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Modeling uncertainty of flexible structures with unknown high-order modal parameters

— A geometric characterization of frequency responses —

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Abstract: A control-oriented uncertainty modeling on frequency domain is presented for a class of spectral systems with unknown high-order modal parameters. At any user-specified frequency, the set of all the frequency responses of the feasible systems is characterized on the complex plane in terms of the convex hull of several circle segments, where the system is said to be feasible if partial modal parameters are given and some other conditions are satisfied by the unknown parameters. We emphasize that such a characterization enables us to quantify the least upper bounds of errors for any nominal models, and to develop further efficient results using some additional information. It is shown that, the dc gain information of the system reduces the size of the feasible set to the half or smaller for all frequencies. The efficiency of the presented scheme is demonstrated by a simple example of ideal flexible beam.

Keywords: feasible sets, Kelvin-Voigt damping, dc gain

1. Introduction

For controller design synthesis in view of robust control theory, it is necessary to specify a nominal model describing essential dynamics of the plant and also bounds of magnitudes of the uncertainty for the plant[1]. On bounding uncertainty, efforts have been made using not only physical knowledge or first principles[2] but also input-output data of the plant. Efficient numerical techniques have been developed for upper and lower bounds of modal parameters[3]. Most bounding results so far have been, however, obtained by evaluating the norm of the error, that is, the size of a ball covering the feasible set, and this may overestimate the uncertainty and cause possible conservatisms on subsequent controller design.

In this paper we present characterization of frequency responses in a geometric fashion on the complex plane, and show that it enables us to develop new results using information like dc gain of the system, which effectively shrink the size of uncertainty. By ideal flexible beam example, we will see that if the plant is physically governed by elastic equation with Kelvin-Voigt damping, then all the parameters needed for the bounding can easily be determined in the process of modal analysis using finite element methods as well as parameter estimation from data.

Notation: By $\mathrm{ch}\,(A)$ we denote a convex hull of a set A on complex plane, that is, the minimum convex set which contains A.

2. The problem and preliminaries

Large flexible structures, plates, and strings, are formulated actually by linear elastic vibrating systems, and it is well known that they are modeled by superposition of simple vibrating modes[4]

$$G(s)=\sum_{i=1}^{\infty}\frac{k_i}{1+2\zeta_i(s/\omega_i)+(s/\omega_i)^2} \qquad (1)$$

where $0<\omega_1<\omega_2<\cdots\to\infty$ and

$$\sum_{i=1}^{\infty}|k_i|\le\rho \qquad (2)$$

for some given $\rho>0$. Here ω_i is a resonant (angular) frequency, k_i a resonant multiplier, and ζ_i a damping factor. We assume that first ℓ triples of (k_i,ω_i,ζ_i) where $i=1,\ldots,\ell$, are known but all the rest $(i=\ell+1,\ldots,)$ unknown. Furthermore, let us assume it is verified that

$$\zeta_i\ge\gamma\omega_i,\text{ and }\omega_i\ge\nu\text{ for }i>\ell \qquad (3)$$

for some given $\gamma>0$ and $\nu\ge\omega_\ell$.

Our problem is, then, to characterize the set of all the possible frequency responses $G(j\omega)$ as a shape on the complex plane at any specified frequency ω.

Denote the ℓ-th partial sum of $G(s)$, the known part, by

$$G_\ell(s):=\sum_{i=1}^{\ell}\frac{k_i}{1+2\zeta_i(s/\omega_i)+(s/\omega_i)^2}, \qquad (4)$$

and by $\mathcal{P}_\ell^{\rho,\nu}$ the set of all the systems of the form equation (1) that satisfy the conditions presented above; that is,

$$\mathcal{P}_\ell^{\rho,\nu} := \left\{ G(s) = \sum_{i=1}^{\infty} \frac{k_i}{1 + 2\zeta_i (s/\omega_i) + (s/\omega_i)^2} \right| \sum_{j=1}^{\infty} |k_j| \leq \rho; \omega_j \geq \nu \ (j > \ell) \Bigg\}. \quad (5)$$

We call the $\mathcal{P}_\ell^{\rho,\nu}$ as a feasible set, and $\mathcal{P}_\ell^{\rho,\nu}(j\omega)$ a feasible set at frequency ω, which is the set of all the frequency responses corresponding to the elements of $\mathcal{P}_\ell^{\rho,\nu}$.

Now, if we define $\overline{\rho}^{(\ell)} := \rho - \sum_{j=1}^{\ell} k_j$, we see

$$\mathcal{P}_\ell^{\rho,\nu} = \left\{ G(s) = G_\ell(s) + \tilde{G}(s) \middle| \tilde{G}(s) \in \mathcal{P}_0^{\overline{\rho}^{(\ell)},\nu} \right\} \quad (6)$$

and the feasible set $\mathcal{P}_0^{\overline{\rho}^{(\ell)},\nu}$ at frequency ω is characterized as

$$\mathrm{ch}\,[\{\pm\overline{\rho}^{(\ell)} H_\theta(j\omega) | \theta \geq \nu\}]$$

where

$$H_\theta(s) = \frac{1}{1 + 2\gamma s + (s/\theta)^2}.$$

This implies that the set $\mathcal{P}_0^{\overline{\rho}^{(\ell)},\nu}(j\omega)$ is depicted on the complex plane as the convex hull of the two circle segments

$$A_1 := \{\overline{\rho}^{(\ell)} H_\theta(j\omega), \theta \geq \nu\}$$
$$A_2 := \{-\overline{\rho}^{(\ell)} H_\theta(j\omega), \theta \geq \nu\}.$$

Remark 1. Based on the above results, we can immediately give alternate proof of the fact shown in [3] that, for each frequency ω, a complex number $G_n(\omega)$ that minimizes

$$\sup_{G \in \mathcal{P}_\ell^{\rho,\nu}} |G(j\omega) - G_n(\omega)|$$

is $G_\ell(j\omega)$, and the minimum

$$\min_{G_n(\omega) \in \mathbb{C}} \sup_{G \in \mathcal{P}_\ell^{\rho,\nu}} |G(j\omega) - G_n(\omega)|$$

is given by

$$\overline{\rho}^{(\ell)} \cdot \sup_{\nu \leq \theta} |H_\theta(j\omega)|$$

where

$$\sup_{\nu \leq \theta} |H_\theta(j\omega)| = \begin{cases} 1/(2\gamma\omega), & \text{for } \omega \geq \nu \\ |H_\nu(j\omega)|, & \text{for } \omega \leq \nu. \end{cases}$$

3. Main results

Generally, the more information about the plant, the less size of feasible set. We consider the case where the dc gain

$$G(0) = \sum_{i=1}^{\infty} k_i (=: d)$$

is given. Many researchers have attacked to improve the precision of reduced order models using such dc gain information so far [5], but to the best of author's knowledge, few discussions have ever been focused on the effectiveness of using dc gain in high frequency range and other theoretical consequences.

The feasible set which corresponds to this case is $\mathcal{P}_\ell^{\rho,\nu,d} := \mathcal{P}_\ell^{\rho,\nu} \cap \mathcal{D}_d$ where $\mathcal{D}_d := \{G(s) | G(0) = d\}$.

We define $\overline{d}^{(\ell)} := d - \sum_{i=1}^{\ell} k_i$, and the relation

$$\mathcal{P}_\ell^{\rho,\nu,d} = \left\{ G(s) = G_\ell(s) + \tilde{G}(s) \middle| \tilde{G}(s) \in \mathcal{P}_0^{\overline{\rho}^{(\ell)},\nu,\overline{d}^{(\ell)}} \right\} \quad (7)$$

reveals that it is enough for our problem to characterize $\mathcal{P}_0^{r,v,\delta}(j\omega)$ for given r, v, and δ since $\mathcal{P}_\ell^{\rho,\nu,d}(j\omega) = \mathcal{P}_0^{\overline{\rho}^{(\ell)},\nu,\overline{d}^{(\ell)}}(j\omega) + G_\ell(j\omega)$.

As the main result, a geometric characterization of $\mathcal{P}_0^{r,v,\delta}(j\omega)$ is presented in the following

Theorem 1. For user-specified frequency ω, let us define $\nu_0^2 := \omega^2/(1 + 4\gamma^2\omega^2)$. Then, the $\mathcal{P}_0^{\rho,\nu,d}(j\omega)$ coincide with the convex hull of the union of the following six circle segments

$$\begin{aligned}
A_{1a} &:= \{(d-\rho)/2 \cdot H_\nu(j\omega) + (d+\rho)/2 \cdot H_\theta(j\omega) | \theta \geq \nu\}, \\
A_{1b} &:= \{(d-\rho/2 \cdot H_\theta(j\omega) + (d+\rho)/2 \cdot H_{+\infty}(j\omega) | \theta \geq \nu\}, \\
A_{2a} &:= \{(d+\rho)/2 \cdot H_\nu(j\omega) + (d-\rho)/2 \cdot H_\theta(j\omega) | \theta \geq \nu\}, \\
A_{2b} &:= \{(d+\rho)/2 \cdot H_\theta(j\omega) + (d-\rho)/2 \cdot H_{+\infty}(j\omega) | \theta \geq \nu\}, \\
A_{1\rho} &:= \{\rho H_\theta(j\omega) + (d-\rho)/(4\gamma j\omega) | \nu \leq \theta \leq \nu_0\}, \\
A_{2\rho} &:= \{-\rho H_\theta(j\omega) + (d+\rho)/(4\gamma j\omega) | \nu \leq \theta \leq \nu_0\}.
\end{aligned}$$

that is, $\mathcal{P}_0^{\rho,\nu,d}(j\omega) = \mathrm{ch}\,[A_{1a} \cap A_{1b} \cap A_{2a} \cap A_{2b} \cap A_{1\rho} \cap A_{2\rho}]$ where the equality is in the sense of set theoretic.

Note that the sets $A_{1\rho}$ and $A_{2\rho}$ become empty for $\nu_0 < \nu$.

A brief sketch of proof We can show the problem is reduced to the "two term" inclusion problems, as in the

Lemma 1. The following relation holds:

$$\mathcal{P}_0^{\rho,\nu,d}(j\omega) = \mathrm{ch}\,[\mathcal{S}_0^{\rho,\nu,d}(j\omega)] \quad (8)$$

where $\mathcal{S}_0^{\rho,\nu,d}(j\omega) := \{k_a H_{\omega_a}(j\omega) + k_b H_{\omega_b}(j\omega) | \omega_b \geq \omega_a \geq \nu, k_a + k_b = d, |k_a| + |k_b| = \rho\}$ is a set of candidate extreme points for $\mathcal{P}_0^{\rho,\nu,d}(j\omega)$.

We can see the convex hulls are characterized by circle segments as shown in the

Lemma 2. $\mathcal{S}_0^{\rho,\nu,d}(j\omega)$ consists of the union of (S1) and (S2) as follows. For $\nu \geq \nu_0$,

(S1) the region enclosed by circle segments A_d, A_{1a} and A_{1b},

(S2) the region by A_d, A_{2a}, A_{2b},

and, for $\nu \leq \nu_0$,

(S1) the region by A_d, A_{1a}, A_{1b}, $A_{1\rho}$,

(S2) the region by A_d, A_{2a}, A_{2b}, $A_{2\rho}$

where $A_d := \{d \cdot H_\theta(j\omega) | \theta \geq \nu\}$.

From the theorem, a result corresponding to Remark 1 follows immediately as in

Corollary 1. For each frequency ω, a complex number $G_n(\omega)$ that minimizes $\sup_{G \in \mathcal{P}_\ell^{\rho,\nu,d}} |G(j\omega) - G_n(\omega)|$ is

$$G_\ell(j\omega) + \frac{\overline{d}^{(\ell)}}{2} \cdot (H_\nu(j\omega) + H_{+\infty}(j\omega)), \text{ for } \nu \geq \nu_0, \text{ and}$$

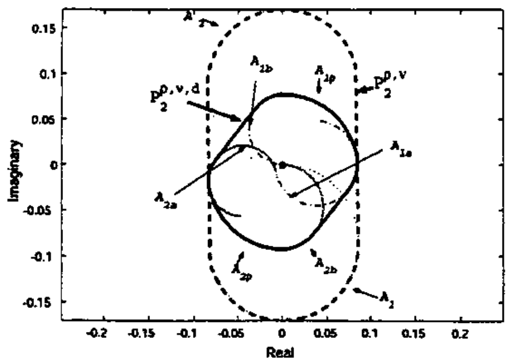

Figure 1: The feasible sets with and without dc information ($\omega = 62$ rad/sec, as in the example).

$$G_\ell(j\omega) + (\overline{d}^{(\ell)})/(4\gamma j\omega), \text{ for } \nu \le \nu_0,$$

and the minimum is given by

$$\min_{G_n(\omega)\in \mathbf{C}} \sup_{G\in \mathcal{P}_\ell^{\rho,\nu,d}} |G(j\omega) - G_n(\omega)|$$

$$= \begin{cases} (\overline{\rho}^{(\ell)}/2)\cdot |H_\nu(j\omega) - H_{+\infty}(j\omega)|, & \text{for } \nu \ge \nu_0, \\ \overline{\rho}^{(\ell)}/(4\gamma\omega), & \text{for } \nu \le \nu_0. \end{cases}$$

We can prove the Corollary by considering two most distant points in $\mathcal{P}_0^{\rho,\nu,d}$. Note that this gives fundamental limitation on the error bounds to any nominal models, and we call the minimum as the radius of the feasible set.

As a further consequence of Theorem 1, a rational nominal model using the dc gain d with explicit least upper bounds of the error is given by the following

Corollary 2. Let $G_n(s) := G_\ell(s) + \overline{d}^{(\ell)} H_{+\infty}(s)$ as a nominal model. Then, the minimum radius of a disk which contains $\mathcal{P}_\ell^{\rho,\nu,d}$ is given by

$$(\overline{\rho}^{(\ell)} + |\overline{d}^{(\ell)}|)/2 \cdot |H_{+\infty}(j\omega) - H_\nu(j\omega)| \text{ for } \nu \ge \nu_0, \text{ and}$$

$$(\overline{\rho}^{(\ell)} + |\overline{d}^{(\ell)}|)/(4\gamma\omega)\cdot |1 - 4\gamma j\omega H_{+\infty}(j\omega)| \text{ for } \nu \le \nu_0.$$

4. Example

We consider an example for modeling of an ideal flexible beam. Dynamics of bending motion of canti-levered beam where sensors and actuators may not be collocated is described as

$$v_{tt}(t,\xi) + 2\gamma v_{\xi\xi\xi\xi t}(t,\xi) + v_{\xi\xi\xi\xi}(t,\xi) = \delta(\xi - \xi_i)u(t) \quad (0 < \xi < 1) \tag{9.a}$$

$$v(t,0) = v_\xi(t,0) = v_{\xi\xi}(t,1) = v_{\xi\xi\xi}(t,1) = 0 \tag{9.b}$$

$$y(t) = v(t,\xi_o) \tag{9.c}$$

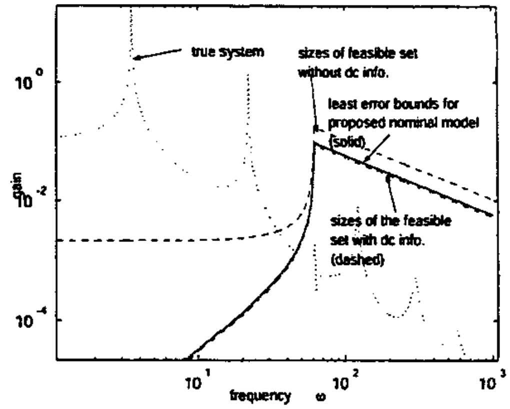

Figure 2: Comparison between sizes of feasible sets and the bound

where $\delta(\xi)$ is Dirac's delta function. Furthermore $\gamma = 1\times 10^{-4}$, and $\xi_i = 1$ and $\xi_o = 0.5$ represent the location of point input and output, respectively.

It is well-known that a countable infinite number of non-trivial solutions to the eigenvalue problem

$$\varphi''''(\xi) = \mu\varphi(\xi) \quad (0 < \xi < 1)$$

$$\varphi(0) = \varphi'(0) = \varphi''(1) = \varphi'''(1) = 0$$

exist, and let the real eigenvalues are ordered as $0 < \mu_1 \le \mu_2 \le \cdots$ and corresponding eigenfunction be $\varphi_i(\xi)$. The transfer function can be written as

$$G(s) = \sum_{i=1}^{\infty} \frac{c_i b_i/\mu_i}{1 + 2\gamma s + s^2/\mu_i} \tag{10}$$

where $c_i = \varphi_i(\xi_o)$ and $b_i = \varphi_i(\xi_i)$. Taking $\omega_i^2 = \mu_i$, $\zeta_i = \gamma\omega_i$, and $k_i = c_i b_i/\mu_i$, this is reduced to (1).

We also have

$$\sum_{i=1}^{\infty} |c_i/\omega_i|^2 = \eta_{\xi_o}(\xi_o)$$

where $\eta_{\xi_o}(\xi)$ is a solution to the following boundary value problem:

$$\eta''''_{\xi_o}(\xi) = \delta(\xi - \xi_o),\ 0 < \xi < 1$$

$$\eta_{\xi_o}(0) = \eta'_{\xi_o}(0) = \eta''_{\xi_o}(1) = \eta'''_{\xi_o}(1) = 0$$

$\eta_{\xi_i}(\xi)$ can be defined similarly, and

$$\sum_{i=1}^{\infty} |b_i/\omega_i|^2 = \eta_{\xi_i}(\xi_i)$$

. We obtain an evaluation

$$\rho = \sqrt{\eta_{\xi_o}(\xi_o)\cdot \eta_{\xi_i}(\xi_i)}$$

as in [3]. We suppose the situation where just first two modes are known as follows: $\ell = 2$, $\omega_1 = 3.516$, $\omega_2 = 22.03$, $k_1 = 0.1099$, $k_2 = -5.88 \times 10^{-3}$, $d = 0.104$, $\rho = 0.118$, $\overline{\rho}^{(\ell)} = 2.11 \times 10^{-3}$, $\overline{d}^{(\ell)} = 1.89 \times 10^{-4}$. The construction of feasible set at frequency $\omega = 62$ rad/sec is depicted inFigure 1. The frequency characteristic of the radii of the feasible set with and without dc gain are compared in Figure 2. Error bounds of the proposed nominal model in Corollary 2 is also plotted in the same figure.

5. Conclusion

In this paper we proposed a modeling of uncertainty in elastic vibrating systems. Here we presented a method to characterize uncertainty as a feasible set in the frequency domain. The shape of the bounded set of all the complex numbers of frequency responses of the systems that satisfy the condision is depicted by several circle segments.

Theoretical limitation was clarified about the minimum additive uncertainty of any nominal models under the information given. The set theoretic characterization enables us to develope new results that the information of dc gain of the system will effectively shrink the size of the feasible set.

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