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# $\gamma$-$\omega$ Hough transform-elimination of quantization noise and linearization of voting curves in the $\rho$-&thetas; parameter space.

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# $\gamma - \omega$ Hough Transform
## — Elimination of quantization noise and linearization of voting curves in the $\rho - \theta$ parameter space. —

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**Abstract**

*It is known that the $\rho - \theta$ parameter space[1] has inherent bias[2], and it has been treated as the appearance of the white noise in the image space[3]. In this paper, we first show that the bias is caused by the uniform quantization of the parameter space. To eliminate the bias, a new parameter $\gamma$ representing a non-uniform quantization along the $\rho-$axis is introduced and the $\gamma - \theta$ parameter space is constructed. In this space, the uniform quantization does not introduce any bias. Then, by a non-linear transformation of $\theta$, the $\gamma - \omega$ parameter space is derived, in which a voting curve becomes a pair of straight lines preserving the unbiasedness.*

## 1 Introduction

The $\rho - \theta$ Hough transform is one of the most effective methods to detect straight lines from noisy images. Votes from a feature point $(X, Y)$ form a curve in the $\rho - \theta$ parameter space defined by the following equation:

$$\rho = X \cos\theta + Y \sin\theta \quad (1)$$

In practice, the parameter space is quantized into cells for the vote accumulation. A cell corresponds to a line in the image space, and the accumulated vote in the cell corresponds to the number of feature points included in the line. However, those cells corresponding to lines which consist of the same number of pixels do not always catch the same number of votes. This means that the parameter space has a *bias*. This is a well known problem of the $\rho - \theta$ parameter space and the bias has been treated as the *performance characteristics* of the parameter space[3]. The performance analysis was conducted to investigate how the white noise in the image space appears in the parameter space considering both image and parameter spaces as continuous.

In this paper, we show another analysis of the bias. Our analysis is based on the quantization process in the parameter space. For digital images, feature points are represented as pixels, that is, the image space is already uniformly quantized. If the quantization of the parameter space is not suitable for digital images, quantization noise is introduced into the parameter space.

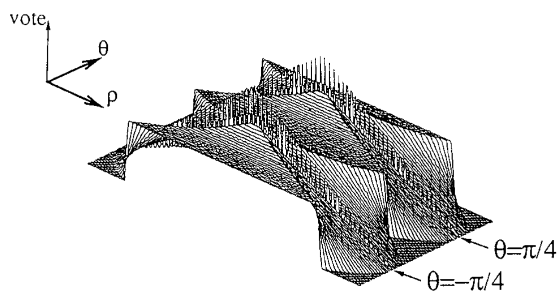


Figure 1: Total vote surface of the $\rho - \theta$ parameter space.

Figure 1 shows the shape of accumulated votes from all pixels, called *total vote surface*, in the uniformly quantized $\rho - \theta$ parameter space. In this figure, quantization noise appears as:

- Global bias along the $\theta-$axis
- Moiré pattern along the $\rho-$axis at $\theta = \pm\pi/4$

It is obvious from this figure that the uniform quantization of the $\rho - \theta$ space is not suitable for digital images.

The first topic in this paper is the elimination of such quantization noise in the $\rho-\theta$ space. By introducing the parameter $\gamma$ which represents a non-uniform quantization along the $\rho-$axis, the $\gamma - \theta$ space is derived. Then, it is shown that a uniform quantization of the parameter space does not introduce any quantization noise.

The second topic is the linearization of voting curves in the $\gamma - \theta$ parameter space. By introducing the parameter $\omega$ representing the non-uniform quantization along the $\theta-$axis, the $\gamma - \omega$ parameter space



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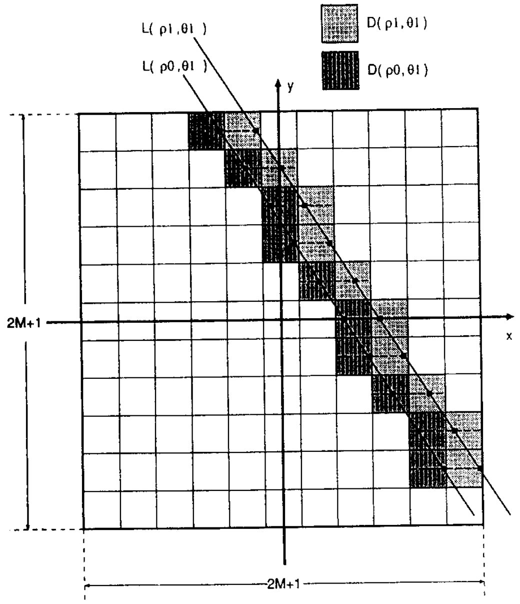


Figure 2: Image space, pixels, analog line, and digital line.

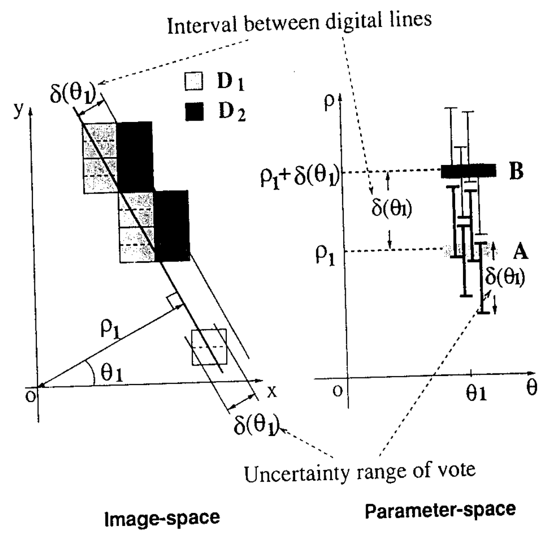


Figure 3: Interval between disjoint digital lines and uncertainty range of vote.

is derived, in which voting curves become piecewise straight lines.

In section 2, some definitions are given. The quantization noise in the $\rho-\theta$ parameter space is analyzed and the $\gamma-\theta$ parameter space is introduced in section 3. In section 4, the gradient of a voting curve in the $\gamma-\theta$ parameter space is analyzed, and the $\gamma-\omega$ parameter space is derived.

## 2 Definitions

First we define *image space*, *pixel*, *analog line*, and *digital line* as follows:

**Definition 1** Image space $P$ *is defined as* $(2M+1)\times(2M+1)$ *2D square region:*

$$P=\left\{(x,y)\in \mathbf{E}^2 \mid -M-\frac{1}{2}\leq x,y<M+\frac{1}{2}\right\}, \quad (2)$$

*where* $M$ *is a positive integer and* $\mathbf{E}^2$ *is the 2D Euclidean space.*

**Definition 2** Pixel $p(X,Y)$ *is defined as a* $1\times 1$ *square region around point* $(X,Y)$ *:*

$$p(X,Y)=\left\{(x,y)\in P \mid X-\frac{1}{2}\leq x<X+\frac{1}{2},\ Y-\frac{1}{2}\leq y<Y+\frac{1}{2}\right\}, \quad (3)$$

*where* $(X,Y)$ *is a integer point in* $P$.

**Definition 3** Analog line $L(\rho,\theta)$ *is defined as follows:*

$$L(\rho,\theta)=\{(x,y)\in P \mid \rho=x\cos\theta+y\sin\theta\}, \quad (4)$$

*where* $-\pi/2\leq\theta<\pi/2$ *.*

**Definition 4** Digital line $D(\rho,\theta)$ *determined by analog line* $L(\rho,\theta)$ *is defined as follows:*

$$D(\rho,\theta)=\begin{cases}\left\{p(X,Y) \mid X-\frac{1}{2}\leq\frac{(\rho-Y\sin\theta)}{\cos\theta}<X+\frac{1}{2}\right\} & ,\mid\theta\mid\leq\pi/4\\ \left\{p(X,Y) \mid Y-\frac{1}{2}\leq\frac{(\rho-X\cos\theta)}{\sin\theta}<Y+\frac{1}{2}\right\} & ,\mid\theta\mid>\pi/4\end{cases} \quad (5)$$

The objects defined above are illustrated in figure 2.

## 3 The $\gamma-\theta$ parameter space

Since a pixel is not a point but a region, the vote from a pixel should be located within the range $\delta(\theta)$ along the $\rho-$axis, which represents the uncertainty of voting location (figure 3). $\delta(\theta)$ is calculated as follows:

$$\delta(\theta)=\max(\mid\sin\theta\mid,\mid\cos\theta\mid) \quad (6)$$

A set of digital lines, which are mutually disjoint and adjacent, can decompose the image space into a disjoint union (figure 5). Such digital lines are located at regular intervals of $\delta(\theta)$ along the $\rho-$axis in the $\rho-\theta$ space (figure 3).

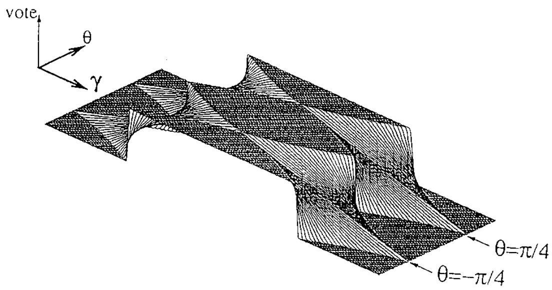


Figure 4: Total vote surface of the $\gamma-\theta$ parameter space.

The uncertainty range of voting location and the interval of disjoint digital lines denote different objects. But, these values happen to take the same value.

The density of disjoint digital lines along the $\rho-$axis is $1/\delta(\theta)$ which varies with parameter $\theta$. It appears as the global bias along the $\theta-$axis.

At $\theta=\pm\pi/4$, the voting location can be calculated as follows:

$$\rho = X\cos\theta + Y\sin\theta = (X \pm Y)/\sqrt{2} \tag{7}$$

Since $X \pm Y$ is an integer value, the voting points are located at regular intervals of $1/\sqrt{2}$. Thus the Moiré pattern appears at $\theta=\pm\pi/4$ unless we use $n/\sqrt{2}$ as the size of cells along the $\rho-$axis ($n=1,2,\cdots$). Unfortunately, if we used this size of cells, another Moiré pattern would appear at $\theta=0,\pm\pi/2$, because at these $\theta$s the voting points are located at regular intervals of 1.

Consequently, we can conclude that these quantization noise in the $\rho-\theta$ parameter space, such as the bias and Moiré, are caused by the mismatch between *the size of cells along the $\rho-$axis* and *the interval of digital lines.* It is obvious that the quantization noise can be eliminated by adopting $\delta(\theta)$ as the size of cells.

Instead of adopting $\delta(\theta)$ as the size of cells, we introduce a new parameter $\gamma$, and vote by the following transformation preserving the uniform quantization of the parameter space:

$$\gamma(\theta) = \frac{X\cos\theta + Y\sin\theta}{\delta(\theta)} \tag{8}$$

We call this parameter space the $\gamma-\theta$ parameter space, in which uniform quantization does not introduce any quantization noise or bias. The total vote surface of the $\gamma-\theta$ parameter space is shown in figure 4.

The essence of the $\gamma-\theta$ parameter space is the decomposition of the image space into disjoint unions, each of which consists of mutually disjoint digital lines as shown in figure 5. This means that a cell always corresponds to a digital line, and the set of cells along

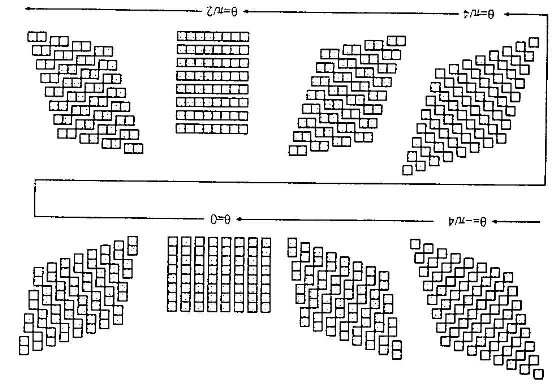

Figure 5: Decomposition of the image space into disjoint digital lines. Each digital line corresponds to a cell in the $\gamma-\theta$ parameter space.

the $\gamma-$axis at any $\theta$ represents a disjoint union of the image space. The total vote of the $\gamma-\theta$ space forms a plain vote surface, because any digital line which comes across the opposite sides of the image frame always includes $2M+1$ pixels.

## 4 The $\gamma-\omega$ parameter space

In the $\gamma-\theta$ space, a voting curve, represented as a sine curve in the $\rho-\theta$ space, becomes a highly nonlinear curve with knots. Hence, it is hard to analyze the relationship between a voting curve and a feature point in the image space.

Here we introduce a new parameter $\omega$ and consider $\theta$ is the value of function $\theta(\omega)$. The gradient $\partial\gamma/\partial\omega$ can be calculated as follows:

$$\frac{\partial\gamma(\theta(\omega))}{\partial\omega} = \begin{cases} \dfrac{d\theta(\omega)}{d\omega}\dfrac{X}{\sin^2\theta(\omega)}, & -\dfrac{\pi}{2} \le \theta(\omega) < -\dfrac{\pi}{4} \\ \dfrac{d\theta(\omega)}{d\omega}\dfrac{Y}{\cos^2\theta(\omega)}, & -\dfrac{\pi}{4} \le \theta(\omega) < \dfrac{\pi}{4} \\ -\dfrac{d\theta(\omega)}{d\omega}\dfrac{X}{\sin^2\theta(\omega)}, & \dfrac{\pi}{4} \le \theta(\omega) < \dfrac{\pi}{2} \end{cases} \tag{9}$$

By setting $d\theta(\omega)/d\omega$ as follows, we can make gradient $\partial\gamma/\partial\omega$ constant in each interval:

$$\frac{d\theta(\omega)}{d\omega} = \begin{cases} \alpha\sin^2\theta(\omega), & -\dfrac{\pi}{2} \le \theta(\omega) < -\dfrac{\pi}{4} \\ \alpha\cos^2\theta(\omega), & -\dfrac{\pi}{4} \le \theta(\omega) < \dfrac{\pi}{4} \\ \alpha\sin^2\theta(\omega), & \dfrac{\pi}{4} \le \theta(\omega) < \dfrac{\pi}{2} \end{cases} \tag{10}$$

, where $\alpha$ is a constant.

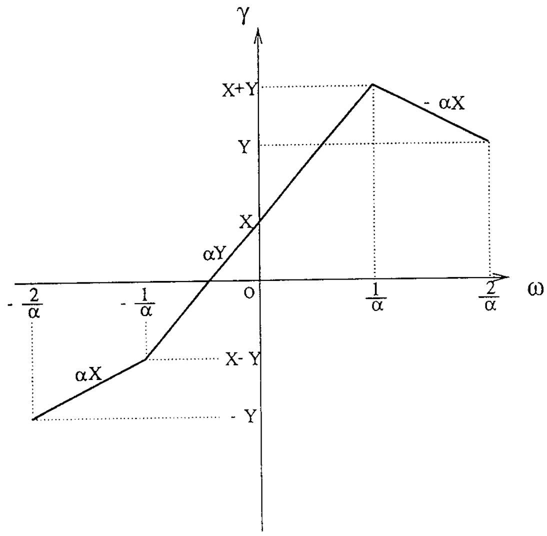


Figure 6: Voting curve in the $\gamma - \omega$ parameter space.

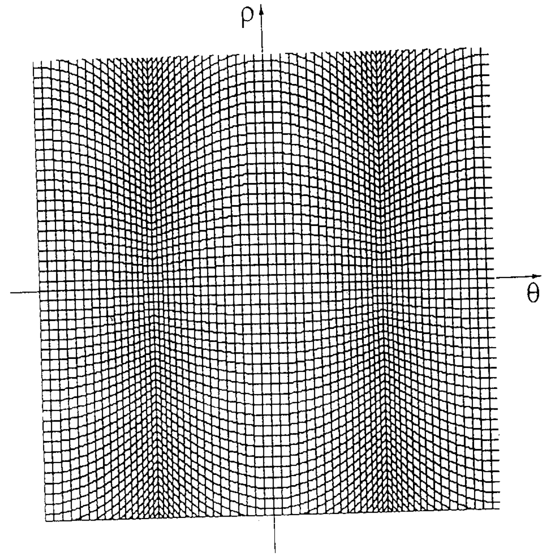


Figure 7: Grid in the $\rho - \theta$ parameter space which is uniform in the $\gamma - \omega$ parameter space.

Solving above differential equation, we obtain $\theta(\omega)$ as follows:

$$\theta(\omega) = \begin{cases} \arctan(\alpha\omega + 2) - \frac{\pi}{2}, & -\frac{2}{\alpha} \leq \omega < -\frac{1}{\alpha} \\ \arctan \alpha\omega, & -\frac{1}{\alpha} \leq \omega < \frac{1}{\alpha} \\ \arctan(\alpha\omega - 2) + \frac{\pi}{2}, & \frac{1}{\alpha} \leq \omega < \frac{2}{\alpha} \end{cases} \tag{11}$$

Using $\omega$ as a new parameter instead of $\theta$, we can construct the $\gamma - \omega$ parameter space, in which voting curves become piecewise linear lines, preserving the unbiasedness (figure 6). If we choose the angle domain as $-1/\alpha \leq \omega < 3/\alpha$ $(-\pi/4 \leq \theta < 3\pi/4)$, a voting curve becomes just a pair of straight lines.

The $\gamma - \omega$ parameter space is equivalent to the non-uniform quantization of the $\rho - \theta$ parameter space as shown in figure 7. This means that the $\gamma - \omega$ space is isomorphic to the $\rho - \theta$ space.

The knots of voting lines in the $\gamma - \omega$ space can be easily calculated from the coordinate value X and Y of feature points by addition and/or subtraction. Hence, voting can be easily realized by drawing straight lines between these knots in the parameter space. This can be done by the following transformation:

$$\gamma(X, Y, \omega) = \begin{cases} \alpha\omega Y + X & -\frac{1}{\alpha} \leq \omega < \frac{1}{\alpha} \\ (2 - \alpha\omega)X + Y & \frac{1}{\alpha} \leq \omega < \frac{3}{\alpha} \end{cases} \tag{12}$$

From equation 12, it is obvious that the parameter $\omega$ is related to gradient of a line in the image space.

The $\gamma - \omega$ parameter space includes the $a - d$ parameter space[4] as a special case $(\alpha = 1/M)$, which was designed to detect lines passing pairs of grid points in the image space. The derivation of the $a - d$ parameters was not based on elimination of the bias, but on linearization of the voting curve. Therefore the property of the $a - d$ parameter space has not been explored enough and its advantage is not well known. On the other hand, $\gamma - \omega$ parameters are derived to make $\rho - \theta$ parameters suitable for digital images. It is obvious that both $a - d$ and $\gamma - \omega$ parameters are refined versions of $\rho - \theta$ parameters. It is also clear that the $a - d$ parameter space belongs to the class of unbiased parameter spaces.

Different derivations have led to almost the same result. This gives us a strong evidence to claim that we should use $\gamma - \omega$ or $a - d$ parameters instead of $\rho - \theta$ for straight line detection from digital images.

## References

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