Universidade de Lisboa

Faculdade de Ciências

Departamento de Matemática

Forward Backward Stochastic Differential Equations

Existence, Uniqueness, a Large Deviations Principle and Connections with Partial Differential Equations

André de Oliveira Gomes

Mestrado em Matemática

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On ne voit bien qu'avec le coeur. L'essentiel est invisible pour les yeux.
Antoine Saint- Exupéry
Le Petit Prince

Ao Pedro. À memória da avó Arminda e à minha mãe.

"il est remarquable qu`une science qui a débuté avec l´examen des jeux de hasard aurait dû devenir le plus important objet de la connaissance humaine"

Pierre Simon Laplace

Theorie Analytique des Probabilitiés

Abstract

We consider Forward Backward Stochastic Differential Equations (FBSDEs for short) with different assumptions on its coefficients.
In a first part we present results of existence, uniqueness and dependence upon initial conditions and on the coefficients. There are two main methodologies employed in this study. The first one presented is the Four Step Scheme, which makes very clear the connection of FBSDEs with quasilinear parabolic systems of Partial Differential Equations (PDEs for short). The weakness of this methodology is the smoothness and regularity assumptions recquired on the coefficients of the system, which motivate the employment of Banach`s Fixed Point Theorem in the study of existence and uniqueness results. This classic analytical tool requires less regularity on the coefficients, but gives only local existence of solution in a small time duration. In a second stage, with the help of the previous work with a running-down induction on time, we can assure the existence and uniqueness of solution for the FBSDE problem in global time.
The second goal of this work is the study of the assymptotic behaviour of the FBSDEs solutions when the diffusion coefficient of the forward equation is multiplicatively perturbed with a small parameter that goes to zero. This question adresses the problem of the convergence of the classical/viscosity solutions of the quasilinear parabolic system of PDEs associated to the system. When this quasilinear parabolic system of PDEs takes the form of the backward Burgers Equation, the problem is the convergence of the solution when the viscosity parameter goes to zero. To study conveniently this problem with a probabilistic approach , we present a concise survey of the classical Large Deviations Principles and the basics of the so-called "Freidlin-Wentzell Theory". This theory is mainly concerned with the study of the Itô Diffusions with the diffusion term perturbed by a small parameter that converges to zero and the richness of properties of the FBSDEs shows us that (even in a coupled FBSDE system) this approach is a good one, since we can extract for the solutions of the perturbed systems a Large Deviations Principle and state convergence of the perturbed solutions to a solution of a deterministic system of ordinary differential equations.

Keywords: Existence and Uniqueness, Forward Backward Stochastic Differential Equations, Gradient Estimates, Quasilinear Equations of Parabolic Type, Four Step Scheme, Large Deviations, Freidlin-Wentzell Theory, Burgers Equations Type, Viscosity Solutions

Contents

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Introduction

Forward-Backward Stochastic Differential Equations (FBSDEs for short) is a subject in Stochastic Analysis that is being aim of increasing study by the the mathematical community since the last years.
Initially FBSDEs become known due to stochastic optimal control theory, with the pioneering work of Bismut [6] in 1973, and, after that, this kind of stochastic equations become well-known thanks to the different applications found in applied and in theoretical fields, such as stochastic optimal control, mathematical finance, among other fields. Mentioning a little of the story, FBSDEs considered initially by Bismut were only of decoupled type, more precisely a system of a forward stochastic differential equation (FSDE for short) with a linear backward one. Only in the 90s, Antonelli, Protter and Young [3] [24] started to study coupled FBSDEs, obtaining Antonelli in his PhD thesis a first result on the solvability of a FBSDE of this type over a small time duration.
The connections between Stochastic Differential Equations and Partial Differential Equations (PDEs for short) are well known since the celebrated Feynman-Kac Formula (see the excelent reference ok Karatzas and Schreve [19] to details in the connections hidden between these two subjects), that gives a representation of an evolution problem as the expectation of a functional of a Brownian Motion, among other probabilistic representation formulas for a huge class of parabolic and elliptic equations. Within this philosophy, Peng [30] realized that a solution of a Backward Stochastic Differential Equation (BSDE for short) could be used as a probabilistic interpretation of the solution to some quasilinear PDE. In this monography we adress this connection, that is obvious when we are confronted with one of the most well-known method to solve a coupled FBSDE- the Four Step Scheme of Ma-Protter-Young [24], which recquires non-degeneracy of the forward diffusion coefficient and the non-randomness of all the coefficients of the system. We only remark that is possible to relax these conditions, allowing the coefficients to be random, with a kind of a generalization of the Four Step

Scheme method, but replacing the connection of the system with a quasilinear parabolic system of deterministic PDEs by a backward stochastic partial differential equation, which can be used to generalize the Feynman-Kac Formula. We refer that it has a lot of applications in mathematical finance in the study of term structure of interest rates (see the reference of Ma-Young and its own references in the subject [25]).
Although the Four Step Scheme approach gives us a very precise and concrete relation between this equations and a quasilinear parabolic system of PDEs, this method recquires very stong regularity assumptions on the coefficients of the FBSDEs, which is something that we can weaken as we will see in the chapter 1 of this work.

Our main interest in this work is to study existence, uniqueness and assymptotic behaviour of a system of stochastic differential equations in the general form:

$$(E)\begin{cases} X_s^{t,x} = x + \int_t^s f(r,\theta(r))\,\mathrm{d}r + \int_t^s \sigma(r,\theta(r))\,\mathrm{d}Br \\ Y_s^{t,x} = h(X_T) + \int_s^T g(r,\theta(r))\,\mathrm{d}r - \int_s^T Z_r \mathrm{d}Br \\ \qquad\qquad \forall\, 0 \leq t \leq s \leq T,\ x \in \mathbb{R}^d,\ \forall\, T > 0 \end{cases}$$

where

$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^d$

$g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$

$\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^{d\times d}$

$h : \mathbb{R}^k \longrightarrow \mathbb{R}^k$

$\theta(r) = (X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$

being the assymptotic study concerned when we replace σ by $\varepsilon\sigma$ and let $\varepsilon \to 0$.

In the first chapter we present several sufficient conditions that will assure local and global existence and uniqueness of solutions for (E). As we will realize, one of the main differences between backward stochastic differential equations and backward deterministic differential equations is the natural recquirement (that is classical in the usual Itô Calculus) of the adaptedness of the solution to the natural filtration considered. This will be an issue to think about that could lead us in other directions, searching a new kind of stochastic calculus without recquirements of adaptedness (for example the *non-anticipating calculus*, which is very well presented in [26]). In this work we will not follow this direction and we mantain the natural recquirement of adaptedness for the concept of solution of (E). It is shown in the first chapter onr example of a non-solvable FBSDE, which motivate the natural study of existence and solution for a general BSDE, before we take care the coupled problem (E). After stating the existence and uniqueness of solution for a BSDE in general form, under classical recquirements of Lipschitz continuity and sublinear growth, we expose the Ma-Protter-Yong Four Step Scheme methodology, and with the help of a result of Ladyzhenskaja [22] in Quasilinear Partial Differential Equations, we will be able to obtain global existence and uniqueness of solution for (E). This is an interesting point of this survey of results on existence and uniqueness issues, that shows how Stochastic Analysis and PDEs connects in a natural way with the study of stochastic differential equations. But one of the problems of the realization of the Four Step Scheme methodology is the question of existence of solution for a system of quasilinear parabolic PDEs of the form

$$(E_*)\begin{cases} \dfrac{\partial u^l}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a_{i,j}\dfrac{\partial^2 u^l}{\partial x_i \partial x_j} + \sum_{i=1}^{d} f_i \dfrac{\partial u^l}{\partial x_i} + g = 0 \\ \hfill u(T,x) = h(x) \\ \hfill l = 1,...,k \end{cases}$$

where $a_{i,j} = (\sigma\sigma^T)_{i,j}$ is a non degenerate matrix.

If we recquire smoothness and boundedness on the coefficients f,g,h, σ, Ladyzhenskaja`s result can be used and we can conclude that $u(t,x) = Y_t^{t,x}$ is a classical $C_b^{1,2}$ solution ( $C^1$ with respect to the time variable and $C^2$ with respect to the spatial variable, with bounded derivatives). This can be relaxed, and we will show Delarue`s procedure [8]. Delarue`s work consist in relax this smoothness conditions and assume only Lipschitz continuity, monotonicity and sublinear growth on the coefficients, and applying Banach`s Fixed Point Theorem, which allow us to assure existence and uniqueness of solution under small time duration. In a second stage, Delarue uses a running-down induction in time, under additional appropriate assumptions (non-degeneracy of the the diffusion matrix and boundedness of the coefficients as functions of the spatial variable x) and get a global result of existence and uniqueness of solution for the problem (E), making use of the estimates on the gradient

of the solution of (E_*) with the Ladyzhenskaja`s result.
But for the purest probabilists, it is a good remark that Delarue gives in [8] a probabilistic proof of existence and uniqueness of solution for the system (E), using Malliavin Calculus tools of Derivation in the Wiener Space, a way of thinking that we do not reproduce in this monography. We prefered to use without discrimination the results of PDEs deterministic literature to our purposes. One of the main connections between (E) and (E_*) is from the powerful result that $Y_s^{t,x} = u(s, X_s^{t,x})$ for a function $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^k$ with some "nice" properties. Specifically, under smoothness and boundedness of the coefficients, $u(t,x)$ is a classical solution of the system (E_*), and if the hypothesis under the coefficients are relaxed, it can be shown that u is a viscosity solution for (E_*).

After discussing existence and uniqueness of solution for the system (E_*^ε) the goal of this work is to study the assymptotic behaviour of:

$$(E^\varepsilon)\begin{cases} X_s^{t,x,\varepsilon} = x + \int_t^s f(r, \theta^\varepsilon(r))\,\mathrm{d}r + \int_t^s \sqrt{\varepsilon}\sigma(r, \theta^\varepsilon(r))\,\mathrm{d}Br \\ Y_s^{t,x,\varepsilon} = h(X_T^\varepsilon) + \int_s^T g(r, \theta^\varepsilon(r))\,\mathrm{d}r - \int_s^T Z_r^\varepsilon \mathrm{d}Br \\ \forall\, 0 \le t \le s \le T,\ x \in \mathbb{R}^d,\ \forall T > 0 \end{cases}$$

when
$\varepsilon \to 0$
.

This is a natural question and it will lead us to the study of the assymptotic behaviour of the corresponding system (E_*^ε) of parabolic quasilinear PDEs associated when the small parameter converges to zero.

$$(E_*^\varepsilon)\begin{cases} \dfrac{\partial u^{l,\varepsilon}}{\partial t} + \dfrac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}\dfrac{\partial^2 u^{l,\varepsilon}}{\partial x_i \partial x_j} + \sum_{i=1}^d f_i \dfrac{\partial u^{l,\varepsilon}}{\partial x_i} + g = 0 \\ u^\varepsilon(T,x) = h(x) \\ x \in \mathbb{R}^d,\ l = 1, ..., k \end{cases}$$

If $f(s, \theta(s)) = y$, $k = d$ and $(a_{ij}) = I_{d\times d}$, (E_*^ε) becomes a backward Burgers Equation

$$\begin{cases} \dfrac{\partial u^{l,\varepsilon}}{\partial t} + \dfrac{\varepsilon}{2}\Delta u^\varepsilon + u^\varepsilon \nabla u^\varepsilon + g = 0 \\ u^\varepsilon(T,x) = h(x) \\ x \in \mathbb{R}^d \end{cases}$$

If u^ε solves the problem above, $v^\varepsilon = -u^\varepsilon(T-t,x)$ solves a Burgers equation

$$\begin{cases} \dfrac{\partial u^{l,\varepsilon}}{\partial t} - \dfrac{\varepsilon}{2}\Delta u^\varepsilon + u^\varepsilon \nabla u^\varepsilon + g = 0 \\ u^\varepsilon(T,x) = h(x) \\ x \in \mathbb{R}^d \end{cases}$$

which is one important simplified model for turbulence and describes the motion of a compressible fluid with viscosity $\frac{\varepsilon}{2}$ under the influence of an external force g.

For example, [37] is one of the examples in the deterministic PDE literature that studies the behaviour of the above evolution problem when $\varepsilon \to 0$. It is well known of PDE deterministic literature (see [23]) that when G is an open bounded set of $\mathbb{R}^3$ with a smooth boundary ∂G, if $g \in L^\infty(0,T;L(G))$, there exists u^ε unique solution of a Burgers forward equation in the intersection of the functional spaces $L^\infty(0,T;L(G)) \cap L^2(0,T;W_0^{1,2}(G))$ and when g is in $L^\infty(0,T;W_0^{3,2}(G))$, then u^ε will converge in $L^2(G)$ to the corresponding solution of the limiting equation on a small non-empty time interval as the viscosity parameter goes to zero [37]. Motivated by this, in chapter 3 we study the convergence of the solutions of (E^ε) when ε goes to zero, and under classical assumptions, we prove that the solutions of (E^ε) converge in a certain functional space to solutions of a deterministic coupled system of ordinary differential equations and $u^\varepsilon(t,x) = Y_t^{t,x}$ classical solution of (E^ε) converges to a function that "solves" an equation type Burgers without viscosity. If we weaken the hypothesis under the coefficients of the system (E^ε), we still can prove that the viscosity solutions of (E_*^ε) converge to a viscosity solution of the limiting equation. In [33] S. Rainero shows the existence of a Large Deviations Principle for the couple $(X_s^{t,x,\varepsilon}, Y_s^{t,x,\varepsilon})_{t\leq s\leq T}$ in the case the FBSDE (E^ε) is decoupled, but the a posteriori property $Y_s^{t,x,\varepsilon} = u(s, X_s^{t,x,\varepsilon})$ almost surely has important implications turning the forward equation an equation only dependant on $X_s^{t,x,\varepsilon}$ and with the works of Azencott[2], Prioret [32] and Baldi-Maurel [3], that extends the so called Freidlin-Wentzell Classical Estimates [15] we will be able to show that (E^ε) obey a Large Deviation Principle even in the coupled case, generalizing the result presented in [33].

In matter of sources, chapter 1 is entirely inspired in the book of Ma-Yong [25] and in the work of Delarue [8] mentioned before; chapter 2 is a concise survey of Large Deviations Principles and the Classical Freidlin-Wentzell Theory, which contains the theory needed to work the Large Deviations Principle of chapter 3 and it is inspired mainly in the coursenotes [16] and in the presentation of the subject due to Dembo-Zeitoni [9] and in the works of [2], [3], [32]. Chapter 3 is the result of the assymptotic study that we had made for (E^ε) and generalizes in a natural way the work [33].

Preliminaries

0.1 Some Remarks

The following monograph assumes the reader has at least basic knowledge in probability theory and in stochastic analysis (stochastic processes, stochastic integration, stochastic differential equations), where "the main character" is Brownian Motion. For references in Stochastic Analysis we mention the excellent works [19], [21], [27]. Everytime we use a result of Stochastic Analysis or some result that we consider of more technical order we specify the reference you can read about it.

0.2 The General Work Environment

For any topological set X and any subset $A \subset X$ we denote $\overline{A}$ the closure, by A° the interior and A^C the complement of A in X.
The infimum of a function over the empty set is defined to be ∞. The derivate of a function ϕ with respect to time will often be denoted by $\dot{\phi}$ or by $\frac{d\phi}{dt}$.
For a topological space X we denote $\mathcal{B}(X)$ the Borel σ-algebra, i.e., the σ-algebra generated by the open sets for the topology given. Furthermore, in a topological space X the neighbourhood N of a set A is any open set N such that A (unless we explicitly note that A should be a closed neigbourhood).
Let (X, d) be a metric space, x_0 and $\delta > 0$. $B(x_0, \delta) := \{x : d(x, x_0) < \delta\}$ is the open ball of radius δ centered at x_0. The distance between a set A and a point $x \in A^C$ is given by $d(x, A) := \inf_{y \in A} d(x, y)$.
We denote $a \vee b = max(a, b)$ and $a \wedge b = min(a, b)$ for every $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. The transpose of a matrix or a vector A will be denoted by A^T.
Let $T > 0$.
For each $n \in \mathbb{N}$, $< ., . >$ and $| \, . \, |$ stand for the euclidian inner product and

the euclidian norm on $\mathbb{R}^n$.

Let Ω be a set and $\mathcal{F}$ be a σ-algebra on Ω. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ will always denote a complete filtered probability space on which we define a n-dimensional Brownian Motion $(B_t)_{0\le t\le T}$ such that $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration of $(B_t)_{0\le t\le T}$ augmented with the collection of the null sets of Ω; $\mathcal{N} := \{A \subset \Omega \ : \exists\, G \in \mathcal{F} \text{such that } \mathbb{P}(G) = 0\}$.

If $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ for some $n, m \in \mathbb{N}$ we refer $\nabla_x u$ the gradient vector of the spatial derivatives of first order and $\nabla_{xx} u$ the matrix of the second order spatial derivatives of u.

In our work it will be important to fix the following functional spaces:

$$\mathrm{H}^2_T(\mathbb{R}^n) := \Big\{\phi : \Omega\times[0,T] \longrightarrow \mathbb{R}^n : \phi \text{ is } \{\mathcal{F}_t\}_{0\le t\le T} \text{ adapted and } \ \mathbb{E}\int_0^T \mid \phi_t \mid^2 \mathrm{d}t < \infty\Big\}$$
(0.2.1)

$$\mathrm{S}^2_T(\mathbb{R}^n) := \Big\{\phi : \Omega \times [0,T] \longrightarrow \mathbb{R}^n : \phi \text{ is } \{\mathcal{F}_t\}_{0\le t\le T} \text{ adapted and }$$
$$\mathbb{E} \sup_{0\le t\le T} \mid \phi_t \mid^2 \ < \infty\Big\}$$
(0.2.2)

which are clearly Banach Spaces with the natural norms given by
$$\|\phi\|^2_{H^2_T(\mathbb{R}^n)} := \mathbb{E}\int_0^T \mid \phi_t \mid^2 \mathrm{d}t$$
$$\|\phi\|^2_{S^2_T(\mathbb{R}^n)} := \mathbb{E} \sup_{0\le t\le T} \mid \phi_t \mid^2$$

Moreover, we define the following well-known functional spaces

$$\mathrm{C}(\ [0,\mathrm{T}],\ \mathbb{R}^n) := \Big\{\phi : [0,T] \longrightarrow \ \mathbb{R}^n : \phi \text{ is continuous }\Big\}$$
$$\mathrm{C}_x = C_x\ ([0,T], \mathbb{R}^n\ ) := \Big\{\phi \in C\ \ ([0,T], \mathbb{R}^n\ ) : \phi_0 = x\Big\}$$
(0.2.3)

These spaces are Banach Spaces under the uniform norm, ie the norm given by:

$\| \phi \|_{\infty} = \sup_{0 \leq t \leq T} | \phi |^2$
(0.2.4)

In particular $C_0 = C_0([0, T], \mathbb{R}^n)$

We introduce now the integrable functional spaces to respect to Lebesgue measure:

$\mathrm{L}^p \ ([0, T], \mathbb{R}^n \,) := \Big\{ \phi : \Omega \times [0, T] \longrightarrow \ \mathbb{R}^n$is measurable to respect the borelian σ-algebras : $\displaystyle\int_0^T | \phi_t |^p \mathrm{d}t < \infty \}$
where $p \geq 1$. (0.2.5)

which is clearly a Banach Space under the norm given by:
$\|\phi\|_{L^p}^p := \displaystyle\int_0^T | \phi_t |^p \mathrm{d}t$

We will need the subspace of the functions in C_0 whose derivate is square integrable, ie,

$H^1 = H^1 \ ([0, T], \mathbb{R}^n \,) := \Big\{ \ \phi \in C_0([0, T], \mathbb{R}^n) \ : \ \displaystyle\int_0^T | \dot{\phi}_t |^2 \mathrm{d}t < \infty \}$
which is a Hilbert Subspace of $C_0([0, T], \mathbb{R}^n)$ with the inner product given by:
$\|\phi\|_{H^1} := \displaystyle\int_0^T | \dot{\phi}_t |^2 \mathrm{d}t.$

Other functional spaces that we will need to our work will be specified under the context they appear.

Chapter 1

Forward Backward Stochastic Differential Equations: Existence and Uniqueness Results

1.1 An informal discussion

Consider on $\mathbb{R}$ the following deterministic terminal value problem:

$$\begin{cases} dX_t = 0, \ t \in [0, T[\\ X(T) = \xi \end{cases} \tag{1.1.1}$$

where $\xi \in \mathbb{R}$ and $T > 0$ is called the terminal time.
We know that there exists a unique solution of (1.1.1), $X_t \equiv \xi \ \forall\, t \in [0, T]$. But if we consider this problem as a stochastic differential equation in Itô´s sense,

$$\begin{cases} dX_t = 0, \ t \in [0, T[\\ X(T) = \xi \end{cases} \tag{1.1.2}$$

where $\xi : \Omega \longrightarrow \mathbb{R}$ is a random variable $\mathcal{F}_T$-measurable with finite second moment, we have (1.1.2) as a stochastic differential problem that does not fit with the definition of solution for a stochastic differential equation. One of the recquirements of the definition of a solution to a stochastic differential equation (SDE for short) is that the solution needs to be adapted to the filtration considered in the probability space. Since the candidate solution (unique) to (1.1.2) is $X_t = \xi, \ \forall\, t \in [0, T]$, which is not necessarily $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ adapted unless ξ is a constant, we do not have a solution in general to (1.1.2) regarded in Itô sense. The problem here is the reversibility in time, which is

not possible for SDEs, due to the problem of the adaptedness recquirement, which marks one of the fundamental differences between the structure of the SDE world and the deterministic ordinary differential equations theory.
We could remove the adaptedness recquirement in the definition of solution to the problem (1.1.2) and begin to develop a new calculus, *the anticipating calculus* (see [26] for information about it), but our approach will be to reformulate the terminal-value problem of an SDE in the way it may allow a solution $\{\mathcal{F}_t\}_{0\leq t\leq T}$-adapted.
Define

$$X_t \equiv \mathbb{E}(\xi \mid \mathcal{F}_t)\, t \in [0,T]. \tag{1.1.3}$$

Using the *Martingale Representation Theorem* (see [19]), there exists a unique $(Z_t)_{0\leq t\leq T}$ stochastic process $\{\mathcal{F}_t\}_{0\leq t\leq T}$-adapted such that $\int_0^T \mathbb{E} \mid Z_t \mid^2 \mathrm{d}t < \infty$ and

$$X_t = X_0 + \int_0^t Z_s \mathrm{d}s, \tag{1.1.4}$$

since (1.1.3) defines $(X_t)_{0\leq t\leq T}$ as a square integrable $\{\mathcal{F}_t\}_{0\leq t\leq T}$-martingale with zero expectation.
So

$$\begin{cases} dX_t = Z_t dB_t,\ t\in[0,T[\\ X(T)=\xi \end{cases} \tag{1.1.5}$$

Considering (1.1.4) with $t = T$

$$X_0 = X_T - \int_0^T Z_s \mathrm{dB}s = \xi - \int_0^T Z_s \mathrm{dB}s \tag{1.1.6}$$

. So (1.1.4) becomes

$$X_t = \xi - \int_t^T Z_s \mathrm{dB}s\,,\, \forall\, t \in [0,T] \tag{1.1.7}$$

. (1.17) will be named throughout this work a backward stochastic differential equation (BSDE for short) and (1.1.5) is its differential formulation.
Applying Itô´s Formula (see reference [21] for example) to $(\mid X_t \mid^2)_{0\leq t\leq T}$

$$\mathbb{E} \mid X_T \mid^2 = \mathbb{E} \mid X_t \mid^2 + \int_t^T \mathbb{E} \mid Z_s \mid^2 \mathrm{d}s, \tag{1.1.8}$$

If we take $(X_1(t), Z_1(t))_{0\leq t\leq T}$ and $(X_2(t), Z_2(t))_{0\leq t\leq T}$ two $\{\mathcal{F}_t\}_{0\leq t\leq T}$ adapted solutions for (1.1.7), $(X_1(t) - X_2(t), Z_1(t) - Z_2(t))_{0\leq t\leq T}$ is an $\{\mathcal{F}_t\}_{0\leq t\leq T}$

adapted solution to (1.1.7) with $\xi = 0$.

Using (1.1.8),
$\mathbb{E} \mid X_1(t) - X_2(t) \mid^2 + \int_t^T \mathbb{E} \mid Z_1(s) - Z_2(s) \mid^2 \mathrm{d}s = 0$
which determines $X_1 \equiv X_2$ and $Z_1 \equiv Z_2$. Then taking ξ a non-random constant, by uniqueness of solution to the deterministic terminal-value problem(1.1.1), we recover $X_t \equiv \xi$ and $Z_t \equiv 0$, $\forall\, t \in [0, T]$.

We now define clearly what we intend to be a solution of a Forward-Backward Stochastic Differential Equation (FBSDE for short).
Let $d, k \in \mathbb{N}$ and let
$f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^d$
$g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$
$\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$
$h : \mathbb{R}^k \longrightarrow \mathbb{R}^k$
be measurable functions with respect to the borelian σ-algebras. Consider $\mathcal{M}[0, T] = S_T^2(\mathbb{R}^d) \times S_T^2(\mathbb{R}^k) \times H_T^2(\mathbb{R}^{k\times d})$, which is a Banach Space under the norm given by
$\| (X_t, Y_t, Z_t)_{0\le t\le T} \|^2 := \mathbb{E} \sup_{0\le t\le T} \mid X_t \mid^2 + \mathbb{E} \sup_{0\le t\le T} \mid Y_t \mid^2 + \mathbb{E} \int_0^T \mid Z_t \mid^2 \mathrm{d}t.$

Given ξ a $\mathbb{R}^d$ valued and $\mathcal{F}_0$-measurable random vector with finite second moment, we are interested in the solutions $(X_t, Y_t, Z_t)_{0\le t\le T} \in \mathcal{M}_{[0,T]}$ of the problem $\forall T > 0$

$$(E) \begin{cases} X_t = \xi + \int_0^t f(s, X_s, Y_s, Z_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\, \mathrm{d}Bs \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \\ \qquad\qquad \forall\, 0 \le t \le T,\ x \in \mathbb{R}^d, \end{cases}$$

We remark that the matrix-function σ does not depend on $(Z_t)_{0\le t\le T}$.
Consider for example the problem:

$$\begin{cases} X_t = x + \int_0^t Z_s\, \mathrm{d}Bs \\ Y_t = X_T - \int_t^T Z_s \mathrm{d}Bs \\ \forall\, 0 \le t \le T,\ x \in \mathbb{R}^d,\ \forall T > 0 \end{cases}$$

The above problem has an infinite number of solutions. In order to prove existence and uniqueness for the problem (E), we will impose restrictions and conditions on the coefficients of the system. So with this example we see that does not make sense to consider σ dependent on the z variable if we want uniqueness of solution for the problem (E).
We can view a FBSDE as a two point boundary value problem for SDE with the extra recquirement of adaptedness to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and we can make advantage of the knowledge that,in general, two point boundary-value problems for ordinary (deterministic) differential equations do not admit necessarily solutions.

Proposition 1.1.1. *Suppose*

$$\begin{cases} \begin{pmatrix} \dot{X}_t \\ \dot{Y}_t \end{pmatrix} & = A(t) \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \end{cases}$$

with the boundary conditions

$$\begin{cases} X_0 = x \in \mathbb{R}^d, \ t \in [0,T[\\ Y(T) = GX_T \end{cases}$$

does not have a solution, where $A : [0,T] \to \mathbb{R}^{(k\times d)(k\times d)}$ is a deterministic integrable function, and $G \in \mathbb{R}^{k\times d}$.
Then for any $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^{d\times d}$ measurable to respect to the borelian σ-algebras, the following FBSDE problem

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = A(t) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + \begin{bmatrix} \sigma(t, X_t, Y_t, Z_t)dB_t \\ Z_t dB_t \end{bmatrix}$$

with the initial-terminal mixed conditions:

$$\begin{cases} X_0 = x \\ Y_T = GX_T \end{cases}$$

do not have an adapted solution to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$.

Proof:
If the given FBSDE had an adapted solution $(X_t, Y_t, Z_t)_{0\leq t\leq T} \in \mathcal{M}_{[0,T]}$, we could take expectations and see that $(EX_t, EY_t)_{0\leq t\leq T}$ is a solution for the given deterministic problem, which contradicts the assumption. □

Example: Consider the following deterministic system of ordinary differential equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ x(0) = x \\ y(\frac{3\pi}{4}) = -x(\frac{3\pi}{4}) \end{cases}$$

A straightforward computation show us that the solution of the first two equations must be of the form

$$\begin{cases} x(t) = c_1 cost + c_2 sint \\ y(t) = -c_1 sint + c_2 cost \end{cases}$$

$c_1, c_2 \in \mathbb{R}$, where $t \in [0, \frac{3\pi}{4}]$.

The condition $y(\frac{3\pi}{4}) = -x(\frac{3\pi}{4})$ implies $x = 0$, which shows that for $x \neq 0$ this is a non solvable ordinary deterministic system of differential equations, giving us the conclusion, by the proposition above, that exists of non-solvable FBSDEs problems.

So, with this brief introduction, we understand the technicalities of the FBSDE problem (E) and the importance to make assumptions on the coefficients of this system, since as we saw in general we do not have existence and uniqueness of solution for a FBSDE problem.

1.2 Backward Stochastic Differential Equations

We now consider the following BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \tag{1.2.1}$$

$\forall\, 0 \leq t \leq T, \ \forall T > 0,$

where ξ is a $\mathcal{F}_T$ k-valued random variable with finite second moment and $g : [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ is measurable to respect to the borelian σ-algebras of the domain and in the range such that:

Assumption 1.2

$\exists L, C > 0 \ \forall\, (t, y, z), (t, \overline{y}, \overline{z}) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\mid g(t, y, z) \mid \leq L(1+ \mid y \mid^2 + \mid z \mid^2) \tag{1.2.2}$$

$$| g(t,y,z) - g(t,\overline{y},\overline{z}) | \leq C(| y-\overline{y} | + | z-\overline{z} |) \tag{1.2.3}$$

Consider the space $\mathcal{N}[0,T] = S_T^2(\mathbb{R}^k) \times H_T^2(\mathbb{R}^{k\times d})$ which is a Banach Space under the norm given by

$$\| (Y_t, Z_t)_{0\leq t\leq T} \|^2 := \mathbb{E} \sup_{0\leq t\leq T} | X_t |^2 + \mathbb{E}\int_0^T | Z_t |^2 \mathrm{d}t.$$

We are seeking for a solution of (1.2.1) in $\mathcal{N}[0,T]$.

Theorem 1.2.1. ***Existence and Uniqueness of solution for a BSDE*** *Under the hypothesis (1.2.2) and (1.2.3) about g, for any $\xi : \Omega \to \mathbb{R}^k$, $\mathcal{F}_T$ measurable with finite second moment, there exists a unique $(Y_t, Z_t)_{0\leq t\leq T}$ in $\mathcal{N}_{[0,T]}$ that solves (1.2.1).*

Proof. Given $(Y_t, Z_t)_{0\leq t\leq T} \in \mathcal{N}[0,T]$ by the assumptions about g, $g(t, Y_t, Z_t)$ is $\{\mathcal{F}_t\}_{0\leq t\leq T}$ adapted and by (1.2.2) $(g(t, Y_t, Z_t))_{0\leq t\leq T} \in H_T^2(\mathbb{R}^k)$. Define

$$W_t := \mathbb{E}\{\xi - \int_0^T g(s)\mathrm{d}s \mid \mathcal{F}_t\} \tag{1.2.4}$$

$$Y_t := \mathbb{E}\{\xi - \int_t^T g(s)\mathrm{d}s \mid \mathcal{F}_t\} \tag{1.2.5}$$

$(W_t)_{0\leq t\leq T}$ is a square integrable $\{\mathcal{F}_t\}_{0\leq t\leq T}$ martingale ($(W_t) \in H_T^2(\mathbb{R}^k)$) and $W_0 = Y_0$.
By the *Martingale Representation Theorem* there exists $(Z_t)_{0\leq t\leq T} \in H_T^2(\mathbb{R}^{k\times d})$ such that:

$$W_t = W_0 + \int_0^t Z_s \mathrm{d}Bs \tag{1.2.6}$$

ξ is $\mathcal{F}_T$ measurable, so

$$W_T = \mathbb{E}\{\xi - \int_0^T g(s)\mathrm{d}s \mid \mathcal{F}_T\} = Y_0 + \int_0^T Z_s \mathrm{d}Bs \tag{1.2.7}$$

$$\mathrm{Y}_t = \mathbb{E}\{\xi - \int_t^T g(s)\mathrm{d}s \mid \mathcal{F}_t\} = \mathbb{E}\{\xi - \int_0^T g(s)\mathrm{d}s + \underbrace{\int_0^t g(s)\mathrm{d}s}_{\mathcal{F}_t measurable} \mid \mathcal{F}_t\} =$$

$$= \mathbb{E}\{\xi - \int_0^T g(s)\mathrm{d}s \mid \mathcal{F}_t\} + \int_0^t g(s)\mathrm{d}s = W_t + \int_0^t g(s)\mathrm{d}s$$

$$= Y_0 + \int_0^t Z_s \mathrm{d}Bs + \int_0^t g(s)\mathrm{d}s = \xi - \int_t^T g(s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s (1.2.8)$$

So $(Y_t, Z_t)_{0\leq t\leq T}$ solves the following equation:

$$\begin{cases} dY_t = g(t, Y_t, Z_t)dt + Z_t dB_t, \ t \in [0, T[\\ Y(T) = \xi \end{cases} \tag{1.2.9}$$

By construction, $(Y_t, Z_t)_{0\leq t\leq T}$ is $\mathcal{F}_t$-adapted and we can see by straightforward computations that $(Y_t, Z_t)_{0\leq t\leq T}$ in $\mathcal{N}[0, T]$.

Now consider $(\overline{y}_t, \overline{z}_t)_{0\leq t\leq T} \in \mathcal{N}[0, T]$ and $(\overline{Y}_t, \overline{Z}_t)_{0\leq t\leq T} \in \mathcal{N}[0, T]$ such that:

$$\begin{cases} d\overline{Y}_t = g(t, \overline{y}_t, \overline{z}_t)dt + \overline{Z}_t dB_t, \ t \in [0, T[\\ \overline{Y}_T = \xi \end{cases} .$$

Applying Itô`s Formula :

$$\mathbb{E} \mid Y_t - \overline{Y}_t \mid^2 + \mathbb{E} \int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \leq$$
$$2\mathrm{C}\ \mathbb{E} \int_t^T \mid Y_s - \overline{Y}_s \mid \Big(\mid y_s - \overline{y}_s \mid + \mid z_s - \overline{z}_s \mid \Big)\mathrm{d}s \ (1.2.10)$$

Setting:

$$\begin{cases} \varphi_t := \sqrt{\mathbb{E} \mid Y_t - \overline{Y}_t \mid^2} \\ \psi_t := \big(\mathbb{E} \mid y_t - \overline{y}_t \mid^2 \big)^{1/2} + \big(\mathbb{E} \mid z_t - \overline{z}_t \mid^2 \big)^{1/2} \end{cases} \tag{1.2.11}$$

we have the following, using (1.2.10):

$$\varphi_t^2 + \mathbb{E} \int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \leq 2C\Big(\int_t^T \varphi_s \psi_s \mathrm{d}s\Big). \tag{1.2.12}$$

Calling $\theta_t := \Big(\int_t^T \varphi_s \psi_s \mathrm{d}s\Big)^2$, we see by (1.2.12) that
$\dot{\theta}_t = -\varphi_t \psi_t \geq -\psi_t \sqrt{2C\theta_t}$, which implies that $\dfrac{d}{dt}\sqrt{\theta_t} \geq -\sqrt{\frac{C}{2}}\psi_t$.
Also $\theta_T = 0$ implies, after integrating the above inequality, that $-\sqrt{\theta_t} \geq$

$-\sqrt{\frac{C}{2}}\int_t^T \psi_s \mathrm{d}s$. So $\theta_t \le \frac{C}{2}\Big\{\int_t^T \varphi_s \mathrm{d}s\Big\}^2 \ \forall\, t\in[0,T]$.
So we get that

$$\varphi_t^2 + \mathbb{E}\int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \le C^2\Big\{\int_t^T \varphi_s \mathrm{d}s\Big\}^2. \qquad (1.2.13)$$

Applying this intermediary conclusion:

$\mathbb{E} \mid Y_t - \overline{Y}_t \mid^2 + \mathbb{E}\int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \le$
$\mathrm{C}^2\Big\{\int_t^T \big(\mathbb{E}\mid y_s - \overline{y}_s \mid^2\big)^{1/2} + \big(\mathbb{E}\mid z_s - \overline{z}_s\mid^2\big)^{1/2}\Big\}^2 \le$
$\mathrm{C}_1(T-t) \parallel \big(y_t - \overline{y}_t, z_t - \overline{z}_t\big)_{t\le s\le T} \parallel^2_{\mathcal{N}[t,T]}$ (1.2.14)

where C_1 is a constant only depending on C, and the space $\mathcal{N}[t,T]$ is the analogue of $\mathcal{N}[0,T]$ with the natural replacement of $[0,T]$ by $[t,T]$ in the definition.

Using *Doob`s Inequality* (see reference [19]):

$$\parallel \big(Y_s - \overline{Y}_s, Z_s - \overline{Z}_s\big)_{t\le s\le T} \parallel^2_{\mathcal{N}[t,T]} \le C_1(T-t) \parallel \big(y_s - \overline{y}_s, z_s - \overline{z}_s\big)_{t\le s\le T} \parallel^2_{\mathcal{N}[t,T]}$$

$\forall\, t\in[0,T]$(1.2.15).

So the map $(y_t, z_t) \mapsto (Y_t, Z_t)$ in $\mathcal{N}(T-\delta, T)$ where (Y_t, Z_t) is the solution of (1.2.9) concerning (y_t, z_t) is a contraction, if we take $\delta = \dfrac{1}{2C_1}$.
We have the conditions to apply the classical *Banach`s Fixed Point Theorem* ([20]) in the Banach Space $\mathcal{N}(T-\delta, T)$ and conclude that this map has a unique fixed point, that is the solution of (1.2.1) in the interval $[T-\delta, T]$by the construction of this application.
We can repeat this argument in $[T-2\delta, T-\delta]$ and recorrently we obtain existence and uniqueness of solution for the problem (1.2.1).

□

In what follows we prove the continuous dependence of the solutions of the BSDE (1.2.1) on the data ξ and in the function g.

Theorem 1.2.2. ***Continuous dependence on the coefficients***
Let $g, \overline{g} : [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \to \mathbb{R}^k$ be measurable functions to the respective borelian σ-algebras in the domain and in the image, $\xi, \overline{\xi}$ be k-valued $\mathcal{F}_T$

measurable random vectors, $(Y_t, Z_t)_{0\le t\le T}$ *and* $(\overline{X}_t, \overline{Y}_t)_{0\le t\le T}$ *be the adapted solutions of (1.2.1) concerning* (g,ξ) *and* $(\overline{g},\overline{\xi})$ *respectively.*
Then:

$$\| \left(Y_t - \overline{Y}_t, Z_t - \overline{Z}_t\right) \|^2_{\mathcal{N}[0,T]} \le C_1 \Big\{ \mathbb{E} \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_0^T \mid g(s,Y_s,Z_s) - g(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s \Big\} \quad (1.2.16)$$

where C_1 *is a constant depending only on* C *and in* T*.*

Proof. Applying the Itô`s Formula to $\mid Y_t - \overline{Y}_t \mid^2$ we obtain:

$$\mid Y_t - \overline{Y}_t \mid^2 + \int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s =$$

$$\mid \xi - \overline{\xi} \mid^2 - 2\int_t^T < Y_s - \overline{Y}_s, g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) > \mathrm{d}s -$$

$$2\int_t^T < Y_s - \overline{Y}_s, Z_s - \overline{Z}_s \mathrm{dB}s > \le$$

$$\mid \xi - \overline{\xi} \mid^2 + 2\int_t^T \mid Y_s - \overline{Y}_s \mid\mid g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid +$$

$$\mathrm{C} \mid Y_t - \overline{Y}_t \mid \big(\mid Y_t - \overline{Y}_t \mid + \mid Z_t - \overline{Z}_t \mid \big) \mathrm{d}s - 2\int_t^T < Y_s - \overline{Y}_s, Z_s - \overline{Z}_s \mathrm{dB}s >$$

$$\le \mid \xi - \overline{\xi} \mid^2 + \int_t^T (1 + 2C + 2C^2) \mid Y_s - \overline{Y}_s \mid^2 + \frac{1}{2} \mid Z_s - \overline{Z}_s \mid^2 + \mid g(s,Y_s,Z_s) -$$

$$\overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s - 2\int_t^T < Y_s - \overline{Y}_s, (Z_s - \overline{Z}_s)\mathrm{dB}s > \quad (1.2.17)$$

Taking expectations,

$$\mathbb{E} \mid Y_t - \overline{Y}_t \mid^2 + \tfrac{1}{2}\mathbb{E}\int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \le$$

$$\mathbb{E} \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_t^T \mid g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s +$$

$$(1+2\mathrm{C}+2\mathrm{C}^2)\mathbb{E}\int_t^T \mid Y_s - \overline{Y}_s \mid^2 \mathrm{d}s \quad \forall t \in [0,T]. \ (1.2.18)$$

From *Gronwall Inequality* (see reference [13] for example)

$$\mathbb{E} \mid Y_t - \overline{Y}_t \mid^2 + \mathbb{E}\int_t^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s \le$$

$$\mathrm{C}_1 \Big\{ \mathbb{E} \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_t^T \mid g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s \Big\}$$ for a new constant C_1 (1.2.19)

Using the *Burkholder-Davis-Gundy`s Inequality*, ([19]) redefining the constant C_1 eventually in the following inequalities:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mid Y_t - \overline{Y}_t \mid^2\right) \leq$$
$$\mathrm{C}_1\left\{\mathbb{E} \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_0^T \mid g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s +\right.$$
$$\left. 2\ \mathbb{E}\sup_{0\leq t\leq T} \mid \int_t^T < Y_s - \overline{Y}_s, (Z_s - \overline{Z}_s)\mathrm{dB}s > \mid \right\}$$
$$\mathrm{C}_1\left\{\mathbb{E} \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_0^T \mid g(s,Y_s,Z_s) - \overline{g}(s,\overline{Y}_s,\overline{Z}_s) \mid^2 \mathrm{d}s +\right.$$
$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mid Y_t - \overline{Y}_t \mid^2\right)^{1/2}\left(\mathbb{E}\int_0^T \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s\right)^{1/2}$$
(1.2.20)

Combining (1.2.19) and (1.2.20) we obtain the inequality (1.2.16) that we want.

□

1.3 The Four Step Scheme Methodology

Ma, Protter and Yong [24] proved, under strong regularity assumptions on the coefficients of (E) and assuming non-degeneracy of the diffusion coefficient of the Forward Equation, that (E) admits an unique solution. Their method is called "Four Step Scheme" since is based in four major steps. The problem of this method is the strong regularity recquired on the coefficients of (E), an issue that we will overdeal with the work of Delarue [8] in the next section.

Presentation of the method:

Let $d, k \in \mathbb{N}$ and let
$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^d$
$g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$
$\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^{d\times d}$
$h : \mathbb{R}^k \longrightarrow \mathbb{R}^k$
be measurable functions with respect to the borelian σ-algebras.
Although in section 1.1 we had seen that it does not make sense to consider

σ dependent on the z variable, if we want to guarantee uniqueness of solution to our system, for now we will consider the general case of σ depending on the z variable and later on we will make the natural assumption of σ independent of the z variable. Given $\xi : \Omega \to \mathbb{R}^d$ a valued random variable square integrable, consider the main problem (E):

$$(\mathrm{E}) \begin{cases} X_t = \xi + \int_0^t f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,\mathrm{d}Bs \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \\ \forall\, 0 \le t \le T, \quad \forall\, T > 0 \end{cases}$$

or in differential form:

$$\begin{cases} dX_t = f(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dBt \\ dY_t = -g(t, X_t, Y_t, Z_t)dt + Z_t dBt \\ X(0) = \xi\,, Y_T = h(X_T) \end{cases} \tag{1.3.1}$$

We underline f, g, h, σ are deterministic. Suppose $(X_t, Y_t, Z_t)_{0 \le t \le T}$ is an adapted solution for (E) and that X_t and Y_t are related by some function u in the way:

$$Y_t = u(t, X_t)\ \ \forall\, t \in [0, T] \quad a.s.\ \mathbb{P} \tag{1.3.2}$$

This functional relation between X_t and Y_t will be vital for our work and it will be presented clearly in the next sections of this chapter. We remark this relation (1.3.2) will be crucial for the establishment of a Large Deviations Principle and in the study of the Assymptotic Behavior of the System (E) when we multiply the forward diffusion coefficient by a small parameter that goes to zero, and it is the key to understand the link between FBSDEs and parabolic quasilinear systems of PDEs.
Assume $u \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, ie u has continuous bounded derivative to respect to time and continuous bounded second order spatial derivatives.
For each $l = 1, ..., k$ denote $Y_t = (Y_t^1, ..., Y_t^k)$ and using Itô`s Formula we have:

$$\mathrm{dY}_t^l = du^l(t, X_t) = \Big\{ \frac{\partial u^l(t, X_t)}{\partial t} + < \nabla_x u^l(t, X_t), f(t, X_t, u(t, X_t), Z_t) > + \frac{1}{2} tr \big[\nabla_{xx} u^l(t, X_t)(\sigma\sigma^T)(t, X_t, u(t, X_t), Z_t) \big] \Big\} dt + \Big\langle\ \nabla_x u^l(t, X_t), \sigma(t, X_t, u(t, X_t), Z_t) dBt \Big\rangle \tag{1.3.3}$$

where $\nabla_{xx}u^l$ represents the Hessian spatial matrix of u^l

With this formula for $du^l(t, X_t)$, $\forall\, l = 1, ..., k$, we see by (1.3.1) that for each $l = 1, ..., k$:

$$\begin{cases} g^l(t, X_t, u(t, X_t) = \Big\{ \dfrac{\partial u^l}{\partial t}(t, X_t) + < \nabla_x u^l(t, X_t), f(t, X_t, u(t, X_t), Z_t) > \\ + \dfrac{1}{2} tr\left[\nabla_{xx} u^l(t, X_t)(\sigma\sigma^T)(t, X_t, u(t, X_t), Z_t)\right] \\ u(t, X_t) = h(X_T) \end{cases}$$

(1.3.4)

and

$$Z_t = \nabla_x u(t, X_t)\sigma(t, X_t, u(t, X_t), Z_t)$$

(1.3.5)

So we are suggested by the following scheme:

Step 1:

Find a function $z(t, x, y, p)$ that satisfies

$$z(t, x, y, p) = p\sigma(t, x, y, z(t, x, y, p)) \;\forall\, (t, x, y, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \tag{1.3.6}$$

Step 2:

Using z in step 1, consider the following parabolic system for $u(t, x)$, that we want to solve:

$$\begin{cases} \dfrac{\partial u^l}{\partial t} + \frac{1}{2} \displaystyle\sum_{i,j=1}^{d} (\sigma\sigma^T)_{i,j}(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) \dfrac{\partial^2 u^l}{\partial x_i \partial x_j} \\ + \displaystyle\sum_{i=1}^{d} f_i(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) \dfrac{\partial u^l}{\partial x_i} \\ + g^l(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) = 0 \\ u(T, x) = h(x) \\ l = 1, ..., k \end{cases} \tag{1.3.7}$$

Step 3:

With z obtained in **step 1** and u obtained on **step 2**, we want to solve the following forward SDE:

$$\begin{cases} dX_t = \overline{f}(t, X_t)dt + \overline{\sigma}(t, X_t)dBt, \ t \in [0, T] \\ X_0 = \xi \end{cases} \quad (1.3.8)$$

where:

$$\begin{cases} \overline{f}(t, x) = f(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) \\ \overline{\sigma}(t, x) = \sigma(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) \end{cases} \quad (1.3.9)$$

Step 4:

Set

$$\begin{cases} Y_t = u(t, X_t) \\ Z_t = z(t, X_t, u(t, X_t), \nabla_x u(t, X_t)) \end{cases}$$

(1.3.10)

If we can realize this scheme, which is extremely useful to compute numerical algorithms that could help to solve the FBSDE (E), $(X_t, Y_t, Z_t)_{0 \le t \le T}$ would give an adapted solution of it. But with this methodology, we are dependent on the realization of all steps. Specially the **step 1** and **step 2**, that are of deterministic order , could not have solution.
We need to impose restrictions on the coefficients of the system.
Consider the following set of assumptions:

Assumption A.1.3

1)
$\exists\, L,\, \Lambda > 0 \ \forall\, (t, x, y, z),\, (\overline{t}, \overline{x}, \overline{y}, \overline{z}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$such that:
$\mid \sigma(t, x, y, z) - \sigma(\overline{t}, \overline{x}, \overline{y}, \overline{z}) \mid \le L\big(\mid t - \overline{t} \mid + \mid x - \overline{x} \mid + \mid y - \overline{y} \mid + \mid z - \overline{z} \mid \big)$
$\mid f(t, x, y, z) - f(\overline{t}, \overline{x}, \overline{y}, \overline{z}) \mid \le L\big(\mid t - \overline{t} \mid + \mid x - \overline{x} \mid + \mid y - \overline{y} \mid + \mid z - \overline{z} \mid \big)$
$\mid g(t, x, y, z) - g(\overline{t}, \overline{x}, \overline{y}, \overline{z}) \mid \le L\big(\mid t - \overline{t} \mid + \mid x - \overline{x} \mid + \mid y - \overline{y} \mid + \mid z - \overline{z} \mid \big)$
$\mid \sigma(t, x, y, z) \mid \le \Lambda\big(1 + \mid x \mid + \mid y \mid + \mid z \mid \big)$
$\mid f(t, x, y, z) \mid \le \Lambda\big(1 + \mid x \mid + \mid y \mid + \mid z \mid \big)$ (1.3.11)

2) Assume σ bounded.

3)
$\exists \beta \in]0,1[: \mid [\sigma(t,x,y,z) - \sigma(t,x,y,\overline{z})]^T \nabla_x u^l(t,x) \mid \leq \beta \mid z - \overline{z} \mid$
$\forall (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k$,z, $\overline{z} \in \mathbb{R}^{k\times d}$ $\forall l = 1,...,k$

(1.3.12)

Theorem 1.3.1. ***Realization of the Four Step Sheme and Existence and Uniqueness of solution for (E)***
If we assume the set of assumptions (A.1.3), if (1.3.6) admits a unique solution $z(t,x,y,p)$ uniformly Lipschitz continuous and with sublinear growth, and if (1.3.7) admits a classical solution $u(t,x) \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$, then the process $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ is an adapted solution for (E).
Furthermore, the adapted solution of the system is unique.

Proof. **Existence of solution:**
Under our assumptions $\overline{f}(t,x)$ and $\overline{\sigma}(t,x)$ are uniformly Lipschitz continuous in x, and setting a $\mathbb{R}^d$ random variable ξ $\mathcal{F}_0$ measurable, we have by the standarb theory of existence of solutions to stochastic differential equations (see reference [27] for example about SDEs) there exists a strong unique solution of (1.3.8). Applying Itô`s Formula, (1.3.1) is checked.
Hence $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ is a solution of (1.3.1).
Uniqueness:
The claim is that any adapted solution $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ of (1.3.1) must be of the form constructed before, using the Four Step Scheme.
Let $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ be any solution of (1.3.1). Define:

$$\begin{cases} \overline{Y}_t = u(t, X_t) \\ \overline{Z}_t = z(t, X_t, u(t,X_t), \nabla_x u(t,X_t)) \end{cases} \tag{1.3.13}$$

By our assumtions (A.1.3), (1.3.6) admits a unique solution. So using (1.3.13)

$$\overline{Z}_t = \nabla_x u(t,X_t)\sigma(t,X_t,\overline{Y}_t,\overline{Z}_t) \text{ a.s } \mathbb{P}\,,\ t \in [0,T] \tag{1.3.14}$$

Applying Itô`s Formula to $u(t,X_t)$, by (1.3.7) and (1.3.10), $\forall l = 1,...,k$
$d\overline{Y}_t^l = du^l(t,X_t) = \Big\{\dfrac{\partial u^l}{\partial t}(t,X_t) + < \nabla_x u^l(t,X_t), f(t,X_t,Y_t,Z_t) > +$
$\dfrac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{i,j}(t,X_t,Y_t,Z_t)\dfrac{\partial^2 u^l}{\partial x_i \partial x_j}\Big\}dt + < \nabla_x u^l(t,X_t), \sigma(t,X_t,Y_t,Z_t)dBt > =$
$\Big\{ < \nabla_x u^l(t,X_t), f(t,X_t,Y_t,Z_t) - f(t,X_t,\overline{Y_t},\overline{Z_t}) > +$

$$\frac{1}{2}\sum_{i,j=1}^{d}\{(\sigma\sigma^T)_{i,j}(t,X_t,Y_t,Z_t)-(\sigma\sigma^T)_{i,j}(t,X_t,\overline{Y}_t,\overline{Z}_t)\}\frac{\partial^2 u^l}{\partial x_i \partial x_j}-$$
$$-g^l(t,X_t,\overline{Y}_t,\overline{Z}_t)\Big\}dt+<\nabla_x u^l(t,X_t),\sigma(t,X_t,Y_t,Z_t)dBt>$$

Using (1.3.14) and (1.3.1), taking expectations,

$$\mathbb{E}\mid \overline{Y}_t - Y_t\mid^2=$$
$$\mathbb{E}\int_t^T\sum_{l=1}^{k}2(\overline{Y}_s^l-Y_s^l)\Big\{<\nabla_x u^l(t,X_s),f(s,X_s,Y_s,Z_s)-f(s,X_s,\overline{Y}_s,\overline{Z}_s)>+$$
$$\frac{1}{2}\sum_{i,j=1}^{d}\{(\sigma\sigma^T)_{i,j}(s,X_s,Y_s,Z_s)-(\sigma\sigma^T)_{i,j}(s,X_s,\overline{Y}_s,\overline{Z}_s)\}\frac{\partial^2 u^l}{\partial x_i \partial x_j}(s,X_s)-$$
$$g^l(s,X_s,\overline{Y}_s,\overline{Z}_s)+g^l(s,X_s,Y_s,Z_s)\Big\}+$$
$$\mid[\sigma(s,X_s,Y_s,Z_s)-\sigma(s,X_s,\overline{Y}_s,\overline{Z}_s)]^T\nabla_x u^l(s,X_s)+\overline{Z}_s-Z_s\mid^2\mathrm{d}s \quad (1.3.15)$$

Using the assumption (A.1.3), the boundedness of $\nabla_x u$ and the uniform Lipschitz continuity of σ, we must obtain:

$$\mid[\sigma(s,X_s,Y_s,Z_s)-\sigma(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s)]^T\nabla_x u^l(s,X_s)+\overline{Z}_s-Z_s\mid^2\geq$$
$$\geq\mid Z_s-\overline{Z}_s\mid^2+\mid[\sigma(s,X_s,Y_s,Z_s)-\sigma(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s)][\nabla_x u^l(s,X_s)]^T\mid^2\geq$$
$$\geq(1-\beta)\mid\overline{Z}_s-Z_s\mid^2-C\mid Y_s-\overline{Y}_s\mid \quad (1.3.16)$$

where $C>0$ depends only on the Lipschitz Constant of σ. So:

$$\mathbb{E}\mid Y_t-\overline{Y}_t\mid^2+(1-\beta)\int_t^T\mathbb{E}\mid Z_s-\overline{Z}_s\mid^2\mathrm{d}s\leq$$
$$\mathrm{C}\int_t^T\mathbb{E}\{\mid Y_s-\overline{Y}_s\mid^2+\mid\overline{Y}_s-Y_s\mid\mid\overline{Z}_s-Z_s\mid\mathrm{d}s \text{ , for a new C eventually}$$
$$\leq C_\varepsilon\int_t^T\mathbb{E}\{\mid Y_s-\overline{Y}_s\mid^2\mathrm{d}s+\varepsilon\int_t^T\mathbb{E}\{\mid Y_s-\overline{Y}_s\mid^2\mathrm{d}s \quad (1.3.17)$$

using Cauchy-Schwartz Inequality with $\varepsilon>0$ where $\varepsilon>0$ fixed arbitrarily and $C_\varepsilon=C(\varepsilon)$. Since $\beta<1$, we can choose $\varepsilon<1-\beta$ and applying Gronwall Inequality we obtain:
$Y_t=\overline{Y}_t$ and $Z_t=\overline{Z}_t$ a.s $\mathbb{P}$ and a.e. $t\in[0,T]$. So, if $(X_t,Y_t,Z_t)_{0\leq t\leq T}$ and

$(\overline{X}_t, \overline{Y}_t, \overline{Z}_t)_{0\leq t\leq T}$ are two solutions of (1.3.1), by the previous argument we have that:

$$\begin{cases} Y_t = u(t, X_t),\ \overline{Y}_t = u(t, \overline{X}_t) \\ Z_t = z(t, X_t, u(t, X_t), \nabla_x u(t, X_t)),\ \overline{Z}_t = z(t, \overline{X}_t, u(t, \overline{X}_t), \nabla_x u(t, \overline{X}_t)) \end{cases} \tag{1.3.18}$$

Since (X_t) and $(\overline{X}_t)$ solves the forward SDE (1.3.8), by the standarb theory of SDEs, under our assumptions, (1.3.8) has a unique strong solution, so $X_t = \overline{X}_t$ a.s $\mathbb{P}$, $t \in [0, T]$ and by (1.3.18) $Y_t = \overline{Y}_t$ and $Z_t = \overline{Z}_t$ a.s $\mathbb{P}$, $t \in [0, T]$ □

Remark:

The existence and uniqueness of solution for (1.3.1) depends as we saw on the solution for (1.3.6) and (1.3.7). Since the assumption (1.3.1) seems to be hard to be verified in general, we must look for a simplification that implies (1.3.6) has a unique solution. This is the case of considering σ independent of the z variable. But another assumption in the previous theorem is the existence and uniqueness of solution of the quasilinear parabolic system (1.3.7). To assure this, we will make use of the results of the deterministic PDEs literature in this kind of systems [22], which is a typical example how the two fields, *Partial Differential Equations* and *Stochastic Analysis* touches.
So we are going to make new assumptions:

Assumption B-1.3

1) f ,g, σ are smooth functions with bounded first derivatives by a new constant L
2) $\sigma = \sigma(t, x, y)$ independent of the $z-$variable
3) $\exists\, \nu : [0, \infty[\to [0, \infty[$, and $\mu > 0$ such that
$\forall\, (t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$:
$\nu(\mid y \mid)I \leq \sigma(t, x, y)\sigma^T(t, x, y) \leq \mu I$
$\mid f(t, x, 0, 0) \mid + \mid h(t, x, 0, 0) \mid \leq \mu$
$\exists\, \alpha \in]0, 1[$: $h$bounded in $C^{2+\alpha}(\mathbb{R}^d)$ (1.3.19)

Note that $C^{2+\alpha}(\mathbb{R}^d)$ is a Holder Space of order $2 + \alpha$. For more information about Holder Spaces see [13] and its references about the subject. We remark that this technical assumptions are needed to use the following

result, that can be found in [22]. We do not specify the exact smoothness recquired, being the meaning of smooth that we have the existence of the partial derivatives needed to our arguments.
σ is independent of z so, (1.3.6) is trivially uniquely solvable for z. So, (1.3.1) turns:

$$\begin{cases} X_t = \xi + \int_0^t f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\,\mathrm{d}Bs \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \\ \forall\, 0 \le t \le T, \;\; \forall T > 0 \end{cases}$$

(1.3.20)

and (1.3.7) takes the form:

$$\begin{cases} \frac{\partial u^l}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{i,j}(t, x, u(t, x))\frac{\partial^2 u^l}{\partial x_i \partial x_j} + \\ < f(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))), \nabla_x u(t, x) > + \\ g^l(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) = 0 \;; \\ 0 \le t \le T\,,\; x \in \mathbb{R}^d \;\;, l = 1, ..., k \\ u(T, x) = h(x),\; x \in \mathbb{R}^d \end{cases} \tag{1.3.21}$$

We are going to use the following result that can be found in [22].

Lemma 1.3.1. *Consider the following terminal boundary value problem :*

$$\begin{cases} \frac{\partial u^l}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t, x, u(t, x))\frac{\partial^2 u^l}{\partial x_i \partial x_j} + \\ < f(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))), \nabla_x u(t, x) > + \\ g^l(t, x, u(t, x), z(t, x, u(t, x), \nabla_x u(t, x))) = 0 \;; \\ 0 \le t \le T\,,\; x \in \mathbb{R}^d \;\;, l = 1, ..., k \\ u(T, x) = h(x),\; x \in B(0, R) \;\;, R > 0 \end{cases} \tag{1.3.22}$$

*Suppose $a_{i,j}, f_i, g_l$, for each $i, j = 1, ..., d; l = 1, ..., k$ are smooth and z is the function determined by **Step 1**, which is clearly smooth, where :*
$a_{i,j}(t, x, y) = \frac{1}{2}\sigma(t, x, y)\sigma^T(t, x, y)$
Suppose also $\forall\,(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k$, $p \in \mathbb{R}^{k\times d}$

$\nu(| y |)I \leq (a_{i,j}) \leq \mu(| y |)I$ (1.3.23)

$| f(t,x,y,z(t,x,y,p)) | \leq \mu(| y |)(1+ | p |)$ (1.3.24)

$| \dfrac{\partial a_{i,j}}{\partial x_l}(t,x,y,p)| + | \dfrac{\partial a_{i,j}}{\partial y_r}(t,x,y,p)| \leq \mu(| y |), \forall\, l = 1,...,d\ , r = 1,...,k$
(1.3.25)

and for some continuous functions μ, ν positive

$| f(t,x,y,z(t,x,y,p)) | \leq [\varepsilon(| y |) + P(| p |,| y |)](1+p^2)$
$where$P$(| p |,| y |) \longrightarrow 0$ as $| p | \to \infty$ and $\varepsilon(| y |)$ small enough (1.3.26)

and
$$\sum_{l=1}^{k} g^l(t,x,y,z(t,x,y,p))y_l \geq -L(1+ | y |^2) \qquad (1.3.27)$$

for some constant $L > 0$.
Suppose also $h \in C^{2+\alpha}(\mathbb{R}^d)$ bounded, for some $\alpha \in]0,1[$.
Then (1.3.22) admits a unique classical solution in $C_b^{1,2}([0,T] \times \mathbb{R}^d)$

Now, this technicall lemma, which asserts existence and uniqueness of a classical solution of (1.3.22) can be used to help us to prove the existence and uniqueness of classical solution for the system (1.3.21) and under the assumptions (B-1.3) which determines the possibility to follow the Four Step Scheme Methodology, we conclude existence and uniqueness of solution for the FBSDE (1.3.1).
Just a little remark about the regularity of the solution:
If $h \in C^{2+\alpha}(\mathbb{R}^k)$, bounded, for some $\alpha \in]0,1[$, the solution of the system (1.3.22) and its partial derivatives are all bounded uniformly in $R > 0$ (this is a technical regularity result of parabolic quasilinear PDEs regularity theory, since only the interior Schauder estimate is used). See [22] for more details.

Theorem 1.3.2. ***Solvability of the Four Step Scheme and Existence and Uniqueness of solution for FBSDE***
If the assumptions (A.1.3) (B.1.3) are in force, (1.3.21) admits a unique classical solution $u(t,x) \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$. Consequently the FBSDE (1.3.1) has a unique solution.

Proof. We just have to check that the technical conditions recquired in the previous lemma are verified. σ does not depend on z (A.1.3), so the function $z(t, x, y, p)$ determined by (1.3.6) satisfies

$$| z(t, x, y, p) | \leq C | p | \ \forall (t, x, y, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \qquad (1.3.28)$$

(1.3.23) and (1.3.25) follows from the assumptions.
(1.3.24) follows from (B.1.3-1); (1.3.26) and (1.3.27) follows from (B.1.3-1) and (B.1.3-3).
Using the lemma (1.3.1) there exists a unique bounded solution $u(t, x, R)$ of (1.3.22) for which u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, $\frac{\partial u}{\partial x_i \partial x_j}$ are bounded uniformly in $| x |= R$ for each $i, j = 1, ..., d$.
With a diagonalization argument, there exists a subsequence $u(t, x, R)$ which converges to $u(t, x)$ as $| R | \to \infty$. So $u(t, x)$ will be, by construction, a classical solution of (1.3.21) and $\frac{\partial u}{\partial t}$, $\nabla_x u$, $\nabla_{xx} u$ as well $u(t, x)$ itself are bounded, which proves the existence and uniqueness of solution in $C_b^{1,2}([0, T] \times \mathbb{R}^d)$ for (1.3.18).
The uniqueness of solution follows from a standarb application of Gronwall Inequality, which can be used with our regularity assumptions.
With this, we conclude the realization of the Four Step Scheme and consequently the existence and uniqueness of solution for (1.3.1). □

Remark:

In this section we were conducted to a methodology which can be very useful for numerical analysis of FBSDEs, and that determines a unique solution for FBSDE problem (1.3.1), but the restrictive assumptions imposed on the coefficients are not optimal to assure existence and uniqueness for our main problem (E). We can relax the conditions recquired for the coefficients f, g, h, σ. That will be discussed in the next section. The counterpart of this relaxation on the assumptions is the existence of solution not in global time, but only in a small time duration. The highlight of this chapter will be the combination of the Four Step Scheme Methodology and the main result of the following section in section five (which can be found with more detail in Delarue`s work [8]), in order to conclude existence and uniqueness of solution for the problem (E) in global time.

1.4 A result of existence and uniqueness for FBSDEs in a small time duration under classical assumptions

We are going to revisit our main problem:

$$\begin{cases} X_t = \xi + \int_0^t f(s, X_s, Y_s, Z_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\, \mathrm{d}Bs \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \\ \forall\, 0 \leq t \leq T, \quad \forall\, T > 0 \end{cases}$$

(1.4.1)

where
$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^d$
$g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$
$\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$
$h : \mathbb{R}^k \longrightarrow \mathbb{R}^k$
are measurable functions with the respective borelian σ algebras. Our main goal is to establish, with classical assumptions (Lipschitz monotone coefficients) under a Fixed Point Argument, a result of Existence and Uniqueness in local time (a enough small time duration $T > 0$)- **theorem 1.4.1**.
In the next section we extend this result to a global one, in time, by means of a running down induction in time, which crucial point will be the control of the lenght of the interval at which the **theorem 1.4.1** asserts existence and uniqueness of solution for (1.4.1).
In order to establish our result we make the following set of assumptions:

Assumption A.1.4
$\exists\, L, \Lambda > 0$, $\forall\, t \in [0,T]$, $\forall\, (x,y,z), (\overline{x}, \overline{y}, \overline{z}) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$:

$| f(t,x,y,z) - f(t,x,\overline{y},\overline{z}) | \leq L(| y - \overline{y} | + | z - \overline{z} |)$
$| g(t,x,y,z) - g(t,\overline{x},y,\overline{z}) | \leq L(| x - \overline{x} | + | z - \overline{z} |)$
$| h(x) - h(\overline{x}) | \leq L(| x - \overline{x} |)$
$| \sigma(t,x,y) - \sigma(t,\overline{x},\overline{y}) |^2 \leq L^2(| x - \overline{x} |^2 + | y - \overline{y} |^2)$

$< x - \overline{x}, f(t,x,y,z) - f(t,\overline{x},y,z) > \leq L \mid x - \overline{x} \mid^2$
$< y - \overline{y}, f(t,x,y,z) - f(t,x,\overline{y},z) > \leq L \mid x - \overline{x} \mid^2$

$| f(t,x,y,z) |\leq \Lambda(1+ | x | + | y | + | z |)$
$| g(t,x,y,z) |\leq \Lambda(1+ | x | + | y | + | z |)$
$| h(x) |\leq \Lambda(1+ | x |)$

u $\mapsto f(t,u,y,z)$
v $\mapsto g(t,x,v,z)$ are continuous mappings (1.4.2)

Theorem 1.4.1. ***Existence and Uniqueness in small time duration***
Assuming f,g,h,σ following the set of assumptions (A.1.4), for every random d-vector ξ $\mathcal{F}_0$ measurable with finite second moment, every solution $(X_t,Y_t,Z_t)_{0\leq t\leq T}$ of the problem (1.4.1) satisfies:

i) $(X_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ are continuous.

ii) $\mathbb{E}\big(\sup_{0\leq t\leq T} | X_t |^2 + \sup_{0\leq t\leq T} | Y_t |^2 \big) < \infty$

Moreover, there exists $C>0$ depending only on L such that for every $T\leq C$, for every random d-vector ξ $\mathcal{F}_0$ measurable with finite second moment, (1.4.2) admits a unique solution.

Proof. We are going to construct the map:

$$\Theta : S^2_T(\mathbb{R}^d)\times S^2_T(\mathbb{R}^k)\times H^2_T(\mathbb{R}^{k\times d}) \to S^2_T(\mathbb{R}^d)\times S^2_T(\mathbb{R}^k)\times H^2_T(\mathbb{R}^{k\times d})$$

$(X_t,Y_t,Z_t)_{0\leq t\leq T} \mapsto (\overline{X}_t,\overline{Y}_t,\overline{Z}_t)_{0\leq t\leq T}$, where $\forall\, 0\leq t\leq T$

$$\begin{cases} \overline{X}_t = \xi + \int_0^t f(s,\overline{X}_s,Y_s,Z_s)\mathrm{d}s + \int_0^t \sigma(s,\overline{X}_s,Y_s)\mathrm{d}Bs \\ \overline{Y}_t = h(\overline{X}_T) + \int_t^T g(s,\overline{X}_s,\overline{Y}_s,Z_s)\mathrm{d}s - \int_t^T \overline{Z}_s\mathrm{d}Bs \\ \mathbb{E}\int_0^T | \overline{Z}_s |^2 \mathrm{d}Bs < \infty \end{cases}$$

$(\overline{X}_t)_{0\leq t\leq T}$ is a solution of the forward equation, and the couple $(\overline{Y}_t,\overline{Z}_t)_{0\leq t\leq T}$ is actually built as a solution of the backward equation.
By the standarb theory of SDEs, under our assumptions, there exists a unique solution for the forward equation, denoted $(\overline{X}_t)_{0\leq t\leq T}$.
By the previous study on BSDEs in the section 2, under our assumptions, there exists an unique solution of the backward equation, denoted $(\overline{Y}_t,\overline{Z}_t)_{0\leq t\leq T}$

(see **theorem (1.2.1)** .
So, the map Θ is well defined.
Our strategy to assure existence and uniqueness of solution for the problem (1.4.1) is to prove Θ is a contraction in the Banach Space $\mathcal{M}[0,T] = S_T^2(\mathbb{R}^d) \times S_T^2(\mathbb{R}^k) \times H_T^2(\mathbb{R}^{k\times d})$ and use the well-known *Banach`s Fixed Point Theorem.*
Suppose first $T \leq 1$.
Consider $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ and $(U_t, V_t, W_t)_{0\leq t\leq T} \in \mathcal{M}[0,T]$ and
$(\overline{X}_t, \overline{Y}_t, \overline{Z}_t)_{0\leq t\leq T} = \Theta(X_t, Y_t, Z_t)_{0\leq t\leq T}$

$(\overline{U}_t, \overline{V}_t, \overline{W}_t)_{0\leq t\leq T} = \Theta(U_t, V_t, W_t)_{0\leq t\leq T}$

We want to see that exists $D < 1$ such that

$$\| (\overline{X}_t, \overline{Y}_t, \overline{Z}_t) - (\overline{U}_t, \overline{V}_t, \overline{W}_t)_{0\leq t\leq T} \|^2_{\mathcal{M}[0,T]} \leq D \| (X_t, Y_t, Z_t) - (U_t, V_t, W_t) \|^2_{\mathcal{M}[0,T]}$$

Using the hypothesis (A.1.4) and Itô´s Formula, and taking expectations,

there exists $D = D(L)$ such that

$$\mathbb{E} \sup_{0\leq t\leq T} | \overline{X}_t - \overline{U}_t |^2 \leq D \Big\{ \mathbb{E} \int_0^T | \overline{X}_s - \overline{U}_s | \big(| \overline{X}_s - \overline{U}_s | + | Y_s - V_s |$$

$$+ | Z_s - W_s | \big) \mathrm{d}s + \mathbb{E} \int_0^T | \overline{X}_s - \overline{U}_s |^2 + | Y_s - V_s |^2 \, \mathrm{d}s \Big\} +$$

$$2\mathbb{E} \sup_{0\leq t\leq T} \int_0^T < \overline{X}_s - \overline{U}_s, (\sigma(s, \overline{X}_s, Y_s) - \sigma(s, \overline{U}_s, V_s)) \mathrm{d}Bs >$$

By *Burkholder-Davis-Gundy Inequality* [19] and modifying $D > 0$ eventually but only depending on L:

$$\mathbb{E} \sup_{0\leq t\leq T} | \overline{X}_t - \overline{U}_t |^2 \leq D \Big\{ \mathbb{E} \int_0^T | \overline{X}_s - \overline{U}_s | \big(| \overline{X}_s - \overline{U}_s | + | Y_s - V_s |$$

$$+ | Z_s - W_s | \big) \mathrm{d}s + \mathbb{E} \int_0^T | \overline{X}_s - \overline{U}_s |^2 + | Y_s - V_s |^2 \, \mathrm{d}s +$$

$$\mathbb{E} \Big(\int_0^T | \overline{X}_s - \overline{U}_s |^2 \big(| \overline{X}_s - \overline{U}_s |^2 + | Y_s - V_s |^2 \, \mathrm{d}s \big)^{1/2} \Big\} \leq$$

$$\leq DT^{1/2} \Big(\mathbb{E} \sup_{0\leq t\leq T} | \overline{X}_s - \overline{U}_s |^2 + \mathbb{E} \sup_{0\leq t\leq T} | Y_s - V_s |^2 +$$

$$+ \mathbb{E} \int_0^T | Z_s - W_s |^2 \, \mathrm{d}s \Big)$$

So, modifying $D > 0$ eventually

$(1 - \mathrm{DT}^{1/2})\mathbb{E} \sup_{0\leq t\leq T} \mid \overline{X}_t - \overline{U}_t \mid^2 \leq DT^{1/2}\mathbb{E} \sup_{0\leq t\leq T} \mid Y_s - V_s \mid^2 +$

$+ \mathbb{E} \int_0^T \mid Z_s - W_s \mid^2 \mathrm{d}s$ (1.4.3)

Using Itô´s Formula, $\forall t \in [0, T]$

$\mid \overline{Y}_t - \overline{V}_t \mid^2 + \int_t^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s =$

$=$

$\mid h(\overline{X}_T) - h(\overline{U}_T) \mid^2 + 2 \int_t^T < \overline{Y}_s - \overline{V}_s, g(s, \overline{X}_s, \overline{Y}_s, \overline{Z}_s) - g(s, \overline{U}_s, \overline{V}_s, \overline{W}_s) > ds$

$- 2 \int_t^T < \overline{Y}_s - \overline{V}_s, (\overline{Z}_s - \overline{W}_s)\mathrm{d}Bs >$ (1.4.4)

Using the estimate

$\mathbb{E}\Big(\int_0^T \mid \overline{Y}_s - \overline{V}_s \mid^2 \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s\Big)^{1/2} \leq$

$\mathbb{E}\Big(\sup_{0\leq t\leq T} \mid \overline{Y}_s - \overline{V}_s \mid^2 + \int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s\Big) < \infty$

and *Burholder-Davis-Gundy Inequality* we see that

$\forall t \in [0, T] \quad \mathbb{E} \int_0^T < \overline{Y}_s - \overline{V}_s, (\overline{Z}_s - \overline{W}_s)\mathrm{d}Bs >= 0$

So, using (A.1.4) there exists D_1 only depending on $L > 0$ such that

$\mathbb{E} \int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s \leq$

$\leq D_1\Big[\mathbb{E} \mid \overline{X}_T - \overline{U}_T \mid^2 +$

$\mathrm{E} \int_0^T \mid \overline{Y}_s - \overline{V}_s \mid (\mid \overline{X}_s - \overline{U}_s \mid + \mid \overline{Y}_s - \overline{V}_s \mid + \mid \overline{Z}_s - \overline{W}_s \mid)\mathrm{d}s\Big]$

Modifying D_1 eventually :

$\mathbb{E} \int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s \leq$

$\leq D_1\Big[(1 + T)\mathbb{E} \sup_{0\leq t\leq T} \mid \overline{X}_s - \overline{U}_s \mid^2 + T\mathbb{E} \sup_{0\leq t\leq T} \mid \overline{Y}_s - \overline{V}_s \mid^2 \Big] +$

$$+ \frac{1}{2}\mathbb{E}\int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s$$

Modifying D_1 if necessary:

$$\mathbb{E}\int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s \leq D_1\Big((1+T)\mathbb{E}\sup_{0\leq s\leq T} \mid \overline{X}_s - \overline{U}_s \mid^2 +$$
$$+ \text{T } \mathbb{E}\sup_{0\leq s\leq T} \mid \overline{Y}_s - \overline{V}_s \mid^2 \Big) \ (1.4.5)$$

By (1.4.4) and still using *Burkholder-Davis-Gundy* there exists $D_2 > 0$ only depending on L such that:

$$\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{Y}_t - \overline{V}_t \mid^2 \leq D_2\Big[\mathbb{E} \mid \overline{Y}_T - \overline{V}_T \mid^2 +$$
$$+ \mathbb{E}\Big(\int_0^T \mid \overline{Y}_s - \overline{V}_s \mid^2 \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s\Big)^{1/2} +$$
$$+ \mathbb{E}\int_0^T \mid \overline{Y}_s - \overline{V}_s \mid (\mid \overline{X}_s - \overline{U}_s \mid + \mid \overline{Y}_s - \overline{V}_s \mid + \mid \overline{Z}_s - \overline{W}_s \mid)\mathrm{d}s\Big]$$

Using (1.4.5) and modifying D_2 eventually:

$$\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{Y}_t - \overline{V}_t \mid^2 \leq D_2\Big[(1+T)\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{X}_t - \overline{U}_t \mid^2 +$$
$$+ \text{T } \mathbb{E}\sup_{0\leq t\leq T} \mid \overline{Y}_t - \overline{V}_t \mid^2 \Big] + \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{Y}_t - \overline{V}_t \mid^2$$

Modifying D_2 if necessary we get:

$$(1\text{- D}_2T)\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{Y}_t - \overline{V}_t \mid^2 \leq D_2(1+T)\mathbb{E}\sup_{0\leq t\leq T} \mid \overline{X}_t - \overline{U}_t \mid^2 \ (1.4.6)$$

Using (1.4.3), (1.4.5) and (1.4.6) we obtain:

$$(1\text{- DT}^{1/2})\mathbb{E}\sup_{0\le t\le T} \mid \overline{X}_t - \overline{U}_t \mid^2 \le DT^{1/2}\Big(\mathbb{E}\sup_{0\le t\le T} \mid \overline{Y}_t - \overline{V}_t \mid^2 +$$
$$\mathbb{E}\int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s\Big) \quad (1.4.7)$$

$$\mathbb{E}\int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s \le D_2\Big[(1+T)\mathbb{E}\sup_{0\le t\le T} \mid \overline{X}_t - \overline{U}_t \mid^2 + T\mathbb{E}\sup_{0\le t\le T} \mid \overline{Y}_t - \overline{V}_t \mid^2\Big] \quad (1.4.8)$$

So there exists $C > 0$ only depending on L such that $\forall\ T \le C$ and D^* only depending on L such that:

$$\mathbb{E}\sup_{0\le t\le T} \mid \overline{X}_t - \overline{U}_t \mid^2 + \mathbb{E}\sup_{0\le t\le T} \mid \overline{Y}_t - \overline{V}_t \mid^2 + \mathbb{E}\int_0^T \mid \overline{Z}_s - \overline{W}_s \mid^2 \mathrm{d}s \le$$
$$\le D^*\Big(\mathbb{E}\sup_{0\le t\le T} \mid Y_t - V_t \mid^2 + \mathbb{E}\int_0^T \mid Z_s - W_s \mid^2 \mathrm{d}s\Big) \quad (1.4.9)$$

which proves that Θ is a contraction in the Banach Space $\mathcal{M}[0,T]$. Using *Banach`s Fixed Point Theorem*, for $T \le C$ there exists an unique $\{\mathcal{F}_t\}_{0\le t\le T}$ adapted solution to the problem (1.4.1) ($\Theta(X_t, Y_t, Z_t)_{0\le t\le T} = (X_t, Y_t, Z_t)_{0\le t\le T}$) . The property i) follows by the assumptions (A.1.4) on the coefficients of the system and from the fact $(X_t, Y_t, Z_t)_{0\le t\le T} \in \mathcal{M}[0,T]$. The property ii) follows by standarb inequalities and from the *Burkholder-Davis-Gundy`s Inequalities* , like the estimates we have done before. □

Remark

Theorem (1.4.1) says that for every $x \in \mathbb{R}^d\ \forall\ T \le C$, the problem

$$\begin{cases} X_t = x + \displaystyle\int_0^t f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\,\mathrm{d}Bs \\ Y_t = h(X_T) + \displaystyle\int_t^T g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}Bs \\ \mathbb{E}\displaystyle\int_0^T \mid\ X_t \mid^2 + \mid Y_t \mid^2 + \mid Z_t \mid^2 \mathrm{d}t \\ \forall\, 0 \le t \le T,\ x\ \in \mathbb{R}^d,\ \forall T > 0 \end{cases}$$

admits a unique $\{\mathcal{F}_t\}_{0\le t\le T}$ adapted solution $(X_t, Y_t, Z_t)_{0\le t\le T}$. So, in particular, Y_0 is a $\mathcal{F}_0$ measurable random vector, by application of *Blumenthal`s 0-1 Law* (see reference [19]) , therefore deterministic.

Theorem 1.4.2. ***Main estimate***

If we are in the conditions of the assumption (A.1.4), there exists $0 < C^* \leq C$ *only depending on L such that for every* $T \leq C^*$ *and for every* $(\overline{f}, \overline{g}, \overline{h}, \overline{\sigma})$ *satisfying (A.1.4) with the same constants* L, Λ *as* (f, g, h, σ)*, for every* $A \in \mathcal{F}_0$*, and for every* $\xi : \Omega \to \mathbb{R}^d$ *measurable with finite second moment, we have the following estimate:*

$$\mathbb{E}1_A \sup_{0\leq t\leq T} |\ \overline{X}_t - X_t\ |^2 + \mathbb{E}\,1_A \sup_{0\leq t\leq T} |\ \overline{Y}_t - Y_t\ |^2 + \mathbb{E}\int_0^T 1_A\ |\ \overline{Z}_s - Z_s\ |^2\, \mathrm{d}s \leq$$

$$\leq \Gamma\Big[\mathbb{E}\,1_A\ |\ \xi - \overline{\xi}\ |^2 + \mathbb{E}\,1_A\ |\ h(X_T) - \overline{h}(\overline{X}_T)\ |^2 +$$

$$\mathbb{E}\int_0^T 1_A\ |\ \sigma - \overline{\sigma}\ |^2 + \mathbb{E}\,\Big(\int_0^T 1_A(|\ f - \overline{f}\ | + |\ g - \overline{g}\ |)\mathrm{d}s\Big)^2\Big] \quad (1.4.10)$$

where Γ *only depends on L and where* $(X_s, Y_s, Z_s)_{0\leq s\leq T}$ *and* $(\overline{X}_s, \overline{Y}_s, \overline{Z}_s)_{0\leq s\leq T}$ *stand for the solutions of the respective problems associated to the coefficients* (f, g, h, σ) *and* $(\overline{f}, \overline{g}, \overline{h}, \overline{\sigma})$ *and with initial conditions* $(0, \xi)$ *and* $(0, \overline{\xi})$*.*

Proof. With the notations of the statement, using Itô`s Formula, omitting the arguments of the functions when there is no danger of confusion.

$$\mathbb{E}1_A \sup_{0\leq t\leq T} |\ \overline{X}_t - X_t\ |^2 \leq \mathbb{E}\,1_A\ |\ \xi - \overline{\xi}\ |^2 + \mathbb{E}\int_0^T 1_A\ |\ \sigma - \overline{\sigma}\ |^2\, \mathrm{d}s +$$
$$2\ \mathbb{E} \sup_{0\leq t\leq T}\int_0^t 1_A < \overline{X}_s - X_s, \overline{f} - f > \mathrm{d}s +$$
$$+\ 2\ \mathbb{E} \sup_{0\leq t\leq T}\int_0^t 1_A < \overline{X}_s - X_s, (\overline{\sigma} - \sigma)\mathrm{d}Bs >$$

Using *Burkholder-Davis-Gundy`s Inequalities*, there exists $\gamma > 0$ such that:

$$\mathbb{E}1_A \sup_{0\leq t\leq T} |\ \overline{X}_t - X_t\ |^2 \leq \mathbb{E}\,1_A\ |\ \xi - \overline{\xi}\ |^2 + \mathbb{E}\int_0^T 1_A\ |\ \sigma - \overline{\sigma}\ |^2\, \mathrm{d}s +$$
$$2\ \mathbb{E} \sup_{0\leq t\leq T}\int_0^t 1_A < \overline{X}_s - X_s, \overline{f} - f > \mathrm{d}s +$$
$$+\ 2\ \gamma\mathbb{E}\,\Big(\int_0^t 1_A\ |\ \overline{X}_s - X_s\ |^2 |\ \overline{\sigma} - \sigma\ |^2\, \mathrm{d}s\Big)^{1/2}$$

Modifying γ if necessary, it follows that

$$\mathbb{E}1_A \sup_{0\leq t\leq T} \mid \overline{X}_t - X_t \mid^2 \leq$$
$$\leq \gamma\Big\{\mathbb{E}1_A \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_0^T 1_A \mid \overline{\sigma}(s, \overline{X}_s, \overline{Y}_s) - \sigma(s, X_s, Y_s) \mid^2 \mathrm{d}s +$$
$$+ \mathbb{E} \sup_{0\leq T\leq T} \Big(\int_0^t 1_A < \overline{X}_s - X_s, \overline{f}(s, \overline{X}_s, \overline{Y}_s, \overline{Z}_s) - f(s, X_s, Y_s, Z_s) > \mathrm{d}s\Big) +$$
$$+ \mathbb{E} \Big(\int_0^T 1_A \mid \overline{f} - f \mid \mathrm{d}s\Big)^2\Big\}$$

Using the assumptions (A.1.4) about Lipschitz continuity on the coefficients, , there exists a different $\gamma > 0$ eventually such that:

$$\mathbb{E}\ 1_A \sup_{0\leq t\leq T} \mid \overline{X}_t - X_t \mid^2 \leq$$
$$\leq \gamma\Big\{\mathbb{E}1_A \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}\int_0^T 1_A(\mid \overline{X}_s - X_s \mid^2 + \mid \overline{Y}_s - Y_s \mid^2)\mathrm{d}s +$$
$$+ \mathbb{E} \int_0^T 1_A \mid \overline{X}_s - X_s \mid\mid \overline{Z}_s - Z_s \mid \mathrm{d}s +$$
$$\mathbb{E} \int_0^T 1_A(\mid \overline{\sigma} - \sigma \mid^2 \mathrm{d}s + \mathbb{E}\Big(\int_0^T 1_A \mid \overline{f} - f \mid \mathrm{d}s\Big)^2\Big\} \quad (1.4.11)$$

For every $0 \leq t \leq T$

$$\mathbb{E}1_A \mid \overline{Y}_t - Y_t \mid^2 + \mathbb{E}\int_0^T 1_A \mid \overline{Z}_s - Z_s \mid^2 \mathrm{d}s = \mathbb{E}1_A \mid \overline{h}(\overline{X}_T) - h(X_T) \mid^2$$
$$+2\mathbb{E} \int_t^T 1_A < \overline{Y}_s - Y_s, \overline{g}(s, \overline{X}_s, \overline{Y}_s, \overline{Z}_s) - g(s, X_s, Y_s, Z_s) > \mathrm{d}s$$

Once again, using *Burkholder-Davis-Gundy`s Inequalities* there exists γ_1 such that:

$$\mathbb{E} \Big(1_A \sup_{0\leq t\leq T} \mid \overline{Y}_t - Y_t \mid^2 \Big) \leq$$
$$\leq \mathbb{E}1_A \mid \overline{h}(\overline{X}_T) - h(X_T) \mid^2 + \gamma_1\mathbb{E}\Big(\int_0^T 1_A \mid Y_s - \overline{Y}_s \mid^2\mid Z_s - \overline{Z}_s \mid^2 \Big)^{1/2}$$
$$+2\ \mathbb{E}\ \Big(\sup_{0\leq t\leq T} \int_t^T 1_A\ <\ \overline{Y}_s - Y_s, \overline{g}(s, \overline{X}_s, \overline{Y}_s, \overline{Z}_s) - g(s, X_s, Y_s, Z_s)\ >\ \mathrm{d}s\Big)$$
(1.4.12)

Modifying γ_1 eventually, with routine estimates as done before, we get:

$$\mathbb{E}\Big(\ 1_A \sup_{0\le t\le T} \mid \overline{Y}_t - Y_t \mid^2 \Big) + \mathbb{E}\Big(\int_0^T 1_A \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s\Big) \le$$

$$\le \gamma_1\Big[\mathbb{E}1_A \mid \overline{h}(\overline{X}_T) - h(X_T) \mid^2 + \mathbb{E}\Big(\sup_{0\le t\le T}\int_t^T 1_A < \overline{Y}_s - Y_s, \overline{g}(s,\overline{X}_s,\overline{Y}_s,\overline{Z}_s) -$$

$$g(s,X_s,Y_s,Z_s) > \mathrm{d}s\Big) + \mathbb{E}\Big(\int_0^T 1_A \mid \overline{g} - g \mid \mathrm{d}s\Big)^2\Big]$$

Using (1.4.11), and the assumptions (A.1.4) modifying γ_1 eventually:

$$\mathbb{E}\ 1_A \sup_{0\le t\le T} \mid \overline{X}_t - X_t \mid^2 + \mathbb{E}\Big(1_A \sup_{0\le t\le T} \mid \overline{Y}_t - Y_t \mid^2 \Big) + \mathbb{E}(\int_0^T 1_A \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s$$

$$\le \gamma_1\Big[\mathbb{E}\, 1_A \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}1_A \mid \overline{h}(\overline{X}_T) - h(X_T) \mid^2 + \mathbb{E}\int_0^T 1_A \mid \overline{X}_s - X_s \mid^2 \mathrm{d}s$$

$$+ \mathbb{E}\int_0^T 1_A \mid \overline{Y}_s - Y_s \mid^2 \mathrm{d}s$$

$$+ \mathbb{E}\Big(\int_0^T 1_A(\mid \overline{f} - f \mid + \mid \overline{g} - g \mid)\mathrm{d}s\Big)^2$$

$$+ \mathbb{E}\int_0^T 1_A \mid \overline{\sigma} - \sigma \mid^2 \mathrm{d}_s$$

$$+ \mathbb{E}\int_0^T 1_A \mid \overline{Z}_s - Z_s \mid (\mid \overline{Y}_s - Y_s \mid + \mid \overline{X}_s - X_s \mid \mathrm{d}s\Big] \quad (1.4.13)$$

So, there exist C^* and Γ such that for every $T \le C^*$:

$$\mathrm{E}\ 1_A \sup_{0\le t\le T} \mid \overline{X}_t - X_t \mid^2 + \mathbb{E}\Big(1_A \sup_{0\le t\le T} \mid \overline{Y}_t - Y_t \mid^2 \Big) + \mathbb{E}(\int_0^T 1_A \mid Z_s - \overline{Z}_s \mid^2 \mathrm{d}s$$

$$\le \Gamma\Big[\mathbb{E}\, 1_A \mid \xi - \overline{\xi} \mid^2 + \mathbb{E}1_A \mid \overline{h}(\overline{X}_T) - h(X_T) \mid^2 + \mathbb{E}\int_0^T 1_A \mid \overline{\sigma} - \sigma \mid^2 \mathrm{d}s$$

$$+ \mathbb{E}\ \Big(\int_0^T 1_A(\mid \overline{f} - f \mid + \mid \overline{g} - g \mid)\mathrm{d}s\Big)^2\Big] \quad (1.4.14)$$

□

Corollary 1.4.1. ***Dependence upon the initial conditions***
Suppose we have the assumption (A.1.4).
For every $T \le C^*$, for every $t \in [0,T]$ and for every $\mathcal{F}_t$ measurable random

vector with finite second moment, we can define the process $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{t\leq s\leq T}$ *as the unique solution of the problem:*

$$\begin{cases} X_t = \xi + \int_t^s f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_t^s \sigma(s, X_s, Y_s)\,\mathrm{d}Bs \\ Y_t = h(X_T) + \int_s^T g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_s^T Z_s \mathrm{d}Bs \\ \forall\, t \leq s \leq T,\ \ \forall\, T > 0 \end{cases}$$

(1.4.15)

extended to the whole interval $[0, T]$ if $\xi = x$ a.s $\mathbb{P}$ putting for $0 \leq s \leq t$:

$\mathrm{X}_s^{t,x} = x$
$\mathrm{Y}_s^{t,x} = Y_t^{t,x}$
$\mathrm{Z}_s^{t,x} = 0$ (1.4.16)

Then the following properties are satisfied:

There exists a constant C_1 depending on L, Λ such that $\forall (t, x) \in [0, T] \times \mathbb{R}^d$:

$$\mathbb{E} \sup_{0\leq s\leq T} \mid X_s^{t,x} \mid^2 + \mathbb{E} \sup_{0\leq s\leq T} \mid Y_s^{t,x} \mid^2 + \mathbb{E} \int_t^T \mid Z_s^{t,x} \mid^2 \mathrm{d}s \leq C_1 (1 + \mid x \mid^2) \tag{1.4.17}$$

There exists $\Gamma_1 > 0$ only depending on L, Λ such that for every $(t, x), (t_1, x_1) \in [0, T] \times \mathbb{R}^d$:

$$\mathbb{E} \sup_{0\leq s\leq T} \mid X_s^{t,x} - X_s^{t_1,x_1} \mid^2 + \mathbb{E} \sup_{0\leq s\leq T} \mid Y_s^{t,x} - Y_s^{t_1,x_1} \mid^2 + \mathbb{E} \int_t^T \mid Z_s^{t,x} - Z_s^{t_1,x_1} \mid^2 \mathrm{d}s$$

$\leq \Gamma \mid x - x_1 \mid^2 + \Gamma_1 (1 + \mid x \mid^2) \mid t - t_1 \mid$ (1.4.18)

Proof. Let us assume $T \leq C^*$. We want to prove (1.4.17). Note by *Blumenthal`s 0-1 Law* for every $(t,x) \in [0,T] \times \mathbb{R}^d$ $(Y_s^{t,x})_{0\leq s\leq t}$ is actually reduced to a deterministic vector, and we have that $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{0\leq s\leq T}$ is solution of:

$$\begin{cases} X_t = x + \int_0^s 1_{[t,T]}(r) f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_0^s 1_{[t,T]}(r)\sigma(0, X_r, Y_r)\,\mathrm{d}Br \\ Y_t = h(X_T) + \int_s^T 1_{[t,T]}(r) g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_s \mathrm{d}Bs \\ \forall\, 0 \leq s \leq T \end{cases}$$

(1.4.19)

Note that $(1_{[t,T]}f, 1_{[t,T]}g, 1_{[t,T]}\sigma, h)$ and $(0,0,0,0)$ satisfy the assumption (A.1.4), the **theorem 1.4.2** gives by the main estimate the property (1.4.17). Again, we just have to observe that $(1_{[t,T]}f, 1_{[t,T]}g, 1_{[t,T]}\sigma, h)$ and $(1_{[t_1,T]}f, 1_{[t_1,T]}g, 1_{[t_1,T]}\sigma, h)$ satisfy the assumption (A.1.4) and we can use again the main estimate- **theorem 1.4.2** to conclude the property (1.4.18). □

Corollary 1.4.2. ***Functional Dependence of the solutions***
Suppose the hypothesis (A.1.4) in force and the notations of the previous result. For each $T \leq C^*$ the map:
$u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k (t,x) \mapsto Y_t^{t,x}$ *satisfies:*

$$| \, u(t,x) \, |^2 \leq C_1(1+ \mid x \mid^2) \tag{1.4.20}$$

$$| \, u(t,x) - u(t_1,x_1) \, |^2 \leq \Gamma \mid x - x_1 \mid^2 + \Gamma_1(1+ \mid x \mid^2) \mid t - t_1 \mid \tag{1.4.21}$$

and for every ξ $\mathcal{F}_t$ measurable with finite second moment, there exists a $\mathbb{P}$ null set $N_Y^{t,\xi} \in \mathcal{F}_0$ such that:

$\forall s \in [t,T] \, \forall \, \omega \notin N_Y^{t,\xi} : \quad Y_s^{t,\xi} = u(s, X_s^{t,\xi}(\omega))$. (1.4.22)

Proof. f we consider $(t,x) \in [0,T] \times \mathbb{R}^d$, from the remark after the **theorem 1.4.1**, the vector $Y_t^{t,x}$ is deterministic, so u is well defined.
(1.4.20) and (1.4.21) follow easily from (1.4.17) and (1.4.18). Let us prove

(1.4.22).
If ξ is a $\mathcal{F}_t$ measurable d-valued random vector with finite second moment, **theorem 1.4.2** (main estimate) shows that for each $\varepsilon > 0$:

$$\mathbb{E}\, 1_{|\xi-x|<\varepsilon} \mid Y_s^{t,\xi} - Y_s^{t,x} \mid^2 \leq \Gamma\, \mathbb{E}\, 1_{|\xi-x|<\varepsilon} \mid \xi - x \mid^2$$

Using the Lispchitz property (1.4.21):

$$\mathbb{E}\, 1_{|\xi-x|<\varepsilon}(\mid u(t,\xi) - Y_t^{t,\xi} \mid^2) \leq$$

$2 \left[\Gamma\, \mathbb{E}\, 1_{|\xi-x|<\varepsilon}(\mid \xi - x \mid^2) + \mathbb{E}\, 1_{|\xi-x|<\varepsilon}(\mid u(t,\xi) - u(t,x) \mid^2) \right]$
$\leq 4\Gamma \mathbb{E}\, 1_{|\xi-x|<\varepsilon}(\mid \xi - x \mid^2)$
So for each $n \in \mathbb{N}$:

$$\sum_{k\in\mathbb{Z}^d} \mathbb{E}\, 1_{|\xi-\frac{k}{n}|_\infty<1/n}(\mid u(t,\xi) - Y_t^{t,\xi} \mid^2) \leq \frac{4}{n^2}\, \Gamma \mathbb{E}\Big(1_{|\xi-\frac{k}{n}|_\infty<1/n} \Big)$$

We deduce for any $n \in \mathbb{N}$ that :

$$\mathbb{E}\, \mid u(t,\xi) - Y_t^{t,\xi} \mid^2 \leq \frac{2^{d+2}}{n^2}\Gamma$$

So, in particular

$\mathrm{Y}_t^{t,\xi} = u(t,\xi)$ a.s $\mathbb{P}$. (1.4.23)

Moreover, for each $w \in [s,T]$, $(X_w^{t,\xi,}, Y_w^{t,\xi,}, Z_w^{t,\xi,})_{s\leq w\leq T}$ is the solution for the problem:

$$\begin{cases} X_w = X_s^{t,\xi} + \displaystyle\int_s^w f(r, X_r, Y_r, Z_r)\, \mathrm{d}r + \int_s^w \sigma(r, X_r, Y_r)\, \mathrm{d}Br \\ Y_t = h(X_T) + \displaystyle\int_w^T g(r, X_r, Y_r, Z_r)\, \mathrm{d}r - \int_w^T Z_r \mathrm{d}Br \\ \forall\, s \leq w \leq T \end{cases}$$

(1.4.24)

(1.4.23) implies that $Y_w^{t,\xi} = u(w, X_w^{t,\xi})$ a.s $\mathbb{P}$.

The continuity of u and of the trajectories of the processes $(X_s^{t,\xi})_{t\leq s\leq T}$, $(Y_s^{t,\xi})_{t\leq s\leq T}$ show that a.s $\mathbb{P}$ $\forall s \in [t,T]$ $Y_s^{t,\xi} = u(s, X_s^{t,\xi})$

□

Proposition 1.4.1. *u depends only on f, g, h, σ and T.*

Proof. This is a technical result, and the scheme used for the proof is the one developed by Yamada and Watanabe to prove that pathwise uniqueness of SDEs solutions implies uniqueness in the sense of the probability law. See details in Delarue`s paper [8] using Roger and Williams presentation of the fact we mentioned in [34]. □

Corollary 1.4.3. ***Dependence upon the coefficients***

*If we assume the hypothesis (A.1.4) and $T \leq C^*$, keeping the notations of **corollary 1.4.2**, let $(f_n, g_n, \sigma_n, h_n)_{n\in\mathbb{N}}$ be a sequence satisfying (A.1.4) with respect to the same constants L, Λ as (f, g, σ, h) and such that:*
$\forall\, a.e\, t \in [0,T]$ $\forall\,(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$
$(f_n, g_n, \sigma_n, h_n)(t,x,y,z) \longrightarrow (f, g, \sigma, h)(t,x,y,z)$ as $n \to \infty$
If for every ξ $\mathcal{F}_0$ measurable with finite second moment, $(X_t^{n,0,\xi}, Y_t^{n,0,\xi,}, Z_t^{n,0,\xi,})$ stands for the solution of:

$$\begin{cases} X_t = \xi + \int_0^t f_n(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma_n(s, X_s, Y_s)\,\mathrm{d}Bs \\ Y_t = h_n(X_T) + \int_t^T g_n(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_s^T Z_s \mathrm{d}Bs \\ \forall\, 0 \leq t \leq T \end{cases}$$

(1.4.25)

then when $n \to \infty$

$$\mathbb{E} \sup_{0\leq s\leq T} \mid X_s^{n,0,\xi} - X_s^{0,\xi} \mid^2 + \mathbb{E} \sup_{0\leq s\leq T} \mid Y_s^{n,0,\xi} - Y_s^{0,\xi} \mid^2 + \mathbb{E} \int_t^T \mid Z_s^{n,0,\xi} - Z_s^{0,\xi} \mid^2 \mathrm{d}s \longrightarrow 0 \quad (1.4.26)$$

In particular, as

n $\to \infty$, $u_n \longrightarrow u$ (1.4.27)

uniformly on every compact set of $[0,T]\times\mathbb{R}^d$, where u_n stands for the map associated by means of **corollary 1.4.22** to the coefficients $(f_n, g_n, \sigma_n, h_n)$.

Proof. Using the main estimate in **theorem 1.4.2** as well *Lebesgue`s Dominated Convergence Theorem*, we prove (1.4.26). In particular the pointwise convergence of u_n to u.
(1.4.26) shows that the maps (u_n) are equicontinuous on every compact set of $[0,T]\times\mathbb{R}^d$. Using *Arzela-Ascoli Theorem* (see for example [20]), the convergence is uniform on every compact set of $[0,T]\times\mathbb{R}^d$. □

1.5 A global time result of existence and uniqueness for FBSDEs

In the last section we proved the main result of existence and uniqueness of solution for the problem (E):

$$\begin{cases} X_s = \xi + \int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_s = h(X_T) + \int_s^T g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r \mathrm{d}Br \\ \forall\, t \le s \le T,\ \forall\, T > 0 \end{cases}$$

(1.5.1)

where $T = 0$

But our proof, based in a fixed point argument, stands the result for a small T, depending on the size of the Lipschitz constant $L > 0$ as we have seen in the **theorem 1.4.1** . One of the key results of the last section was **corollary 1.4.2**, specially concerning the property (1.4.22)
But in order to get existence and uniqueness of solution for the system (E) in all $[0,T]$, with $T > 0$ stated initially , we will make use of the functional relation between $(X_s^{T,\xi}, Y_s^{T,\xi})_{t\le s\le T}$, $Y_s^{t,\xi}(\omega) = u(s, X_s^{t,\xi})(\omega)$.
If the coefficients f, g, σ, h are sufficiently regular, we know by the study presented in the third section of this chapter - **Four Step Sheme** of Ma-Yong ([24], [25]) that the solutions of (E) are really connected by means of

u with the quasilinear parabolic system of PDEs:

$$\begin{cases} \dfrac{\partial u^l}{\partial t}(t,x) + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a_{i,j}(t,x,u(t,x))\dfrac{\partial^2 u^l}{\partial x_i \partial x_j}(t,x) + \\ \displaystyle\sum_{i=1}^{d} f_i(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x,u(t,x)))\dfrac{\partial u^l}{\partial x_i} + \\ +g^l(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x,u(t,x))) = 0 \\ t \in [0,T] \quad x \in \mathbb{R}^d \quad ,l = 1,...,k \\ u(T,x) = h(x) \ x \in \mathbb{R}^d \end{cases}$$

(1.5.2)

With $a_{i,j} = \sigma\sigma^T$.
Assuming the non-degeneracy of the diffusion coefficient of the forward equation and with strong regularity assumptions on the coefficients, the Four Step Scheme proves that (E) admits an unique solution, with the wise use of Ladyzhenskaja`s results of deterministic parabolic quasilinear PDE [22].
Another link between (E) and (1.5.2) is that under appropriate assumptions, eventually with σ degenerate, the solution of (E) provides a viscosity solution to the problem (1.5.2) (see the works of Pardoux-Tang [29]), as we are going to present in the last chapter of this work.
So, it is our intention to show that the local result of existence and uniqueness of solution for (E) and the functional dependence between $(X_s^{t,\xi}, Y_s^{t,\xi})_{t\leq s\leq T}$ can be extended, with a tecnhique of running-down induction on time. Based on the work of Ma-Yong [24], and using some estimates of the gradient of solutions of quasilinear parabolic systems of PDEs presented in Ladyzhenskaja`s work [22], Delarue [8] proved under appropriate assumptions (non-degeneracy of σ and boundedness of the coefficients as functions of x) a global result of existence and uniqueness for (E).
We need a new set of assumptions on the coefficients:

Assumptions A.1.5

We assume that f, g, h, σ satisfy (A.1.4) and also there exists $L_1, \Lambda, \lambda > 0$ such that $\forall\, t \in [0,T]\ \forall\,(x,y,z),(x_1,y_1,z_1) \in \mathbb{R}^d \times \mathbb{R}^k$:

$$\mid \sigma(t,x,y) - \sigma(t,x_1,y_1) \mid^2 \leq L_1(\mid x - x_1 \mid^2 + \mid y - y_1 \mid^2)$$

$| h(x) - h(x_1) |\leq L_1 | x - x_1 |$

$| f(t,x,y,z) |\leq \Lambda(1+ | y | + | z |)$
$| g(t,x,y,z) |\leq \Lambda(1+ | y | + | z |)$
$| \sigma(t,x,y) |\leq \Lambda(1+ | y |)$
$| h(x) |\leq \Lambda$

$<\xi, a(t,x,y,z)\xi >\geq | \xi |^2 \quad \forall\, \xi \in \mathbb{R}^d$
where $a(t,x,y) = \sigma\sigma^T(t,x,y)$.

σ is continuous on its definition set . (1.5.3)

Lemma 1.5.1. *Assume* $\overline{f}, \overline{g}, \overline{\sigma}, \overline{h}$ *are bounded* C^∞ *with bounded derivatives and satisfying the hypothesis (A.1.4) and (A.1.5) with respect to the constants* L, L_1, Λ, λ. *Setting* $a = \sigma\sigma^T$, *the following system of PDEs:*

$$\begin{cases} \dfrac{\partial u^l}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a_{i,j}(t,x,u(t,x))\dfrac{\partial^2 u^l}{\partial x_i \partial x_j} + \sum_{i=1}^{d} f_i(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x,u(t,x)))\dfrac{\partial u^l}{\partial x_i} + \\ +g^l(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x,u(t,x))) = 0 \\ \forall (t,x) \in\ [0,T] \times \mathbb{R}^d \ \forall\, l = 1,...,k \\ \forall\ \ x \in \mathbb{R}^d \ u(t,x) = h(x) \end{cases}$$

(1.5.4)

admits a unique solution $u \in\ C_b^{1,2}([0,T] \times \mathbb{R}^d)$.

In addition there exists a constant κ only depending on Λ, T and two constants $\kappa_1, \kappa_2 > 0$ only depending on L, L_1, Λ, d, k, T such that :

$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} | u(t,x) |\leq \kappa$ (1.5.5)

$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} | \nabla_x u(t,x) |\leq \kappa_1$ (1.5.6)

$\forall\ t, t_1 \in \mathbb{R}^d\ \forall x \in \mathbb{R}^d\ | u(t_1,x) - u(t,x) |\leq \kappa_2 | t - t_1 |^{1/2}$ (1.5.7)

$\forall t \in [0,T]$ and for every $\mathcal{F}_t$ measurable d-valued random vector ξ with finite second moment, the SDE:

$$
\mathrm{X}_s = \xi + \int_t^s f(r, X_r, Y_r, u(r, X_r), \nabla_x u(r, X_r)\sigma(r, X_r, u(r, X_r)))\mathrm{d}r
$$
$$
+ \int_t^s \sigma(r, X_r, u(r, X_r))\mathrm{d}Br \quad \forall\, t \leq s \leq T
$$
(1.5.8)

admits a unique solution, denoted by $(X_s^{t,\xi})_{t\leq s\leq T}$ and the process $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{s\leq t\leq T}$ given by:

$$
\begin{cases} Y_s^{t,\xi} = u(s, X_s^{t,\xi}) \\ Z_s^{t,\xi} = \nabla_x u(s, X_s)\sigma(s, X_s, u(s, X_s)) \end{cases} \tag{1.5.9}
$$

satisfies the FBSDE associated with (f, g, σ, h) and to the initial condition (t, ξ).

Proof. The proof of this crucial and vital lemma is due to Ma-Yong work [24] and to Ladyzhenskaja`s result **(lemma 1.3.1**) (see [22] for more information) and to **theorem 1.3.2**.
Delarue delivers a probabilistic proof of (1.5.4), proving the bounds (1.5.5) (1.5.6) and (1.5.7) probabilistically too in [8]. The probabilistic proof of existence and uniqueness of solution of (1.5.4) uses more sofisticated tools of Stochastic Analysis that we avoid here, such as the notion of Malliavin Derivative on the Wiener Space (see [26] for a good introduction to the subject). □

Proposition 1.5.1. ***Approximation***
Under the hypothesis (A.1.5) , there exists a sequence of C^∞ functions $(f_n, g_n, \sigma_n, h_n)_{n\in\mathbb{N}}$ satisfying (A.1.5) for every $n \in \mathbb{N}$ with respect to the constants $L + 4\Lambda, 2\Lambda, \frac{\Lambda}{2}$ such that:
$(f_n, g_n, \sigma_n, h_n) \to (f, g, \sigma, h)$ as $n \to +\infty$
a.e $t \in [0, T]$ and for every $(x, y, z) \in \mathbb{R}^d$.
Moreover , letting $a_n = \sigma_n\sigma_n^T$ for every $n \in \mathbb{N}$, the following system of PDEs:

$$\begin{cases} \frac{\partial u_n^l}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}(a_n)_{i,j}(t,x,u_n(t,x))\frac{\partial^2 u_n^l}{\partial x_i \partial x_j} + \\ \sum_{i=1}^{d}(f_n)_i(t,x,u_n(t,x),\nabla_x u_n(t,x)\sigma_n(t,x,u_n(t,x)))\frac{\partial u_n^l}{\partial x_i} + \\ +g_n^l(t,x,u_n(t,x),\nabla_x u_n(t,x)\sigma_n(t,x,u_n(t,x))) = 0 \\ \forall (t,x) \in [0,T]\times\mathbb{R}^d \ \forall\, l=1,...,k \\ u(T,x) = h(x) \ \forall \ x\in\mathbb{R}^d \end{cases}$$

(1.5.10)

admits a unique bounded solution $u_n \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. In addition, there exists a constant κ only depending on Λ, T and two constants $\kappa_1, \kappa_2 > 0$ only depending on L, L_1, Λ, d, k, T such that , for all $n \in \mathbb{N}$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mid u_n(t,x) \mid \leq \kappa \tag{1.5.11}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mid \nabla_x u_n(t,x) \mid \leq \kappa_1 \tag{1.5.12}$$

$$\forall\ t,t_1 \in \mathbb{R}^d\ \forall x \in \mathbb{R}^d \mid u_n(t_1,x) - u_n(t,x) \mid \leq \kappa_2 \mid t - t_1 \mid^{1/2} \tag{1.5.13}$$

Proof. For details see Delarue [8].
This is a technical proof with the classical analytical approximation procedure. In order to do it, we introduce the following objects:
i) $(\rho_n), (\rho_n^1), (\rho_n^2), (\rho_n^3)$ four mollifiers on $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^k$, $\mathbb{R}^{k\times d}$ respectively defined by:
$\rho_n(.) = c_n\phi(n \mid . \mid)$
$\rho_n^1(.) = c_n^1\phi(n^d \mid . \mid)$
$\rho_n^2(.) = c_n^2\phi(n^k \mid . \mid)$
$\rho_n^3(.) = c_n^3\phi(n^{k\times d} \mid . \mid)$
where $\phi(x) = e^{-\frac{1}{x^2-1}} 1_{]-1,1[}(x)\ \ \forall\ x\in\mathbb{R}$
and c, c_1, c_2, c_3 are fours constants of normalization.
ii) $\forall\ N \in \mathbb{N}, \forall r > 0$ we define the following map:
$\tau_r^N\ x \mapsto \frac{r}{r\vee \mid x \mid}x$
which is 1-Lipschitz and satisfies for each $x \in \mathbb{R}^N$

$| \tau_r^N(x) | \leq r \wedge | x |$ and for every $r > 0$ set
$\tau_r^1 = \tau_r^d$, $\tau_r^2 = \tau_r^k$, $\tau_r^3 = \tau_r^{k\times d}$
Let also

$$\pi_r : \mathbb{R}^+ \to \mathbb{R}^+$$

$x \mapsto 1_{[0,r]}(x) + \dfrac{2r - x}{r} 1_{[r,2r]}(x)$which is $\frac{1}{n}$ - Lipschitz and satisfies for every $x \in \mathbb{R}$ $0 \leq \pi_r(x) \leq 1$
iii) We extend the functions (f, g, σ) to $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ putting for every $(x, y, z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$
$(f, g, \sigma)(t, x, y, z) = (f, g, \sigma)(0, x, y, z)$ if $t < 0$
$(f, g, \sigma)(t, x, y, z) = (f, g, \sigma)(T, x, y, z)$ if $t > T$
iv) For each $n \in \mathbb{N}$ we denote $\omega_n$the modulus of continuity of σ on the compact set $[0, T] \times \{x \in \mathbb{R}^d :| x | \leq n\} \times \{y \in \mathbb{R}^k :| y | \leq n\}$
So for every $n \in \mathbb{N}$ there exists integers $p_n \geq 2$ such that:

$\sup_{|x|\leq n\,,|y|\leq n} | \sigma(t, x, y) | \omega_n(\dfrac{4}{p_n}) \leq \dfrac{\lambda}{2n}$ where $(p_n)_{n\in\mathbb{N}}$ is chosen strictly increasing and growing up to ∞.

v) For every $n \in \mathbb{N}$ we define:

$$f_n(t, x, y, z) = \int f(t-s, x-u, \tau_n^2(y-v), \tau_n^3(z-w))\rho_{p_n}(s)\rho^1_{p_n}(u)\rho^2_{p_n}(v)\rho^3_{p_n}(w)\mathrm{d}s\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$g_n(t, x, y, z) = \int \pi_n(| y-v |)g(t-s, x-u, y-v, \tau_n^3(z-w)\rho_{p_n}(s)\rho^1_{p_n}(u)\rho^3_{p_n}(w)\mathrm{d}s\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$h_n(t, x, y, z) = \int h(x - u)\rho^1_{p_n}(u)\mathrm{d}u$$

$$\sigma_n(t, x, y, z) = \int \sigma(t - s, \tau_n^1(x - u, \tau_n^2(y - v)\rho_{p_n}(s)\rho^1_{p_n}(u)\rho^2_{p_n}(v)\mathrm{d}s\mathrm{d}u\mathrm{d}v$$

It can be proved that under this construction, we are in the conditions to apply the result of the **lemma 1.5.1** to conclude the result.

□

In what follows we keep the notations: $\overline{K} = max(L_1, L + 4, \Lambda, \kappa_1)$ and $\gamma = C_*$ that works in **theorem 1.4.2** for the constants above.

Corollary 1.5.1. *Assuming (A.1.5), under the latter notations, there exists an integer* $N > 0$ *given by* $N = [\frac{T}{\gamma}+1]$*, and* $N + 1$ *real numbers* $(t_i)_{0\leq i\leq N}$*, defined as follows:*
$t_0 = 0$, $T_i = T - (N - i)\gamma$ *(*$t_N = T, t_{N-1} = T - 2\gamma, ...$*) such that for every* $n \in \mathbb{N}_0$ *and for each* $0 \leq i \leq N$*, for every* $t \in [t_i, t_{i+1}]$ *and for every* $\mathcal{F}_t$

measurable random d -vector ξ with finite second moment, the problem:

$$\begin{cases} X_t = \xi + \int_t^s f_n(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma_n(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_t = u_n(t_{i+1}, X_{t_{i+1}}) + \int_s^{t_{i+1}} g_n(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_{i+1}} Z_r \mathrm{d}Br \\ \forall\, t_i \leq s \leq t_{i+1}, \end{cases}$$

(1.5.14)

admits a unique solution in $\mathcal{M}[t_i, t_{i+1}]$
$(X_s^{n,t,i,\xi}, Y_s^{n,t,i,\xi}, Z_s^{n,t,i,\xi})_{t_i \leq s \leq t_{i+1}}$ that satisfies a.s $\mathbb{P}$

$$\begin{cases} Y_s^{n,t,i,\xi} = u_n(s, X_s^{n,t,i,\xi}) \\ Z_s^{n,t,i,\xi} = \nabla_x u_n(s, X_s^{n,t,i,\xi})\sigma_n(s, X_s^{n,t,i\xi}, Y_s^{n,t,i\xi}) \\ \mid Z_s^{n,t,i\xi} \mid \leq \Gamma_1 \end{cases} \tag{1.5.15}$$

where Γ_1 depends on L, L_1, Λ, d, k, T.

Proof. We only give the idea of the proof.
Fix $n \in \mathbb{N}$, $i = 0, ..., N-1$, $t \in [t_i, t_{i+1}]$ and
ξ $\mathcal{F}_t$ measurable with second order moment finite, due to **lemma 1.5.1**, with u replaced by u_n and σ by σ_n we can associate to $n \in \mathbb{N}$, $(X_s^{n,t,\xi}, Y_s^{n,t,\xi}, Z_s^{n,t,\xi})_{t \leq s \leq T}$ satisfying both (1.5.9) and (1.5.14).
Using the result of existence and uniqueness of solution for small time duration- **theorem 1.4.1** , from our choice of γ we see that this solution is unique. From (1.5.11), (1.5.12), (A.1.5) we conclude the property (1.5.15) and the proof is complete. □

Proposition 1.5.2. *Under the assumption (A.1.5) and keeping the notations of the* ***proposition 1.5.1*** *there exists a map $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^k$ such that:*

$$u_n \to u$$

uniformly on every compact set of $[0, T] \times \mathbb{R}^d$ *as* $n \to \infty$(1.5.16)

$\forall\ (t, x) \in [0, T] \times \mathbb{R}^d \quad \mid u(t, x) \mid \leq \kappa$ (1.5.17)

$\forall\ (t, x), (t_1, x_1) \in [0, T] \times \mathbb{R}^d \mid u(t_1, x_1) - u(t, x) \mid \leq \kappa_1 \mid x - x_1 \mid + \kappa_2 \mid t - t_1 \mid$
(1.5.18)

$\forall\ x\ \in \mathbb{R}^d\ u(T,x) = h(x)$ (1.5.19)

$\forall\, i = 0, ..., N-1\ \forall t \in [t_i, t_{i+1}]\ \forall \mathbb{F}_t$ measurable random d- vector ξ with finite second order moment, the problem:

$$\begin{cases} X_s = \xi + \int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_s = u(t_{i+1}, X_{t_{i+1}}) + \int_s^{t_{i+1}} g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_{i+1}} Z_r \mathrm{d}Br \\ \forall\, t \le s \le t_{i+1} \end{cases} \tag{1.5.20}$$

admits a unique solution $(X_s^{t,i,\xi}, Y_s^{t,i,\xi}, Z_s^{t,i,\xi})_{t \le s \le t_{i+1}}$ that satisfies:
$\mathbb{P}\Big(\forall s \in [t, t_{i+1}]\ Y_s^{t,i,\xi} = u(s, X_s^{t,i,\xi})\Big) = 1$ and
$\mathbb{P} \otimes \mu\Big((\omega, s) \in \Omega \times [t, t_{i+1}] :\mid Z_s^{t,i,\xi}(\omega) \mid > \Gamma_1\Big) = 0$
($\otimes$ means the product of the two measures)

Proof. We build the map u using a running-time down induction. Thanks to the **theorem 1.4.1** we show that for every $t \in [t_{N-1}, T[$ and for every $\xi\, \mathcal{F}_t$ measurable with finite second order moment, the problem:

$$\begin{cases} X_s = \xi + \int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_t = h(X_T) + \int_s^T g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r \mathrm{d}Br \\ \forall\, t \le s \le T \end{cases}$$

admits a unique solution $(X_s^{t,N-1,\xi}, Y_s^{t,N-1,\xi}, Z_s^{t,N-1,\xi})_{t \le s \le T}$.
Following the previous chapter, we define the map:

$$\begin{cases} u : [t_{N-1}, T] \times \mathbb{R}^d \to \mathbb{R}^k \\ (t,x) \mapsto u(t,x) = Y_t^{t,N-1,x} \end{cases} \tag{1.5.21}$$

From **corollary 1.4.2** we know a.s $\mathbb{P}$ that $Y_s^{t,N-1,\xi} = u(s, X_s^{t,N-1,\xi})$. From **corollary 1.4.3** and from **corollary 1.5.1** we know that the maps $u_n \to u$ as $n \to \infty$ uniformly on every compact set of $[t_{N-1}, T] \times \mathbb{R}^d$.
In particular, (1.5.12) and (1.5.13) gives that for every $(t,x), (t_1, x_1) \in [t_{N-1}, T] \times \mathbb{R}^d$
$\mid u(t_1, x_1) - u(t,x) \mid \le \kappa_1 \mid x - x_1 \mid + \kappa_2 \mid t - t_1 \mid$.
Corollary 1.5.1 also proves that for every $t \in [t_{N-1}, T]$
$\mathbb{E} \int_t^T \mid Z_s^{n,t,N-1,\xi} - Z_s^{t,N-1,\xi} \mid \mathrm{d}s \to 0$ as $n \to \infty$
From **corollary 1.4.3** we deduce that

$\mathbb{P}\otimes\mu\Big((\omega,s)\in\Omega\times[t,t_{i+1}]:\mid Z_s^{t,i,\xi}\mid>\Gamma_1\Big)=0.$
So we proved the result on $[t_{N-1},T]$.
We can do the same in $[t_{N-2},t_{N-1}]$ and using a running-down induction we build a map satisfying the assertion of the proposition. □

Corollary 1.5.2. *If (A.1.5) is assumed, and keeping the notations of the* ***proposition 1.5.2****, for every ξ $\mathcal{F}_t$ measurable with finite second moment , every solution $(X_s,Y_s,Z_s)_{t\le s\le T}$ in $\mathcal{M}[t,T]$ of the problem:*

$$\begin{cases} X_s=\xi+\displaystyle\int_t^s f(r,X_r,Y_r,Z_r)\,\mathrm{d}r+\int_t^s\sigma(r,X_r,Y_r)\,\mathrm{d}Br\\ Y_s=h(X_T)+\displaystyle\int_s^T g(r,X_r,Y_r,Z_r)\,\mathrm{d}r-\int_s^T Z_r\mathrm{d}Br\\ \forall\, t\le s\le T\end{cases}$$

(1.5.22)

satisfies for every $i\le j\le N-1$

$$\mathbb{E}\sup_{\bar t_j\le s\le\bar t_{j+1}}\mid X_s-X_s^{\bar t_j,j,X_{\bar t_j}}\mid^2=\mathbb{E}\sup_{\bar t_j\le s\le\bar t_{j+1}}\mid Y_s-Y_s^{\bar t_j,j,X_{\bar t_j}}\mid^2=$$
$$\mathbb{E}\int_{\bar t_j}^{\bar t_{j+1}}\mid Z_s-Z_s^{\bar t_j,j,X_{\bar t_j}}\mid^2\mathrm{d}s=0\ (1.5.23)$$

where $i=0,...,N-1$ is the unique integer such that $t\in[t_i,t_{i+1}]$; $(\bar t_j)_{i\le j\le N}$ stand for the real numbers defined as follows:
$\overline{t_i}=t,\ \overline{t_j}=t_j$ if $j>i$.
In particular

$$\mathbb{P}\Big(\forall s\in[t,T]\ Y_s=u(s,X_s)\Big)=1\qquad(1.5.24)$$

$$\mathbb{P}\otimes\mu\Big((\omega,s)\in\Omega\times[t,T]:\mid Z_s(\omega)\mid>\Gamma_1\Big)=0\qquad(1.5.25)$$

Proof. Apply the same running-down induction method as in the previous proposition. □

Theorem 1.5.1. ***Existence and Uniqueness of solution***
If (A.1.5) is satisfied, keeping the notations of the ***proposition 1.5.2*** *then:*

1) for every $T>0$ for every $\mathcal{F}_0$ measurable random vector ξ with finite second order moment, the problem (E) admits n unique solution in $\mathcal{M}[0,T]$(1.5.26)

2) for every $t \in [0,T]$ and for every ξ $\mathcal{F}_t$ measurable with finite second order moment , the unique solution of the problem in $\mathcal{M}[t,T]$

$$\begin{cases} X_s^{t,\xi} = \xi + \displaystyle\int_t^s f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\xi}, Y_r^{t,\xi})\,\mathrm{d}Br \\ Y_s^{t,\xi} = h(X_T^{t,\xi}) + \displaystyle\int_s^T g(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})\,\mathrm{d}r - \int_s^T Z_r^{t,\xi}\mathrm{d}Br \\ \forall\, t \le s \le T, \end{cases} \tag{1.5.27}$$

satisfies

$$\mathbb{P}\Big(\forall s \in [t,T]\ Y_s^{t,\xi} = u(s, X_s^{t,\xi})\Big) = 1 \tag{1.5.28}$$

$$\mathbb{P}\otimes\mu\Big((\omega,s) \in \Omega\times[t,T] :\mid Z_s^{t,\xi} \mid > \Gamma_1\Big) = 0 \tag{1.5.29}$$

In particular, there exist versions of the processes $(Y_s^{t,\xi})_{t\le s\le T}$ and $(Y_s^{t,\xi})_{t\le s\le T}$ whose trajectories are uniformly bounded.

Remark:

The same reasoning of the **proposition 1.4.1** shows that the map u only depends on f, g, h, σ and T.

Proof. **Existence of solution:**

Consider ξ a $\mathcal{F}_0$ measurable d- vector with finite second moment. Let us show existence of a solution for (E). Using a running- up induction, thanks to **proposition 1.5.2** the problem:

$$\begin{cases} X_t = \xi + \displaystyle\int_0^t f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_0^t \sigma(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_s = u(t_1, X_{t_1}) + \displaystyle\int_t^{t_1} g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_t^{t_1} Z_r\mathrm{d}Br \\ \forall\, 0 \le t \le t_1, \end{cases}$$

admits a unique solution in $\mathcal{M}[0,t_1]$ denoted by $(X_t^{0,0,\xi}, Y_t^{0,0,\xi}, Z_t^{0,0,\xi})_{0\le t\le t_1}$ (using the notations of **corollary 1.5.1**) which we denote upon now by $(X_t^0, Y_t^0, Z_t^0)_{0\le t\le t_1}$.

Aplying again **proposition 1.5.2** the problem

$$\begin{cases} X_t = X^0_{t_1} + \displaystyle\int_{t_1}^{t} f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_{t_1}^{t} \sigma(r, X_r, Y_r)\,\mathrm{d}Br \\ Y_s = u(t_2, X_{t2}) + \displaystyle\int_{t}^{t_2} g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_{t}^{t_2} Z_r \mathrm{d}Br \\ \forall\, t_1 \le t \le t_2, \end{cases}$$

admits a unique solution in $\mathcal{M}[t_1, t_2]$, which we will denote by $(X^1_t, Y^1_t, Z^1_t)_{t_1 \le t \le t_2}$

It satisfies $X^1_{t_1} = X^0_{t_1}$ and $Y^1_{t_1} = u(t, X^0_{t_1}) = Y^0_{t_1}$ a.s $\mathbb{P}$.

Using a simple inductive argument, the processes $\big((X^k_t, Y^k_t, Z^k_t)_{t_k \le t \le t_{k+1}}\big)_{k=0,\dots,N-1}$ for every $k = 0, .., N-1$, are solutions to the problems in $\mathcal{M}[t_k, t_{k+1}]$:

$$\begin{cases} X_t = X^{k-1}_{t_k} + \displaystyle\int_{t_k}^{t} f(r, X^k_r, Y^k_r, Z^k_r)\,\mathrm{d}r + \int_{t_k}^{t} \sigma(r, X^k_r, Y^k_r)\,\mathrm{d}Br \\ Y_s = u(t_{k-1}, X^k_{t_{k-1}}) + \displaystyle\int_{t}^{t_{k+1}} g(r, X^k_r, Y^k_r, Z^k_r)\,\mathrm{d}r - \int_{t}^{t_{k+1}} Z^k_r \mathrm{d}Br \\ \forall\, t_k \le t \le t_{k+1}, \end{cases}$$

We have that for every $0 \le k \le N-1$

$X^k_{t_k} = X^{k-1}_{t_k}$ and $Y^k_{t_k} = u(t_k, X^{k-1}_{t_k}) = Y^{k-1}_{t_k}$ a.s $\mathbb{P}$.

This proves the processs $(X_t, Y_t, Z_t)_{0 \le t \le T}$ defined as follows: $\forall\ k = 0, ..., N-1\ \forall\ t \in [tk, t_{k+1}] : X_t = X^k_t$, $Y_t = Y^k_t$, $Z_t = Z^k_t$ is actually a solution to the problem in $\mathcal{M}[0, T]$.

Uniqueness of solution:
Consider $(U_t, V_t, W_t)_{0 \le t \le T}$ a solution to (E) in $\mathcal{M}[0, T]$. By **corollary 1.5.2**

$$\mathbb{E} \sup_{0 \le t \le t_1} \mid U_t - X_t \mid^2 = \mathbb{E} \sup_{0 \le t \le t_1} \mid Y_t - V_t \mid^2 = \mathbb{E} \int_{t_0}^{t_1} \mid W_t - Z_t \mid^2 \mathrm{d}t = 0.$$

In particular $U_{t1} = X_{t1}$, $Y_{t1} = V_{t1}$, $W_{t1} = Z_{t1}$ a.s $\mathbb{P}$. A new induction procedure completes the proof of the uniqueness property.
(2) is a direct consequence of **corollary 1.5.2**.

So, the main result of this chapter is now completely proved.

□

Chapter 2

Large Deviations Principles and the Freidlin-Wentzell Theory

"... a particularly convenient way of stating assymptotic results that, on the one hand, are accurate enough to be useful and, on the other hand, are losse enough to be correct."
Amir Dembo, Ofer Zeitoni, "Large Deviations Techniques and Applications"

2.1 Informal Ideas

In a general way, Large Deviations Theory is concerned with the study of the probabilities of very rare events, formatting the heuristic ideas of concentration of measures and widely generalizing the notion of convergence of probability measures.

Rougly speaking, Large Deviations Theory concerns itself with the exponential decay of the probability measures of certain kinds of extreme or tail events, as the number of obervations grow arbirarily large. The first rigourous results concerning Large Deviations Theory are due to the Swedish mathematician Harold Cramer, who applied them to model the insurance business, although a clear unified formal definition was introduced only in 1966 by the 2007 Abel Prize Srinivasa Varadhan.

But what is meant by rare?

As Dembo and Zeitoni says, a theory of rare events should provide an analysis of the rarity of these events. As Deuschel and Strook says in their book [10] there is no real theory of Large Deviations, but we present in this introduction a classical situation in Probability Theory that lead us to Large

Deviations Ideas.
Let $X_1, ..., X_n$ be a sequence of independent, standarb normal real valued random variables and consider the empirical mean

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim Gaussian(0, \frac{1}{\sqrt{n}})$$

By Central Limit Theorem (see [20]) for each $\delta > 0$ we have

$$\mathbb{P}\big(\mid S_n \mid \geq \delta\big) \longrightarrow 0 \ as \ n \to \infty \tag{2.1.1}$$

and for any A interval

$$\lim_{n\to\infty} \mathbb{P}\big(\sqrt{n}S_n \in A\big) = \frac{1}{\sqrt{2\pi}}\int_A e^{\frac{-x^2}{2}} \mathrm{d}x \tag{2.1.2}$$

Noting that:

$$\mathbb{P}\big(\mid S_n \mid \geq \delta\big) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{\frac{-x^2}{2}} \mathrm{d}x \tag{2.1.3}$$

therefore

$$\frac{1}{n}\log \mathbb{P}\big(\mid S_n \mid \geq \delta\big) \to \frac{-\delta^2}{2} \text{as } n \to \infty \tag{2.1.4}$$

(2.1.4) is an example of a Large Deviations statement: the typical value of S_n is by (2.12) of the order $\frac{1}{\sqrt{n}}$, but with small probability (of the order $e^{-\frac{n\delta^2}{2}}$) $\mid S_n \mid$ takes relatively large values.
Since both (2.1.1) and (2.1.2) remain valid as long as $\{X_i\}$ are independent, identically distributed (iid) random variables of zero mean and with unit variance, it could be asked wheter (2.1.3) also holds for non-Gaussian $\{X_i\}$, which is answered by the classical *Cramer`s Theorem*. For more information see the reference [9] of Dembo and Zeitoni.
But our motivation to study Large Deviations is that we want to understand the behaviour of strong solutions to stochastic differential equations in $\mathbb{R}^d$ of the form:

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t \\ X_0^\varepsilon = x \end{cases} \tag{2.1.5}$$

where we assume $\sqrt{\varepsilon}$ small and $(B_t)_{0\leq t\leq T}$ is a d-dimensional Brownian Motion.
For sufficiently small ε the perturbed process (X_t^ε) should be close to the

solution of the deterministic ordinary differential equation:

$$\begin{cases} dX_t = b(X_t)dt \\ X_0 = x \end{cases} \tag{2.1.6}$$

And indeed if b is Lipschitz continuous with Lipshitz constant L, then we have that:

$$| X_t^\varepsilon - X_t | \leq L \int_0^t | X_s^\varepsilon - X_s | \, \mathrm{d}s + \sqrt{\varepsilon} \, | B_t | \tag{2.1.7}$$

Applying *Gronwall`s Inequality* (see [13] for example) this leads to the estimate:

$$\sup_{0 \leq t \leq T} | X_t^\varepsilon - X_t | \leq \sqrt{\varepsilon} \sup_{0 \leq t \leq T} | B_t | \, e^{LT}. \tag{2.1.8}$$

In other words, the behaviour of $| X_t^\varepsilon - X_t |$ for $t \in [0, T]$ can be estimated if we know the behaviour of the d-dimensional Brownian Motion $(B_t)_{0 \leq t \leq T}$:

$$\mathbb{P}\Big(\sup_{t \in [0,T]} | X_t^\varepsilon - X_t | \geq \delta \Big) \leq \mathbb{P}\Big(\sup_{t \in [0,T]} | B_t | \geq \frac{\delta}{\sqrt{\varepsilon}} e^{-LT} \Big) \tag{2.1.9}$$

So, to measure the event that " X_t^ε deviates away from X_t during $[0, T]$" we basically need to know the probability that (B_t) leaves a ball of some radius r before the time T. This can be estimated using **lemma 2.1.1** below.

$$\mathbb{P}\Big(\sup_{t \in [0,T]} | X_t^\varepsilon - X_t | \geq \delta \Big) \leq 4d \exp\Big\{ \frac{-\delta^2 exp\{-2LT\}}{2d\varepsilon T} \Big\} \tag{2.1.10}$$

Let us take a closer look at this estimate.
As might have been expected, the probability of leaving a δ neighbourhood of the deterministic solution increases with T and ε and decreases as δ grows. For example, for increasing δ or decreasing ε the probability of leaving the δ neighbourhood of X_t decays exponentially.
If we choose $A \in \mathcal{B}(C([0, T], \mathbb{R}^d))$ such that no path $(\varphi_t) \in A$ remains inside the δ neighbourhood of the deterministic solution for all $t \in [0, T]$, then the path $X^\varepsilon = (X_t^\varepsilon)_{0 \leq t \leq T}$ of the perturbed equation (2.1.5) satisfies the inequality:

$$\mathbb{P}\big(X^\varepsilon \in A \big) \leq 4d \frac{-\delta^2 e^{-2LT}}{2d\varepsilon T} \to 0$$

as $\varepsilon \to 0$.(2.1.11)

The event $X^\varepsilon \in A$ for sets A as described above and $\varepsilon \to 0$ is what we call a Large Deviations: the expected behaviour would of course be $X^\varepsilon \notin A$ for any such A, because a typical path X^ε should remain near the deterministic solution for small $\varepsilon > 0$ enough.
Our aim is to find the rate at which the probability (2.1.11) tends to zero as $\varepsilon \to 0$ depending on the choice of A. To achieve this, we have to find a better estimate for the probability, which takes into account the choice of A. Our aim is to find a rate function $I : C([0,T], \mathbb{R}^d) \to [0, +\infty]$ such that
$\mathbb{P}\Big(\parallel X^\varepsilon - \phi \parallel_\infty < \delta\Big) \approx exp\{-\frac{I(\phi)}{\varepsilon}\}$
Let us prove the estimate for the Brownian Motion in $\mathbb{R}^d$ which we have used before. Set $B_t^\varepsilon = \sqrt{\varepsilon} B_t$

Lemma 2.1.1. ***Large Deviations for*** $(B_t^\varepsilon)_{0\le t\le T}$
The following estimate holds:

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mid B_t^\varepsilon \mid \ge \delta\Big) \le 4d \; \exp\big\{\frac{-\delta^2}{2dT\varepsilon}\big\} \tag{2.1.12}$$

Proof. Fix $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $\alpha > 0$ such that
$\big\{x \in \mathbb{R}^d : \mid x \mid^2 \ge \alpha\big\} \subset \bigcup_{i=1}^{d} \{x \in \mathbb{R}^d : \mid x_i \mid^2 \ge \frac{\alpha}{d}\}$
Denote (W_t^1) a one dimensional Brownian Motion. We get the estimate:
$\mathbb{P}\Big(\sup_{t\in[0,T]} \mid B_t^\varepsilon \mid \ge \delta\Big) = \mathbb{P}\Big(\sup_{t\in[0,T]} \mid B_t \mid^2 \ge \frac{\delta}{\varepsilon}\Big) \le d\,\mathbb{P}\Big(\sup_{t\in[0,T]} \mid W_t^1 \mid^2 \ge \frac{\delta}{d\varepsilon}\Big)$
Once the laws of (W_t^1) and $(\sqrt{T} W^1_{\frac{t}{T}})_{0\le t\le T}$ are identical, this estimate and time-scaling leads to:

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mid B_t^\varepsilon \mid \ge \delta\Big) \le d\,\mathbb{P}\Big(\sup_{t\in[0,1]} \mid W_t^1 \mid \ge \frac{\delta}{\sqrt{Td\varepsilon}}\Big) \tag{2.1.13}$$

The distribution of the Brownian Motion is symmetric so,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mid W_t^1 \mid \ge \eta\Big) \le 2\,\mathbb{P}\Big(\sup_{t\in[0,1]} W_t^1 \ge \eta\Big) \le \mathbb{P}\Big(B_1^1 \ge \eta\Big) \le 4e^{-\frac{n^2}{2}} \tag{2.1.14}$$

where we have used the Reflection Principle (theorem of Désire-André). See [9]. Combining this estimate with (2.1.13) we get the proof complete. □

Inspired in [9] and using mainly [16] we want in the next chapters to expose the classical results and statetements of Large Deviations Theory that we will need to apply in our assymptotic study of the problem mentioned in the introduction.

2.2 The Basic Tools of Large Deviations Principles

Let X be a topological space and $\mathcal{B}$ a σ algebra over X such that $\mathcal{B}(X) \subset \mathcal{B}$. Consider $\{ \mu_\varepsilon\}_{\varepsilon>0}$ a family of probability measures on the measurable space $(X, \mathcal{B})$. We want to describe the behaviour of the measures μ_ε as $\varepsilon \to 0$ by a rate function I, more precisely, we will formulate assintotically the exponential bounds on the values that μ_ε assigns to sets from $\mathcal{B}$ in terms of a rate function.
Recall that $f : X \to \overline{\mathbb{R}}$ is said to be lower semicontinuous if $\{x \in X : f(x) \leq \alpha\}$ is a closed set in X for every $\alpha \in \mathbb{R}$.

Definition 2.2.1. Rate function
A lower semicontinuous function $f : I \to X$ is named a rate function.
In the classic Freidlin-Wentzell book [15] rate functions are called action functionals.
If for every $\alpha \in \mathbb{R}$ the level sets $\phi_I(\alpha) = \{x \in X : I(x) \leq \alpha\}$ are compact in X, we call I a good rate function.
We remark that the level sets of lower semicontinuous functions are always closed by the lower semi-continuity property.

Definition 2.2.2. Large Deviations Principle
The family of measures $\{ \mu_\varepsilon\}_{\varepsilon>0}$ satisfies a Large Deviations Principle (LDP for short) with the rate function I if for all $A \in \mathcal{B}$ the following holds:

$$- \inf_{x \in A^\circ} I(x) \leq \lim_{\varepsilon \to 0} \inf \varepsilon log \mu_\varepsilon(A) \leq \lim_{\varepsilon \to 0} \sup \varepsilon log \mu_\varepsilon(A) \leq - \inf_{x \in \overline{A}} I(x) \quad (2.2.1)$$

Remark:
The use of interior and closure of A in the formulation of (2.2.1) is explicitly necessary if we assume $\{ \mu_\varepsilon\}_{\varepsilon>0}$ to be non-atomic probability measures, ie, $\mu_\varepsilon\{x\} = 0 \; \forall \, x \in X \; \forall \, \varepsilon > 0$.
We will show this for the lower bound:
Assume in (2.2.1) the lower limit holds for A instead of A°. Let $\{ \mu_\varepsilon\}_{\varepsilon>0}$ be non-atomic probability measures. Then for each $x \in X$

$-I(x) = -\inf_{x\in\{x\}} I(x) \leq \lim_{\varepsilon\to 0}\inf \varepsilon log\mu_\varepsilon(\{x\}) = -\infty.$
Then $\forall\ x \in X\ I(x) = \infty$. But $X = \overline{X}$ and the upper bound in (2.2.1) implies
$0 = \lim_{\varepsilon\to 0}\sup \varepsilon \log \mu_\varepsilon(X) \leq -\inf_{x\in X} I(x) \leq -\infty$ which is false!

Proposition 2.2.1. ***Some facts about rate functions.*** *Let I be a good rate function.*
i) since I has compact level sets $\{I \leq \alpha\}$, the infimum $\inf_{x\in A} I(x)$ is achieved over any non-empty closed sets $A \subset X$.
ii) Let $\{F_\delta\}_{\delta>0}$ be a nested family of closed sets, ie, for any $\delta < \delta_1$ we assume that $F_\delta \subset F_{\delta 1}$.
If we set $F_0 = \bigcap_{\delta>0} F_\delta$, then $\inf_{x\in F_0} I(y) = \lim_{\delta\to 0}\inf_{y\in F_\delta} I(y)$
iii) On a metric space (X, d), $\inf_{y\in\overline{A}} I(y) = \lim_{\delta\to 0}\inf_{y\in A^\delta} I(y)$
where $A^\delta = \{y \in X : d(y, A) \leq \delta\}$ is the closed blow-up of A.

Proof. It uses only properties of lower semi-continuity. See the reference [9] - lemma 4.1.6 for the proof. □

So we want now to discuss existence and uniqueness properties of Large Deviations Principle (LDP for short). For example, if the space X under consideration has a coarse topology, the information provided by a Large Deviations Principle may be relatively poor; e.g. if the topology of X is given by $\{X, \emptyset\}$, then the Large Deviations Principle on the space $(X, \mathcal{B}(X))$ only implies that

$$\inf_{x\in X} I(x) = 0$$

and nothing more.
Hence, if we intend to prove uniqueness of the rate function, we have to make further assumptions on the topology: we will be able to show that uniqueness of the rate function holds if X is a regular Hausdorff Space.
X is called a *Hausdorff Space*, if for every pair of points x, y, $x \neq y$ of X we cand find disjoint open sets $A, B \subset X$ such that $x \in A, y \in B$ (we say that x, y are separated by open neighbourhoods). Furthermore, X is named *regular* if for any closed set $F \subset X$ and any point $x \in F^C$ we can find disjoint open sets $A, B\ \subset X$ such that $F \subset A$, $x \in B$.

Remark: Some elementary facts about regular Hausdorff Spaces

Let X be a regular Hausdorff space and $x \in X$.
i) For any neighbourhood A of x there exists a neighbourhood B of x such that $\overline{B} \subset A$.
ii) Every metric space is a regular Hausdorff space. Furthermore, every real topological vector space with the Hausdorff property is regular.
iii) Any lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ satisfies
$f(x) = \sup\left\{ \inf_{y \in A} f(y) : A \text{ is a neighbourhood of } x\right\}$
This implies that for every $y \in X$ and any (arbirarily small) $\delta > 0$ we can find a neighbourhood $G(y, \delta)$ of y such that $(f(y) - \delta) \wedge \frac{1}{\delta} \leq \inf_{z \in G(y,\delta)} f(z)$
Now (i) allows us to select a neighbourhood $F(y, \delta)$ of y such that $\overline{F(y, \delta)} \subset G(y, \delta)$ and we obtain
$\inf_{z \in \overline{F(y,\delta)}} f(z) \geq \inf_{z \in G(y,\delta)} f(z) \geq (f(y) - \delta) \wedge \frac{1}{\delta}$
In metric spaces, sets of the form $G(y, \delta)$ might be selected as balls $B(y, \delta)$ with small radius δ (which does not have to be equal to δ).

Lemma 2.2.1. ***Uniqueness of the rate function***
Let X be a regular Hausdorff space. A family $\{\mu_\varepsilon\}_{\varepsilon>0}$ on X can not have more than one rate function associated with its Large Deviations Principle.

Proof. Assume there are two rate functions I_1, I_2 such that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP with both of them. Suppose without loss of generality that exists $x_0 \in X$ such that $I_1(x_0) > I_2(x_0)$. Fix $\delta > 0$ and consider A an open set such that $x_0 \in A$ and $\inf_{y \in \overline{A}} I_1(y) \geq (I_1(x_0) - \delta) \wedge \frac{1}{\delta}$.
Such a set A exists by (iii) of the previous remark. The assumed LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$ implies that

$$-\inf_{y \in \overline{A}} I_1(y) \geq \limsup_{\varepsilon \to 0} \varepsilon log \mu_\varepsilon(A) \geq \liminf_{\varepsilon \to 0} \varepsilon log \mu_\varepsilon(A) \geq -\inf_{y \in A} I_2(y) \quad (2.2.2)$$

and

$$I_2(x_0) \geq \inf_{y \in A} I_2(y) \geq \inf_{y \in \overline{A}} I_1(y) \geq (I_1(x_0) - \delta) \wedge \frac{1}{\delta} \quad (2.2.3)$$

Since $\delta > 0$ can be arbitrarily small, this contradicts the assumption $I_1(x_0) > I_2(x_0)$. □

Remark:
In any topological space X, a subset A of the topology is named a base of

the topology if any open open set (i.e any element of the topology) is a union of sets from A.

Definition 2.2.3. Weak Large Deviation Principle
The family $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a Weak Large Deviation Principle with a rate function I if the upper bound holds for all compact sets $A \in \mathcal{B}$ and the lower bound holds for all $A \in \mathcal{B}$.

Theorem 2.2.1. ***Existence of the Weak Large Deviations Principle***
Let X be a topological space and $\mathcal{A}$ be a basis of the topology on X. Furthermore, let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of measures on $(X, \mathcal{B})$ and define for any $A \in \mathcal{A}$, $\mathcal{L}_A = -\liminf_{\varepsilon\to 0} \varepsilon log\mu_\varepsilon(A)$.
Finally set for any $x \in X$, $I(x) = \sup\{\mathcal{L}_A : A \in \mathcal{A}$ such that $x \in A\}$.
If now, for all $x \in X$,
$I(x) = \sup\{-\limsup_{\varepsilon\to 0} \varepsilon log\mu_\varepsilon(A) : A \in \mathcal{A}$ such that $x \in A\}$
holds, then the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a weak LDP with the rate function $I(x)$.

Proof. See theorem 4.1.1 of [9]. □

Remark: **Weak does not imply Full**.
The definition of a weak and a full Large Deviations Principle raises the question how the two are related. While it is obvious full implies weak, the opposite implication does not hold in general. If we consider the Dirac measures $\mu_\varepsilon := \delta_n, n \in \mathbb{N}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies a weak LDP with a good rate function, where $\varepsilon = \frac{1}{n}$ Let $F \in \mathcal{B}(\mathbb{R})$ be a compact set and n large enough. Then $\mu_n(F) = 0$. Hence the upper bound in (2.2.1) holds for the rate function $I := \infty$. At the same time, this rate function makes the lower bound in (2.2.1) trivial for any $F \in \mathcal{B}(\mathbb{R})$. In the other hand, if we choose $F = [1, +\infty[$ then we see that $\limsup\limits_{n\to\infty} \frac{1}{n} log\mu_n(F) = 0 > -\infty = \inf\limits_{x\in\overline{F}} I(x)$, which contradicts the upper bound in (2.2.1).
But the weak LDP can imply a full LDP if the family of the probability measures satisfies an extra condition.

A family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on X is called *exponentially tight* if for any (arbitrarily big) $\alpha < \infty$ there exists a compact set K_α such that $\limsup\limits_{\varepsilon\to 0} \varepsilon log\mu_\varepsilon(K_\alpha^C) < -\alpha$, ie, it can be specified that as $\varepsilon \to 0$ the probability measures are concentrated on K_α.

Theorem 2.2.2. *Assume $\{\mu_\varepsilon\}_{\varepsilon>0}$ is an exponentially tight collection of probability measures on $(X, \mathcal{B})$. If I is a good rate function and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a weak LDP with the rate function I, then I is a good rate function and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a full LDP with rate function I.*

Proof. See lemma 1.2.18 of [9]. □

For our work we will move from a LDP of $B_t^\varepsilon = \sqrt{\varepsilon} B_t$ to a LDP for the laws of the solution $(X_t^\varepsilon)_{0\le t\le T}$ of

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon} dB_t \\ X_0^\varepsilon = x \end{cases} \tag{2.2.4}$$

But we need to have a tool which could be able to transefer a LDP. This is the topic that we want to present next. There exists a very well known tool in the literature that we can use to do this transfer: *The Contraction Principle.*

First, we present the so-called continuous version of *The Contraction Principle.*

Theorem 2.2.3. *Contraction Principle- the Continuous Version*
Let X, Y be topological spaces, $I : X \to [0, \infty]$ a good rate function and $f : X \to Y$ continuous. We define the function $I_1 : X \to [0, \infty]$ by
$I_1(y) = \inf\{I(x) : x \in X \text{ such that } f(x) = y\}$.
Then I_1 is a good rate function on Y.
If I governs a LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$, I_1 governs a LDP for the image measures $\{\mu_\varepsilon \circ f^{-1}\}_{\varepsilon>0}$ on Y.
This result may also be applied if X, Y are the same space but equipped with different topologies.

Proof. We first show I_1 is a good rate function.
$I_1 \ge 0$ is obvious by its definition. Since I is a good rate function,for all $y \in f(X)$ the infimum in the definition of I_1 is obtained for (at least) one $x \in X$. Hence
$\phi_{I_1}(\alpha) = \{y \in Y : I_1(y) \le \alpha\} = \{I(x) : I(x) \le \alpha, x \in X\} = f(\phi_I(\alpha))$
where $\phi_I(\alpha)$ are the level sets of I.
The compactness of the level sets of I in X implies the same for the level sets of I_1 in Y, which makes I_1 a good rate function, since f is continuous.
If we can show for all $A \in \mathcal{B}(Y)$

$$-\inf_{y\in A^\circ} I_1(y) \le \lim_{\varepsilon\to 0}\inf \varepsilon log(\mu_\varepsilon \circ f^{-1})(A) \le \lim_{\varepsilon\to 0}\sup \varepsilon log(\mu_\varepsilon \circ f^{-1})(A) \le -\inf_{y\in \overline{A}} I_1(y) \tag{2.2.5}$$

the proof is complete.
By the definition of I_1, for all $A \subset Y$ $\inf_{y \in A} I_1(y) = \inf_{x \in f^{-1}(A)} I(x)$.
To show the lower bound of (2.2.5) , we have to prove that for all open sets $A \in \mathcal{B}(Y)$

$$-\inf_{x \in f^{-1}(A)} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon log(\mu_\varepsilon \circ f^{-1})(A) \tag{2.2.6}$$

But, by continuity $f^{-1}(A)$ is open in X for all $A \subset Y$ open and the LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$ implies that
$-\inf_{x \in f^{-1}(A)} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon log(\mu_\varepsilon(f^{-1}(A))$ which proves (2.2.6).
The proof of (2.2.5) is completed by a similar argument for closed sets, utilizing the upper bound of the LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$. □

Our aim is to generalize the contraction principle from continuous functions to functions which can be approximated in some sense by continuous ones.

First we define the concept of *exponential equivalence of measures.*

Definition 2.2.4. Exponential Equivalence of Measure
Let (Y, d) be a metric space and consider families $\{\mu_\varepsilon^1\}_{\varepsilon>0}$ and $\{\mu_\varepsilon^2\}_{\varepsilon>0}$ of probability measures on Y.
The two families are called *exponentially equivalent* if there exists a family $(\Omega, \mathcal{B}_\varepsilon, \mathbb{P}_\varepsilon)_{\varepsilon>0}$ of probability spaces and two families $\{Z_\varepsilon^1\}_{\varepsilon>0}$ and $\{Z_\varepsilon^2\}_{\varepsilon>0}$ of Y-valued random variables with joint distributions $\{\overline{\mu}_\varepsilon\}_{\varepsilon>0}$ and marginal distributions $\{\mu_\varepsilon^1\}_{\varepsilon>0}$ and $\{\mu_\varepsilon^2\}_{\varepsilon>0}$ respectively such that the following holds:
$\forall \delta > 0$ $\left\{\omega \in \Omega : (Z_\varepsilon^1(\omega), Z_\varepsilon^2(\omega)) \in \Gamma_\delta\right\} \in \mathcal{B}_\varepsilon$ and $\lim_{\varepsilon \to 0} \varepsilon log \overline{\mu}_\varepsilon(\Gamma_\delta) = -\infty$
where the set $\Gamma_\delta := \left\{(y, y_1) \in Y \times Y : d(y, y_1) > \delta\right\}$
. Families of $\{Z_\varepsilon^1\}_{\varepsilon>0}$, $\{Z_\varepsilon^2\}_{\varepsilon>0}$ which fulfill these conditions are also named exponentially equivalent.

Remark:
If Y is separable, the recquired measurability automatically holds (page 114 [9]).

Now we need the following concept:

Definition 2.2.5. Exponentially Good Approximations
Let $(Y.d)$ be a metric space and Γ_δ defined as above. For each $\varepsilon > 0$, $m \in \mathbb{N}$, let $(\Omega, \mathcal{B}_\varepsilon, \mathbb{P}_\varepsilon)_{\varepsilon>0}$ be a probability space and let the Y - valued random variables Z_ε and $Z_{\varepsilon,m}$ be distributed according to the joint distributions $\overline{\mu}_{\varepsilon,m}$ with marginal distributions μ_ε and $\mu_{\varepsilon,m}$ respectively.
The random variables $\{Z_{\varepsilon,m}\}_{m\in\mathbb{N},\varepsilon>0}$ are called exponentially good approximations of $\{Z_\varepsilon\}_{\varepsilon>0}$ if for every $\delta > 0$ we have
$\left\{\omega \in \Omega : (Z_\varepsilon(\omega), Z_{\varepsilon,m}(\omega)) \in \Gamma_\delta\right\} \in \mathcal{B}_\varepsilon$
for every $m \in \mathbb{N}$ and $\lim_{m\to\infty} \limsup_{\varepsilon\to 0} log\overline{\mu}_{\varepsilon,m}(\Gamma_\delta) = -\infty$.
We call the families of the measures $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ *exponentially good approximations* of the family $\{\mu_\varepsilon\}$ if we construct a family $(\Omega, \mathcal{B}_\varepsilon, \mathbb{P}_\varepsilon)$ of probability spaces above.

Theorem 2.2.4. ***Large Deviations under Exponentially Good approximations***
Let (Y, d) be a metric space and suppose that for any $m \in \mathbb{N}$ the family $\{\mu_{\varepsilon,m}\}_{m\in\mathbb{N}}$. Furthermore, assume that $\{\mu_{\varepsilon,m}\}_{m\in\mathbb{N}}$ are exponentially good approximations of $\{\mu_\varepsilon\}_{\varepsilon>0}$. Then the following holds:
i) $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a weak LDP with the rate function
$I(y) := \sup_{\delta>0} \liminf_{m\to+\infty} \inf_{z\in B(y,\delta)} I_m(z)$.

ii) If I is a good rate function, and for every closed set F in Y we have: $\inf_{y\in F} I(y) \leq \limsup_{m\to\infty} \inf_{y\in F} I_m(y)$, then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a full Large Deviations Principle with the rate function I.

Proof. See theorem 4.2.16 of [9]. □

And with this result we can generalize the continuous version of the Contraction Principle.

Theorem 2.2.5. ***Contraction Principle***
Let (Y, d)be a metric space and X a Hausdorff Space. Let $\{\mu_{\varepsilon,m}\}_{m\in\mathbb{N}}$ be a family pf probability measures on X that satisfies the LDP with a good rate function I. For any $m \in \mathbb{N}$ let $f_m : X \to Y$ be a continuous mapping. If there exists a measurable map $f : X \to Y$ such that for all $\alpha < \infty$

$$\limsup_{m\to\infty} \sup\{d(f_m(x), f(x)) : x \in \phi_I(\alpha)\} = 0$$

and if $\{\overline{\mu}_{\varepsilon,m}\}_{m\in\mathbb{N}}$ is a family of probability measures on Y for which $\{\mu_\varepsilon \circ f^{-1}\}_{m\in\mathbb{N},\varepsilon}$ are exponentially good approximations, then $\{\overline{\mu}_{\varepsilon,m}\}_{m\in\mathbb{N}}$ satisfies a LDP with the good rate function

$$I_1(y) = \inf\{I(x) : x \in X, f(x) = y\}$$

Proof. See theorem 4.2.23 of the reference [9]. □

In what follows in the next chapter we use a simplified version of the contraction principle for functions approximated by $(f_\varepsilon)_{\varepsilon>0}$ measurables, a family of functions indexed on the same parameter ε of the probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$.

Corollary 2.2.1. ***Approximations with dependence on*** ε
If $f : X \to Y$ is a continuous mapping from a topological vector space X to a metric space (Y, d) and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a LDP with a good rate function $I : X \to [0, \infty]$ and $\varepsilon > 0$ and $f_\varepsilon : X \to Y$ measurable functions such that for each $\delta > 0$

$$\Gamma_{\varepsilon,\delta} = \{x \in X : d(f(x), f_\varepsilon(x)) > \delta\}$$

is measurable and

$$\lim_{\varepsilon\to 0} \sup \varepsilon \log \mu_\varepsilon(\Gamma_{\varepsilon,\delta}) = -\infty$$

So $\{\mu_\varepsilon \circ f_\varepsilon^{-1}\}_{\varepsilon>0}$ satisfies a LDP with the good rate function

$$I_1(y) = \inf\{I(x) : x \in X \text{ such that } f(x) = y\}$$

Proof. See corollary 4.2.21 of [9] □

2.3 Sample Paths Large Deviations for Brownian Motion

Now we want to present the classical Large Deviations result for Brownian Motion first proved by Schilder. If $(B_t)_{0\le t\le T}$ is a Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathbb{R}^d$, fixing $B_0 = 0$ for $\varepsilon > 0$ we define the scaled Brownian Motion

$$B_t^\varepsilon = \sqrt{\varepsilon} B_t$$

The following theorem states a LDP for the distribution of this scaled Brownian Motion as $\varepsilon \to 0$. We recall $C_0 = \left\{\varphi \in C([0, T], \mathbb{R}^d) : \varphi_0 = 0\right\}$

Theorem 2.3.1. ***Schilder`s Theorem, 1966***
The family $\{\mathbb{P} \circ (B^\varepsilon)^{-1}\}_{\varepsilon>0}$ *of probability measures on* $(C_0, \mathcal{B}(C_0))$ *satisfies a LDP with the good rate function:*

$$I(\varphi) = \begin{cases} \frac{1}{2} \parallel \varphi \parallel^2_{H^1}, \varphi \in H^1(0,T;\mathbb{R}^d) \\ \infty \ \textit{otherwise} \end{cases}$$

Before we present the proof of this result, we state some remarks about its conclusions.
The theorem above states the relation:

$$-\inf_{\varphi \in \Gamma^\circ} I(\varphi) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(B^\varepsilon \in \Gamma) \geq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(B^\varepsilon \in \Gamma) \leq -\inf_{\varphi \in \overline{\Gamma}} I(\varphi)$$
(2.3.1)

holds for all $\Gamma \in \mathcal{B}(C_0)$ and the rate function I has compact level sets $\{I \leq \alpha\}$.

Remarks:

i) Since the paths of a Brownian Motion are almost surely of unbounded variation, we have that $B^\varepsilon \notin H^1(0,T,\mathbb{R}^d)$ a.s $\mathbb{P}$. Hence , for all $\varepsilon > 0$ $I(B^\varepsilon) = \infty \ a.s \ \mathbb{P}$.

ii) In the case of **Schilder`s Theorem**, I is a good rate function and there exists φ such that $I(\varphi) = 0$; e.g. $\varphi(t) \equiv 0$, by (2.3.1) any set containing this φ has maximal probability with respect to $\mathbb{P} \circ (B^\varepsilon)^{-1}$ as $\varepsilon \to 0$. In other words, B^ε concentrates near the zero function.

iii) We want to specify the probability that $B^\varepsilon \in C([0,T],\mathbb{R}^d)$ leaves a ball of radius δ around the origin.

$$\text{Set } B = B(0,\delta) = \{\varphi \in C([0,T],\mathbb{R}^d) : \parallel \varphi \parallel_\infty < \delta\}.$$

Since the typical spreading of Brownian Motion scales with $\sqrt{t}$ we expect that B^ε remains inside $B(0,\delta)$ as long as $T \ll \frac{\delta^2}{2}$. Here, $\inf_{\varphi \in B^C} I(\varphi)$ is obtained for any φ of the form $\varphi_s = \dfrac{s}{T} x$ where x is such that $\mid x \mid = \delta$.

$$\inf_{\varphi \in B^C} I(\varphi) = I\big(\frac{s}{T}x\big) = \frac{1}{2}\int_0^T (\frac{\delta}{T})^2 \mathrm{d}t = \frac{\delta^2}{2T}$$

Schilder`s Theorem implies that $\mathbb{P}(B^\varepsilon \notin B)$ decays like $\exp(-\frac{\delta^2}{2\varepsilon T})$.

Proof. We structure the proof by steps.

Step 1:

We want to prove that the level sets of I , $\phi_I(\alpha) = \{\phi \in C_0 : I(\phi) \leq \alpha\}$, where $\alpha \in [0, \infty[$ are compact.
For every $\varphi \in \phi_I(\alpha)$, where $\alpha \in [0, \infty[$,

$$\int_0^T \mid \dot{\phi}_s \mid^2 \mathrm{d}s = \parallel \varphi \parallel_{H^1}^2 \leq 2\alpha$$

For every $t \in [0, T]$

$$\mid \varphi_t \mid = \mid \varphi_0 + \int_0^T \dot{\varphi}_s \mathrm{d}s \mid \leq \mid \varphi_0 \mid + \sqrt{T \int_0^T \mid \dot{\varphi} \mid_s^2 \mathrm{d}s} \leq \mid \varphi_0 \mid + \sqrt{T2\alpha}$$

Consequently all $\varphi \in \phi_I(\alpha)$ are uniformly bounded for any $\alpha \in [0, \infty[$. So for any $\varphi \in \phi_I(\alpha)$ and for every t, h such that $\{t, t+h\} \subset [0, T]$

$$\mid \varphi_{t+h} - \varphi_t \mid \leq \int_t^{t+h} \mid \dot{\varphi}_s \mid \mathrm{d}s \leq \sqrt{h \int_t^{t+h} \mid \dot{\varphi}_s^2 \mid \mathrm{d}s} \leq \sqrt{h \int_0^T \mid \dot{\varphi}_s^2 \mid \mathrm{d}s} \leq \sqrt{h2\alpha} \to 0 \text{ as } h \to 0$$

ie, all elements of $\phi_I(\alpha)$ are equicontinuous for any $\alpha \in [0, \infty[$. The compactness of the level sets $\phi_I(\alpha)$ follows from the *theorem of Arzela-Ascoli.*

Next we want to prove the upper bound and the lower bound of (2.3.1). In order to do it, we want to prove an auxiliary result.

Step 2:

We are going to prove that

$$\forall \delta > 0 \; \forall \gamma > 0 \; \forall k > 0 \; \exists \varepsilon_0 = \varepsilon_0(\delta, \gamma, k, T) > 0 \; \forall \varepsilon \leq \varepsilon_0 \;\; \forall \varphi \in C_0 \; I(\varphi) < K$$
$$\mathbb{P}\Big(\parallel B^\varepsilon - \varphi \parallel_\infty < \delta \Big) \geq \exp\big[- \frac{1}{\varepsilon}(I(\varphi) + \gamma) \big]$$

Applying *Girsanov Theorem* (see [19])

$$\mathbb{P}\ (\ \| B^\varepsilon - \varphi \|_\infty\ ) =$$

$$= \exp -\frac{1}{2\varepsilon}\int_0^T \mid \dot\varphi_s \mid^2 \mathrm{d}s \int_{B\in B(0,\frac{\delta}{\sqrt{\varepsilon}})} \exp\Big(\frac{-1}{\sqrt{\varepsilon}}\int_0^T < \dot\varphi_s, \mathrm{d}Bs > \Big)\mathrm{d}\mathbb{P}(\omega)$$

We now split the domain of integration in two parts:

If $C = \sqrt{I(\varphi)\frac{4}{\varepsilon}}$ we define:

$$\mathrm{A}_C = \Big\{\omega \in \Omega : \frac{-1}{\sqrt{\varepsilon}}\int_0^T < \dot\varphi_s, \mathrm{d}Bs > \leq -C\Big\}$$

To obtain a precise lower bound, we want to base our estimate on those $\omega \in \Omega$ when the integrand is not too small, ie, in A_C^c. Thus we want show firts thatA_C is small:
Using *Chebychèv Inequality* (see [19]), and from our choice of C we get that

$$\mathbb{P}(A_C) = \frac{1}{2}\mathbb{P}\Big(\mid \frac{-1}{\sqrt{\varepsilon}}\int_0^T < \dot\varphi_s, \mathrm{d}Bs > \mid\ \geq -C\Big) \leq$$

$$\frac{1}{2\varepsilon C^2}\mathbb{E}\Big[\Big(\int_0^T < \dot\varphi_s, \mathrm{d}Bs > \Big)^2\Big] \leq \frac{1}{2\varepsilon C^2}\int_0^T \mid \dot\varphi_s \mid^2 \mathrm{d}s = \frac{1}{\varepsilon C^2}I(\varphi) = \frac{1}{4}$$

On other hand,

$$\mathbb{P}\Big(\ \{\ \| B^\varepsilon - \varphi \|_\infty < \delta\} \cap A_C^C\} \geq \exp(-\tfrac{I(\varphi)}{\varepsilon})e^{-C}\mathbb{P}\Big(\Big\{B \in B(0,\frac{\delta}{\sqrt{\varepsilon}})\Big\} \cap A_C^C\Big)$$

$$\geq \exp\big[-\tfrac{I(\varphi)}{\varepsilon} - C\big]\big(\mathbb{P}(A_C^C) - \mathbb{P}(B \in B(0,\tfrac{\delta}{\sqrt{\varepsilon}})^C)\big) \quad (2.3.2)$$

since $\mathbb{P}(A_C^c) \geq \dfrac{3}{4}$, there exists $\varepsilon > 0$ small enough such that:

$$\mathbb{P}\ (\ \mathrm{A}_C^C) - \mathbb{P}\Big(B \in B(0,\frac{\delta}{\sqrt{\varepsilon}})\Big)^C \geq \frac{1}{2}.$$

By definition of C, we finally get for any small enough ε ($\varepsilon < \varepsilon_0(\delta,\gamma,K,T)$) that

$$\mathbb{P}\ (\ \| B^\varepsilon - \varphi \|_\infty < \delta\big) \geq \mathbb{P}\big(\{\ \| B^\varepsilon - \varphi \|_\infty < \delta\} \cap A_C^C\big) \geq \exp -\frac{I(\varphi)+\gamma}{\varepsilon}$$

So we are ready to prove the lower bound in (2.3.1).

Step 3:

Select an arbitrary open set $G \subset C_0$. If $\inf_{\varphi\in G} I(\varphi) = \infty$, the lower bound is trivial. So assume $\inf_{\varphi\in G} I(\varphi) < \infty$.

Since I i a good rate function, this allows us to choose $\varphi \in G$ such that $I(\varphi) < \infty$. G is opens, so we choose r_φ such that $B(\varphi, r_\varphi) \subset G$.
From the above step

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(B^\varepsilon \in G) \geq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(B^\varepsilon \in B(\varphi, r_\varphi)) \geq -I(\varphi)$$

So we can conclude taking the infimum over all $\varphi \in G$.
In order to prove the upper bound in (2.3.1) we need to prove a new assertium.

Step 4:

$$\forall \delta > 0 \, \forall \gamma > 0 \, \forall \alpha_0 > 0 \, \exists \varepsilon_0 \, \forall \varepsilon \leq \varepsilon_0 \, \forall \alpha \leq \alpha_0 \, , \mathbb{P}\Big(dist(B^\varepsilon, \phi_I(\alpha)) \geq \delta\Big)$$
$$\leq \exp[-\frac{\alpha - \gamma}{\varepsilon}] \text{ (2.3.3)}$$

where $dist(\varphi, \phi_I(\alpha)) = \min_{\psi \in \phi_I(\alpha)} \parallel \varphi_t - \psi_t \parallel_\infty$

If we want to prove it, we face the main problem that $I(B^\varepsilon) = \infty$, so in order to escape from that we have to approximate the scaled Brownian Motion by functions from $H^1([0,T];\mathbb{R}^d)$. We use random polygons to solve this problem.
To construct an approximating random polygon $x^{n,\varepsilon}$ for B^ε, divide $[0,T]$ into parts of identical lenght $\Delta > 0$. We specify Δ later. Assume for now $\frac{T}{\Delta} \in \mathbb{N}$.
The approximating polygon $x^{n,\varepsilon}$ shall have the vertices $(0,0), (\Delta, X_a^\varepsilon)$, $(a\Delta, X_{2A}^\varepsilon), ..., (T, X_T^\varepsilon)$.
In order to prove the upper bound in the statement, with the help of this approximation, consider two events: "$x^{n,\varepsilon}$ is a bad approximation of B^ε" or "$x^{n,\varepsilon}$ is a good approximation", that it leaves $\phi_I(\alpha)$ whenever B^ε leaves the δ-neighbourhood of the level set.

$$\mathbb{P}\Big(dist(B^\varepsilon, \phi_I(\alpha)) \geq \delta\Big) \leq \underbrace{\mathbb{P}\Big(\parallel B^\varepsilon - x^{n,\varepsilon} \parallel_\infty \geq \delta\Big)}_{1} + \underbrace{\mathbb{P}\Big(I(x^{n,\varepsilon}) > \alpha\Big)}_{2} \text{ (2.3.5)}$$

First we prove an upper bound for 1 in (2.3.5), which is the probability of $x^{n,\varepsilon}$ is a bad approximation. In what follows we use the fact that the distances $\mid B_S^\varepsilon - x_s^{n,\varepsilon} \mid$ considered on different time intervals $[k\Delta, (k+1)\Delta[$ are identically distributed .

$$\mathbb{P}\Big(\parallel B^{\varepsilon}-x^{n,\varepsilon}\parallel_{\infty}\geq\delta\Big)=\mathbb{P}\Big\{\sup_{0\leq s\leq T}\mid B^{\varepsilon}_{s}-x^{n,\varepsilon}_{s}\mid\geq\delta\Big\}$$

$$\leq\tfrac{T}{\Delta}\mathbb{P}\Big(\sup_{0\leq s\leq\Delta}\mid B^{\varepsilon}_{s}-x^{n,\varepsilon}_{s}\mid\geq\delta\Big)\leq\mathbb{P}\Big(\sup_{0\leq s\leq\Delta}\mid B^{\varepsilon}_{s}\mid\geq\delta\Big)\underbrace{\leq}_{lemma2.1.1}\frac{4dT}{\Delta}\exp-\frac{\delta^{2}}{2d\varepsilon\Delta}$$

Choosing $\Delta=\dfrac{\delta^{2}}{2d\alpha_{0}}$ we get for all $\varepsilon\leq\varepsilon_{0}=\varepsilon_{0}(T,\delta,\gamma,\alpha_{0})$

$$\mathbb{P}\Big(\parallel B^{\varepsilon}-x^{n,\varepsilon}\parallel_{\infty}\geq\delta\Big)\leq\frac{1}{2}\exp-\frac{\alpha_{0}-\varepsilon\log\frac{4d^{2}T\alpha_{0}}{\delta^{2}}}{\varepsilon}\leq\frac{1}{2}\exp-\frac{\alpha_{0}-\gamma}{\varepsilon}$$

Now we estimate (2) which specifies the probability that the approximation $x^{n,\varepsilon}$ leaves the level set. Since $x^{n,\varepsilon}$ is a polygon,

$$\mathrm{I}(\mathrm{x}^{n,\varepsilon})=\frac{1}{2}\sum_{l=1}^{\frac{T}{\Delta}}\int_{(l-1)\Delta}^{l\Delta}\frac{\mid\sqrt{\varepsilon}B_{l\Delta}-\sqrt{\varepsilon}B_{(l-1)\Delta}\mid^{2}}{\Delta^{2}}\mathrm{d}s=\frac{\varepsilon}{2}\sum_{l=1}^{\frac{T}{\Delta}}\frac{\mid B_{l\Delta}-B_{(l-1)\Delta}\mid^{2}}{\Delta}$$

This has a distribution equal to the distribution of $\sum\xi_{i}^{2}$ over the squares of $\dfrac{dT}{\Delta}$ independent, where $\xi_{i}\sim Gaussian(0,1)$, which can be estimated by *Chebychév`s Inequality*. So, for $k\in]0,\frac{1}{2}[$

$$\mathbb{P}\Big(\mathrm{I}(\mathrm{x}^{n,\varepsilon})>\alpha\Big)=\mathbb{P}\Big(\sum_{i=1}^{\frac{dT}{\Delta}}\xi_{i}^{2}>\frac{2\alpha}{\varepsilon}\Big)$$

$$\leq\exp-\frac{2K\alpha}{\varepsilon}\Big(\mathbb{E}\big(e^{k\xi_{1}^{2}}\big)\Big)^{\frac{dT}{\Delta}}\leq(1-2k)^{-\frac{dT}{2\Delta}}\exp\frac{-2K\alpha}{\varepsilon}$$

Now choose $K=\frac{1}{2}\big(1-\frac{\gamma}{2\alpha}\big)$ then for small enough $\varepsilon>0$

$$\mathbb{P}\Big(\mathrm{I}(\mathrm{x}^{n,\varepsilon})>\alpha\Big)=\big(\tfrac{\gamma}{2\alpha}\big)^{\frac{-dT}{2\alpha}}\exp-\frac{\alpha-\frac{\gamma}{2}}{\varepsilon}\leq\frac{1}{2}\exp-\frac{\alpha-\gamma}{\varepsilon}$$

Now we are ready to prove the upper bound in (2.3.1).

Step 5:

Choose F closed arbitrarily. The result is trivial if $\inf_{\varphi\in F}I(\varphi)=0$. So we assume $\inf_{\varphi\in F}I(\varphi)>0$ and choose $\gamma>0$ such that $\alpha=\inf_{\varphi\in F}I(\varphi)-\gamma>0$.
I is a good rate function , so $\phi_{I}(\alpha)$ is compact. By definition of α, $\phi_{I}(\alpha)\cap F=\emptyset$. If we take $\delta=dist(F,\phi_{I}(\alpha))>0$, by the previous step and by definition of α we get

$\mathbb{P}\Big(\mathrm{B}^{\varepsilon}\in F\Big)\leq\mathbb{P}\Big(dist(B^{\varepsilon},\phi_{I}(\alpha))\geq\delta\Big)\leq\exp\dfrac{\inf_{\varphi\in F}I(\varphi)-2\gamma}{\varepsilon}$ which completes the proof. □

2.4 General Freidlin-Wentzell Theory: Sample Path Large Deviations for Strong Solutions of Stochastic Differential Equations

Our purpose now is to understand the behaviour of the strong solution $(X_t^\varepsilon)_{0\le t\le T}$ of the stochastic differential equation:

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dBt, \ t\in[0,T] \\ X_0^\varepsilon = x \end{cases} \tag{2.4.1}$$

in $\mathbb{R}^d$, where we assume the existence of an unique solution $(X_t^\varepsilon)_{0\le t\le T}$, by assuming b,σ bounded and uniformly Lipschitz continuous (the general conditions to assure existence and uniqueness of strong solutions to SDE). Our aim in this section is to prove a LDP for the laws of the solution $\mu_\varepsilon = \mathbb{P}\circ(X_\varepsilon)^{-1}$ Consider now $\sigma = 1$ and $x_0^\varepsilon = 0$.
In this special case, we can use the continuous version of the Contraction Principle to obtain a LDP for the distributions of $(X_\varepsilon)_{\varepsilon>0}$ from **Schilder`s Theorem**.
Let f be the unique solution in C_0 of the integral equation (it exists in C_0 by the standarb theory of deterministic ordinary differential equations)

$$f(t) = \int_0^t b(f(s))\mathrm{d}s + g(t), \ g\in C_0$$

and define

$$F: C_0 \to C_0$$

$$\mathrm{g} \mapsto F(g) := f$$

Note that $F(\sqrt{\varepsilon}B) = X^\varepsilon$.
To apply the **Contraction Principle** using F, we have to prove that F is continuous. Choose $g_1, g_2 \in C_0$ and denote $f_1 = F(g_1)$ and $f_2 = F(g_2)$. For every $t\in[0,T]$

$$\| f_1 - f_2 \|_\infty \le L\int_0^t \sup_{r\in[0,s]} | f_1(r) - f_2(r) | \,\mathrm{d}s + \| g_1 - g_2 \|_\infty$$

where L is a Lipschitz constant such that $\| b(x) - b(y) \|_\infty \le L \| x - y \|_\infty$
Gronwall`s Lemma now implies that:

$$\| f_1 - f_2 \|_\infty \leq e^{LT} \| g_1 - g_2 \|_\infty$$

The continuity of F follows from there.
By **Schilder`s Theorem** $(\mathbb{P} \circ (B_t^\varepsilon))_{\varepsilon>0}$, where $B_t^\varepsilon = \sqrt{\varepsilon} B_t$,satisfies a LDP with the good rate function

$$I(\varphi) = \begin{cases} \frac{1}{2} \| \varphi \|_{H^1}^2, \ \varphi \in H^1(0,T,\mathbb{R}^d) \\ +\infty \text{ otherwise} \end{cases}$$

Applying the *continuous version of the Contraction Principle* (see **theorem 2.2.3**), $X^\varepsilon = F(B^\varepsilon)$ satisfies a LDP with the good rate function:

$$\begin{aligned} I_1(f) &:= \inf\{I(g) \mid g \in C_0 \text{ and } F(g) = f\} = \\ &= \inf\{\frac{1}{2} \| g \|_{H^1}^2 \mid g \in C_0 : F(g) = f\} \end{aligned}$$

Finally we want to identify I_1.

If $g \notin H^1(0,T,\mathbb{R}^d)$ $f = F(g) \notin H^1(0,T,\mathbb{R}^d)$.

If $g \in H^1(0,T,\mathbb{R}^d)$ f is a.s $\mathbb{P}$ differentiable with:

$$\begin{cases} \dot{f}(t) = b(f(t)) + \dot{g}(t) \\ f(0) = 0 \end{cases}$$

Then $\exists\, B > 0$ such that $\forall\, t \in [0,T]$

$$| \dot{f}(t) | \leq B \int_0^t | \dot{f}(s) | \, \mathrm{d}s + | b(0) | + | \dot{g}(t) | \tag{2.4.2}$$

Gronwall`s Lemma implies that if $g \in H^1(0,T,\mathbb{R}^d)$ then $f \in H^1(0,T,\mathbb{R}^d)$. Thus

$$\mathrm{I}_1(f) = \begin{cases} \frac{1}{2} \int_0^t | \dot{f}(s) - b(f(s)) |^2 \, \mathrm{d}s, \ g \in H^1(0,T,\mathbb{R}^d) \Rightarrow f \in H^1(0,T,\mathbb{R}^d) \\ \infty \ \ g \notin H^1(0,T,\mathbb{R}^d) \Rightarrow f \notin H^1(0,T,\mathbb{R}^d) \end{cases}$$

We now discuss the less simple case.

Let X^ε be the unique solution of (2.4.1). We want to understand the Large Deviations behaviour of this stochastic process. The first idea is to

apply the same tools as we did above- construct some continuous transference F and use the Contraction Principle. However the map defined by X^ε on $C([0,T],\mathbb{R}^d)$ does not necessary have to be continuous.
It can be shown that if we replace the Brownian Motion B_t by its polygonal approximation (hence a continuous approximation) the solution of (2.4.1) differs in the limit from X^ε by a non-zero correction term (so called wong-Zakai correction term). See the reference [19] for more information about it. The existence of this non-zero correction term contradicts the assumption of continuity. Hence we may not use the continuous version of the Contraction Principle. But in other hand, the mentioned correction term is of order ε, so we may expect that it will not influence Large Deviations results. Consequently we guess that, even though we have just realized that the proof will not work as above, the rate function for this situation might in principle be the same as above:

$$\mathrm{I}_1(f) = \inf\{\frac{1}{2} \parallel g \parallel^2_{H^1}: g \in H^1(0,T,\mathbb{R}^d) : f(t) = x + \int_0^t b(f(s))\mathrm{d}s + \\ + \int_0^t \sigma(f(s))\dot{g}(s)\mathrm{d}s\} \ (2.4.3)$$

The following theorem confirms this guess.

Theorem 2.4.1. *Consider the stochastic differential equation (2.4.1). Assume that b, σ are bounded and uniformly Lipschitz continuous, and X^ε be the solution of (2.4.1). Then $\{\overline{\mu}_\varepsilon\} = \{\mathbb{P} \circ (X^\varepsilon)^{-1}\}$ satisfies a LDP with the good rate function I_1 as defined above in (2.4.3).*

Proof. See theorem 5.6.7 of [9]. □

2.5 More General Results in Freidlin-Wentzell Theory

For our purposes in the next chapter we will need to study a more general result of LDP for strong solutions of Stochastic Differential Equations, when the drift and the diffusion coefficient depends on ε too.
Our framework will be:

$$\begin{cases} dX_t^\varepsilon = b^\varepsilon(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma^\varepsilon(t, X_t^\varepsilon)dBt, \ \ t \in [0, T] \\ X_0^\varepsilon = x \end{cases} \qquad (2.5.1)$$

We want to study the assymptotic behaviour of the strong solution of (2.5.1) when $\varepsilon \to 0$ and establish a Large Deviations Principle.
This general results states estimates for b^ε, σ^ε when there exists dependence possibly on ε but under the classical assumptions of boundedness and Lipschitz continuity. This section is based entirely in the results of Prioret [32], Azencott [2] and Baldi-Maurel [3], following very closely this last one.

Azencott`s original idea [2] was to remark that the Itô`s mapping, associating the Brownian Motion path to the corresponding path of the solution of a SDE is not in general continuous, but it is regular enough to develop a kind of Contraction procedure in the spirit of what we have done before in the previous section.
The *Schilder`s Theorem* (**theorem 2.3.1**) states that $B_t^\varepsilon = \sqrt{\varepsilon}B_t$ satisfies a LDP with the good rate function:

$$I(\varphi) = \begin{cases} \frac{1}{2} \parallel \varphi \parallel_{H^1}^2, \varphi \in H^1(0, T; \mathbb{R}^d) \\ \infty \text{ otherwise} \end{cases}$$

If $\varepsilon > 0$ let

$$b^\varepsilon : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$$
$$\sigma^\varepsilon : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$$

be families of vector and matrix fields respectively.

We will make the following assumption **Assumption A.2.5**

a) There exist a vector field $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and a matrix field $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ such that:

$\forall\, h \in H^1([0, T], \mathbb{R}^d), \ x \in \mathbb{R}^d$ the ordinary differential equation

$$\begin{cases} \dot{g}_t = b(t, g_t) + \sigma(t, g_t)\dot{h}_t \\ g_0 = x \end{cases} \qquad (2.5.2)$$

has a unique solution on [0,T].

b) Let $S_x(h)$ denote the solution of (2.5.2).
So $S_x : H^1(0,T;\mathbb{R}^d) \to C([0,T],\mathbb{R}^d)$. For any $a > 0$ let S_x^a be the restriction of S_x to $K_a = \{h \in H^1(0,T;\mathbb{R}^d) :\| h \|_{H^1} \leq a\}$. Suppose S_x^a be continuous with respect to the uniform norm: for every $\{h_n\}_n \subset K_a$ such that $\| h_n - h \| \to 0$ as $n \to \infty$ with $h \in K_a$, then $\| S_x(h_n) - S_x(h) \|_\infty \to 0$ as $n \to \infty$.

c) **The Quasi-Continuity Property**
For every $R > 0$, $\rho > 0$, $a > 0$, $c > 0$ there exist $\varepsilon_0 > 0$ $\alpha > 0$ such that if $\varepsilon < \varepsilon_0$

$$\mathbb{P}\Big(\| X^\varepsilon - g \|_\infty > \rho, \| B^\varepsilon - h \|_\infty \leq \alpha \Big) \leq \exp\big(- \tfrac{R}{\varepsilon}\big)$$

uniformly for $\| h \|_{H^1} \leq a$ and $| x | \leq c$ where $g = S_x(h)$.

We remark that the Assumption A2.5 (c) means that if the Brownian path is such that $\| B^\varepsilon - h \|_\infty \leq \alpha$, then the corresponding path of the diffusion $(X_t^\varepsilon)_{0\leq t\leq T}$ is near the path $g = S_x(h)$, with a probability converging to 1 as $\varepsilon \to 0$ at a high exponential rate.
It can be viewed as a weak continuity property of Itô`s mapping. It will be necessary that the coefficients b^ε, σ^ε converge in a suitable sense to b and σ respectively.

Theorem 2.5.1. *Suppose that $b^\varepsilon, \sigma^\varepsilon$ are Lipschitz continuous and the SDE (2.5.1) has a strong solution for every $\varepsilon \to 0$. Then if (A.2.5) holds, the family $\{X_t^\varepsilon\}_{\varepsilon>0}$ satisfies a LDP in $C_x([0,T],\mathbb{R}^d)$ with the good rate function:*

$$I_1(\varphi) := \inf\{\frac{1}{2} \| h \|_{H^1}^2 | \, g \in C_0 : S_x(h) = \varphi\}$$

We remark that this means that:

$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq - \inf_{\psi\in F} I_1(\psi)$ (2.5.3)

$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq - \inf_{\psi\in G} I_1(\psi)$ (2.5.4)

for every closed set $F \in C_x([0,T],\mathbb{R}^d)$ and open set $G \in C_x([0,T],\mathbb{R}^d)$ and that the level sets of I_1 are compact.

Proof. For more technical details see [2] or [3]. Thanks to (A.2.5 (b)) in the definition of $I_1(\varphi) := \inf\{\frac{1}{2} \parallel h \parallel_{H^1}^2 \mid g \in C_0 : S_x(h) = \varphi\}$, the infimum is attained unless $I(\varphi) = \infty$.So, if $I(\varphi) = a$ then we have also
$I_1(\varphi) := \inf\{\frac{1}{2} \parallel h \parallel_{H^1}^2 \mid g \in C_0 : S_x(h) = \varphi, \frac{1}{2} \parallel h \parallel_{H^1(0,T,\mathbb{R}^d)} \leq a+1\}$
and it suffices now to remark that in the uniform norm the set $\{S_x(h) = g\}$ is closed thanks to A.2.5 (b) and the function $h \mapsto \frac{1}{2} \parallel h \parallel_{H^1(0,T,\mathbb{R}^d)}^2$ is lower semicontinuous.

The same argument proves that I_1 is lower semicontinuous with compact level sets, as $\{I_1 \leq a\}$ turns out to be the image of
$C_a = \{\frac{1}{2} \parallel h \parallel_{H^1(0,T,\mathbb{R}^d)}\}$ which is compact in the uniform norm, through the transformation S_x, whose restriction to C_a is continuous in the uniform norm.

Now we will prove the lower and upper bounds:

If for every Borel set $A \subset C_x(0,T,\mathbb{R}^d)$, define $\Lambda(A) := \inf_{g \in A} I_1(g)$.
Then we reformulate (2.5.3) and (2.5.4) as

$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -\Lambda(F)$ (2.5.5)
$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \geq -\Lambda(G)$ (2.5.6)

for every closed set $F \subset C_x(0,T,\mathbb{R}^d)$ and open set $G \subset C_x(0,T,\mathbb{R}^d)$.

<u>Lower Bound:</u>
Let $\delta > 0$ and $g \in G$ such that $I_1(g) \leq \Lambda(G) + \delta$ and $h \in H^1(0,T,\mathbb{R}^d)$ such that $S_x(h) = g$ and $\frac{1}{2} \parallel h \parallel_{H^1(0,T,\mathbb{R}^d)}^2 = I_1(g)$.
Thus if $\rho > 0$ is such that the neighbourhood of radius ρ of g in $C_x(0,T,\mathbb{R}^d)$ is contained in G, then for every $\alpha > 0$

$$\mathbb{P}\Big(X^\varepsilon \in G\Big) = \mathbb{P}\Big(\parallel X^\varepsilon - g \parallel_\infty < \rho\Big)$$
$$\geq \mathbb{P}\Big(\parallel X^\varepsilon - g \parallel_\infty < \rho, \parallel B^\varepsilon - h \parallel_\infty < \alpha\Big) = \mathbb{P}\Big(\parallel B^\varepsilon - h \parallel_\infty < \alpha\Big) -$$
$$\mathrm{P}\ \Big(\parallel X^\varepsilon - g \parallel_\infty > \rho; \parallel B^\varepsilon - h \parallel_\infty < \alpha\Big)$$

Now, for every $\alpha > 0$, thanks to the classical Schilder estimates:

$$\lim_{\varepsilon \to 0} \varepsilon \mathbb{P}\Big(\parallel B^\varepsilon - h \parallel_\infty < \alpha\Big) \geq -\frac{1}{2} \parallel h \parallel_{H^1}^2 = -I_1(g) \geq -\Lambda(G) - \delta$$

Using (A.2.5(c)), **the quasi-continuity property**, for $\alpha > 0$ small enough:

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(\parallel X\varepsilon - g \parallel_\infty > \rho, \parallel B^\varepsilon - h \parallel_\infty < \alpha \Big) < -R$$

with $R > \Lambda(G) + 1$, so that
$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big(X^\varepsilon \in G\big) \geq -\Lambda(G) - \delta$
which δ being arbitrarily allows to conclude.

Upper Bound:
If $\Lambda(F) = 0$, there is nothing to prove.
Otherwise let $0 < a < \Lambda(F)$ and consider the compact sets in $C_x([0,T], \mathbb{R}^d)$ and $C_0([0,T];\mathbb{R}^d)$ respectively:

$$K_a = \big\{ g \in C_x([0,T], \mathbb{R}^d) : I_1(g) \leq a \big\}$$

$$C_a = \big\{ h \in C_0([0,T];\mathbb{R}^d) : \tfrac{1}{2} \parallel h \parallel^2_{H^1} \leq a \big\}$$

Then $K_a \cap F = \emptyset$ and F being closed and K_a compact, for every $g \in K_a$ there exists $\rho = \rho_g$ such that $B(g,\rho) \cap F = \emptyset$
For every $h \in C_a$, $g = S_x(h)$ is a path belonging to K_a and by A.2.5 (c)- there exists $\alpha = \alpha_h$ such that

$$\mathbb{P}\Big(\parallel X^\varepsilon - g \parallel_\infty > \rho, \parallel B^\varepsilon - h \parallel_\infty \leq \alpha \Big) \leq \exp - \frac{R}{\varepsilon}$$

The balls $B(h,\alpha_h), h \in C_a$ form an open cover of C_a which is compact; so that there exist $h_1, ..., h_r$ such that $\{B(h_i,\alpha_i)\}_{i=1,\ldots,r}$ is a finite subcover of C_a. Let $A = \bigcup_{i=1}^r B(h_i,\alpha_i)$ and $g_i = S_x(h_i)$.

Then $\mathbb{P}\Big(X^\varepsilon \in F\Big) \leq \mathbb{P}\Big(X^\varepsilon \in F, B^\varepsilon \in A\Big) + \mathbb{P}\Big(B^\varepsilon \in A^C\Big)$

Now again thanks to Schilder Estimates, as A^C is a closed set such that $\frac{1}{2} \parallel h \parallel^2_{H^1([0,T];\mathbb{R}^d)} \geq a$

$$\mathbb{P}\big(B\varepsilon \in A^C\big) \leq e^{-\frac{a}{\varepsilon}}$$

for small ε ;

If $g_i = S_x(h_i)$ $\forall i = 1, ..., r$

$$\mathbb{P}\big(X^\varepsilon \in F, B^\varepsilon \in A\big) \leq \sum_{i=1}^{r} \mathbb{P}\big(X^\varepsilon \in F, \| B^\varepsilon - h_i \|_\infty < \alpha_i\big)$$
$$\leq \sum_{i=1}^{r} \mathbb{P}\big(\| X^\varepsilon - g_i \|_\infty > \rho_i, \| B^\varepsilon - h_i \|_\infty < \alpha_i\big)$$

So that , again for small ε and a possibly smaller α_i $i = 1, ..., r$

$\mathbb{P}\ \big(\ \mathrm{X}^\varepsilon \in A\big) \leq re^{-R/\varepsilon} + e^{-a/\varepsilon}$ which for $R > a$ gives $\limsup_{\varepsilon \to 0} \varepsilon \mathbb{P}\big(X^\varepsilon \in A\big) \leq -a$ for every $a < \Lambda(F)$ which allows to conclude the upper bound (2.5.3). $\square$

Now we want to give conditions that assure that *A.2.5 (a)* and *A.2.5 (b)* holds.

Lemma 2.5.1. *If* b, σ *are Lipschitz continuous, with sublinear growth, bounded in time, then A.2.5(a) and A.2.5(b) hold. Moreover, for every* $K \subset \mathbb{R}^d$ *compact and* $a > 0$ *there exists* $H > 0$ *such that:*

$$\sup_{x \in K} \sup_{\|h\|_{H^1}} \| S_x(h) \|_\infty \leq H$$

(2.5.7)

Proof. Existence and uniqueness of the solution for (2.5.2) are standarb facts from the theory of ODEs, under the hypothesis stated about b, σ.
Let us prove (2.5.7) applying the Gronwall Lemma. Let $C_0 \geq$ be such that :

$$| b(t, x) | \leq C_0(1 + | x |)$$
$$| \sigma(t, x) | \leq C_0(1 + | x |)$$

Setting $g = S_x(h)$, by *Cauchy-Schwartz Inequality*

$$| g_t | \leq | x | + \int_0^t (1 + | g_s |) \mathrm{d}s + C_0 \int_0^t (1 + | g_s |) \mid \dot{h}(s) \mid \mathrm{d}s$$
$$\leq | x | + C_0 \sqrt{T} \Big(\int_0^t (1 + | g_s |)^2 \mathrm{d}s \Big)^{1/2} + C_0 a \Big(\int_0^t (1 + | g_s |)^2 \mathrm{d}s \Big)^{1/2}$$

taking in consideration that $\| h \|_{H^1} \leq a$.

Denoting R the radius of a ball $B(0, R) \supset K$

$| g_t |^2 \leq 2 | x |^2 + 2C_0^2(\sqrt{T}+a)^2 \int_0^t (1+ | g_s |)^2 \mathrm{d}s \leq 2R^2 + 4C_0^2 T(\sqrt{T}+a)^2 + 4C_0^2 T(\sqrt{T}+a)^2 \int_0^t | g_s |^2 \mathrm{d}s$

So, using Gronwall Inequality
$| g_t |^2 \leq (2R^2 + 4C_0^2 T(\sqrt{T}+a)^2) \exp\big(4C_0^2(\sqrt{T}+a)^2 T\big) := H \ \forall t \in [0,T]$

In order to prove A.2.5(b), we will prove an intermediary statement:

Let ψ be a bounded Lipschitz function, with the bound M and the Lipschitz constant L, and let $h_1, h_2 \in H^1(0,T,\mathbb{R}^d)$ with the bound $\| h_i \|_{H^1} \leq a$, $i = 1,2$. and $g_2 = S_x(h_2)$ Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\| h_1 - h_2 \|_\infty \leq \delta$

$$| \int_0^t \psi(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s | \leq \varepsilon$$

In order to prove this, suppose at first ψ differentiable. As $| \dot{\psi}(s) | \leq L$, by the Lispschitz property, we may integrate by parts and have

$$
\begin{aligned}
&| \int_0^t \psi(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s | = \\
&= | \psi(g_2(t))(h_1(t) - h_2(t)) - \int_0^t \frac{d}{ds} | \psi(g_2(s))(h_1(s) - h_2(s))\mathrm{d}s | \leq \\
&\leq M \| h_1 - h_2 \|_\infty + L \| h_1 - h_2 \|_\infty \int_0^t | \dot{g_2}(s) | \mathrm{d}s) \leq \\
&\leq \| h_1 - h_2 \|_\infty (M + LMT + LM \int_0^t | \dot{h}_2(s) | \mathrm{d}s) \\
&\leq M \| h_1 - h_2 \|_\infty (1 + LT + L\sqrt{T} \| h_2 \|_{H^1}) \leq \\
&\leq M \| h_1 - h_2 \|_\infty \ (1 + L(T + a\sqrt{T}))
\end{aligned}
$$
(2.5.8)

and the result is proved.

In general, if ψ is not differentiable, we can approximate it with a regular function. Let $\phi \in C^\infty$ such that $\int_{\mathbb{R}^d} \phi \mathrm{d}x = 1$ and $\phi(x) = 0$ if $| x | > 1$ and $0 \leq \phi \leq 1$. We can construct it with a partition of the unity (see [20]). For $\eta > 0$ set $\phi_\eta(x) = \frac{1}{\eta^d}\phi(\frac{x}{\eta})$. ϕ_η is called a mollifier and if we set:

$$\psi_\eta(x) = \psi * \phi_e ta(x) = \int_{\mathbb{R}^d} \psi(z)\phi_\eta(x-z)\mathrm{d}z =$$

$$= \int_{\mathbb{R}^d} \psi(x-z)\phi_\eta(z)\mathrm{d}z$$

(2.5.9)

then ψ_η is differentiable ($\mathrm{C}^\infty actually).\psi_\eta$ is still Lipschitz continuous with the same constant L and also bounded with the same bound as ψ. So, by the first part of the proof of this statement,

$$\mid \int_0^t \psi_\eta(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s \mid \leq$$

$\leq M \parallel h_1 - h_2 \parallel_\infty (1 + L(T + a\sqrt{T}))$ (2.5.10)

It can be easily checked that $\mid \psi_\eta(x) - \psi(x) \mid \leq L\eta$ and

$$\mid \int_0^t \psi(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s - \int_0^t \psi_\eta(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s \mid \leq$$

$$\int_0^t \mid \psi(g_2(s)) - \psi_\eta(g_2(s)) \mid\mid \dot{h}_1 - \dot{h}_2 \mid \mathrm{d}s \leq L\sqrt{T}\eta \parallel h_1 - h_2 \parallel_{H^1} \leq$$

$$\leq 2\eta La\sqrt{T}$$

and η being arbitrary completes the proof.

So we can proceed in order to prove A2.5(b).
Let $h_1, h_2 \in K_a = \{\parallel h \parallel_{H^1} \leq a\}$ and let $g_i = S_x(h_i)$ for $i = 1, 2$.
From $\sup_{x\in K} \sup_{\|h\|_{H^1}\leq a} \parallel S_x(h) \parallel_\infty \leq H$ we have $\parallel g_i \parallel \leq H$.
Recall that b, σ are bounded by a constant M and Lipschitz continuous (with a constant L) on $B(0, H)$. Then,

$$\mathrm{g}_1(t) - g_2(t) = \int_0^t (b(s, g_1(s)) - b(s, g_2(s)))\mathrm{d}s + \int_0^t (\sigma(s, g_1(s)) - \sigma(s, g_2(s))\dot{h}_1(s)\mathrm{d}s +$$

$$\int_0^t \sigma(s, g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\mathrm{d}s \quad (2.5.11)$$

By the statement above, if $\parallel h_i \parallel_{H^1} \leq a$ for $i = 1, 2$ for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\parallel h_1 - h_2 \parallel_\infty < \delta$ we have

$$| \int_0^t \sigma(g_2(s))(\dot{h_1}(s) - \dot{h_2}(s))\mathrm{d}s | \leq \varepsilon \ (2.5.12)$$

that yields

$$| g_1(t) - g_2(t) | \leq \varepsilon + L \int_0^t | g_1(s) - g_2(s) | \mathrm{d}s + L \int_0^t | g_1(s) - g_2(s) || \dot{h_1}(s)\mathrm{d}s \leq$$
$$\leq \varepsilon + L \int_0^t | g_1(s) - g_2(s) | \mathrm{d}s +$$
$$\mathrm{L} \Big(\int_0^t | g_1(s) - g_2(s) |^2 \mathrm{d}s \Big)^{1/2} \Big(\int_0^t | \dot{h_1}(s) |^2 \Big)^{1/2} \leq$$
$$\leq \varepsilon + L(\sqrt{T} + a) \Big(\int_0^t | g_1(s) - g_2(s) |^2 \mathrm{d}s \Big)^{1/2}$$

If $\| h_1 - h_2 \|_\infty < \delta$ we get

$$| g_1(t) - g_2(t) |^2 \leq 2\varepsilon^2 + 2L^2(\sqrt{T} + a)^2 \int_0^t | g_1(s) - g_2(s) |^2 \mathrm{d}s$$

By Gronwall Inequality,

$$| g_1(t) - g_2(t) |^2 \leq 2\varepsilon^2 \exp\big(2L^2(\sqrt{T} + a)^2 t\big)$$

which allows to conclude. □

Our next step is to give reasonable conditions under which A2.5(c) holds.

A natural hypothesis is:

Assumption B.2.5

b, σ are locally Lipschitz continuous, have a sublinear growth at infinity, and:

$$\lim_{\varepsilon \to 0^+} | b^\varepsilon(s, y) - b(s, y) | = 0 \ (2.5.13)$$

$$\lim_{\varepsilon \to 0^+} | \sigma^\varepsilon(s, y) - \sigma(s, y) | = 0 \ (2.5.14)$$

uniformly in compact sets.

We want to prove that B.2.5 $\Rightarrow$ A.2.5 (c)

Lemma 2.5.2. *Let $c, c_\varepsilon : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be vector fields such that*

$\mid c_\varepsilon(s,x) \mid + \mid c(s,x) \mid \leq \phi(s)\ 0 \leq s \leq T$ (2.5.15)

$\mid c(s,y) - c(s,z) \mid \leq \psi(s) \mid y - z \mid \quad 0 \leq s \leq T$ (2.5.16)

for some $\phi \in L^2([0,T], \mathbb{R}^d)$ and $\psi \in L^1([0,T], \mathbb{R}^d)$ respectively such that $\lim_{\varepsilon \to 0} \int_0^T \sup_y \mid c_\varepsilon(s,y) - c(s,y) \mid \mathrm{d}s = 0$ (2.5.17)

Let $\sigma_\varepsilon, \sigma$ be $k \times d$ matrix fields such that σ is Lipschitz continuous, bounded by $M{>}0$, and such that (2.5.13)- (2.5.14) holds uniformly in $y \in \mathbb{R}^d$- Let $(X_t^\varepsilon)_{0 \leq t \leq T}$, $(\gamma_t)_{0 \leq t \leq T}$ be the solutions of

$$\begin{cases} X_t^\varepsilon = x + \int_0^t c_\varepsilon(s, X_s^\varepsilon)\mathrm{d}s + \sqrt{\varepsilon}\int_0^t \sigma_\varepsilon(s, X_s^\varepsilon)\mathrm{d}Bs \\ \gamma_t = x + \int_o^t c(s,\gamma_s)\mathrm{d}s \end{cases} \tag{2.5.18}$$

respectively. Then for all $R > 0$, $\rho > 0$, there exists $\varepsilon_0 > 0, \alpha > 0$ such that for every $x \in \mathbb{R}^d$ and ϕ, ψ such that $\parallel \phi \parallel_{L^2} \leq a_1$ and $\parallel \psi \parallel_{L^1} \leq a_1$ we have

$$\mathbb{P}\big(\parallel X^\varepsilon - \gamma \parallel_\infty > \rho, \parallel B^\varepsilon \parallel_\infty \leq \alpha \big) \leq e^{-R/\varepsilon} \text{ for all } \varepsilon < \varepsilon_0.$$

For more details see Baldi [2] and Chaleyat-Maurel [3].

Proof. We have

$$X_t^\varepsilon - \gamma_t = \int_0^t c_\varepsilon(s, X_s^\varepsilon) - c(s, X_s^\varepsilon)\mathrm{d}s + \int_0^t c(s, X_s^\varepsilon) - c(s,\gamma_s)\mathrm{d}s \\ + \sqrt{\varepsilon}\int_0^t \sigma_\varepsilon(s, X_s^\varepsilon)\mathrm{d}Bs$$

For small $\varepsilon > 0$ due to (2.5.17),

$$\mid \int_0^t c_\varepsilon(s, X_s^\varepsilon) - c(s, X_s^\varepsilon)\mathrm{d}s \mid \leq \frac{\rho}{2} e^{-a_1 T}.$$

(2.5.16) gives

$$| \int_0^t c(s,X_s^\varepsilon) - c(s,\gamma_s)\mathrm{d}s \,| \leq \int_0^t |\, \psi(s) \,||\, X_s^\varepsilon - \gamma_s \,| \,\mathrm{d}s \leq$$
$$\leq \int_0^t |\, \psi(s) \,| \sup_{0\leq r\leq s} |\, X_r^\varepsilon - \gamma(r) \,| \,\mathrm{d}s$$

So if $U_\varepsilon(t) = \sqrt{\varepsilon}\int_0^t \sigma_\varepsilon(s,X_s^\varepsilon)\mathrm{d}Bs$

$$\sup_{0\leq s\leq t} |\, X_s^\varepsilon - \gamma_s \,| \leq \frac{\rho}{2}e^{-a_1T} + \|\, B^\varepsilon \,\|_\infty + \int_0^t |\, \psi(s) \,| \sup_{0\leq r\leq s} |\, X_r^\varepsilon - \gamma(r) \,| \,\mathrm{d}s$$

From *Gronwall`s Lemma*, for $\|\, \psi \,\|_{L^1} \leq a_1$

$$\|\, X^\varepsilon - \gamma \,\|_\infty \leq \frac{\rho}{2} + \|\, U_\varepsilon \,\|_\infty \, e^{a_1T}.$$

Thus

$$\mathbb{P}\big(\, \|\, X^\varepsilon - \gamma \,\|_\infty > \rho, \|\, B^\varepsilon \,\|_\infty \leq \alpha\big) \leq \mathbb{P}\big(\, \|\, U_\varepsilon \,\|_\infty > \frac{\rho}{2}e^{-a_1T}, \|\, B^\varepsilon \,\|_\infty \leq \alpha\big)$$

The conclusion follows from the next statement.

□

claim 2.5.1.

$$\forall R > 0 \;\; \forall \rho > 0, \exists \varepsilon_0 > 0 \;\; \alpha > 0 \; \textit{such that for } \varepsilon < \varepsilon_0$$
$$\varepsilon \log \mathbb{P}\big(\, \|\, U^\varepsilon \,\|_\infty > \rho, \|\, B^\varepsilon \,\|_\infty \leq \alpha\big) \leq -R$$

Proof. For every $n \in \mathbb{N}$ let $t_0 = 0$, $t_1 = \frac{T}{N}$,..., $t_k = \frac{kT}{n}$, ..., $t_n = T$ be a discretization of $[0,T]$ and define the approximations:

$$X_t^{\varepsilon,n} = X_{t_k}^\varepsilon \text{ if } t_k \leq t \leq t_{k+1}$$

We have that $\big\{\, \|\, U^\varepsilon \,\|_\infty > \rho, \|\, B^\varepsilon \,\|_\infty \leq \alpha\big\} \subset A \cup B \cup C$

where

A= $\big\{\, \|\, X^\varepsilon - X^{\varepsilon,n} \,\|_\infty > \beta\big\}$

B = $\big\{ \sup_{t\leq T} |\, \sqrt{\varepsilon}\int_0^t (\sigma_\varepsilon(s,X_s^\varepsilon) - \sigma_\varepsilon(s,X_s^{\varepsilon,n})\mathrm{d}Bs \,| > \frac{\rho}{2}, \|\, X^\varepsilon - X^{\varepsilon,n} \,\|_\infty \leq \beta\big\}$

C= $\big\{ \sup_{t\leq T} |\, \sqrt{\varepsilon}\int_0^t \sigma_\varepsilon(s,X_s^{\varepsilon,n})\mathrm{d}Bs \,| > \frac{\rho}{2}, \|\, X^\varepsilon - X^{\varepsilon,n} \,\|_\infty \leq \beta, \|\, B^\varepsilon \,\|_\infty \leq \alpha\big\}$

Split $B = B_1 \cup B_2 \cup B_3$ where:

$$B_1 = \{ \sup_{t \leq T} \mid \sqrt{\varepsilon} \int_0^t (\sigma_\varepsilon(s, X_s^\varepsilon) - \sigma(s, X_s^\varepsilon) \mathrm{d}Bs \mid > \frac{\rho}{6} \}$$
$$B_2 = \{ \sup_{t \leq T} \mid \sqrt{\varepsilon} \int_0^t (\sigma(s, X_s^\varepsilon) - \sigma_\varepsilon(s, X_s^{\varepsilon,n}) \mathrm{d}Bs \mid > \frac{\rho}{6}, \| X^\varepsilon - X^{\varepsilon,n} \|_\infty \leq \beta \}$$
$$B_3 = \{ \sup_{t \leq T} \mid \sqrt{\varepsilon} \int_0^t (\sigma_\varepsilon(s, X_s^{\varepsilon,n}) - \sigma(s, X_s^{\varepsilon,n}) \mathrm{d}Bs \mid > \frac{\rho}{6}, \}$$

Due to (2.5.13) and (2.5.14) for all $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$

$$\sup_y \mid \sigma_\varepsilon(s, y) - \sigma(s, y) \mid \leq \eta \ \forall s$$

The exponential inequality for martingales [19] gives for small η:

$$\mathbb{P}(B_1) \leq 2d \exp - \frac{\rho^2}{72T\eta^2} \frac{1}{\varepsilon}$$
$$< e^{-R/\varepsilon}$$
$$\mathbb{P}(B_3) \leq 2d \exp - \frac{\rho^2}{72T\eta^2} \frac{1}{\varepsilon}$$
$$< e^{-R/\varepsilon}$$

As σ is supposed to be Lipschitz continuous, with a constant L, on B_2 holds
$\mid \sigma(s, X_s^\varepsilon) - \sigma_\varepsilon(s, X_s^{\varepsilon,n}) \mid \leq L\beta$
and again the exponential inequality for martingales gives that

$$\mathbb{P}\ (B_2) \leq 2d \exp - \frac{\rho^2}{72TL^2\beta^2} \frac{1}{\varepsilon}$$
$$< e^{-R/\varepsilon}$$

So $\mathbb{P}(B) \leq 3e^{-\frac{R}{\varepsilon}}$ for $\varepsilon < \varepsilon_0$ and small β independently of $n \in \mathbb{N}$.
As for C, on the set $\{\| B^\varepsilon \|_\infty \leq \alpha\}$ it holds

$$\mid \sqrt{\varepsilon} \int_0^t \sigma_\varepsilon(X_s^{\varepsilon,n}) \mathrm{d}Bs \mid = \mid \sqrt{\varepsilon} \sum_{k=0}^{n-1} \sigma_\varepsilon(X_{t_k}^\varepsilon)(B_{t_{k+1} \wedge t} - B_{t_k \wedge t}) \mid \leq 2Mn\alpha$$

which gives $C = \emptyset$ if $\alpha < \dfrac{\rho}{4Mn_0\eta}$ for some $n_0 \in \mathbb{N}$
For A:

$$\mathbb{P}\ (\ \| X^\varepsilon - X^{\varepsilon,n} \|_\infty > \beta) = \mathbb{P}\Big(\bigcup_{k=0}^{n-1} \Big\{ \sup_{t_k \leq t \leq t_{k+1}} \mid X_t^\varepsilon - X_{t_k}^\varepsilon \mid > \beta \Big\} \Big) \leq$$
$$\leq \sum_{k=0}^{n-1} \mathbb{P}\Big(\sup_{t_k \leq t \leq t_{k+1}} \mid \int_{t_k}^t c_\varepsilon(s, X_s^\varepsilon) \mid \mathrm{d}s + \sqrt{\varepsilon} \sup_{t_k \leq t \leq t_{k+1}} \mid \int_{t_k}^t \sigma_\varepsilon(s, X_s^\varepsilon) \mathrm{d}Bs \mid > \beta \Big)$$

By *Cauchy-Schwartz Inequality*

$$| \int_{t_k}^{t} c_\varepsilon(s, X_s^\varepsilon) \mathrm{d}s \,| \leq \sqrt{\frac{T}{n}} \Big(\int_0^T \mid \phi(s) \mid^2 \mathrm{d}s \Big)^{1/2} \leq a_1 \sqrt{\frac{T}{n}}$$

So, $\forall n \geq n_0$ large enough independently of ε, the events

$$\Big\{ \sup_{t_k \leq t \leq \_k+1} \mid \int_{t_k}^{t} c_\varepsilon(s, X_s^\varepsilon) \mid \mathrm{d}s > \frac{\beta}{2} \Big\} = \emptyset \; \forall \; k = 0, ..., n-1$$

Applying again the exponential inequality of martingales:

$$\mathbb{P}\Big\{ \sqrt{\varepsilon} \sup_{t_k \leq t \leq \_k+1} \mid \int_{t_k}^{t} c_\varepsilon(s, X_s^\varepsilon) \mid \mathrm{d}s > \frac{\beta}{2} \Big\} \leq 2d \exp\big[\frac{-n\beta^2}{8M^2T} \frac{1}{\varepsilon} \big]$$

Furthermore, for $n > n_0$ and $\varepsilon > 0$

$\varepsilon \log \mathbb{P}\big(\parallel x^\varepsilon - X^{\varepsilon,n} \parallel_\infty > \beta \big) \leq \varepsilon \log(2nd) - \dfrac{n\beta^2}{8M^2T} \leq -R$ for a possible larger value of n_0 and $\varepsilon < 1$ With the fact $\mathbb{P}(B) \leq 3e^{-\frac{R}{\varepsilon}}$ we conclude the statement. □

In this moment we are ready to state the final theorem that completes the proof that $(\mathbb{P} \circ (X^\varepsilon) - 1)_{\varepsilon > 0}$, ie the laws of the process solution of (2.4.1), satisfies a LDP with the good rate function I under the assumption of b, σ Lipschitz continuous with sublinear growth at infinity , and with the property of the Assumption B.

For more details and deep information about the more general Frendell-Wentzell estimates and some other complements and alternatives, such as Laplace Principle, see [2] and [3].

Theorem 2.5.2. *Under the assumption B.2.5, for every $R > 0$, $\rho > 0$, $a > 0$, $C > 0$, there exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that for every $\varepsilon < \varepsilon_0$*

$$\mathbb{P}\big(\parallel X^\varepsilon - g \parallel_\infty > g, \parallel B^\varepsilon - h \parallel_\infty \leq \alpha \big) \leq e^{-R/\varepsilon}$$

where $h \in H^1([0,T], \mathbb{R}^d)$, $g = S_x(h)$, uniformly for $\parallel h \parallel_{H^1} \leq a$, $\mid x \mid \leq C$. Moreover, if b, σ are bounded and the convergence in (2.5.13), (2.5.14) is uniform in y, then the quasi-continuity property is uniform in x (the starting point).

Proof. Let

$$L_\varepsilon = \exp\Big(\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{h}(s)\mathrm{d}Bs - \frac{1}{2\varepsilon}\int_0^T \mid \dot{h}(s)\mid^2 \mathrm{d}s\Big)$$

and $\mathbb{P}^\varepsilon$ the probability on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ having density L_ε with respect to $\mathbb{P}$. By *Girsanov`s Formula* [19], under $\mathbb{P}^\varepsilon$ the process $W_t^\varepsilon = B_t - \dfrac{1}{\sqrt{\varepsilon}}h_t$ is a Brownian Motion for $0 \le t \le T$ and let X^ε such that:

$$\begin{cases} \mathrm{dX}_t^\varepsilon = c_\varepsilon(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}dW_t^\varepsilon \\ X_0^\varepsilon = x \end{cases}$$

where $c^\varepsilon(t,x) = b^\varepsilon(t,x) + \sigma^\varepsilon(t,x)\dot{h}_t$. We set $c(t,y) = b(t,y) + \sigma(t,y)\dot{h}_t$ and suppose that b, σ are bounded by $M > 0$ and have the same constant L of Lipschitzian continuity and that the convergence in (2.5.13) - (2.5.14) is uniform. Then:

$\mid c_\varepsilon(s,x)\mid + \mid c(s,x)\mid \le$
2M $(1 + \mid \dot{h}_s\mid) \mid c(s,y) - c(s,x)\mid \le L(1+ \mid \dot{h}_s\mid) \mid y - z\mid$

and

$$\sup_y \mid c_\varepsilon(s,y) - c(s,y)\mid$$
$$\le (1+ \mid \dot{h}_s\mid) \sup_y \big\{ \mid b_\varepsilon(s,y) - b(s,y)\mid + \mid \sigma_\varepsilon(s,y) - \sigma(s,y)\mid \big\}$$

Then by the hypothesis of the previous lemma, that are verified, for every $R_1 > 0$ there exists $\varepsilon_0 > 0$ $\alpha > 0$ such that if $\varepsilon < \varepsilon_0$ and

$\mathrm{A}_\varepsilon = \big\{ \parallel X^\varepsilon - g\parallel_\infty > \rho \parallel \sqrt{\varepsilon}W^\varepsilon \parallel_\infty \le \alpha\big\}$ then

$\mathbb{P}^\varepsilon(A_\varepsilon) < \exp\big(-\frac{R_1}{\varepsilon}\big)$

We have that

$$\frac{d\mathbb{P}}{d\mathbb{P}^\varepsilon} = L_\varepsilon^{-1} = \exp\Big(-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{h}(s)\mathrm{d}Bs + \frac{1}{2\varepsilon}\int_0^T \mid \dot{h}(s)\mid^2 \mathrm{d}s\Big)$$

From Cauchy-Schwartz`s Inequality,

$$\mathbb{P}(A_\varepsilon) = \int_{A_\varepsilon} L_\varepsilon^{-1}\mathrm{d}\mathbb{P}^\varepsilon \le \mathbb{P}^\varepsilon(A_\varepsilon)^{1/2}\mathbb{E}^\varepsilon\big[(L_\varepsilon^{-1})^2\big]^{1/2},$$

being $\mathbb{E}^\varepsilon$ the expectation under $\mathbb{P}^\varepsilon$. And we get also that

$$\mathbb{E}^{\varepsilon}(L_{\varepsilon}^{-1})^2 = \mathbb{E}^{\varepsilon}\exp\Big(-\frac{2}{\sqrt{\varepsilon}}\int_0^T \dot{h}(s)\mathrm{d}Bs + \frac{1}{\varepsilon}\int_0^T \mid \dot{h}(s)\mid^2 \mathrm{d}s\Big) =$$

$$= \underbrace{\mathrm{E}^{\varepsilon}\exp\Big(-\frac{2}{\sqrt{\varepsilon}}\int_0^T \dot{h}(s)\mathrm{d}Bs + \frac{2}{\varepsilon}\int_0^T \mid \dot{h}(s)\mid^2 \mathrm{d}s\Big)}_{=1} \times \exp\big(\tfrac{-1}{\varepsilon} \parallel h \parallel_{H^1}^2\big)$$

$$= \exp\big(\tfrac{-1}{\varepsilon} \parallel h \parallel_{H^1}^2\big)(2.5.19)$$

We take that for every $\parallel h \parallel_{H^1} \leq a$

$\mathbb{P}^{\varepsilon}(A_{\varepsilon}) < \exp\Big\{-\dfrac{R_1 - a^2}{2}\dfrac{1}{\varepsilon}\Big\}$ which proves (A.2.5 (c)).

It remains to drop the assumptions of boundedness and global Lipschitz continuity for b, σ and also the uniformity of the convergence in (2.5.13)-(2.5.14).

This can be done easily by a localization procedure. The idea is that the event $\{\parallel X^{\varepsilon} - g \parallel_{\infty} > \rho\}$ only depends on the value of the coefficients in a neighbourhood of the path g, therefore in a bounded set, where they are Lispchitz continuous and bounded.

By the **lemma 2.5.1** the set of paths that solve (2.5.2) as h varies in $\{\parallel h \parallel_{H^1} \leq a\}$ and x in a compact set $K \subset \mathbb{R}^d$ remains inside an open ball of radius H and centered at the origin of $\mathbb{R}^d$. Let

$$\overline{b}_{\varepsilon}(t,x) = \begin{cases} b^{\varepsilon}(t,x) \text{ if } \mid x \mid < H + 2\rho \\ b^{\varepsilon}\big(t, \frac{x}{|x|}H\big) \text{ if } \mid x \mid \geq H + 2\rho \end{cases} \qquad (2.5.20)$$

and in a similar way $\overline{b}, \overline{\sigma}_{\varepsilon}, \overline{\sigma}$. These new coefficients are trivially bounded, Lipschitz continuous and

$$\lim_{\varepsilon \to 0} \mid \overline{b}_{\varepsilon}(t,x) - \overline{b}(t,x) \mid = \lim_{\varepsilon \to 0} \mid \overline{\sigma}_{\varepsilon}(t,x) - \overline{\sigma}(t,x) \mid = 0 \qquad (2.5.21)$$

uniformly in $x \in \mathbb{R}^d$ and in $t \in [0,T]$ Moreover, if $X^{\varepsilon,*}$ and g^* denote the solutions of:

$$\mathrm{X}_t^{\varepsilon,*} = x + \int_0^t \overline{b}^{\varepsilon}(s, X_s^{\varepsilon,*})\mathrm{d}s + \int_0^t \sqrt{\varepsilon}\,\overline{\sigma}^{\varepsilon}(s, X_s^{\varepsilon,*})\mathrm{d}Bs$$

$$\mathrm{g}_t^* = x + \int_0^t \overline{b}(s, g_s^*)\mathrm{d}s + \int_0^t \overline{\sigma}(s, g_s^*)\dot{h}_s\mathrm{d}s$$

Of course $g^* \equiv g$ and as $\overline{b}^\varepsilon \equiv b^\varepsilon$ and $\overline{\sigma}^\varepsilon \equiv \sigma^\varepsilon$ in the ball of radius $H + 2\rho$, X^ε and $X^{\varepsilon,*}$ coincide up the exit from this ball and

$$\left\{ \| X^\varepsilon - g \|_\infty > \rho \right\} = \left\{ \| X^{\varepsilon,*} - g \|_\infty > \rho \right\}$$

and then

$$\left\{ \| X^\varepsilon - g \|_\infty > \rho \| B^\varepsilon - h \|_\infty < \delta \right\} = \left\{ \| X^{\varepsilon,*} - g \|_\infty > \rho \| B^\varepsilon - h \|_\infty \leq \delta \right\}$$

which concludes the proof

□

Chapter 3

Assymptotics, Connections with Quasilinear Parabolic Partial Differential Equations and a Large Deviations Principle

3.1 The Main Task

The main purpose of this chapter is to discuss the following:

$$\begin{cases} X_t^{\varepsilon,t,x} = x + \int_t^s f(r, X_r^{\varepsilon,t,x}, Y_r^{\varepsilon,t,x})\,\mathrm{d}s + \sqrt{\varepsilon}\int_t^s \sigma(r, X_r^{\varepsilon,t,x}, Y_r^{\varepsilon,t,x})\,\mathrm{d}Bs \\ Y_t^{\varepsilon,t,x} = h(X_T^{\varepsilon,t,x}) + \int_s^T g(r, X_r^{\varepsilon,t,x}, Y_r^{\varepsilon,t,x}, Z_r^{\varepsilon,t,x})\,\mathrm{d}s - \int_t^T Z_r^{\varepsilon,t,x}\mathrm{d}Bs \\ \forall\, 0 \le t \le s \le T,\ x\ \in \mathbb{R}^d \end{cases}$$
(3.1.1)

under the hypothesis of the first chapter.
(3.1.1) is the natural perturbation of the FBSDE (E) with a small parameter $\sqrt{\varepsilon}$ in the diffusion coefficient of the forward equation. Our goal is to consider $\varepsilon \to 0$ and study what happens to the solution of the problem.
In the first chapter we have assured sufficient conditions to conclude that (3.1.1) has a unique solution in $\mathcal{M}[t,T]$ $(X_t^{\varepsilon,t,x}, Y_t^{\varepsilon,t,x}, z_t^{\varepsilon,t,x})_{t\le s\le T}$ $\forall s \in [t,T]$. so, now our question is to study the convergence of this solution when $\varepsilon \to 0$. Under the classical assumptions, the boundedness of h, Lipschitz continuity

property, sublinear growth and monotonicity; assuming σ bounded too and letting $a = \sigma\sigma^T$, we will see that defining

$$u(t,x) = Y_t^{t,x} \ t \in [0,T]$$

u is a viscosity solution of

$$\begin{cases} \dfrac{\partial u^{l,\varepsilon}}{\partial t} + \dfrac{\varepsilon}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u^{l,\varepsilon}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} f_i \frac{\partial u^{l,\varepsilon}}{\partial x_i} + g^l = 0 \\ u^{\varepsilon}(T,x) = h(x) \\ x \in \mathbb{R}^d, \ l = \ 1, ..., k \end{cases} \tag{3.1.2}$$

And in addition, $(X_t^{\varepsilon,s,x}, Y_t^{\varepsilon,s,x})_{s\leq t\leq T}$ obeys a Large Deviations Principle.

3.2 FBSDEs and Viscosity Solutions

Consider the following FBSDE

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,\mathrm{d}s + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})\,\mathrm{d}Bs \\ Y_s^{\varepsilon,s,x} = h(X_T) + \displaystyle\int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,\mathrm{d}s - \int_s^T Z_s Bs \\ \forall\, 0 \leq t \leq s \leq T, \ x \ \in \mathbb{R}^d \end{cases}$$

(3.2.1)

Our purpose is to show that if the FBSDE (3.2.1) has a unique adapted solution denoted $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{t\leq s\leq T}$, then $u(t,x) \equiv Y_t^{t,x}$ is a viscosity solution to a quasilinear PDE.
We are going to do the following assumptions- **Assumption A.3.2**

i) The coefficients f, g, σ, h are under the assumption (A.1.3) of the first chapter and in order to simplify the presentation, we consider $k = 1$, ie $(Y_t^{s,x})_{t\leq s\leq T}$ is one-dimensional.

The proof we will present here (inspired in the work of Ma-Yong [25) extends easily to systems of quasilinear second order PDEs of parabolic type. However for the notion of viscosity solution make sense, we need to make restrictions on the dependence of the coefficients f, g upon the variable z:

ii) f does not depend on z.

iii) $\forall 1 \leq l \leq k$ the l-th coordinate g_k of g depends only on the k-th row of the matrix z.

iv) The functions f, g are differentiable in z.

The coefficient $\sigma = \sigma(t, x)$ independent of y garantees the uniqueness of viscosity solution when the solution class is restricted to, for example, bounded, continuous functions that are uniform Lipschitz in x and Holder $-1/2$ in t. For more details in viscosity solutions of second order parabolic equations see [7].

Under suitable additional conditions (assumption A.1.5) the system (3.2.1) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$. We recall that, by the remark after **theorem 1.4.1**, resulting of the application of Blumenthal 0-1 Law, $u(t, x) = Y_t^{t,x}$ is deterministic.

We define the following differential operator :

$$(L\,\varphi)(t, x, y, z) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x, y) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(t, x) + + < f(t, x, y, z), \nabla\varphi(t, x) >$$

$\forall \varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^k)$, $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$

where $a_{ij}(t, x) = \sigma\sigma^T(t, x)\ \forall i, j = 1, ..., d$

We claim that $u(t, x) = Y_t^{t,x}$ is a viscosity solution of the following backward quasilinear second order PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (Lu)(t, x, u(t, x), \nabla_x u(t, x)\sigma(t, x, u(t, x)))+ \\ g(t, x, u(t, x), \nabla_x u(t, x)\sigma(t, x)) = 0 \\ u(T, x) = h(x),\ \ x \in \mathbb{R}^d,\ \ t \in [0, T] \end{cases}$$

(3.2.2)

Let us recall the definition of a viscosity solution for the PDE (3.2.2) (see [7]).

Definition 3.2.1. Viscosity Solution
Let $u \in C^{1,2}([0, T] \times \mathbb{R}^d)$, satisfying $u(T, x) = h(x)$, $x \in \mathbb{R}^d$. u is called a

viscosity subsolution (respectively supersolution) of the PDE (3.2.2) if for every $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^k)$, and (t,x) is a local minimun (respectively maximum) of $\varphi - u$ we have

$$\frac{\partial u}{\partial t} + (Lu)(t,x,u(t,x), \nabla_x \varphi(t,x)\sigma(t,x,u(t,x)))$$
$$+\ g(t,x,u(t,x), \nabla_x \varphi(t,x)\sigma(t,x,u(t,x))) \geq 0 \ (3.2.3)$$

(respectively ≤ 0).

u is called a viscosity solution of the PDE (3.2.2) it it is both a viscosity sub and supersolution.

We are going to prove the following:

Theorem 3.2.1. *Assuming the assumptions A.3.2 on the coefficients of the system (3.2.2) (ie lipschitz continuity, sublinear growth with boundedness of h and differentiability on z of all of them), (3.2.1) has an adapted solution* $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *and the function* $u(t,x) = Y_t^{t,x}$ $t \in [0,T]$ *is a viscosity solution of the quasilinear PDE (3.2.2).*

Proof. The existence and uniqueness of the solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ of the FBSDE (3.2.1) is standarb, due to the work developed in chapter 1.
We will prove that u is a viscosity subsolution of (3.2.2) . The proof that u is a supersolution is identical. Note by the chapter 1, $u(t,x) = Y_t^{t,x}$ is continuous on $[0,T] \times \mathbb{R}^d$ Lipschitz continuous in x and Holder continuous $-1/2$ in t.
Let $(t,x) \in [0,T] \times \mathbb{R}^d$; let $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be such that (t,x) is a local minimum point of $\varphi - u$ such that $u(T,x) = \varphi(T,x) = h(x)$.
Modifying slightly " at infinity" if necessary, we assume without loss of generality that $\varphi, \nabla_x \varphi$ are uniformly bounded, thanks to the uniform Lipschitz property of u in x.
$\forall\, 0 \leq t \leq \tau \leq T$, $u(\tau, X_\tau^t) = Y_\tau^t$ thanks to the pathwise uniqueness of the FBSDE (3.2.1)
Therefore
$u(t,x)$= $\mathrm{u}(\tau, X_\tau^t) + \int_t^\tau g(s, X_s^t, Y_s^t, Z_s^t)\mathrm{d}s - \int_t^\tau Z_s^t \mathrm{d}Bs$ (3.2.4)

Applying Itõ`s Formula to $\varphi(., X_\tau^t)$ from t to τ:

$\varphi(\tau, X_\tau^t) = \varphi(t,x) + \int_t^\tau \frac{\partial \varphi_t}{\partial t}(s, X_s^t)\mathrm{d}s + \int_t^\tau < \nabla_x \varphi(s, X_s^t), f(s, X_s^t, \varphi(s, X_s^t), Z_s^t) >$
$\mathrm{d}s + \int_t^\tau \frac{1}{2} tr\big[\sigma\sigma^T(s, X_s^t, Y_s^t)\nabla_{xx}\varphi(s, X_s^t)\big]\mathrm{d}s + \int_t^\tau < \nabla_x\varphi(s, X_s^t)\sigma(s, X_s^t, Y_s^t)\mathrm{d}Bs >$
(3.2.5)

Writing

g(s, $X_s^t, Y_s^t, Z_s^t) = g(s, X_s^t, Y_s^t, [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t)) +$
$<\alpha(s), Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t) >$
f(s, $X_s^t, Y_s^t, Z_s^t) = f(s, X_s^t, Y_s^t, [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t))) + \beta(s)\big[Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t)\big]$
(3.2.6)

where

$$\begin{cases} \alpha(s) = \int_0^1 \frac{\partial g}{\partial z}(s, X_s^t, Y_s^t, Z_s^{t,\theta})\mathrm{d}\theta \\ \beta(s) = \int_0^1 \frac{\partial f}{\partial z}(s, X_s^t, Y_s^t, Z_s^{t,\theta})\mathrm{d}\theta \\ Z_s^{t,\theta} = \theta Z_s^t + (1-\theta)\sigma^T(s, X_s^t, Y_s^t)\nabla_x\varphi(s, X_s^t) \end{cases}$$
(3.2.7)

By the asumptions α, β are bounded adapted processes.
Subtracting (3.2.5) of (3.2.4), combined with (3.2.6) and (3.2.7), noting that $u(t,x) = \varphi(t,x)$ and $u(\tau, X_\tau^t) = \varphi(\tau, X_\tau^t)$, we obtain:

$0 \geq u(\tau, X_\tau^t) - \varphi(\tau, X_\tau^t) = \int_0^\tau \Big\{ -\frac{\partial\varphi}{\partial t}(s, X_s^t) - g(s, X_s^t, Y_s^t, [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t)) -$

$<Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t), \alpha(s) - \nabla_x\varphi(s, X_s^t)\beta(s) > \Big\}\mathrm{d}s +$
$\int_t^\tau < Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t), \mathrm{d}Bs >$ (3.2.8)

Defining $\theta(s) = \alpha(s) + \nabla_x\varphi(s, X_s^t)\beta(s), s \in [t,T]$
we see that $\theta(s)$ is uniformly bounded and the following process is a $\mathbb{P}$-martingale on $[t,T]$

$$\Theta_s^t = \exp\Big\{ -\int_t^s < \theta_r, \mathrm{d}Br > -\frac{1}{2} \mid \theta(r) \mid^2 \mathrm{d}r\Big\}, s \in [t,T]$$

By Girsanov\`s Theorem, let $\mathbb{Q}$ a new measure of probability, such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = \Theta_T^t$ and $W_t^s = B(s) - B(t) - \int_t^s \theta(r)\mathrm{d}r$ is a $\mathbb{Q}$ Brownian Motion on $[t,T]$.
Since $(X^{t,x}, Y^{t,x}, Z^{t,x})$ is on $\mathcal{M}[t,T]$, ie,

$$\mathbb{E}\sup_{t\le s\le T} \mid X_s \mid^2 + \mathbb{E}\sup_{t\le s\le T} \mid Y_s \mid^2 + \mathbb{E}\int_t^T \mid Z_s \mid^2 \mathrm{d}s < \infty$$

With the boundedness of $\nabla_x\varphi$ and the uniform Lipschitz property of σ, we have that exists $C > 0$ such that:

$$\mathbb{E}^{\mathbb{Q}}\Big\{\int_t^T \mid Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t Y_s^t) \mid^2 \mathrm{d}s\Big\}^{1/2} \le$$
$$\le C\mathbb{E}\Big\{\Theta_T^t\Big(\int_t^T (1+ \mid X_s^t \mid^2 + \mid Y_S^t \mid^2 + \mid Z_s^t \mid^2)\mathrm{d}s\Big)^{1/2}\Big\} \le$$
$$\le C\Big[\mathbb{E}(\Theta_T^t)^2\Big]^{1/2}\Big\{\mathbb{E}\int_t^T 1+ \mid X_s^t \mid^2 + \mid Y_S^t \mid^2 + \mid Z_s^t \mid^2)\mathrm{d}s\Big\}^{1/2} < \infty \quad (3.2.9)$$

In other words,

$$\mathrm{M}^t(r) = \int_t^r < Z_s^t - [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t), \mathrm{d}W_s > \ r \in [t, T] \quad (3.2.10)$$

is a local $\mathbb{Q}$ martingale on $[t, T]$ satisfying $\mathbb{E} < M_t >_T^{1/2} < \infty$, so the Burkholder-Davis-Gundy Inequality shows that process is a $\mathbb{P}$ martingale on $[t, T]$. For details about compensators see [19] or [21].
Taking the expectation $\mathbb{E}^{\mathbb{Q}}$ on both sides of (3.2.8) we obtain:

$$0 \ge \mathbb{E}^{\mathbb{Q}}\Big\{\int_t^\tau -\frac{\partial\varphi}{\partial t}(s, X_s^t) - (L\varphi)(s, X_s^t, Y_s^t, [\sigma^T\nabla_x\varphi](s, X_s^t, Y_s^t) -$$
$$\mathrm{g}(\mathrm{s},\ \mathrm{X}_s^t, Y_s^t, \nabla_x\varphi(s, X_s^t)\sigma(s, X_s^t, Y_s^t))\mathrm{d}s\Big\} (3.2.11)$$

Divide both sides by τ and send $\tau \to 0$ to conclude and obtain the inequality (3.2.3).

□

3.3 The Assymptotic Study and a Large Deviation Principle

Recall our main FBSDE problem:

$$\begin{cases} X_s^{t,\varepsilon,x} = x + \int_t^s f(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x})\mathrm{d}r + \sqrt{\varepsilon}\int_t^s \sigma(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon})\mathrm{d}Br \\ Y_s^{t,\varepsilon,x} = h(X_t^{t,x,\varepsilon}) + \int_s^T g(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}, Z_r^{t,\varepsilon,x})\mathrm{d}r - \int_t^s Z_r^{t,\varepsilon,x}\mathrm{d}Br \\ x \in \mathbb{R}^d \;\; 0 \le t \le s \le T \;\; , l = 1, ..., k \end{cases}$$

(3.3.1)

where

$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$

$g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$

$\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^{d\times d}$

$h : \mathbb{R}^k \to \mathbb{R}^k$

are measurable functions to respect to the considered borelian fields in the euclidian spaces in case, obeying the usual assumptions (A.1.5) of Lipschitz continuity, sublinear growth, monotonicity, with the new assumption of *boundedness on* σ.

We know (3.3.1) has a unique solution in $\mathcal{M}[t,T]$ $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t\le s\le T}$, that we will denote sometimes in a simplified way, without danger of confusion, by $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)_{t\le s\le T}$. By the **theorem 3.2.1**, we know that $u^\varepsilon(t,x) = Y_{t,x}^t$ is a viscosity solution of the associated quasilinear parabolic PDE system:

$$\begin{cases} \frac{\partial (u^\varepsilon)^l}{\partial t}(t,x) + \frac{\varepsilon}{2}\sum_{i=1}^d a_{ij}(t,x,u^\varepsilon(t,x))\frac{\partial^2 (u^\varepsilon)^l}{\partial x_i \partial x_j}(t,x) + \\ \sum_{i=1}^d f^l(t,x,u^\varepsilon(t,x))\frac{\partial (u^\varepsilon)^l}{\partial x_i}(t,x) + \\ g^l(t,x,u^\varepsilon(t,x), \nabla_x u^\varepsilon(t,x)\sigma^\varepsilon(t,x,u^\varepsilon(t,x))) = 0 \\ u^\varepsilon(T,x) = h(x) \;\; x \in \mathbb{R}^d \;\; t \in [0,T] \\ l = 1, ..., k \end{cases}$$

(3.3.2)

Using the **proposition 1.4.1** we know that for each $\varepsilon > 0$ $u^\varepsilon[0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ is only dependent on f, g, h, σ, so the properties below hold uniformly in ε.

Using the **theorem 1.5.1** :

$$\mathbb{P}\big(\forall\, s \in [t,T] : u^\varepsilon(s, X_s^{t,\xi}) = Y_s^{t,\xi}\big) = 1 \qquad (3.3.3)$$

$$\mathbb{P} \otimes \mu\big(\{(\omega, s) \in \Omega \times [t,T] :\mid Z_s^{t,\varepsilon,x}(\omega) \mid \geq \Gamma_1\}\big) \tag{3.3.4}$$

In particular, there exists continuous versions and uniformly bounded (in ε too) of $(Y_s^{t,x}, Z_s^{t,x})_{t\leq s\leq T}$. But if we assume more regularity on the coefficients of the FBSDE (3.3.1), the regularity enough to use the **lemma 1.5.1**, we deduce that :

$$\mid u^{\varepsilon}(t,x) \mid \leq \kappa \tag{3.3.5}$$

$$u^{\varepsilon} \in C_b^{1,2}([0,T] \times \mathbb{R}^d) \tag{3.3.6}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mid \nabla_x u^{\varepsilon}(t,x) \mid \leq \kappa_1 \tag{3.3.7}$$

and

$$Z_s^{t,\varepsilon,x} = u^{\varepsilon}(t, X_s^{t,\varepsilon,x})\sigma(s, X_s^{t,\varepsilon}) \tag{3.3.8}$$

where u^{ε} solves uniquely (3.3.2)
Our first result is:

Theorem 3.3.1. ***Assymptotic Behaviour***
Under the previous assumptions, in particular if σ bounded:

1. The solution $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t\leq s\leq T}$ converges in $\mathcal{M}[t,T]$, when $\varepsilon \to 0$ to $(X_s, Y_s, 0)_{t\leq s\leq T}$, being $(X_s, Y_s)_{t\leq s\leq T}$ solution of the deterministic ordinary coupled system of differential equations:

$$\begin{cases} \dot{X}_s = f(s, X_s, Y_s) \\ \dot{Y}_s = -g(s, X_s, Y_s),\ t \leq s \leq T \\ X_t = x, Y_T = h(X_T) \end{cases}$$

(3.3.9)

2. u is bounded, continuous Lipschitz in x and uniformly continuous in time.

3. $u(t,x) = Y_t^{t,x}$, limit in $\varepsilon \to 0$ of $Y_t^{t,\varepsilon,x}$ is a viscosity solution of

$$\begin{cases} \dfrac{\partial u^l}{\partial t} + \displaystyle\sum_{i=1}^{d} f_i(t,x,u(t,x))\frac{\partial u^l}{\partial x_i}(t,x) + g^l(t,x,u(t,x), \nabla_x u(t,x)\sigma(t,x,u(t,x))) = 0 \\ u^{\varepsilon}(t,x) = h(x)\ x \in \mathbb{R}^d\ t \in [0,T] \\ l = 1, ..., k \end{cases}$$

(3.3.10)

4. Furthermore, u is the unique classic solution continuous Lipschitz in x and uniformly continuous in time of (3.3.1) if $u \in C_b^{1,1}([0,T]\times\mathbb{R}^d)$ and under the hypothesis that (3.3.9) has a unique continuous solution.

Proof. To simplify notations in the computations that follows, we write
$f^\varepsilon(s) = f(s, X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x})$
$g^\varepsilon(s) = g(s, X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})$
$\sigma^\varepsilon(s) = \sigma(s, X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x})$
$h^\varepsilon = h(X_T^{t,\varepsilon,x})$
By an estimate like the main estimate of the section 4 of the first chapter, **theorem 1.4.2**, we have that :

$$\mathbb{E}\sup_{t\le s\le T} \mid X_s^\varepsilon - X_s^{\varepsilon_1} \mid^2 + \mathbb{E}\sup_{t\le s\le T} \mid Y_s^\varepsilon - Y_s^{\varepsilon_1} \mid^2 +$$
$$\mathbb{E}\int_t^T \mid Z_s^\varepsilon - Z_s^{\varepsilon_1} \mid^2 \mathrm{d}s \le \gamma\Big\{\mathbb{E} \mid h^\varepsilon - h^{\varepsilon_1} \mid + \mathbb{E}\int_t^T \mid \sqrt{\varepsilon}\sigma^\varepsilon(s) - \sqrt{\varepsilon_1}\sigma^{\varepsilon_1}(s) \mid$$
$$\mathrm{d}s + \mathbb{E}\Big(\int_t^T \mid Y_s^\varepsilon - Y_s^{\varepsilon_1} \mid + \mid g^\varepsilon(s) - g^{\varepsilon_1}(s) \mid \mathrm{d}s\Big)^2\Big\} \quad (3.3.11)$$

where $\gamma > 0$ only depends on the Lipschitz constants of the coefficients of the FBSDE system and $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)_{t\le a\le T}$ solves uniquely (3.3.1) and $(X_s^{\varepsilon_1}, Y_s^{\varepsilon_1}, Z_s^{\varepsilon_1})_{t\le a\le T}$ solves the equation (3.3.1) but with ε replaced by ε_1 .
We can estimate:

$$\mathbb{E} \mid h(X_T^\varepsilon) - h(X_T^{\varepsilon_1}) \mid \le L^2\mathbb{E}sup_{t\le s\le T} \mid X_s^\varepsilon - X_s^{\varepsilon_1} \mid^2 \quad (3.3.12)$$

where L is the constant Lipschitz of h. Another estimate is:

$$\mathbb{E}\int_t^T \Big(\mid \sqrt{\varepsilon}\sigma^\varepsilon(s) - \sqrt{\varepsilon_1}\sigma^{\varepsilon_1}(s) \mid \Big)\mathrm{d}s \le \mathbb{E}\int_t^T \mid \sqrt{\varepsilon} - \sqrt{\varepsilon_1} \mid \underbrace{M}_{bound|\sigma|} + \varepsilon_1(\mid X_s^\varepsilon - X_s^\varepsilon \mid^2 +$$
$$\mid Y_s^\varepsilon - Y_s^{\varepsilon_1} \mid^2) + 2\sqrt{\varepsilon_1} \mid \sqrt{\varepsilon} - \sqrt{\varepsilon_1} \mid 2M\mathrm{d}s$$

$$\leq c_1 \mid \sqrt{\varepsilon} - \sqrt{\varepsilon_1} \mid + c_2 \mathbb{E} \int_t^T \mid X_s^{\varepsilon} - X_s^{\varepsilon_1} \mid^2 + \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 \mathrm{d}s \quad (3.3.13)$$

where c_1, c_2 are independent of $\varepsilon, \varepsilon_1$ only dependent on the bound of $\mid \sigma \mid$, T and the Lipschitz constant of σ.
Furthermore, for some $c_3 > 0$

$$\mathbb{E}\Big(\int_t^T \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid + \mid g^{\varepsilon}(s) - g^{\varepsilon_1}(s) \mid \mathrm{d}s \Big)^2 \leq$$
$$\mathbb{E} \int_t^T \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 + \mid g^{\varepsilon} - g^{\varepsilon_1} \mid^2 + 2 \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid\mid g^{\varepsilon} - g^{\varepsilon_1} \mid \mathrm{d}s \leq$$
$$\mathrm{c}_3 \; \mathbb{E}\Big[\int_t^T \mid X_s^{\varepsilon} - X_s^{\varepsilon_1} \mid^2 + \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 + \mid Z_s^{\varepsilon} - Z_s^{\varepsilon_1} \mid^2 \Big] \leq$$
$$\mathrm{c}_3 \left\{ \mathbb{E} \sup_{t \leq s \leq T} \mid X_s^{\varepsilon} - X_s^{\varepsilon_1} \mid^2 + \mathbb{E} \sup_{t \leq s \leq T} \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 + \mathbb{E} \sup_{t \leq s \leq T} \mid Z_s^{\varepsilon} - Z_s^{\varepsilon_1} \mid^2 \right\}$$
(3.3.14)

using Jensen Inequality and using the Lipschitz property of g, and modifying c_3 throught the lines.

By (3.3.9) we have that

$$\mathbb{E} \int_t^T \mid Z_s^{\varepsilon} \mid^2 \mathrm{d}s \leq \mathbb{E} \sup_{t \leq s \leq T} \mid Z_s^{\varepsilon} \mid^2 \leq M \varepsilon \Gamma \to 0 \text{ as } \varepsilon \to 0 \quad (3.3.15)$$

By the Cauchy property of the space $H_T^2(\mathbb{R}^{k \times d})$, we know by the above, once Z_s^{ε} converges to 0 as $\varepsilon \to 0$, that

$$\mathbb{E} \int_t^T \mid Z_s^{\varepsilon} - Z_s^{\varepsilon_1} \mid^2 \mathrm{d}s \to 0 \text{ as } \varepsilon \to 0 \quad (3.3.16)$$

Furthermore, using *Burkholder-Davis-Gundy`s Inequalities*

$$\mathbb{E} \sup_{t \leq s \leq T} \mid X_s^{\varepsilon} - X_s^{\varepsilon_1} \mid^2 + \mathbb{E} \sup_{t \leq s \leq T} \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 + \mathbb{E} \int_t^T \mid Z_s^{\varepsilon} - Z_s^{\varepsilon_1} \mid^2 \mathrm{d}s \leq$$
$$\gamma \mathbb{E} \int_t^T \mid X_s^{\varepsilon} - X_s^{\varepsilon_1} \mid^2 + \mid Y_s^{\varepsilon} - Y_s^{\varepsilon_1} \mid^2 + \mid Z_s^{\varepsilon} - Z_s^{\varepsilon_1} \mid^2 \mathrm{d}s$$
$$\leq \gamma \mathbb{E} \int_t^T \sup_{t \leq r \leq s} \mid X_r^{\varepsilon} - X_r^{\varepsilon_1} \mid^2 + \sup_{t \leq r \leq s} \mid Y_r^{\varepsilon} - Y_r^{\varepsilon_1} \mid^2 + \sup_{t \leq r \leq s} \mid Z_r^{\varepsilon} - Z_r^{\varepsilon_1} \mid^2 \mathrm{d}s$$
(3.3.17)

Moreover, for new $\gamma_1, \gamma_2 > 0$, eventually, using (3.3.12)- (3.3.15) we get

$$\mathbb{E} \sup_{t\leq s\leq T} \mid X_s^\varepsilon - X_s^{\varepsilon_1} \mid^2 + \mathbb{E} \sup_{t\leq s\leq T} \mid Y_s^\varepsilon - Y_s^{\varepsilon_1} \mid^2 + \mathbb{E}\int_t^T \mid Z_s^\varepsilon - z_s^{\varepsilon_1} \mid^2 \mathrm{d}s \leq$$

$$\gamma_1 \mid \sqrt{\varepsilon} - \sqrt{\varepsilon_1} \mid + \gamma_2 \mathbb{E}\int_t^T \sup_{t\leq r\leq s} \mid X_r^\varepsilon - X_r^{\varepsilon_1} \mid^2 + \sup_{t\leq r\leq s} \mid Y_r^\varepsilon - Y_r^{\varepsilon_1} \mid^2 +$$

$\sup_{t\leq r\leq s} \mid Z_r^\varepsilon - Z_r^{\varepsilon_1} \mid^2 \mathrm{d}s$ (3.3.18)

Using *Gronwall Inequality*,

$$\mathbb{E}\int_t^T \sup_{t\leq r\leq s} \mid X_r^\varepsilon - X_s r^{\varepsilon_1} \mid^2 + \sup_{t\leq r\leq s} \mid Y_s^\varepsilon - Y_s^{\varepsilon_1} \mid^2 \leq$$

$\leq C \mid \sqrt{\varepsilon} - \sqrt{\varepsilon_1} \mid \mathbb{E} \sup_{t\leq s\leq T} \mid Z_s^\varepsilon - Z_s^{\varepsilon_1} \mid^2 \to 0$ as $\mid \varepsilon - \varepsilon_1 \mid \to 0$ (3.3.19)

For some $C > 0$ independent of ε. We conclude, by the previous computations that the pair $(X_s^{t,\varepsilon}, Y_s^{t,\varepsilon})_{t\leq s\leq T}$ converges in $S_T^2(\mathbb{R}^d) \times S_T^2(\mathbb{R}^k)$ by the completeness of the normed spaces concerned. Call $(X_s, Y_s)_{t\leq s\leq T t\leq s\leq T}$ its limit. So, $(X_s, Y_s, 0)_{t\leq s\leq T}$ is the limit in $\mathcal{M}[t,T]$ of $(X_s^{t,\varepsilon}, Y_s^{t,\varepsilon}, 0)_{t\leq s\leq T}$ when $\varepsilon \to 0$.

Now, considering the forward equation on (3.3.1), if we take the limit pointwise when $\varepsilon \to 0$, and using the boundedness of σ and the continuity of f, we have

$\mathrm{X}_s^t = x + \int_t^s f(r, X_r, Y_r)\mathrm{d}r$ a.s $\mathbb{P}$ (3.3.20)

Similarly, taking the limit on the backward equation when $\varepsilon \to 0$, using the continuity of the functions h, g and the fact that $\mathbb{E}\big(\int_s^T Z_r^{t,\varepsilon}\mathrm{d}Br\big)^2 = \mathbb{E}\int_s^T \mid Z_r^{t,\varepsilon} \mid^2 \mathrm{d}r \to 0$ as $\varepsilon \to 0$ implies $\int_s^T Z_r^{t,\varepsilon}\mathrm{d}Br \to 0$, *a.s* $\mathbb{P}$

$\mathrm{Y}_s^t = h(X_T^t) + \int_s^T g(r, X_r, Y_r, 0)\mathrm{d}r$ a.s $\mathbb{P}$ (3.3.21)

In conclusion, $(X_s, Y_s)_{t\leq s\leq T}$ solves the following deterministic problem of ordinary (coupled) differential equations,

$$\begin{cases} \dot{X}_s = f(s, X_s, Y_s) \\ \dot{Y}_s = -g(s, X_s, Y_s),\ t \leq s \leq T \\ X_t = x, Y_T = h(X_T) \end{cases}$$

(3.3.22)

2. u^ε converges uniformly in $[0,T] \times K$ for all K compact of $\mathbb{R}^d$; an analogue estimate of the main estimate of the **theorem 1.4.2** shows, such as the **corollary 1.4.3**, that $(u^\varepsilon)$are equicontinuous; and we can apply *Arzela`s Ascoli Theorem* and conclude the uniform convergence of (u^ε) in the compact sets of $[0,T] \times \mathbb{R}$.

$u(t,x) = Y_t^{t,x}$ is continuous by the type of convergence of u^ε , using the inequality (3.3.11). The boundedness of u and the continuous Lipschitz condition in x and the uniform continuity of u in time follows from the type of convergence of u^ε and of the uniform bounds (3.3.7)- (3.3.7) in ε for u^ε.

3. Using the **theorem 3.2.1** from the previous chapter, we see clearly since the hypothesis of the coefficients of the concerned FBSDE problem , that u^ε is a viscosity solution in $[0,T] \times \mathbb{R}^d$. of (3.3.2).

But (3.3.2) can be written by

$$- \frac{\partial (u^\varepsilon)^l}{\partial t}(t,x) + G_l^\varepsilon(t,x,u^\varepsilon t,x), \nabla_x u^\varepsilon(t,x), \nabla_x x u^\varepsilon(t,x)) = 0 \quad l = 1,...k \quad (3.3.23)$$

and (3.3.10) by

$$- \frac{\partial u^l}{\partial t}(t,x) + G_l(t,x,u^\varepsilon t,x), \nabla_x u(t,x), \nabla_x x u(t,x)) = 0 \quad l = 1,...k \quad (3.3.24)$$

where $G^\varepsilon(t,x,y,p,q) = - < f(t,x,y), p > - \frac{\varepsilon}{2} tr(a(t,x,y)q)$
$- g(t,x,y,\sqrt{\varepsilon} p \sigma(t,x,y)))$ in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$ and
$G(t,x,y,p,q) = - < f(t,x,y), p > - g(t,x,y,0)$ in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$.

Clearly, since the coefficients of the system are Lipschitz continuous, it is easy to conclude that G^ε converge uniformly to G in all compact sets of $[0,T] \times \mathbb{R}^d$, and we know that once u^ε converges to u in every compact set of$[0,t] \times \mathbb{R}^d$. In these conditions, knowing the properties of viscosity solutions (see [7]), we conclude that u is a viscosity solution of (3.3.10).

4. If $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ is a $C_b^{1,1}([0,T], \mathbb{R}^k)$ solution, continuous Lipschitz in x and uniformly continuous in t for (3.3.10), fixing $(t,x) \in$

$[0,T]\times\mathbb{R}^d$, we can consider the following function:
$\psi:[t,T]\to\mathbb{R}^k$ $\psi(s):=v(s,X_s^{t,x})$, computing its time derivative:

$$\frac{d\psi}{ds}(s)=\frac{\partial v}{\partial s}(s,X_s^{t,x})+\sum_{i=1}^{d}\frac{\partial v}{\partial x_i}(t,X_s^{t,x})\frac{\partial(X_s^{t,x})}{\partial t}=$$
$$=\ \frac{\partial v}{\partial s}(s,X_s^{t,x})+\sum_{i=1}^{d}\frac{\partial v}{\partial x_i}(t,X_s^{t,x})f(s,X_s^{t,x},Y_s^{t,x})=$$
$$-g(s,X_s^{t,x},v(x,X_s^{t,x}),\nabla_x v(x,X_s^{t,x})\sigma(s,X_s^{t,x},v(s,X_s^{t,x})))$$

$\psi(T)=v(s,X_T^{t,x})=h(x)$

Consequently $v(t,x)=v(t,X_t^{t,x})=u(t,x)$, if we have uniqueness of solution for the ordinary system of differential equations (3.3.9).
So, we have a uniqueness property for (3.3.10) under the class of $C_b^{1,1}([0,T]\times\mathbb{R}^d)$, which are Lipschitz continuous in x and uniformly in t.

□

Our second result is a Large Deviations Principle for the couple $(X_s^{t,\varepsilon,x},Y_s^{t,\varepsilon,x})_{t\leq s\leq T}$.

Theorem 3.3.2. ***A Large Deviations Principle***
When $\varepsilon\to 0$, $(X_s^{t,\varepsilon})_{t\leq s\leq T}$ obeys a LDP in $C([t,T],\mathbb{R}^d)$ with the good rate function

$$I(\varphi)=\inf\Big\{\frac{1}{2}\int_t^T\mid\dot g_s\mid^2\mathrm{d}s:g\in H^1([t,T],\mathbb{R}^d),$$
$$g_s=x+\int_t^s f(r,g_r,u^\varepsilon(r,g_r))\mathrm{d}r+\int_t^s\sigma(r,g_r)\dot\varphi_r\mathrm{d}r\ \ s\in[t,T]\Big\}\quad(3.3.25)$$

for all $\varphi\in C([t,T],\mathbb{R}^d)$

and $(Y_s^{t,\varepsilon})_{t\leq s\leq T}$ obeys a LDP in $C([t,T],\mathbb{R}^k)$ with the good rate function

$$\mathrm{J}\ (\psi)=\inf\Big\{I(\varphi):F(\varphi)=\psi\text{ if }\psi\in H^1([0,T],\mathbb{R}^d)\Big\}\quad(3.3.26)$$

where $F(\varphi)(s)=u(t,\varphi_t)$ for all $\varphi\in C([t,T],\mathbb{R}^d)$

The main property employed to establish the LDP for $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x})_{t\leq s\leq T}$, is $Y_s^{t,\varepsilon,x} = u^\varepsilon(s, X_s^{t,\varepsilon,x})$, that helps us to generalize the LDP recognized in [33] for the FBSDE system decoupled, ie, when $f(t,x,y) = f(t,x)$.

Proof. Since $Y_s^{t,\varepsilon,x} = u^\varepsilon(s, X_s^{t,\varepsilon,x})$, the first equation on the FBSDE (3.3.1) is in the differential form given by (we omit ^{t,x})

$$\begin{cases} dX_s^\varepsilon = \underbrace{f(s, X_s^\varepsilon, u^\varepsilon(s, X_s^\varepsilon))}_{b^\varepsilon(s,X_s^\varepsilon)}\,\mathrm{d}s + \sqrt{\varepsilon}\,\underbrace{\sigma(s, X_s^\varepsilon, u^\varepsilon(s, X_s^\varepsilon))}_{\sigma_1^\varepsilon(s,X_s^\varepsilon)}\,\mathrm{d}Bs;\ t\leq s\leq T \\ X_t^\varepsilon = x \end{cases}$$

(3.3.27)

So we have a typical setting in which we can apply the Freidlin-Wentzell theory results of the section 4 of the second chapter;defining:
$b^\varepsilon : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k \to \mathbb{R}^d$
$b^\varepsilon(t,x) = f(t,x,u^\varepsilon(t,x))$
$\sigma_1^\varepsilon : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k \to \mathbb{R}^{d\times d}$
$\sigma_1^\varepsilon(t,x) = \sigma(t,x,u^\varepsilon(t,x))$ noting that b^ε and σ_a^ε are clearly Lipschitz continuous, with sublinear growth

and (3.3.27) is given by

$$\begin{cases} dX_s^\varepsilon = b^\varepsilon(s, X_s^\varepsilon)\mathrm{d}s + \sqrt{\varepsilon}\sigma_1^\varepsilon(s, X_s^\varepsilon)\mathrm{d}Bs;\ t\leq s\leq T \\ X_t^\varepsilon = x \end{cases}$$

(3.3.28)

Since $\lim_{\varepsilon\to 0} \mid u^\varepsilon(t,x) - u(t,x) \mid = 0$ uniformly in all compact sets of $[0,T]\times\mathbb{R}^d$ as we remarked in the **theorem 3.3.1** before, we are conducted to the obvious conclusion, by the Lipschitz property

$$\lim_{\varepsilon\to 0} \mid b^\varepsilon(t,x) - b(t,x) \mid = \lim_{\varepsilon\to 0} \mid \sigma_1^\varepsilon(t,x) - \sigma_1(t,x) \mid = 0. \qquad (3.3.29)$$

uniformly in all compact sets of $[0,T]\times\mathbb{R}^d$. Here b and σ_1 are the corresponding limit coefficients of (3.3.29) when $\varepsilon = 0$. By the standarb theory of ODEs, since b and σ (considering $\varepsilon = 0$) are Lipschitz continuous, the problem

$$\begin{cases}\dot{g}_s = b(s,g_s)\mathrm{d}s + \sigma(s,g_s)\dot{h}_s; \quad h \in H^1([0,T],\mathbb{R}^d)\\ g_t = x\end{cases}$$
(3.3.30)

has a unique solution in $C([t,T],\mathbb{R}^d)$.

We will denote it $g = S_x(h)$.
In this way, we are defining an operator $S_x : H^1([t,T],\mathbb{R}^d) \to C([t,T],\mathbb{R}^d)$ which is uniformly continuous (ie continuous for the supremum norm) and by (3.3.29), with the fact that $b^\varepsilon, \sigma^\varepsilon, b, \sigma$ are Lipschitz continuous, with sublinear growth, we know, applying the **theorem 2.5.1** , that $(\mathbb{P}\circ(X^\varepsilon)^{-1})_{\varepsilon>0}$ obeys a LDP with the good rate function

$$\mathrm{I}(\varphi) = \inf\Big\{\frac{1}{2}\int_t^T \mid \dot{g}_s \mid^2 \mathrm{d}s : g \in H^1([t,T],\mathbb{R}^d),$$
$$\mathrm{g}_s = x + \int_t^s f(r,g_r,u^\varepsilon(r,g_r))\mathrm{d}r + \int_t^s \sigma(r,g_r)\dot{\varphi}_r\mathrm{d}r \;\; s \in [t,T] \text{if } \; \varphi \in H^1([t,T],\mathbb{R}^d)\Big\}$$
(3.3.31)

In what follows, in order to prove a LDP to $(Y_s^\varepsilon)_{t\le s\le T}$ we consider the following operator:

$$\mathrm{F}^\varepsilon : C([t,T],\mathbb{R}^d) \to C([t,T],\mathbb{R}^k)$$
$$\mathrm{F}^\varepsilon(\varphi)(s) = u^\varepsilon(s,\varphi_s)$$
(3.3.32)

We observe that $Y_s^\varepsilon = F^\varepsilon(X_s^\varepsilon)$ for all $s \in [t,T]$.
To establish a LDP for $(\mathbb{P}\circ(Y^\varepsilon)^{-1})_{\varepsilon>0}$ we want to use the **corollary 2.2.1**, that we presented in chapter 2. It is easy to see that it is sufficient to prove that $(F^\varepsilon)_{\varepsilon>0}$ is a family of continuous functions and that $F^\varepsilon \to F$ in $C([t,T],\mathbb{R}^d)$ in all compact sets.

In order to prove *the continuity of* F^ε*:*

Let $\varepsilon > 0$ and $x \in C([t,T],\mathbb{R}^d)$. We will prove the continuity of the function by means of sequential continuity, since $C([t,T],\mathbb{R}^d), C([t,T],\mathbb{R}^k)$ are metric spaces.Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $C([t,T],\mathbb{R}^d)$ converging to x in the uniform norm. Fix $\delta > 0$. Since $\| x_n - x \|_\infty \to 0$, there exists $M > 0$

such that $\| x_n \|_\infty$, $\| x_n \|_\infty \leq M$.
u^ε is a continuous function in $[0,T] \times \mathbb{R}^d$ (**theorem 3.3.1**) and u^ε is uniformly continuous in $[t,T] \times K$ where $K = \overline{B(0,M)} \subset \mathbb{R}^d$.

There exists $\eta > 0$ such that for $s, s_1 \in [t,T]$ and $z, z_1 \in K$ $\mid s - s_1 \mid < \eta$ and $\mid z - z_1 \mid < \eta$, we have $\mid u^\varepsilon(s,z) - u^\varepsilon(s_1,z_1) \mid < \delta$.

Since $x_n \to x$ in $C([t,T],\mathbb{R}^d)$, fix $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\| x_n - x \|_\infty < \eta$.

For all $r \in [t,T]$ and for all $n \geq n_0$ $x_n(r), x(r) \in K$. and
$\mid u^\varepsilon(r, x(r)) - u^\varepsilon(r, x_n(r)) \mid < \delta$.

So we conclude that $F^\varepsilon(x_n) \to F^\varepsilon(x)$, which proves the continuity of F^ε in the point x.

The next step is to see the uniform convergence, in the compact sets of $C([t,T],\mathbb{R}^d)$ of F^ε, to F when $\varepsilon \to 0$, where $F(\varphi)(s) = u(t,\varphi_t)$ for all $\varphi \in C([t,T],\mathbb{R}^d)$.
Thanks to the point 1 of the **theorem 3.3.1** we see that for all $x \in \mathbb{R}^d$

$\mathbb{E} \sup_{t\leq s\leq T} \mid Y_s^{\varepsilon,t,x} - Y_s^{t,x} \mid^2 + \mathbb{E}\int_t^T \mid Z_s^{\varepsilon,t,x} \mid^2 \mathrm{d}s \to 0.$ (3.3.33)

Consider K a compact set of $C([t,T],\mathbb{R}^d)$ and let
$A := \{\varphi_s : \varphi \in K, s \in [t,T]\}$.
It is obvious that A is a compact set of $\mathbb{R}^d$.
By (3.3.33) above we see that

$\sup_{\varphi\in K} \| F^\varepsilon(\varphi) - F(\varphi) \|_\infty^2 = \sup_{\varphi\in K} \sup_{s\in[t,T]} \mid u^\varepsilon(s,\varphi_s) - u(s,\varphi_s) \mid^2 =$
$\sup_{\varphi\in K} \sup_{s\in[t,T]} \mid Y_s^{\varepsilon,s,\varphi_s} - Y_s^{\varepsilon,s,\varphi_s} \mid^2$
$\leq \sup_{x\in A} \sup_{s\in[t,T]} \mid Y_s^{\varepsilon,s,\varphi_s} - Y_s^{\varepsilon,s,\varphi_s} \mid^2 \to 0$ as $\varepsilon \to 0$ a.s $\mathbb{P}$ (3.3.34)

using the uniform convergence of u^ε to u in the compact sets of $[0,T] \times \mathbb{R}^d$.
So, using the **corollary 2.2.1** we conclude that $(\mathbb{P} \circ (Y^\varepsilon)^{-1})_{\varepsilon>0}$ satisfies a LDP principle with the good rate function

$$J(\psi) = \inf\Big\{ I(\varphi) : F(\varphi) = \psi; \varphi \in H^1([t,T],\mathbb{R}^d) \Big\} \tag{3.3.35}$$

for all $\psi \in C([t,T],\mathbb{R}^k)$

$\square$

Bibliography

[1] Antonelli F., *Backward Forward Stochastic Differential Equations*, Annals of Applied Probability 3 , 777-793, 1993.

[2] Azencott R., *Grandes Déviations et Applications, École d´Été de Probabilitiés de Saint-Flour*, 1-76, Lecture Notes in Mathematics, vol.774, Springer 1980.

[3] Baldi P., Chaleyat-Maurel M., A*n Extension of Ventzell-Freidlin Estimates, Stochastic Analysis and Related Topics*, Silivri 1986, 305-327, Lecture Notes in Math., 1316, Springer, Berlin, 1988.

[4] Barles G., Buckdahn R., Pardoux E., *Backward Stochastic Differential Equations and Integral-Partial Differential Equations*, Stochastics and Stochastics Reports, 60, 57-83, 1997

[5] Bensoussan A., *Stochastic Maximum Principle for Distributed Parameter Systems*, J. Franklin Institute, 315, 387-406, 1983

[6] Bismut J.M., *Théorie Probabilistique du Contrôle des Diffusions*, Mem. Amer. Math. Society, 176, Providence, Rhode Island,1973

[7]Crandall M., Ishii H., Lions P.L. , *User`s Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bull. Amer. Math. Soc. 27, 1-67, 1982

[8] Delarue F., *On the Existence and Uniqueness of Solutions to FBSDEs in a Non Degenerate Case*, Stoch. Processes and their Applications, 99, 209-286, 2002

[9] Dembo A. , Zeitoni O., *Large Deviations Techniques and Applications*,

2nd edition, Jones and Bartlett, 1998

[10] Deuschel J.D., Strook. D., *Large Deviations*, AMS, 2000

[11] Donnati L., Roualt A., Y. M., Zani M., *Large Deviations for Squares of Bessel and Ornstein-Uhlenbeck Processes*, Prob. Theory and Related Fields, 129(2), 261-289, 2004

[12] Doss H. , Rainero S., *Sur l´Existence, l´Unicité, la Stabilité et les Proprietétés de Grandes Déviations des Solutions d´Équations Différentielles Stochastiques Rétrogrades à Horizon Aléatoire. Application à des Problèmes de Perturbations Singulières*, Bull. Sciences Math. 13, 99-174, 2007

[13] Evans, L. C, *Partial Differential Equations*, AMS, 1998

[14] Fleming H., Soner H. M., *Controlled Markov Processes and Viscosity Solutions*, Applications of Mathematics 25, Springer-Verlag, NY, 1993

[15] Freidlin M. , Wentzell D.A., *Random Perturbations of Dynamical Systems*, 2nd ed., Springer, 1998

[16] Gentz B., *Random Perturbations of Dynamical Systems, Lecture Notes,* MPI Leipzig, 2003 (see http://www.math.uni-bielefeld.de/ gentz/)

[17] Hu Y., Peng S., *Adapted Solutions of Backward Stochastic Equations*, Stoch. Analysis and Applications,9, 445-459, 1991

[18] Hu Y., Peng S., *Solution of Forward-Backward Stochastic Differential Equations*, Prob. Theory and Related Fields, 103, 273-283, 1995

[19] Karatzas I., Schreve S. E., *Brownian Motion and Stochastic Calculus*, 2nd edition, 2005, GTM, Springer.

[20] Knapp, Anthony W., *Basic Real analysis*, Birkhauser, 2005

[21] Kuo H., *Introduction to Stochastic Integration*,Universitext, Springer Verlag, 2006

[22] Ladyzhenskaja O.A., Solonnikov V.A., Ural´ceva N.N., *Linear and Quasi-*

linear Equations of Parabolic Type, AMS, Providence, RI, 1968

[23] Lions J.L, *Quelques Méthodes de Résolution aux Problèmes Non Linéaires*, Dunod, 2002

[24] Ma J., Protter P., Young J., *Solving Forward-Backward Stochastic Differential Equations Explicitly - a Four Step Scheme*, Probability Theory and Related Fields, 98, 339-359

[25] Ma J., Young J., *Forward-Backward Stochastic Differential Equations*, Lecture Notes in Mathematics, Springer, 2007

[26] Nualart D., *Malliavin Calculus and Related Topics*,Probability and Related Topics, Springer, 2010

[27] Oksendal B., *Stochastic Differential Equations: An Introduction with Applications*, Universitext, 6th ed, 2010, Springer

[28] Pardoux E., *Backward Stochastic Differential Equations and Viscosity Solutions of Systems of Semilinear Parabolic and Elliptic PDEs of 2nd order*, Stoch. Analysis and Related Topics, The Geilo Workshop, 79-127, Birkhauser,1996

[29] Pardoux E., Tang S., *Forward Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations*, Prob. Theory and Related Fields, 114, 123-150,1999

[30] Peng S., *Probabilistic Interpretation for Systems of Semilinear Parabolic PDEs*, Stochastics and Stochastic Reports, 37, 61-74, 1991

[31] Peng S., *A Generalized Dynamic Programming Principle and Hamilton-Jacobi-Bellman Equation*, Stochastics and Stochastics Reports, 38, 119-134, 1992

[32] Prioret P., *Remarques sur les Petites Perturbations de Systèmes Dynamiques*, Seminar on Probabilités, XVI, 920, 184-200, Lecture Notes in Mathematics, Springer Berlin 1982

[33] Rainero S., *Un Principe de Grandes Déviations pour une Equation Dif-*

férentielle Stochastique Progressive-Rétrógrade, Comptes Rendus Mathematiques, Acad. Sci. Paris, I-343, 141-144, 2006

[34] Rogers L. C. G., Williams D., *Diffusions, Markov Processes and Martingales: Itô Calculus*, Wiley, UK, 1987

[35] Schilder M., *Some Assymptotic Formulas for Wiener Integrals*, Transl. AMS, 125, 63-85, 1966

[36] Strook D., *An Introduction to Large Deviations Theory*, Springer, 1984

[37] Ton, B.A., *Non Stationary Burger Flows with Vanishing Viscosity in Bounded Domains in* $\mathbb{R}^3$, Math. Z., 145, 69-79, 1975

[38] Varadhan S.R.S., *Large Deviations and Applications*, Society for Industrial and Applied Mathematics SIAM, 1984