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***Abstract*—This note gives a new method for computing a feedback gain which achieves state deadbeat control. From systems given in the staircase form, this method derives the deadbeat gain in a numerically reliable way. It is also shown that the gain turns out to be LQ optimal for some weightings.**

I. Introduction

As is widely recognized, it is by no means trivial to compute state deadbeat gains in a numerically reliable way. In addition to the conventional method using the reachability matrix, a number of approaches have been proposed in literature (see, e.g., [4]).

Emami–Naeini [3] has given a method based on the linear quadratic (LQ) control theory. Van Dooren [2] has computed the gain by repeating conversion of a given system. These two methods are numerically stable, and each of them has its own merit. In [3], the deadbeat gain is computed as an LQ regulator for a specially chosen pair of weightings, which gives an elegant interpretation of the deadbeat gain. In [2], the norm of the gain is optimized in addition to the minimal time deadbeat property.

In this note, we propose a new method for computing a deadbeat gain. The obtained gain turns out to be LQ optimal for some weightings, and hence it shares the advantage with the method by [3]. Yet the present approach does not require solving the Riccati equation, and hence is algorithmically more transparent. LQ optimality is shown by using the result on the inverse regulator problem.

II. Main Result

Consider a reachable system

$$x(t+1) = F_o x(t) + G_o u(t), \qquad t = 1, 2, \cdots \tag{1}$$

where $F_o \in \mathbb{R}^{n\times n}$, $G_o \in \mathbb{R}^{n\times m}$. No assumption is made on nonsingularity of F_o. As in the existing work [2], [3], we start by converting the pair (F_o, G_o) into the staircase form

$$G = \begin{bmatrix} \Delta_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1\mu} \\ \Delta_2 & F_{22} & & \vdots \\ & \ddots & \ddots & \\ 0 & & \Delta_\mu & F_{\mu\mu} \end{bmatrix} \tag{2}$$

where μ is the reachability index, Δ_p has full row rank r_p, the diagonal element matrices F_{pp} are $r_p \times r_p$, and the rest are of compatible sizes. It is well known that the form (2) can be obtained via orthogonal similarity transformations, and hence this process is numerically reliable.

Now we compute the deadbeat gain directly from (2). In order to explain the recursive nature of the algorithm, let us first define a sequence of submatrices

$$\hat{G}_p := \begin{bmatrix} \Delta_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \hat{F}_p := \begin{bmatrix} F_{pp} & & \cdots & F_{p\mu} \\ \Delta_{p+1} & \ddots & & \vdots \\ & \ddots & \ddots & \\ 0 & & \Delta_\mu & F_{\mu\mu} \end{bmatrix} \tag{3}$$

for $p = 1, \cdots, \mu$. Then we have $F = \hat{F}_1$, $G = \hat{G}_1$, and

$$\hat{F}_p = \begin{bmatrix} F_{pp} & * \\ \hat{G}_{p+1} & \hat{F}_{p+1} \end{bmatrix}, \qquad p = 1, \cdots, \mu - 1. \tag{4}$$

Namely, $\hat{F}_{p+1}$ is embedded into the preceding $\hat{F}_p$.

With these notions, the deadbeat gain K is given by the simple recursive formula:

$$\begin{cases} K_\mu := \Delta_\mu^- \hat{F}_\mu, \\ K_p := \Delta_p^- [I \quad K_{p+1}] \hat{F}_p, \qquad p = \mu - 1, \cdots, 1, \\ K := K_1 \end{cases} \tag{5}$$

Here, A^- denotes any right inverse of the matrix A.

Remark: For each p, K_p in the formula (5) is chosen so that

$$\hat{F}_p - \hat{G}_p K_p = \begin{bmatrix} 0 & -K_{p+1} \\ 0 & I \end{bmatrix} \hat{F}_p.$$

Note that in the right-hand side, the r_p columns of the first factor matrix are all zeros. An effect of taking such K_p is indicated by the following theorem.

Theorem 1: The feedback gain (5) achieves minimal time deadbeat control.

Proof: In view of (2), (4), and (5) we compute

$$\begin{aligned} \left(\hat{F}_1 - \hat{G}_1 K_1\right)^\mu &= \begin{bmatrix} 0 & -K_2 \\ 0 & I \end{bmatrix} \hat{F}_1 \cdots \begin{bmatrix} 0 & -K_2 \\ 0 & I \end{bmatrix} \hat{F}_1 \\ &= \begin{bmatrix} 0 & -K_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & * \\ 0 & \hat{F}_2 - \hat{G}_2 K_2 \end{bmatrix}^{\mu-1} \hat{F}_1 \\ &= \begin{bmatrix} 0 & -K_2\left(\hat{F}_2 - \hat{G}_2 K_2\right)^{\mu-1} \\ 0 & \left(\hat{F}_2 - \hat{G}_2 K_2\right)^{\mu-1} \end{bmatrix} \hat{F}_1. \end{aligned} \tag{6}$$

We can proceed this process recursively with respect to the suffix p. Furthermore, in the final suffix $p = \mu$, we have $\hat{F}_\mu - \hat{G}_\mu K_\mu = 0$ by (5). By induction, we thus obtain $(F - GK)^\mu = (\hat{F}_1 - \hat{G}_1 K_1)^\mu = 0$, as expected. □

III. LQ Optimality

Now we show that the deadbeat gain obtained by (5) is optimal for some weightings. To this end, we give a criterion for optimality in a more general form.

Theorem 2: In the system (2), suppose that G has full column rank. (Then Δ_1 is square and invertible.) If a stabilizing feedback gain K is written as

$$K = \Delta_1^{-1} [I \quad L] F \tag{7}$$

for some L, then K is LQ optimal for the performance index

$$J := \sum_{t=0}^{\infty} x^T(t) Q x(t), \qquad Q := H^T H, \qquad H := \Delta_1^{-1} [I \quad L]. \tag{8}$$

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Proof: Denoting $Z(z) := (zI - F)^{-1}G$, the return difference matrix for the gain (7) is

$$\begin{aligned} W(z) &:= I + KZ(z) \\ &= \Delta_1^{-1}[I \quad L]\begin{bmatrix}\Delta_1 \\ 0\end{bmatrix} + \Delta_1^{-1}[I \quad L]FZ(z) \\ &= z\Delta_1^{-1}[I \quad L]Z(z) \end{aligned} \tag{9}$$

by (2) and (7). Hence we have, in view of (8),

$$W^T(z^{-1})W(z) = Z^T(z^{-1})QZ(z). \tag{10}$$

On the other hand, let $\bar{K}$ be the LQ optimal gain for the performance index (8), and let $\bar{W}(z)$ be its return difference matrix. Then it is well known ([1], [5], [7]) that $\bar{W}(z)$ satisfies the Kalman equation

$$\bar{W}^T(z^{-1})G^T\bar{\Pi}G\bar{W}(z) = Z^T(z^{-1})QZ(z) \tag{11}$$

where $\bar{\Pi}$ is the solution of the corresponding Riccati equation. In view of (10) and (11), we can readily show that $K = \bar{K}$ by slightly modifying the technique in [5, prop. 3.1]. □

Remark: Note that the control weighting $R = 0$ in the performance index (8). The existence of the Riccati solution is, however, ensured in this case (see [6]).

We see that the deadbeat gain by formula (5) satisfies the condition (7) by putting $L = K_2$. We thus have the following result.

Corollary: The deadbeat gain K by (5) is LQ optimal.

IV. An Example

Consider the system

$$F_o = \text{diag}(1, 2^{-1}, \cdots, 2^{-n+1}), \qquad g_o = (1, \cdots, 1)^T. \tag{12}$$

Applying the staircase algorithm, we obtain the form (2). If $n = 4$, for example, we have

$$F = \begin{bmatrix} 4.7e-1 & -3.4e-1 & 5.3e-17 & -1.3e-17 \\ -3.4e-1 & 6.9e-1 & -2.3e-1 & 5.4e-17 \\ 0 & -2.3e-1 & 4.5e-1 & 1.2e-1 \\ 0 & 0 & 1.2e-1 & 2.7e-1 \end{bmatrix},$$

$$g = \begin{bmatrix} -2.0e+0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

Then the deadbeat gain by the formula (5) is

$$K_{st} = (3.0e+0, -1.3e+0, 1.7e-1, -6.0e-3) \tag{14}$$

while the gain K_c computed via the reachability matrix is the same.

In the latter method, we have used the toolbox "ctrbf" of MATLAB. Due to the special algorithm of this software, it is feasible to calculate reachability matrices without causing the well-known numerical instability. Even so, as n increases, the error of the latter grows rapidly and, as an extreme case, when $n = 16$ we have obtained

$$\begin{aligned} \|(F_o - g_oK_{st})^n\| &= 9.2e-28, \\ \|(F_o - g_oK_c)^n\| &= 2.3e+79. \end{aligned} \tag{15}$$

Here, the computation was carried out by MATLAB of NEC-PC version.

V. Conclusion

We have shown that the gain (5) achieves deadbeat control and is LQ optimal. Although we do not have to solve the Riccati equation for any weightings, it is ensured that the obtained feedback gain belongs to the class of optimal control.

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A Newton-Squaring Algorithm for Computing the Negative Invariant Subspace of a Matrix

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***Abstract*—By combining Newton's method for the matrix sign function with a squaring procedure, a basis for the negative invariant subspace of a matrix can be computed efficiently. The algorithm presented is a variant of multiplication-rich schemes for computing the matrix sign function such as the well-known inversion-free Schulz method which requires two matrix multiplications per step. However, by avoiding a complete computation of the matrix sign and instead concentrating only on the negative invariant subspace, the final Newton steps can be replaced by steps which require only one matrix squaring each. This efficiency is attained without sacrificing the quadratic convergence of Newton's method.**

I. Introduction

The need to compute invariant subspaces of a matrix $A \in \mathbb{C}^{n_A \times n_A}$ arises in a variety of settings. For example, a common procedure for solving the algebraic Riccati equation involves finding the negative invariant subspace of an associated Hamiltonian matrix [25], [26]. In recent years there has been renewed interest [1]–[4], [7], [8], [10]–[13], [18]–[22], [25], [27]–[29] in using the matrix sign function to solve such problems, especially for large matrices.

The matrix sign function is a generalization of the scalar sign function for complex numbers. If z is a complex number which is not on the imaginary axis, then sgn(z) is -1 if z has negative

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