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Abstract - A simple mixed triangular edge element is proposed for the finite elements, with which inhomogeneously-loaded and arbitrarily shaped waveguides are analyzed. The shape functions used for approximating the fields are found analytically to be divergence-free. The formulation has been found to encounter spurious-free solutions. As an evidence, the non-physical solutions that appeared in the longitudinal component finite element formulation are shown to be absent in the present formulation. A comparison with another mixed element is furnished here in order to demonstrate the advantages provided by the present element.

I. INTRODUCTION

The finite element method has been widely used for the analysis of waveguide components and is considered to be one of the most powerful and versatile methods for the solution of a wide variety of waveguide problems. However, the finite element analysis of electromagnetic problems is well known to be plagued by the occurrence of non-physical or spurious solutions. One of the earliest reports on these spurious solutions was by Daly [1], who used two components (axial) FEM for waveguide analysis. The origin of these spurious modes lies in the fact that they don't satisfy the divergence-free condition implied by the Maxwell's equation. In this connection, methods like Penalty [2] and Lagrange multiplier method [3] have been proposed which incorporate the constraint $\nabla \cdot E = 0$ in the original formulation. This constraint has empirically been found to suppress the spurious modes, or at least push them out of the region of interest. Another approach introduced by Kobelansky and Webb [4] is the use of basis functions in which the fields are exactly divergence-free. These divergence-free basis functions are obtained by solving an auxiliary eigenmatrix equation in which intensive computations are required since at least tens of basis functions are needed to be calculated. For triangular edge elements, this paper proposes the use of shape functions which are analytically divergence-free.

II. BASIC EQUATION AND VARIATIONAL FORMULATION

Considering an inhomogeneously loaded waveguide with arbitrary cross-section Ω, the source-free Maxwell's equations with time dependence of $\exp(j\omega t)$ being implied are given by

$$\nabla \times E = -j\omega\mu_0\mu_r H \tag{1}$$

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$$\nabla \times H = j\omega\varepsilon_0\varepsilon_r E \tag{2}$$

where ω is the angular frequency, ε_0 and μ_0 are the permittivity and permeability of free space, respectively, and ε_r and μ_r are the corresponding relative material properties. By taking the curl of both sides of (1) and then substituting (2) into (1), the following vectorial wave equation for E is derived:

$$\mu_r^{-1}\nabla \times (\nabla \times E) - k_0^2\varepsilon_r E = 0 \quad \text{in } \Omega \tag{3}$$

where $\quad k_0^2 = \omega^2\varepsilon_0\mu_0$

Corresponding to (3), the following functional (4) can easily be obtained when there is no energy flow across the boundary, and it has been proved that the functional is stationary about the correct solution :

$$F(E) = \int_\Omega \{(\nabla \times E^*)\cdot(\mu_r^{-1}\nabla \times E) - k_0^2\varepsilon_r E^* \cdot E\}\, d\Omega \tag{4}$$

where the asterisk denotes the complex conjugate.

III. FINITE ELEMENT FORMULATION

The cross-section of a waveguide is divided into the finite elements of simple geometric shape to approximate the domain. For edge elements, the elements are connected to each-other by sharing the common edges on the boundaries of the elements. We here use triangular mixed edge elements where the shape functions chosen are very simple and analytically solenoidal. However, even though Koshiba's element [5] is also analytically divergence-free, the present shape function can be presented by expression which are simpler compared (see appendix) to Koshiba's. The finite element discretization is carried out by approximating E for each element as

$$E(x,y,z) = [F]^T\{\phi\}_e \tag{5}$$

$$\text{where } [F] = \begin{bmatrix} \{F_x\} & \{F_y\} & \{0\} \\ \{0\} & \{0\} & j\{N\} \end{bmatrix}$$

$$\text{with } \{F_x\} = [\,N_{12}^x + N_{21}^x \quad N_{23}^x + N_{32}^x \quad N_{31}^x + N_{13}^x\,],$$
$$\{F_y\} = [\,N_{12}^y + N_{21}^y \quad N_{23}^y + N_{32}^y \quad N_{31}^y + N_{13}^y\,] \tag{6}$$

and

$$\{\phi\}_e = \begin{bmatrix} \{\phi_t\}_e \\ \{\phi_z\}_e \end{bmatrix} \quad \text{with} \quad \{\phi_t\}_e = \begin{bmatrix} \phi_{t1} \\ \phi_{t2} \\ \phi_{t3} \end{bmatrix} \quad \text{and} \quad \{\phi_z\}_e = \begin{bmatrix} \phi_{z1} \\ \phi_{z2} \\ \phi_{z3} \end{bmatrix}. \tag{7}$$

In (7), the set of unknowns of tangential components ϕ_{t1}, ϕ_{t2} and ϕ_{t3} and axial components ϕ_{z1}, ϕ_{z2} and ϕ_{z3} are defined on the edges and the nodes respectively of the triangle as shown in Fig. 1. Besides, N^x_{ij} and N^y_{ij}, the components of $\{F_x\}$ and $\{F_y\}$ respectively, are x and y directed components of the vector shape function $\mathbf{N}_{ij}$, which are given by

$$\mathbf{N}_{23} = \frac{\mathbf{n}_3}{\mathbf{a}_1 \cdot \mathbf{n}_3} N_2,\ \mathbf{N}_{31} = \frac{\mathbf{n}_1}{\mathbf{a}_2 \cdot \mathbf{n}_1} N_3,\ \mathbf{N}_{12} = \frac{\mathbf{n}_2}{\mathbf{a}_3 \cdot \mathbf{n}_2} N_1$$
$$\mathbf{N}_{32} = \frac{\mathbf{n}_2}{\mathbf{a}_1 \cdot \mathbf{n}_2} N_2,\ \mathbf{N}_{13} = \frac{\mathbf{n}_3}{\mathbf{a}_2 \cdot \mathbf{n}_3} N_3,\ \mathbf{N}_{21} = \frac{\mathbf{n}_1}{\mathbf{a}_3 \cdot \mathbf{n}_1} N_1 \tag{8}$$

where 'a's are the unit vectors defined along the edges and 'n's are the unit normals defined on the corresponding edges of the triangle as shown in Fig. 1, and 'N's are the area coordinates. It is easy to show $\nabla \cdot [F] = 0$ with present interpolation function for the element. This can be shown with Koshiba's element [5] too. Here, the electric field is assumed to have a z dependence as $\boldsymbol{E}(x,y) \exp(-j\beta z)$, where β is the propagation constant. Substituting (5) - (8) in (4), and making the functional stationary for element e, we obtain

$$[K]_e \{\phi\}_e - k_0^2 [M]_e \{\phi\}_e = \{0\} \tag{9}$$

with

$$[K]_e = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix},\ [M]_e = \begin{bmatrix} [M_{11}] & [0] \\ [0] & [M_{22}] \end{bmatrix}$$

where

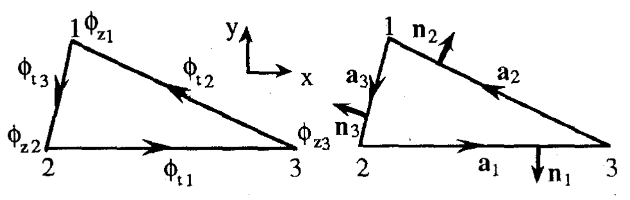

Fig. 1. Present triangular edge element showing the location of tangential and axial unknowns and unit vectors along the edges and normal to the edges.

$$[K_{11}] = \int_\Omega [(\frac{\partial\{F_y\}}{\partial x} - \frac{\partial\{F_x\}}{\partial y})(\frac{\partial\{F_y\}}{\partial x} - \frac{\partial\{F_x\}}{\partial y})^T + \beta^2 (\{F_x\}\{F_x\}^T + \{F_y\}\{F_y\}^T)]\, d\Omega$$

$$[K_{12}] = \int_\Omega [\beta (\{F_x\}\frac{\partial\{N\}^T}{\partial x} + \{F_y\}\frac{\partial\{N\}^T}{\partial y})]\, d\Omega = [K_{21}]^T$$

$$[K_{22}] = \int_\Omega [\frac{\partial\{N\}}{\partial x}\frac{\partial\{N\}^T}{\partial x} + \frac{\partial\{N\}}{\partial y}\frac{\partial\{N\}^T}{\partial y}]\, d\Omega$$

$$[M_{11}] = \int_\Omega [\{F_x\}\{F_x\}^T + \{F_y\}\{F_y\}^T]\, d\Omega$$

$$[M_{22}] = \int_\Omega [\{N\}\{N\}^T]\, d\Omega$$

After carrying out some algebraic operations, (9) can be rewritten by a system of matrix equations for all the elements as follows :

$$\begin{aligned} [S_{tt}]\{\phi_t\} - \beta[S_{tz}]\{\phi_z\} - \beta^2[M_{tt}]\{\phi_t\} &= 0 \\ -\beta[S_{zt}]\{\phi_t\} + [S_{zz}]\{\phi_z\} &= 0 \end{aligned} \tag{10}$$

where $\{\phi_t\}$ and $\{\phi_z\}$ are the global unknowns, and the system matrices are given by

$$[S_{tt}] = \sum_e \int_{\Omega_e} [k_0^2 (\{F_x\}\{F_x\}^T + \{F_y\}\{F_y\}^T) - (\frac{\partial\{F_y\}}{\partial x} - \frac{\partial\{F_x\}}{\partial y})(\frac{\partial\{F_y\}}{\partial x} - \frac{\partial\{F_x\}}{\partial y})^T]\, dxdy$$

$$[S_{tz}] = \sum_e \int_{\Omega_e} [(\{F_x\}\frac{\partial\{N\}^T}{\partial x} + \{F_y\}\frac{\partial\{N\}^T}{\partial y})]\, dxdy = [S_{zt}]^T \tag{11}$$

$$[S_{zz}] = \sum_e \int_{\Omega_e} [(k_0^2 \{N\}\{N\}^T - \frac{\partial\{N\}}{\partial x}\frac{\partial\{N\}^T}{\partial x} - \frac{\partial\{N\}}{\partial y}\frac{\partial\{N\}^T}{\partial y})]\, dxdy$$

$$[M_{tt}] = \sum_e \int_{\Omega_e} [\,(\{F_x\}\{F_x\}^T + \{F_y\}\{F_y\}^T\,)]\,dx\,dy$$

By eliminating $\{\phi_z\}$ from the two matrix equations of (10), the following eigenvalue equation can be obtained where the unknowns to be resolved are consist of tangential components only.

$$([S_{tt}] - \beta^2 [K_{tt}])\{\phi_t\} = 0 \qquad (12)$$

$$\text{where } [K_{tt}] = [S_{tz}][S_{zz}]^{-1}[S_{zt}] + [M_{tt}]$$

Note that for simplicity, material properties are not shown in the above formulation. Material properties are properly incorporated while considering dielectric loaded waveguides.

IV. Comparison of Present Element with Koshiba's

In order to demonstrate the different distinct features the present element has, it is compared with Koshiba's element [5] which has also been found to be analytically divergence-free. For comparison, in the appendix, the shape functions for both the elements are discretized down to their most simplified expression which are given in terms of the coordinates. Comparing (A3) and (A6), it is obvious that the present element is simpler than Koshiba's. For example, as shown in (A1), expression of F_{x1} for the present element needs only the length A_1 of the triangle-side along which F_{x1} is defined, y coordinate of the vertex opposite to A_1 and the area of the triangle. In contrast, to get the expression for F_{x1} for Koshiba's element, first, one needs to calculate the angles at the mid-points of the triangle-sides, then obtain Δ from (A5), and finally, insert the values of Δ and the coordinates of the corners and mid-points of the sides of the triangle in (A6).

Another advantages provided by the present shape function is that it facilitates the formulation rendering an easy approach for integrations and other algebraic operations. This is quite evident when one investigates the over all shape function employed to approximate the fields. As given in (6), the over all shape function is composed of vector shape function **F** and area coordinates N, which are used to approximate the transverse and axial components of the field respectively. But again, as shown in (8), the components of **F** comprise area coordinates N and some unit vectors, leaving N as the variable part and the unit vectors (**a**, **n**) as a constant for an element. Therefore, the matrices of (11) involve inter-operations between area coordinates only, so that the integration scheme and algebraic operations provided by the area coordinate system is used in obtaining the system matrices very easily. We investigated and employed both present and Koshiba's element in the same waveguiding formulation, and found that the degree of accuracy provided by both the elements are of same order, only that, the element matrices obtained with the present shape functions are much simpler compared to the one obtained with Koshiba's.

V. Numerical Examples

To justify the validity of the mixed edge element developed in the previous sections, the numerical analyses are carried out by employing the mixed elements for some sample problems. The first example, the structure of which is shown in Fig. 2, is a waveguide where the outer surface is a perfect conductor and a strip of zero thickness is placed in the center. The dispersion curve for the dominant (quasi-TEM) mode obtained by present and Daly's [1] formulation is furnished in Fig. 3. For symmetry, half of the structure is only considered. The dotted lines for Daly's solutions indicate the deviation due to singularity. The spurious modes appeared in the Daly's solutions have been found to be eliminated completely in the present solutions. Next, a rectangular waveguide, half of which is loaded with a dielectric slab as shown in Fig. 4, is considered. The dispersion curve obtained for the longitudinal section magnetic (LSM) mode by employing the present element is compared in Fig. 5 with the one obtained by Angkaew [6]. The present element offers better accuracy than the Angkaew's for the same number of element divisions (8×4). Finally, to test the applicability of the element to a waveguide with curved boundaries, a hollow circular waveguide of radius 'a' is analyzed. The computation is carried out with the mesh shown in Fig. 6, where a quarter of the waveguide is considered with 32 triangular elements. The dispersion characteristics for TM_{01} and TE_{21} modes are compared in Fig. 7. Angkaew [6] employed 36 first order

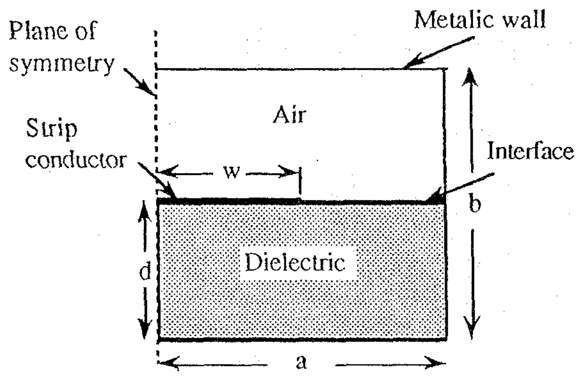

Fig. 2. Cross-section of the closed microstrip waveguide.

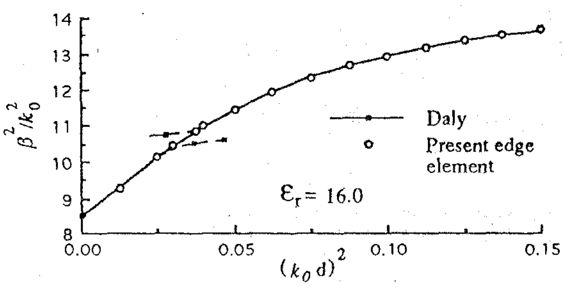

Fig. 3. Comparison of present and Daly's solutions.

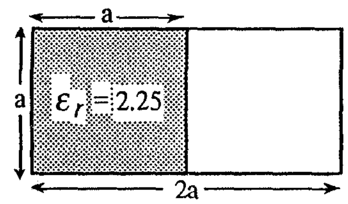

Fig. 4. Cross-section of half loaded dielectric waveguide.

linear triangular elements. The present elements have been found to yield spurious-free solutions and exhibit better accuracy even for fewer number of elements.

VI. Conclusions

A simple mixed triangular edge element that provides divergence-free shape functions is proposed, and its capability is examined for inhomogeneously-loaded arbitrarily shaped waveguiding problem. The formulation has been found to encounter spurious-free solutions. Another merit is that the shape functions facilitate the formulation by providing an easy approach for integration, algebraic computation and construction of the element matrices. The element also provide good accuracy and convergence characteristics.

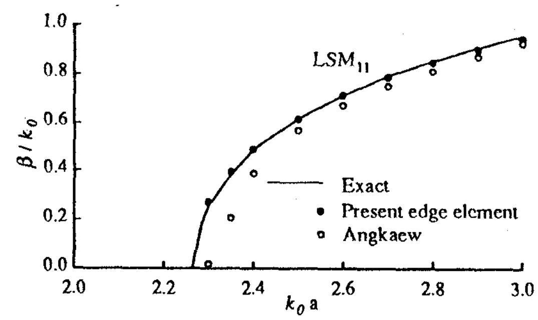

Fig. 5. Comparison of present and Angkaew's solutions.

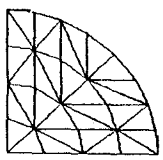

Fig. 6. Finite element mesh of a quarter circular waveguide.

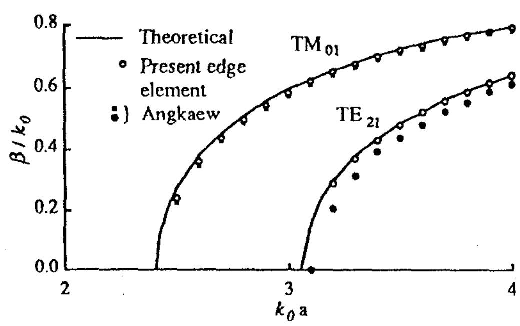

Fig. 7. Dispersion curves for a hollow circular waveguide.

Appendix

The vector shape functions employed to approximate the transverse field components can be presented by simple expression. For example, if $\{F_x\}$ of (6) is expressed as $\{F_x\} = [\,F_{x1}, F_{x2}, F_{x3}\,]$, then F_{x1} is given by

$$F_{x1} = (y_1 - y) \times A_1 / d \qquad \text{(A1)}$$

where A_1 is the length of the triangle-side along which F_{x1} is defined; y_1 is the y coordinate of the vertex corresponding to the base A_1 and d equals twice the area of the triangle. The derivation of (A1) is obtained from (6) and (8) as follows :

$$F_{x1} = \frac{\mathbf{n}_3 \cdot \mathbf{i}}{\mathbf{a}_1 \cdot \mathbf{n}_3} N_2 \quad + \quad \frac{\mathbf{n}_2 \cdot \mathbf{i}}{\mathbf{a}_1 \cdot \mathbf{n}_2} N_3 \qquad \text{(A2)}$$

After several algebraic simplification F_{x1} is finally derived as

$$F_{x1} = (y_1 - y)\,[\,(x_3 - x_2)^2 + (y_3 - y_2)^2\,]^{1/2} / d \qquad \text{(A3)}$$

The other shape functions are also given by similar expression as (A3). For Koshiba's element, the edge shape functions are given by

$$\{F_x\} = [\ f_1 + g_1 y \quad f_2 + g_2 y \quad f_3 + g_3 y\ ] \qquad \text{(A4)}$$

where $f_k = [(y_{m+3}\cos\theta_{m+3} - x_{m+3}\sin\theta_{m+3})\sin\theta_{l+3} - (y_{l+3}\cos\theta_{l+3} - x_{l+3}\sin\theta_{l+3})\sin\theta_{m+3}] / \Delta$

$g_k = [\,\cos\theta_{l+3}\sin\theta_{m+3} - \cos\theta_{m+3}\sin\theta_{l+3}\,] / \Delta$

$$\Delta = \sum_{k=1}^{3} (y_{k+3}\cos\theta_{k+3} - x_{k+3}\sin\theta_{k+3})(\cos\theta_{l+3}\sin\theta_{m+3} - \cos\theta_{m+3}\sin\theta_{l+3}) \qquad \text{(A5)}$$

After carrying out some trigonometric operations :

$$F_{x1} = f_1 + g_1 y = \frac{[(x_3 - x_1)(y_2 - y_3)(y_6 - y) - x_6(y_3 - y_1)(y_2 - y_3) - (x_2 - x_3)(y_3 - y_1)(y_5 - y) + x_5(y_2 - y_3)(y_3 - y_1)]}{\Delta\,[\,(x_2 - x_3)^2 + (y_2 - y_3)^2\,]^{1/2}\,[\,(x_3 - x_1)^2 + (y_3 - y_1)^2\,]^{1/2}} \qquad \text{(A6)}$$

The other shape functions also arrive at similar expressions.

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