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# Fat tail phenomena in a stochastic model of stock market : the long-range percolation approach

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## 1 Introduction

It is well known from empirical data on stock markets that distributions of stock returns or stock price changes show a slow asymptotic decay deviating from a Gaussian distributions [4, 6].

In this article we present a model, where trading strategies and market sentiments are described as interaction energies of Gibbs distribution and a long–range percolation communication system between traders is introduced to derive a jump process producing fat tails in the distribution of the logarithm of stock returns.

In the theory of *econophysics* [7], various approaches [6, 10, 3] are made to explain the "fat tail" in the distribution of the logarithm of stock returns. Mandelbrot [6] explained the fat tailed phenomenon in a cotton market by using a probability density function of Lévy distribution.

Stauffer [10] and Cont and Bouchaud [3] proposed to use percolation models to illustrate the herd behavior of a stock market participants.

Usually, traders are rather rational in the sense that traders determine their trading positions by analyzing the past data on the stock market and take their trading strategies into account. However, sometimes traders do not look at the past data on the market and follow an advice of an investment adviser scrupulously that is, traders sharing the same advice behave in the same way. This herd behavior causes a large fluctuation and derive a distribution of stock returns deviating from Gaussian and having fat tails.

We consider two types of traders called Group A and Group B ; Group A traders determine their trading positions by analyzing past market data and their trading strategy, on the other hand, Group B traders determine their trading positions by an advice which is randomly reached from an investment adviser through a long–range percolation system. When Group B traders receive an advice to buy (sell) stocks, they make buy (sell) orders. If any advice does not reach to a trader of Group B, he (or she) does not participate in the trading. We regard Group A traders as chartists and Group B traders as retail traders.

We consider one kind of stocks and assume that each Group A trader can make an order to buy or sell a unit number of stocks at each discrete time $u \in \{1, \ldots, n\}$.

We denote by $w_u(i)$ an order made by a trader $i$, $w_u(i) = +1(-1)$ means buy (sell) order and $w_u(i) = 0$

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means that trader $i$ does not participate in the trading at time $u$. We denote by $w_u = (w_u(1), ..., w_u(N))$ a configuration of orders for $N$ traders, and by $w = (w_1, ..., w_n)$ a configuration of orders from time 1 to $n$. The totality of $w$ is denoted by $\Omega_n$.

Group A traders check the past data $(w_{u-a}, ..., w_u)$ on stock market to determine their trading positions $w_u$. A probabilistic intensity that $w_u$ is realized at time $u$ is described by interaction energies, which are defined precisely in section 2. With these interaction energies we define a Gibbs distribution on the configuration space $\Omega_n$.

In Group B, we assume that an investment adviser is in the origin of $\mathbf{Z}$ and countably many Group B traders are located in $\mathbf{Z} \setminus \{0\}$. At each time, the investment adviser receives a good, bad and no news with probabilities $\gamma^+$, $\gamma^-$ and $1 - \gamma^+ - \gamma^-$, respectively. When the investment adviser received a good (bad) news, he sends an advice to buy (sell) stocks to Group B traders through the long–range percolation system. If a communication channel between two traders located at $x$ and $y$ is open, they can share the same information. Let us remark that a trader could receive the advice indirectly through other traders, even if he (or she) does not receive it from the investment adviser directly. All traders who receive the advice through open communication channels behave in the same way. We assume the probability whether a communication channel is open or closed follow the long–range percolation model, that is

$$P_p \text{ (channel between } x \text{ and } y \text{ is open)} = \begin{cases} p, & (|x-y| = 1), \\ \beta |x-y|^{-2}, & (|x-y| \geq 2). \end{cases} \tag{1.1}$$

We suppose $p = p(u) \in [0, 1)$ depends on time u and we call it the *nearest neighbor percolation probability*. We increase the nearest neighbour percolation probability $p(u)$ at every time $u = T_k$ when an accumulated number of Group B traders who received advices exceeds some points. The precise definition of stopping times $\{T_k\}$ will be given in section 2. The interaction energies of Group A traders also change at every $u = T_k$. This means that the trading behavior of Group B traders affects the probability distribution of Group A trading positions. In other words, the trading strategies of Group A traders changes as the total amount of information for Group B traders exceeds some points. We introduce the notions of surplus orders $\langle w_u \rangle$ and $\langle \hat{w}_u \rangle$ for Group A and Group B traders at time $u$ (see (2.1) (2.2) for precise definition), and define a stock price change at time $u$ by

$$\frac{S_{u+1}}{S_u} = e^{c_0(\langle w_u \rangle + \langle \hat{w}_u \rangle)}$$

where $c_0$ is a constant called *marketdepth*.

It is known ([8, 2, 5]) that there exists a critical value $p_0(\beta) \in (0, 1)$ for the long–range percolation model such that there is a unique infinite open cluster almost surely if $p \geq p_c(\beta)$, otherwise there is no infinite open cluster. Moreover this critical phenomenon shows the first order phase transition, in the sense that the probability that the origin belongs to an infinite open cluster is a discontinuous function of $p$ at critical value $p_c(\beta)$.

We make the nearest neighbor percolation probability $p(u)$ increase to $p_c(\beta)$. Once $p(u)$ reaches $p_c(\beta)$, infinitely many traders receive the advice. Thereby orders from Group B traders will increase dramatically and a financial discontinuity will be caused.

Repeating the above procedures independently and taking the scaling limit of the process, we obtain a continuous time stock price process $\tilde{S}_t = \tilde{S}_0 \exp\{c_0 X_t\}$, where $X_t$ is a Lévy process given by

$$X_t = \int_0^t (\mu_A(v) + \mu_B(v))dv + \int_0^t \sigma_A(v)dB_v + Y_t,$$

where $B_v$ is a standard Brownian motion. Group A contributes to the trend function $\mu_A(v)$ and the volatility function $\sigma_A^2(v)$, both of them are described in terms of the polymer functionals in the theory of cluster expansion. Group B contributes to the trend function $\mu_B(v)$ and the jump term $Y_t$ which is a compound Poisson process, given by

$$Y_t = \int_{[0,t]} \int_{(-\infty,\infty)\setminus\{0\}} xN_p(ds\, dx),$$

where $N_p(ds\, dx)$ is a Poisson random measure. The Lévy measure of $Y_t$ is $\mu(dx) = c\,\rho(dx)$ with $c > 0$ and $\rho\left((-\infty,\infty)\right) = 1$.

## 2 Description of Model

As we mentioned in the introduction, we introduce a Gibbs measure with interactions between the past positions and present positions to describe a probability distribution of trading positions for Group A traders. However, these interactions are influenced by the trading behavior of Group B traders.

We first state a definition of a configuration space $\Omega_n$ of trading positions for Group A traders, then define a long –range percolation model for Group B traders, and finally define a Gibbs distribution for Group A traders.

### 2.1 Configuration space for Group A

Group A consists of $N$ traders. At each time $u = 1, ..., n$, each trader of Group A can give an order to buy (sell) unit numbers of stocks, or does not participate in the trading. We say he (or she) takes buy position, sell position and neutral position if he (or she) gives an order to buy, to sell unit numbers of stocks and does not participate in the trading, respectively. Let $\{w_u(i); i = 1, ..., N, u = 1, ..., n\}$ be a family of random variables taking values in the set $\{+1, -1, 0\}$ with probability given later. The random variable $w_u(i)$ stands for the types of trading positions which a Group A trader $i$ takes at time $u$. We denote by $w_u(i) = +1$, $-1$ and $0$ for a buy position, a sell position and neutral position, respectively. The configuration space of trading positions of Group A is given by $\Omega_n = \{+1, -1, 0\}^{n\times N}$. We write $w_u^+$ or $w_u^-$ the number of traders in Group A who make a buying or selling order at time $u$, respectively. Let us fix a positive constant $d_0$. We denote the *modified number of market participants* or *activity* of Group A by

$$|w_u| = \begin{cases} w_u^+ + w_u^- - d_0, & \text{if } w_u^+ + w_u^- > d_0, \\ 0, & \text{otherwise.} \end{cases}$$

*The modified surplus orders for Group A traders* is given by

$$\langle w_u \rangle = \begin{cases} w_u^+ - w_u^- - d_0, & \text{if } w_u^+ - w_u^- > d_0, \\ -(w_u^- - w_u^+ - d_0), & \text{if } w_u^- - w_u^+ > d_0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

We say $w_u$ is active if $|w_u| \neq 0$, otherwise, we say it is *static*. Note that if $w_u$ is static, then $\langle w_u \rangle = 0$ and it implies that the trading behavior of Group A traders at time $u$ does not cause any change in stock price process.

## 2.2 Configuration space for Group B and long-range percolation model

Group B traders are located in $\mathbf{Z}$ and *an investment adviser* is in its origin $0 \in \mathbf{Z}$.
Let $\{\overline{w}_u ; u = 1, ..., n\}$ be a sequence of random variables taking values in the set $\{+1, -1, 0\}$ with probability given later. The random variable $\overline{w}_u$ stands for the type of news the investment adviser receives at time $u$. We denote by $\overline{w}_u = +1, -1$ or $0$, if a good, a bad or no news is received, respectively. The configuration space of the types of news is denoted by $\overline{\Omega}_n = \{+1, -1, 0\}^n$. We write $\mathrm{B} = \{\{x, y\} ;\ x, y \in \mathbf{Z}\}$ the set of all pairs of Group B traders. Let $\{\tilde{w}_u(x, y); \{x, y\} \in \mathrm{B}, u = 1, ..., n\}$ be a family of random variables taking values in the set $\{0, +1\}$ with probability given also later. We denote by $\tilde{w}_u(x, y) = +1\ (0)$, if the channel between $x$ and $y$ is open (closed). The configuration space of communication system is denoted by $\tilde{\Omega}_n = \{0, +1\}^{n \times \mathrm{B}}$.
The configuration space of Group B is given by $\hat{\Omega}_n = \overline{\Omega}_n \times \tilde{\Omega}_n$. We denote an element of $\hat{\Omega}_n$ by $\hat{w}_u = (\overline{w}_u, \tilde{w}_u)$.

We say a pair $\{x, y\} \in \mathrm{B}$ of traders belongs to the same *open component* (at time $u$) if there is a sequence of traders $x = x_0, x_1, ..., x_k = y \in \mathbf{Z}$ such that $\tilde{w}_u(\{x_{l-1}, x_l\}) = +1$ for all $l = 1, ..., k$. The event that a pair $\{x, y\} \in \mathrm{B}$ of traders belongs to the same open component is denoted by $x \leftrightarrow y$.
At each time $u = 1, ..., n$, if the news is good (bad), the investment adviser sends an advice to buy (sell) the stocks to the traders belonging to the same open component with him. The set of all traders who receive the advice is

$$C_\infty = C_\infty(0) = \{x \in \mathbf{Z} ; 0 \leftrightarrow x\}.$$

We also denote by $C_\infty(x)$ the open component including $x \in \mathbf{Z}$.
Put $B_{N_n} = [-N_n, N_n] \cap \mathbf{Z}$, where $N_n$ is a positive integer given in (4.3). We assume that only traders in $B_{N_n}$ can participate in the trading, we call them the *selected traders*. A set of the selected traders who receive the advice is

$$C_{N_n} = \{x \in B_{N_n} ; 0 \leftrightarrow x\}.$$

Also, we assume that each Group B trader can trade $1/|B_{N_n}|$ unit of stocks at each time. As all traders in $C_{N_n}$ behave in the same way according to the type $\overline{w}_u$ of news, *the modified surplus orders for Group B traders* is given by

$$\langle \hat{w}_u \rangle = \frac{\overline{w}_u |C_{N_n}|}{|B_{N_n}|}. \tag{2.2}$$

The stopping times $T_k$ and $U_k$ are defined by

$$T_0(\hat{w}) = 0,\ T_k(\hat{w}) = \min\left\{ u \geq 1; \sum_{l=1}^{u} < \hat{w}_{U_{k-1}+l} > \geq n^{\delta} \right\}\ (k \geq 1),$$

$$U_0(\hat{w}) = 0,\ U_k(\hat{w}) = U_{k-1}(\hat{w}) + T_k(\hat{w})\ (k \geq 1),$$

where $\delta$ is a constant satisfying $\frac{3}{8} < \delta < \frac{1}{2}$. Let $q = 1/2 - \delta (< \delta)$. We decompose the set of discrete times $\{1, \ldots, n\}$ into random intervals $I_n^{\hat{w}}(1), ..., I_n^{\hat{w}}(n^q + 1)$ where

$$I_n^{\hat{w}}(k) = \begin{cases} \{U_{k-1}(\hat{w}) + 1, ..., U_k(\hat{w})\}, & \text{for } k = 1, ..., n^q, \\ \{U_{n^q}(\hat{w}) + 1, ..., n\}, & \text{for } k = n^q + 1. \end{cases}$$

For each time $u = 1, ..., n$, there is a unique number $k = k(u)$ such that $u \in I_n^{\hat{w}}(k)$. We note that the number $k(u)$ is determined by the past trading data $\{\hat{w}_1, ..., \hat{w}_{u-1}\}$ of Group B traders.

Let $\{\gamma^+{}_k, \gamma^-{}_k \in [0, 1]; \gamma^+{}_k + \gamma^-{}_k \leq 1, k = 1, ..., n^q + 1\}$ be a family of numbers specified next section. We call $\gamma^+{}_k$ and $\gamma^-{}_k$ *news parameters* at time $u$. A probability distribution of the type of news $\overline{w}_u$ is given by

$$\hat{\mathrm{P}}(\overline{w}_u = +1 \mid \hat{w}_1, ..., \hat{w}_{u-1}) = \gamma_k^+,$$
$$\hat{\mathrm{P}}(\overline{w}_u = -1 \mid \hat{w}_1, ..., \hat{w}_{u-1}) = \gamma_k^-,$$
$$\hat{\mathrm{P}}(\overline{w}_u = 0 \mid \hat{w}_1, ..., \hat{w}_{u-1}) = 1 - \gamma_k^+ - \gamma_k^-,$$

provided that $u \in I_n^{\hat{w}}(k)$.

We assume that the advice from the investment adviser spreads over Group B via the longrange percolation model defined in (1.1). It is known that the long–range percolation model exhibits the first order phase transition. We state some known results on this model as follows.

**Theorem 2.1** ([2], [5], [8]) *For any $\beta > 1$, the following statements holds :*

(1) *There exists a critical value $p_c(\beta) \in (0, 1)$ depending on $\beta$ such that*

$$P_p(|C_\infty| = \infty) \begin{cases} = 0, & (p < p_c(\beta)), \\ \geq \beta^{-1/2}, & (p \geq p_c(\beta)). \end{cases}$$

(2) *For any $p \geq p_c(\beta)$, there is a unique infinite cluster almost surely.*

(3) *For any $p < p_c(\beta)$, there is a constant $c_0(p, \beta) < \infty$ depending on p and $\beta$ such that*

$$\tau(x, y) \leq c_0(p, \beta)|x - y|^{-2} \quad \text{for any} \quad x, y \in \mathbf{Z}^2$$

*where $\tau(x, y) = P_p(x \leftrightarrow y)$ is a connectivity function.*

Let $\{p_k ; k = 0, 1, ..., n^q + 1\}$ be a increasing sequence defined by

$$p_k = \begin{cases} p_0 + \dfrac{p_c(\beta) - \varepsilon(n) - p_0}{n^q} k, & \text{if } k \le n^q, \\ p_c(\beta) & \text{if } k = n^q + 1, \end{cases}$$

where $p_0$ and $\varepsilon(n)$ are some positive numbers. The precise definition of $p_0$ and $\varepsilon(n)$ is given in (4.15) and (4.1). Note that $p_{n^q} = p_c(\beta) - \varepsilon(n)$ approaches to $p_c(\beta)$ as $n$ tends to infinity. We call $p_k$ the *nearest neighbor percolation probability*. Let $\beta > 1$ be a fixed constant. A probability distribution of the states $\tilde{w}_u(\{x, y\})$ of communication channel $\tilde{w}_u(\{x, y\})$ is given by

$$\hat{\mathrm{P}}\big(\tilde{w}_u(\{x, y\}) = +1 \mid \hat{w}_1, ..., \hat{w}_{u-1}\big) = \begin{cases} p_k, & (|x - y| = 1), \\ \beta |x - y|^{-2}, & (|x - y| \ge 2), \end{cases}$$

and

$$\hat{\mathrm{P}}\big(\tilde{w}_u(\{x, y\}) = 0 \mid \hat{w}_1, ..., \hat{w}_{u-1}\big) = 1 - \hat{\mathrm{P}}\big(\tilde{w}_u(\{x, y\}) = +1 \mid \hat{w}_1, ..., \hat{w}_{u-1}\big),$$

provided that $u \in I_n^{\hat{w}}(k)$. Also, we assume that $\{\overline{w}_u, \tilde{w}_u(x, y); (x, y) \in \mathrm{B}\}$ are independent for each fixed $u$.

### 2.3 Gibbs distribution for Group A

A *Gibbs measure* on $\Omega_n$ with respect to $\hat{w} \in \hat{\Omega}$, which reflects the trading strategy of the traders in a Group A, is defined by

$$\mathrm{P}^{\hat{w}}(w) = \frac{1}{Z_n^{\hat{w}}} \exp\Big[- H^{\hat{w}}(w)\Big], \tag{2.3}$$

where $Z_n^{\hat{w}}$ is a normalization constant. A *Hamiltonian* is given by

$$H^{\hat{w}}(w) = \sum_{u=1}^{n} H^{u,\hat{w}}(w). \tag{2.4}$$

A local Hamiltonian which describes the traders' behavior in Group A at time $u$ in random interval $I_n^{\hat{w}}(k)$ is given by

$$\begin{aligned} H^{u,\hat{w}}(w) &= \beta_1 |w_u|^2 + \beta_2 \Phi\big(w_u \mid w_{u-a}^k, ..., w_{u-1}^k\big) \\ &\quad - \beta_3 f_1\Big(|w^k|_{u,a}, k/n^q\Big) |w_u| - \frac{\beta_4}{\sqrt{n}} f_2\Big(\langle w^k \rangle_{u,a}, k/n^q\Big) \langle w_u \rangle, \end{aligned} \tag{2.5}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are positive constants, $a$ is a fixed positive integer, $w^k$ is a restriction of $w \in \Omega_n$ to $I_n^{\hat{w}}(k)$, $f_1(x, t)$, $f_2(x, t)$ are real valued functions continuous in $t$, and

$$|w^k|_{u,a} = \sum_{\substack{1 \le l \le a \\ u-l \in I_n^{\hat{w}}(k)}} \big|w_{u-l}^k\big|, \quad \langle w^k \rangle_{u,a} = \sum_{\substack{1 \le l \le a \\ u-l \in I_n^{\hat{w}}(k)}} \big\langle w_{u-l}^k \big\rangle.$$

Note that $|w^k|_{u,a}$ and $\langle w^k \rangle_{u,a}$ are the total amounts of market activities and modified surplus orders in the past $\{u-a, u-a+1, ..., u-1\} \cap I_n^{\hat{w}}(k)$. If $u-l \notin I_n^{\hat{w}}(k)$, we do not count the term $w_{u-l}^k$ in $|w^k|_{u,a}$ and $\langle w^k \rangle_{u,a}$. The first term in (2.5) controls the activities of Group A. The second term expresses the trading strategies of traders who analyse the past data $\{w_{u-l}; 1 \le l \le a-1, u-l \in I_n^{\hat{w}}(k)\}$ in $I_n^{\hat{w}}(k)$. The third term plays a role to generate a volatility of the stock price process. If $f_1(\cdot, \cdot) > 0$ then the activity is increasing and a large volatility is obtained, otherwise the activity is decreasing and a small volatility is obtained. The forth term derives a drift (or trend) of the stock price process. If $f_2(\cdot, \cdot) > 0$ then the stock price process is in an up trend, otherwise is a down trend.

We assume that the local Hamiltonian satisfies the following conditions (A.1)–(A.4),

(A.1) If $w_u$ is static, that is $|w_u| = 0$, then

$$\Phi(w_u \mid w_{u-a}^k, ..., w_{u-1}^k) = 0.$$

(A.2) For any $w \in \Omega_n$,

$$\Phi(-w_u \mid -w_{u-a}^k, ..., -w_{u-1}^k) = \Phi(w_u \mid w_{u-a}^k, ..., w_{u-1}^k),$$

where $-w_u = (-w_u(1), ..., -w_u(N))$.

(A.3) There are positive constants $c_1$, $c_2$ and $c_3$ such that

$$|\Phi(w_u \mid w_{u-a}^k, ..., w_{u-1}^k)| \le c_1 |w_u|^2,$$

$$|f_1(x,t)| \le c_2 x \ \text{ for any } x > 0,$$

$$|f_2(x,t)| \le c_3 |x| \ \text{ for any } x.$$

(A.4) For any $x$ and $t$, $f_2(x,t) + f_2(-x,t) \ge 0$, and there exists an interval $J$ such that $f_2(x,t) + f_2(-x,t) \ge \varepsilon_0$ for any $x \in J$, where $\varepsilon_0 > 0$.

The coupling measure P on $\Omega \times \hat{\Omega}$ is defined by

$$\mathrm{P}(w, \hat{w}) = \mathrm{P}^{\hat{w}}(w)\, \hat{\mathrm{P}}(\hat{w}).$$

When the total amount of modified surplus orders $\langle w_u \rangle + \langle \hat{w}_u \rangle$ is positive, it is expected that there is a strong driving activity on the part of buyer and the stock price is going to move in upper direction. On the other hand, market is going to fall when $\langle w_u \rangle + \langle \hat{w}_u \rangle$ is negative. We define the stock price change at time $u$ by

$$\frac{S_u}{S_{u-1}} = e^{c_0(\langle w_u \rangle + \langle \hat{w}_u \rangle)}, \tag{2.6}$$

where $c_0 > 0$ is a constant called the *market depth*. Note that we think $S_u$ is the *closing price* other than opening price. This recurrence formula implies that

$$S_u = S_0 \exp\left\{ c_0 \sum_{l=1}^{u} (\langle w_l \rangle + \langle \hat{w}_l \rangle) \right\}, \tag{2.7}$$

where $S_0$ is the initial stock price at time 0.

## 3 Statement of results

We consider the processes

$$W_u = \sum_{l=1}^{u} \langle w_l \rangle, \quad \hat{W}_u = \sum_{l=1}^{u} \langle \hat{w}_l \rangle.$$

Then by (2.7), the stock price process is described as $S_u = S_0 \, e^{c_0 (W_u + \hat{W}_u)}$.

Note that for every $\hat{w}$ and $t \in (0,1]$ there exists a unique $k = k(n,t) = 1, ..., n^q + 1$ such that $[nt] \in I_n^{\hat{w}}(k(n,t))$. A scaled process $\{W_t^{(n)}\}_{t \in [0,1]}$ of $\{W_u\}_{u=1}^{n}$ is given by $W_0^{(n)} = 0$ and

$$W_t^{(n)} = \begin{cases} \dfrac{1}{\sqrt{n}} W_{U_{k(n,t)}}, & \text{if } k(n,t) \le n^q, \\ \dfrac{1}{\sqrt{n}} W_{U_{n^g} + n^{1-\lambda}}, & \text{otherwise,} \end{cases} \quad \text{for } t \in (0,1],$$

where $\lambda$ is a constant satisfying $\frac{1}{4} < \lambda < \frac{1}{2}$. A scaled process $\left\{\hat{W}_t^{(n)}\right\}_{t \in [0,1]}$ of $\{\hat{W}_u\}_{u=1}^{n}$ is also given in a similar way.

We define a process by

$$X_t^{(n)} = W_t^{(n)} + \hat{W}_t^{(n)}.$$

Let $\tau \in (0,1)$ be a fixed time and $f_3(t) > 0$ be a continuous function on [0, 1] such that

$$\int_0^1 \frac{1}{f_3(x)} dx = \tau.$$

A continuous function $s(t)$, $t \in [0, \tau]$ is defined implicitly by

$$\int_0^{s(t)} \frac{1}{f_3(x)} dx = t. \tag{3.1}$$

Since $f_3(t)$ is a positive function, $s(t)$ is well–defined.

**Theorem 3.1** *For $\frac{3}{8} < \delta < \frac{1}{2}$, $\frac{1}{4} < \lambda < \frac{1}{2}$ and $q = \frac{1}{2} - \delta$, the process $X_t^{(n)}$ converges in finite dimensional distribution to the process*

$$X_t = \int_0^t \big(\mu_A(v) + \mu_B(v)\big) dv + \int_0^t \sigma_A(v)\, dB_v + h\, 1_{\{t=\tau\}}, \quad \textit{for all } t \in [0, \tau], \tag{3.2}$$

*where $B_t$ is a standard Brownian motion and $h$ is a jump length given in* (4.16). *Trend terms and the volatility term of the limit price process are described in terms of the polymer functionals in the theory of cluster expansion as follows.*

$$\mu_A(t) = \beta_4 \sum_{i(A)=0} \langle A \rangle f_2(A,\ s(t)) \phi_0(A) e^{\beta_3 f_1(A,\ s(t))} \frac{\alpha^T(A)}{A\,!} f_3(s(t)), \tag{3.3}$$

$$\mu_B(t) = f_3(s(t)), \tag{3.4}$$

$$\sigma_A^2(t) = \sum_{i(A)=0} \langle A \rangle^2 \phi_0(A) e^{\beta_3 f_1(A,\, s(t))} \frac{\alpha^T(A)}{A\,!} f_3(s(t)). \tag{3.5}$$

We shall summarize a method of cluster expansion in the section 5. The precise definition of polymer functionals are given in (5.4), (5.5), (5.6), (5.7), (5.8).

Let $\{\tau_i\,; i = 1, 2, ...\}$ be i.i.d. sequence of exponential holding times with mean 1/c, and we write $\tau_0 = 0$. When $\tau_i \le 1$, the stock price is continuous on each random interval $(\tau_{i-1}, \tau_i)$, and it jumps at each random time $\tau_i$, and jumps are i.i.d. with distribution $\rho$. The stock price process on $(\tau_{i-1}, \tau_i]$ behaves just like on $(0, \tau_1]$. Then by using the same argument in the proof of Theorem 3.1 repeatedly, we will obtain the following.

Theorem 3.2 *The scaled process $X_t^{(n)}$ converges in finite dimensional distribution to the process*

$$X_t = \int_0^t (\mu_A(v) + \mu_B(v))\, dv + \int_0^t \sigma_A(v)\, dB_v + Y_t, \quad \textit{for all} \quad t \in [0, 1],$$

*where the jump term $Y_t$ is a compound Poisson process, that is*

$$Y_t = \int_{[0,t]} \int_{(-\infty,\infty)\setminus\{0\}} x N_p(ds\, dx),$$

*where $N_p(ds\ dx)$ is a Poisson random measure. The Lévy measure of $Y_t$ is $\mu(dx) = c\rho(dx)$ with $c > 0$ and $\rho((-\infty, \infty)) = 1$.*

## 4 The long–range percolation model

In this section, we define the initial nearest neighbor percolation probabilities $p_0$ and the news parameters $\gamma_k^+$, $\gamma_k^-$ by using some known results on the long–range percolation model. We also estimate the stopping times $T_k$. We sometimes denote the probability measure on $\hat{\Omega}$ defined in the previous section by $\hat{\mathrm{P}}_p$ with the nearest neighbor percolation probability $p$ instead of $\hat{\mathrm{P}}$.

We set $C_1(p, \beta) = p + \beta/4$ and

$$\varepsilon(n) = \inf\left\{ \varepsilon > 0; \frac{16C_0(p_c(\beta) - \varepsilon, \beta)^3}{C_1(p_c(\beta) - \varepsilon, \beta)^2} \le n^{2\lambda - 1/2} \right\} + \frac{1}{n}, \tag{4.1}$$

for each $n \in \mathrm{N}$ Since $16C_0(p_c(\beta) - \varepsilon, \beta)^3 / C_1(p_c(\beta) - \varepsilon, \beta)^2 < \infty$ for each $p < p_c(\beta)$ and $\lim_{n\to\infty} n^{2\lambda-1/2} = \infty$, we have

$$\varepsilon(n) \to 0, \quad \text{as} \quad n \to \infty.$$

We note that

$$\frac{16C_0\,(p_c(\beta) - \varepsilon(n), \beta)^3}{C_1(p_c(\beta) - \varepsilon(n), \beta)^2} \frac{1}{n^{2\lambda}} \le \frac{1}{\sqrt{n}} \tag{4.2}$$

holds for each $n \in \mathrm{N}$.

Number of selected Group B traders is given by

$$N_n = \inf\left\{N \in \mathrm{N}; \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_N|}{|B_N|}\right] \leq n^{-\lambda}\right\}. \tag{4.3}$$

Proposition 4.1 *If n is sufficiently large, then we have*

$$\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] = \frac{1}{n^{\lambda}} - o\left(\frac{1}{n^{\lambda}}\right). \tag{4.4}$$

$$\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|^2}{|B_{N_n}|^2}\right] \leq \frac{1}{\sqrt{n}}. \tag{4.5}$$

Lemma 4.2 *For any $p < p_c(\beta)$ and $N \geq 2$,*

$$0 < 2C_1(p,\beta) \leq \hat{\mathrm{E}}_p\left[|C_N|\right] \leq \hat{\mathrm{E}}_p\left[|C_\infty(0)|\right] \leq 4c_0(p,\beta) < \infty, \tag{4.6}$$

Proof. From Theorem 2.1 (3),

$$\hat{\mathrm{E}}_p\left[|C_N|\right] = \sum_{x\in B_N} \tau(0,x) = 2\sum_{l=1}^{N} \tau(0,l) \geq 2\sum_{l=1}^{N} p(l) \geq 2c_1(p,\beta) > 0,$$

and

$$\hat{\mathrm{E}}_p\left[|C_N|\right] \leq \hat{\mathrm{E}}_p\left[|C_\infty(0)|\right] = 2\sum_{l=1}^{\infty} \tau(0,l) \leq 2c_0(p,\beta)\sum_{l=1}^{\infty} l^{-2} \leq 4c_0(p,\beta) < \infty. \qquad \square$$

Proof of Proposition 4.1 By the definition of $N_n$ we have,

$$\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \leq n^{-\lambda} \tag{4.7}$$

$$\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n-1}|}{|B_{N_n-1}|}\right] > n^{-\lambda} \tag{4.8}$$

$$\begin{aligned}
&\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n-1}|}{|B_{N_n-1}|}\right] - \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right]\\
&= \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n-1}|}{|B_{N_n-1}|}\right]\frac{|B_{N_n}| - |B_{N_n-1}|}{|B_{N_n}|} - \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}| - |C_{N_n-1}|}{|B_{N_n}|}\right]\\
&\leq \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n-1}|}{|B_{N_n-1}|}\right]\frac{2}{|B_{N_n}|}\\
&\leq \frac{2}{|B_{N_n}|}\frac{1}{n^{\lambda}}
\end{aligned}$$

From (4.6) and (4.7), we have

$$2C_1(p_c(\beta)-\varepsilon(n),\beta)n^{\lambda} \le |B_{N_n}|. \tag{4.9}$$

From this and (4.8), we have (4.4).

By the tree graph inequality (see [1] Prop. 4.1),

$$\hat{\mathrm{P}}(x_1 \leftrightarrow x_2 \leftrightarrow x_3) \le \sum_{y\in\mathbf{Z}} \tau(y,x_1)\tau(y,x_2)\tau(y,x_3) \tag{4.10}$$

From this we have

$$\begin{aligned}
\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[|C_{N_n}|^2\right] &= \sum_{x_1,x_2\in B_{N_n}} \hat{\mathrm{P}}(0 \leftrightarrow x_1 \leftrightarrow x_2) \\
&\le \sum_{x_1,x_2\in B_{N_n}} \sum_{y\in\mathbf{Z}} \tau(y,0)\tau(y,x_1)\tau(y,x_2) \\
&\le \sum_{y\in\mathbf{Z}}\left[\tau(y,0)\left\{\sum_{x\in B_{N_n}} \tau(y,x)\right\}^2\right] \\
&\le \sum_{y\in\mathbf{Z}}\left[\tau(y,0)\left\{\sum_{x\in\mathbf{Z}} \tau(0,x)\right\}^2\right] \\
&= \hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[|C_\infty(0)|\right]^3.
\end{aligned} \tag{4.11}$$

Thus, by (4.7), (4.6), (4.9) and (4.2)

$$\begin{aligned}
\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|^2}{|B_{N_n}|^2}\right] &\le \frac{\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[|C_\infty(0)|\right]^3}{\hat{\mathrm{E}}_{p_c(\beta)-\varepsilon(n)}\left[|C_{N_n}|\right]^2 n^{2\lambda}} \\
&\le \frac{\left(4C_2\left(p_c(\beta)-\varepsilon(n),\beta\right)\right)^3}{\left(2C_1\left(p_c(\beta)-\varepsilon(n),\beta\right)\right)^2 n^{2\lambda}} = \frac{16C_0\left(p_c(\beta)-\varepsilon(n),\beta\right)^3}{c_1\left(p_c(\beta)-\varepsilon(n),\beta\right)^2}\frac{1}{n^{2\lambda}} \le \frac{1}{\sqrt{n}}.
\end{aligned} \tag{4.12}$$

Hence, we have (4.5). □

**Proposition 4.3** *For every $\beta > 1$, if $p \ge p_c(\beta)$ then we have*

$$\hat{\mathrm{E}}_p\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \ge \beta^{-1} \tag{4.13}$$

$$\hat{\mathrm{E}}_p\left[\frac{|C_{N_n}|^2}{|B_{N_n}|^2}\right] \ge \beta^{-3/2} \tag{4.14}$$

**Proof of Proposition 4.3** The first assertion obeys from Theorem 2.1 (1) (2) and the FKG inequality as

$$
\begin{aligned}
\hat{\mathrm{E}}\big[|C_{N_n}|\big] &= \sum_{x\in B_N} \hat{\mathrm{P}}(0 \leftrightarrow x) \\
&\geq \sum_{x\in B_N} \hat{\mathrm{P}}\big(0 \leftrightarrow x, |C_\infty(0)| = +\infty, |C_\infty(x)| = +\infty\big) \\
&= \sum_{x\in B_N} \hat{\mathrm{P}}\big(|C_\infty(0)| = +\infty, |C_\infty(x)| = +\infty\big) \\
&\geq |B_{N_n}|\hat{\mathrm{P}}\big(|C_\infty(0)| = \infty\big)^2 \\
&\geq |B_{N_n}|\beta^{-1}.
\end{aligned}
$$

Similarly, we see the second assertion

$$
\begin{aligned}
\hat{\mathrm{E}}\Big[|C_{N_n}|^2\Big] &= \sum_{x_1,x_2\in B_N} \hat{\mathrm{P}}(0 \leftrightarrow x_1 \leftrightarrow x_2) \\
&\geq \sum_{x_1,x_2\in B_N} \hat{\mathrm{P}}\big(0 \leftrightarrow x_1 \leftrightarrow x_2, |C_\infty(0)| = +\infty, |C_\infty(x_1)| = +\infty, |C_\infty(x_2)| = +\infty\big) \\
&= \sum_{x_1,x_2\in B_N} \hat{\mathrm{P}}\big(|C_\infty(0)| = +\infty, |C_\infty(x_1)| = +\infty, |C_\infty(x_2)| = +\infty\big) \\
&\geq |B_{N_n}|^2\hat{\mathrm{P}}\big(|C_\infty(0)| = \infty\big)^3 \\
&\geq |B_{N_n}|^2\beta^{-3/2}.
\end{aligned}
$$

□

We define an initial percolation probability by

$$
p_0 = \inf\left\{p > 0; \hat{\mathrm{E}}_p\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \geq \frac{\sup\{f_3(t); t \in [0,1]\}}{\sqrt{n}}\right\}. \tag{4.15}
$$

By noticing that from Proposition 4.1 we have

$$
\frac{\sup\{f_3(t); t \in [0,1]\}}{\sqrt{n}} \leq \hat{\mathrm{E}}_{p_k}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \leq \frac{1}{n^\lambda} \quad \text{for} \quad k \leq n^q,
$$

we take the news parameters $\gamma^+(u) = \gamma_k^+$ and $\gamma^-(u) = \gamma_k^-$ at time $u$ satisfying

$$
\hat{\mathrm{E}}\big[\langle\hat{w}_u\rangle\big] = (\gamma_k^+ - \gamma_k^-)\hat{\mathrm{E}}_{p_k}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] = \begin{cases} \dfrac{1}{\sqrt{n}} f_3\left(\dfrac{k}{n^q}\right), & \text{if } k \leq n^q, \\ \dfrac{h}{n^{1/2-\lambda}}, & \text{if } k = n^q + 1. \end{cases} \tag{4.16}
$$

provided that $u$ belongs to $I_n^{\hat{w}}(k)$, where $h \in \mathrm{R}$ is a constant. We note that ${\gamma^+}_k > {\gamma^-}_k$ whenever $k \leq n^q$.

**Proposition 4.4** *If*

$$
0 < \varepsilon < \delta/2, \tag{4.17}
$$

*then for any* $k = 1, \ldots, n^q$,

$$\hat{\mathrm{P}}_{p_k}\left(\left|T_k - \frac{n^{\delta+1/2}}{f_3(k/n^q)}\right| > n^{\delta/2+1/2+\varepsilon}\right) \le \frac{c}{n^{2\varepsilon}} \tag{4.18}$$

*for a sufficiently large n, where c is a positive constant.*

**Lemma 4.5** *For* $k = 1, ..., n^q$, *we have* $\hat{\mathrm{P}}_{p_k}(T_k < u) \le 2\hat{\mathrm{P}}_{p_k}\left(\hat{W}_u > n^{\delta}\right)$.

**Proof.** We will make a use of the coupling argument. Let $U(u)$ be an independent random variable, uniformly distributed on the interval [0, 1], which is independent of $\{\tilde{w}_u\}$. We can make a relation $U(u)$ and $\overline{w}_u$ by $\overline{w}_u = \left(-1_{[0,\gamma^-(u)]} + 1_{(\gamma^-(u),\gamma^+(u)+\gamma^-(u)]}\right)(U(u))$. We define $h(u) = \left(-1_{[0,\gamma^-(u)]} + 1_{(\gamma^-(u),2\gamma^-(u)]}\right)(U(u))$. Since $\gamma^-(u) < \gamma^+(u)$, it is clear that

$$\hat{\mathrm{P}}(\overline{w}_u \ge h(u) \text{ for any } u) = 1. \tag{4.19}$$

For each $u = 1, 2, ..., n$ let $\hat{\eta}(u) = h(u)\tilde{w}_u$, $Z(u) = \sum_{l=1}^{u} \hat{\eta}(l) + \hat{\eta}(0)$, where $\hat{\eta}(0) = 0$ a.s. Then $Z(u)$ is a symmetric random walk on R. From (4.19), $Z(u)$ is stochastically dominated by $\hat{W}_u$, that is

$$\hat{\mathrm{P}}(\hat{W}_u \ge Z(u) \text{ for any } t) = 1.$$

We define the filtration as $\mathcal{F}_u = \sigma\left(U(l), \tilde{w}_l; l \le u\right)$. Let $\hat{\mathrm{P}}^{n^{\delta}}$ is a shifted measure of $\hat{\mathrm{P}}$ with

$$\hat{\mathrm{P}}^{n^{\delta}}\left(W(0) = n^{\delta}, Z(0) = n^{\delta}\right) = \hat{\mathrm{P}}^{n^{\delta}}\left(w(0) = n^{\delta}, \eta(0) = n^{\delta}\right) = 1.$$

Then by the strong Markov property,

$$\begin{aligned}
\hat{\mathrm{P}}\left(T_k < u, \hat{W}_u < n^{\delta}\right) &= \hat{\mathrm{E}}\left[\hat{\mathrm{P}}\left(\hat{W}_u < n^{\delta} \mid \mathcal{F}_{T_k}\right)1_{\{T_k < u\}}\right] \\
&= \hat{\mathrm{E}}\left[\hat{\mathrm{P}}^{W(T_k)}\left(W(u-T_k) < n^{\delta}\right)1_{\{T_k<u\}}\right] \\
&\le \hat{\mathrm{E}}\left[\hat{\mathrm{P}}^{n^{\delta}}\left(W(u-T_k) < n^{\delta}\right)1_{\{T_k<u\}}\right] \\
&\le \hat{\mathrm{E}}\left[\hat{\mathrm{P}}^{n^{\delta}}\left(Z(u-T_k) < n^{\delta}\right)1_{\{T_k<u\}}\right] = \frac{1}{2}\hat{\mathrm{P}}(T_k < u).
\end{aligned}$$

Therefore

$$\begin{aligned}
\hat{\mathrm{P}}(T_k < u) &= \hat{\mathrm{P}}\left(T_k < u, \hat{W}_u \ge n^{\delta}\right) + \hat{\mathrm{P}}\left(T_k < u, \hat{W}_u < n^{\delta}\right) \\
&= \hat{\mathrm{P}}\left(\hat{W}_u \ge n^{\delta}\right) + \hat{\mathrm{P}}\left(T_k < u, \hat{W}_u < n^{\delta}\right) \\
&\le \hat{\mathrm{P}}\left(\hat{W}_u \ge n^{\delta}\right) + \frac{1}{2}\hat{\mathrm{P}}(T_k < u).
\end{aligned}$$

□

Proof of Proposition 4.4 From Lemma 4.5 and Chebyshev's inequality, we have

$$\hat{\mathrm{P}}\left(\left|T_k - \frac{n^{\delta+1/2}}{f_3(k/n^q)}\right| > n^{\delta/2+1/2+\varepsilon}\right)$$

$$\leq 2\hat{\mathrm{P}}\left(\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}-n^{\delta/2+1/2+\varepsilon}} > n^\delta\right) + 2\hat{\mathrm{P}}\left(\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}+n^{\delta/2+1/2+\varepsilon}} < n^\delta\right)$$

$$\leq 2\hat{\mathrm{P}}\left(\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}-n^{\delta/2+1/2+\varepsilon}} - E\left[\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}-n^{\delta/2+1/2+\varepsilon}}\right] > f_3(k/n^q)n^{\delta/2+\varepsilon}\right)$$

$$+ 2\hat{\mathrm{P}}\left(\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}+n^{\delta/2+1/2+\varepsilon}} - E\left[\hat{W}_{\frac{n^{\delta+1/2}}{f_3(k/n^q)}+n^{\delta/2+1/2+\varepsilon}}\right] < -f_3(k/n^q)n^{\delta/2+\varepsilon}\right)$$

$$\leq 2\frac{\frac{n^{\delta+1/2}}{f_3(k/n^q)} - n^{\delta/2+1/2+\varepsilon}}{f_3(k/n^q)^2 n^{\delta+2\varepsilon+1/2}} + 2\frac{\frac{n^{\delta+1/2}}{f_3(k/n^q)} + n^{\delta/2+1/2+\varepsilon}}{f_3(k/n^q)^2 n^{\delta+2\varepsilon+1/2}} = \frac{4}{f_3(k/n^q)^3 n^{2\varepsilon}} \leq \frac{c}{n^{2\varepsilon}},$$

since $f_3$ is a strictly positive function.

We call $\mathbf{r} = \{r_k\}_{k=1}^{n^q}$ an *admissible sequence* if

$$\left|r_k - \frac{n^{\delta+1/2}}{f_3(k/n^q)}\right| \leq n^{\delta/2+1/2+\varepsilon} \quad \text{for all} \quad k = 1, \dots, n^q.$$

For $k = 1, \dots, n^q$, we set $s_k = r_1 + r_2 + \cdots + r_k$ and let $I(k) = I_n(k, r)$ be a set of consecutive numbers $\{s_{k-1}+1, s_{k-1}+2, \dots, s_k\}$, and we write $I(n^q+1) = \{s_{n^q}+1, \dots, n\}$. For an admissible sequence $\mathbf{r} = \{r_k\}_{k=1}^{n^q}$, we have

$$\begin{aligned} r_1 + \cdots + r_{n^q} &= \sum_{k=1}^{n^q} \frac{n^{\delta+1/2}}{f_3(k/n^q)} + o(n) \\ &= n^{\delta+1/2} n^q \sum_{k=1}^{n^q} \frac{1}{n^q} \frac{1}{f_3(k/n^q)} + o(n) \\ &= n \int_0^1 \frac{dx}{f_3(x)} + o(n) = n\tau + o(n). \end{aligned}$$

Corollary 4.6 *If*

$$0 < q/2 < \varepsilon < \delta/2, \tag{4.20}$$

*then we have*

$$\hat{\mathrm{P}}\left(\{T_k\}_{k=1}^{n^q} \text{ is an admissible sequence}\right) \to 1 \ (n \to \infty). \tag{4.21}$$

## 5 Method of cluster expansion

In this section, we summarize an algebraic formalism of cluster expansion for Gibbs distribution developed in the mathematical theory of phase transition. We introduce a notion of polymer $\xi$ and decompose $w \in \Omega_n$ into a set of polymers $\{\xi^1, ..., \xi^m\}$ in such a way that (i) the stock price changes only in $\xi^i$ and (ii) $\xi^i$ and $\xi^i \{i \neq j\}$ are independent.

A *polymer* $\xi$ with respect to $\hat{w} \in \hat{\Omega}$ is a collection $(\eta, b(\xi), k(\xi))$ with the following four conditions :

(P.1) $k(\xi) \in \{1, 2, ..., n^q + 1\}$ and $b(\xi)$ is a set of consecutive numbers in $I_n^{\hat{w}}(k(\xi))$

(P.2) $\eta = \left\{\eta_u \in \{+1, -1, 0\}^N ; u \in b(\xi)\right\}$.

(P.3) For each $u \in b(\xi)$, there exists $l = 0, ..., a$ satisfying $u + l \in b(\xi)$ and $w_{u+l}$ is active.

(P.4) Let $u_0$ be the left end point of $b(\xi)$. If $u_0 \neq U_{k(\xi)-1} + 1$, then $\eta_{u_0+a}$ is active and $\eta_{u_0+l}$ is static for $l = 0, ..., a-1$.

We denote by $\mathcal{D}$ the set of all polymers.

A pair of polymers $\xi$ and $\xi'$ is said to be *compatible* if $b(\xi) \cap b(\xi') = \emptyset$. A family of polymers $\{\xi'\}$ is said to be compatible if each pair of polymers $\xi^i$ and $\xi^j (i \neq j)$ is compatible. A pair of polymers $\xi$ and $\xi'$ is said to be *incompatible* if it is not compatible.

Let $m_0$ be the number of static elements in $\{+1, -1, 0\}^N$, which can be expressed by

$$m_0 = \sum_{k_1 + k_2 \leq d_0} \frac{N!}{k_1! k_2! (N - k_1 - k_2)!}.$$

We introduce a statistical weight of a polymer $\xi = (\eta, b(\xi), k(\xi))$ by

$$\mathcal{W}(\xi) = \exp\left\{- |b(\xi)| \log m_0 - \sum_{u \in b(\xi)} H^{u, \hat{w}}(\eta)\right\}, \tag{5.1}$$

here, we regard $\eta$ as an element in $\Omega_n$ with $\eta_u = 0$ for $u \notin b(\xi)$, so that the expression $H^{u,\hat{w}}(\eta)$ makes sense.

For given $\hat{w} \in \hat{\Omega}$ and for $u = 1, 2, ..., n$, we write $k(u) = 1, ..., n^q + 1$ if $u \in I_n^{\hat{w}}(k(u))$. Take $w \in \Omega_n$, a set

$$\bigcup_{\substack{u = 1, ..., n \\ w_u \text{ is active}}} \{u - a, ..., u - 1, u\} \cap I_n^{\hat{w}}(k(u))$$

is decomposed into the sets $W_1, ..., W_m$ of consecutive numbers and there is a increase sequence $\{k_i\}_{i=1}^m$ of positive integers such that $I_n^{\hat{w}}(k_i)$ is a unique random interval includes $W_i$. Then for any $w \in \Omega$, a collection of triples $\{\xi^i\}_{i=1}^m = \left\{\left(\{w_u\}_{u \in W_i}, W_i, k_i\right)\right\}_{i=1}^m$ is a compatible family of polymers, we say $w$ forms $\{\xi^i\}_{i=1}^m$. For any compatible family of polymers $\{\xi^i\}_{i=1}^m$, we write

$$\mathrm{P}^{\hat{w}}(\xi^1, ..., \xi^m) = \mathrm{P}^{\hat{w}}\left(w \text{ forms} \{\xi^i\}_{i=1}^m\right) = \frac{1}{Z_n^{\hat{w}}} \Xi_n^{\hat{w}}(\xi^1, ..., \xi^m),$$

where

$$\Xi_n^{\hat{w}}(\xi^1, ..., \xi^m) = \sum_{\substack{w \in \Omega_n \\ w \text{ forms} \{\xi^i\}_{i=1}^m}} \exp\left\{- H^{\hat{w}}(w)\right\}. \tag{5.2}$$

For any compatible family of polymers $\{\xi^i = (\eta^i, b(\xi^i), k(\xi^i))\}$ and any $w$ which forms $\{\xi^i\}_{i=1}^m$, if $u \in \{1, ..., n\} \setminus \bigcup_{i=1}^m b(\xi^i)$, then $w_u$ is static, so that $H^{u,\hat{w}}(w) = 0$, and if $u \in b(\xi^i)$, then $w_u$ coincides with $\eta_u^i$ for $i = 1, ..., m$. Hence we have

$$\Xi(\xi^1, ..., \xi^m) = \left( \prod_{i=1}^m \exp\left\{ - \sum_{u \in b(\xi^i)} H^{u,\hat{w}}(\eta^i) \right\} \right) \times m_0^{n - \sum_{i=1}^m |b(\xi^i)|}.$$

This implies the polymer representation

$$\mathrm{P}_n^{\hat{w}}(\xi^1, ..., \xi^m) = \frac{1}{\tilde{\mathrm{Z}}_n^{\hat{w}}} \prod_{i=1}^m \mathcal{W}(\xi^i), \tag{5.3}$$

where $\tilde{Z}_n^{\hat{w}} = Z_n^{\hat{w}} / m_0^n$.

We denote by $\chi$ the space of mappings $A$ from $\mathcal{D}$ to $\mathrm{N} \cup \{0\}$ satisfying

$$|A| := \sum_{\xi \in \mathcal{D}} A(\xi) < \infty.$$

We denote by $suppA = \{\xi \in \mathcal{D} \mid A(\xi) \neq 0\}$ , $A! = \prod_{\xi \in suppA} A(\xi)!$, $b(A) = \sum_{\xi \in suppA} b(\xi)$ and define $\alpha(A)$ by

$$\alpha(A) = \begin{cases} 1, & \text{if } A! = 1 \text{ and any pair of } \xi^{\mathrm{i}}, \xi^{\mathrm{i}} \in suppA \text{ are compatible,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $G(A)$ be a graph whose vertex set is $suppA$ and edge set is all incompatible pairs in $suppA$.
For any graph $G$, we denote by $|G|$ a number of edges. The *Ursell function* $\alpha^T(A)$ is defined by

$$\alpha^T(A) = \sum_{G' \subset G(A)} (-1)^{|G'|}, \tag{5.4}$$

where the summation is over all connected subgraph $G'$ of $G$ whose vertex set is also $suppA$. We observe that if $\alpha^T(A) \neq 0$, then for each $\xi^i, \xi^j \in suppA$ there exists a chain $\{\xi_1, \xi_2, ..., \xi_m\} \subset suppA$ such that $\xi_1 = \xi^i$, $\xi_m = \xi^j$ and $b(\xi_l) \cap b(\xi_{l+1}) \neq \emptyset$ for $l = 1, ..., m$, which implies there exits a unique $k(A) = 1, 2, ..., n^q + 1$ such that $b(A) \subset I_n^{\hat{w}}(k(A))$. Hence, for any $A \in \chi$ with $\alpha^T(A) \neq 0$ we define

$$f_1(A, k(A)/n^q) = \sum_{\xi \in \mathcal{D}} \sum_{\xi \in b(\xi)} f_1\left( |\eta^{k(\xi)}|_{u,a}, k(A)/n^q \right) |\eta_u| A(\xi), \tag{5.5}$$

$$f_2(A, k(A)/n^q) = \sum_{\xi \in \mathcal{D}} \sum_{\xi \in b(\xi)} f_2\left( \langle \eta^{k(\xi)} \rangle_{u,a}, k(A)/n^q \right) \langle \eta_u \rangle A(\xi). \tag{5.6}$$

For any $A \in \chi$, we write

$$\langle A \rangle = \sum_{\xi \in \mathcal{D}} \sum_{\xi \in b(\xi)} \langle \eta_u \rangle A(\xi), \qquad \langle A \rangle^2 = \sum_{\xi \in \mathcal{D}} \sum_{\xi \in b(\xi)} \langle \eta_u \rangle^2 A(\xi). \tag{5.7}$$

A function space $\mathcal{L}$ is given by

$$\mathcal{L} = \left\{ \varphi : \chi \to \mathrm{C};\ \sup_{|A|=n} |\varphi(A)| < \infty \text{ for any } n \right\}.$$

An element $\varphi \in \mathcal{L}$ is said to be *multiplicative* if

$$\varphi(A_1 + A_2) = \varphi(A_1)\varphi(A_2) \text{ for all } A_1, A_1 \in \chi.$$

For any $\xi = (\eta, b(\xi), k(\xi)) \in \mathcal{D}$, put

$$\phi_0(\xi) = \exp\left[ -|b(\xi)| \log m_0 - \sum_{u \in b(\xi)} \left\{ \beta_1 |\eta_u|^2 + \beta_2 \Phi\left( \eta_u \mid \eta_{u-a}^{k(\xi)}, \ldots, \eta_{u-1}^{k(\xi)} \right) \right\} \right],$$

$$\phi_1(\xi) = \exp\left[ \beta_3 \sum_{u \in b(\xi)} f_1\left( |\eta^{k(\xi)}|_{u,a}, k(\xi)/n^q \right) |\eta_u| \right],$$

$$\phi_3(\xi) = \exp\left[ \frac{\beta_4}{\sqrt{n}} \sum_{u \in b(\xi)} f_2\left( \langle \eta^{k(\xi)} \rangle_{u,a}, k(\xi)/n^q \right) \langle \eta_u \rangle \right].$$

For any $A \in \chi$, we set

$$\phi_0(A) = \prod_{\xi \in \mathcal{D}} \phi_0(A)^{A(\xi)}, \quad \phi_1(A) = \prod_{\xi \in \mathcal{D}} \phi_1(A)^{A(\xi)}, \quad \phi_2(A) = \prod_{\xi \in \mathcal{D}} \phi_2(A)^{A(\xi)}. \tag{5.8}$$

Note that $\phi_0$, $\phi_1$ and $\phi_2$ are multiplicative functions. If $\alpha^T(A) \neq 0$, then

$$\phi_1(A) = \exp\{\beta_3 f_1(A, k(A)/n^q)\}, \quad \phi_2(A) = \exp\left\{ \frac{\beta_4}{\sqrt{n}} f_2(A, k(A)/n^q) \right\},$$

**Proposition 5.1** (1) *If a multiplicative function $\varphi \in \mathcal{L}$ satisfies that*

$$\sum_{A \in \chi} |\varphi(A)| \frac{|\alpha^T(A)|}{A!} < \infty$$

*then we have*

$$\sum_{A \in \chi} \varphi(A)\alpha(A) = \exp\left[ \sum_{A \in \chi} \varphi(A) \frac{\alpha^T(A)}{A!} \right].$$

(2) *There exists $\beta_0 > 0$ such that for any $\beta_1 > \beta_0$, we have*

$$\sum_{\substack{A \in \chi \\ 0 \in b(A)}} \phi_0(A)\phi_1(A)\phi_2(A) \frac{|\alpha^T(A)|}{A!} \leq g_1(\beta_1), \tag{5.9}$$

*where $g_1(\beta_1) \to 0$ as $\beta_1 \to \infty$.*

(3) *For any $0 < c < 1$ and any $\beta_1 > \beta_0/(1-c)$, we have*

$$\sum_{\substack{A \in \chi \\ 0 \in b(A), |A|_2 \geq k}} \phi_0(A)\phi_1(A)\phi_2(A) \frac{|\alpha^T(A)|}{A!} \leq g_2(\beta_1, c) e^{-c\beta_1 k}, \tag{5.10}$$

*where $g_2(\beta_1, c) \to 0$ as $\beta_1 \to \infty$ and*

$$|A|_2 = \sum_{\xi = (\eta, b(\xi)) \in \mathcal{D}} \sum_{u \in b(\xi)} |\eta_u|^2 A(\xi).$$

It follows from (1) and (2) that

$$\tilde{Z}_n^{\tilde{w}} = \exp\left[\sum_{A \in \chi} \phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A\,!}\right]. \tag{5.11}$$

**Lemma 5.2** *Let $\beta_2, \beta_3, \beta_4 > 0$ be fixed. For any $\beta_1 > 0$, put*

$$\theta(\beta_1) := \sum_{\substack{\xi \in \mathcal{D} \\ 0 \in b(\xi)}} \mathcal{W}(\xi)\exp\big[|b(\xi)|\big]. \tag{5.12}$$

*Then $\theta(\beta_1)$ tends to zero as $\beta_1$ tends to $\infty$.*

**Proof.** For any polymer $\xi = (\eta, b(\xi), k(\xi))$, we set

$$G(\eta_u) = \exp\Big\{-m_0 + 1 - \big(\beta_1 - c\beta_2 - c\beta_3 - c\beta_4\big)|\eta_u|^2\Big\},$$

for $u \in b(\xi)$, then the assumptions (A.1)–(A.5) imply

$$\mathcal{W}(\xi)\exp\big[|b(\xi)|\big] \le \prod_{u \in b(\xi)} G(\eta_u).$$

For any $k = 1, ..., N - d_0$, let $m_k$ be the number of the elements in $\{+1, -1, 0\}^N$ such that its modified number of market participants is $k$. It is obvious that $m_k \le 3^N$. Then we have

$$\sum_{\substack{\eta_u \in \{+1,-1,0\}^N \\ \eta_u \text{ is active}}} G(\eta_u) \le r, \qquad \sum_{\eta_u \in \{+1,-1,0\}^N} G(\eta_u) \le 1 + r,$$

where

$$r = \sum_{k=1}^{N-d_0} \exp\Big\{-m_0 + 1 - N\log 3 - \big(\beta_1 - c\beta_2 - c\beta_3 - c\beta_4\big)k^2\Big\},$$

which tends to zero as $\beta_1$ tends to $\infty$. Note that the number of $u \in b(\xi)$ such that $\eta_n$ is active is at least $\lceil |b(\xi)|/a \rceil$, where $\lceil x \rceil$ is the least integer not less than $x$. Hence we have

$$\begin{aligned}\sum_{\substack{\xi \in \mathcal{D} \\ 0 \in b(\xi)}} \mathcal{W}(\xi)\exp\big[|b(\xi)|\big] &= \sum_{l=1}^{\infty} \sum_{\substack{\xi \in \mathcal{D} \\ 0 \in b(\xi), |b(\xi)| = l}} \mathcal{W}(\xi)\exp\big[|b(\xi)|\big] \\ &\le \sum_{l=1}^{\infty} \big\{a(1+r)^{a-1}r\big\}^{\lceil l/a \rceil}.\end{aligned}$$

This implies our assertion. □

**Proof of Proposition 5.1.** Lemma 3.5 of [9] and Lemma 5.2 imply the first assertion. The second assertion immediately follows from the first assertion. The third assertion is obtained from Lemma 3.1 of [9] and the first assertion. □

## 6 Proof of the main Theorem

Proof. (First step) : We show the convergence of $X_t^{(n)}$ in one dimensional distribution. Let $\varphi_t(z) = \mathrm{E}\left[\exp\{iz\, X_t^{(n)}\}\right]$ be a characteristic function of $X_t^{(n)}$. We decompose it into two parts as follows,

$$\begin{aligned}\varphi_t(z) &= \mathrm{E}\left[e^{iz\, X_t^{(n)}}; \{T_j\}_{j=1}^{n^q} \text{ is admissible}\right] + \mathrm{E}\left[e^{iz\, X_t^{(n)}}; \{T_j\}_{j=1}^{n^q} \text{ is not admissible}\right] \\ &= I_1 + I_2.\end{aligned}$$

If $\delta > 1/2 - 2\varepsilon$, then it follows from Proposition 4.4 that

$$|I_2| \leq \frac{4n^q}{n^{2\varepsilon}} = 4n^{-(2\varepsilon - 1/2+\delta)} \to 0 \ \ (n \to \infty).$$

For the sequence of stopping times $\mathbb{T} = \{T_j\}_{j=1}^{n^q}$ and an admissible sequence $\mathbf{r} = \{r_j\}_{j=1}^{n^q}$, we write $\mathbb{T} = \mathbf{r}$ if $T_j = r_j$ for all $j = 1, ..., n^q$. Using this terminology we have

$$I_1 = \sum_{\mathbf{r}:\text{admissible}} \mathrm{E}^{\mathbf{r}}\left[\mathrm{e}^{izW_t^{(n)}}\right]\hat{\mathrm{E}}\left[\mathrm{e}^{iz\hat{W}_t^{(n)}} \,|\, \mathbb{T} = \mathbf{r}\right]\hat{\mathrm{P}}\,(\mathbb{T} = \mathbf{r}),$$

where $\mathrm{E}^{\mathrm{r}}[\cdot] = \mathrm{E}^{\hat{w}}\left[\cdot \mid \mathbb{T}(\hat{w}) = \mathbf{r}\right]$.

From (3.1), we have $s'(t)/f_3(s(t)) = 1$, thus we obtain

$$s(t) = \int_0^t f_3(s(x))\, dx. \tag{6.1}$$

Also it follows from (3.1) that

$$\sum_{u=1}^{s(t)n^q} \frac{1}{f_3(u/n^q)} \frac{1}{n^q} = t + o(1), \quad \text{as } n \to \infty.$$

On the other hand, for any $t \leq \tau$,

$$\sum_{u=1}^{k(n,t)} \frac{n^{\delta+1/2}}{f_3(u/n^q)} - k(n,t)n^{\delta/2+1/2+\varepsilon} \leq [nt] \leq \sum_{u=1}^{k(n,t)} \frac{n^{\delta+1/2}}{f_3(u/n^q)} + k(n,t)n^{\delta/2+1/2+\varepsilon}$$

Divide by $n$, since $k(n,t) \leq n^q$ and $\varepsilon < \delta/2$,

$$\sum_{u=1}^{k(n,t)} \frac{1}{f_3(u/n^q)} \frac{1}{n^q} = t + o(1).$$

Since $f_3(t) > 0$, there is a constant $c > 0$ such that

$$o(1) = \frac{1}{n^q}\left|\sum_{u=1}^{k(n,t)} \frac{1}{f_3(u/n^q)} - \sum_{u=1}^{s(t)n^q} \frac{1}{f_3(u/n^q)}\right| \geq \frac{c}{n^q}\,|k(n,t) - s(t)n^q|.$$

Hence we have

$$k(n,t) = s(t)n^q + o(n^q). \tag{6.2}$$

First, we consider the case $t < \tau$. We shall show the following convergences

$$\hat{\mathrm{E}}\Big[\mathrm{e}^{iz\hat{W}_t^{(n)}}\,|\mathbb{T}=\mathbf{r}\Big] \to \exp\Big\{iz\int_0^t \mu_B(v)\,dv\Big\}, \tag{6.3}$$

$$\mathrm{E}^{\mathbf{r}}\Big[\mathrm{e}^{iz\hat{W}_t^{(n)}}\Big] \to \exp\Big\{iz\int_0^t \mu_A(v)\,dv - \frac{1}{2}z^2\int_0^t \sigma_A^2(v)\,dv\Big\}. \tag{6.4}$$

Combining (6.3) and (6.4) with Corollary 4.6, we obtain for $t < \tau$,

$$\varphi_t(z) \to \exp\Big\{iz\int_0^t \big(\mu_A(v)+\mu_B(v)\big)\,dv - \frac{1}{2}z^2\int_0^t \sigma_A^2(v)\,dv\Big\}.$$

To see (6.3), recall that $I(k) = \{s_{k-1}+1,\dots,s_k\}$ where $s_k = \sum_{j=1}^k r_j$, we write

$$\hat{\mathrm{E}}\Big[\mathrm{e}^{iz\hat{W}_t^{(n)}}\,|\mathbb{T}=\mathbf{r}\Big] = \prod_{k=1}^{s(t)n^q}\hat{\mathrm{E}}_{I(k)}\Big[\exp\Big\{\frac{iz}{\sqrt{n}}\sum_{u\in I(k)}\langle\hat{w}_u\rangle\Big\}\,|\mathbb{T}=\mathbf{r}\Big]$$

Since $|\langle\hat{w}_u\rangle| \le 1$, we see

$$n^\delta \le \sum_{u=1}^{r_j}\Big|\langle\hat{w}_{u+s_j}\rangle\Big| \le n^\delta + 1$$

Divide the above inequality by $\sqrt{n}$, noticing that $\delta + q = 1/2$,

$$\frac{1}{\sqrt{n}}\sum_{u=1}^{r_j}\Big|\langle\hat{w}_{u+s_j}\rangle\Big| = \frac{1}{n^q} + O\Big(\frac{1}{\sqrt{n}}\Big).$$

From (6.1) and (3.4), we have

$$\begin{aligned}\hat{\mathrm{E}}\Big[\mathrm{e}^{iz\hat{W}_t^{(n)}}\,|\mathbb{T}=\mathbf{r}\Big] &= \exp\Big\{izs(t) + O\Big(\frac{n^q}{\sqrt{n}}\Big)\Big\}\\ &= \exp\Big\{iz\int_0^t \mu_B(v)\,dv + O\Big(\frac{1}{n^\delta}\Big)\Big\}.\end{aligned}$$

Hence we obtain (6.3).

As $W_t^{(n)}$ is a scaled process of a sum of independent random variables in $\{I_n^{\hat{w}}(k)\}_{k=1}^{k(n,t)}$ one can write

$$\mathrm{E}^{\mathbf{r}}\Big[\mathrm{e}^{izW_t^{(n)}}\Big] = \prod_{k=1}^{k(n,t)}\mathrm{E}_{I(k)}\Big[\exp\Big\{\frac{iz}{\sqrt{n}}W_{I(k)}\Big\}\Big].$$

We apply the method of cluster expansion (Proposition 5.1) and obtain the following formula,

$$\mathrm{E}_{I(k)}\Big[\exp\Big\{\frac{iz}{\sqrt{n}}W_{I(k)}\Big\}\Big] = \exp\Big\{\sum_{A\subset I(k)}\Big(\exp\Big\{\frac{iz}{\sqrt{n}}\langle A\rangle\Big\}-1\Big)\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A\,!}\Big\}.$$

We define a mirror image $\overline{\xi}$ of a polymer $\xi = (\eta, b(\xi), k(\xi)) \in \mathcal{D}$ by $\overline{\xi} = (-\eta, b(\xi), k(\xi))$ and define a reflection $\overline{A}$ of $A \in \chi$ by $\overline{A}(\xi) = A(\overline{\xi})$. Using a Taylor expansion and taking the fact that $\langle \overline{A} \rangle = -\langle A \rangle$, $\phi_0(\overline{A}) = \phi_0(A)$ and $\phi_1(\overline{A}) = \phi_1(A)$ into account, we have

$$\begin{aligned}
&\sum_{A \subset I(k)} \left( \exp\left\{ \frac{iz}{\sqrt{n}} \langle A \rangle \right\} - 1 \right) \phi_0(A)\phi_1(A)\phi_2(A) \frac{\alpha^T(A)}{A!} \\
&= \sum_{A \subset I(k)} \left( \frac{iz}{\sqrt{n}} \langle A \rangle - \frac{z^2}{2n} \langle A \rangle^2 + O\left( \frac{1}{n\sqrt{n}} \right) \right) \\
&\quad \left( 1 + \frac{\beta_4}{\sqrt{n}} f_2(A, k/n^q) + O\left( \frac{1}{n} \right) \right) \phi_0(A)\phi_1(A) \frac{\alpha^T(A)}{A!} \\
&= \frac{iz}{\sqrt{n}} \sum_{A \subset I(k)} \langle A \rangle \phi_0(A)\phi_1(A) \frac{\alpha^T(A)}{A!} - \frac{z^2}{2n} \sum_{A \subset I(k)} \langle A \rangle^2 \phi_0(A)\phi_1(A) \frac{\alpha^T(A)}{A!} \\
&\quad + \frac{iz\beta_4}{n} \sum_{A \subset I(k)} \langle A \rangle f_2(A, k/n^q) \phi_0(A)\phi_1(A) \frac{\alpha^T(A)}{A!} + O\left( \frac{n^{\delta+1/2}}{n\sqrt{n}} \right) \\
&= -\frac{z^2}{2n^q} \sum_{i(A)=0} \langle A \rangle^2 \phi_0(A) e^{\beta_3 f_1(A, k/n^q)} \frac{\alpha^T(A)}{A!} \\
&\quad + \frac{iz\beta_4}{n^q} \sum_{i(A)=0} \langle A \rangle f_2(A, k/n^q) \phi_0(A) e^{\beta_3 f_1(A, k/n^q)} \frac{\alpha^T(A)}{A!} + O\left( \frac{1}{n^{1-\delta}} \right).
\end{aligned}$$

Noticing that $s'(t) = f_3(s(t))$ and (3.5), we have

$$\begin{aligned}
&\frac{1}{n^q} \sum_{k=1}^{s(t)n^q} \sum_{i(A)=0} \langle A \rangle^2 \phi_0(A) e^{\beta_3 f_1(A, k/n^q)} \frac{\alpha^T(A)}{A!} \\
&\to \int_0^{s(t)} \sum_{i(A)=0} \langle A \rangle^2 \phi_0(A) e^{\beta_3 f_1(A, v)} \frac{\alpha^T(A)}{A!} \, dv \\
&= \int_0^t \sum_{i(A)=0} \langle A \rangle^2 \phi_0(A) e^{\beta_3 f_1(A, s(v))} \frac{\alpha^T(A)}{A!} s'(v) \, dv = \int_0^t \sigma_A^2(v) \, dv.
\end{aligned}$$

The same argument leads to

$$\begin{aligned}
&\frac{\beta_4}{n^q} \sum_{k=1}^{s(t)n^q} \sum_{i(A)=0} \langle A \rangle f_2(A, k/n^q) \phi_0(A) e^{\beta_3 f_1(A, k/n^q)} \frac{\alpha^T(A)}{A!} \\
&\to \int_0^t \frac{\beta_4}{n^q} \sum_{i(A)=0} \langle A \rangle f_2(A, v) \phi_0(A) e^{\beta_3 f_1(A, v)} \frac{\alpha^T(A)}{A!} \, dv = \int_0^t \mu_A(v) \, dv.
\end{aligned}$$

Hence we obtain (6.4).

Next we consider the case $t = \tau$. We will show the followings :

$$\hat{\mathrm{E}}\left[ \mathrm{e}^{iz\hat{W}_\tau^{(n)}} \,|\, \mathbb{T} = \mathbf{r} \right] \to \exp\left\{ iz \int_0^\tau \mu_B(v) \, dv + izh \right\}, \tag{6.5}$$

$$\mathrm{E}^{\mathbf{r}} \left[ \mathrm{e}^{iz\hat{W}_\tau^{(n)}} \right] \to \exp\left\{ iz \int_0^\tau \mu_A(v) \, dv - \frac{1}{2} z^2 \int_0^\tau \sigma_A^2(v) \, dv \right\}. \tag{6.6}$$

In view of Corollary 4.6, it follows from (6.5) and (6.6) that

$$\varphi_\tau(z) \to \exp\left\{ iz \int_0^\tau \big(\mu_A(v) + \mu_B(v)\big)\, dv - \frac{1}{2} z^2 \int_0^\tau \sigma_A^2(v)\, dv + izh \right\}.$$

Set $J(n^q+1) = \left\{U_{n^q}+1, ..., U_{n^q}+n^{1-\lambda}\right\}$. We see $|J(n^q+1)| = n^{1-\lambda} = o(n^{\delta+1/2})$, since $1-\lambda < 3/4 < 7/8 < \delta + 1/2$.

To see (6.5), we observe

$$\hat{\mathrm{E}}_{I(n^q+1)}\left[\exp\left\{\frac{iz}{\sqrt{n}} \sum_{u\in J(n^q+1)} \langle \hat{w}_u \rangle\right\} | \mathbb{T} = \mathbf{r}\right] = \hat{\mathrm{E}}_{I(n^q+1)}\left[\exp\left\{\frac{iz}{\sqrt{n}} \langle \hat{w}_u\rangle\right\}\right]^{n^{1-\lambda}}$$
$$= \left(1 + \frac{iz}{\sqrt{n}} \hat{\mathrm{E}}_{I(n^q+1)}\big[\langle \hat{w}_u\rangle\big] + O\left(\frac{1}{n}\right)\right)^{n^{1-\lambda}} = \left(1 + \frac{iz}{\sqrt{n}} \frac{h}{n^{1/2-\lambda}} + O\left(\frac{1}{n}\right)\right)^{n^{1-\lambda}}$$
$$= \left(1 + \frac{izh}{n^{1/2-\lambda}} + O\left(\frac{1}{n}\right)\right)^{n^{1-\lambda}} \to \exp\{izh\}.$$

Hence, following the same argument as (6.3), we obtain (6.5).

To see (6.6), note that $O(1/n^{1-\delta}) = o(1/n^q)$, since $q < 1/2 < 1-\delta$. By cluster expansion we have

$$\log \mathrm{E}_{I(n^q+1)}\left[\exp\left\{\frac{iz}{\sqrt{n}} W_{J(n^q+1)}\right\}\right]$$
$$= -\frac{z^2 n^{1-\lambda}}{2n} \sum_{i(A)=0} \langle A\rangle^2 \phi_0(A) e^{\beta_3 f_1(A,k/n^q)} \frac{\alpha^T(A)}{A!}$$
$$+ \frac{\beta_4 n^{1-\lambda}}{n} \sum_{i(A)=0} \langle A\rangle f_2(A, k/n^q) \phi_0(A) e^{\beta_3 f_1(A,k/n^q)} \frac{\alpha^T(A)}{A!} + O\left(\frac{1}{n^{1-\delta}}\right)$$
$$= o\left(\frac{1}{n^q}\right).$$

since $n^{1-\lambda}/n = o(n^{\delta+1/2}/n) = o(1/n^q)$. Consequently, we have (6.6).

(Second step) : We show the convergence of multi–dimensional distribution.

For $0 = t_0 < t_1 < t_2 < ... < t_m = \tau$, let $\varphi_{t_1}, ..., t_m(z_1, ..., z_m)$ be the characteristic function of the joint distribution $\left(X_{t_1}^{(n)}, ..., X_{t_m}^{(n)}\right)$. For a sequence of stopping times $\mathbb{T} = \{T_j\}_{j=1}^{n^q}$, we write

$$\varphi_{t_1}, ..., t_m(z_1, ..., z_m) = \mathrm{E}\left[\exp\left\{i \sum_{l=1}^m z_l X_{t_l}^{(n)}\right\}\right]$$
$$= \mathrm{E}\left[\exp\left\{i \sum_{l=1}^m z_l X_{t_l}^{(n)}\right\} : \mathbb{T} \text{ is admissible}\right] + \mathrm{E}\left[\exp\left\{i \sum_{l=1}^m z_l X_{t_l}^{(n)}\right\} : \mathbb{T} \text{ is not admissible}\right]$$
$$= I_3 + I_4.$$

If $\delta > 1/2 - 2\varepsilon$, then

$$|I_4| \le \frac{4n^q}{n^{2\varepsilon}} = 4n^{-(2\varepsilon - 1/2+\delta)} \to 0 \quad (n \to \infty).$$

We rewrite $I_3$ as,

$$I_3 = \sum_{\mathbf{r}:\text{admissible}} \mathrm{E}^{\mathbf{r}}\left[\exp\left\{ i \sum_{l=1}^{m} z_l W_{t_l}^{(n)} \right\}\right] \hat{\mathrm{E}}\left[\exp\left\{ i \sum_{l=1}^{m} z_l \hat{W}_{t_l}^{(n)} \right\} | \mathbb{T} = \mathbf{r}\right] \hat{\mathrm{P}}(\mathbb{T} = \mathbf{r}).$$

We will show the followings :

$$\hat{\mathrm{E}}\left[\exp\left\{ i \sum_{l=1}^{m} z_l \hat{W}_{t_l}^{(n)} \right\} | \mathbb{T} = \mathbf{r}\right] \to \exp\left\{ i \sum_{l=1}^{m} z_l \int_0^{t_l} \mu_B(v)\, dv + i z_m h \right\}, \tag{6.7}$$

$$\mathrm{E}^{\mathbf{r}}\left[\exp\left\{ i \sum_{l=1}^{m} z_l W_{t_l}^{(n)} \right\}\right] \to \exp\left\{ i \sum_{l=1}^{m} z_l \int_0^{t_l} \mu_A(v)\, dv - \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} z_k z_l \int_0^{t_k \wedge t_l} \sigma_A^2(v)\, dv \right\}. \tag{6.8}$$

As in the same argument as we proved the convergence in one dimensional case, we obtain (3.2) from (6.7) and (6.8).

Recall that $k(n,t)$ is a unique number satisfying $[nt] \in I(k(n,t))$ and $k(n,0) = 0$. First, we note that

$$\begin{aligned} \sqrt{n} \sum_{l=1}^{m} z_l W_{t_l}^{(n)} &= \sum_{l=1}^{m} z_l \sum_{k=1}^{k(n,t_l)} W_{I(k)} = \sum_{l=1}^{m} z_l \sum_{p=1}^{l} \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)} W_{I(k)} \\ &= \sum_{p=1}^{m} \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)} \sum_{l=p}^{m} z_l W_{I(k)}. \end{aligned} \tag{6.9}$$

We also have

$$\sqrt{n} \sum_{l=1}^{m} z_l \hat{W}_{t_l}^{(n)} = \sum_{p=1}^{m} \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)} \sum_{l=p}^{m} z_l \hat{W}_{I(k)}. \tag{6.10}$$

To prove (6.7), using (6.10) we have

$$\hat{\mathrm{E}}\left[\exp\left\{ i \sum_{l=1}^{m} z_l \hat{W}_{t_l}^{(n)} \right\} | \mathbb{T} = \mathbf{r}\right] = \prod_{p=1}^{m} \prod_{k=k(n,t_{p-1})+1}^{k(n,t_p)} \hat{\mathrm{E}}_{I(k)}\left[\exp\left\{ \frac{i}{\sqrt{n}} \sum_{l=p}^{m} z_l \hat{W}_{I(k)} \right\} | \mathbb{T} = \mathbf{r}\right]. \tag{6.11}$$

In the same way as (6.3) and (6.5), we see that the right hand side of (6.11) converges to

$$\begin{aligned} &\left[\prod_{p=1}^{m} \exp\left\{ i \sum_{l=p}^{m} z_l \int_{t_{p-1}}^{t_p} \mu_B(v)\, dv \right\}\right] e^{i z_m h} = \exp\left\{ i \sum_{p=1}^{m} \sum_{l=p}^{m} z_l \int_{t_{p-1}}^{t_p} \mu_B(v)\, dv + i z_m h \right\} \\ &= \exp\left\{ i \sum_{l=1}^{m} \sum_{p=1}^{l} z_l \int_{t_{p-1}}^{t_p} \mu_B(v)\, dv + i z_m h \right\} = \exp\left\{ i \sum_{l=1}^{m} z_l \int_0^{t_p} \mu_B(v)\, dv + i z_m h \right\}. \end{aligned}$$

Hence we have (6.7).

To prove (6.8), using (6.9) and applying the method of cluster expansion we see that

$$\mathrm{E}^{\mathrm{r}}\left[\exp\left\{i\sum_{l=1}^{m} z_l W_{t_l}^{(n)}\right\}\right] = \prod_{p=1}^{m}\prod_{k=k(n,t_{p-1})+1}^{k(n,t_p)} \mathrm{E}_{I(k)}\left[\exp\left\{\frac{i}{\sqrt{n}}\sum_{l=p}^{m} z_l W_{I(k)}\right\}\right]$$

$$= \prod_{p=1}^{m}\prod_{k=k(n,t_{p-1})+1}^{k(n,t_p)} \exp\left\{\sum_{A\subset I(k)}\left(\exp\left\{\frac{i}{\sqrt{n}}\sum_{l=p}^{m} z_l W_{I(k)}(A)\right\}-1\right)\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}\right\}. \tag{6.12}$$

Remind that $W_{I(k)}(A) = \langle A\rangle$ if $A \subset I(k)$, we have from Taylor's expansion that

$$\begin{aligned}
&\mathrm{E}^{\mathrm{r}}\left[\exp\left\{i\sum_{l=1}^{m} z_l W_{t_l}^{(n)}\right\}\right] \\
&= \prod_{p=1}^{m}\exp\left\{\frac{i}{\sqrt{n}}\sum_{l=p}^{m} z_l \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)}\sum_{A\subset I(k)}\langle A\rangle\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}\right. \\
&\left.-\frac{1}{2n}\left(\sum_{l=p}^{m} z_l\right)^2 \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)}\sum_{A\subset I(k)}\langle A\rangle^2\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}+O\left(\frac{1}{n^{1-\delta}}\right)\right\} \\
&= \prod_{p=1}^{m}\exp\left\{\frac{i\beta_4}{n^q}\sum_{l=p}^{m} z_l \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)}\sum_{i(A)=0}\langle A\rangle\phi_0(A)e^{\beta_3 f_1(A,k/n^q)}f_2(A,k/n^q)\frac{\alpha^T(A)}{A!}\right. \\
&\left.-\frac{1}{2n^q}\left(\sum_{l=p}^{m} z_l\right)^2 \sum_{k=k(n,t_{p-1})+1}^{k(n,t_p)}\sum_{i(A)=0}\langle A\rangle^2\phi_0(A)e^{\beta_3 f_1(A,k/n^q)}f_2(A,k/n^q)\frac{\alpha^T(A)}{A!}+O\left(\frac{1}{n^{1-\delta}}\right)\right\} \\
&\to \prod_{p=1}^{m}\exp\left\{i\beta_4\sum_{l=p}^{m} z_l\int_{t_{p-1}}^{t_p}\mu_A(v)\,dv-\frac{1}{2}\left(\sum_{l=p}^{m} z_l\right)^2\int_{t_{p-1}}^{t_p}\sigma_A^2(v)\,dv\right\}.
\end{aligned} \tag{6.13}$$

For any function $f(p)$ we have the following relation from simple calculation

$$\begin{aligned}
\sum_{p=1}^{m}\left(\sum_{l=p}^{m} z_l\right)^2 f(p) &= \sum_{p=1}^{m}\sum_{k=p}^{m}\sum_{l=p}^{m} z_k z_l f(p) = \sum_{k=1}^{m}\sum_{p=1}^{k}\sum_{l=p}^{m} z_k z_l f(p) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{m} z_k z_l\sum_{p=1}^{k\wedge l} f(p).
\end{aligned} \tag{6.14}$$

Using this relation (6.14), we obtain (6.8) from (6.13). □

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# Fat tail phenomena in a stochastic model of stock market : the long–range percolation approach

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Using a Gibbs distribution developed in the theory of statistical physics and a long–range percolation theory, we present a new model of a stock price process for explaining the fat tail in the distribution of stock returns.

We consider two types of traders, Group A and Group B : Group A traders analyze the past data on the stock market to determine their present trading positions. The way to determine their trading positions is not deterministic but obeys a Gibbs distribution with interactions between the past data and the present trading positions. On the other hand, Group B traders follow the advice reached through the long–range percolation system from the investment adviser. As the resulting stock price process, we derive a Lévy process.

Keywords : stock price process, Lévy process, Gibbs distribution, long–range percolation, fat tail