.

First, in the formula of Prop. 3.1, one can specialize $\mathbf{x}, \mathbf{y}$ to arbitrary roots of unity to obtain formulas in $\mathrm{GL}_2(\hat{\mathbb{Z}} \otimes \mathbb{Q}_{\rm ab})$. In particular, for $\sigma \in \widehat{GT}$ we know $\mathbb{B}_\sigma(-1,-1) = (-1)^{\frac{\lambda_\sigma - 1}{2}} \lambda_\sigma$ and $\lim_{x\to 1} \dfrac{\mathbb{B}_\sigma(x,-1)-1}{x-1} = 2\rho_2(\sigma)$, where $\rho_2 : \widehat{GT} \to \hat{\mathbb{Z}}$ is a suitable extension of the Kummer 1-cocycle on $G_{\mathbb{Q}}$ along the $\{\sqrt[n]{2}\}_{n\geq 1}$. Therefore, we obtain, for example,

$$f_\sigma\left(\left(\begin{smallmatrix}1&2\\0&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\2&1\end{smallmatrix}\right)\right) = \begin{pmatrix} (-1)^{\frac{\lambda_\sigma-1}{2}}\lambda_\sigma & (-1)^{\frac{\lambda_\sigma-1}{2}}\lambda_\sigma - 1 \\ 1-(-1)^{\frac{\lambda_\sigma-1}{2}}\lambda_\sigma & 2-(-1)^{\frac{\lambda_\sigma-1}{2}}\lambda_\sigma \end{pmatrix}, \tag{4.1}$$

$$f_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)\right) = \begin{pmatrix} 1 & -4\rho_2(\sigma) \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

for all $\sigma \in \widehat{GT}$. These add new knowledge to (but do not generalize) our previous list of the values of $f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))$, $f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}))$, $f_\sigma((\begin{smallmatrix}1&1\\0&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}))$ on $G_{\mathbb{Q}}$ obtained in [N3, Part I], [NS], [NT] respectively.

Next, we shall consider an effect of Prop. 1.1. Combining it with $(\mathrm{IV}'_{\rm bis})$ (cf. [NT]) yields

$$\begin{aligned} f_\sigma(\tau_1, \tau_1\tau_2) &= (\tau_1\tau_2)^{-\rho_2(\sigma)} f_\sigma(\tau_1^2, \tau_1\tau_2)\tau_1^{2\rho_2(\sigma)} \\ &= (\tau_1\tau_2)^{3\rho_2(\sigma)-3\rho_3(\sigma)} f_\sigma(\tau_1^3, \tau_1\tau_2)\tau_1^{6\rho_3(\sigma)-6\rho_2(\sigma)} \end{aligned}$$

in $\hat{B}_3$ for $\sigma \in G_{\mathbb{Q}}$. Then, by applying the usual specialization $\hat{B}_3 \to \mathrm{GL}_2(\hat{\mathbb{Z}})$ ($\tau_1 \mapsto (\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$, $\tau_2 \mapsto (\begin{smallmatrix}1&0\\-1&1\end{smallmatrix})$), we obtain the relations:

$$\begin{aligned} f_\sigma((\begin{smallmatrix}1&1\\0&1\end{smallmatrix}), (\begin{smallmatrix}0&1\\-1&1\end{smallmatrix})) &= (\begin{smallmatrix}0&1\\-1&1\end{smallmatrix})^{\rho_2(\sigma)} \cdot f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}), (\begin{smallmatrix}0&1\\-1&1\end{smallmatrix})) \cdot (\begin{smallmatrix}1&2\rho_2(\sigma)\\0&1\end{smallmatrix}) \\ &= (-1)^{\rho_2(\sigma)-\rho_3(\sigma)} \cdot f_\sigma((\begin{smallmatrix}1&3\\0&1\end{smallmatrix}), (\begin{smallmatrix}0&1\\-1&1\end{smallmatrix})) \cdot (\begin{smallmatrix}1&6\rho_3(\sigma)-6\rho_2(\sigma)\\0&1\end{smallmatrix}). \end{aligned} \tag{4.3}$$

So, if one of the above sides gets an explicit description, then all the other sides do together. The new point is that the matrix addressed at the second variable of f_σ is torsion, in that circumstance f_σ has not been given explicit matrix expressions in our works so far (except in those cases coming from Prop. 3.1

above). Actually, we have encountered with a similar situation at the equation (V) of [NT]:

$$f_\sigma(\tau_1, \tau_1\tau_2\tau_1) = (\tau_1\tau_2\tau_1)^{\rho_3(\sigma)-2\rho_2(\sigma)} f_\sigma(\tau_1^2, \tau_1\tau_2\tau_1)\tau_1^{6\rho_2(\sigma)-3\rho_3(\sigma)} \qquad \text{(V)}$$

which connects $f_\sigma((\begin{smallmatrix}1&1\\0&1\end{smallmatrix}),(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}))$ and $f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}))$. By [NT] ($GF_0$)-($GF_1$), these terms are also linked to $g_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))$. In [NT2], by using the Anderson-Ihara adelic beta function, we give the explicit evaluation of the last one in the form:

$$g_\sigma\left((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})\right) = \pm\mathbb{B}_\sigma\left((\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}),(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix})\right)\cdot(\begin{smallmatrix}\lambda_\sigma^{-1}&-8\rho_2(\sigma)\lambda_\sigma^{-1}\\0&\pm1\end{smallmatrix}).$$

Finally, let us make a specialization of the equation in Prop. 2.1. The triangle group $\Delta(\infty,4,2)$ has a representation as the Hecke group $\mathfrak{G}(2\cos(\frac{\pi}{4}))$ in $\mathrm{PSL}_2(\mathbb{R})$ ([He]). A standard system of generators is given by $X=(\begin{smallmatrix}1&\sqrt{2}\\0&1\end{smallmatrix}), Y=(\begin{smallmatrix}\sqrt{2}&1\\-1&0\end{smallmatrix}), Z=(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})$ operating on the complex upper half plane $\mathfrak{H}$ with relations $XYZ=1, Y^4=-1, Z^2=-1$ in $\mathrm{SL}_2(\mathbb{R})$. Since we work only algebraically, we may modify these matrices as $x:=\zeta_8^5X$, $y:=\zeta_8Y$, $z:=\zeta_8^2Z$ with $\zeta_8=e^{\frac{2\pi i}{8}}$ so that truly the triangle relation $xyz=y^4=z^2=1$ holds in $\mathrm{GL}_2(\mathbb{Q}(\zeta_8))$. Then, we have the specialization homomorphism of the profinite completion of $\Delta(\infty,4,2)$ to the (finite) adelic group $\mathrm{GL}_2(\mathbb{Q}(\zeta_8)_f)$ by sending the corresponding generators in the obvious way. Moreover, combined with the conjugation by $(\begin{smallmatrix}\sqrt{2}&1\\0&1\end{smallmatrix})$, the specialization mapping finally takes the form:

$$x\mapsto\zeta_8^5\begin{pmatrix}1&2\\0&1\end{pmatrix}, y^2xy^{-2}\mapsto\zeta_8^5\begin{pmatrix}1&0\\-2&1\end{pmatrix}, y\mapsto\zeta_8\begin{pmatrix}\cos(\frac{\pi}{4})&\sin(\frac{\pi}{4})\\-\sin(\frac{\pi}{4})&\cos(\frac{\pi}{4})\end{pmatrix}.$$

Using this and recalling that $g_\sigma(S,S)=S^{2\rho_2(\sigma)}$ ([NS] Prop. 2.4), we obtain from Prop. 2.1 the following equation:

$$g_\sigma\left((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})\right) = (\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})^{\rho_2(\sigma)} f_\sigma\left((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}\sqrt{2}/2&\sqrt{2}/2\\-\sqrt{2}/2&\sqrt{2}/2\end{smallmatrix})\right)(\begin{smallmatrix}1&2\\0&1\end{smallmatrix})^{4\rho_2(\sigma)} \qquad (4.4)$$

for σ lying in the image of $G_\mathbb{Q}\hookrightarrow\widehat{GT}$. This can be understood in the smaller adelic group $\mathrm{GL}_2(\mathbb{Q}(\sqrt{2})_f)$.

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