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Integral cohomology and Chern classes of the special linear group over the ring of integers

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Abstract

This paper is devoted to the complete calculation of the additive structure of the 2-torsion of the integral cohomology of the infinite special linear group $SL(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$. This enables us to determine the best upper bound for the order of the Chern classes of all integral and rational representations of discrete groups.

1. *Introduction*

The Hopf algebra structure of the mod 2 cohomology of the infinite special and general linear groups $SL(\mathbb{Z})$ and $GL(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$ has been completely determined in [**AMNY**] as a module over the Steenrod algebra. For instance, $H^*(SL(\mathbb{Z});\mathbb{Z}/2) \cong H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ is the tensor product of a polynomial algebra with an exterior algebra:

$$\begin{aligned} H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) &\cong H^*(BSO;\mathbb{Z}/2) \otimes H^*(SU;\mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[w_2, w_3, \ldots, w_j, \ldots] \otimes \Lambda_{\mathbb{Z}/2}(u_3, u_5, \ldots, u_{2k-1}, \ldots), \end{aligned}$$

where $\deg(w_j) = j$ and $\deg(u_{2k-1}) = 2k-1$.

The first goal of this paper is to investigate the mod 2 cohomological Bockstein spectral sequence

$$E_1^*(BSL(\mathbb{Z})^+) \cong H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) \Longrightarrow (H^*(BSL(\mathbb{Z})^+;\mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}/2$$

of the space $BSL(\mathbb{Z})^+$ (see [**Brd**, sections 1–5]). By using the mod 2 Bockstein spectral sequences of the spaces BSO and $BSL(\mathbb{F}_p)^+$ (for a prime $p \equiv 5 \bmod 8$) and the maps h: $BSL(\mathbb{Z})^+ \to BSO$ and f_p: $BSL(\mathbb{Z})^+ \to BSL(\mathbb{F}_p)^+$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and by the reduction mod p respectively, we compute the terms $E_r^*(BSL(\mathbb{Z})^+)$ and the differentials d_r for all $r \geqslant 1$ (see Theorem 4·3 and Corollary 4·4). Of course, this detects the 2-torsion of the integral cohomology $H^*(BSL(\mathbb{Z})^+;\mathbb{Z}) \cong H^*(SL(\mathbb{Z});\mathbb{Z})$ of the special linear group $SL(\mathbb{Z})$. Theorem 4·7 actually provides an explicit additive presentation of the 2-torsion of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ by generators and relations (see also Remark 4·8 for some partial information on the multiplicative structure). It turns out that $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ contains no cyclic direct summand of order 4 and that the set of all non-trivial elements of $\Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$ is in one-to-one correspondence with an additive basis of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})/\text{torsion}$ (see Corollary 4·6). Moreover, we are able to understand the effect of the induced homomorphisms h^*: $H^*(BSO;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ and f_p^*: $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+,\mathbb{Z})$ on the 2-torsion elements: Theorem 4·9 asserts in particular that h^* is injective on the elements of order 2 and that f_p^* is injective on all cyclic direct summands of order 2^r with $r \geqslant 3$.

Notice that it is easy to extend these results to the integral cohomology $H^*(GL(\mathbb{Z});\mathbb{Z}) \cong H^*(BGL(\mathbb{Z})^+;\mathbb{Z})$ of the general linear group $GL(\mathbb{Z})$ because of the homotopy equivalence $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$ (see for example [**Ar1**, lemma 1·2]).

As a consequence, we obtain the exact order of all Chern classes $c_n(SL(\mathbb{Z})) \in H^{2n}(SL(\mathbb{Z});\mathbb{Z})$ of the inclusion $SL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$ (see Proposition 5·2 and Theorem 5·3) and deduce the best upper bound for the order of the Chern classes of all integral and rational representations of discrete groups (see Corollary 5·6).

The paper is organized as follows. Sections 2 and 3 present the mod 2 Bockstein spectral sequence for the spaces BSO and $BSL(\mathbb{F}_p)^+$ respectively. The mod 2 Bockstein spectral sequence and the 2-torsion of the integral cohomology of $BSL(\mathbb{Z})^+$ are computed in Section 4. Finally, Section 5 is devoted to the investigation of the order of the Chern classes of integral and rational representations of discrete groups.

2. *The mod 2 Bockstein spectral sequence for BSO*

The determination of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ is based on cohomological calculations involving the pull-back diagram

$$\begin{tikzcd} (BSL(\mathbb{Z})^+)\hat{}_2 \arrow[r, "h"] \arrow[d, "f_p"] & BSO\hat{}_2 \arrow[d, "c"] \\ (BSL(\mathbb{F}_p)^+)\hat{}_2 \arrow[r, "b"] & BSU\hat{}_2, \end{tikzcd}$$

where $(-)\hat{}_2$ denotes the completion at the prime 2, p any prime $\equiv 3$ or 5 mod 8, h the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, f_p the map induced by the reduction mod p : $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$, c the complexification and b the Brauer lift, and where the homotopy fibres of both horizontal maps are $SU\hat{}_2$ (for details of that construction, see [**AMNY**], where the argument is presented for $GL(\mathbb{Z})$ instead of $SL(\mathbb{Z})$, or [**Au**, chapter 3]; notice also that an unstable version of this computation is given in [**He**]). In order to go through the mod 2 Bockstein spectral sequence for $BSL(\mathbb{Z})^+$, we shall first consider the mod 2 Bockstein spectral sequence for the spaces BSO and $BSL(\mathbb{F}_p)^+$, and the homomorphisms induced in cohomology by the maps h and f_p.

Let us start by looking at the mod 2 Bockstein spectral sequence

$$\begin{aligned} E_1^*(BSO) &\cong H^*(BSO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \ldots, w_j, \ldots] \\ &\Longrightarrow (H^*(BSO;\mathbb{Z})/\mathrm{torsion}) \otimes \mathbb{Z}/2\,. \end{aligned}$$

Its first differential is $d_1 = Sq^1$ and we know by Wu's formula (see for instance [**MT**, part I, p. 138, theorem 5·12]) that for $i \geqslant 1$, $Sq^1(w_{2i}) = w_{2i+1}$, $Sq^1(w_{2i+1}) = 0$, and that $Sq^1(w_j^2) = 0$ for $j \geqslant 1$. Thus, we may deduce that

$$E_2^*(BSO) \cong \mathbb{Z}/2[w_2^2, w_4^2, \ldots, w_{2i}^2, \ldots]$$

is concentrated in degrees $\equiv 0$ mod 4. Since d_r is of degree 1, it is then obvious that $d_r = 0$ for all $r \geqslant 2$. Consequently, we have proved the following result.

Proposition 2·1.

(*a*) *The* mod 2 *Bockstein spectral sequence for BSO has the property that* $E_1^*(BSO) \cong \mathbb{Z}/2[w_j, j \geqslant 2]$ *and* $E_r^*(BSO) \cong E_\infty^*(BSO) \cong \mathbb{Z}/2[w_{2i}^2, i \geqslant 1]$ *for all* $r \geqslant 2$.

(*b*) *All non-trivial* 2*-torsion elements of* $H^*(BSO;\mathbb{Z})$ *have order exactly equal to* 2.

(*c*) $H^*(BSO;\mathbb{Z})/torsion \cong \mathbb{Z}[p_{4i}, i \geqslant 1]$, *where* p_{4i} *is of degree* $4i$ *and represents an element of* $H^{4i}(BSO;\mathbb{Z})$ *whose reduction* mod 2 *is* $w_{2i}^2 \in H^{4i}(BSO;\mathbb{Z}/2)$.

Remark 2·2. The additive and multiplicative structures of the 2-torsion of $H^*(BSO;\mathbb{Z})$ has been obtained a long time ago in [**Brn**, theorem 1·5], and [**F**, theorem 1] (see also [**Bo**, theorem 24·7 and proposition 25·6], [**CV**, theorem 1], and [**ThE**, theorem A]). For completeness, let us recall here its additive structure, which can also be determined by the argument we shall use in the next sections (see Lemma 3·6 and Theorems 3·7 and 4·7): if Ψ denotes the graded $\mathbb{Z}$-algebra $\mathbb{Z}[q_{2i+1}, i \geqslant 1] \otimes \mathbb{Z}[p_{4i}, i \geqslant 1]$ with $\deg(q_{2i+1}) = 2i+1$ and $\deg(p_{4i}) = 4i$, then the 2-torsion subgroup of $H^*(BSO;\mathbb{Z})$ is additively isomorphic to the graded Ψ-module generated by

$$\{G_A \mid A \text{ running over all non-empty finite subsets of } N_1 = \mathbb{N} - \{0\}\}$$

with relations generated by

$$\left\{ 2G_A\,,\ \sum_{i \in A} q_{2i+1} G_{A-\{i\}} \right\}.$$

Here, the element $G_{\{i_1,\ldots,i_t\}} \in H^*(BSO;\mathbb{Z})$ is of degree $2(\sum_{j=1}^t i_j) + 1$ and reduces mod 2 to the class $\sum_{j=1}^t w_{2i_j+1} w_{2i_1} \cdots w_{2i_{(j-1)}} w_{2i_{(j+1)}} \cdots w_{2i_t} \in H^*(BSO;\mathbb{Z}/2)$.

3. *The mod* 2 *Bockstein spectral sequence for* $BSL(\mathbb{F}_p)^+$

Let us consider the space $BSL(\mathbb{F}_p)^+$ for any prime number $p \equiv 5$ mod 8. Its mod 2 cohomology is

$$H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[c_2, c_3, \ldots, c_k, \ldots] \otimes \Lambda_{\mathbb{Z}/2}(e_2, e_3, \ldots, e_k, \ldots)$$

with $\deg(c_k) = 2k$ and $\deg(e_k) = 2k - 1$ (see [**Q**, theorem 1]). The first differential of its mod 2 Bockstein spectral sequence

$$\begin{aligned} E_1^*(BSL(\mathbb{F}_p)^+) \;&\cong\; \mathbb{Z}/2[c_k, k \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(e_k, k \geqslant 2) \\ &\Longrightarrow (H^*(BSL(\mathbb{F}_p)^+; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}/2 \end{aligned}$$

is trivial since $d_1(c_k) = Sq^1(c_k) = 0$ and $d_1(e_k) = Sq^1(e_k) = 0$ for all $k \geqslant 2$ according to [**Ar4**, lemmas 3 and 4]. Thus,

$$E_2^*(BSL(\mathbb{F}_p)^+) \cong E_1^*(BSL(\mathbb{F}_p)^+) \,.$$

In order to understand the higher differentials d_r, let us recall the definition of d_r (see [**Brd**, section 1]). If $x \in E_r^n(BSL(\mathbb{F}_p)^+)$, then there is an element $\widetilde{x} \in H^n(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2^r)$ such that the homomorphism θ_r: $H^n(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2^r) \to H^n(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2)$ induced by the natural surjection $\mathbb{Z}/2^r \twoheadrightarrow \mathbb{Z}/2$ sends $\widetilde{x}$ onto x. Let β_r: $H^n(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2^r) \to H^{n+1}(BSL(\mathbb{F}_p)^+; \mathbb{Z})$ denote the Bockstein homomorphism associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2^r} \mathbb{Z} \longrightarrow \mathbb{Z}/2^r \longrightarrow 0$$

and red_2: $H^{n+1}(BSL(\mathbb{F}_p)^+; \mathbb{Z}) \to H^{n+1}(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2)$ the reduction mod 2. Then, the differential d_r: $E_r^n(BSL(\mathbb{F}_p)^+) \to E_r^{n+1}(BSL(\mathbb{F}_p)^+)$ is defined by

$$d_r(x) = \mathrm{red}_2(\beta_r(\widetilde{x})) \,.$$

Let us apply this to the case of the space $BSL(\mathbb{F}_p)^+$.

Definition 3·1. For any integer $r \geqslant 2$, let $N_r = \{k \in \mathbb{N} \mid v_2(k) = r - 2\}$, where $v_2(-)$ is the 2-adic valuation (in other words, $N_r = \{k = 2^{r-1}i + 2^{r-2} \mid i \geqslant 0\}$).

Remark 3·2. For any prime $p \equiv 5 \bmod 8$ and any integer $r \geqslant 2$, $N_r = \{k \in \mathbb{N} \mid v_2(p^k - 1) = r\}$. In order to check this, it is sufficient to show that $v_2(p^k - 1) = v_2(k) + 2$ for any positive integer k. Let us write $p = 4m + 1$ with m odd. Then $p^k - 1 = \sum_{t=1}^{k} \binom{k}{t} 4^t m^t$. For $t \geqslant 2$, one has

$$\begin{aligned} v_2\left(\binom{k}{t} 4^t m^t\right) &= 2t + v_2\left(\frac{k(k-1)\cdots(k-t+1)}{t!}\right) \\ &\geqslant 2t + v_2(k) - v_2(t!) \geqslant v_2(k) + t + 1 \geqslant v_2(k) + 3 \,, \end{aligned}$$

since $v_2(t!) \leqslant t - 1$. This implies that $v_2(p^k - 1) = v_2(4km) = v_2(k) + 2$.

LEMMA 3·3. *Let p be any prime $\equiv 5 \bmod 8$ and r be any integer $\geqslant 2$. If $k \in N_r$, then the class e_k belongs to $E_s^{2k-1}(BSL(\mathbb{F}_p)^+)$ for all $s \leqslant r$ and $d_r(e_k) = c_k \in E_r^{2k}(BSL(\mathbb{F}_p)^+)$.*

Proof. If k belongs to N_r, then $r = v_2(p^k - 1)$ by Remark 3·2. Thus, according to [**Q**, section 3], e_k is the image of an element $\widetilde{e}_k \in H^{2k-1}(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2^r)$ under the homomorphism θ_r and consequently, $e_k \in E_r^{2k-1}(BSL(\mathbb{F}_p)^+)$. Moreover, it follows from [**Q**, lemma 5] that $d_r(e_k) = \mathrm{red}_2(\beta_r(\widetilde{e}_k)) = c_k$.

We get the complete calculation of the mod 2 Bockstein spectral sequence for the space $BSL(\mathbb{F}_p)^+$.

THEOREM 3·4. *For any prime $p \equiv 5$ mod 8, the* mod 2 *Bockstein spectral sequence for $BSL(\mathbb{F}_p)^+$ satisfies*:

(*a*) $E_2^*(BSL(\mathbb{F}_p)^+) \cong E_1^*(BSL(\mathbb{F}_p)^+) \cong \mathbb{Z}/2[c_k, k \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(e_k, k \geqslant 2)$ *and* $d_2(e_k) = c_k$ *whenever* $k \in N_2 = \{$*odd positive integers*$\}$;

(*b*) *for any* $r \geqslant 3$, $E_r^*(BSL(\mathbb{F}_p)^+) \cong \mathbb{Z}/2[c_k,\ k \in N_s \text{ for } s \geqslant r] \otimes \Lambda_{\mathbb{Z}/2}(e_k,\ k \in N_s \text{ for } s \geqslant r)$ *and* $d_r(e_k) = c_k$ *whenever* $k \in N_r$;

(*c*) $E_\infty^*(BSL(\mathbb{F}_p)^+) = 0$.

By looking at the differential graded $\mathbb{Z}/2$ algebras

$$F_s = \mathbb{Z}/2[c_k, 2 \leqslant k \in N_s] \otimes \Lambda_{\mathbb{Z}/2}(e_k, 2 \leqslant k \in N_s)\,, \quad \delta_s(e_k) = c_k\,, \quad \text{for } s \geqslant 2\,,$$

one can write the E_r terms of the mod 2 Bockstein spectral sequence of $BSL(\mathbb{F}_p)^+$ as follows.

COROLLARY 3·5. *For any prime* $p \equiv 5 \mod 8$ *and for any integer* $r \geqslant 2$, $E_r^*(BSL(\mathbb{F}_p)^+) \cong \bigotimes_{s \geqslant r} F_s$ *with the differential* $d_r = \delta_r$ *on* F_r *and* $d_r = 0$ *on* F_s *when* $s > r$.

The knowledge of the mod 2 Bockstein spectral sequence determines the additive structure of the 2-torsion of the integral cohomology $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ since the elements of the image of d_r detect the elements of order 2^r in $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$. Let us start with the following observation.

LEMMA 3·6. *Consider a set* N *and the differential graded* $\mathbb{Z}/2$ *algebra*

$$D_N = \mathbb{Z}/2[x_n, n \in N] \otimes \Lambda_{\mathbb{Z}/2}(y_n, n \in N)\,,$$

where the differential is a derivation δ *given by* $\delta(y_n) = x_n$. *Then the image of* δ *is the* $\mathbb{Z}/2[x_n, n \in N]$ *module generated by*

$$\left\{ H_A = \sum_{a \in A} x_a \prod_{b \in A - \{a\}} y_b \mid A \text{ running over all non-empty finite subsets of } N \right\}$$

with the relations generated by

$$\left\{ \sum_{a \in A} x_a H_{A-\{a\}} \mid A \text{ running over all non-empty finite subsets of } N \right\}.$$

Proof. Let us write P for the polynomial tensor factor $\mathbb{Z}/2[x_n, n \in N]$ of D_N. Since $\delta^2 = 0$, one has $\delta(P) = 0$ and the fact that δ is a derivation shows that $D_N \twoheadrightarrow \mathrm{Im}(\delta)$ is a morphism of P-modules. The elements of the form $\prod_{a \in A} y_a$, where A runs over all non-empty finite subsets of N, generate D_N as a P-module. Therefore, the image of δ is generated, as a P-module, by the elements $H_A = \delta(\prod_{a \in A} y_a)$. We get obviously the relations $\sum_{a \in A} x_a H_{A-\{a\}} = \delta^2(\prod_{a \in A} y_a) = 0$ and there are no other relations because $H_*(D_N, \delta) \cong \mathbb{Z}/2$.

Let us deduce the following explicit description of the additive structure of the 2-torsion of $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$. Observe in particular that there is no direct summand of order 2 in $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$.

THEOREM 3·7. *Let* p *be any prime* $\equiv 5 \mod 8$ *and consider the graded* $\mathbb{Z}$ *algebra*

$$\Phi = \mathbb{Z}[a_k, k \geqslant 2] \otimes \Lambda_{\mathbb{Z}}(b_k, k \text{ even } \geqslant 2)\,,$$

where $\deg(a_k) = 2k$ *and* $\deg(b_k) = 2k - 1$.

As a graded abelian group, the 2-torsion subgroup of $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ is additively isomorphic to the graded Φ-module generated by

$$\{H_{A,r} \mid r \geqslant 2\,,\ A \text{ running over all non-empty finite subsets of } N_r\}$$

with relations generated by

$$\begin{cases} 2^r H_{A,r}, & \\ a_k H_{A,r} & \text{for all } k \in N_s \text{ with } 2 \leqslant s < r, \\ b_k H_{A,r} & \text{for all } k \in N_s \text{ with } 2 \leqslant s \leqslant r, \\ \displaystyle\sum_{k\in A} a_k H_{A-\{k\},r} & \text{for all } r \geqslant 2,\ A \subset N_r. \end{cases}$$

The element $H_{\{k_1,\dots,k_t\},r} \in H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ is of degree $2(\sum_{j=1}^t k_j) - t + 1$ and reduces mod 2 *to the class $\sum_{j=1}^t c_{k_j} e_{k_1} \cdots e_{k_{(j-1)}} e_{k_{(j+1)}} \cdots e_{k_t} \in H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}/2)$.*

Proof. Let us denote by P_r the polynomial tensor factor $\mathbb{Z}/2[c_k, 2 \leqslant k \in N_r]$ of F_r for $r \geqslant 2$. According to Corollary 3·5, the image of d_r is

$$\mathrm{Im}(d_r) = \mathrm{Im}(\delta_r) \otimes \left(\bigotimes_{s>r} F_s\right)$$

and Lemma 3·6 implies that $\mathrm{Im}(d_r)$ is the $(P_r \otimes (\bigotimes_{s>r} F_s))$-module generated by the $H_{A,r}$s, where A runs over all non-empty finite subsets of N_r, with the relations given by Lemma 3·6. By definition of Φ, the generators a_k and b_k of the $\mathbb{Z}$ algebra Φ are in one-to-one correspondence with the classes c_k and e_k respectively, which generate the $\mathbb{Z}/2$ algebra

$$P_2 \otimes \left(\bigotimes_{s>2} F_s\right) \cong \mathbb{Z}/2[c_k, k \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(e_k, k \text{ even} \geqslant 2)\,.$$

The assertion then follows by gluing together the information on $\mathrm{Im}(d_r)$ for $r \geqslant 2$.

Remark 3·8. The additive structure of $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ has been already calculated in [**Hu**], but in a completely different way.

4. *The mod 2 Bockstein spectral sequence for $BSL(\mathbb{Z})^+$*

Finally, let us investigate the mod 2 Bockstein spectral sequence

$$\begin{aligned} E_1^*(BSL(\mathbb{Z})^+) &\cong \mathbb{Z}/2[w_j, j \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(u_{2k-1}, k \geqslant 2) \\ &\Longrightarrow (H^*(BSL(\mathbb{Z})^+;\mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}/2 \end{aligned}$$

for the mod 2 cohomology of the space $BSL(\mathbb{Z})^+$. Since the induced homomorphism $h^*\colon H^*(BSO;\mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ sends the Stiefel–Whitney classes $w_j \in H^j(BSO;\mathbb{Z}/2)$ onto the corresponding classes $w_j \in H^j(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$, we have again $Sq^1(w_{2i}) = w_{2i+1}$, $Sq^1(w_{2i+1}) = 0$ for $i \geqslant 1$ and we know from [**AMNY**, lemma 12] that $Sq^1(u_{2k-1}) = 0$ for $k \geqslant 2$. Therefore, we obtain the E_2-term as follows:

$$E_2^*(BSL(\mathbb{Z})^+) \cong E_2^*(BSO) \otimes \Lambda_{\mathbb{Z}/2}(u_{2k-1}, k \geqslant 2) \cong \mathbb{Z}/2[w_{2i}^2, i \geqslant 1] \otimes \Lambda_{\mathbb{Z}/2}(u_{2k-1}, k \geqslant 2)\,.$$

Because of the naturality of the mod 2 Bockstein spectral sequence with respect to h^*, we may deduce from Section 1 that all higher differentials d_r are trivial on $\mathbb{Z}/2[w_{2i}^2, i \geqslant 1]$, $r \geqslant 2$.

LEMMA 4·1. *For all positive integers r and i, one has $d_r(u_{4i+1}) = 0$ in $E_r^{4i+2}(BSL(\mathbb{Z})^+)$.*

Proof. This is true if $r = 1$. For $r \geqslant 2$, $d_r(u_{4i+1})$ is of the form $d_r(u_{4i+1}) = \sum_s w(s) \otimes u(s)$, where $w(s)$ is a product of classes w_{2i}^2 $(i \geqslant 1)$ and $u(s)$ a product of classes u_{2k-1} $(k \geqslant 2)$. According to [**AMNY**, proposition 7], the classes u_{2k-1} $(k \geqslant 2)$ are primitive cohomology classes, in other words, $\mu^*(u_{2k-1}) = u_{2k-1} \otimes 1 + 1 \otimes u_{2k-1}$, where μ^* is the coproduct $H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^+ \times BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ provided by the H-space structure of $BSL(\mathbb{Z})^+$. In particular, it follows from the fact that $E_r^*(BSL(\mathbb{Z})^+)$ is a differential Hopf algebra (see [**Brd**, proposition 4·7]) that $\mu^*(d_r(u_{4i+1})) = d_r(\mu^*(u_{4i+1}))$ and consequently that $d_r(u_{4i+1})$ is primitive. However, for $\sum_s w(s) \otimes u(s)$ to be primitive, it is necessary to have $u(s)$ primitive, which is only possible if $u(s) = 1$ or $u(s) = u_{2k-1}$ for some k. In both cases, the element $w(s) \otimes u(s)$ cannot lie in degree $4i + 2$ since $\deg(w(s)) \equiv 0 \mod 4$. Consequently, the sum must be empty and we get $d_r(u_{4i+1}) = 0$.

Now, let us consider any prime $p \equiv 5 \bmod 8$ and the homomorphism

$$f_p^*\colon H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[c_k, k \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(e_k, k \geqslant 2) \longrightarrow H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$$

induced by the reduction $\bmod p$. We shall replace the generators u_{4i-1} (for $i \geqslant 1$) of the exterior subalgebra $\Lambda_{\mathbb{Z}/2}(u_{2k-1}, k \geqslant 2)$ of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ by

$$\varepsilon_{4i-1} = f_p^*(e_{2i}) = u_{4i-1} + \sum_{j=2}^{2i-2} w_j^2 u_{4i-2j-1} \in H^{4i-1}(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$$

(see [**AMNY**, theorem 13]). Thus, the first two terms of the mod 2 Bockstein spectral sequence of $BSL(\mathbb{Z})^+$ can be expressed as

$$\begin{aligned} E_1^*(BSL(\mathbb{Z})^+) &\cong H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[w_j, j \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{4i-1}, i \geqslant 1) \otimes \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)\,, \\ E_2^*(BSL(\mathbb{Z})^+) &\cong \mathbb{Z}/2[w_{2i}^2, i \geqslant 1] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{4i-1}, i \geqslant 1) \otimes \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)\,. \end{aligned}$$

Let us consider again the sets of integers $N_r = \{k = 2^{r-1}i + 2^{r-2} \mid i \geqslant 0\}$ introduced in Definition 3·1.

LEMMA 4·2. *For any $r \geqslant 3$, if $k \in N_r$, then ε_{2k-1} belongs to $E_s^*(BSL(\mathbb{Z})^+)$ for all $s \leqslant r$ and $d_r(\varepsilon_{2k-1}) = w_k^2$.*

Proof. By naturality of the mod 2 Bockstein spectral sequence with respect to f_p^*, this follows from the equality

$$d_r(e_k) = c_k \quad \text{for } k \in N_r$$

given by Lemma 3·3, from the definition $\varepsilon_{2k-1} = f_p^*(e_k)$ (where k is even since $k \in N_r$ with $r \geqslant 3$) and the formula $f_p^*(c_k) = w_k^2$ (see [**Ar3**, Lemma 1·4]).

This argument implies also the vanishing of d_2, because $w_k^2 = 0$ in $E_2^*(BSL(\mathbb{Z})^+)$ when k belongs to $N_2 = \{\text{odd positive integers}\}$. Let us summarize the information we obtain on $E_r^*(BSL(\mathbb{Z})^+)$.

THEOREM 4·3. *The* mod 2 *Bockstein spectral sequence for $BSL(\mathbb{Z})^+$ satisfies*:

(*a*) $E_1^*(BSL(\mathbb{Z})^+) \cong H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_j, j \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{4i-1}, i \geqslant 1) \otimes$

$\Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$ *and* $d_1(w_{2i}) = w_{2i+1}$, $d_1(w_{2i+1}) = 0$, *and* d_1 *is trivial on all classes* ε_{4i-1} *and* u_{4i+1} $(i \geqslant 1)$.

(*b*) $E_2^*(BSL(\mathbb{Z})^+) \cong E_3^*(BSL(\mathbb{Z})^+) \cong \mathbb{Z}/2[w_k^2, k \text{ even} \geqslant 2] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{2k-1}, k \text{ even} \geqslant 2) \otimes \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$.

(*c*) *For any* $r \geqslant 3$,

$$E_r^*(BSL(\mathbb{Z})^+) \cong \mathbb{Z}/2[w_k^2\,,\, k \in N_s \text{ for } s \geqslant r] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{2k-1},\, k \in N_s \text{ for } s \geqslant r) \otimes \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$$

and $d_r(\varepsilon_{2k-1}) = w_k^2$ *whenever* $k \in N_r$.

(*d*) $E_\infty^*(BSL(\mathbb{Z})^+) \cong \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$.

The calculation of the differentials in that mod 2 Bockstein spectral sequence enables us to split its E_1-term as a tensor product of differential graded $\mathbb{Z}/2$-algebras:

$$E_1^*(BSL(\mathbb{Z})^+) \cong D_1 \otimes \left(\bigotimes_{s \geqslant 3} D_s \right) \otimes D_\infty\,,$$

where

$$D_1 = \mathbb{Z}/2[w_{2i+1}, i \geqslant 1] \otimes \Lambda_{\mathbb{Z}/2}(w_{2i}, i \geqslant 1)\,, \quad \delta_1(w_{2i}) = w_{2i+1}\,,$$
$$D_s = \mathbb{Z}/2[w_k^2, k \in N_s] \otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{2k-1}, k \in N_s)\,, \quad \delta_s(\varepsilon_{2k-1}) = w_k^2\,, \quad \text{for } s \geqslant 3\,,$$
$$D_\infty = \Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)\,, \quad \delta_\infty = 0\,.$$

The spectral sequence can then be described in the following simple way.

COROLLARY 4·4. *The mod 2 Bockstein spectral sequence for* $BSL(\mathbb{Z})^+$ *satisfies*:

(*a*) $E_1^*(BSL(\mathbb{Z})^+) \cong D_1 \otimes (\bigotimes_{s \geqslant 3} D_s) \otimes D_\infty$ *and the first differential is* $d_1 = \delta_1$ *on* D_1 *and* $d_1 = 0$ *on* D_s *when* $3 \leqslant s \leqslant \infty$.

(*b*) $E_2^*(BSL(\mathbb{Z})^+) \cong E_3^*(BSL(\mathbb{Z})^+)$ *and for* $r \geqslant 3$, $E_r^*(BSL(\mathbb{Z})^+) \cong (\bigotimes_{s \geqslant r} D_s) \otimes D_\infty$ *with the differential* $d_r = \delta_r$ *on* D_r *and* $d_r = 0$ *on* D_s *when* $r < s \leqslant \infty$.

(*c*) $E_\infty^*(BSL(\mathbb{Z})^+) \cong D_\infty$.

Remark 4·5. Let us mention that the mod 2 Bockstein spectral sequence for the group $SL_3(\mathbb{Z}[\frac{1}{2}])$ has been recently computed (see [**He**, Section 4·3]).

The following interesting observations are immediate consequences of Theorem 4·3(*b*) and (*d*) and Corollary 4·4(*b*) and (*c*).

COROLLARY 4·6.

(a) *There is no cyclic direct summand of order* 4 *in* $H^*(BSL(\mathbb{Z})^+; \mathbb{Z})$.

(b) *The set of all non-trivial elements of* $\Lambda_{\mathbb{Z}/2}(u_{4i+1}, i \geqslant 1)$ *is in one-to-one correspondence with an additive basis of* $H^*(BSL(\mathbb{Z})^+; \mathbb{Z})/\text{torsion}$.

By applying Lemma 3·6 again, we get an explicit description of the additive structure of the 2-torsion of the integral cohomology $H^*(BSL(\mathbb{Z})^+; \mathbb{Z})$. In order to formulate the main result of this section, let us use again the notation introduced in Remark 2·2 and Definition 3·1: $N_1 = \mathbb{N} - \{0\}$ and $N_r = \{k = 2^{r-1}i + 2^{r-2} \mid i \geqslant 0\}$ for $r \geqslant 3$.

THEOREM 4·7. *Consider the graded* $\mathbb{Z}$*-algebra*

$$\Omega = \mathbb{Z}[\omega_{k,1}, k \in N_1] \otimes \mathbb{Z}[\omega_{k,r}, k \in N_r \text{ with } r \geqslant 3] \otimes \Lambda_{\mathbb{Z}}(z_{2k-1}, k \geqslant 2)\,,$$

where $\deg(\omega_{k,1}) = 2k+1$, $\deg(\omega_{k,r}) = 2k$ *when* $r \geqslant 3$ *and* $\deg(z_{2k-1}) = 2k-1$.

As a graded abelian group, the 2-torsion subgroup of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ is additively isomorphic to the graded $\mathbf{\Omega}$-module generated by

$$\{J_{A,r}\,|\,r = 1 \text{ or } r \geqslant 3\,,\ A \text{ running over all non-empty finite subsets of } N_r\}$$

with relations generated by

$$\begin{cases} 2^r J_{A,r}, & \\ \omega_{k,s} J_{A,r} & \text{for all } s < r \text{ and all } k \in N_s,\ \text{when } r \geqslant 3, \\ z_{2k-1} J_{A,r} & \text{for all } k \in N_s \text{ with } 3 \leqslant s \leqslant r,\ \text{when } r \geqslant 3, \\ \sum_{k\in A} \omega_{k,r} J_{A-\{k\},r} & \text{for } r = 1 \text{ and } r \geqslant 3,\ A \subset N_r. \end{cases}$$

The element $J_{\{k_1,\dots,k_t\},1} \in H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ is of degree $2(\sum_{j=1}^t k_j) + 1$ and reduces mod 2 *to the class $\sum_{j=1}^t w_{2k_j+1} w_{2k_1} \cdots w_{2k_{(j-1)}} w_{2k_{(j+1)}} \cdots w_{2k_t} \in H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$. For $r \geqslant 3$, the element $J_{\{k_1,\dots,k_t\},r} \in H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ is of degree $2(\sum_{j=1}^t k_j) - t + 1$ and reduces* mod 2 *to the class $\sum_{j=1}^t w_{k_j}^2 \varepsilon_{2k_1-1} \cdots \varepsilon_{2k_{(j-1)}-1} \varepsilon_{2k_{(j+1)}-1} \cdots \varepsilon_{2k_t-1} \in H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$.*

Proof. For $r = 1$ or $r \geqslant 3$, let us call P_r the polynomial tensor factor of D_r. Because of the splitting of $E_r^*(BSL(\mathbb{Z})^+)$ given by Corollary 4·4, the image of d_r is

$$\mathrm{Im}(d_r) = \mathrm{Im}(\delta_r) \otimes \left(\bigotimes_{s>r} D_s\right) \otimes D_\infty$$

and Lemma 3·6 implies that $\mathrm{Im}(d_r)$ is the $(P_r \otimes (\bigotimes_{s>r} D_s) \otimes D_\infty)$ module generated by the $J_{A,r}$s, where A runs over all non-empty finite subsets of N_r, with the relations provided by Lemma 3·6. Now, let us denote by $\mathbf{\Omega}$ the graded $\mathbb{Z}$-algebra $\mathbf{\Omega} = \mathbb{Z}[\omega_{k,1}, k \in N_1] \otimes \mathbb{Z}[\omega_{k,r}, k \in N_r \text{ with } r \geqslant 3] \otimes \Lambda_{\mathbb{Z}}(z_{2k-1}, k \geqslant 2)$ whose generators are in one-to-one correspondence with those of

$$P_1 \otimes \left(\bigotimes_{s\geqslant 3} D_s\right) \otimes D_\infty \cong \mathbb{Z}/2[w_{2k+1}, k \geqslant 1] \otimes \mathbb{Z}/2[w_k^2, k \text{ even} \geqslant 2]$$
$$\otimes \Lambda_{\mathbb{Z}/2}(\varepsilon_{2k-1}, k \text{ even} \geqslant 2) \otimes \Lambda_{\mathbb{Z}/2}(u_{2k-1}, k \text{ odd} \geqslant 3)$$

as follows: $\omega_{k,1}$ corresponds to w_{2k+1}, $\omega_{k,r}$ to w_k^2 when $r \geqslant 3$ and $k \in N_r$, z_{2k-1} to ε_{2k-1} when k is even and to u_{2k-1} when k is odd. The assertion of the theorem then follows by gluing together the information on the elements of order 2^r in $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ given by the determination of $\mathrm{Im}(d_r)$ for $r \geqslant 1$.

Remark 4·8. The isomorphism established in Theorem 4·7 is an additive isomorphism: for instance, for $3 \leqslant s < r$, $k \in N_s$ and A a non-empty finite subset of N_r, the product of the elements of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ corresponding to $J_{\{k\},s}$ and $J_{A,r}$ under that isomorphism is non-trivial, even if the reduction mod 2 of $J_{\{k\},s}$ is w_k^2, which is the generator of D_s corresponding to the generator $\omega_{k,s}$ of $\mathbf{\Omega}$ (that product is actually an element of order 2^s in $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ which reduces mod 2 to the reduction mod 2 of $(\sum_{k\in A} \omega_{k,r} \prod_{i\in A-\{k\}} z_{2i-1})\, J_{\{k\},s}$, where $(\sum_{k\in A} \omega_{k,r} \prod_{i\in A-\{k\}} z_{2i-1}) \in \mathbf{\Omega}$). More generally, the above mod 2 Bockstein spectral sequence calculation provides the following multiplicative relations mod 2 between the additive generators of the 2-torsion subgroup of $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ given by Theorem 4·7.

If $1 \leqslant s < r$, then it is obvious that

$$J_{A,r}J_{B,s} \equiv \left(\sum_{k \in A} \omega_{k,r} \prod_{i \in A-\{k\}} z_{2i-1} \right) J_{B,s} \mod 2\,.$$

If $r \geqslant 3$ and if red_2 denotes again the reduction mod 2, one has

$$\begin{aligned}\mathrm{red}_2(J_{A,r}J_{B,r}) &= \mathrm{red}_2(J_{A,r})\,\delta_r\left(\prod_{j\in B}\varepsilon_{2j-1}\right)\\ &= \delta_r\left(\mathrm{red}_2(J_{A,r})\prod_{j\in B}\varepsilon_{2j-1}\right)\\ &= \delta_r\left(\sum_{k\in A} w_k^2 \prod_{i\in A-\{k\}}\varepsilon_{2i-1}\prod_{j\in B}\varepsilon_{2j-1}\right).\end{aligned}$$

The fact that the classes ε_{2i-1} are exterior implies that

$$\begin{aligned}\mathrm{red}_2(J_{A,r}J_{B,r}) &= \delta_r\left(\sum_{\substack{k\in A \text{ such that}\\ A-\{k\}\cap B=\varnothing}} w_k^2 \prod_{i\in A-\{k\}\cup B}\varepsilon_{2i-1}\right)\\ &= \sum_{\substack{k\in A \text{ such that}\\ A-\{k\}\cap B=\varnothing}} w_k^2\,\delta_r\left(\prod_{i\in A-\{k\}\cup B}\varepsilon_{2i-1}\right)\end{aligned}$$

and finally that

$$J_{A,r}J_{B,r} \equiv \sum_{\substack{k\in A \text{ such that}\\ A-\{k\}\cap B=\varnothing}} \omega_{k,r}\, J_{A-\{k\}\cup B,r} \mod 2\,.$$

If $r = 1$, the formula

$$J_{A,1}J_{B,1} \equiv \sum_{\substack{k\in A \text{ such that}\\ A-\{k\}\cap B=\varnothing}} \omega_{k,1}\, J_{A-\{k\}\cup B,1} \mod 2$$

does still hold but the classes ε_{2i-1} should be replaced by w_{2i} in the argument.

Finally, the above computation helps us to understand, at the prime 2, the homomorphisms $h^*\colon H^*(BSO;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ and $f_p^*\colon H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and by the reduction mod p, when $p \equiv 5 \bmod 8$.

THEOREM 4·9.

(a) *The homomorphism* $h^*\colon H^*(BSO;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ *is injective on the torsion classes of* $H^*(BSO;\mathbb{Z})$.

(b) *For every generator of infinite order* p_{2k} *in* $H^{2k}(BSO;\mathbb{Z})$ *with* k *even,* $h^*(p_{2k})$ *is an element of order* 2^r *if* k *belongs to* N_r, $r \geqslant 3$ (*up to odd torsion*).

(c) *For* $p \equiv 5 \mod 8$, *the image of any generator of any cyclic direct summand of order* 4 *in* $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ *under the homomorphism* $f_p^*: H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$ *has order* 2 *in* $H^*(BSL(\mathbb{Z})^+;\mathbb{Z})$.

(d) *For any* $r \geqslant 3$, *the homomorphism* f_p^* (*with* $p \equiv 5 \bmod 8$) *is injective on all cyclic direct summands of order* 2^r *in* $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$.

Proof. Assertion (a) is obvious since the Stiefel–Whitney classes $w_j \in H^j(BSO;\mathbb{Z})$ correspond to the elements $w_j \in H^j(BSL(\mathbb{Z})^+;\mathbb{Z})$ via h^*. The fact that the reduction mod 2 of p_{2k} is $w_k^2 \in H^{2k}(BSO;\mathbb{Z}/2)$, by Proposition 2·1$(c)$, and that $d_r(\varepsilon_{2k-1}) = w_k^2$ if $k \in N_r$, according to Theorem 4·3(c), implies (b). According to Theorem 3·7, the generators of the cyclic direct summands of order 2^r in $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z})$ belong to the Φ-module generated by the $H_{A,r}$'s, where $A = \{k_1, \ldots, k_t\}$ is a finite subset of N_r and one has:

$$\begin{aligned} f_p^*(\mathrm{red}_2(H_{A,r})) &= f_p^*\left(\sum_{j=1}^{t} c_{k_j} e_{k_1} \cdots e_{k_{(j-1)}} e_{k_{(j+1)}} \cdots e_{k_t}\right) \\ &= \sum_{j=1}^{t} w_{k_j}^2 \varepsilon_{2k_1-1} \cdots \varepsilon_{2k_{(j-1)}-1} \varepsilon_{2k_{(j+1)}-1} \cdots \varepsilon_{2k_t-1} \end{aligned}$$

since $f_p^*(c_{k_j}) = w_{k_j}^2$ by [**Ar3**, lemma 1·4]. If $r = 2$, $N_2 = \{\text{odd positive integers}\}$ and Assertion (c) follows from the equality

$$\begin{aligned} d_1(w_{k_j-1} w_{k_j} \varepsilon_{2k_1-1} \cdots \varepsilon_{2k_{(j-1)}-1} \varepsilon_{2k_{(j+1)}-1} \cdots \varepsilon_{2k_t-1}) \\ = w_{k_j}^2 \varepsilon_{2k_1-1} \cdots \varepsilon_{2k_{(j-1)}-1} \varepsilon_{2k_{(j+1)}-1} \cdots \varepsilon_{2k_t-1} \end{aligned}$$

when k_j is odd. If $r \geqslant 3$, then Assertion (d) is a consequence of

$$d_r\left(\prod_{j=1}^{t} \varepsilon_{2k_j-1}\right) = \sum_{j=1}^{t} w_{k_j}^2 \varepsilon_{2k_1-1} \cdots \varepsilon_{2k_{(j-1)}-1} \varepsilon_{2k_{(j+1)}-1} \cdots \varepsilon_{2k_t-1}$$

in $E_r^*(BSL(\mathbb{Z})^+)$.

5. *Chern classes of integral representations of groups*

For $n \geqslant 1$, let us call $c_n(SL(\mathbb{Z})) \in H^{2n}(SL(\mathbb{Z});\mathbb{Z})$ the nth Chern class of the inclusion $\sigma\colon SL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$, i.e. $c_n(SL(\mathbb{Z})) = \sigma^*(\overline{c}_n)$, where $\overline{c}_n$ denotes the nth universal Chern class of degree $2n$ in $H^*(BGL(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_n, \ldots]$ and $\sigma^*\colon H^*(BGL(\mathbb{C});\mathbb{Z}) \to H^*(BSL(\mathbb{Z});\mathbb{Z}) \cong H^*(SL(\mathbb{Z});\mathbb{Z})$ the homomorphism induced by σ. The order of $c_n(SL(\mathbb{Z}))$ has been only determined up to a factor 2.

Definition 5·1. For any positive even integer n, let E_n be the denominator of B_n/n, where B_n is the nth Bernoulli number; for instance, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and $E_2 = 12$, $E_4 = 120$, $E_6 = 252$.

Proposition 5·2. *The Chern classes $c_n(SL(\mathbb{Z}))$ are all torsion classes. If n is odd, then $c_n(SL(\mathbb{Z}))$ is of order 2 in $H^{2n}(SL(\mathbb{Z});\mathbb{Z})$. If n is even, then the order of $c_n(SL(\mathbb{Z}))$ in $H^{2n}(SL(\mathbb{Z});\mathbb{Z})$ is equal to $2E_n$ when $n \equiv 2$ mod 4, and to E_n or $2E_n$ when $n \equiv 0$ mod 4.*

Proof. See [**EM1**, section 5], [**EM2**, main theorem], and [**Ar1**, Einleitung and Korollar 2·5].

The determination of the exact order of $c_n(SL(\mathbb{Z}))$ in the case where $n \equiv 0$ mod 4 now follows from the mod 2 Bockstein spectral sequence calculations presented in Theorem 4·3 and Corollary 4·4.

THEOREM 5·3. *For any positive even integer n, the Chern class $c_n(SL(\mathbb{Z}))$ is a torsion class of order $2E_n$ in $H^{2n}(SL(\mathbb{Z});\mathbb{Z})$.*

Proof. The odd-primary part of the order of $c_n(SL(\mathbb{Z}))$ is given by Proposition 5·2. For all integers $r \geqslant 3$, we know from Lemma 4·2 that $d_r(\varepsilon_{2n-1}) = w_n^2$ when $n \in N_r$. Therefore, the fact that $\mathrm{red}_2(c_n(SL(\mathbb{Z}))) = w_n^2$ for all $n \geqslant 1$ (see [**MT**, part I, p. 137, theorem 5·11]) implies that for n even, $c_n(SL(\mathbb{Z}))$ is of order 2^r in $H^n(SL(\mathbb{Z});\mathbb{Z})$ when $n \in N_r$ ($r \geqslant 3$). On the other hand, according to von Staudt's theorem (see [**BS**, p. 384, theorem 4]), 2^t divides E_n if and only if 2^{t-1} divides n. Thus, if $n \in N_r = \{n \in \mathbb{N} \mid v_2(n) = r-2\}$, then the 2-primary part of E_n is 2^{r-1}. Consequently, the 2-primary parts of the order of $c_n(SL(\mathbb{Z}))$ and of the integer $2E_n$ coincide for any even integer $n \geqslant 2$.

Remark 5·4. The same result holds for the Chern classes $c_n(GL(\mathbb{Z}))$ of the general linear group $GL(\mathbb{Z})$ as there is a homotopy equivalence $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$ (see for instance [**Ar1**, lemma 1·2]).

The knowledge of the order of the Chern classes of $SL(\mathbb{Z})$ produces the following result on the Chern classes of the linear groups over the field of rationals $\mathbb{Q}$.

COROLLARY 5·5. *The Chern classes $c_n(SL(\mathbb{Q}))$ and $c_n(GL(\mathbb{Q}))$ are all torsion classes. If n is odd, they are of order 2. If n is even, the order of $c_n(SL(\mathbb{Q}))$ and of $c_n(GL(\mathbb{Q}))$ is equal to $2E_n$.*

Proof. Since the order of $c_n(SL(\mathbb{Q}))$ and of $c_n(GL(\mathbb{Q}))$ is a positive multiple of the order of $c_n(SL(\mathbb{Z}))$, a lower bound for it is given by Proposition 5·2 and Theorem 5·3. The assertion then follows from [**Ar2**, theorem 11].

For any complex representation $\rho\colon G \to GL(\mathbb{C})$ of any discrete group G, the Chern classes of ρ are $c_n(\rho) = \rho^*(\bar{c}_n) \in H^{2n}(G;\mathbb{Z})$, where ρ^* is the induced homomorphism $H^{2n}(BGL(\mathbb{C});\mathbb{Z}) \to H^{2n}(BG;\mathbb{Z}) \cong H^{2n}(G;\mathbb{Z})$. Of course, the above calculations produce the following consequence for any integral representation $\rho\colon G \to GL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$ or any rational representation $\rho\colon G \to GL(\mathbb{Q}) \hookrightarrow GL(\mathbb{C})$ of any discrete group G.

COROLLARY 5·6. *The best upper bound for the order of the* nth *Chern class $c_n(\rho)$ of any integral or rational representation ρ of any discrete group G is equal to 2 when n is odd and to $2E_n$ when n is even.*

Proof. Since ρ is an integral representation, respectively a rational representation, $c_n(\rho)$ is the image of $c_n(GL(R))$ under the induced homomorphism $H^{2n}(GL(R);\mathbb{Z}) \to H^{2n}(G;\mathbb{Z})$, where $R = \mathbb{Z}$, respectively $R = \mathbb{Q}$. Consequently, the order of $c_n(\rho)$ divides the order of $c_n(GL(R))$ which has been obtained in Proposition 5·2, Theorem 5·3, Remark 5·4 and Corollary 5·5. It turns out that the order of $c_n(SL(\mathbb{Z}))$ is the best possible upper bound since one can choose $G = SL(\mathbb{Z})$ and ρ the inclusion into $GL(\mathbb{C})$.

Remark 5·7. The assertion of Corollary 5·6 is of particular interest because the best upper bound for the order of the nth Chern class of any integral representation of any finite group is smaller, i.e. only equal to E_n (see [**EM1**, theorem 4·12], and [**ThC**, p. 89]).

We wish to dedicate this paper to Guido Mislin on the occasion of his sixtieth birthday

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