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Introducing Choice Sequences Into Mathematical Ontology

Josiano Cláudio Oliveira Nereu

MESTRADO EM FILOSOFIA

(Epistemologia e Metafísica)

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Orientadores: Prof. Dr. Fernando Ferreira e Prof. Dr. Pedro M. S. Alves

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Dedico este trabalho aos meus pais Clara e Manuel, às minhas tias Elisabete e Luísa e à minha irmã Liliana.

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INTRODUCING CHOICE SEQUENCES INTO MATHEMATICAL ONTOLOGY

Josiano Nereu

September 29, 2011

Resumo

A ideia de objectos matemáticos que estão em permanente desenvolvimento no tempo foi pela primeira vez avançada por L.E.J. Brouwer. Na matemática intuicionista estes objectos são concebidos como sequência infinitas de números naturais que em qualquer estágio do seu crescimento têm apenas um número finito de valores, além disso, tais valores podem ser livremente escolhidos, no sentido em que a sua produção não necessita de ser determinada por nenhuma regra matemática definida. Tais objectos são denominados de *sequências de escolha*. O presente trabalho tem como objectivo fornecer uma resposta à seguinte questão: são as sequências de escolha legítimos objectos matemáticos? A resposta que iremos propor e à qual iremos argumentar favoravelmente é a seguinte: tais objectos não podem ser considerados objectos matemáticos legítimos. Com esta tese em vista, iremos discutir as propriedades intrínsecas das sequências de escolha relativamente à maneira como são incorporadas no contexto matemático e as suas implicações. Seguindo esta metodologia pretendemos atingir um cabal entendimento filosófico das consequências em que incorremos ao aceitarmos sequências de escolha como objectos da ontologia matemática e das razões que temos para não as aceitarmos como tal.

Abstract

The idea of mathematical objects which are in a permanent state of growth in time was by the first time defended by L.E.J. Brouwer. In intuitionistic mathematics these objects are conceived as infinite sequences of natural numbers that at any stage of growth have only finitely many values defined. Additionally, these values may be freely chosen, in the sense that their generation has not to follow any determinate mathematical rule. These objects are called *choice sequences*. The present work aims at providing the answer to the following question: are choice sequences legitimate mathematical objects? The answer that we will propose and argue for is a negative one: that they cannot be considered legitimate mathematical objects. In order to do this we will discuss the intrinsic features of choice sequences concerning the way they are incorporated into a mathematical framework and their implications. Following this methodology we expect to achieve a good philosophical understanding of the consequences of accepting choice sequences into our mathematical ontology and of the reasons that we have not to accept them as such.

Contents

Introduction

In the beginning of the 20^{th} century three main schools in the Philosophy of Mathematics emerged to deal with the recent discovered paradoxes into the foundations os mathematics at the time. Hilbert's Formalism, Frege and Russell's Logicism and Brouwer's Intuitionism. Hilbert's formalistic programme, whose objective was to prove that all of mathematics was a conservative extention of consistent axiomatic theories, was refuted by the inconsistency theorems due to Kurt Gödel. Frege's Logicism, whose objective was to show that the concept of natural number was founded on logical concepts, was also refuted by Russell's paradox due to Bertrand Russell. And Russell's Logicism, whose objective was the same as Frege's, proved incapable of justifying the axioms of mathematics only through logical principles. From the three philosophical schools enunciated above only Brouwer's Intuitionism remained practically intact untill nowadays.

One of the main reasons that established Intuitionism as a valid account for mathematics untill nowadays is that Intuitionism is not concerned with the foundations of classical mathematics. Intuitionistic mathematics was developed without the concern whether its results happen to coincide to classical mathematical ones and, more important, it was not developed to stand as a suitable foundation for classical mathematics. The main concern of intuitionistic mathematics was that the resultant mathematics was to be viewed as the *right* mathematics. For this reason, many of the principles and methods of classical mathematics proved to be intuitionistically invalid. The principle of excluded middle is the most proeminent classical principle that is not valid on intuitionistic grounds. But there are many other classical facts of mathematics that also proved to be intuitionistically false or just untenable.

It is widely accepted by philosophers of mathematics that the fundamental reason that led intuitionistic mathematics to be inconsistent with classical mathematics is due to its particular ontology. Like classical mathematics, intuitionism deals with infinite collections. But the kind of infinite objects that pertain to intuitionism do not fit classical theories. When Brouwer started (around 1913) with his intuitionistic reconstruction of the theory of the *continuum*, he found that the notion of *choice sequence* could be regarded as a legitimate intuitionistic notion, and as a means of retaining the advantages of an arithmetic theory of the *continuum*. However, the notion of choice sequence is not a simple one. It has particularities that lead some mathematicians and philosophers to regard it as a dubious mathematical notion.

A *choice sequence* α of natural numbers may be viewed as an unfinished, ongoing process of choosing values $\alpha(0), \alpha(1), \alpha(2), ...$ by an ideal mathematician. At any stage of his activity the ideal mathematician has determined only finitely many values, plus, possibly, some restrictions on future choices. Choice sequences are unfinished objects in the sense that they are permanently growing thus being *potential* infinite objects. This feature of choice sequences is persistently pointed as one of the reasons to reject choice sequences as legitimate mathematical objects. But there are other properties of choice sequences that motivate the same conclusion. Choice sequences are time-dependent objects since they are at permanent state of growth and hence they cannot be seen as finished objects like classical mathematical objects. But despite the fact that choice sequences are obbjects of a peculiar sort, their use in mathematics have advantages over the classical ones. In particular, they allow us to mathematically model the *continuum* and its properties in a more natural and intuitive way than the way it is conceived in classical mathematics. In fact, say the proponent of choice sequences as mathematical objects, this is the main reason to introduce choice sequences into our mathematical ontology. In what follows we will discuss all these (and more) properties of choice sequences. We will try to provide a detailed discussion of the pros and cons of accepting choice sequences as legitimate objects of mathematical theories.

In first chapter we provide a detailed characterization of choice sequences and of the way how they can be incorporated into a mathematical framework. We also discuss the motivation for introducing them into our mathematical ontology. Some major results of intuitionistic real analysis are presented to illustrate how the *continuum* can be intuitively built using choice sequences. At this point a fundamental principle governing choice sequences is presented and the relation between it and the previous results is established. This principle is the *Weak Continuity Principle*. We also present a very important consequence of this principle in which a direct consequences of the law of excluded middle is refuted by it. After this is done we turn to the logical aspects of choice sequences. A particular emphasis is given to that fact that this principle makes impossible to use classical Logic to deal with choice sequences because it refutes some classical principles. Therefore, a new logic is needed. This new logic is the intuitionistic Logic, which does not have Double Negation as a universally valid law. The reason lies in the interpretation of the meaning of intuitionistic logical operators, which have a non-classical or *contructive* meaning. After all, choice sequences are meant to be contructive, non-classical objects. Due to the constructive character of choice sequences we find important to establish the link between choice sequences and the notion of a *constructive proof*. We will find that this relation is not unproblematic. Then we reach the high point of the first chapter: the problems of identity and individuation for choice sequence. Contrary to classical objects, it is problematic to serch for individuation criteria for choice sequences. And this happens because it is not also clear how do we establish identity for them (in particular for a specific class of choice sequences, the *lawless* sequences). We conclude this paragraph discussing whether the use of choice sequences allow us to conclude that a new *paradigm* of mathematical reasoning is found, a paradigm dealing with potential infinite objects that do not exist in the classical ontology of mathematics.

In second chapter our aim is to present three major accounts of choice sequences. Brouwer's philosophical context in which he introduces them into mathematical reasoning, Troelstra's axiomatic treatment of lawless sequences and Weyl's *quasi-eliminativist* account of quantification over choice sequences. We start by presenting the original setting were choice sequences, for the first time, were introduced as possible mathematical objects: the semi-intuitionistic conception of Borel (who rejected them as legitimate mathematical objects). Then it is shown how Brouwer, contrary to Borel and despite his criticism, reintroduces choice sequences as legitimate objects of intuitionistic mathematics (and not as classically definable objects, as Borel had thinked of). Brouwer's philosophy of mathematics has two distinctive (chronological) moments, which are called the *Two acts of intuitionism*. We discuss his conception of mathematics, whose principles, concepts and methods arise as consequences of the mind free creating activity. A special emphasis is given to the second act, where he recognizes choice sequences as mathematical objects and proposes the weak continuity principle as the fundamental principle governing them. After this is done we turn to the axiomatic account of choice sequences where the axioms for lawless sequences are formulated and motivated by Troelstra (and Kreisel). The theory buitl is called **LS**. We present and discuss individually in some detail the four axioms of **LS**. We also discuss the possible interpretations of the interesting *Elimination theorem* proved within **LS**. Finally, and motivated by the general case of the eliminativist interpretation of the Elimination theorem, we present Weyl's account of choice sequences and discuss some of his claims about how we should deal with *individual* choice sequences. This chapter is completed with a critical discussion of the weak continuity principle, whose philosophical justification is not clear. We also show that none of the three acconts presented constitute a completely satisfactorily account for choice sequences.

Due to the incomplete character of the previous accounts for choice sequences we turn to the recent developed *phenomenology* of choice sequences. The phenomenological setting of choice sequences is chiefly proposed and developed by Mark van Atten. And the third chapter is totally devoted to present it in some detail. We start by establishing the connection between Intuitionism and Phenomenology. The main purpose of van Atten is to show that Intuitionism and Phenomenology are convergent in the sense that Brouwer's theses concerning the ontology of mathematics (which are philosophically deficient) have sufficient aspects in commom such that Brouwer's intuitionism can be incorporated into and justified by Husserl's transcendental phenomenology. In particular, van Atten argues that choice sequences can be phenomenologically *constituted* as individual mathematical objects. And that this fact justifies the introduction of choice sequences into our mathematical ontology. There are, however, aspects of Husserl's personal view on the foundations of mathematics that do not agree with Brouwer's philosophy. Van Atten's try to surpass such problems showing that some of Husserl's claims are not phenomenologically grounded as they were supposed to be. We finish this chapter by presenting van Atten's phenomenologically-based argument in favour of weak continuity principle for choice sequences.

The last two chapters are devoted to present our arguments against the claim that choice sequences are legitimate mathematical objects. In fourth chapter we develop a critical discussion of van Atten's phenomenology of choice sequences. In particular we develop a set of arguments whose objective is to show that the class of lawless (choice) sequences cannot be constituted as *individual* mathematical objects. In the last chapter we start by discussing some aspects of the non-eliminativist interpretation of the elimination theroem, with emphasis on van Atten's arguments. Then, we advance our ouw view on choice sequences as mathematical objects. Our view is close to Weyl's quasi-eliminativist interpretation, in which existential quantification over lawless sequences is to paraphrased into quantification over unproblematic individual mathematical objects and universal quantification is interpreted as a way of referring to choice sequences as a notion. For this, we start by discussing Weyl's own account of choice sequences in some detail, establishing an important parallelism concening the way Weyl and Husserl conceive mathematical existence. Then, we discuss a technical result that, we think, represents the thesis we advocate.

Chapter 1

Choice sequences and their implications

1.1 What are choice sequences?

Choice sequences are potentially infinite sequences of mathematical objects, chosen one after the other; in the process of choosing, free choices may play a part.[1] Imagine that we have at our disposal a collection of mathematical objects, e.g. the collection of natural numbers. Make a choice, pick a number, any one, and see the result. Put it back into the collection. Choose a number again, see the result and put it back into the collection. Now let's see the resulting sequence of these two choices. Maybe the result has been $\langle 5, 12 \rangle$. Making further choices, we can arrive at $\langle 5, 12, 45, 2, 32 \rangle$ and we can continue from here. *A choice sequence is what we get if we think this process of choosing as never ending, or, as potentially infinite*. The potential infinity of this sequence is indicated by the three dots, $\langle 5, 12, 45, 2, 32, ... \rangle$. The first two sequences given above are *initial segments* of the choice sequence and they are always finite sequences. In fact, we cannot make an actually infinite number of choices, i.e., at any stage of growth of a choice sequence we just have a finite sequence. However, we can always extend an initial segment by making a further choice. (*OnB*, p. 30.)

Information about a particular choice sequence is divided into *extensional* information and *intensional* information. When a choice sequence is given to us we have access to an initial segment of the sequence this is the extensional information about the choice sequence; but we also have access, for example, to the *process* by which it is generated and to possible *restrictions* it is subjected to in each stage, this is the intensional information of the choice sequence. (*OnB*, pp. 33-34) Let's take into consideration the following choice sequence, $\alpha = \langle 0, 2, 4, 6, 8, 10, ... \rangle$. As extensional information of α we have the initial segment $\langle 0, 2, 4, 6, 8, 10 \rangle$ given to us; as intensional information of α we have the law or rule '$2n$' that generates it and the possible restriction that after the 100^{th} choice, all choices has the result 1, i.e. that for any $n > 100$, it is the case that $\alpha(n) = 1$.

The collection of all choice sequences is constitued by different kinds of

[1] Mark van Atten, *On Brouwer*, Wadsworth, Belmont, 2004, p. 6. From now on quoted as '*OnB*'.

choice sequences. There is a *spectrum* in which the extremes are constitued by choices sequences called *lawlike* and the ones called *lawless*. And between the oposite sides there are many possible combinations of choice sequences. (*OnB*, p. 33) Lawlike sequences are those whose process of generation is totally determined in advance by a finite law or rule. Given this law, it is always possible to *calculate* the next element of the sequence in question. Consider the choice sequence given by the process $n + 1$. Given any initial segment of the resulting sequence, say, $\langle 1, 2, 3, 4, 5, ... \rangle$ we can make a calculation that presents to us the next element of the sequence, $\langle 1, 2, 3, 4, 5, 6, ... \rangle$. The idea of a lawlike sequence is that of a choice sequence *restricted* to a law at any stage of the process, i.e., generated by a process of choosing elements of a domain in a way that does not involve any kind of arbitrariness from one stage of the process to the next one. Lawlike sequences are analogous (and to some extend, equivalent[2]) to *recursive* functions in classical mathematics, with the aditional clause that they are not finished objects like in classical mathematics.

Lawless sequences are those choice sequences for which no definable law or restriction (in a strong sense) is present in the basis of their process of generation, i.e., sequences for which we don't have any specific law to calculate the next element of a given initial segment. About lawless sequences it is often said that they are *freely* generated sequences. Lawless (choice) sequences are also known as 'free choice sequences', since they are generated by the *free will* of the individual subject who generates them. The choice sequence given above is an example of a lawless sequence, since it has no law assigned to it and there is no restriction to follow from one stage of generation to the next; we have choosen the elements of it in a random or arbitrary way. Contrary to lawlike, lawless sequences don't have an analogous notion in classical mathematics nor does any clasical concept remotely resemble it.

Between lawlike and lawless sequences there exists a myriad of possible combinations. For example, the sequence given by the following process: if the input is an even natural number, then the sequence is totaly free and if the input is an odd number, then the sequence is determined by a law. This sequence is neither lawlike nor lawless. The most proeminent combination, however, is the type of choice sequences known as *hesitant* sequences. A hesitant sequence is a process of choosing elements of a domain such that at any stage we either decide to conform to a law in determining future values, or, if we have not already decided to conform to a law at any earlier stage, we freely choose a new element of the domain.[3] Thus at any stage of the process of generation of a hesitant sequence we can decide to *transform* it into a lawlike sequence or into a lawless sequence. But there are other possible combinations; the number of possible combinations is endless.

Let's make a distinction between *first-order* rules and *second-order* rules (or equivalently, *first-order* and *second-order* restrictions). The first-order rules are applied directly upon the generation of the elements of the sequence and the second-order rules are applied to the first-order rules. A first-order

[2]An intuitionist might object to this claim: it would possibly say that the only prescription for a sequence to be lawlike is that its values are generated by a law *in a broad sense*, leaving open the precise sense or definition of 'lawlikeness', thus leaving open also the hypothesis that lawlikeness would concide with recursiveness.

[3]Troelstra & van Dalen, *Constructivism in Mathematics: an Introduction*, vol. I, North-Holland, Amsterdam, 1988, p. 208. From now on quoted as '*CinM I*'.

rule/restriction on a choice sequence α is, for example, to determine that after the k^{th} choice the elements of α are always 0. But we may constrain the generation of a choice sequence α by not allowing any restrictions on the choices of its values; this is a second-order restriction: 'no first-order restrictions are allowed upon α'. Notice that this second-order restriction defines precisely the lawless sequences. So even a lawless sequence is subjected to a restriction, but one of second order, not first. (*OnB*, p. 34)

However, if we accept general second-order restrictions upon choice sequences, a possible tension is raised. If we accept (second-order) restrictions like the above mentioned one, how can we say that some choice sequences (the lawless ones) are *totally free* generated? After all, the restriction that no first-order restrictions are to be allowed is a restriction.

1.2 The distinctive features of choice sequences

The first main feature of choice sequences, as seen above, is that they are *potentially infinite* objects. The potential infinity of choice sequences indicates that the process of choosing the next element of the sequence never ends, i.e., that the sequence is in a state of permanent growth. This is how the concept of 'choice sequence' encapsulates the notion of *potential* infinite as oposed to the classical notion of *completed* or *actual* infinite. The 'potentiality' of choice sequences is a consequence of a more general property that choice sequences possess: they are *dynamic* objects, i.e., objects that at some moment have parts added on them. Classic mathematical objects are then considered as being *static* objects.[4] The potentiality of choice sequences is the property that at some moment of their growth, in fact constantly, they have parts added to them (but not removed). Often, in philosophical literature, the term 'potential' has different meanings attributed to it. In particular, there is an interpretation (the more common) that equates the term 'potential' with 'possible'; under this interpretation, potential objects are objects whose existence is possible, opposed to those objects whose existence is a comprovated fact. Notice, however, that in discussing the properties of choice sequences, we are here using the term 'potential' in a specific sense. By saying that choice sequences are potential objects we do not mean to say that they are mere possible objects. What is intended to say is that they are unfinished objects. Choice sequences are potential in the sense that they are in a permanent state of growth, and they cannot be considered completed objects.

Another main feature of choice sequences is their *intensional* character. What is it that we have in mind when talking about the intensional character of choice sequences? What we do have in mind is how choice sequences are given to us, how a choice sequence comes to existence. In classical mathematics an object is defined by its extensional properties: a set is determined by the totality of its elements (Axiom of extensionality). So, in classical mathematics, the way an object is determined is always by *extensional identity*. Choice sequences however are determined by *intensional identity*, i.e., a choice sequence is given to us by specifying a *construction* process for it, a process that produces (in a strong sense) its elements. The primary notion of identity for choice sequences

[4]Mark van Atten, *Brouwer meets Husserl: The Phenomenology of Choice Sequences*, Springer, Dordrecht, 2007, p. 16. From now on quoted as '*BmH*'.

is that of intensional identity: if two choice sequences are given by the same construction they are intensionally identical, otherwise different, even though in some relevant ways they may still be the same (e.g., they may have the same outputs). (*OnB*, p. 2) Thus, the specific intensional character of choice sequences refers to the fact that they are objects whose identity is determined by, or depends on, the way they are given. Choice sequences are highly intensional objects in the sense just explained: their identity is determined through their mode of presentation, not through their 'elements'.

When we say that choice sequences are given by specifying a construction for them, we have to ask *who* specifies such a construction. According to the choice sequence proponent, choice sequences depend on individual choices upon the mathematical domain we have in advance, normally the domain of natural numbers which is viewed as primitive. A choice sequence is constructed by picking out elements of the the domain of natural numbers; we can decide to pick them in a determined and mechanical way (the case of lawlike sequences) or we can decide to pick them in a somehow arbitrary way (the case of lawless sequences). When this second situation happens, we say that we *freely* choose the elements of the domain. But who make the choices? The choices, says the proponent choice sequences, are always made by a *virtual subject*, an idealization of real subjects; it is he who carries the construction of a particular choice sequence. In other words, an individual choice sequence has all of its properties solely in virtue of a particular subject's decision to construct it in a certain way. (*OnB*, p. 34) This subject is an idealization of real mathematicians and was called by Brouwer the *creating subject* ('*scheppend subject*' in the original ducth). There are costs, however. The alleged introduction of a *subjective* character in Mathematics that is absent from classical mathematics. Classically, subjective choices in constructing mathematical objects do not play any role, because in classical mathematics objects are not constructed by any subject, even an idealized one. In classical mathematics the particular subjects, the real mathematicians, just grasp or discover the objects that are already existent in the *mathematical world*. These objects are not constructed in the sense that they come into existence by a *mental* operation carried and started by the idealized mathematician, as choice sequences are.

This leads us to the most intrincated feature of choice sequences, their *temporality*. Choice sequences are not objects that exists irrespectively of time. They start to exist at the moment the process that generates them starts to operate; they have a beginning in time. But they also evolve through time. Since the moment they start to exist, choice sequences do not remain the same, they growth with time: at each moment (or stage) a new choice plays a part and this process does not have an end. So, choice sequences are temporal objects in two different ways: they have a beginning at a specific moment of time and they evolve (grow) through time. These temporal aspects of choice sequences makes them the dynamic objects they are.

1.3 Motivation for choice sequences

At the beginning of the 20^{th} century the notion of *continuum infinite*, after several centuries of obscurity, achieved finally its state of conceptual clarity within Mathematics. For instance, the continuum can be defined as an infinite collection whose elements are so-called Dedekind cuts.[5] Note that Dedekind cuts are infinite sets (i.e., each real is an infinite set). Cantor's Set Theory made all these notions mathematically rigorous. For example, it made more clear the distinction beetwen the set of rationals and the set of reals: the rationals are *countable* (they are in a one-one relation with the natural numbers), the reals are not (the reals are more in number then the natural numbers); the set of real numbers is closed under convergence (any Cauchy sequence[6] of real numbers converges to a real number), the set of rational numbers is not (there are Cauchy sequences of rational numbers that do not converge to any rational number, e.g., the rational sequences that converge to π or $\sqrt{2}$). But more important of all, it established the property that essentially distinguishes between the reals and the rationals: the set of real numbers respects the *least upper bound principle*. Further, Cantor's set theory with its theory of transfinite cardinality established that the set of real numbers has a cardinality that surpasses the cardinality of the set of natural numbers.[7]

Though Brouwer, the father of Intuitionism both as a philosophy and as mathematics, had embraced the drift to a higher abstractionism performed in the beginning of the 20th century, he did not agree however with Cantor's analysis of the continuum. The problem was not in Cantor's use of notions like infinite sets or sequences, but in the fact that they could be considered as *actual* or *finished* infinite sets or sequences. For Brouwer, the notion of an actual infinite totality makes no sense at all. Actual totalities can only be of a finite character. For him, infinite totalities are always potential or unfinished. In the notion of an *actual infinite totality* lies the intrinsic platonist character of classical mathematics.

Brouwer, because of his repudiation of the platonist character of classical mathematics, was the first to show how to rectify this situation. A legitimate infinite object had to be given by a process (law or rule). But this process had not to be entirely *deterministic*, i.e., in such a way that, for instance, only objects determinated by computable functions were to be allowed. If this were the case, Brouwer would be stuck to the *reduced continuum*, i.e., the continuum made only of lawlike sequences which is only countable (and hence of measure zero). Brouwer realized that such process had to accommodate some *freedom* or *flexibility*. He allows some flexibility on the process of choosing the elements of infinite sequences because he wanted to retain both fundamental ideas which go to form the classical conception of the continuum, admitting not only infinite sequences determined in advance by an effective rule for computing their terms (lawlike sequences), but also sequences which admit free selection of their elements (lawless sequences).[8] Choice sequences are the prototype and the paradigm of such infinite objects. In intuitionist mathematics, as advanced by the first time by Brouwer, the continuum is constructed on the basis of such objects: certain choice sequences of nested intervals of rational numbers.[9]

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[5] A Dedekind cut is an initial segment from the set of rationals such that: it is not empty, it has an upper bound and it has not a maximal element.

[6] A sequence $(x_n)_n$ of real numbers is a Cauchy seqquence iff $\forall \epsilon > 0 \exists r \forall n, m > r \, |x_n - x_m| < \epsilon$.

[7] See Carl Posy, 'Intuitionism and Philosophy' in *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford University Press, New York, 2005, p. 321.

[8] Michael Dummett, *Elements of Intutionism*, 2nd edition, Clarendon Press, Oxford, 2000, p. 45. From now on quoted as '*EI*'.

[9] See *OnB*, pp. 1-2; Carl Posy, 'Intuitionism and Philosophy', in *The Oxford Handbook of Philosophy*

For Brouwer, choice sequences are the paradigmatic objects for constructing the continuum because they have the unfinished or potential character of any legitimate infinite magnitude.

1.4 Some logico-mathematical consequences of choice sequences

Intuitionist mathematics contains a series of interesting as unusual results (theorems). All the results presented in this section are directly or indirectly consequences of the introduction of choice sequences into Mathematical reasoning and of the principles that rule them, most notably the *Weak Continuity Principle*. Some of these results have a more mathematical character and others a more logical one, but all of them are corollaries of the use of choice sequences as, for example, the basis by which the real numbers are defined or simply by being a logical domain of quantification.[10]

Definition: A function $f : A \subset \mathbb{R} \longmapsto \mathbb{R}$ is **continuous** at a point $a \in A$ iff $\lim_{x \longrightarrow a} f(x) = f(a)$, i.e., iff

$$\forall \delta > 0 \exists \epsilon > 0 \forall x \in A(|x - a| < \epsilon \rightarrow \left|f(x) - f(a)\right| < \delta);$$

f is **uniformly** continuous on A iff

$$\forall \delta > 0 \exists \epsilon > 0 \forall x, y \in A(\left|x - y\right| < \epsilon \rightarrow \left|f(x) - f(y)\right| < \delta).$$

In other words, a function f from a set A to a set B is **continuous at a point** $a \in A$ iff, for every sequence $\alpha_n \in A$ converging to a, $f(\alpha_n)$ converges to $f(a)$; and f is **uniformly** continuos when its continuity is registered by a rule that calculates uniformly from an open interval i around $f(a)$, an interval j around a that f maps into i. From these definitions applied to intuitionistic real analysis we get the following results:

Proposition 1 (**Brouwer's Theorem**). *Every function mapping of the closed unit interval* $[0, 1]$ *into* $\mathbb{R}$ *is uniformly continuous.*

From the truth of Brouwer's Theorem, one can infer the following:

Proposition 2 (**Continuity Theorem**). *Every total function from* $\mathbb{R}$ *to* $\mathbb{R}$ *is continuous.*[11]

Another quite interesting result of the intuitionist theory of real numbers that follows from the Continuity Theorem is the following one:

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of Mathematics and Logic, pp. 321-322; A. S. Troelstra & D. Van Dalen, *CinM II*, pp. 639-641. For a detailed exposition of the construction of the reals by choice sequences see A. S. Troelstra & D. Van Dalen, *CinM I*, ch.5; see also M. Dummett, *EI*, ch.2 and D. Vellemman & A. George, Blackwell Publishers, Massachusetts, 2002, ch. 5.

[10]For a sinoptic and non-technical presentation of these results see D.C. McCarty, 'Intuitionism in Mathematics' in *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp.365-369.

[11]In Brouwer's exposition he presents **Proposition 2** as a corollary of **Proposition 1**. In fact **Proposition 2** is a consequence of **Proposition 1**, but it can be derived without reference to it. The proof for **Proposition 1** uses two important principles of intuitionistic mathematics: WC-N and Fan Theorem, while for the proof of **Proposition 2** we only need WC-N (show by Wim Veldman). (We will explore the implications of WC-N at the next chapter; Fan Theorem will not be discussed by us in this essay.)

Proposition 3 (**Unsplitability of the continuum**). *The continuum cannot be divided into two non-trivial subsets, i.e. if* $\mathbb{R} = A \cup B$ *and* $A \cap B = \emptyset$*, then* $A = \mathbb{R}$ *or* $B = \mathbb{R}$.

This property of the intuitionist continuum is called 'viscosity' of the continuum. This terminology is inspired in Aristotle's analysis of the continuum, who initiated an image of fluid motion that remainded the hallmark of all continuous magnitudes for millennia. Aristotle's conceptual analysis of the continuum and his accompanying image of viscosity (we can't cut the continuous medium) gained precision in the topological notion of 'unsplitability' or connectivity. It is a property that characterizes the intuitionistic continuum but that, according to intuitionists, is lost in the classical version of the continuum.[12] Of course, this fact is an important (not mentioned before) motivation for the introduction of choice sequences into mathematical ontology, precisely because they reintroduce into real analysis the non-discreteness of reals that is lost in classical analysis of the reals.

These theorems of the intuitionistic real analysis are false in the classical real analysis. In classical mathematics a function whose domain is $\mathbb{R}$ is not always continuous in $\mathbb{R}$, it can be discontinuous. For the same reason, in classical mathematics, the continuum is separable in two (or more) non-trivial subsets.

Still in the domain of the real numbers, we have the following property of the continuum, which follows from the unsplitability of the continuum:

Proposition 4 (**Equality is indecidable on** $\mathbb{R}$). *It is not the case that for all* $a, b \in \mathbb{R}$*, either* $a = b$ *or* $a \neq b$.

These results illustrate the radical consequences of the introduction of choice sequences into mathematical ontology. All these results are possible due to the so called *Weak Continuity Principle* (WC-N). The Weak Continuity Principle is the most proeminent principle that rules the behavior of choice sequences and one of the main axioms of the Axiomatic Theory of Choice Sequences. Without WC-N, the notion of choice sequence would be just an intelectual curiosity, without any mathematical interest. WC-N is what makes possible to talk about choice sequences in a mathematical context. Consider a two-place predicate $A(\alpha, x)$, where α ranges over choice sequences and x over natural numbers; let's also assume that A is an extensional property.[13] Then the formulation of WC-N runs as follows:

WC-N: $\forall\alpha\exists x A(\alpha, x) \rightarrow \forall\alpha\exists n\exists x\forall\beta(\bar{\beta}n = \bar{\alpha}n \rightarrow A(\beta, x))$,

where α and β range over choice sequences, n and x over natural numbers and $\bar{\alpha}n$, $\bar{\beta}n$ stand for the initial segments of α and β, respectively, with length n. The principle says that if we have a construction (rule or law) that assigns a number to each choice sequence, all the information it needs to do so is an initial segment of that sequence. More information (intensional properties) may be given, but is not required. (*OnB*, p. 103) Notwithstanding the fact that choice sequences are highly intensional objects, WC-N makes possible, according to intuitionism, to use just extensional information about choice sequences (initial

[12]See D.C. McCarty, 'Intuitionism in Mathematics' in *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp.345-347.

[13]The rigorous definition is: $\forall\alpha, \beta(\forall n(\alpha n = \beta n) \wedge A(\beta, n) \rightarrow A(\alpha, n))$ (see *EI*, p. 46).

segments) to talk about them, in particular, to establish when two choice sequences have the same extensional property or not. (We will talk more closely of WC-N later in a section devoted to it.)

WC-N does not allows us to prove just the results presented above, it allows to prove something much more drastic: WC-N allows us to refute a direct consequence of the *Principle of the Excluded Middle* (PEM). PEM is the logical law that asserts the following:

PEM: For any (mathematical) proposition P, $P \vee \neg P$; i.e., for any proposition P, P is true or the negation of P is true.

This principle is the cornerstone of classical mathematics and, according to intuitionists, the full manifestation of the platonistic character of classical mathematics that they repudiate. The direct consequence of PEM that WC-N refutes is ∀-PEM, which says the following:

∀-PEM: $\forall\alpha(\forall n(\alpha n = 0) \vee \neg\forall n(\alpha n = 0))$,

where α is ranging over choice sequences and n over natural numbers. ∀-PEM is a direct consequence of PEM in the following way: we just have to put $\forall n(\alpha n = 0)$ in place of P, for P stands for any mathematical proposition, and quantify over α, a trivial logical operation. This principle states that for any choice sequence α, we can decide either for all n $\alpha n = 0$ or not. In other words, ∀-PEM states that the property 'to be the null sequence' (=having all values equal to 0) is generally *decidable* in the domain of choice sequences.

Given the explanation above of the concepts in question we can now express the following result:

Proposition 5 (**WC-N refutes ∀-PEM**). *WC-N* $\Rightarrow \neg$*∀-PEM.*

This theorem states that, assuming WC-N as being true, we can prove that ∀-PEM is false. That is, if we accept the introduction of choice sequences in our mathematical ontology and, along with them, WC-N as a principle that rules their behavior, then we cannot accept general decidability for sentences about choice sequences as being valid in all cases. So, in the domain of choice sequences, *tertium non datur* (**PEM**) has not the universal validity that characterizes its presence in classical mathematics.

1.5 The proper logic for choice sequences

As we have seen above, the law of excluded middle does not hold for the domain of choice sequences. It is not generally true that for every property A of an arbitrary choice sequence α, $A\alpha$ or $\neg A\alpha$. In particular, if we define $A\alpha$ as $\forall n(\alpha n = 0)$, we cannot prove the disjunct $\forall n(\alpha n = 0) \vee \neg\forall n(\alpha n = 0)$. This result shows that an essential law of classical logic is inadequate to deal with objects like choice sequences and their properties. Let's see why this happens.

When we are dealing with classical objects like sets, for example, we are dealing with a sort of objects whose behavior is rather different from choice sequences. Sets are finished objects. All properties that a particular set has are established once and for all, that is, sets are not objects that can gain new properties through time like choice sequences. So, in principle, all we need to settle for a particular set x to have a property A is to show that the proposition Ax is true. *Bivalent truth* is the key semantic notion in classical logic. In classical logic, to have a proof of a proposition P consists of having a succession of chained (true) statements or formulas by which the truth of P is established, being P the last element of the succession. And, generally, to establish the truth of a proposition P is equivalent to prove that the supposed falsity of P originates an absurdity and to settle this is logically equivalent to prove that $\neg P$ entails a proposition of the form $A \wedge \neg A$, a logical contradiction. This kind of proof is called *reductio ad absurdum*, for it settles the truth of a proposition by showing that its falsity leads to a contradiction. This kind of logical reasoning is said to be an *indirect* kind of proof, for it doesn't prove the truth of P by means of an argument showing directly that P holds. For the same reason this kind of proof is also called a *non-constructive* proof: we don't prove P by an explicitation of a concrete way of showing that P holds but by showing that $\neg P$ cannot hold.

The explanation given above of what generally counts as a proof for a proposition on the basis of the semantic notion of truth motivates the classical interpretation of the logical constants, also called the *platonistic* or *bivalent* interpretation of the logical constants.

1. $\neg A$ is true iff A is false;
2. $A \wedge B$ is true iff both A and B are true;
3. $A \vee B$ is true iff A is true or B is true;
4. $A \rightarrow B$ is true iff whenever A is true, then B is also true;
5. $\forall x Ax$ is true iff for all x in the domain D (D is the intended range of the variable x), Ax is true;
6. $\exists x Ax$ is true iff there exists (at least one) x in D (D is the intended range of the variable x) such that Ax is true.

In classical logic, every proposition is seen as a description of (a physical or abstract) reality. The truth or falsity of propositions are determined by the correspondence of the propositions with reality[14]: a proposition A is true iff there is the state of things described by A; if such a state of things does not exist, then A is false. In classical logic all entities are seen as complete, somehow finished, and so is the reality whose correspondence with turns the propositions into true or false ones. The correspondence with a reality makes $A \vee \neg A$ a universally valid law. There is not a third hypothesis; in the sense that a proposition cannot be neither non-true nor non-false simultaneously, i.e., any proposition is either true or false[15].

[14]There is a sense in which an intuitionist would accept too that every proposition is a description of reality, namely in the exact extent that such a reality is to be interpreted as a *mental* reality, not as an extra-mental (physical or abstract) reality. Another way of putting the same issue, assuming that a physical or abstract reality is a *static* reality, consists in saying that for an intuitionist too it is through a description of reality that propositions are true or false, but in his case such a reality is *dynamic* and not static.

[15]As we will see below, in intutionistic logic not all propositions are either true or false: there are cases where it is not possible to prove that a given proposition is determinately true or determinately false.

Because of the meanings of logical operators, the notion of an indirect or non-constructive proof gains its full expression within classical reasoning. Suppose we want to prove a statement of the form $\neg\forall x \neg Ax \rightarrow \exists x Ax$. The natural way to prove statements of this form is to suppose the falsehood of the implication, i.e, to suppose that $\neg(\neg\forall x \neg Ax \rightarrow \exists x Ax)$. But to suppose the falsehood of the implication is to suppose that $\neg\forall x \neg Ax$ is true but $\exists x Ax$ false. This step of the proof is legitimated by the meaning of the logical operator '$\rightarrow$'. The falsity of $\exists x Ax$ is equivalent to assert $\neg\exists x Ax$, allowed by the meaning of '$\neg$'. If $\neg\exists x Ax$ holds, then there is not any x belonging to some domain D for which Ax is the case. But, by the meaning of '$\exists$', this is equivalent to say that for any x belonging to D, it is not the case that Ax, i.e., $\forall x \neg Ax$. This, however, contradicts $\neg\forall x \neg Ax$ whose truth was assumed at the beginning of the proof. So, by achieving a contradiction, we can now state that $\neg(\neg\forall x \neg Ax \rightarrow \exists x Ax)$ is false, leading to the desired result.

Notice that the proof of the claim $\neg\forall x \neg Ax \rightarrow \exists x Ax$ does not guarantee that we can find a particular a such that Aa. In fact, we may be unable to find any particular object x in the domain that satisfies the property Ax. This fact, however, does not forces the truth of such a claim to loose its strenght, since truth or falsity is established by the (classical) meaning of the logical connectives and quantifiers, not by the fact that we are able or not to find a particular element of the domain that possesses the desired property. Let's define $A\alpha$ as $\forall n(\alpha n = 0)$, with n ranging over natural numbers and α over choice sequences. Then we have the following claim: $\neg\forall\alpha\neg\forall n(\alpha n = 0) \rightarrow \exists\alpha\forall n(\alpha n = 0)$. This claim states the following: if it is false that for all choice sequence α, α is not the null sequence, then there is an α such that α is the null sequence. For a classical proof of this statement it suffices to show that assuming the truth of $\neg\forall\alpha\neg\forall n(\alpha n = 0)$ and the falsity of $\exists\alpha\forall n(\alpha n = 0)$ we get a contradiction. But from the fact that the negation of the implication leads to a contradiction, it does not follow that we can find an α such that for all $n \in \mathbb{N}$, $\alpha n = 0$. Since, in the case of lawless sequences, we may be unable to find a particular α that is the null sequence, by the same reason given above: there is no general method that, for a particular choice sequence α, warrants that $\alpha n = 0$ for all n. The only cases where we are able to give a method for finding a particular α in that condition are the lawlike sequences. But the domain of choice sequences contains both lawlike and lawless sequences and, in the case of lawless sequences it can happens that we are unable to find a particular α in such conditions.

The argument presented above belong to what are called *counterexamples* and they were introduced by Brouwer for the purpose of showing that certain classicaly acceptable statements are constructively unacceptable. The counterexamples are divided into *weak* and *strong* counterexamples. The weak counterexamples were introduced (at least implicitly) in the so called *first act of intuitionism*, whose aim was to justify why he (Brouwer) had *abstained* from using PEM and related principles as logical laws. (*CinM I*, pp.8-9) Instances of weak counterexamples are cases where generally we are unable to provide a construction method for a sentence like $\exists x Ax$ or where we are unable to decide between A or B in a statement like $A \vee B$, even although they are classically seen as being true. The strong counterexamples were introduced in the *second act of intuitionism*, where Brouwer does not just abstains from using principles as PEM but, in fact, *proved* that using such principles would produce false statements. An instance of a strong couterexample is $\neg\forall$-PEM.

These couterexamples lead to the rejection of the notion of 'bivalent truth' as the key semantic notion for objects like choice sequences and consequently to reject the classical interpretation of the logical constants. So, a new interpretation for the logical constants is needed if we want to deal with choice sequences. This new interpretation of the logical constants is called the *BHK-interpretation* (Brouwer-Heyting-Kolmogorov interpretation) for intuitionist logic:

1. A proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction ($\bot$);
2. A proof of $A \wedge B$ is a construction that presents a proof of A and a proof of B;
3. A proof of $A \vee B$ is a construction that presents a proof of A or a proof of B (and the proof has to be *explicitly* a proof of A or a proof of B);
4. A proof of $A \rightarrow B$ is a construction that transforms any proof of A into a proof of B;
5. A proof of $\forall x A x$ is a construction that assigns to every $a \in D$ (D is the intended range of the variable x) a proof of Aa;
6. A proof of $\exists x A x$ is a construction that selects an $a \in D$ and that presents a proof of Aa.

The key semantic notion for intuitionist logic is that of a 'construction'. The truth of a proposition P is no more than a construction method that evidently shows that P. One may have noticed that negation is defined as implication of falsehhod: $\neg P := P \rightarrow \bot$. Thus, intuitionistic negation is not merely a reversal of truth values, but involves the possession of a certain construction method. This fact justifies the rejection of some classical rules of deduction as Double Negation (DN), the rule that states the equivalence between A and $\neg\neg A$, i.e., that $A \leftrightarrow \neg\neg A$ (from A we can infer $\neg\neg A$ and from $\neg\neg A$ we can infer A). Suppose we want to prove A, then an obvious classical way to do that is to show that $\neg A$ leads to a contradiction, inferring from that the conclusion $\neg\neg A$. DN allows to infer A from $\neg\neg A$, establishing the desired result. In fact, DN is the formal deductive step that allows to conclude all the results discussed above that are not acceptable intuitionisticaly. In intuitionist logic the step from $\neg\neg A$ to A is not allowed, for the fact that $\neg A$ leads to a contradiction doesn't necessarily provides a construction method to prove A; the reverse step is, however, intuitionistically valid. Intuitionisticaly we can have a construction method to prove $\neg\neg A$ but fail to find a construction that proves A. But if we have construction method that proves A, then we have a construction method that shows the absurdity of $\neg A$, leading us to be justified, by the definiton of intuitionistic negation, to assert $\neg\neg A$.

We have seen that intuitionist logic is needed when we are talking of choice sequences. But we can argue for the correctness of the intuitionistic interpretation of the logical constants over classical interpretation without even talking about choice sequences. That is, we can argue for the correctness of intuitionistic logic irrespectively of accepting choice sequences as legitimate mathematical objects or not. From a neutral standpoint over the introduction of choice sequences into mathematical reasoning, we can choose to argue for or against the correctness of intuitionistic claims. In fact, this is the path choosen by Michael Dummett for instance. He argues for intuitionistic logic on the basis of semantical arguments only, not on the basis of ontological ones like we are doing here.[16] When, however, we start talking about things like choice sequences, i.e., when we start to discuss about the legitimacy of the introduction of unfinished objects into mathematical theories, to argue for intuitionist logic is not a question of choice, it is a necessity because classical logic would render the theory of choice sequences contradictory.

1.6 Choice sequences and constructivity

We have seen that to deal with choice sequences we have to reject the classical concept of proof based on the notion of truth. This notion is substituted by the more acceptable, according to the intuitionist, notion of constructive proof. But in fact, the concept of constructive proof constitutes a philosophical (and rather technical) problem that intuitionists have to deal with. The notion of a constructive proof is not, however, restricted solely to intuitionistic or other forms of constructivist mathematics. The distinction between constructive and non-constructive proofs arises in classical logic and mathematics in an entirely intelligible way. This distinction arises mainly for proofs of existential and disjunctive statements. A proof of a statement P provides always something more than the truth of the conclusion P, it provides additional information about how we came to the truth of P. A proof of a closed statement of the form $\exists x Ax$, x ranging over $\mathbb{N}$ for example, is classically constructive just in case it either proves that, for a specific $n \in \mathbb{N}$ An holds, or at least it yields an effective mean, even though in principle, for finding a proof of An. Likewise, a proof of a closed statement of the form $A \vee B$ is constructive if it is either a proof of A or of B, or at least if it gives origin to a effective method for obtaining a proof of one or another disjunct. For a $\forall\exists$-statement, i.e., a statement of the form $\forall x \exists y A(x, y)$, with no free variables and x, y rangin over $\mathbb{N}$, then its proof is constructive if it yields an effectively calculable function f (given by an algorithm) such that $A(n, f(n))$ holds for each n. A proof of a statement of the form $\forall x(Ax \vee Bx)$ (with no free variables) is classically constructive if it yields an effective way for finding, for each n, a proof either of An or of Bn. (*EI*, pp. 6-7.)

A non-constructive proof of any of the statements above is a proof of the falsity of its negation. And in classical logic and mathematics both kinds of proofs are acceptable. But the intuitionist is not in this position; the BHK-interpretation of the logical constants entails that he must have a constructive proof because the intuitionistic interpretation of the conclusion is always such that no non-constructive proof could count as a proof of it. In classical logic and mathematics both methods of proof are acceptable because the truth of a proposition is independant of the fact that we have a proof for it or not. The truth or falsity of a proposition is something pre-established once and for all (the platonistic character of classical mathematics), a proof just makes known which of the two are the case for that proposition. In the intuitionistic case, it is quite the different. A proposition is considered to be true if we have a (constructive) proof of it. So, the primitive notion to be argued for is that of 'proof'; a much

[16]See *EI*, ch. 1 or 'The Philosophical Basis of Intuitionistic Logic' in *Truth and Other Enigmas*, Harvard University Press, Cambridge, 1978, pp. 215-247.

more harder task than to argue for the legitimacy of the primitive notion of 'truth', we think.

When we are dealing with lawlike sequences the notion of constructive proof is quite intuitive because of their determinacy, viz. the law that generates their values. Notwihstanding the fact that not always the existence of such a law provides the correspondent constructive proof (e.g., $\neg\forall$-PEM goes the same for lawlike sequences, or the Halting problem in classic theory of recursive functions), a constructive proof for a proposition involving lawlike sequences is relatively easy to describe. A proof of a disjunctive proposition $A\alpha \vee \neg A\alpha$, where α is a lawlike sequence, is to show that we have a method to decide that for a particular α, $A\alpha$ hold or that $\neg A\alpha$ hold. By the fact that α is lawlike, i.e., given by a finite law or rule that computes all the elements of the sequence, the existence of such a method *is not ruled out*. Because we can calculate all the elements of α via the law that generates its values in a completely determined way, it is not ruled out that we can decide if all the elements of the sequence have the property A or not. So, it possible in principle to have a method that warrants one of the disjuncts. And it turns out to be the same for the existencial claim, for an existence claim of the form $\exists\alpha A\alpha$ or for an $\forall\exists$-statement. If we have an existencial proposition $\forall\alpha\exists x A(\alpha, x)$ in which α is a lawlike sequence and x ranges over $\mathbb{N}$, then we may have in advance a method that makes possible in principle to show if for an arbitrary α there exists a value x such that $A(\alpha, x)$.

For lawless sequences, however, the notion of a constructive method is not as clear as for lawlike ones. For lawless sequences there is not any law or rule given in advance that generates their values since they are freely generated by the subject who starts them. Generally, we *cannot* have a method that computes whether for a lawless sequence α, $A\alpha$ or $\neg A\alpha$. The only way we have to decide this is to successively pick initial segments of α (finite parts of α) and look whether they agree with A or not. (This situation is reducible to decidability method for lawlike sequences.) Suppose this (quasi-empirical) 'method' allows to decide affirmatively that, for some initial segment $\bar{\alpha}n$ of α, $\bar{\alpha}n$ is according to A, then can we infer that $A\alpha$? This 'method' does not allow to infer that $A\alpha$; since, for some initial segment $\bar{\alpha}m$ bigger than $\bar{\alpha}n$ it may happen that $\bar{\alpha}m$ does not agree with A. For this reson, for lawless sequences, contrary to lawlike sequences, the idea of a contructive method to prove $A\alpha \vee \neg A\alpha$ does not make sense at all, even if in principle. Such 'method' is simply *impossible to exist*. So, we can ask, which notion of constructivity the class of lawless sequences entails, given that they are meant to be constructive objects?

1.7 Identity for choice sequences

An immediate consequence of the last paragraph of the previous section is that, in general, we cannot know whether two lawless sequences are the same or are different ones. For any two lawless sequences α and β, the proposition $\alpha = \beta$, defined as $\forall n(\alpha n = \beta n)$, is not provable in constructive grounds. For lawlike sequences we can prove that $\alpha = \beta$ by proving that the laws f_α and f_β that generates their elements respectively, give raise to the same outputs. For lawless sequences, however, this method does not work since for lawless sequences their outputs are freely chosen by the subjects who generates them. In the case of lawless sequences, once more, what we have is a method to decide whether two initial segments $\bar{\alpha}n$ and $\bar{\beta}n$ of α and β, respectively, are identical or not. But, once more, when $\bar{\alpha}n = \bar{\beta}n$, for some n, this does not guarantee that $\alpha = \beta$; if we choose initial segments $\bar{\alpha}m$ and $\bar{\beta}m$ bigger than $\bar{\alpha}n$ and $\bar{\beta}n$, respectively, we cannot guarantee that $\bar{\alpha}m$ and $\bar{\beta}m$ will continue to be equal; they may diverge in some point, establishing that $\alpha \neq \beta$ after all.

The discussion above motivates the following distinction for choice sequences. Let's denote the property of two choice sequences α and β be *extensionally equal*, i.e., $\forall n(\alpha n = \beta n)$, as '=' and the property of α and β being *intensionaly identical*, i.e., α and β refer to the same (identical) process of generation, as '≡'. For lawlike sequences, if $\alpha \equiv \beta$, then $\alpha = \beta$. For, if α and β are given by the same process, then all the values of α will coincide with the values of β because they are generated in the same way. For lawlike sequences, extensional equality is implied by intensional identity. Inversely, if two lawlike sequences are equal, i.e., $\alpha = \beta$, it can be assumed that the laws generating them are equivalent in the sense that they give raise to the same outputs. For lawless sequences, however, the second case is a little different. We cannot assume without good arguments that equal lawless sequences are given by the same process of generation, since the distinction between extensional and intensional properties will vanish. With respect to this, in the next chapter we will present two arguments trying to establish that $\alpha = \beta \rightarrow \alpha \equiv \beta$, advanced by Troelstra and Dummett respectively, in the context of a particular formal theory for lawless sequences (**LS**).

Lawless sequences, because of their *sui generis* nature, creates an unsurpassable gap between extensional and intensional properties about them. In particular, and at first sight, extensional and intensional identity do not coincide for lawless sequences. Suppose we have two lawless sequences α and β, and that $\alpha = \beta$, i.e., they are extentionally equal. Can we infer that $\alpha \equiv \beta$, i.e., can we infer that they are intensionally identical sequences? Intuitively, the answer is negative. From the fact that $\forall n(\alpha n = \beta n)$ nothing is said about how α and β are generated, nothing is said about the processes that generates them. In particular, no information is given that allows us to tell whether the processes that generates both α and β are the same process or different ones. Suppose now, we have that $\alpha \equiv \beta$, i.e., the elements of α and β, respectively, are generated by the same intensional process. Can we infer that $\alpha = \beta$? This inference seems to be more likely, for an individual choice sequence has all of its properties solely in virtue of the subject's decision to construct it in a particular way. But let's see closer. Suppose a particular mathematician (the real world constructors of choice sequences) starts two choice sequences α and β at a given moment of time but chooses their elements (through undeterminate processes, respectively) such that $\alpha n = \beta n$ indefinitely. Are we justified to infer from this that for *all* the values of α and β, they will coincide up to the infinite with no further clauses? The natural answer is no. Since, choice sequences are potential objects, growing freely through time, and at any stage of growth we have access only to a (sufficiently long, pehaps, but) finite initial segment. So, we cannot, in general, know whether α and β will mantain the coincidence of their elements or, at some stage of growth of α and β, they will diverge. The only thing we can know is that, if at some stage of growth of α and β they produced different outputs, then they are different choice sequences; but their identity generally stays an open question.

Due to the problem in finding a criterion for the identity of choice sequences, we can ask if such a criterion is really needed at all. Notice that choice sequences are essentially intensional objects and, as such, a criterion of identity cannot be an extensional one, but intensional. This same problem is faced by the second order logic,, where quantification over predicates is allowed. Predicates are also intensional objects and there is not an identity criterion for them too. What we can show is that two different predicates have the same extension: the predicates 'to possess a heart' and 'to possess a kidney' have the same extension (all the animals that possesses a heart possesses also a kidney). But they are obviously different predicates. This fact, however, need not limit in any way the consistent development of theories that uses second order logic,, i.e., does not limits the way we talk about predicates. The proponent of choice sequence may ask: are not the theories that allow quantification over choice sequences at the same level?

1.8 Individuation for choice sequences

Until now we have talked about the (identity of) processes that generates choice sequences, when two choice sequences are generated by the same process or not. We have seen that for two (or more) choice sequences identity is not in general easy to settle. But individuation is a much more difficult fact to settle. Suppose we have an individual lawless sequence α, how do we come to know that α is the particular choice sequence it is? How do we distinguish α from all the other possible choice sequences? For a classical object like a set individuation is settled by its extensional character: the self-identity of a set (finite or infinite) is given by its elements, i.e., a particular set is the set *it is* because of the totality of its elements, that are given in advance once and for all. (*Mutatis mutandis*, it goes the same for lawlike sequences.) A lawless sequence, however, is an unfinished object, the totality of its elements are never given to us. At each stage of growth of a choice sequence we only have access to an initial segment of it, an incomplete (finite) part of the totality of its elements. So, extentional information about a particular choice sequence is not sufficient to function as a basis for its individuation.

How about intensional information? Think about the *subject(s)* who generates choice sequences, about the precise *moment* in time which a particular choice sequence is started or about the restrictions imposed on choice sequences; all these are also possible intensional information about choice sequences. We have told that all properties an individual choice sequence has is due to the particular way the subject decides to construct it. So, it is natural to think that the precise moment in time an individual choice sequence is started by a subject warrants its individuality. Or that the simple fact that an individual choice sequence is started by one particular subject, says A, instead of another, says B, warrants its individuality.

Consider two choice sequences α and β such that all their elements coincide indefinitely and they were started by the subject A, possibly constructed by the same process. Suppose now that the only explicit difference between α and β is that α was started at a moment t_n and β was started at a subsequent moment t_{n+1}; for all the other properties α and β are indistinguishable from each other. Does the fact that α and β were started at different moments, t_n and t_{n+1} respectively, establish a sufficient condition to individualize each of them?

In fact, it seems to work, since if all the properties a choice sequence has is due to the particular way the subject decides to construct it, then the fact that he has choosen to start α and β at different moments in time shows that α and β have at least one intentional property that does not coincide. But think now if α and β were started at the same moment t_n. Notice that α and β may come to diverge at some stage but this information is not available to us. How to procede in this situation? How do we individualize α and β when both come into existence at the same moment in time and are indistinguishable with respect to other intensional (and extentional) properties? It is also reasonable to think that α, the same choice sequences started at t_n, could have been started at different moment, t_m. Should this contingent fact have as consequence that α started at t_n is different from α started at t_m? This seems hardly sustainable.

Suppose now α and β such that all their elements coincide indefinitely. Suppose the only intensional information that distinguishes from α and β is the fact that α was started by the subject A and β was started by the subject B. The fact that α and β have different subjects, A and B respectively, constitutes sufficient condition for the individuation both of α and β? Does subjective aspects of choice sequences makes α and β the choice sequences they are, each of them individually distinguishable? This state of affairs is sustained by the own nature of the notion of choice sequence; but it naturally seems that something is wrong. In fact it is hard to see how the fact that α and β have different bearers, *ceteris paribus*, can individuate them.

Consider this last argument against the subjective character of choice sequences. Suppose a subject A starts a lawless sequence α at some moment. Suppose, in addition, A wants to share information about α with another subject B. How should this be done? How should A talk about α with B since B has not access to any information about α except, possibly, to an initial segment of α, reported to him by A? Lawlike sequences can always be shared by two or more subjects since their process of construction is describable in a (finite) mathematical code. A lawless sequence, however, does not possess such a process of construction and is not describable in a mathematical code. How the proponent of choice sequences should deal with this issue?

Lastly, consider a pragmatic counter-argument against both the temporal and subjective aspects of choice sequences. Is it reasonable, for mathematical purposes, to consider that the different moments they were started or the fact they have different bearers makes α and β distinct choice sequences? After all, it seems natural to think that temporal and subjective aspects of mathematical objects are mathematically irrelevant. For mathematical purposes all we need to know are the mathematical conditions that make two objects be the same or not, distinguishable or not. Temporality and subjectivism are extra-mathematical properties that do not influence the truth conditions or the (constructive) proof conditions for a mathematical theorem. How to reconcile choice sequences's extra-mathematical features as temporality and subjectivism with an acceptable mathematical theory about them or just involving them? This is a real problem which the proponent of choice sequences as legitimous mathematical objects has to deal with.

1.9 A new paradigm of mathematical reasoning?

Notwithstanding the fact that the notion of choice sequence is a very intuitive one (we can imagine the series of natural numbers as a choice sequence given by the iteration of the process $n + 1$, for example), some mathematicians and philosophers refuse to accept choice sequences as legitimate mathematical objects. Because of their dependence on the subject's mental processes and on time, some have considered choice sequences to be empirical objects, and hence as having no place in pure mathematics (Wittgenstein and Gödel, for example). Another cause for this refusal to accept choice sequences as mathematical objects is motivated by the fact that the behavior of choice sequences does not respect a well established property of the (classical) mathematical universe of objects, the law of excluded middle. In the first place, choice sequences are dependent on the subjects's particular way of construction for them, which turns the notion of choice sequence into a subjective notion. In the second place, choice sequences are potential or dynamic objects, they evolve through time; that makes them temporal and unfinished objects. Traditionally, none of these properties are attributed of mathematical objects. In the third place, with the introduction of choice sequences into our mathematical ontology comes WC-N, a principle that allows the proof of several new results as the unsplitability of the continuum or the fact that any total funcion from $\mathbb{R}$ to $\mathbb{R}$ is continuous, or the more drastic result that decidability, in general, is not granted; putting PEM and the classical notion of mathematical proof under suspicion. Why should we accept any mathematical objects for which the identity and indivuation conditions are unclear? In one word, choice sequences are too much *exotic* to be considered as mathematical objects. This is why the opponent of choice sequences does not consider choice sequences as mathematical objects.

With the mentioned results of intuitionitic logic and mathematics, it becomes obvious that the introduction of choice sequences into mathematical ontology is not innocuous. The introduction of choice sequences into mathematical ontology gives rise to an unsurpassable exclusive disjunction, at least from the philosophical standpoint: either we accept the law of the excluded middle as universally valid, and with it classical mathematics, or we accept the introduction of choice sequences into our mathematical universe, which forces us to accept WC-N and intuitionistic mathematics.

In reply to the last consideration, the proponent of choice sequences says that, in fact, intuitionistic and classical mathematics are not in equal positions. They are not talking about the same things, they do not employ the same concepts or methods. So, in some sense, they are not comparable. (The classical mathematician also may say the same thing.) It is true that intuitionistic and classical mathematics attribute different meanings, for example, to the logical constants or to the notion of truth. In fact, the ontologies of classical and intuitionistic mathematics are different: classical mathematics deals with sets (finished objects) and intuitionistic mathematics deals with choice sequences (unfinished objects). On the other hand, however, we can say that both intuitionistic and classical mathematics have as primary task to present a mathematical model of infinite magnitudes, like the continuum or the collection of natural numbers. And for this reason they are not so incomparable as the proponent of choice sequences says. But the proponent of choice sequences will insist that his notion of infinite magnitudes is different, since in intuitionistic mathematics, all infinity is potential infinity: there is no completed infinite, as in classical mathematics. Contrary to classical mathematics, for an intuitionist to grasp an infinite structure is to grasp the process which generates it, to refer to such a structure is to refer to that process, and to recognize such structure as being infinite is to recognize that the process will not come to an end. In intuitionistic mathematics, there is no sense in which we can have any conception of an infinite magnitude as a totality except by the process of generation. For this reason, the proponent of choice sequences can state that classical and intuitionistic mathematics are not comparable, since the concepts involved in classical mathematics are simply unintelligible for him. So, he can say that intuitionistic mathematics and logic, both justified by the introduction of choice sequences into mathematical ontology, constitutes a wholly *new paradigm of mathematical reasoning*, since the concepts, the methods and, especially, the mathematical objects are entirely new ones. (Though wholly new, it should have sufficiently much in common with classical mathematics if both are to be called mathematics.) This statement constitutes, we think, the only really indisputable thesis advanced by the proponent of choice sequences as mathematical objects. Irrespectively whether classic mathematics or intuitionistic mathematics is the right way to do mathematics, it is unquestionable true that intuitionistic mathematics and logic, with their new and exotic concepts, methods, and objects constitute a new and attractive form of mathematical reasoning.

If it is true that the new paradigm introduced by choice sequences is innovative and interesting, it is also true that this new paradigm of mathematical reasoning gives rise to new and difficult epistemological issues. How do we philosophically solve the problems raised by the introduction of choice sequences? How do we justify the introduction of dynamic objects into the mathematical universe? How do we deal with the problem of identity or with the problem of individuation for choice sequences? What should count as a constructive proof for propositions involving lawless (choice) sequences? How should we deal with the unnatural hypothesis that assigns temporal and subjective aspects to putative mathematical objects? All these are philosophical problems that the proponent of choice sequences has to face if he wants to guarantee the introduction of choice sequences into mathematical ontology. In the continuation of this essay, we will see how these problems are faced by several proponents of the introduction of choice sequences into mathematical reasoning.

Chapter 2

Three accounts of choice sequences

2.1 The origin of the concept of choice sequence

Troelstra (following Heyting) calls any mathematician an intuitionist who subscribes the following two claims: (a) mathematics is not only formal, but also has content; (b) mathematical objects are grasped directly by the thinking mind, hence mathematical knowledge does not depend on (an outside world) experience. Then he observes that the second claim may be interpreted in at least two different ways: (b1) one thinks of a mathematical object as having an existence independent of our thinking, but we can only conclude its existence and investigate it by means of a mental activity reconstructing the mathematical object in our mind; and (b2) mathematical objects exist *only* as mental constructions, at least we cannot base a mathematical argument on their existence independent of our knowledge. Heyting calls the point of view represented by (a)+(b1) semi-intuitionism; the point of view represented by (a)+(b2) corresponds to Brouwerian intuitionism.[1]

Around the beginning of the 20^{th} century a group of (mostly French) mathematicians held views wich might be described as semi-intuitionistic in the above sense: E. Borel, H. Lebesgue, R. Baire, N. Lusin. H. Poincaré is often referred as a precursor of the semi-intuitionist school. Borel is considered the most proeminent of the semi-intuitionists and it is claimed that Brouwer learnt of the views of the semi-intuitionists through his writings. (*Ibidem.*)

Every form of constructive mathematics has to cope with the continuum, with $\mathbb{R}$. The standard treatments of the continuum proceed by 'arithmetization': the collection of reals is described via sequences or sets of rationals, or sequences of rational intervals. From a constructivist point of view, there is a conceptual difficulty here, which led the semi-intuitionists, and in particular Borel, to adopt the continuum as a primitive notion, to be understood as a whole (i.e.,

[1] A.S. Troelstra, 'On the origin and development of Brouwer's concept of choice sequence', in *The L.E.J. Brouwer Centenary Symposium*, North-Holland, Amsterdam, 1982, p. 466. Unfortunately we did not have access to Arend Heyting's book *Les fondements des mathématiques. Intuitionisme, Théorie de la démonstration*, quoted by Troelstra and in which this schematic division is originally made.

the continuum is more than the collection of its elements). This holistic view of the continuum is referred to as the "geometric continuum", whereas the continuum as made up of reals given by fundamental (Cauchy) sequences is called the "arithmetical continuum". (*CinM II*, pp. 639-640.)

The conceptual difficulty is the following one. For the semi-intuitionists, any specific real number should be given by a complete definition of (say) a representing fundamental sequence; and since the linguistic means at our disposal permit only countably many descriptions, there would exist only countably many reals. Cantor's diagonal argument shows that these cannot exhaust the continuum. Notice that the semi-intuitionists, while accepting classical logic (by (b1)), had to accept this argument. Thus one could not very well conceive the continuum as being made up of its definable elements (the 'practical continuum', or 'reduced continuum' when these elements are lawlike sequences). (*CinM II*, p. 640.)

The concept of a choice sequence was however already known to the semi-intuitionists. In Borel's thinking the adoption of the arithmetical continuum was directly connected with the acceptation of countable sequences of arbitrarily chosen objects as legitimate objects of mathematics, which was to him 'highly debatable'. The notion, however, of an *uncountable* infinity of choices (a choice sequence), was to him 'entirely meaningless'. These claims were explicitly asserted in Borel's writings:

> It is necessary to say something on the notion of a continuum, the only well-known example of an uncountable set (...). I regard this notion as obtained from geometrical intuition; one knows that the completely arithmetical concept of the continuum requires that one admits the legitimacy of a countable infinity of sucessive choices. This legitimacy seems to me to be highly debatable, but neverthless one should distinguish between this legitimacy and the legitimacy of an uncountable infinity of (sucessive or simultaneous) choices. The latter concept seems to, as I have remarked before, entirely meaningless (...). (Émile Borel, 'Sur les principes de la théorie des ensembles', 1909.)[2]

To Borel the meaningless of talking about choice sequences was not because the concept in itself was unintelligible (in fact he admits that in some areas of mathematics it could have use), but because when talking about individual choice sequences we would not be able to settle questions of identity (the very same problem we are dealing in this thesis):

> On my part, I regard it as possible to ask questions of probability concerning decimal numbers obtained in this way, by choosing digits, either entirely arbitrarily, or imposing some restrictions which leave some arbitrariness. But I regard it as impossible to talk about a single individual such number since if one denotes such number by a, different mathematicians, in talking about a, will never be sure

[2]Quoted in A.S. Troelstra, 'On the origin and development of Brouwer's concept of choice sequence', p. 467. Unfortunately we also did not have access to any of Borel's articles we are quoting here, so they are taken directly from Troelstra's article cited above.

to be talking about the same number. (Émile Borel, 'La philosophie mathématique et l' infini', 1912.)[3]

In an early stage of his thinking, Brouwer was in agreement with Borel's points of view and consequently faced the same difficulties about the continuum. He first assumed the continuum as a primitive notion. But soon (around 1917) he realized that this holistic view of the continuum was not implied by his intuitionism ((a)+(b2)); in particular by his criticism and rejection of some classic logical principles. From an intuitionistic point of view there was no objection against considering sequences obtained by successive choices instead of being completely defined in advance. What seemed illegitimate in the (so called) 'empirical' view of Borel, fitted very well in his view of mathematics as created in the mind of an ideal mathematician. This move enabled Brouwer to reinstate the arithmetical account of the continuum, but now in an intuitionistic context. Brouwer's new arithmetical theory of the continuum based on choice sequences is *analytic* in contrast to the earlier holistic theory where the geometric continuum had to be understood as a whole. In the new theory, reflection on what it means to be given a choice sequence led Brouwer accept new mathematical entities and to settle new mathematical principles. We will now consider how Brouwer's mathematical path was determined by his philosophical ideas; and in particular the so called 'two acts of intuitionism': two moments in Brouwer's philosophical activity that explain (among other things) his passage from the geometric conception of the continuum to the arithmetic conception.

2.2 Brouwerian intuitionism

Let's think of a basic mathematical operation: $\{2, 3\} \rightsquigarrow 2 + 3$, for example. One intuitively knows that the result is 5, i.e., we can prove it by simple mental steps. Let's construct the number 2 in our mind by abstracting from two every day objects and keep it in memory. In the same way, we can construct the number 3 and keep it in memory. Now we can construct the number 5. If we compare it with the result of the sum we have made, we will see that they are the same, so we can judge that $2 + 3 = 5$. According to Brouwer, all mathematics is like this, i.e., all mathematical reasoning is an *intuitive mental construction*. By 'mental' Brouwer wants to stress that mathematical results are obtained exclusively by means of a subject's introspective reflection. A mathematical theorem is achieved by the subject only when he is capable of constructing it by means of some intrinsic mental process. So, mathematical theorems are mental entities, not extra mental things. By 'construction' Brouwer means that in the (mental) process of establishing a mathematical theorem each stage of reasoning is based on the previous one and retains its degree of accuracy. The construction of a mathematical theorem entails that each step on the path to the final conclusion (the theorem itself) can be clearly analysed and reduced to the most elementary insights of the subject's mind. By 'intuitive' he wants to emphazise that in the process of establishing a mathematical theorem each stage of the mental construction is allowed by an intuition, i.e., by an evident and

[3]Quoted in A.S. Troelstra, 'On the origin and development of Brouwer's concept of choice sequence', p. 467.

necessary (inner) experience/perception. This inner perception is what allows or forbids a construction, i.e., is what determines the validity of a construction. Let's see a quote of Brouwer:

> Intuitionism considers the falling apart of moments of life into qualitatively dierent parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intelect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-ity.[4]

For Brouwer, all thinking (inner life) is based on the phenomenon of *time*, i.e, all moments or acts of reasoning are connected between them by moments of time. Brouwer makes a sharp distinction between subjective time (inner time counsciousness) and objective time (time measured by physics). The phenomenon that connects all acts of reasoning is the subjective time. So it becomes natural, according to Brouwer, to base mathematics on the subjective time. This is done by abstracting from the content of the moments connected by time (the emotional content) and taking into consideration only the inner time by itself, so taking into consideration the pure form of subjective time, i.e., the schematic character of time as a mental operation. This schematic character of time is what Brouwer calls the two-ity, the "basal intuition of mathematics". (*ibidem*.)

> This intuition of two-ity (...) creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-ity may be thought of as a new two-ity, which process may be repeated indefintely; this gives raise still further to the smallest infinite ordinal number ω. Finally, this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives raise immediately to the intuition of the linear continuum, i.e., of the 'between', which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. (*Ibidem*.)

For Brouwer, it is from the intuiton of two-ity that all primary notions of mathematics gets their origin. The two-ity creates the numbers 1 and 2 simultaneously by reference to the pure form of time 'then-now'. The 'then' corresponds to 1, the 'now' to 2. The number 3 and subsequent numbers are created by indefinetly repetition of this mental operation of two-ity, when an element of the two-ity falls into a new two-ity, embedding this new two-ity into the old one. Brouwer points out that it is necessary to have a two-ity as starting point and not a unity. The insight that this process of embedding two-ityes can be iterated give rise to the infinite ordinal ω and, consequentely, to the series of natural numbers $\mathbb{N}$. This infinite object has to be thought of as potential and not as actual. This is so because the subject can start a potentially infinite sequence of objects but it cannot complete it. (*OnB*, pp. 5-6.)

The basic intuition of mathematics, the two-ity, doesn't gives rise only to the discrete magnitudes, it originates also the continuous ones; in fact it gives raise

[4] L.E.J. Brouwer, 'Intuitionism and formalism', in *Philosophy of Mathematics*, Cambridge University Press, New York, 1983, p. 80.

to magnitudes such as $\mathbb{R}$. Continuity is what is 'between' the elements of a two-ity; it is the abstract form of time that flows between 'then' and 'now'. (*OnB*, p. 6.) The 'then' and 'now' are discrete moments, what is between them are not discrete but continuous moments. For Brouwer, the 'then-now', the discrete, and the 'between', the continuous, are irreducible and complementary notions. They are directly apprehended by the mind with full evidence. This early notion of the continuum as given by intuition is due to the first act of intuitionism:

> The first act of intuitionism separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e., the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.[5]

The first act of intuitionism separates mathematics of mathematical language, i.e., it settles mathematics as an intrinsic mental activity that is not reducible to the formal or grammatical character of simbolic languages (claim (a)). In particular it separates mathematics as a mental activity from logic as a pure mechanical activity devoided of intuition. This is so because, according to Brouwer, the intuition of the two-ity, in which all mathematics is based, is not dependent of any linguistic skills. The intuition of the two-ity is essentially a cognitive experience that does not pressuposes any symbolic or formal kind of reasoning. It is part of pre-linguistic phenomenon. And it is not even necessarily shareable among subjects. In fact, according to Brouwer, it is quite the opposite:

> [Logic is] to be a faithful, automatic, stenographic copy of the *language of mathematics*, which itself *is not mathematics*, but no more than a defective expedient for men to communicate mathematics to each other and to aid their memory for mathematics. (...) Mathematics is created by a free action independent of experience; it develops from a single aprioristic basic intuition (...). A logical [or linguistic] construction of mathematics, independent of the mathematical intuition, is impossibe – for by this method no more is obtained than a linguistic structure, which irrevocably remains separated from mathematics (...).[6]

By this quotation it becomes clear that for Brouwer linguistic considerations about mathematics is a "second order" activity, i.e., something that takes into consideration a pre-established and primitive mental activity. Mathematics,

[5] L.E.J. Brouwer, 'Historical background, principles and methods of intuitionism', in *South African Journal of Science*, volume 39, 1952. We did not have access to this article of Brouwer, so the respective quotes are taken from *BmH* where the original reference is present; this particular quote is in *BmH*, p. 24.

[6] L.E.J. Brouwer, *On the foundations of mathematics* (PhD thesis), in *Collected works I*, North-Holland, Amsterdam, 1975, pp. 92 and 97.

therefore, is essentially languageless. It studies non-linguistic constructions out of something that is not of a linguistic nature. For Brouwer, in this sense, linguistic considerations about mathematical activity are inessential to the development of pure mathematical reasoning.

These considerations about mathematics and mathematical language are directed against the formalist project of Hilbert. For Hilbert, a mathematical assertion is a sequence of simbols whose meaning is given by the symbols themselves present in the sequence. So, the symbolic character of mathematics is what makes mathematics what it is, i.e., the symbolic character of mathematics is what grounds mathematics as a valid intellectual activity. Hilbert has used this insight to legitimate classical mathematics proposing that all classical mathematics was a sucession of conservative extensions of *finitary* axiomatic theories.[7]

Brouwer's considerations about language and mathematics stress a concern about the proper locus for certainty in mathematics. For a platonist the proper locus of certainty about mathematics is in an independent realm of abstract objects; for a formalist, it is on the symbols written on a paper; for an intuitionist, however, certainty is only to be found in the mind. For an intuitionist, the certainty of our mathematical thought is not accounted for in terms of grasping some abstract entities outside the mind or by apprehending the scheme of a sequence of symbols written in a paper, but in terms of the mind's own structure. (*OnB*, p. 3.)

Another relevant difference between the intuitionist and the platonist and formalist is that the platonist and the formalist try to give a philosophical account of classical mathematics, a pre-existent body of mathematical reasoning. The intuitionist however is indifferent with respect to whether mathematics as founded on his notion of construction happens to coincide with classical mathematics or not. What counts for the intuitionist is the solidity of the philosophical basis in which mathematics gains its validity. Brouwer erected his mathematics on the basis of his own philosophy, not the contrary. He was not seeking a philosophical basis for an already existent mathematical theory, he was looking for a mathematics that was philosophically sound.

In the first act of intuitionism the continuum is conceived as a whole, that 'therefore can never be thought of as a mere collection of units'. The 'continuous' and the 'discrete' are seen as primitive and complementary, i.e., one cannot be constructed upon the other. Brouwer explicitly states this view in the following:

> The *continuum as a whole* was given to us by intuition; a construction for it, an action which would create from the mathematical intuition 'all' its points as individuals, is inconceivable and impossible. The mathematical intuition is unable to create other than denumerable sets of individuals. But it is able, after having created a scale of order type η, to superimpose upon it a *continuum as a whole*. (*On the foundations of mathematics* (PhD thesis), p. 45.)

So, in this stage of Brouwer's thinking the arithmetical continuum does not play a role because the continuum 'is not exhaustible by the interposition of new units'. This is so because in this stage of philosophical development Brouwer

[7] See David Hilbert, 'On the infinite', in *Philosophy of Mathematics*, pp. 183-201.

had many new ideas about the nature of mathematics (it has genuine content, it is a languageless activity, it is deeply related to time and mind, etc.) but he still was assuming the objects of mathematical theories as being the classical ones (viz., definable infinite sequences). The notion of choice sequence was dismissed by Borel as inadequate for mathematical purposes. But in the case of Borel he was committed to classic logic. Brouwer however did not share such commitment and hence he was not forced (as Borel) to reject the notion of choice sequence as mathematical inadequate. Eventually, Brouwer realized that from an intuitionistic point of view non-classic objects as choice sequences were not forced to be abolished. In fact Brouwer realized that, from an intuitionistic point of view, choice sequences and other non-classic objects were allowed to play a role in mathematical theories. Brouwer called this new stage of his philosophical conception of mathematics the *second act of intuitionism*.

> The second act of intuitionism (...) recognizes the possibility of generating new mathematical entities: firstly in the form of infinitely proceeding sequences $p_1, p_2,...$, whose terms are chosen more or less freely from mathematical entities previously acquired; secondly in the form of mathematical *species*, i.e., properties supposable for mathematical entities previously acquired.[8]

The second act of intuitionism recognizes the possibility of introducing new entities or new kind of entities into the mathematical realm. In particular it allows the introduction of new kind of entities such as *choice sequences* and *species* that are not present at the usual classical realm of mathematical objects. A choice sequences is a potentially infinite sequence whose terms are freely chosen from a domain of mathematical entities previously acquired (e.g., the domain of natural numbers). Such a sequence is never finished and may be subjected to various conditions. Species are an intuitionistic analogue to classical (cantorian) sets. One may think of a species as a set of elements singled out from a previously constructed totality by a property. In a species, entities that share a certain property are collected. But what Brouwer means by 'property' is a very peculiar notion:

> Often it is quite simple to construct inside such a structure, independently of how it is originated, new structures, as the elements of which we take elements of the original structure or systems of these, arranged in a new way, but bearing in mind their original arrangement. The so called 'properties' of a system express the possibility of constructing such new systems having a certain connection with the given system. (*On The Foundations of Mathematics* (PhD thesis), p. 52.)

Suppose we have a collection or 'structure' of mathematical objects (defined by a law). Then, if we can establish certain mental relations between some elements of the given collection, we get a species, i.e., a new collection of objects that are connected by a property they share. Species can also have other species as elements, giving rise to species of higher order. An important

[8] L.E.J. Brouwer, 'Historical background, principles and methods of intuitionism', in *South African Journal of Science*, quoted in *BmH*, p. 24.

difference with classical sets is that species are intensional objects, i.e., objects that come to exist solely in virtue of some property defined in such a way that it gathers their elements as objects sharing that property. By this fact, species cannot be identified with their elements. Species are collections originated by a previous defined mental property. Two species can have the same elements but be different species, because the properties that gathers the two collections can be different ones. In the case of sets, their identity is determined by their elements (axiom of extentionality); if two sets have the same elements, then they are the same set irrespectively whether they are given by different properties or not. On the other hand the identity of a species consists in the way it is defined, not in the elements they gather.

With the second act of intuitionism, new objects as choice sequences and species are introduced into the (intuitionistic) mathematical universe. The introduction of choice sequences in particular enabled Brouwer to reinstate the arithmetical account of the continuum, but in an intuitionistic context. The new intuitionistic account of the continuum provided a much more satisfactory grasp of the continuum than to postulate the continuum as a primitive intuition. $\mathbb{R}$ is now defined as an infinite collection of *species*, each species gathering *choice sequences* sharing some property.[9]

2.3 Weak Continuity Principle (WC-N)

Even if the concept of choice sequence is philosophically interesting, how can it be put to use? How can we introduce such putative objects into mathematical theories and methods? The concrete problem is the following: if we want to apply a function to a choice sequence, or evaluate whether a predicate holds of it, a sequence will have to act as an argument, to which then a method is applied to calculate the function's output or value. But we cannot construct the argument in its entirety, as a choice sequence is always in state of growth, i.e., is an unfinished object. There will never be a moment, then, at which the argument is fully constructed and the subject can go on to apply the function or predicate to it. (*OnB*, p. 35.)

In some cases it is obvious that not the whole choice sequence is needed; for example, if the function returns to the 10^{th} element in the sequence or if the choice sequence is given by a law. But what about the general case, when the choice sequence is lawless? For Brouwer, the answer is simple: we never need the whole sequence, an initial segment of the sequence is sufficient to determine the output of the function in which the lawless sequence is an argument. The formalization of this thesis is given by WC-N:

WC-N: $\forall\alpha\exists x A(\alpha, x) \rightarrow \forall\alpha\exists n\exists x\forall\beta(\bar{\beta}n = \bar{\alpha}n \rightarrow A(\beta, x))$

(where A is an extensional predicate, α and β range over choice sequences, n and x over natural numbers and $\bar{\alpha}n$, $\bar{\beta}n$ stand for the initial segments of α and β respectively of length n). The principle says that if we have a construction that assigns a number to every choice sequence, all the information it needs to do so is an initial segment of that sequence. More information (intensional

[9]For some mathematical detail see, e.g., *EI*, pp. 22 et seqs.

properties) may be given, but is not required. The antecedent of the WC-N says that to each choice sequence α is related a natural number x; the consequent says that there is a natural number n such that any choice sequence β whose first n choices coincides with the first n choices of α is related to x in the same way as α is related to x. Thus, sharing the first n values with α is a sufficient condition for any choice sequence β to be related to that x; no more properties of β need to be considered.

Intuitionistically, the assignment of a function value to an argument requires a construction. When the argument is a choice sequence, two types of information about the argument are available to the subject. First, there is the initial segment of choices made so far. This is extentional information: it doesn't matter how the sequence is defined, just what choices are made at what stage. Second, there is the set of restrictions that the subject may have imposed so far. This is intentional information: it depends on how the sequence is defined. Both kinds of information are of a finite character. Out of these, the function value has to be constructed. The weak continuity principle claims that, if the function if defined for any choice sequences (the general case), only the first type of information is needed.

The weak continuity principle has always been held plausible and Brouwer used it freely. But he never gave a (suitable) justification for it notwithstanding the fact that without a principle such as WC-N, choice sequences could hardly be more than a mere intelectual curiosity or excentricity. Let's see why justification is required for WC-N.

The universe of choice sequences is inhabited (among other) by lawlike and lawless sequences and WC-N has to be true of both kinds of sequences; in other words WC-N has to held for the entire universe of choice sequences. Limited to the lawless sequences, the validity of WC-N seems to be plausible: as there are no first-order restrictions, there simply is no information about the choices except for the values chosen so far (the initial segment). So any other sequence with the same initial segment, when used as an argument for A (the binary predicate in WC-N), must give the same result. (*OnB*, p. 35.)

WC-N doesn't seems however to work as well for lawlike sequences. Let's see the following argument. Assume we have a proof of $\forall\alpha\exists xA(\alpha,x)$. That means that if we take a particular α, we can begin to construct a proof of $\exists xA(\alpha,x)$. When the proof is completed, only finitely many values of α have been chosen, as proofs are completed in finite amount of time. Therefore, for any β with the same initial segment we have $\exists xA(\beta,x)$. The problem with this argument is that the possible presence of intensional information (always present in lawlike sequences) has been neglected. For example, suppose that of the sequence α four values have been chosen so far, but also, at the fourth step, a restriction has been posed that from now on α is constant. Then one can immediately say what the 100^{th} value in α will be. However, only four values of α have been chosen, and it is certainly not the case that any β that begins with the same four values will also agree with α on the 100^{th} position. One could try to save the argument by allowing only numerical information about choice sequences in the construction of proofs. In effect, this is to treat all sequences as if they were lawless. But once more the principle would be established only for the class of the lawless sequences, which is not what we asked for. *What is required is an argument to the effect that, even if one is allowed to use intensional information, just an initial segment would suffice.* (*OnB*, p. 36.)

One way to obtain a rigorous argument for the validity of WC-N for the universe of choice sequences would be to derive it formally as a logical principle for choice sequences. Another way would be to justify it directly as an axiom for a theory of choice sequences or to derive it from a set of less controversial axioms for a theory of choice sequences. The first hypothesis is not possible and we will present an argument that supports this fact. The argument is based on a conclusion of Michael Dummett stated in *Elements of intuitionism*. While Dummett is discussing the problem of (intuitionistic) quantification over choice sequences in comparison to the classical quantification, he presents the following case as a puzzling one (for intuitionists). Suppose we want to prove the truth of a $\forall\exists$-statement in which choice sequences occurs, a statement of the form $\forall\alpha\exists n A(\alpha, n)$ ($A(\alpha, n)$ is an extensional predicate). Notice that the consequent of WC-N is a statement of this kind. So, the statement implies that there must be a uniform effective procedure for finding, for each given α, an n such that $A(\alpha, n)$. As α is a choice sequence, it cannot be 'given' in its entirety, so the procedure for finding n must operate upon some finite amount of information about α that we may possess at some stage. In this particular case, the claim is made that n may be computed from some sufficiently long initial segment of α, without further regard to possible restrictions upon future choices of terms that may have been imposed by that stage. In other words, if a statement of the form $\forall\alpha\exists n A(\alpha, n)$ is to hold, we must have an effective rule by which we can decide, for every finite sequence, whether or not it is sufficient to determine an n such that $A(\alpha, n)$ holds for every α of which that finite sequence is an initial segment, and which enables us to compute such an n if the sequence is sufficiently long; and every choice sequence α must have some initial segment from which the rule will compute such an n. (*EI*, p. 46.)

After giving this explanation, Dummett concludes that "that this is implied by a statement of the form '$\forall\alpha\exists n A(\alpha, n)$', where $A(\alpha, n)$ is extensional, is not an immediate consequence of the meanings of the [intuitionostic] quantifiers, but needs to be argued for on the basis of a more exact analysis of the notion of a choice sequence." (*Ibidem.*) What is implied in Dummett's affirmations is that the meanings of the intuitionistic quantifiers when used for quantifying over choice sequences are not sufficient to settle the truth of $\forall\exists$-statements, i.e., these type of sentences cannot be derived in a pure formal basis, just through the meanings of (intuitionistic) logical operators. As (both the antecedent and) the consequent of WC-N are sentences of that form, it goes the same for it: WC-N cannot be proved to be true just through the meanings of the logical operators in it, so it cannot be formally derived. And this is so because its strength (what it claims to be true) is based on specific *insights* about (quantification over) choice sequences which are not consequence of the meanings of the quantifiers. So this hypothesis of justification of WC-N is ruled out. How about to justify WC-N within the framework of a theory of choice sequences, i.e., in a framework where we restrict the quantification only to lawless sequences? In fact this second hypothesis (where we can say more about the quantifiers than in the general case) is taken more seriously and there has been a strong effort to justify WC-N as a principle for choice sequences within a particular axiomatic theory for choice sequences. In particular, Anne Troelstra has done major work on this field of research and we will now consider his account of choice sequences.

2.4 Troelstra's analytical approach to lawless sequences

In the task of justifying WC-N as an axiom or as a derivated principle for choice sequences, it is useful to make reference of an ideia coined by Kreisel under the recommendation of Gödel: the ideia of *informal rigour* when one is performing the conceptual analysis of an object or a class of objects.

> The 'old-fashioned' idea is that one obtains rules and definitions by analyzing intuitive notions (...) Informal rigour wants (i) to make this analysis as precise as possible (...) and (ii) to extend this analysis, in particular not to leave undecided questions which can be decided by full use of evident properties of these intuitive notions.[10]

So, the path traced by this method is to seek to obtain axioms for a certain sort of objects by studying the way these objects are given in intuition, i.e., by studying the way these objects's properties come to appear to us in our mind. In particular it stresses a way to seek certain axioms for choice sequences by studying the way the properties of choice sequences are given to us. This aproach to the study of choice sequences is called by Troelstra *the analytical approach*:

> The method of justifying axioms for choice sequences by reflection on what it means to be given a choice sequence ('conceptual analysis of the notion') can be carried considerably farther than is done in Brouwer writings, and leads to interesting insights and results. (*CinM II*, p. 643)

Besides the analytical approach to the study of choice sequences, there are two more approaches: the *holistic approach* and the *'figure of speech' approach*. The holistic approach rules out individual choice sequences as objects of study, it only considers *the notion* of choice sequence as an object of study and the point of departure of this approach is to grasp this notion as a *whole*. The "figure of speech" approach tries to *explain* what means to quantify over choice sequences without explicit reference to individual choice sequences. In a more formal version, it tries explains, within the context of a given language, what it means to quantify over choice sequences by translating sentences involving choice quantyifiers ($\forall\alpha$, $\exists\alpha$) into sentences not involving such quantifiers (a 'contextual definition' of $\forall\alpha$, $\exists\alpha$). And then, the characteristic principles for choice sequences are translated into true sentences. (*CinM II* p. 644.) The differences between the holistic and the 'figure of speech' approaches to choice sequences are the following. A holistic approach to the study of choice sequences would not allow us to speak of *individual* choice sequences. Sentences with quantification over choice sequences would be regarded as sentences stating properties about the *notion* of choice sequence, since the idea of an 'arbitrary individual choice sequence' cannot be properly defined. What is intended is to grasp the notion of choice sequence as a whole. (About this see the discussion on Weyl's account of choice sequences below.) The 'figure of speech' account is akind to the holistic approach, inasmuch the idea of an arbitrary individual choice

[10]Georg Kreisel, 'Informal rigour and completeness proofs', in *Problems in the philosophy of mathematics*, North-Holland, Amsterdam, 1967, pp. 138-139.

sequence is also rejected. The method to argue this mutual claim is, however, rather different: one admits quantification over individual choice sequences and then tries to show how these sentences can be translated into equivalent sentences without quantification over choice sequences. In the 'figure of speech' approach the point of departure is an *explanation* of what means to talk about choice sequences. Speech involving choice sequences can be paraphrased into a speech involving only mathematical objects whose epistemic status is not disputed (e.g., lawlike sequences).

Troelstra stresses the fact that the three approaches to the subject of choice sequences "can merge gradually and almost imperceptibly into each other." (*Ibidem.*) But the analytical approach is the one to be mainly pursued by him because "choice sequences are not only a good example of the possibilities (and limitations) of conceptual analysis, but are also of interest in themselves, as demonstrating the possibilities of coherent reasoning about [individual] incomplete objects"; i.e., the analytical approach enables us to study the role of intensional aspects of choice sequences in mathematics. (*CinM II*, p. 643.) For Troelstra, an example for the cogency of conceptual analysis over choice sequences is Brouwer's weak counterexamples using lawlike sequences showing that classical logic is 'unreliable' in intuitionistic mathematics, while lawless sequences allow to refute the laws of classical logic (on intuitionistic grounds), since WC-N has consequences as ¬∀-PEM. (*Ibidem.*) One source of limitations to the analytical approach is in the fact that, in building a theory, we unavoidably idealize, i.e., in thinking about choice processes carried out by an ideal mathematician, certain aspects of an actual choice process (mood, time, etc.) are automatically left out, are abstacted from and regarded as mathematically irrelevant. Another source of limitations to the analytical approach is that we can try to be as precise and rigorous as possible in the justifications of principles valid for such notion, but we cannot expect absolute rigour: in presenting the justification for certain principles in terms of a conceptual analysis, we may have to accept that sometimes we have to make *intuitive jumps*, namely when a new insight seems to be required, while we are unable (at the moment) to analyze the matter any further. In such cases, Troelstra accepts that the best we can do is to show as clearly as possible what it is that we have to accept. (*CinM II*, pp. 643-644.)

We will now compactly state Troelstra's set of axioms for the domain of choice sequences and then we will present a critical discussion for each one. **LS** is an axiomatic theory for lawless sequences with the following four axioms[11]:

LS1: $\forall n \exists \alpha (\alpha \in n)$

LS2: $\forall \alpha \forall \beta (\alpha \neq \beta \vee \alpha = \beta)$

LS3: $\forall \alpha [(\neq (\alpha, \vec{\beta}) \rightarrow A(\alpha, \vec{\beta})) \rightarrow \exists n (\alpha \in n \wedge \forall \gamma \in n A(\gamma, \vec{\beta}))]$

[11]The language of **LS** contains the following variables: $x, y, z, u, v, w, ...$ for natural numbers, $a, b, c, d, e, f, ...$ for lawlike sequences, $\alpha, \beta, \gamma, \delta, ...$ for lawless sequences, and in addition a constant K for a species/class of sequences; K is the inductively defined class of (lawlike) neighbourhood functions. The following abbreviations are used: $\vec{\alpha} := (\alpha_1, \alpha_2, ..., \alpha_n)$; $\forall \vec{\alpha} := \forall \alpha_1, \forall \alpha_2, ..., \forall \alpha_n$. The following definitions are introdued: $\neq (\alpha, \vec{\beta}) := \alpha \neq \beta_1 \wedge ... \wedge \alpha \neq \beta_n$; $\not\approx \vec{\alpha} := \forall i, \forall j (\alpha_i \neq \alpha_j)$. The notation '$\alpha \in n$' means that a finite sequence n is an initial segment of α and the notation '$\forall \alpha \in n$' stands for 'all lawless sequences α with the (same) initial segment n'.

LS4: $\forall\vec{\alpha}\exists a[\not\approx\vec{\alpha} \to A(\vec{\alpha}, a)] \to \exists e \in K\forall n[en \neq 0 \to \exists a\forall\vec{\alpha} \in n(\not\approx\vec{\alpha} \to A(\vec{\alpha}, a))]$

Troelstra starts by saying that there is not a single type of choice sequence, but there are many (lawlike, lawless, hesitant, etc.) according to the *type of data* (information) which can be known about the sequences in the course of their construction. (*CinM II*, p. 645.) Let's remind that for a lawless sequence the choice of future values is at any stage completely free and that distinct lawless sequences are completely independent, i.e., the values of any such sequence are not determined or restricted relative to the values of other lawless sequences. So we may think of a lawless sequence α as a process of generating values $\alpha(0), \alpha(1), \alpha(2), ...$ in $\mathbb{N}$ without any general restriction, such that at any stage of the process only finitely many values of α are known (and the further choices completely left free). (*CinM II*, p. 645.)

The complete freedom of choice in lawless sequences suggests that the following principle is true for them:

LS1: $\forall n\exists\alpha(\alpha \in n)$.

The intuitive content of this axiom, the density axiom, is that each possible initial segment n occurs as an initial segment of some lawless sequence α, i.e., any finite sequence is an initial segment of a lawless sequence. (*Ibidem*.) For Troelstra, a good intuitive representation of this idea is that the sequence of the casts of a die (except that that now the values have to come, not from $\mathbb{N}$, but from $\{1, 2, 3, 4, 5, 6\}$).[12] At no stage we know more than an initial segment of this sequence. However, as the values are not freely chosen, but determined by throwing the die, we cannot be certain that all initial segments will accur, no matter how many dice we take into consideration. To get a better aproximation, we should permit a number of deliberate placings of the die before we starting throwing; this ensures the validity of (the analogue for dice of) the density axiom. (*Ibidem*.)

Let's now present Troelstra's *critical discussion* of **LS1**. He starts by saying that there is a slight difficulty in the preceeding discussion of this axiom: if we rigorously stick to the idea that at any stage a single value is to be chosen, and no further restrictions are to be made, can we then be certain that any possible initial segment will occur as initial segment of *some* lawless sequence? How can we be certain that a particular initial segment of lenght 10 (for example) will occur? (*CinM II*, p. 646.) Bringing again the analogy with de casts of a die, we can put this difficultyt in another terms: as a thought experiment, we may envisage all possible sequences of casts of dice started simultaneously, we *expect* but cannot be *certain* that all possible initial segments from $\{1, 2, ..., 6\}$ will occur. As to the lawless sequences themselves, the assertion that a particular initial segment will not occur (formally $\neg\exists\alpha(\alpha \in n)$) intuitively conflicts with the freedom of making choices, so we convinced that at least it is true that $\neg\neg\exists\alpha(\alpha \in n)$, but that is not enough to justify the density axiom. (*Ibidem*.) Let's remember that in intuitionistic logic the principle $\neg\neg A \to A$ is not valid, so we

[12] Troelstra likes this idea, but it is also known that Brouwer already objected to it. (Remark by Freudenthal, reported in Dirkvan Dalen's introduction to *Brouwer's Cambridge Lectures on Intuitionism*, Cambridge University Press, Cambridge, 1981.)

cannot justify (passing from $\neg\neg\exists\alpha(\alpha \in n)$ to) $\exists\alpha(\alpha \in n)$, asserted in the density axiom.

Troelstra's idea to move over this 'slight difficulty' is to modify the initial idea of a lawless sequence permitting a better analogy between lawless sequences and casts of a die. So, to get a better analogy, says Troelstra, we should for lawless sequences permit *the specification of an arbitrary initial segment in advance*. Then we may think of our universe of lawless sequences as containing at *least* some $\alpha_n \in n$, for each n, i.e., containing at least an arbitrary lawless sequence with an initial segment n given in advance. Our universe may contain other lawless sequences besides α_n. Thus the density axiom seems to be guaranteed for this slightly modified idea of lawless sequence, permitting the stipulation of certain initial segments *a priori*. And it is this new idea of lawless sequence that Troelstra adopts in the further conceptual analysis. To avoid confusion between the original and the new (modified) idea of a lawless sequence, he introduces the designation "*proto-lawless sequence*" to the original unmodified idea and mantains the designation "lawless sequence" to the new modified one. Based on this, we may think of the proto-lawless sequences as forming part of the universe of lawless sequences, consisting of those lawless sequences for which no initial segment was specified in advance; and in fact, he says, we have no reasons to uphold the density axiom for the proto-lawless sequences. (*Ibidem*.)

If we write $\alpha \equiv \beta$ to indicate that α and β really refer to the *same* (intensional) process, then, translated in terms of "casts of a die", $\alpha \equiv \beta$ means that α and β are the same sequences of casts of the same die. By this definition of intensional identity we have, for all properties A,

$A(\alpha) \wedge \alpha \equiv \beta \rightarrow A(\beta)$.

If α has a property A and α is given by the same intensional process that generates β, then β also has the property A; in other words, of α and β are given by the same process, we can substitute one by the other in any formula in which one of them occurs. Another property of intensional identity is that it is always decidable in intuitionistic mathematics:

$\alpha \equiv \beta \vee \neg(\alpha \equiv \beta)$,

since we *know* whether α and β refer to the same process or not (in terms of dice: we know whether we are referring to the same die or not). (*CinM II*, pp. 646-647.) Generally, intensional identity is decidable because it depends solely upon the way the objects are given to us, and we must be able to tell whether the way in which an object is presented to us on one occasion is or is not the same as that in which an object is presented to us on another. (*EI*, p. 215.)

The intensional identity of lawless sequences already implies extensional equality of lawless sequences:

(1) $\alpha \equiv \beta \rightarrow \alpha = \beta$,

where as usual $\alpha = \beta$ is formally defined as $\forall x(\alpha(x) = \beta(x))$. If two sequences are given by the same process, then their values will coincide. Otherwise, it would be possible for two identical lawless sequences to had different initial segments, for instance; which is absurd.

The reciprocal case,

(2) $\alpha = \beta \to \alpha \equiv \beta$,

is not so intuitive. Does the fact of two lawless sequences conicide in its values means that they are generated by the same intensional process? Is it not reasonable to think of two lawless sequences that has the same values even though being intensional different processes? They can accidentally have the same values. Suppose we have $\alpha = \beta$ and $\alpha \not\equiv \beta$. Notice that the distinctness of α and β ($\alpha \not\equiv \beta$) conflicts with $\alpha = \beta$ since $\alpha = \beta$ implies that β is completely determined by α and $\alpha \not\equiv \beta$ means that α and β are independant of each other. Formally, we can state that $(\alpha \not\equiv \beta) \to \alpha \neq \beta$, which conflicts with the assumption that $\alpha = \beta$.

Troelstra says that one may be tempted to try to refute it by the following 'counter example'. Suppose sequences α and β are given to us by means of values produced by two distinct *black boxes* (a black box is simply a process the workings of which are unknown to us), one for α and one for β. To us (subjectively) α and β therefore appear as (proto-)lawless sequences. But suppose we are told, after a long time, that there is a hidden connection between the boxes, such that they will keep turning out the same values; then suddenly we know that $\alpha = \beta$. We could not decide this beforehand, nor were we given, at the start, a guarantee that the decision could ever be made in the future. So we have no right to assert the axiom schema of decidability of equality generally. However, says Troelstra, this 'counterexample' starts from the wrong picture of lawless sequences. In fact, we should look at the example in a different way. Initially, α and β were given to us as distinct processes; later, we have learn that $\alpha = \beta$, and this is a type of information which was not permitted by our description of a lawless sequence. If, though considering α and β as distinct ($\alpha \not\equiv \beta$), we have to reckon with the possibility that later we will learn that $\alpha = \beta$, then α and β cannot be considered as lawless from the beginning, but falling under another notion of sequence. This is argued by Troelstra because choice sequences initially given as distinct and lawless should remain *independent*. (*CinM II*, pp. 647.)

There is another argument for **LS2**, presented by Dummett (*EI*, pp. 215 et seq.) that makes use of a major principle for lawless sequences, the principle or schema of open data. This principle states that for lawless sequences any property which can be asserted must depend on initial segments of these sequences only. If A is a property of lawless sequences, not containing any further choice parameters, we can formulate the one-variable or schematic form of the axiom of open data:

$$A(\alpha) \to \exists n(\alpha \in n \wedge \forall \beta \in n A(\beta)).$$

The intuitive content of this principle is that if α has a property A, then exists an initial segment n of α such that for any β with n, β also as the property A.

(*CinM* vol. II, p. 648.) If we introduce a parameter β[13] stating that α and β are in the relation A plus the clause that they are generated by different processes, we get the following version of the open data schema:

$$A(\alpha, \beta) \wedge (\alpha \not\equiv \beta) \rightarrow \exists n \forall \gamma \in n((\gamma \not\equiv \beta) \rightarrow A(\gamma, \beta))$$

This version of the schema of open data states that for any two *independant* sequences α, β that bare the relation A, we only need an initial segment of the first to establish that a third sequence γ sharing the same initial segment and being independant of β is in the same relation A to β.

Given this, Dummett's argument runs as follows. It was shown that intensional identity verifies the principle of excluded middle: $\alpha \equiv \beta \vee \alpha \not\equiv \beta$. We know also that intensional identity implies extensional equality: $\alpha \equiv \beta \rightarrow \alpha = \beta$. Then, to prove the reciprocal, it is enough to show that the conjunction of $\alpha = \beta$ and $\alpha \not\equiv \beta$ leads to a contradiction (since $\neg\neg(\alpha \equiv \beta)$ implies $\alpha \equiv \beta$ by the principle of excluded middle for intensional identity). Suppose we have $\alpha = \beta$ and $\alpha \not\equiv \beta$ as the absurd hypothesis. If we settle that $A(\alpha, \beta)$ is the extensional equality $\alpha = \beta$, we have $A(\alpha, \beta) \wedge \alpha \neq \beta$. Hence, by the (modified) principle of open data, there exists a finite sequence n such that, for all $\gamma \in n$ (i.e., γ coincides with α up to $n - 1$), we have $A(\gamma, \beta)$. Let's take (by the density axiom) a γ in such conditions, so that $\gamma(n) \neq \beta(n)$. Then $\gamma \neq \beta$, which is a contradiction.

Using the density axiom and the (modified) priciple of open data it was shown that $\alpha = \beta \leftrightarrow \alpha \equiv \beta$. This result allows the replacement of intensional identity by extensional equality in **LS**. Then, by substitutivity and decidability of intensional identity, we can assert the following:

Axiom schema of decidability of equality: $\alpha = \beta \vee \alpha \neq \beta$.

By simple quantification over the free variables α, β we get

LS2: $\forall \alpha \forall \beta(\alpha \neq \beta \vee \alpha = \beta)$

The replacement of intensional identity by extensional equality is technically advantageous and makes **LS** a more economic and elegant theory.

Dummett's argument for **LS2** gives us the motto for discussing **LS3**. If we introduce the abbreviation

$$\not\equiv (\alpha, \beta_0, ..., \beta_n) := (\alpha \not\equiv \beta_0 \wedge \alpha \not\equiv \beta_1 \wedge ... \wedge \alpha \not\equiv \beta_n),$$

we can also easily state the general form of open data with parameters:

Axiom of open data: $A(\alpha, \vec{\beta}) \wedge \not\equiv (\alpha, \vec{\beta}) \rightarrow \exists n[\alpha \in n \wedge \forall \gamma \in n(\not\equiv (\gamma, \vec{\beta}) \rightarrow A(\gamma, \vec{\beta}))]$

This version of the schema of open data intuitively states that for every *independent* lawless sequences in the relation A, we only need an initial segment of them to establish whether another lawless sequence, different from them, bares the same relation to A. Notice that $\neq (\alpha, \vec{\beta})$, in virtue of the identification

[13] We use this slightly modified version of open data for an easier explanation of the argument. Dummett uses the general form with a n-variable parameter.

of = and ≡, expresses that α is independent (distinct from) of the $\vec{\beta}$ and this condition is really necessary, for otherwise it could give raise to conflicts with the density axiom. As a fact of matter, Troelstra shows an instance of a sentence originated by the axiom of open data without the condition $\neq(\alpha,\vec{\beta})$ that is refuted by the density axiom). Suppose we have $A(\alpha,\beta) := \forall n(\alpha n = \beta n)$ and apply to it the axiom of open data without $\neq(\alpha,\vec{\beta})$, $\neq(\gamma,\vec{\beta})$. We find then that if $\alpha = \beta$, $\exists m(\alpha \in m \wedge \forall\gamma \in m(\gamma = \beta))$, which is obviously false and also refutable by the density axiom: take any $\gamma \in \alpha m * \langle \beta m + 1\rangle$. (*CinM II*, p. 648.)

On the critical discussion of the principle of open data, Troelstra starts by indicating a possible objection. For the proto-lawless sequences, the open data's schema seems to him to be irreproachable, but as to its general validity for (the modified) lawless sequences, it is possible to raise an objection: there is one piece of information concerning a lawless sequence α which cannot be read of from an initial segment of α, namely the length of the initial segment that was fixed in advance. In a more rigorous account, let Φ_I be the operation assigning to each lawless sequence α its initially specified segment. Then for any α there is an n such that $\Phi_I(\alpha) = n$, i.e., such that Φ_I assigns n to α. This originates a counterexample to open data's schema: applying the open data's schema to $\Phi_I(\alpha) = n$, we find that, for some extension m of n the following is the case, $\forall\beta \in m(\Phi_I(\beta) = n)$, and this is easily refuted, e.g. by considering a β with $\Phi_I(\beta) = m * \langle 0\rangle$. So, if we wish to maintain the open data's schema, we should only consider properties A which do not refer to Φ_I. A more natural way of looking at the matter, says Troelstra, is perhaps the following one. Just as in the actual process of generating values of a lawless sequence, we have abstracted from irrelevant circumstances accompanying a process of choosing in the physical world (time between choices, the weather, etc.), it seems to be mathematically irrelevant which part of an individual lawless sequence was determined *a priori*, and which part generated by further free choices. Thus for the sake of open data's schema we have to restrtict $A(\alpha)$ to properties different from Φ_I. (*CinM II*, p. 649.)

As to the axiom of open data (with parameters), whenever we assert at some stage $A(\alpha,\vec{\beta})$, there are three types of data regarding α on which our assertion could be based: (1) an initial segment of α, (2) Φ_I, which was already excluded by the preceeding analysis, (3) assertions of the form $\alpha = \beta_i, \alpha \neq \beta_i$, i.e., the identity or non-identity of α with one of the other parameters. Under the assumption of $\neq(\alpha,\vec{\beta})$ only the first type of data is relevant, which leads to the axiom. On the other hand, however, at any stage there is more information than just the initial segment determined so far: we also know the stages at which the values were choosen, or equivalently, how many stages where added at each preceeding stage. But just as in the case of Φ_I above, this ought to be regarded as irrelevant; we abstract from such details of the generation process. Given the previous analysis, and by quantification over the free variables of the axiom of open data with parameters, we are justified to assume

LS3: $\forall\alpha[(\neq(\alpha,\vec{\beta})) \to A(\alpha,\vec{\beta}) \to \exists n(\alpha \in n \wedge \forall\gamma \in nA(\gamma,\vec{\beta}))]$

as an axiom of **LS**. (*CinM II*, pp. 649-650.)

WC-N can now be derived from **LS3**. Assume (1) $\forall\alpha\exists xA(\alpha,x)$. As an instance of **LS3** we have (2) $A(\alpha,x) \to \exists n(\alpha \in n \wedge \forall\beta \in nA(\beta,x))$. By isolating the consequent of (2) we get (3) $\exists n(\alpha \in n \wedge \forall \beta \in nA(\beta, x))$. Then, by quantifying over the free variables of (3), we get (4) $\forall \alpha \exists x \exists n \forall \beta(\alpha n = \beta n \to A(\beta, x))$. Finally, (1)-(4) allows us to formulate WC-N.

LS3 is a form of continuity that enables us to assert that properties of lawless sequences depend on initial segments of the sequences only. But it does not exhaust our intuitions about continuity and, in fact, stronger forms of *continuity axioms* can be argued for. Suppose $\Phi \in \mathbb{N}^{\mathbb{N}} \longmapsto \mathbb{N}$ is a continuous functional in $\mathbb{N}^{\mathbb{N}}$. Then, says Troelstra, it seems plausible to assume that "we actually *know* whether an initial segment of α suffices to compute $\Phi\alpha$." (*CinM I*, p.211. Troelstra's italics.) This is to say that we can actually associate with ϕ a *lawlike* function $\varphi \in \mathbb{N} \longmapsto \mathbb{N}$ such that

$\varphi(\bar{\alpha}n) = 0 \Leftrightarrow \bar{\alpha}n$ is not sufficiently long to compute $\Phi\alpha$ from it; and

$\varphi(\bar{\alpha}n) = m + 1 \Leftrightarrow \bar{\alpha}n$ is sufficiently long to compute $\Phi\alpha$ from it, and $\Phi\alpha = m$.

From this definition of φ follows two immediate results: (i) for any sequence α it is allways possible to determine that there is a initial segment sufficiently long to compute it, i.e., $\forall \alpha \exists x(\varphi(\alpha x) \neq 0)$; and that after we have found the element n that suffices to compute α, any other initial segment m longer than αn yield the same result, i.e., $\forall n \forall m(\varphi n \neq 0 \to \varphi n = \varphi(n * m))$. The first condition is a consequence of Φ being defined everywhere and the second condition is due to the fact that the value computed should not change if more information about the argument becomes available. Conversely, any φ satisfying these two conditions defines a continuous functional Φ.

We are able now to define K_0 as the class of (neighbourhood) lawlike functions by

$\varphi \in K_0 := \forall \alpha \exists x(\varphi(\alpha x) \neq 0) \wedge \forall nm(\varphi n \neq 0 \to \varphi n = \varphi(n * m))$.

The function $\varphi \in K_0$ assigns an initial segment (of length) n to each α; so we can say that φ determines *an* initial segment which compute α. We can now state a form of **LS4** without parameters:

LS4$_0$: $\forall \alpha \exists a A(\alpha, a) \to \exists e \in K_0 \forall n(en \neq 0 \to \exists a \forall \alpha \in nA(\alpha, a))$.

LS4' is a final strengthening of continuity and it is obtained by substituting in **LS4$_0$** K_0 by the class K of *inductive defined* (lawlike) neighbourhood functions:

LS4': $\forall \alpha \exists a A(\alpha, a) \to \exists e \in K \forall n(en \neq 0 \to \exists a \forall \alpha \in nA(\alpha, a))$.

To describe K and discuss how it relates to K_0 we need to introduce in advance some definitions.[14] We define a *tree* T as an inhabited, decidable set of finite sequences of natural numbers closed under initial segments; so T is a tree iff $\langle\rangle \in T$, $\forall n(n \in T \vee n \notin T)$ and $\forall n \forall m(n \in T \wedge m \prec n \to m \in T)$. T is a *regular* tree if $\forall s \in T$, s is a *terminal node*, i.e., $\neg\exists x(s * \langle x \rangle \in T)$ or $\forall x(s * \langle x \rangle \in T)$.

[14] '$\langle\rangle$' is the null sequence, '$x \prec y$' means that x is the *predecessor* of y (with x, y finite sequences) and '$x * y$' can be read as 'x followed by y' (x, y are numbers, finite sequences or infinite sequences, which becomes clear due to notation).

A regular tree T is *well-founded* when all branches (sequences) 'leave' the tree, i.e., $\forall \alpha \in \mathbb{N}^{\mathbb{N}} \exists n(\alpha n \notin T)$.

Let's write $\mathscr{T}$ as the class of *regular well-founded trees*. Thus, $\mathscr{T}$ is precisely the class of sequences determined by elements of K_0. Let's now define the class $\mathscr{I}\mathscr{T}$ as the class of regular well-founded trees *inductively* defined. The following two clauses generate regular well-founded trees: (1) the single node $\{\langle\rangle\}$ is a regular tree; (2) if $T_0, T_1, T_2, ...$ is an effective sequence of such trees, we can combine them into a new regular well-founded tree saying that their *sum* $\sum_n T_n$ is a regular well-founded tree. If a regular tree is generated by repeated application of (1) and (2) only, we call it an inductive tree. The class $\mathscr{I}\mathscr{T}$ of such inductive trees is then generated by the following reformulations of (1) and (2):

(1′) $\{\langle\rangle\} \in \mathscr{I}\mathscr{T}$,

(2′) $\forall x(T_x \in \mathscr{I}\mathscr{T}) \to \sum_x T_x \in \mathscr{I}\mathscr{T}$.

$\mathscr{I}\mathscr{T}$ can also be described as the least class satisfying (1′) and (2′), since we only put something in $\mathscr{I}\mathscr{T}$ by virtue of its construction from (1′) and (2′); so, if $\mathscr{X}$ is any class of sets of finite sequences satisfying (1′) and (2′), then also $\mathscr{I}\mathscr{T} \subseteq \mathscr{X}$. By this reason, the inductively defined trees contitute a subset of the regular well-founded trees, $\mathscr{I}\mathscr{T} \subseteq \mathscr{T}$. And if we assume that the functions generated by the elements of $\mathscr{I}\mathscr{T}$ form the class K, then we also have $K \subseteq K_0$.

The point in using K in **LS4** instead of K_0 is due to the following fact: the identification of K_0 and K amounts to the assumption that all the regular well-founded trees (all sequences) are in fact generated by the above special method of induction. Intuitionisticaly however, notwithstanding the fact that we can prove $K \subseteq K_0$, the converse case is not formally established, and thus, it has to be assumed as a principle that $K_0 \subseteq K$. Classically we can prove $K_0 \subseteq K$, but the proof uses the principle of excluded middle, and is therefore of no use in obtaining an intuitionistic justification of $K_0 = K$. The assumption of $K_0 = K$ is equivalent to state that *all* sequences are in fact generated (by induction). This principle is called *bar induction* and it is an essential assumption to the development of intuitionistic mathematics.[15]

The identification of K_0 and K is necessary in order to state the final version of **LS4** with parameters:

LS4: $\forall \vec{\alpha} \exists a[\not\approx \vec{\alpha} \to A(\vec{\alpha}, a)] \to \exists e \in K \forall n[en \neq 0 \to \exists a \forall \vec{\alpha} \in n(\not\approx \vec{\alpha} \to A(\vec{\alpha}, a))]$.

This princple is, in Troelstra's words, "the biggest single 'jump' in the justification of the axioms for lawless sequences" (*CinM II*, p. 655). The intuitive motivation for equating K_0 and K is that we are equating two notions of well-foundedness: the ordinary well-foundedness and the inductive well-foundedness for regular trees. This assumption, however, is only plausible if we assume K to contain all choice sequences, not only the lawlike ones.

[15]See Fernando Ferreira, '*Grundlagenstreit* e o Intuicionismo Brauweriano' in *Boletim da Sociedade Portuguesa de Matemática*, Nž 58, 2008, pp. 1-23. This paper was also very useful to understand the distinction between the first and the second act of intuitionim.

2.4.1 The Elimination theorem

Let's now discuss (in a not too much technical way) a very interesting theorem about lawless sequences, proved by Kreisel and Troelstra within **LS**: the *elimination theorem* for lawless sequences.(*CinM* vol. II, pp. 658-665; in these pages Troelstra presents the theorem in more technical and detailed terms.) Troelstra describes at some detail a subtheory **S** of **LS**: **S** is the lawlike fragment of **LS**, i.e., the part of **LS** not involving lawless variables. Given this the following result holds:

Elimination theorem: *There is a syntactically effective translation* τ (see *op. cit.*, pp. 663-664), *mapping formulas of* **LS** *without free lawless variables to formulas of* **S** *such that:*

(i) $A \Leftrightarrow \tau(A)$ *for A without lawless variables*,

(ii) **LS** $\vdash A \leftrightarrow \tau(A)$,

(iii) **LS** $\vdash A \Leftrightarrow$ **S** $\vdash \tau(A)$.

The part (i) of the theorem is simply the result that the translation τ permitts to show the equivalence between sentences involving quantified lawless variables (A) and sentences not involving such lawless variables ($\tau(A)$). Part (ii) shows that, within the context of **LS**, sentences involving lawless variables are equivalent to their *translated* counterparts. And part (iii) of the theorem states that any sentence involving lawless variables is provable in **LS** if and only if its translation is provable in **S**.

The elimination theorem has two possible conceptual interpretations concerning lawless sequences. The proponents of (lawless) choice sequences as legitimate mathematical objects look at the theorem as showing that speech about choice sequences has mathematical coherence. The translation yields equivalent statements of the form $A \leftrightarrow \tau(A)$ where A quantifies over lawless sequences and $\tau(A)$ does not. But as A is a component of the statement $A \leftrightarrow \tau(A)$ as a whole, the lawless sequences are still involved. So, being part of a genuine mathematical statement, A cannot be regarded as illusory. (*BmH*, p. 41.) For this reason, and under the suggestion of Kreisel, the translations are usually understood as giving a *complete analysis of lawless sequences* via contextual definition and, therefore they cannot be regarded as *eliminating lawless sequences*. Therefore, this theorem does not eliminate (in the strong sense of the word) lawless sequences but *explains* them in terms of less controversial mathematical entities, viz. lawlike sequences. And, for the proponent of (lawless) choices sequences, this fact supplies lawless sequences with the same degree of coherence as lawlike sequences.

Another possible way of looking to the elimination theorem takes the 'figure of speech' account of choice sequences at face value, i.e., *literally*. Let's call this interpretation of the elimination theorem the *eliminativist* account of lawless sequences. The eliminativists would say that the elimination theorem shows that the discourse about lawless sequences is nothing but an *illusory* way of talking about lawless sequences; for, in fact, what we are talking about, within the context of **LS**, are lawlike sequences, i.e., as the quantification over lawless sequences is eliminated, the only things which we quantify over (which we talk about) are lawlike sequences. So, the eliminativists can say that we are mislead in talking about lawless sequences and that they don't occour as primitive individual objects in **LS**. There are very interesting analogies in the history of philosophy of mathematics that can elucidates this thesis. (1) For Frege, mathematics (number theory) does not deal with numbers but with logic ('logical concepts') and it is his motivation in trying to show that the arithmetical laws are in reality logical laws. (2) For Hilbert, the real objects of mathematics were empirical signs ('marks on paper') and not sets. (3) And for Wittgenstein, mathematics was no more than a prescription of rules for constructing (mathematical) sentences ('language games'). The same can be stated by an eliminativist: **LS** does not deal with lawless sequences but with lawlike ones; even using variables (in an illusory way) for lawless sequences what we are really talking about are lawlike sequences. (*BmH*, pp. 40-41.)

There is, however, a not so radical eliminativist interpretation of the elimination theorem. It is a mixture of the '*figure of speech*' and the *holistic* approaches to choice sequences. In the eliminativist literal interpretation of the 'figure of speech' account, the quantification over lawless sequences is regarded as illusory because we can translate the sentences with quantification over lawless sequences into equivalent ones in which such quantificaton does not occur. The holistic account of choice sequences accepts the universe of choice sequences as *a single primitive notion* (as a whole), i.e., quantification over choice sequences is permitted but only as a primitive (non definable) notion. So, in the holistic account of choice sequences there is not such thing as a contextual definition of quantification over lawless sequences because quantification over choice sequences is seen as a primitive notion. This interpretation seems to be the one favored by Weyl. We turn to Weyl's interpretation now.

2.5 Weyl's asymmetric interpretation of quantifiers for choice sequences

The third interpretation of the elimination theorem joins together these two accounts of insigths on quantification over choice sequences in a very specific way. To explain this possible interpretation of the elimination theorem we will present Hermann Weyl's ideas on the subject of choice sequences, exposed in the context of his general theory of the *continuum*, with particluar emphasis on his original and somewhat eccentric ideas on what it means to quantify over mathematical entities. Weyl's particular view on quantification over choice sequences is an interpretation of his writings advanced, among others, by Troelstra when he says that in "Weyl's discussion of choice sequences (...) there is a faint echo of the semi-intuitionist's holistic view of the continuum, but on the other hand some of his formulations tend towards a 'figure of speech' interpretation." (*CinM II*, p. 644.) For the exposition of Weyl's ideas on choice sequences and in order to understande how its own view deviates from Brouwer's we will follow the exposition of the subject presented by v. Atten, D. v. Dalen and R. Tieszen in the article 'Brouwer and Weyl: the phenomenology and mathematics of the intuitive continuum'.[16] The exposition of Weyl's ideas on this paper is an elegant reproduction of the arguments presented on Weyl's famous 1921 article 'On The New Foundational Crisis of Mathematics'.[17]

Weyl holds that there is a use in mathematics for the *notion* of lawless sequence, in that it allows us to conceptualize the continuum in the right way. "It is one of the fundamental insights of Brouwer that number sequences, developing through free acts of choice, are possible objects of concept formation." ('New Crisis', p. 94.) On the other hand however, he claims that "the expression 'There is' commits us to Being and law, while 'every' releases us into Becoming and Freedom." ('New Crisis', p. 96.) The explanation given by Weyl of his *semantics* for the quantifiers seems to be that it is essential to pure mathematics that its objects can be *coded* in, or represented by natural numbers:

> Every application of mathematics must set out from certain objects that are to be subjected to mathematical treatment, and that can be distinguished from one another by means of a number character. The characters are natural numbers. The connection to pure mathematics and its constructions is achieved by the symbolic method, which replaces these objects by their characters. The point geometry on the straight line is, in this way, based on the system of the above-mentioned dual intervals, which we are able to identify by means of two whole number characters." ('New Crisis' pp. 100-101.)

The restriction that any mathematically acceptable entity has to be coded in natural numbers is based in Weyl's claim that "the sequence of natural numbers, and the intuition of iteration underlying it, are ultimate foundations of mathematical thought." ('New Crisis' p. 91.) For Weyl, *application of mathematics* does not always mean what it normally would, and does not necessarily contrasts with what is usually understood as pure mathematics. For him point geometry (for example) already is an application of mathematics, whose coded objects are rational segments. This example illustrates that the substitution of numbers for objects in the process of symbilization cannot be a mere labeling of those objects. Some information needs to be preserved, depending on what we want to use the mathematics for. When that is done we are back in the realm of 'pure mathematics and its constructions'.

Since laws are finitely specifiable (countable), they can be coded in natural numbers and therefore, by the criterion above, Weyl accepts lawlike sequences as genuine individual objects of (pure) mathematics. Lawless sequences, however, cannot in general be coded by natural numbers, hence they are not to be regarded as individual mathematical objects. So, although he recognizes a role for lawless sequences in conceptualizing the continuum, in the end mathematics is only about numerically codifiable objects, viz. lawlike sequences. (*Ibidem*.)

[16] M. van Atten, D. v. Dalen and R., 'Brouwer and Weyl: the phenomenology and mathematics of the intuitive continuum'. We had access to this article through the URL: www.phil.uu.nl/~dvdalen/articles/Brouwer-Weyl-page.pdf. This article was also published in *Philosophia Mathematica*, volume 10, 2002, pp. 203-226; but unfortunately we did not have access to it. For commodity we will refer to this paper as 'Brouwer and Weyl' only.

[17] H. Weyl, 'On The New Foundational Crisis of Mathematics', in *From Brouwer to Hilbert. The debate on the foundations mathematics*, pp. 86-118. For commodity we will refer to this paper just as 'New Crisis'

Based on his approach to mathematical entities, Weyl advances the thesis that from conceptual truths about lawless sequences, one arrives at genuine mathematical statements by substituting lawless for lawlike sequence variables:

> The proper judgments that can be gained from these universal judgments come into being (...) in the case of the freely developing choice sequence, by substituting for it a law ϕ that determines an individual number sequence in infinitum. ('New Crisis', p. 100.)

This means that the concept of lawless sequence is meaningful, but only if in the course of mathematical application we are able to substitute lawless vriables by lawlike variables in the sentences they occur. In fact, individual lawless sequences are beyond the reach of Weyl's methods of construction: only objects that can be coded into natural numbers can be constructed in the strong sense. ('Brouwer and Weyl', p. 14.) Weyl tries however, by a disputed argument, to show that, albeit lawless sequences are not mathematical objects, they have mathematical applicability. He says that "this fact is supported by the possibility of establishing correspondences between them [and lawlike sequences]" ('New Crisis', p. 94.) He implicitly postulates that we can choose a lawless sequence as to follow some lawlike one. As all lawlike sequences can be embedded in the continuum of lawless sequences ('New Crisis', p. 100) and it may be part of the meaning of a lawless sequence that it does or does not possess a property E ('New Crisis', p. 96), then there will be general judgment directions for lawlike sequences ('New Crisis', p. 100). The idea behind this postulate is that we will be able to choose the elements of a lawless sequence such that it happens to coincide extensionally with the elements of a lawlike sequence and, doing so, we will guarantee that some properties of lawlike sequences would be inheritated by the lawless sequences. And so, we will be able to make general (universal) judgments for lawlike sequences. In fact, what Weyl is doing is to restrict the domain of lawless sequences which we can deal with by taking into account only the class of lawless sequences that coincidentally follow some lawlike sequence.

There are those who regard this postulate as unacceptable because to specify a lawless sequence α by saying that it follows some lawlike sequences a would contradict the lawless character of α. As lawless sequences are necessarily unfinished objects, one cannot say that a law and a sequence of free choices may simply be alternative ways to describe the same completed infinite sequence. ('Brouwer and Weyl', p. 15.) It can be argued however that what is postulated by Weyl is not that the (idealized mathematician's) generation of a lawless sequence is *determined* to follow a lawlike one, but that *by chance* the completely free generated lawless sequence coincides extensionaly with a lawlike one. Suppose the idealized mathematician starts a lawless sequence and, after some amount of time, by looking back to the sequence generated so far he realizes that (by a twist of fate) the lawless sequence he is generating embodies a law he was unaware of until now. Suppose now that the idealized mathematician, freely generating the (same) sequence, is not interested in looking back and just continues generating new elements of the sequence, that by chance will continue to coincide extensionally with a lawlike sequence. In these conditions, we cannot say that the idealized mathematician is specifying a lawless sequence by following a law (which conflicts with the free character of lawless sequences);

but that he is simply generating a sequence by free acts of choice that by chance coincide extensionally with a lawlike sequence.[18] And then it can be generally accepted that there is a class of such lawless sequences. Perhaps, Weyl's postulate can be consistently paraphrased in this way.

Weyl's semantics for universal and existential quantifiers runs as follows. Universal quantification is allowed to range over lawless and lawlike variables. Existential quantification, however, is only allowed to range over lawlike variables. Further, with respect to universal quantification, a sentence of the form $\forall \alpha A(\alpha)$ means that a certain property A is true for the *notion* of lawless sequence as a whole. When it comes to instanciate an individual with such property, we can only assert it if we can present a lawlike sequence such that $A(a)$. So, generally, we are able to assert $\forall \alpha A(\alpha)$, but when we eliminate the universal quantifier we are not allowed to assert $A(\alpha)$, only $A(a)$. With respect to existential quantification, a sentence of the form $\exists a A(a)$ is allowed for lawlike objects only. A sentence of the form $\exists \alpha A(\alpha)$ would be seen as ill-formed in Weyl's semantics. The universal quantifier ranges over lawless sequences, but statements about individual sequences must be about lawlike ones. In the subject of choice sequences, this semantics for universal quantification is the clearest sign that Weyl rules out lawless sequences as individuals. ('Brouwer and Weyl', p. 16.) Besides this consequence, Weyl's asymmetric quantification over choice sequences is the formal counterpart of the postulate we are discussing: universal statements about choice sequences ($\forall \alpha A(\alpha)$) are meaningful with the clause that when we try to state the same property A as a property of an individual sequence, we have to pass from the lawless variable α to a lawlike variable a such that $A(a)$.

Although Weyl's accout of choice sequences can, to some extent, be argued to suffer from severe difficulties (for eample, it can be argued that Weyl's notion of choice sequence is misleaded and does not agrees with the right Brouwerian notion), we can say that its general idea about choice sequences, or some of his insights about the role of choice sequences in mathematical ontology, are vindicated by the elimination theorem. The theorem shows that, in some sense, to talk about lawless sequences is equivalent to talk about lawlike ones. And (we can extrapolate to a *Weylian* insight that) this can be a sign that we don't need, or have not to accept lawless sequences into our mathematical theories, even though we can accept that the notion of lawless sequence has a role in the intuitive pre-theoretical mathematical discourse.

2.6 The need of further philosophical explanation

We have presented in this part of the thesis three accounts of choice sequences and debated the problems raised by the most important principle that governs them. We have seen that Brouwer's *genetic* account of choice sequences is very original and appelative but he doesn't argues for the validity of WC-N, he just asserts it as being unproblematic, with no further justification. Then we saw why we have to look for a justification for WC-N: it seems to fail for the entire universe of choice sequences, in particular it seems to fail for the lawlike

[18]Notice that in Brouwer's setting, such fact is impossible since it would be a fact about mathematical objects that transcends the creating subject's knowledge. But, of course, nothing guarantees that Brouwer's conception of the creating subject is the right one, or the only possible one.

sequences. Besides this, it cannot be formally derived as a logical theorem because it expresses properties of the quantifier combination $\forall\alpha\exists n$ not already implicit in the BHK-explanation, which are seen to hold by reflection on the way choice sequences are given. This conclusion forces us to look a justification for WC-N in the context of **LS**.

The task of justifying WC-N within **LS** is (mostly) followed by Troelstra who, in the path to achieve a set of axioms for choice sequences, is forced to modify the Brouwerian original idea of a lawless (choice) sequence due to the analytical approach he assumes. However, even with this slight, but necessary, modification of the notion of a lawless sequence, we have seen that difficulties continue to arise: (1) the density axiom doesnt't uphold for the original idea of choice sequence and decidability of equality demands impositions on choice sequences (such as *a priori* independence among them) that don't seems natural; open data schema seems to fail in preserving (intensional) properties of choice sequences, that are in the heart of the very idea of a choice sequence, while for open data with parameters (**LS3**), we have to restrict it from applying to certain kind of properties (such as Φ_I). We have also seen that **LS4** is only plausible under the assumption of $K_0 = K$, otherwise it would not be possible to generate the domaiin of all choice sequences.

Besides these difficulties to erect a suitable (set of axioms for the) theory of choice sequences, there is the elimination theorem holding that (for the modified, but not necessarily for the original idea of) lawless sequences can be *paraphrased* into lawlike ones. An uncontroversial interpretation of this theorem states that it only proves the (initially stated) coherency of mathematical reasoning (and acceptance) about lawless sequences by showing that sentences about them can be formally *reconducted* to sentences whose degree of foundation is unquestionable. We have seen, however, that it is possible to argue in favor of a more controversial interpretetion of the theorem, by means of the eliminativist(s) framework(s) of it. In particular, we showed that a Weylian account of the theorem, in contraposition to the more radical figure of speech approach, can be presented as a possible way of interpreting it, notwhitstanding the (frequently appointed) flaws of the Weylian account of choice sequences.

As to WC-N in the framework of **LS**, it can be derived as a theorem. **LS3** already implies WC-N and **LS4** strengthens it. These formalities, however, do not makes WC-N immune to the problems we resented in section 3. In fact, the difficulties posed by WC-N are, in the context of **LS**, consequences of **LS3**. The prblems discussed about open data axioms, *mutatis mutandis*, go the same for WC-N: how to face the need to make irrelevant the intensional properties of choice sequences? An argument is needed for accepting that the intrinsic intensional character of choice sequences is irrelevant to establish mathematical relations among them. This remains an unsolved problem.

Through the analysis of choice sequences presented so far, we conclude that the legitimacy of regarding choice sequences as genuine mathematical entities remains a problematic matter. In fact, the philosophical problems raised by the subject are explicit in the mathematical difficulties in settling a suitable theory for them. First, there is a need for showing the generally accepted *mathematical character* of such eccentric entities: what is the property possessed by choice sequences (besides the fact that they can be generated from natural numbers) that makes them mathematical objects? How can the intensional character of choice sequences like potentiality, temporality and subjectivity be incorporated in a natural way into mathematics? These are questions that mathematics itself cannot answer. The answers to these questions are of a philosophical nature and need to have a good philosophical account. A philosophical framework advanced to deal with these problems will be presented in the next chapter.

Chapter 3

The phenomenological account of choice sequences

3.1 Intuitionism and Phenomenology

In this part of the thesis we will focus mostly on Mark van Atten's chapters 3-6 of *Brouwer meets Husserl: On the Phenomenology of Choice Sequences*. The aim of the book is to use phenomenology to justify Brouwer's choice sequences as mathematical objects. And the thesis to establish is the following:

> One correct, phenomenological argument on the issue whether mathematical objects can be dynamic (e.g., choice sequences) is not Husserl's (negative) argument, but a reconstruction of Brouwer's (positive) one. (*BmH*, p. 5.)

The thesis involves a meeting of the thoughts of Brouwer and Husserl. It is uncontroversial that there is a *conceptual* and factual connection between Husserl's phenomenology and Brouwer's intuitionism (or constructivism in general). To van Atten, this conceptual proximity is not surprising, for in both lines of thought the main principle is that all genuine knowledge refers back, direct or indirectly, to *intuitions*, i.e., experiences in which the objects are given as themselves. To make explicit this unsurprising connection, van Atten refers to three philosophers of mathematics who have used Husserl's phenomenology to justify intuitionism, or parts of it: Oskar Becker (Husserl's disciple), Arend Heyting (Brouwer's disciple) and Hermman Weyl. A paradigmatic example is Heyting's well-known interpretation of the logical constants ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$) where he uses the phenomenological concepts *intention* and *fulfilment* (of an intention) to analyze intuitionistic ideas about meaning. (*BmH*, p. 6.)

Notwithstanding the connections between Brouwer's and Husserl's thoughts there are also some conflicting positions; the main divergence is their conceptions about the nature of the mathematical universe in relation to time. For both philosophers, mathematical objects are related to time. However, according to Husserl, the mathematical universe is *finished* and the mathematical truths (theorems), like the objects themselves, are *omnitemporal*. (*Experience and Judgment*, section 64c.)[1] In contrast, Brouwer conceives the mathematical universe as *the mathematician's construction*. Therefore, for him, the mathematical universe is not omnitemporal and finished but *dynamic* in the sense that some objects (viz. choice sequences) are open-ended and are developed in time. Van Atten calls the brouwerian universe *intratemporal* with respect to time. With the aim of showing the correctness of his vision on the mathematical universe, Brouwer appeals to notions such as *construction acts* and *free choice* (of the idealized mathematician). On the other side, says van Atten, Husserl affirms without arguing, and never mentioning choice sequences, that mathematical objects are static objects. He simply considers that part of the meaning of mathematical assertions is that mathematical objects possess this property. But this is precisely what Brouwer contests in his arguments. And the fact that, in these arguments, Brouwer makes appeal to certain acts (of consciousness) makes Brouwer seem, in this matter, the real phenomenologist of the two, says van Atten. (*Ibidem*.)

These considerations suggest that it might be possible to find an argument in favor of the dynamic universe, reconstructing Brouwer's argument within Husserl's phenomenology. This reconstruction, nevertheless, will defend some thesis that were explicitly rejected by Husserl. According to van Atten, there are three possible conclusions: (1) all mathematical objects are omnitemporal (defended by Husserl); (2) no mathematical object is omnitemporal (defended by Brouwer); (3) some mathematical objects are omnitemporal and some are not (van Atten's thesis of the co-existence of both type of objects in the mathematical realm). For van Atten, choice sequences are an example of dynamic objects, therefore not omnitemporal. Whether there exists or not other dynamic mathematical objects is left open, not compromising Brouwer's strongest thesis. In relation to Husserl's thesis, it suffices to give a counterexample to refute it. This is the motive, says van Atten, why he talks of *a* correct argument and not of *the* correct argument. (*BmH*, p. 7.) However, although van Atten is arguing directly against a particular thesis of Husserl, his purpose is to try to do justice to some of Brouwer ideas without compromising phenomenology as a (good) philosophical method to settle philosophical issues.

So, assuming that Husserl's and Brouwer's thesis are contradictory, van Atten has to gather evidence for the following claims he makes:

1. Measured by phenomenological standards, Husserl's (negative) argument is not correct.

2. Transcendental phenomenology provides the full ontology for the *a priori* sciences. Or, in other words,

3. In transcendental phenomenology, ontological questions in the *a priori* sciences are decidable. Therefore,

4. Brouwer's (positive) argument can be reconstructed in phenomenology. (*BmH*, p. 9.)

The arguments for claims 1, 2 and 3 will be presented in the following sections 2-5 of this chapter and the 4^{th} claim will be discussed in section 6 dedicated to the phenomenological constitution of choice sequences, carried out

[1] E. Husserl, *Experience and Judgment, Investigations in a Genealogy of Logic*, Routledge & Kegan Paul, London, 1973. We will refer to this book from now on as '*EJ*'.

by van Atten at the chapter 6 of his book. In section 7 we present an application of van Atten's phenomenological constitution of choice sequences: an argument for WC-N. Through the following sections of this chapter, we will also present the motives why van Atten accuses Husserl of falling into *dogmatism* by not considering possible non-static objects as mathematical genuine entities. As the autor says, he will use Husserl's own method against him.

3.2 The heterogeneous mathematical universe

At the end of the fourth chapter of *Brouwer meets Husserl* (*BmH*, pp. 51-52) van Atten advances an old but striking proposal: the *heterogeneous* mathematical universe. What this means and how it is defensible is the task of this section. According to van Atten, both Brouwer and Husserl had a conception of the mathematical universe as *homogeneous*, but in relevant diferent ways. By 'homogeneous' the author wants to emphasize the *temporal* aspect of the mathematical universe. A homogeneous mathematical universe in this context can be seen as an ontological domain whose objects are all of the *same* fundamental temporal kind. They share some essential temporal feature that groups them as belonging to that domain. In the Husserlian case, the temporal property shared by the objects of the mathematical universe is the *omnitemporality*, and in the Brouwerian case is the *intratemporality*. Van Atten's purpose is to show that a (kind of) Brouwerian conception of the mathematical universe is defensible from the point of view of Husserlian phenomenology. That is done by separating Husserl's personal view of the mathematical universe (which contradicts Brouwer's conception directly) from what can be proved from the standpoint of phenomenolgy about the mathematical universe (which according to van Atten can accomodate the Brouwerian universe).

Husserl holds that all mathematical objects are omnitemporal, i.e., they are static and exist at every moment in time. (*EJ*, section 64c.) An object is static exactly if at no moment are parts added to it, or removed from it (*BmH*, p 16). So, for Husserl, mathematical objects are temporal objects exisiting in every moment of time and always the same. It becomes clear that, although Husserl repudiates the classical 'natural attitude' that characterizes (mathematical) realism, his conception of the mathematical universe is, in an important way, classical. It is classical insofar as for him mathematical objects can be considered finished objects that do not grow in time. This immutability of the mathematical objects is the feature that makes the Husserlian mathematical universe homogeneous.

However, it is precisely the opposite feature that makes the Brouwerian universe also homogeneous. For Brouwer, all mathematical objects have a beginning in time. Some of them have an ending, are bounded[2], the finite ones, and others are open-ended or unbounded, the infinite ones. For him, the infinite totalities are in a process of constant growth in time since the moment of their beginning. Such view of the infinite totalities implies a conception of them as potential, as not finished. This kind of mathematical objects obviously have an essential relation to time, they are temporal objects but not by the way of omnitemporality, like in Husserl's view. They are intratemporal objects, i.e.,

[2]For the definitions of 'bounded' and 'unbounded' objects see *BmH*, p. 16.

they do not exist in every moment of time, and they are dynamic, i.e., they have parts added (but not removed) at some moment of time. So, for Brouwer, "all mathematical objects are intratemporal, and some of them are dynamic and unbounded" (*BmH*, p. 52). Choice sequences are paradigmatic intratemporal, dynamic and unbounded objects.

The proposal of an heterogeneous universe advanced by van Atten (*BmH*, p. 51) is the following:

HU: Some mathematical objects are omnitemporal, some are not.

Both Husserl and Brouwer would dissent from this claim by obvious reasons. However, the purpose of van Atten is to prove that the proposal of the heterogeneous universe "can be fitted into Husserl's general philosophy" (*BmH*, p. 52). In order to do so, Van Atten has to show that the following two claims are true:

1. Omnitemporality of mathematical objects is not a necessary consequence of phenomenology as a philosophical method;

2. Phenomenology does not obstruct an intratemporal (at least partially) account of the mathematical universe.

3.3 Phenomenology and revisionism in philosophy of mathematics

In the beginning of chapter 5 of *Brouwer meets Husserl*, Van Atten introduces the term 'revisionism' (in mathematics), which he defines as "the term that applies to any philosophical standpoint which reserves the potential right to sanction or modify pure mathematical practice" (*BmH*, p. 52).[3] By 'pure mathematical practice' van Atten doesn't mean everyday mathematical practice but the practice of the specialist mathematician that works in the foundations of mathematics; by 'pure' he is referring to the mathematician that seeks to give to mathematics the *rigth* foundations, to justify it in solid and indubitable grounds.

A distinction between *weak* and *strong* revisionsm is made by the author. Weak revisionism "potentialy sanctions [or limits] a subset of this [mathematical] practice" and strong revisionism "potentialy not only limits but extends it in different directions." (*BmH*, p. 53) Weak revisionism does not extend the actual pure mathematical practice, it just attempts to justify the alresdy existent body of theories as being correct. But it can limits it in the sense that possible some parts of the existent body of theories may be dropped. Strong revisionism however, by reserving the rigth to limit or extend certain mathematical practices, implies that "certain combinations of limitation and extension *may* lead to a mathematics that is no longer compatible with the unrevised one." (*Ibidem*; our underline.) It may lead, no that it necessarily leads, because it is a question of 'reserving rights', as the author says. Examples of non-revisionism, weak revisionism and strong revisionism are respectively, Wittgenstein's, Hilbert's and Brouwer's philosophies of mathematics (*BmH*, pp. 53-54).

[3]This definition of 'revisionism' is borrowed from Crispin Wright, *Wittgenstein on the Foundations of Mathematics*, Duckworth, London, 1980, p. 117.

According to van Atten, Husserl's view is a defense of a weak revisionism. The revisionism described by Husserl is not concerned with the content of mathematics but with its epistemological methodology. It is not concerned with what mathematical principles and fundamental concepts are the right ones but more with the task of ensuring that we achieve a clear *insigth* into the meaning of *actual* mathematical concepts and principles, establishing in this way that mathematics is not just a useful tool, a technique, but genuine theoretical knowledge. According to van Atten, Husserl's view in *Logical Investigations, Prolegomena to Pure Logic* (section 71) is that phenomenological philosophy should provide the foundation and insight for actual or classical mathematics (*BmH*, p. 55).

This view is the view defended by the so called 'early' Husserl. However, says van Atten, the 'late' Husserl doesn't change essentialy his view on this matter. In *Ideas III*[4], the stress is once more on methodology, like in *Logical Investigations, Prolegomena to Pure Logic*. But this time, says van Atten, a new and essential idea was introduced in Husserl's treatment of this matter, namely, that of transcendental phenomenology as providing the *universal ontology* (*BmH*, p. 55).

This claim is assured by the (phenomenological) 'method of clarification' of concepts (*Ideas III*, sections 18-20) that, according to van Atten, aims at *transcendental constitution* of ontological entities (*BmH*, pp. 58 et seqs.). According to the author, this novelty means that for Husserl "transcendental phenomenology forms a (particular) unity with the ontologies of the particular sciences" (*BmH*, p. 54). This interpretation of trancendental phenomenology is based, among others, on the following passage of *Idées III*:

> Everything that the sciences of the onta, the rational and empirical sciences, offer us (in the enlarged sense they can all be called 'ontologies', insofar as it becomes apparent that they are concerned with unities of the 'constitution'), resolves itself into something phenomenological (...). (*Ideas III*, section 14, pp. 66-67.)[5]

This new idea on the relation of phenomenology and ontology makes all the difference when we come to relate what is Husserl's own view and what can be stated from the point of view of phenomenology as philosophical method. This is the one of the cornerstones of van Atten's discussion on the relations between choice sequences and the phenomenological consequences on mathematical ontology. According to van Atten, the concept of clarification used by the 'late' Husserl has a stronger meaning that the idea of getting insight on mathematics defended by the 'early' Husserl. This is so because the concept of clarification of meaning "is a retrogressive inquiry back to sense-conferring living intention" (*BmH*, p. 58), in other words, it is a *genetic analysis* on the meaning of concepts back to the act (of consciousness) that conferred the meaning of the concept in

[4]Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy, Third Book: Phenomenology and the Foundations of Science*, Martinus Nijhoff Publishers, 1980; translated by Ted E. Klein and William E. Pohl. Referred to as '*Ideas III*'.

[5]In the original: Alles, was uns die Wissenschaften von den Onta, die rationalen und empirischen Wissenschaften (im erweiterten Sinn Können sie alle "Ontologien" heißen, sofern es sich zeigt, daß sie auf Einheiten der "Konstitution" gehen) darbieten, "löst sich in Phänomenologisches auf (...)". Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Drittes Buch, Die Phänomenologie und die Fundamente der Wissenschaften*. Ed. M. Biemel, Husserliana Band V, Kluwer Academic Publishers Netherlands, 1997.

question, his true living meaning. This is nothing more than a transcendental constitution of the sense of that concept.

These considerations make clear that phenomenology relates to mathematical ontology in the following manner: the mathematical entities get their ontological sense from intentional or transcendental constitution (*BmH*, p. 56), and the task of phenomenological philosophy is, with regard to mathematics, to transform it from a merely technique to genuine (i.e., insightful) knowledge by clarification of its concepts. Or as Husserl himself says:

> The point is to lead the sciences back to their origin, which demands insight and rigorous validity, and to transform them into systems of cognition based on insight by work that clarifies, makes distinct, and grounds ultimately, and to trace the concepts and statements back to conceptual essences, themselves apprehensible in Intuition, and the objective data themselves, to which they give appropriate expression insofar as they are actually true. (*Ideas III*, section 18, p. 83.)[6]

Husserl's views on philosophy of mathematics fit into a weak revisionism because his attempt is not to modify the mathematical practice (of the time) but just to make the mathematical concepts and principles more clear, to make their sense obvious and epistemically rigorous. His purpose was not to defend that some parts of mathematics were incorrect or meaningless (but he keeps open the possibility that upon phenomenological analysis some parts of existing practice cannot be upheld); this was the purpose of Brouwer, who says, for example, that the law of excluded middle is not generally valid; this comes into conflict with classical mathematics. Van Atten wants to show that, on phenomenological grounds, there is no reason to pressupose that the method of clarification only takes the form of an epistemological inquiry into the concepts and method of mathematics without more dramatic consequences besides making the actual mathematical concepts, principles and practice more intuitive. For van Atten, the method of clarification has deep consequences with respect to mathematical ontology. With the phenomenological clarification of the meaning of concepts "comes the possibility of rejecting supposed objects" as genuine mathematical objects. (*BmH*, p. 58)

3.4 The strong revisionism implied by transcendental phenomenology

In the previous section we saw how Husserl's personal view on the foundations of mathematics can be said to suport a weak revisionism, as defined by van Atten. We discussed how van Atten's claims that Husserl's mature conception of meaning clarification of the scientific concepts differs essentially from his

[6]In the original: Es gilt, die Wissenschaften auf ihren Einsicht und strenge Geltung verlangenden Ursprung zurückzuführen und sie in Systeme einsichtier Erkenntnis zu verwandeln durch klärende, verdeutlichende, letzt-begründende Arbeit, die Begriffe und Sätze auf in der Intuition faßbare begriffliche Wesen selbst und die sachlichen Gegebenheiten selbst zurückzuführen, denen sie angemessenen Ausdruck geben, soweit sie wirklich wahr sind. Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Drittes Buch, Die Phänomenologie und die Fundamente der Wissenschaften*, pp. 96-97.

first conception of that notion and that, for the author, that difference is not innocuous with respect to the ontology of mathematics. Now we will present van Atten's arguments that attempt to show how this difference gives rise to a strong revisionism in the phenomenological account of the mathematical universe.

In order to prove that Husserl's phenomenology implies a strong revisionism, van Atten has to argue in favor of four crucial claims:

1. Existence is the objective correlate of transcendental constitution;
2. for formal (or purely categorial) objects, transcendental possibility implies existence;
3. in purely formal sciences, the capacity for clarification is exactly the capacity for transcendental constitution;
4. the objects figuring in actual mathematical practice need not exhaust the totality of objects that are possible according to essence. (*BmH*, p. 59)

The first claim is argued for in p. 57, where the author gathers some textual evidence in Husserl's *Cartesian Meditations*, section 26[7]. The second claim gets his source of validity essencially from *Experience and Judgement*, section 96c.[8] These two claims are in an intimate relation with each other: for formal or categorial objects, transcendental possibility is synonimous of transcendental constitution, and once that transcendental constitution has existence as correlate, this warrants that transcendental possibility implies existence. This means that for categorial objects, existence comes from the mere possibility of constitution.

But as it became clear in the previous section, clarification aims at constitution of the meaning of concepts that are already figuring in actual practice and it does not exclude the possibility of dismissing concepts figuring in actual practice and of accepting new ones. It follows that the objects figuring in our actual practice can be excluded as illegitimate and that new ones can be included in that practice as consistent possibilities for which there is phenomenological evidence (*BmH*, p. 64). The first conclusion comes from the fact that, if by clarification of a concept present in the actual practice we cannot achieve a satisfactory account of its constitution, then we cannot say whether it exists or not. The second one comes from the fact that if clarification of the meaning of a concept not pertaining to the actual practice amounts to a constitution of its meaning as a possible mathematical concept, then we can introduce it in our actual mathematical practice as a legitimate object. According to van Atten, this is so because "the particular feature of mathematical essences is that they govern *a priori* possibilities" (*BmH*, p. 62), possibilities that are independent of particular sense data. Stating this proposition more clearly, the valid mathematical meanings are those for wich the possibility of fulfilment is not *a priori* excluded, for which intensional fulfilment is ideally possible *BmH*, p. 63).

[7]Edmund Husserl, *Cartesian Meditations: An Introduction To Phenomenology*, Martinus Nijhoff, traslated by Dorion Cairns, 1982 (7^{th} impression). Referred to as '*CM*'.

[8]We are not quoting the respectives texts here because (1) we will quote and discuss them at some length in the next chapter and (2) it seems not be necessary to the task of explaining van Atten's claims at this point.

Based on these observations, says van Atten, “the same methods that enable Husserl to exercise his weak revisionism in fact make a strong revisionism possible” (*BmH*, p. 64), since clarification has the power to ‘re-create’ formal objects that already figure in actual mathematical practice as it has also the power to ‘create’ formal objects that do not. This conclusion opens the door to criticize Husserl’s ‘dogmatic’ conception of the mathematical universe as omnitemporal.

3.5 That omnitemporality does not follows from phenomenology

The constitution of any genuine object must be constitution *in time*. This means that to any object “a temporal form belongs to it as the noematic mode of its mode of givenness” (*EJ*, section 64c, p. 258). However, this temporal mode of givenness of mathematical objects in particular does not belong to their essence. Temporality is not a part of the ‘noematic essence’ of mathematical objecs. This is so because, for Husserl, mathematical objects do not get their identity (are not constitued) by temporal determination, i.e., the temporal determination in the noema (the specific binding to time) is not part of what makes these objects the mathematical objects they are, because at any time they are given as identically the same. Therefore they are omnitemporal. (*BmH*, pp. 70-71)

For van Atten, this last claim of omnitemporality is too strong. He says that he can accept Husserl’s argument insofar “as it says that mathematical objects are temporal, as opposed to atemporal.” (*BmH*, p. 71.) But it cannot accept it as a cogent argument for omnitemporality, since omnitemporality would forbid choice sequences (non-lawlike objects), whose unfinished character requires a temporal determination for their individuation. For this reason, van Atten has to show that Husserl’s claim is wrong, but also preserve the validity of phenomenology as the correct method to settle philosophical questions. The form of doing this is obvious: to argue that Husserl’s own methodology suggests a way to criticise the omnitemporality claim. (*BmH*, p. 71.)

For Husserl, the omnitemporality of mathematical objects is a matter of essence, i.e., it is part of the essence of the categorial region ‘mathematical object’ that temporality does not influence their identity: mathematical objects are always the same irrespectively of time. But how did Husserl disclose such an essence? According to phenomenology, there is just one method to disclose essences: *eidetic variation*. Eidetic variation consists in (when looking to a certain region of objects) searching for the property(ies) that gather them, i.e., the property(ies) that all members of the region have in common, imagining all the possible variations of the objects belonging to that region. The purpose of these variations is to find the *invariant*.

> While what differentiates the variants remain indifferent to us, this form stands out in the practice of voluntary variation, and as an absolutely identical content, an invariable *what*, according to which all variables coincide: *a general essence*. *EJ*, section 87a, p. 341.)[9]

[9] In the original: Sie hebt sich in der Übung willkürlicher Variation, und während uns das Differierende der Varianten gleichgültig ist, als ein absolut identischer Gehalt, ein invariables Was

It is by reflection on all the eidetic possibilities of a certain kind of object that we disclose the essence that groups them in that kind. However, says van Atten, Husserl didn't follow his own method. For he left out of the eidetic variation a purported mathematical object: choice sequences. An object whose (intra)temporal character is essential to its identity, i.e., is a part of its essence. This position is explicitly made in the following passage:

> What I want to suggest is that Husserl's eidetic variations were one-sided, caused by an ontological prejudice that mathematical objects must be finished objects. Husserl neither then nor later considered choice sequences. His eidetic variations are done (or taken for granted) only on finished objects (...).(*BmH*, p. 71)

For van Atten, Husserl's variations only included classical objects, and therefore the essence arrived at is not necessarily that of the mathematical region in general, but just that of a subregion of the total mathematical region: that of finished mathematical objects. The same argument is applicable to the temporal homogeneity or heterogeneity: Husserl doesn't arrive to an heterogeneous conception of the mathematical universe because he was just looking for the homogeneous classical one.

For van Atten, this criticism of Husserl's conception of the mathematical universe is an inequivocal motive to suppose that the mathematical universe is heterogeneous and, consequently, to make plausible that there exists some mathematical objects that are intratemporal and dynamic, viz. choice sequences. But this criticism provides only the *motive* to suppose that the mathematical universe is heterogeneous; it does not provides the *confirmation* of that. The confirmation that mathematical universe is heterogeneous can only be provided by proving that some dynamic intratemporal object in fact *exists*. And to prove that a dynamic intratemporal object exists in the realm of mathematical objects is equivalent to perform its trancendental constitution. This is the task of van Atten at the 6^{th} chapter of his book. We will now present it.

3.6 Phenomenological constitution of choice sequences as mathematical objects

With respect to the task of showing that phenomenology can establish choice sequences as genuine mathematical objects, it was argued above how phenomenology is capable of ontological judgments with regard to pure mathematics. Generally, complete justification for asserting the existence of a supposed object consists in giving a strict constituition analysis for that object, and what is specific to the case of pure mathematics is that the laws governing strict constitution of its objects are precisely the laws of categorial formation: *for formal objects, transcendental possibility and existence are equivalent*. What has to be shown by the author is that choice sequences can be strictly constitued as formal, or categorical, objects. He continues, explaining that this will be attempted in two steps: (step 1) he has to show that choice sequences can be constitued as

heraus, nach dem hin sich alle Varianten decken: ein allgemeines Wessen. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der logik*, p. 411; Felix Meiner Verlag, Hamburg 1985, revised and Edited by Ludwig Landgrebe.

objects at all and (step 2) that this is constitution of purely categorical objects. (*BmH*, p. 85.)

3.6.1 Choice sequences as objects

A choice sequence is begun at a particular moment in time, and then grows as we choose further numbers. This process is generally open-ended and may be continued forever. For van Atten, there is a clear, implicit understanding in this way of speech that a choice sequence is an object of some sort. In fact, he says, the way we usually speak about choice sequences *manifests* an implicit thesis: that our discourse is about some sort of objects. So, it is his aim to make this thesis explicit, i.e., to disclose that in virtue of what we have such a discourse that implicitly assumes that a series of choices forms a unity, in virtue of what such a discourse constitutes an object. (*BmH*, p. 89.)

In general, to constitute an object by a descriptive analysis we have to take two steps: first, we have to identify the *activity* in which the putative object is given in (the passive synthesis of) consciousness; then, we have to identify the invariant by an *objectifying act* of (the active synthesis of) consciousness, i.e., we have to carry an act that thematizes the invariant. (*EJ*, section 13, p. 72.) The first step is the pre-constitution of the object and the second step is the constitution itself. An example is the constitution of the series of natural numbers: first we identify the activity by which they are given in counsciensness, the activity of counting (one, two, three, four, etc.), then we carry out an objectifying act over the invariant (the sequence $\langle 1, 2, 3, 4, ...\rangle$) by identifying it as a unity ($\mathbb{N}$) through the several differents acts in which it is thematized. For the constitution of choice sequences we also have to carry these two steps. First we have to identify the activity in virtue of which choice sequences are given in passive consciousness; then we have to identify the invariant, implicitly established in the self-giveness of choice sequences.

Van Atten says that the activity that founds the (self-)giveness of choice sequences is that of *choosing*. (*BmH*, p. 89.) Choice sequences are given to us, or come to appear in our counsciensness, by the activity of choosing elements of $\mathbb{N}$. In this activity choice sequences are pre-constitued, meaning that after this activity one only needs to carry out an objectifying act to constitute them as objects at all. (*Ibidem*.) For van Atten, the fact that choice sequences change over time not only does not rule out they are invariant over time in some respect, but actually pressupose such an invariance. And this is so because only some substrate that remains identical can change (or remain the same) over time. In the case of choice sequences, what is this substrate, this invariant?

Before answering this question, van Atten advances two alternatives; both regarding by him as not working. The first alternative supposes that the *concept* of choice sequence is the invariant. Against this hypothesis he argues that "the invariant is not any concept, however specific, that a sequence would fall under." (*Ibidem*.) For, a concept and an object falling under that concept are two different things. In the case of choice sequences, says van Atten, this distinction between concept and object (falling under it) becomes clear when we consider the concept of a choice sequence and the sequence itself: a sequence consists of linearly ordered parts, while a concept governing it does not. Also, he continues, a non-lawlike sequence unfolds in time, a concept does not (i.e., a concept does not change through time). The second alternative supposes that the *initial segment* is the invariant. Indeed, once values are chosen for an initial segment they cannot be changed later on. But again, he says that "[initial] segments cannot be the invariant that are the evidence for choice sequences as genuine objects." (*BmH*, p. 90.) And this is so because different choice sequences may have the same initial segment. What differentiates choice sequences among them are their intensional properties, and since initial segments are extensional properties of choice sequences, they cannot be the invariant for choice sequences. We can think of two lawless sequences, one started at t_0 and other at t_1, with the same initial segment. This is not sufficient to settle the two as being the same sequence, for it may happen that they come to differ at some stage of growth. What makes choice sequences be different (or the same) is their intensional properties, and (in the example above) such a difference need not consist in more than having been begun at different moments of time ($t_0 \neq t_1$).

We have seen that, for van Atten, neither concepts nor initial segments are the invariants for choice sequences. But, he says, the reason why the second alternative does not work provides the clue for the right invariant for choice sequences. Van Atten argues that "the 'and so on' that indicates continued choices is recognized by Husserl as a categorical form (Urteilsform)" (*BmH*, p. 90) in the following quote of *Experience et Jugement*:

> There appears here the new form of determination: "and so on", a basic form in the sphere of judgement. The "and do on" enters into the forms of judgement or it does not, depending on how far the thematic interest in S [the object of which our experience is to be explicated] extends; therefore, it produces differences in the forms of judgements themselves. (*EJ*, section 51b, p. 218.)[10]

After announcing Husserl's point of view on the notion of '*and so on*' with respect to judgments, and based on it, van Atten presents another argument for the incorrectness of regarding initial segments as the invariant for choice sequences and a suggestion to determine the invariant for choice sequences:

> In considering initial segments as the identity-constituting invariant, precisely this horizon 'and so on' is not thematized, is left out of our interest. In doing so, we miss the fact that choice sequences are unfinished objects. I will now work out the suggestion that what remains invariant is the character of the sequence as a developing sequence, a development that started at a particular point in time. (*BmH*, p. 90.)

To carry out this task, van Atten begins by presenting three (sort of) definitions of the notion of *and so on*: '[a] mental process with several members, progressing in an orderly manner [that] carries with it such an open horizon' (Husserl); 'the indefinite repetition of *the same* thing or operation [that] may be defined in a complex way' (Brouwer); 'the concept of successive applications of an operation' (Wittgenstein). To apply this idea to choice sequences, he says,

[10]In the original: Es tritt hier die neue Bestimmungsform des "und so weiter" auf, eine Grundform in der Urtelssphäre. Das "Undsoweiter" geht in die Urteilsgestalten ein oder nicht, je nachdem, wie weit das thematische Interesse an *S* reicht; es schafft also Differenzen in dem Urteilsformen selbst. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der logik*, p. 259.

we have to consider the *free choice of a number* as an operation. In fact, Brouwer's first act of intuitionism introduces the notion of *and so on* as a primitive mental operation (a categorical form in Husserl's terms), the sucessive iteration of a rule defined in a complex manner or not. (*BmH*, p. 91.)

At this point, van Atten recalls the thesis presented above that the process of choosing initiates and extends the same object, i.e., it is the same, identical object that is extended at every stage of growth. And this implies that some substrate must remain the same during the process of growth. In other words, the idea of an *unchangeable substrate* is necessary to the consistency of the idea of change or growth of an object. (*Ibidem.*)

Another thesis presented by the author is that there is a distinction between the process of generating a choice sequence and the choice sequence itself:

> There is the the identity of the process, and the identity of the sequence. These are not the same. The process may have all kinds of characteristics that the sequence does not share. (*BmH*, p. 91.)

For example, the process consists of acts (of choice) and the sequence does not. He says that a notable consequence of this difference is that the time span between sucessive acts of choice does not show up in the sequence, i.e., it plays no representing role beyond the mere ordering. But there are more differences between the process and the sequence. The process may involve revising intensional properties (e.g., abandoning a provisional restriction) that need not leave a trace in the sequence itself. (*Ibidem.*)

The identity of the sequence and the identity of the process are different things. But, says van Atten, there is a close relation between the two: the identity of the sequence is founded on the identity of the process. In other words, there is no identity of the sequence without the identity of the process. And this is why both sequence and process are individuated by the same moment of beginning: the sequence only comes into being at the moment the process of growth starts. This is also the reason why, for van Atten, the individuation of open-ended choice sequences is determined by the particular moment the process initiates. With respect to lawless sequences, "its particular moment of occurence cannot be abstracted from, because that might lead to identification of different non-lawlike sequences that share their initial segments so far." (*BmH*, p. 92.)

If the sequence is founded on the process, how do we go from the process to the sequence itself? The answer is: an act of abstraction from process to sequence. Some aspects of the process, says van Atten, are abstracted from in all cases. The temporal intervals between sucessive choices are an example of what can be abstracted from. But what can be abstracted from varies with the kinds of sequence we have: the temporal aspects of lawlike sequences (given by lawlike processes) can be abstracted from without loss of the sense of identity of the sequences; in the case of lawless sequences, however, we have seen that some temporal aspects (as the moment of beginning) cannot be abstracted from. (*Ibidem*)

Now that we have seen the explication of the main aspects about the constitution of choice sequences as individual objects we will present it in a more orderly manner, by stages, and then explain them in detail:

1. Keeping in retention (in memory), or recollectiing, the process as it has developed till now.

2. Re-presentation of the process.

3. Choice of the next element.

4. Sinking back into retention of the sequence, and return to stage 1.

   (founded on these four acts, and occurring between going back from 4 to 1, there is:)

5. Apprehension of the identity of the process through its categorical form of ongoing process.

At the first stage we consider the process as been given *now* for the first time, i.e., we consider the sucessive choices that we have obtained until now. And the first step then is to locate that process in memory: going back to the first moment the process has been presented to us and has begun to generate the sequence. In the case we have chosen the previous number only a moment ago, this locating does not involve active recollecting because the process is still held in memory and has not sunk back in the past yet. (*BmH*, p. 92.)

The second stage of constitution consists in thematize (re-present) the process not as having been extendable once, but as being extendable now. This means that we have to make actual, or present, the temporal horizon that the process had at the moment we stopped it by keeping the process in memory (stage 1). This is so because if we remember the process as open-ended after the most recent choice, is has to be given as open-ended again now. In other words, we have to make actual the (anticipatory or protentive) intentions associated to the process. (Such intentions may be 'empty' in the sense of not prescribing any specific element(s); this is different for lawlike sequences.) (*Ibidem*)

The third stage is precisely the fulfillment of the intentions made actual in the previous stage: the choice of the next element. If this next element satisfies all the (possible) restrictions imposed to the process, the choice fulfils the previous intention (made actual at stage 1) and gives rise to a new intention directed at an immediately new element. This is also the stage where restrictions can be lifted, revised or added. (*BmH*, pp. 92-93.)

The fourth and fifth stages are retrospective ones. The fourth stage is the stage where we return to stage 1 and the sequence obtained till now sinks back into retention giving place to a new cycle, and to new intentions to be fulfilled (i.e., to new choices to be made). The last stage (founded on the four previous acts) is the stage where we apprehend the identity of the process going back from stage 4 to stage 1. This identity functions as the foundation for the identity of the choice sequence, i.e., where we go from the identity of the process to the identity of the sequence by an act of abstraction from the underlying process. (*BmH*, p. 93.)

3.6.2 Choice sequences as mathematical objects

Once performed the constitution of choice sequences as objects in general, van Atten has to show that they are mathematical objects. He starts by pointing two doubts that may threaten the coherence of choice sequences as mathematical objects, from the traditional (and husserlian) point of view. The first is that choice sequences are unfinished and hence time-dependent objects, they start at a particular moment in time, and grow from there on. The second doubt derives from the subject-dependency of choice sequences, as we think about the real mathematicians and the choice sequences generated by them (the threat of psychologism). (*BmH*, pp. 95-96.)

Against these two doubts, van Atten presents an argument based on Kreisel-Troelstra τ translation. The translation shows that sentences quantifying over lawless sequences are equivalent to other sentences that do not, and whose mathematical nature goes unquestioned (sentences only involving natural numbers, constructive functions and inductively defined neighbourhood functions). And because of that, says van Atten, the sentences that are translated must also be mathematical ones. For van Atten, translations suffice to show that the concept of (lawless) choice sequence is mathematically coherent. (*BmH*, p. 96.)

Using the translation, van Atten argues that the psychological or subjectivist character of choice sequences does not threaten the possibility of their being mathematical objects: mathematical statements do imply statements that quantify over choice sequences, for the equivalences that the translation yields works both ways (from sentences involving choice sequences to sentences not involving them and from sentences not involving choice sequences from sentences involving them). (*Ibidem*)

This general argument for the mathematical coherence of the concept of choice sequence, by itself, does not compel us to acccept that there are objects falling under that concept, says van Atten. However, he says, in the presence of the constitution analysis, that would make no sense, for after this constitution they are given to us as genuine individual objects. In this stage, the acceptance of the intentional properties of choice sequences as mathematically innocuous is only dependent on a satisfactory contextualization. This is the task of the author in subsections 6.3.1-6.3.3 (*BmH*, pp. 96.101). Van Atten resumes each subsection in the following way:

1. It is not their reference to time *simpliciter* that distinguishes choice sequences from other mathematical objects (for all mathematical objects have a relation to time), but the particular way in which they refer to time (intratemporality).

2. This particular reference to time does not threaten their status as formal objects.

3. Their subject-dependency (part of which is the freedom of generation) poses no problems for mathematics. (*Ibidem*)

The temporality of choice sequences

Van Atten shares with Brouwer and Husserl the idea that all mathematical objects refer to time. But he does not agree with them when one says that *all* mathematical objects are intratemporal (Brouwer) and the other says that *all* mathematical objects are omnitemporal (Husserl). For him, there are omnitemporal mathematical objects (the finished ones or the unfinished ones but determined by a law) and intratemporal mathematical objects (the unfinished and lawless ones). For Husserl, all mathematical objects are omnitemporal because, as they exist at all times, what is proven about them (theorems) is proven once and for all, i.e, is proven for all times, past, present and future. But, says van Atten, what can be proven about choice sequences is also proven once and for all:

> But choice sequences, instead of being necessarily infinitely temporal in two directions (and thereby omnitemporal), would be so in just one direction: the future. In that case still what is proven once, is proven forever. But (...) it was this feature of mathematics that lead Husserl to say that mathematical objects are static and omnitemporal. (...) *It follows that infinite temporality only in the direction of the future preserves the original motivation for the thesis of omnitemporality*.(*BmH*, p. 97; our italics.)

In this quote van Atten argues that the same argument used by Husserl to establish the thesis of omnitemporality for mathematical objects can be used, *mutatis mutandis*, to establish his thesis of intratemporality for lawless sequences. In fact, van Atten is performing his claim that Husserl, in respect to the particular relation of mathematical objects to time, remains 'dogmatic'.

Lawlike sequences, although unfinished mathematical objects, are given by a law that, followed at any other point in time, would yield the same elements in the sequence; so we have not to force the intratemporality thesis upon them, i.e., they can be considered omnitemporal objects without loss of their characteristic intensional features; that is, lawlike sequences, albeit being essentially intratemporal objects, can in practice be tretaed as omnitemporal ones. For van Atten, this fact proves that they are in fact intratemporal mathematical objects. Therefore, the specific relation of choice sequences to time is not an obstruction for the thesis that they are mathematical objects.

Both lawlike and lawless sequences are intratemporal objects, but Husserl only admits lawlike sequences as mathematical objects because, in the specific case of lawlike sequences, intratemporality *coincides* with omnitemporality. So they can (wrongly, but harmless for mathematical purposes) be seen as omnitemporal objects. And this is exactly what Husserl does, because for him all mathematical objects are omnitemporal. If Husserl had formulated the thesis of intratemporality (like Brouwer did) he would have realized that some mathematical objects might be intratemporal, and then he would not be able to force the omnitemporality thesis on phenomenological grounds.

The formal character of choice sequences

For (the late) Husserl, the non-formal objects are those that have in their constitution *sensuous* elements, i.e., those whose constitution lies upon physical sensations. For example, the constitution of a (particular) tree depends on sensations such as color, shape, extension, etc. But, as mathematical objects, these kind of objects also refer to time: their constitution is contitution *in* time. So, says van Atten, the relation to time does not excludes the formal character of an object, precisely because:

> (...) The constitution of any identical object presupposes the absolute flow of time. The reference to time does not make choice sequences

> from otherwise pure into mixed [sensations-dependent] categorical objects. (*BmH*, p. 98.)

What determines the categorical or non-categorical features of an object is this dependency-relation to sense data, not the relation to time. Because the temporal aspects of choice sequences are not sensuous, the particular way they refer to time does not rule out that they are formal or categorical objects. To make this claim stronger, van Atten refers to a quote of Husserl saying that not all irreal (=ideal) objects are free of sensuous elements (e.g., the geometrical objects have sensuous components as shape, extension, etc.), only the formal-mathematical ones.[11] (*Ibidem*)

The subject-dependency of choice sequences

A lawless sequence is *bound* to the particular subject generating it in a very specific way: it is essential to a lawless sequence to have a bearer or owner, and such a sequence cannot be handed over from one subject to the other. On the other hand, lawlike sequences can be reproduced or 'cloned' from one bearer to the other. But the fact that particular lawless sequences are unshareable, says van Atten, does not lead to unshareable truths, i.e., does not imply that true sentences about them are also unshareable among different subjects. (*BmH*, p. 98.)

If the implication was the case, the subject-dependency of choice sequences would consist in a major obstruction to admit them as mathematical objects. The following quote presents the way how van Atten rules out this putative obstruction:

> A non-lawlike sequence develops in stages, and at any stage the only information the subject has consists of an initial segment and, possibly, a number of intensional properties such as self-imposed restrictions on future choices. Initial segments and sets of restrictions are finite, therefore all information can be shared. All the subject knows about a particular sequence is *intersubjectively* accesssible. And (...) it is equally important that the categorical form 'choice sequence' can be shared between different subjects.[12] (*BmH*, pp. 98-99.)

Albeit choice sequences are generated by particular subjects, they are given to us in the basis of some information: initial segments and possible restrictions. Both kind of information are of a finite character, and so are shareable by the subjects who generates them. Therefore, the truths about them have to be shareable because the information on which these truths are founded are shareable too. Besides this, there is the fact that they are categorical objects, therefore making them objects of any possible consciousness in which they can be sized upon in a perfectly adequate way. This is possible because choice sequences are not absolute arbitrary creations of the mind; they are constructions according to the laws of categorical formation. And those laws are not arbitrary ones nor they are unshareable among subjects.

[11] Husserl argues this in *Experience and Judgement*, sections 64-65.

[12] Our italics.

Two further objections may arise against the subjective nature of choice sequences. First, based on the fact that initial segments (and possible resctrictions) are sufficient to determine the truths about choice sequnces and that we never have to construct an individual choice sequence in its entirety (in a mathematical proof), "what do we need choice sequences for, instead of just all finite sequences?" (*BmH*, p. 99.) Second, "we may wonder whether the freedom in generating choice sequences is really possible and, if so, whether this does not make everything arbitrary; we may be misguided, we may think that "they seem to involve free choice, but, metaphysically, they are really lawlike as well. (*Ibidem*)

To the first objection, van Atten advances a pragmatic argument. He says that the use of just initial segments will rule out the possibility of constructing the intuitive continuum (we will have instead a 'reduced' continuum), the motivation and one of the most sucessful applications of choice sequences. The categorical form of choice sequences ('and so on') is the key element to capture the intuitive continuum, to capture the continuum as potential. So, the categorical character of choice sequences makes the substitution of lawless sequences by finite sequences inadequate. Against the second objection, van Atten says that it pressupposes a world of 'things in themselves' (or noumena, in Kantian terminology). Such a notion as a noumenon is ruled out by Husserl's *transcendental idealism* where he performs the *phenomenological reduction* (or $\epsilon\pi o\chi\eta$).[13] Through the phenomenological reduction, the dichotomy between phenomenon and noumenon is rejected out and replaced by the correlation object/consciousness-of-the-object (noema/noesis). And because of this correlation, within a phenomenological framework, sequences that for all we can know are lawless but are really lawlike, are ruled out. (*BmH*, pp. 99-100.)

3.7 A phenomenological argument for WC-N

After concluding the phenomenological constitution of choice sequences as mathematical objects, van Atten applies the phenomenological analysis to justify WC-N (*BmH*, pp. 103-110.). Van Atten's strategy to justify WC-N is the following one: instead of given a direct justification for WC-N, he will try to justify a slight modified version of WC-N, namely GWC-N ('G' for graph-extensional), that works as well as WC-N for choice sequences and its applications (in real analysis) but does not pose the same problems as WC-N.

Van Atten says that a special case of WC-N is one where the predicate A is required to be graph-extensional, i.e., A refers to the choice sequence only through its values (=the graph of the choice sequence). So, GWC-N is (the same as) WC-N for a graph-extensional predicate A. The formalization of GWC-N runs as follows:

GWC-N $\forall\alpha\exists x A_{GExt}(\alpha, x)) \Rightarrow \forall\alpha\exists n\exists x\forall\beta(\bar{\beta}n = \bar{\alpha}n \rightarrow A(\beta, x))$,

where $A_{GExt}(\alpha, x)$ means that A is graph-extensional.

[13] For more details on the phenomenological reduction see Husserl's *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy, First Book: General Introduction to a Pure Phenomenology*, section 32.

Notice that the predicate A in WC-N is already an extensional predicate, the only change is the additional clause that this extensionality is given by a graph. The consequent of GWC-N remains identical to the consequent of WC-N; therefore, the desired conclusion remains basically the same: given the graph-extensionality of A, we only need α and β to share the same initial segment to determine if β has the same property A. But an important difference between WC-N and GWC-N is that, in WC-N the possible role of intensional properties is not excluded, while in GWC-N we only accepts properties that depend on the graph of α and β; therefore, in GWC-N the possible role of intensional properties is ruled out by definition.

An example makes this distinction clear. Assume that the creating subject generates choice sequences as individual objects, and can therefore *enumerate* the sequences generated so far. Then WC-N does not hold generally for extensional predicates. Suppose we have an operation F that enumerates the choice sequences. Then we have that $\forall\alpha\exists n(\alpha = F(n))$. Let's assume that the operation F is encapsulated by the (extensional, but not graph-extensional) predicate $G(\alpha, n)$, i.e., $G(\alpha, n) := (\alpha = F(n))$. We have then $\forall\alpha\exists nG(\alpha, n)$. Now, applying WC-N, it follows that $\forall\alpha\exists m\exists n\forall\beta(\beta m = \alpha m \rightarrow G(\beta, n))$. But this means that the same n will be paired to different choice sequences, which conflicts with the notion of 'enumeration'. Therefore, in the presence of an enumeration, WC-N does not hold generally. This is possible because the predicate G is not graph-extensional. (*BmH*, p. 104.)

There is however a feature about the notion of graph-extensionality to be considered. The notion of extensionality can be formalised as follows: A is an extensional predicate iff $\forall n(\alpha n = \beta n) \rightarrow (A(\alpha) \leftrightarrow A(\beta))$. But it is not clear how to formalise the correlated notion of graph-extensionality, or whether this is even possible. (*Ibidem*) On the other hand and contrary to the notion of extensionality, a graph-extensional predicate cannot be based on an intensional property. From this, van Atten says that "one might conjecture that graph-extensionality (...) has to be taken as primitive."(*Ibidem*)

For the intuitionistic reconstruction of analysis, it suffices to have GWC-N instead of WC-N. The main theorems of intuitionistic analysis, like the continuity theorem, the unsplitability of the continuum, the uniform continuity theorem are obtained using GWC-N.

Van Atten has showed the advantages of GWC-N over of WC-N. But still, GWC-N requires the same kind of justification that is required for WC-N: it has to be justified within the theory of choice sequences. For van Atten, however, the task of give a philosophical justification for GWC-N seems easier that doing the same for WC-N.

Van Atten's strategy is to take the analysis of the intuitive notion of choice sequence and extend it by making explicit the application of Husserl's principle of the noetic-noematic correlation to GWC-N:

> Roughly, this principle states that the structure of the way an object is given to us (the noema) is parallel to the structure of the acts in which that object is intended (the noesis). In the case of choice sequences, this leads to the question: in what ways can the freedom the subject enjoys in the process of generation be reflected in the intensional properties of the sequences themselves? That gave the idea of provisional restrictions, and it also gives rise to the already

familiar concepts of lawlike and lawless sequences. (*BmH*, p. 106.)

Looking at a particular description by Brouwer of choice sequences as freely generated objects where he says that "the freedom to proceed with the choice sequence can after every choice be arbitrarily restricted" and that "the arbitrary nature of this restriction, permitted at each new choice as long as the possibility to proceed is retained, is an essential element of the free becoming of the [choice sequence], as is the possibility to link to every choice a restriction of the freedom to make further restrictions of freedom"[14] we can see the key concept to be used by van Atten to justify GWC-N: the concept of a *provisional restriction*.

There are two kinds of restrictions:

- **definitive restrictions**, which have the form 'from now on, restriction R holds, and it will not be revised anymore';
- **provisional restrictions**, which have the form 'for an unspecified number of stages, restriction R holds'.

Both kinds of restrictions should be decidable, says van Atten. Otherwise, "the subject will not be able to conduct its choices in accordance with them." (*BmH*, p. 107.) So, in this framework, we can look at the process of generating a choice sequence as choosing, at each stage, not only an element n, but also a (finite) number k of restrictions of different orders, $R_0, ..., R_k$. Van Atten advances that, given this condition, the subject can make any revision in its ideas about how to go on generating a choice sequence, but in accordance with the following clauses:

1. this revision does not go against any definitive intensional properties of the sequence,
2. this revision admits the existing initial segment, and
3. after the revision, it is still possible to extend any initial segment of the sequence.

Van Atten's idea with this distiction and subsequent clauses to it is to 'force' the subject (who generates a sequence) to be explicit about lifting or not a provisional restrcition at each stage, because "it is up to the subject's own choice to make an intensional property [or restriction] either provisional or definitive". (*BmH*, p. 108.) In other words, as long as a restriction of a particular sequence is not definitive, that restriction can be changed at some latter stage; and, unless a provisional restriction is explicitely lifted, it remains in effect at subsequent stages. This requirement of explicitness, says van Atten, is a consequence of the creating subject's full responsability for the objects in intuitionistic mathematics and, in particular, for the generation of choice sequences. (*Ibidem*)

> In this circumstance lies the the answer to the following question: 'If I know that you imposed a provisional restriction at some earlier stage, what more do I know that I should not have known if you had

[14]This quotation is from a note related to Brouwer's manuscripts for the 1927 Berlin lectures. Troelstra's 'On the origin and developement of Brouwer's concept of choice sequence', in *The L.E.J. Brouwer Centenary Symposium*, pp. 465-486, Amsterdam, 1982 (eds. Troelstra and van Dalen).

imposed no restriction?' The extra information that a provisional restriction gives you consists in the following. If I tell you that I will not begin by lifting the provisional restriction, then you do know something about my next choice of a term, namely, that it has to respect this restriction. Had I imposed no restrictions, then you would not have known this. (*BmH*, p. 108.)

In this quote lies the *rationale* to make explicit the importance of provisional restrictions upon choice sequences. In using provisional restrictions upon choice sequences, all the information that we must have about them (an initial segment and possible restrictions) turns out to be *essentially* of an extensional character; in fact, turns out to be of a graph-extensionally character. (For initial segments, the conclusion is trivial.) The extra information that a provisional restriction alows us to know is that *the next* element of the sequence should respect such and such conditions, so the generation of the sequence is not absolutely arbitrary but has to respect some (finite) conditions, the set of restrictions imposed in some stage. However, the provisional character does not obstruct the freedom of the sequence because it can be lifted at any subsequent stage of growth of the sequence.

In traditional literature about choice sequences there is a particular type of choice sequence that can be interpreted as an application of the distinction between provisional and definitive restrictions: the *hesitant sequence*. A hesitant sequence β is a process of generating values $\beta 0, \beta 1, \beta 2, ...$ such that at any stage we either decide that henceforth we are going to conform to a law in determining future values, or, if we have not already decided to conform to a law at an earlier stage, we freely choose a new value of β. The decision whether or not to conform to a law may stay open indefinitely. Applying the distinction made above, a hesitant sequence is, from the start, a provisionally lawless sequence (which the subject at any stage may decide to turn into a definitively lawlike sequence).

Let's see now how, in van Atten terms, from the recognition of the two types of restrictions, definitive and provisional, may be derived an argument for GWC-N. Basically, the argument runs as follows. Any intensional property that might be useful in constructing a choice sequence should already be included in the (first order) restrictions, as only these have a direct relation to the graph of the sequence. But on a hesitant sequence, the restrictions are precisely the provisional ones. The provisional restrictions are open to revision, i.e., they can be lifted at any stage of generation of the sequence; so, the intensional properties that a sequence has at any stage do not necessarily characterise that sequence also at all later stages. The only non-provisional information left is the initial segment of the sequence. Therefore, the only information we have and need to determine possible properties of the sequence, in particular whether it coincides with a second sequence, is the initial segment. This is exactly what GWC-N asserts.

Roughly speaking, GWC-N asserts that all the information we need to determine possible properties of a (lawless) sequence is of a graph-extensional character. Graph-extensional information are limited to initial segments. All information we have acess to determine possible properties of a sequence are (1) initial segments and (2) first order restrictions; and these restrictions may be definitive or provisional ones. First order restrictions are the only intensional information we need to determine possible properties of sequences. Hesitant sequences are those sequences that only allow provisional restrictions. So, for hesitant sequences, we only need initial segments and provisional restrictions to determine all of its possible properties. If we assume that lawless sequences are precisely the hesitant sequences for which it is left open-ended the decision whether or not to conform it to a restriction, then provisional restrictions are the only intensional information we need to determine possible properties of the sequences. So far we have realized that all information we have to determine future properties of (provisionalized) lawless sequences are initial segments and provisional restrictions. But – and this is the key step in this argument– provisional restrictions cannot be used to determine future properties of choice seaquences because (by definition) they can be lifted at any stage of growth of the sequence; what makes them (according to van Atten's analysis) *inessential*, and hence dispensable, to determine the properties of choice sequences. So, the only reliable information that can be used to determine the properties of choice sequences are their initial segments, graph-extensional information.

One last remark is done by van Atten. A condition for this argument to work is that the universe of choice sequences contains, for each element α, the 'provisionalised' version of α, say α'. α' is a provisionalised version of α iff they have the same initial segment and are subject to the same restrictions, but in the case of α' all of these restrictions are provisional. Van Atten says that in the universe of all choice sequences, this requirement is met by definition. This means that in the universe of all choice sequences the provisionalised version of any sequence exists by definition; otherwise, the universe of *all* choice sequences would not be the universe of all choice sequences (contradiction).

Chapter 4

Critical discussion of the phenomenological account of choice sequences

The previous chapter was devoted to present in some detail van Atten's phenomenological account of the ontological problem of accepting choice sequences as legitimate mathematical objects. In fact, this account of choice sequences can be seen as the most complete philosophical analysis about this subject until now. Van Atten's phenomenology of choice sequences is successful where Brouwer's philosophy is not: it presents a plausible justification for the introduction of choice sequences as mathematical objects, viz. the phenomenological constitution of choice sequences as mathematical objects; and it presents a justification for a suitable substitute of WC-N, viz. GWC-N. It is however a fact that some claims advanced by van Atten are not so evident as they seem. And it is our task now to make this clear through a critical discussion of his arguments.

Van Atten's analysis can be divided into two main moments: a *deconstructive* one and a *constructive* one. The main purppose of van Atten is to show that a heterogeneous mathematical universe is possible within phenomenological standards, precisely by showing that choice sequences are legitimate mathematical objects. Hence that some dynamic objects are mathematical objects. The deconstructive moment aims to show that some of Husserl's views on mathematical ontology are stronger than he claims. The constructive moment aims to show that choice sequences can be phenomenologically *constituted* as legitimate mathematical objects. The deconstructive moment can be divided into two main claims: (1.1) Husserl's phenomenology is able to decide ontological questions in mathematics and (1.2) Husserl's phenomenology implies a strong revisionism in mathematics. The constructive moment can in turn be divided into three claims: (2.1) choice sequences can be constituted as individual objects in general; (2.2) choice sequences can be constituted as individual mathematical objects; (2.3) then, Husserl's phenomenology implies an intratemporal account of mathematical objects. We will try now discuss each of these five claims separately (insofar as is possible to separate them).

The argument for the implied strong revisionism of phenomenology runs as follows: for formal objects, transcendental possibility implies existence; in purely formal sciences, the capacity for clarification is exactly the capacity for transcendental constitution; so, the objects figuring in actual mathematical practice need not exhaust the totality of objects that are possible according to essence, moreover new ones can be introduced by the process of phenomenological constitution.

The notion of *clarification* is introduced by Husserl as a necessary task to perform on the most fundamental concepts of science (*Ideas III*, chapter IV). Sometimes the fundamental notions of a particular science, due to a extensive but uncritical application, become unclear or confused. Their original sense is no longer taken into account with evidence in the applications, and obscurity may arise. An example of such obscurity is the discovery of the paradoxes in the foundations of mathematics in the beginning of the 20^{th} century due to the wide uncritical application of the concept of *set*. So, to avoid such problems within science, it is necessary to make clear the original meanings of the fundamental concepts; this clarification has as its goal to make explicit the initial or original sense of concepts. And there is only one way to make their original sense explicit: making clear the steps that led to the their constitution as valid intentional objects. In Husserl's terms: "Clarification must follow precisely the stages of the constitution of the exemplary object of intuition in question." (*Ideas III*, section 20, p. 88.)

Clarification presupposes constitution, as van Atten argues. In fact, it is re-constitution since, in analyzing the constitution of a concept or object, its original sense is restated or re-validated. By doing this, a criticism of what is *authentic* and of what is *inauthentic* to the original sense of the concept in question is achieved. The authentic is what is essential to the constitution of its sense, the inauthentic is what is not. (*Formal and Transcendental Logic*, Introduction.) And to determine what is authentic in the sense of a concept is precisely to confer its right sense, i.e., to constitute it.

Claim 1.2 implicitly makes use of the notion of clarification. It expresses the possibility that mathematical concepts may be revised through phenomenological clarification. And claim 2.3 states that the current mathematical concepts do not exhaust the totality of possible mathematical concepts. This is possible by an act of clarification of concepts already belonging to actual mathematical practice. In van Atten's view, the clarification of actual concepts lead to two conclusions: (1) that some actual concepts do not have any correspondent evident sense or any correspondent sense at all, therefore they can be rejected as mathematically invalid or inauthentic (*BmH*, pp. 60, 62, 63); and (2) that new concepts can be introduced into the mathematical practice when, by clarifying their constitution process, it becomes evident that they are authentic mathematical concepts (*BmH*, p. 64). To the conjunction of these two statements van Atten calls strong revisionim. He argues that with strong revisionism comes the possibility of introducing new mathematical concepts that are no longer compatible with the current mathematical practice. To do so it is only needed to show that such concept is mathematically valid, i.e., that it can be constituted as a mathematical genuine concept.

4.1 Does intentional constitution implies existence?

What does van Atten means when he states that ontological questions are decidable by phenomenological standards? To decide an ontological question for a particular area of science is to determine which objects can be assumed to *exist* within the framework of a particular theory describing that area of knowledge. In mathematical theories, if an object exists, then we are able to quantify over it, i.e., it has to stand as a bounded variable of a proposition (the Quinean criterion). In other words, such an objects has to be within a domain of quantification. It can be argued that what van Atten has in mind is that by phenomenological considerations it can be determined which domains of objects we can assume to quantify over. And that to determine which domains of objects we can quantify over is to perform an intentional constitution of that domain. This claim is justified by what van Atten calls *the principle of transcendental idealism*: in phenomenology, existence is the objective correlate of constitution. (*BmH*, pp. 57, 59.) The textual source for asserting this is *Cartesian Meditations*, section 26 (quoted below). This principle gives rise to a slightly different claim when restricted to mathematical (or formal) objects: for mathematical objects, possibility implies existence. (*BmH*, pp. 59 and seqs.) The main textual source where van Atten bases this claim is *Experience and Judgement*, section 96c (also quoted below).

The idea that phenomenology is capable of ontological assertions, says van Atten, is based (among others) in the following quotation from Husserl: "All entities get their ontological sense from intentional constitution."[1] This quotation is a version of the principle of transcendental idealism and it means that all entities acquire their ontological status in the process of constitution, i.e., it is the process of constitution of an intentional object that determines whether it belongs to this or that ontological region. Implicit in this quotation, however, is that *only* through intentional constitution an object becomes a legitimate ontological entity of some sort. But what van Atten calls 'principle of transcendental idealism' goes further: it claims that, in general, intentional constitution is suficient to establish *existence*, i.e., constitution stands as a criterion for the existence (or non-existence) of any kind of object whatsoever.

For Husserl there are two fundamental kinds of entities: the *real* and the *ideal* entities. The real entities, for example, are those objects given in world experience, the physical objects. The ideal entities are objects which can only be thought; Husserl also calls them *irreal* objects. There are several criteria for distinguishing between real/ideal entities and, perhaps, the most general is the relation to time: real objects are temporally individuated and ideal objects are temporally non-individuated. Ideal objects can be divided into sensual-dependent and sensual-independent objects. An example of an ideal sensual-dependent objects is the abstract entity (or universal) 'redness': it is not a physical object like a 'red car' but it depends on particular red colored objects, or on sensations of red colored objects. Ideal sensual-independent entities are called by Husserl *formal* or *categorical* objects, since they have no material or *hyletic* parts as constituents of their being. Examples of categorical objects are the logical operators of propositional logic, or the purely mathematical concepts

[1] E. Husserl, 'The Encyclopedia Britannica article' in *Psychological and Transcendental Phenomenology and the Confrontation with Heidegger*, p. 150.

like ‘set’, ‘function’, ‘relation’, ‘class’, etc.

An important difference between real and ideal objects is the following one: in the constitution of the sense of real objects, the *apprehension* of the corresponding object of constitution is not part of their sense. (*EJ*, section 63.) The intentional sense of a real object does not ensure that such an intention is already *fulfilled*, i.e., that to such intention corresponds an adequate examplar. Such a fulfillment has to be carried out by a confirmation or verifying synthesis (*CM*, section 24), presenting to the (transcendental) subject *a possible* state of affairs (a possible world, a model, etc.) in which such intention is exemplified. (*CM*, section 25.) For ideal objects however, Husserl argues that, *in some way*, their apprehension as a unity is part of their constitution. (*EJ*, section 63.) Since they are pre-constituted in the *predicative* activity, they are in advance presented or given as elements of judgements. That is, the constitution of ideal objects implies that they are also apprehended as substantives of assertions. This means that in the sense of ideal objects is already implicit that they can be *thematized*, i.e., that they are to be treated as singular intentional objects. For instance, in the sense of the intentional object ‘redness’ is already implicit that it can be subjected to predicative statements such as ‘redness is a colour property’, for example. The same goes for categorical objects: altogether with the sense of (logical) ‘conjunction’ comes the possibility to assert, for example, that ‘conjunction is a truth-function’. Thus the particular way ideal objects are intentionally constituted implies that they can be thematized as singular objects of possible judgements.

Does what we have presented so far in any way justifies van Atten’s claim that, in general, constitution implies existence? Let’s investigate in which context he assumes this principle as valid. When interpreting Husserl’s quotation above, besides the claim that it is a version of the transcendental idealism principle, van Atten also says that:

> It singles out a special group among all transcendental phenomena, namely those experiences in which the intentional object is given as itself, *as existing*. In *Cartesian Meditations*, Husserl calls the corresponding group of constitution processes cases of strict (‘prägnant’) constitution [*CM*, section 23]. There are correspondences between the essence of an object and the strict constitution of such an object. That is, the way in which an object is given as an identity through various acts is characterised by a rule that is specific for that kind of object [*Ideas I*, section 142], [*FTL*[2], section 98]. Strict constitution is constitution according to this rule. *Finally, existence is the objective correlate of strict constitution* [*CM*, section 26]. (*BmG*, pp. 56-57. Our emphases.)

In this quotation van Atten explains what we have exposed above: that for a special group of objects, constitution implies the intuitive donation of the constituted object as a singular unity. And he reports to the constitution of these kind of objects as ‘strict’ constitution. Then he restates Husserl’s claim that for this kind of objects the constitution is performed by a specific rule that links the constitution to the intuitive apprehension of the object as a singular general

[2]Edmund Husserl , *Formal and Trancendental Logic*, Martinus Nijhoff, The Hague, 1969; translated by Dorian Cairns.

essence (*the invariant* in Husserl's terms) and that this is strict constitution. In the end of the quotation, van Atten states that strict constitution has existence as objective correlate. That is, any possible object for which we have a strict constitution exists. He refers to *Cartesian Meditations*, section 26, for justification of this claim; let's see how. In this section, Husserl seeks to stress that evidence is the only source of validity. Or, in Husserl's own terms:

> It is clear that truth or the true actuality of objects is to be obtained only from *evidence*, and that it is evidence alone by virtue of which an *"actually" existing* [*wirklich seinder*], true, rightly accepted object of whatever form or kind has sense for us (...).(*CM*, section 26. p. 60.)[3]

Husserl uses the designation '*wirklich seinder*' (which we think a better translation would be '*effective being*'[4]) in the composition of this thesis, but also (for him) equivalent terms as 'true' and 'valid'. He does not uses the word 'existence' ('*Existenz*', in german) to characterize the correlate of evidence. This is a clue to dismiss possible misinterpretations of the meaning of 'existence' in this context. It is also a clue to the possibility that such word must not be assumed to mean what it usually means in the context of modern theories of ontological commitment.[5] In fact, a few lines further he defines what he understands by 'effective':

> *Actually existing* object [*Wirklich seiender Gegenstand*] indicates a particular system within this multiplicity, the system of evidences relating to the object and belonging together in such a manner that they combine to make up one (though perhaps an infinity) total evidence. (*CM*, section 29, p. 63.)[6]

Thus, an effective object (folowing the proposed translation) is defined as (the outcome of) an ordered sequence or 'system' of evident acts or intuitions, syntethically associated. Further, this sequence may be infinite. Such a notion of an effective object has a very specific meaning, that does not coincides with the usual modern notion of existence, which we think van Atten has in mind. Therefore, van Atten's interpretation of existence does not coincide with Husserl's analog and it may be considered as a forced interpretation. In Husserl's conception, '*Wirklich*' is equivalent, in a specific way, to *being true*, *being valid* or *being effective*. Based on this, the principle of transcendental idealism can be restated as follows: the constitution of an object implies its *effectiveness*, i.e., intentional constitution turns out mere intentional objects into

[3]In the original: Es ist klar, daß Wahrheit bzw. wahre Wirklisckeit von Gegenständen, nur aus der Evidenz zu schöpfen ist, und daß sie es allein ist, wodurch *wirklich* seinder, wahrhafter, rechtmäßig geltender Gegenstand, welcher Form oder Art immer, für uns Sinn hat, und mit all den ihm für uns unter dem Titel wahrhaften Soseins zugehöringen Bestimmungen. Edmund Husserl, *Cartesianische Meditationem*, Husserliana VI, Band 8, p. 61.

[4]This translation was proposed to me by Pedro M.S. Alves.

[5]By 'modern theories of ontological commitment' we are referring to the Quinean slogan that *to be (to exist) is to be the value of a bound variable* in the context of first-order predicate logic.

[6]In the original: Wirklich seiender Gegenstand indiziert innerhalb dieser Mannigfalttigkeit ein Sondersystem, das System auf ihn bezogener Evidenzen, derart Synthetisch zusammengehörig, daß sie sich zu einer, wenn auch vielleicht unendlichen Totalevidenz zusammenschließen. Edmund Husserl, *Cartesianische Meditationem*, Husserliana VI, Band 8, p. 64.

effective intentional objects. And if we interpret the claim that all entities get their ontological sense from intentional constitution as saying that intentional constitution provides an ontology of effective objects, then it is a fact that (as we advanced below) ontology in phenomenological terms does not coincide with modern theories of ontological commitment. Therefore, it is not obvious that ontological questions are decidable, in the modern sense, in the context of Husserlian phenomenology. If we assume a weaker sense of ontological decidability, namely that the intentional constitution of an object determines to what ontological or material region it belongs, then we can say that some decidability is possible by the constitution of an object. This, however, is not the sense of ontological decidability that van Atten is arguing for.

4.2 Does formal possibility implies existence?

Let's now discuss the version of the principle of transcendental idealism for mathematical or formal objects: for mathematical objects, possibility implies existence. (*BmH*, pp. 59 and seqs.) Van Atten defines transcendental possibility as follows:

> An object is transcendentally possible exactly if it is conceptually possible [i.e., if it implies no formal nor material contradiction] and moreover can be strictly constituted (that is, ideally, with full evidence). (BmH, p.60.)

Thus, transcendental possibility pressuposes logical possibility, conceptual possibility and constitution. As we have seen in the previous section, constitution implies existence, in van Atten's view, therefore transcendental possibiblity also implies existence. The term 'possibility' is here introduced by van Atten due to Husserl's claim that

> All existential judgments of mathematics, as *a priori* existential judgments, are in truth judgments of existence about possibilities; all particular mathematical judgments are direct particular judgments about possibilities – but about possibilities of particular judgments concerning mathematics. (*EJ*, section 96c, p. 371.)[7]

For Husserl this means that mathematical assertions such as 'there is an x with property A' are not assertions about (f)actual entities but *modal* assertions; they are assertions about possible objects, as opposed to impossible ones. So the assertion above can be paraphrased as 'it is possible that there is an x with property A'. For Husserl, existential mathematical propositions determine what is *a priori thinkable* as having this or that property. If a determinate concept is hence *a priori* thinkable as having some property, then it is possible. Notice that to be possible is not the same as to be factual, or effective in the terminology introduced in previous section. In fact, Husserl makes clear this distinction:

[7]In the original: Alle Existenzialurteile sind in Wahrheit Existenzialurteile von Möglichkeiten; alle mathematischen partikulären Urteile sind unmittelbar partikuläre Urteile von Möglichkeiten, aber von Möglichkeiten partikulärer Urteile über Mathematisches. Edmund Husserl, Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik, p. 450.

> But, as we said, these [judgments of existence] are *not particularities pure and simple*, actual particularities, but *a priori possibilities of particularity*. (...) The true sense [of mathematical existence] is not simply a «there is», but rather: *it is possible a priori that there is*. (*EJ*, section 96c, p. 370.)[8]

It is clear that the meaning of 'existence' in mathematics is, for Husserl, to be correlated with *validity*. Mathematical entities are precisely the valid, or possible ones as opposed to the invalid, or impossible ones. Talking about geometry and existential statements in mathematics, he claims the same in the negative form:

> Negative existential propositions have the function of separating out the invalid concepts, the expressions corresponding to no essence. *Ideas III*, section 15, p. 71.)

When we are dealing with formal objects we are dealing with objects that have no material or sensuous properties as part of their meaning, we are dealing with categories, i.e., with schematic forms. These schematic forms have a function of their own: to serve as *generators*. For example, the categorial object '$\wedge$' is a generator of logical conjunctions, propositions of the form $A \wedge B$. And it is part of the sense of '$\wedge$' that it is an *a priori* valid scheme whenever it is applied (assuming that the propositions to which it is applied are true). The particular propositions generated by '$\wedge$' are of an infinite number. So, not all the particular cases of conjunctions are presented to us in fact, but they remain as valid possibilities, through the sense of '$\wedge$'. That is, it is intrinsic to the meaning of '$\wedge$' that: if A and B are true propositions, then it is also true that $A \wedge B$. And this is valid for any (true) A and B whatsoever. It is in this sense that Husserl states that mathematical assertions are about *a priori* possibilities, i.e., about validity.

How can we, in the context of Husserlian phenomenology, interpret van Atten's claim that possibility implies existence? Van Atten's interpretation is due to the way how he defines possibility: an object is possible if it is free from formal and material contradiction, and if it can be strictly constituted. For van Atten, strict constitution implies existence. Therefore, possibility also implies existence. We have seen however how existence is to be (rightly) interpreted in Husserl's phenomenological conception: as effectiveness or validity. So, we believe that existence, in the phenomenological sense, does not correspond to the actual sense van Atten attributes to it.

4.3 Omnitemporality *versus* Intratemporality

Does Husserl's claim of omnitemporality for mathematical objects suffers from inconsistency? It is a suggestion of van Atten that the strong revisionism of Husserl's phenomenology rules out omnitemporality as the only temporal

[8] In the original: Aber es sind, wie gesagt, nicht Partikularitäten schlechthin, wirkliche Partikularitäten, sondern apriorische Möglichkeiten von solchen. (...) Der wahre Sinn ist nicht schlechthin ein, "es gibt", sondern: es ist apriori möglich, daß es gibt. Edmund Husserl, Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik, p. 450.

mode of mathematical objects. Is this claim accurate? Van Atten defines omnitemporal objects as those temporal objects that are static (do not change) and exists at every moment of time, past, present and future. (*BmH*, p. 16.) This definition of omnitemporality has as consequence that omnitemporal objects are to be seen as identically the same at *every* moment of time, since they do not change and they exist (unchanged) at every moment whatsoever. Let's now see how Husserl defines omnitemporality:

> It [the omnitemporal object] is referred to at all times; or correlatively, to whatever time it may be referred, is always absolutely the same; it sustains no temporal differentiation, and, what is equivalent to this, no extension, no expansion in time, and this in the proper sense. (*EJ*, section 64c, p. 259.)[9]

For Husserl, omnitemporality is defined as invariance trough time in a very definite sense, i.e., an ideal object is omnitemporal iff it can be recognized as the same object at any given moment of time in which it could be presented. This has as consequence that the particular moment in which it is presented as the same is not relevant to its sense, albeit such an object can only be constituted, or presented as such, in a moment of time. This means that the act through which the ideal object is constituted, the noesis, can only be performed in a particular moment in time; but the noetic mode of giveness in time, however, is not essential to the ideal object itself, i.e., to its sense present in the noema. And Husserl explains why the time-relation of such objects are not essential to their sense:

> It [the ideal object] is contingently (*kata symbebékos*) in time, insofar as it can «be» the same in any time. The different times do not extend its duration, and ideally this is arbitrary. This implies that, properly speaking, it has no duration as a determination belonging to its essence. [For the ideal objects] (...) *once* they have been actualized or «realized», they are also localized spatiotemporally, but in such a way, to be sure, that this localization does not actually individualize them. (*EJ*, section 64c, pp. 259-260.)[10]

In another words:

> Irreal objectivities make their spatiotemporal appearence in the world, but they can appear simultaneously in many spatiotemporal positions and yet be numerically identical as the same. It belongs essentially to their appearence that they are subjective formations, therefore localized in worldliness (spatiotemporality) by

[9]In the original: Es ist auf alle Zeiten bezogen, oder auf welche auch immer bezogen, immerfort absolut dasselbe; es erfhart keine zeitliche Differenzieerung und, was damit äquivalent ist, keine Ausdehnung, Ausbreitung in de Zeit, und das im eigentlichen Sinne. Edmund Husserl, Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik, p. 311.

[10]In the original: Es liegt zufällig (*κατα συμβεβηχς*) in der Zeit, sofern es, dasselbe, in jeder Zeit "liegen" kann. Die verschiedenen Zeiten verlängern nicht seine Dauer, und ideell ist diese beliebig. Das sagt: eigentlich hat es keine Dauer als eine zu seinem Wesen gehörige Bestimmung. (...) Aber jedenfalls, wenn sie aktualisiert worden sind, oder "realisiert", sind sie auch raum-zeitlich lokalisiert, und freilich so, daß diese Lokalization sie nicht wirklich individuiert. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik*, pp. 311-312.

> the localization of the subject. But they can be produced in different moments of time of the same subject as the *same*, as the same in relation to their repeated productions and as in relation to the productions of different subjects. (*EJ*, section 64c, p. 260.)[11]

Let's now discuss some possible relations between van Atten's and Husserl's definitions of omnitemporality. What we want to stress is the following: does these two definitions of omnitemporality coincide with each other? We write A-omnitemporality and H-omnitemporality for van Atten's and Husserl's definitions, respectively. Notice that in both definitions the identity of the object is preserved in different moments of time. But does this consequence follows from both definitions by the same reason? Our answer is negative, and we will now show why.

An object is A-omnitemporal if it is static and exists at every moment of time and it is H-omnitemporal if it is referred to at any moment of time as the same object. So, if an object is static and exists at every moment of time, then it is to be referred as the same at any moment of time; i.e., A-onmnitemporality implies H-omnitemporality. But if it is referred as the same at any moment of time, then is it also static and exists at every moment of time? That is, does H-omnitemporality implies A-omnitemporality? In other words, is an object that is not static or does not exists at every moment of time (or both) necessarilly a non H-omnitemporal object? Is it possible to have conjointly an H-omnitemporal but not A-omnitemporal object?

Suppose we have the following object: a finite sequence S with k elements (k is finite and constant) whose domain is $\{0, 1\}$, i.e., each element of S is either 0 or 1. Finally, let's add the clause that at any different moment of time at least one element of S is permuted in the follwing manner: if it is 0, replace it by 1 and if it is 1, replace it by 0. Doing this, we can assume that at least one element of S is always different at any moment of time due to a rule. For example, let's assume $k = 3$, then we can stipulate that at moment t_0 $S = \langle 0, 0, 0\rangle$; and thus define a rule such that at any subsequent moment of time one of the elements of S is to be replaced in a certain order such that, for example, at moment t_1 $S = \langle 1, 0, 0\rangle$, at moment t_2 $S = \langle 1, 1, 0\rangle$, at moment t_3 $S = \langle 1, 1, 1\rangle$, etc. The point is that at any moment of time S has parts added or/and removed from it (van Atten's definition of a non-static object, *BmH*, p. 16.). Therefore, at any moment t_n at least one element of S will be removed from S (in the sense that it will be replaced by another numerical distinct object). However, S can be presented as the same object at any particular moment of time, since (as we have defined) each element of S at that particular moment is generated by a rule. Moreover, the temporal character of S is not part of its sense, since at any time it would be given as the same because it is generated by a law (or a collection of laws). Given this we conclude that the non-static object S fits into H-omnitemporality but not into A-omnitemporality. Another example is the case of lawlike sequences. Lawlike sequences are examples of

[11] In the original: Irreale Gegenständlichkeit haben in der Welt raum-zeitliches Aufreten, aber sie Können an vielen Raum-Zeitstellen Zugleich und doch numerisch identisch als dieselben aufreten. Wesensmäß ig gehört zu ihrem Aufreten, daß sie subjektive Gebilde sind, also in der Weltlichkeit (Raum-Zeitlichkeit) durch die Lokalität der Subjekte lokalisiert sind. Aber sie können in verschiedenen Zeitstellen desselben Subjektes als dieselben erzeugt werden, als dieselben gegenüber den wiederholten Erzeugungen, und ebenso als dieselben gegenüber den Erzeugungen verschiedener Subjekte. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik, ibidem.*

non-static unfinished objects that are able to be fitted into Husserl's notion of omnitemporality. (We present concrete arguments for this claim below.) Any lawlike sequence can be recognized as being the same, or as being generated by the same rule, at any particular moment of time. (See *EJ*, section 51b.) The only difference between lawlike sequences and the example above is that one is a finite object and the other an infinite one. But both are H-omnitemporal objects that do not fit A-omnitemporality.

How about existence? Does H-omnitemporality implies that an object has to exist at every moment of time? Our answer is negative, for two reasons. The first reason is that mathematical existence in Husserl's terms is to be interpreted as mathematical validity. Therefore, H-omnitemporality for mathematical objects means that they are to be recognized as mathematically valid at any moment of time whatsoever, not as existent at any moment of time, in van Atten's sense.

The second reason is that, when Husserl speaks about recognition of an (omnitemporal) object as being the same at any moment of time, he is not stating that it is to be, in fact, recognized as being the same at *every single* particular moment of time, in a positive or categorically sense. He is saying instead that if at an *arbitrary* particular moment of time it were presented, then it would have to be recognized as being the same identical object. Husserl's statement of omnitemporality seems to be of a conditional character, while van Atten's claim of omnitemporality is of a categorical character. What H-omnitemporality advances is that if it is presented at a moment, then it should be recognized as identically the same object. Once more, we suggest, Husserl's claim should be interpreted as having a (weaker) conditional character, differently from the (stronger) categorical character of van Atten's claim. The conditional character of H-omnitemporality becomes evident taking into account the following quote of Husserl concerning ideal and, in particular, mathematical objects:

> But afterwards we say: even before they were discovered, they were already "valid"; or we say that they can be assumed – provided that the subjects which have the ability to produce them are present and conceivable – to be producible precisely at any time, and that they have this mode of omnipresent existence: *in all possible modes of productions they would be the same*. (*EJ*, p. 260; our emphasis.)[12]

It is obvious now that the way van Atten defines omnitemporality (and then draw the implications) does not correspond to Husserl's conception of it. We showed that H-omnitemporality does not rule out non-static objects like A-omnitemporality does. Non-static objects as lawlike sequences and the example S can be fitted into H-omnitemporality. And we also showed that Husserl's conception of omnitemporality does not imply that ideal objects have to exist at all moments of time as A-omnitemporality stipulates. Therefore, we conclude that the particular concept of omnitemporality against which van Atten argues is not the Husserlian concept of omnitemporality.

[12]In the original: Aber hinterher heißt es: auch ehe sie entdeckt worden sind, haben sie schon "gegolten", oder sie sind in jeder Zeit – wofern in ihr Subjekte da und denkbar sind, die sie zu erzeugen das Vermögen häten – als erzeugbar eben anzunehmen und haben diese Weise allzeitlichen Daseins: in allen möglichen Erzeugungen wären sie dieselben. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik*, p. 312.

At this point it is obvious that van Atten is wrong in asserting that Husserl rejects choice sequences because they are not omnitemporal objects. We have proposed that lawlike sequences, that are dynamic objects, can be fitted into Husserl's view. The problem with lawless sequences is not the fact that they are intratemporal, and hence dynamic objects. The problem that would lead Husserl to reject lawless sequences as mathematical objects is, we think, related to the fact that they have not a determinate law attached to them. In Husserl's terminology, they do not have a determinate *noema* or sense that makes possible to recognize them as the same object at any moment of time. In other words, the problem is not essentially in the mode of temporality of choice sequences but in the fact that, in general, choice sequences are not individually determinate in a distinctive mathematical manner. This, we think, is what would forbid Husserl to admit choice sequences as mathematical objects.

Does the motivation for omnitemporality motivates also intratemporality? For Husserl, albeit all mathematical objects refer to time in an omnitemporally way, their individuation as mathematical objects do not come from the particular way they refer to time. Mathematical objects are the same at any moment of time. Choice sequences however are intratemporal objects, the particular moment they start to grow constitutes an essential part of their individuality. For lawlike sequences, Husserl's claim is innocuous. Lawlike sequences, albeit intratemporal objects, can additionally be individuated by the particular law that generates their values. So, for lawlike sequences, intratemporality can be incorporated into the Husserlian omnitemporal account. For lawless sequences however, their intratemporality is an essential and irreducible characteristic. For lawless sequences, the nature of their process of generation does not allow individuation as happens with lawlike sequences; further, sharing the same initial segment does not suffice to determine that two lawless sequences are the same, since they may diverge at some point of growth. Therefore, for lawless sequences, the only criterion of individuation is the precise moment they are initiated.

But the fact that choice sequences are intratemporal objects does not rule them out as mathematical objects. Since:

> Instead of being necessarily infinitely temporal in two directions (and thereby omnitemporal), would be so in just one direction (the future). In that case still what is proven once, is proven forever. (...) It follows that infinite temporality only in the direction of the future preserves the original [Husserlian] motivation for the thesis of omnitemporality. (*BmH*, p. 97.)

Husserl's original motivation for omnitemporality of mathematical objects, in van Atten's view, is that what is proven for them is proven once and for all, is proven for any time. Van Atten's claim is that this motivation is also true for intratemporal objects. For lawlike sequences, the fact that they are attached to a determinate law guarantee that what is true for them is true at all moments, since the law that generates their values is the same at any moment of time. For lawless sequences, argues van Atten, it turns to be the same: what is once proven true for them is proven once and for all. But in this case we have to state an additional clause: that it is the same in the direction of future, i.e., once a true property of a lawless sequence is proven, then it remains true at any future moment. For van Atten, intratemporality can be viewed as a *particular case* of omnitemporality when restricted to the future. Therefore, the motivation that lead Husserl to admit classical omnitemporal objects as mathematical objects remains valid for accepting intratemporal non-classical objects as mathematical objects from a phenomenological standpoint.

For van Atten the motivation for omnitemporality is that once some property of a mathematical object is proved, then it is proved once and for all; in the case of lawless sequences, it is also proved once and for all but with respect to the future. Let's now change a bit the perpective. If we adopt a modal account it is possible to show that, contrarily to van Atten's claim, the Husserlian motivation for omnitemporality does not work for the intratemporality *of* choice sequences. The argument is based on the application of the concept of *counterfactuals*. An omnitemporal object in Husserl's sense, besides being referred as the same at any moment of time, has also to be referred as the same at any counterfactual situation (at any possible world). At any other counterfactual situation it could not be thinkable as having other properties than the ones it already has. For example, the number π is always referred as the same at any counterfactual situation, i.e., there is not a possible world where π has a decimal expansion different from the usual one ($\pi = 3.14159...$). So, the name 'π' is a *rigid designator*, i.e., the name'π' designates always the same object, viz. the number π, at any counterfactual situation. Let's see the behavior of lawless sequences under the framework of counteractual situations.

We can imagine a lawless sequence α having some property $A(\alpha)$ at a moment t_k in the (f)actual situation, but having a different property, say $B(\alpha)$ (notice that we can choose $B(\alpha)$ as to imply $\neg A(\alpha)$), in a counterfactual situation at the same moment t_k. The idealized mathematician in the counterfactual situation (who is the same in the actual situation) has the freedom to choose the values of α differently than it had chosen. Assuming this possibilty, then the designation 'α' is not a rigid designator. It is not a rigid designator because one of the two is the case: either 'α' designates different lawless sequences in different counterfactual situations or it designates the same lawless sequence with different properties in different couterfactual situations. Anyway, we have to assume that the lawless sequences generated at different counterfactual situations, but at the same moment t_k, are not (exactly) the same. This possibility, however, goes against the character of necessity of mathematical objects in Husserl's sense. Mathematical objects cannot have different properties in different counterfactual situations. Notice, however, that the free character of the idealized mathematician entails that in different situations it has the freedom to choose the values of α differently. Otherwise it would be *determined* in advance to choose the values of α as it actually did (contradiction?). It would be determined in the following sense: let's assume that a lawless sequence α is in the same position that π, i.e., that 'α' is a rigid designator for the sequence α. Then at any possible world (or, which is the same, in any counterfactual situation) the creating subject could not have chosen the values of α differently from the ones he had chosen in the actual world. This means, for example, that if at the actual world and in moment t_k the idealized mathematician had chosen the number 100 as the 1000^{th} value of α, then the idealized mathematician at world w_k (who is the same at the actual world) and in the same moment t_k, has to choose the number 100 as the 1000^{th} value of α. The assumption, however, that 'α' is a rigid designator for the sequence α entails that the idealized mathematician *is* *not free* to choose the values of α differently from the choices it actually made; at any possible world w_n the idealized mathematician has to choose 100 as the 1000^{th} value of α.

Do we have reached a dilemma? Do we have to choose between the necessary character of mathematical objects (and omnitemporal objects in general) and the freedom of the creating subject? The natural choice, for the proponent of choice sequences anyway, is the second one: lawless sequences have to be conceived as freely constructed step-by-step through the free acts of choices (not determined in any way) of the creating subject. If we are to assume choice sequences as mathematical objects constructed by the creating subject, then we have to accept the fact that the creating subject could have chosen the values of a lawless sequence differentlly. This is precisely what van Atten states:

> When a mathematical construction depends on free choices, we have to reckon with the fact that, when at different times the subject is confronted with the same situation in which to choose, it can do so differently each time, and hence these objects cannot be omnitemporal. (CCM, p. 70.)[13]

Therefore, in accepting the creating subject's free choices, *a fortiori* we have to accept either that, albeit the properties of a lawless sequence became 'crystallized' when proved, they are, after all, *contingent*. The lawless sequence α above could have had other properties than the ones it actually has (e.g., the creating subject could have chosen the number 500 as the 1000^{th} value of α).

On the other hand, Husserlian mathematical objects cannot have their properties contingently in the sense above. Therefore, we think that the real motivation of Husserl for assuming omnitemporality for mathematical objects is not just that once a property is proved it is proved once and for all. Additionaly to the fact that what is proved is proved once and for all, Husserlian omnitemporality is motivated by the fact that a mathematical object have its properties in a non-contingent way and, by this reason, at any moment of time. And this is the case irrespectively whether we assume omnitemporality for past, present and future, or only for the future (intratemporality). Omnitemporality implies that what is proved is proved once and for all, but also that what is *provable* is provable in any time. Mathematical objects do not acquire new properties with the flow of time nor at a counterfactual situation. All the properties of the object are already present in its noema, irrespectively whether they are proved at a determinate moment or at a subsequent moment, or even if they are not proved at all. The respective noema of a mathematical object already contains the properties of the object, even if they are not yet explicitly brought to intuition through an act of consciousness. In Husserlian terminology, all properties of a mathematical object are already present in the *internal horizon* of the noema, are implicitly present in the noema, and they are *explicated* (*explizieren*), i.e., brought to evidence.[14] However, the explication of an object, the intrinsic temporal determination of an object, the disclosure of its internal or imanent properties does not attribute to it new (non-existent-yet or not-present-before)

[13] Mark van Atten, 'Construction and Constitution in Mathematics', in *The New Yearbook for Phenomenology and Phenomenological Philosophy* vol. X, 2010, pp. 43-90. For comodity we will refer to this paper as 'CCM'.

[14] For this new terminology see, for example, *EJ*, section 24.

properties, bul only makes evident what was already present, yet implicitly, in its original intended sense. In Husserl's own words, "the object, *every* object, has its peculiarities, its internal determinations." (*EJ*, section 24a, p. 112.) There is a fundamental difference between the properties of an object (presented in the noema) and the temporal way (through the respective noesis) in which they are presented to a possible consciusness: through the acts of explication of an object we do not attribute properties to the object that were not part of its noema, we only reveal them and make them evident.

Lawlike sequences, albeit intratemporal, satisfy this condition, because, due to the particular law generating their values, all their properties are not contingent in the sense above (the designators of lawlike sequences are rigid designators). Lawless sequences, however, do not satisfy this condition, by the reason presented above. Therefore, this *modified* motivation for omnitemporality rules out intratemporality *per se* and, consequently, rules out lawless sequences as Husserlian mathematical objects. The original and this modified Husserlian motivation for omnitemporality of mathematical objects cannot be separated from each other. For this reason, van Atten's argument fails to establish its conclusion mainly because the non-contingent character of omnitemporal objects is not discussed by him.

4.4 The form 'and so on': substrate or determination?

On pages 71-72 of *Brouwer meets Husserl* van Atten accuses Husserl of being dogmatic. Husserl conceives a method to 'disclose essences', i.e., a method to obtain essences with maximum generality: the *eidetic variation*. (*EJ*, sections 86-93.) This method consists of, from a particular example falling under a concept, to modify it at will by *free imagination* into new examplar cases in order to achieve the concept, the essence in Husserl's terms, in its full generality. This is done by overlaping the different variations. When we overlap the possible variants, something identical in all of them will coincide and arise: the *invariant*, what necessarily they all have in common. This invariant is the characteristic feature that makes them all be the same kind of object, and so we obtain the essential feature of the concept under which all fall. (*EJ*, section 87a-e.)

Van Atten's suggestion is that Husserl's eidetic variations to disclose the essence 'mathematical object' are not neutral, i.e, that Husserl does not consider all possible variations, and so that Husserl's eidetic variations "were one-sided" (*BmH*, p. 71). Further, he says that Husserl never considered non-omnitemporal objects, viz. choice sequences, among his variations. And because of this short-sightedness the essence arrived at by Husserl is not necessarily of 'mathematical object' but that of 'classical mathematical object'. For van Atten to prove this claim he has to show that choice sequences are indeed mathematical objects; and to do so, within a phenomenological framework, is to be able to show that choice sequences can be intentionally constituted as mathematical objects with full evidence. We will now precisely analyze his attempt to achieve such constitution.

The constitution of an ideal object can be divided into two major moments: (1) the identification of the proper act by which it is presented in consciousness (pre-constitution in passive synthesis) and (2) the determination (or thematization) of the invariant presented in that act (constitution in active synthesis). For van Atten, the proper act by which choice sequences are given to the consciousness is the act of *choosing*:

> The activity that founds the self-giveness of choice sequences is that of choosing. In this activity choice sequences are pre-constituted (...), meaning that after this activity one only needs to carry out an objectifying act to constitute the object. Such an objectifying act thematises the invariant that established itself in passive synthesis. (*BmH*, p. 89.)

So, for van Atten, the activity of choosing pre-constitutes choice sequences in the passive experience, meaning that the originary and fundamental process by which choice sequences are initially presented to our minds is the process of choosing. After the identification of the act by which choice sequences are given to consciousness as the act of choosing we only need to determine the invariant presented by such activity and show that it is always the same, i.e., that it is identical in all possible choice sequences. Van Atten advances three possible candidates to figure as the invariant present in all choice sequences: that it is a *concept*, that it is the *initial segment*, or that it is the categorical form '*and so on*'. The first and second candidates are dismissed by him as incorrect; then he argues that the correct invariant is the categorical form 'and so on'.

That the invariant is not a concept nor an initial segment. A concept is not the invariant, since concepts and objects are different things: a concept and an object falling under it are two different things. The concept may have properties that the object falling under it does not possess, and vice-versa. In the case of choice sequences the object 'choice sequence', i.e., the sequence, consists of linearly ordered parts, while a concept governing it does not. Also the temporal aspects of the concept and of the sequence does not coincide. The sequence is intratemporal, while the concept governing it is an omnitemporal ideality: the sequence only comes to existence when the first choice is made and that happens at a particulr moment in time; while the concept (what it means) is the same at every moment of time. A choice sequence changes through time, the concept does not. Initial segments also are not the invariant, since different choice sequences may have the same initial segment. Initial segments are extensional properties of choice sequences and what makes choice sequences different among them is their intensional properties (e.g., having been begun at different moments of time). (*BmH*, pp. 89-90.)

That the invariant is the categorical form 'and so on'. A second reason why initial segments are not the invariant is that, by assuming that they are, an important feature of choice sequences is left behind: the character of a choice sequence as a *developing* sequence. The developing character of choice sequences is not contained in initial segments; these are finished entities, they do not develop trough time as the choice sequence does. Choice sequences are determined as such by being indefinitely developing objects. They are objects that extend continually, but this indefinite extension does not transform them into new objects different from the initial one. Because, says van Atten, it is the same object that is extended, the same object that is growing. Therefore, some substrate must remain identical if the idea of growth is to make sense. This substrate is the categorical form, or operation '*and so on*'. The categorical form 'and so on' is the substrate making possible that we can talk of the *same* object that grows, making possible to talk about choice sequences. A last observation has to be made. The process of constituting choice sequences as objects is not complete if, stresses van Atten, we do not consider that the act of choosing (pre-constituted in the passive synthesis) is, in fact, an act of *free* choosing. In order to do this, he says, we have to consider "the more or less free choice of a number an operation too." (*BmH*, pp. 90-91.)

The act that presents choice sequences as intentional objects is the act of choosing. The invariant present in all choice sequences is the operation 'and so on' (sucessive application of an act). When we combine these two features of choice sequences we get the new operation of *choosing repeatedly*, that characterize choice sequences as open-ended. Further, we have to assume that this new operation is *free*, i.e., that it is an operation of successive free choices. Thus we have so far four (cumulative) elements in the process of constituting choice sequences as objects: the single act of choosing (the founding act), the operation 'and so on' (the invariant), the new operation of successively choosing (the combination of both) and the final operation of successive free choices (the assumption).

The act of choosing as the act through which choice sequences are intentionally given does not poses any problem, in our view; it is, from a phenomenologically poit of view, completely sustainable. That the *specific* substrate which characterizes all choice sequences as such is the categorical form 'and so on' is, however, questionable. The categorical form 'and so on' is in fact recognized by Husserl as such. However, it is not assumed by him that it is restricted to any concept or type of object in particular, but that it is an applicable operation to any substrate whatsoever:

> (...) every substrate of determination is originally always already passively pregiven as something determinable, as something with a horizon of indeterminate determinability and known in conformity with a most general type. In the course of the explication [originary disclosure of the properties of the substrate] this prescription is increasingly fulfilled, but there still constantly remains a *horizon* beyond the succession of actually constituted determinations and *open to new properties which must be expected*. (*EJ*, section 51b, p. 217.)[15]

For Husserl the 'and so on', the *undetermined horizon of determinability*, is a feature applicable to all possible substrates, meaning that we can conceive any substrate (or any object in general) as being determinable (or predicable) *in infinitum*. In fact, in Husserl's context, the operation 'and so on' is not even primarily seen as substrate, nor determining a specific domain of substrates, but instead as a possible *form of determination* applicable to all substrates. Of course, any determination can be turned into a substrate, i.e., it is possible to

[15]In the original: (...) Abschnitt gesehen, daß jedes Bestimmungssubstrat ursprünglich immer schon passiv vorgegeben ist als bestimmbares, als dem allgemeinstein Typus nach bekanntes Etwas mit dem Horizont unbestimmter Bestimmbarkeit. Im Verlauf der Explikation erfüllt sich diese Vorzeichnung immer mehr, aber ständig bleibt noch über die Folge der aktuell konstituierten Bestimmungen hinaus ein offener Horizont für zu erwatende neue Eigenheiten. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik*, pp. 257-258.

turn a determination into a substrate with other (higher order) determinations. But the relevant point in this discussion is that the operation 'and so on' is not seen from the standpoint of Husserl's phenomenology as specific to any particular substrate or kind of substrates, since it is a form of determination of any possible substrate and not (primarily) the substrate itself nor a form of determination exclusive to any kind of substrate.

However, it seems in fact that choice sequences possess the 'and so on' character intrinsically, i.e., that in the case of choice sequences it is not a mere possible form of determination but an essential form of determination. Choice sequences cannot be (correctly) conceived without this feature. But now we ask: is the 'and so on' character of choice sequences a (form of) determination of some substrate or it is the substrate itself? As a matter of fact, it cannot be both at the same time, the substrate and the determination of the substrate simultaneously. And it cannot be both by the same reason why van Atten assumes there is a difference between a concept and an object falling under it: *the substrate* and *the form of determination* of the substrate may have different properties. The substrate behaves (develops) in a manner stipulated by the form of determination 'and so on', but the form of determination 'and so on' does not behaves itself in the same way. The form of determination 'and so on' as such does not 'develop' through time, instead it prescribes such a developing character to some substrate.

Based on the discussion above we think that a distinction between the *invariant substrate* of choice sequences and (let's say) the *invariant form of determination* of choice sequences is possible, and that van Atten inaccurately equates both in the wrong way. In colloquial terms, there is a difference between the identical object subjected to a condition and the condition itself. In our view, to ask for the invariant of choice sequences, as van Atten formulates the question, has two possible interpretations:

1. to ask for the invariant character that makes choice sequences be what they are, whose answer is: the invariant form of determination 'and so on'. In this case, the invariant is not an object but a condition: that, if there are any objects to be called 'choice sequences', then they invariably have to possess the character 'and so on' as a form of determination;

2. to ask for what remains identical in the process of growing of an individual choice sequence, i.e., for what identical component of the individual choice sequence is behaving in conformity to the condition 'and so on', whose answer is still open.

The first question asks for the general essence of choice sequences (the invariant condition 'and so on') and the second question asks for the identity of a particular choice sequence through the process of growth, i.e., the identical object or substrate that grows. They are different questions and have different answers, albeit van Atten seems to not distinguish both. In fact, in our view, what van Atten advances as the answer to the second question (intended by him as the right question) is the right answer for the first question.

4.5 How can be apprehended an individual choice sequence?

We now turn to the following question: how are choice sequences given to consciousness?, how are choice sequences apprehended as individual objects? What means for a choice sequence to be originally given as an intentional singular object? The giveness of an intentional object *as itself* in intuition, the presentation of the intended object as an identical substrate or, more generally, the transcendental experience of an ideal object is the goal of the process of constitution. The process of constitution aims at this final achievement: the originary and evident intuition of the object itself as identical by the transcendental subjectivity. Is this possible for choice sequences? If it is, how? How can be phenomenologically accomodated the intuition of an object like a choice sequence? Van Atten's answer to this question is to be explicitely found in *CCM*:

> With an eye on Brouwer's infinitely proceeding sequences [choice sequences], it should be noted that ideal, adequate giveness of a potentially infinite sequence does not consists in its being given as an actually infinite sequence, for that would contradict the essence of the object qua potentially infinite. Rather, it consists in the giveness of the whole finite *initial segment* generated so far, however large the number of its elements may be, together with the *open horizon* [the categorial form 'and so on'] that adequately indicates the ever present possibility to construct additional elements of the sequence. The absence of such further elements from an intuition of the sequence at a given moment does not render that intuition inadequate, because they do not yet even exist. (CCM, p. 65.)

An individual choice sequence is adequately given to consciousness when (1) its respective initial segment is given to consciousness, (2) together with the form 'and so on'. Now we ask: how do these two factors constitute the giveness of a singular unity, viz. the individual choice sequence? The giveness of an individual choice sequence should present it as a single object, as a unity. Instead, what we have are two distinct moments: the act that presents the initial segment *per se* and the further act that presents the initial segment *as* a developing object. For van Atten the conjunction of these two acts should present the individual choice sequence as an individual entity. Notice that the initial segment and the form 'and so on' separately do not give rise to the choice sequence. They can be seen as independent entities: in one hand the initial segment and the form 'and so on' in the other hand, i.e., they can be thematized separately as independent objects. Therefore, how can they together make us apprehend a *third* individual entity, different from both? Notice that the originary apprehension of an object is something essentially *simple*, i.e., it is a non-complex act or intuition concernig the nature of the apprehended objects themselves. In Husserl's own words, such an act is:

> The contemplaive intuition which *precedes* all explication [temporal determination], the intuition which is directed toward the object «taken as a whole». This *simple apprehension and contemplation* is

> the *lowest level* of common, objectifying activity, the lowest level of the unobstructed exercise of perceptual interest. (*EJ*, section 22, p. 104.)[16]

The most fundamental mode of giveness of any object whatsoever is such that the object is intended *as a whole*, as an indecomposable unity and presented to consciousness in a simple act. At this level of consciousness activity, in which the intentional objects are originarily given in themselves, there is no place for any kind of complexity. Even the (also lower) activity of determination of the intuited substrate is placed at a higher level of consciousness, viz. at the level of the "true *explicative contemplation* of the object (...) [where] the first apprehension and initial simple contemplation already has its horizons (...) which are immediately coawakened." (*EJ*, section 22, pp. 104-105.) At this level of consciousness, in which the object is originarily and evidently apprehended or intuited, is the object itself, in its entirety, that is apprehended. It is not apprehended through the previous apprehension of its *parts* or determinations, as if the apprehension of the original object is a (kind of) second-order apprehension. It is the object itself that is given in the first place as a substrate of possible determinatios, which are not yet determined or explicitated; its parts and determinations are to be posteriorly brough to consciousness, not the reverse (see *EJ*, section 24a).

It is clear that van Atten's conception of the intuitive mode of giveness of individual choice sequences does not fit the Husserlian conception. For van Atten, the giveness of an individual choice sequence is accomplished by the giveness of its initial segment, i.e., by the giveness of a finite *part* of the choice sequence, plus the giveness of the form of determination 'and so on' applied to the finite initial segment. Assuming the cogency of the arguments presentd by us for the incorrectness of admitting the form 'and so on' as the substrate of individual choice sequences, then it is obvious that the giveness of these two elements is not sufficient for the intuitive apprehension of the individual choice sequence in itself. This is so because both the initial segment and the form 'and so on' are not the substrate of the individual choice sequence but its determinations; and the substrate is not originary given through its determinations but in itself. Let's analyse the modes of giveness of the initial segment and of the form 'and so on' separately of each other (as far as possile) to gain some insight on what was just said.

Apprehension of the initial segment. The initial segment is a finite part of the choice sequence. It is a detachable or independent part of the choice sequence, which Husserl calls a *fragment* in respect to a whole (see *EJ*, section 31). The fact that the initial segment is seen as a fragment of the choice sequence does not means that the particular choice sequence to which it belongs can be conceived without the respective initial segment. It means the opposite: that the initial segment can be considered an indvidual object *per se*, independently of the choice sequence to which it belongs, i.e., that it can be thematized as a substrate having its own determinations. (This possibility is confirmed, for example, by the technical fact that within the theory of lawless sequences **LS**,

[16] In the original: Die betrachtende Anschauung, die auf den Gegenstand "im Ganzen" gerichtet ist. Diese schlichte Erfassung und Betrachtung ist die unterste Stufe niederer objektivierender Aktivität, die unterste Stufe ungehemmter Auswirkung des Wahrnehmunsinteresses. Edmund Husserl, *Erfahrung und Urteil, Untersuchungen zur Genealogie der Logik*, p. 114.

but of course, also within the theory of lawlike sequences, we can deal with finite segments and establish properties about them only.) This independence of initial segments means that they are able to be apprehended in themselves as finite totalities. But now we ask: are all initial segments at the same level concerning their apprehension? Does not the *size* of initial segments matter in the way they are apprehended?

Can we apprehend any initial segment in itself "however large the number of its elements may be"? (*CCM, ibidem*.) Is the apprehension of an initial segment with 10 elements at the same level of the apprehension of an initial segment with 10^{10} elements. Notice that both initial segments are finite sequences. But, albeit of a finite size, can the second initial segment of a possible choice sequence be really brought to the mind in itself? Are we capable of apprehending a sequence of 10^{10} elements as we are capable of apprehending a sequence with only 10 elements? It is always possible to determine a rule that computes any finite sequence no matter its size. In such cases, however, it is not the finite sequence that is apprehended in itself but the *rule* which computes the values of the finite sequence. This fact leaves to conjecture that the only way we have to conceive the apprehension of a initial segment is through a rule, i.e., that in general (considering any initial segment and not just the 'small ones') they are apprehended through the sense encapsulated in the rule that generates their values. What is given to consciousness in itself is the rule that generates the values of the sequence. Therefore, for finite totalities (at least the large ones) it is clear that this is the only way we can apprehend them in themselves as a whole with full evidence. Thus, for initial segments, we have just seen that their apprehension is analogous to the apprehension of lawlike sequences.

A possible way out of this crticism would consist in claiming that for the idealized mathematician (as a transcendental subject) the intuition of large finite totalities in themselves is in some way possible, and hence, does not give rise to such problems. This, however, seems barely sustainable since the idealized mathematician is an abstraction from the real mathematicians in which only memory and time limitations are disregarded. The idealized mathematician does not possess a more powerful intuitive capacity than the real ones.

Apprehension of the form 'and so on'. Contrary to the initial segments, the form 'and so on' is not a fragment of individual choice sequences. It is not a detachable finite part of a totality, instead it pervades the choice sequence through all its stages. Also contrary to the initial segment, the form 'and so on' is the same identical entity present in any choice sequence, it cannot be something numerically different in each choice sequence. For example, the categorical form of logical conjunction '$\wedge$' is not a different entity in each particular case it takes place: the same categorial form is present in different propositions $A \wedge B$ and $C \wedge D$, but it is the same form that is present in them, namely the schematic form '... $\wedge$ ...'. Contrary, for example, to the universal *redness* that is not present on every red-coloured thing, but that is an abstraction from the several *tokens* of redness present in red-coloured objects, the categorial form '$\wedge$' is not present in every logical conjunction as a token but in itself, is the same identical object '$\wedge$' that is present in every concrete logical conjunction. Like the logical conjunction, the form 'and so on' does not become something different in each choice sequences; it is numerically the same object that is present in every choice sequence. And if we assume (as van Atten proposes) that the categorial form 'and so on' is the substrate for individual choice sequences, then we are forced to admit that it is the same numerically identical substrate that is present in any choice sequence. This is also a reason why it cannot stand as the substrate of individual choice sequences: different individual objects must have different substrates.

The intuition of a categorial object presents the intendend object as a formal entity, a scheme that is identically the same in each particluar unity of sense in which it is present. The categorial form '$\wedge$' is identically the same object in $A \wedge B$ and in $C \wedge D$. In itself the categorial form '$\wedge$' does not differenciate neither $A \wedge B$ nor $C \wedge D$ since it is present as the same in every logical conjunction. Therefore, the categorial intuition of '$\wedge$', although implied by it, does not determine the individual apprehension of $A \wedge B$ in itself or as being different from $C \wedge D$ since it is the same categorial object ('$\wedge$') that is present in the two distinct propositions. We will argue below that the categorial form 'and so on' behaves in the same way the categorial form '$\wedge$' does. The categorial intuition of the form 'and so on' (although implied by it) does not determine the individual apprehension of a lawless sequence α in itself, nor *as* being different from another lawless sequence β.

Assuming the categorial nature of the form 'and so on', how can its apprehension concur to the apprehension of individual choice sequences? How can an individual choice sequence be given to consciousness through the apprehension of its form 'and so on'? Notice that neither the initial segment nor the form 'and so on' are the substrate of the intended choice sequence (although for different reasons). So, how can the substrate be aprehended through their mutual aprehension? The kind of apprehension proper to categorial objects is the *categorial intuition*. The aprehension of a categorial object (a state-of-affairs, crystalized in a judgement) is not performed at the pre-predicative level but at a higher level of consciousness activity: the predicative activity. (*EJ*, section 58.) For this reason the categorial entities cannot be given or apprehended in the same way, or at the same level, an individual simple object is aprehended. Notwithstanding this difference between the apprehension of a simple object and of a categorial object, a categorial object *is founded* or presupposes the individual objects apprehended in the pre-predicative consciousness. (*EJ*, section 59.) This means, for example, that the apprehension of the logical conjunction '$\wedge$' in the proposition $A \wedge B$ presupposes the apprehension of A and B in themselves (not necessarily as actual or already performed apprehensions, but as possible ones). In other words, the constitution of categorial objectivities presupposes the constitution of the non-categorial substrates taking place (through a specific type of compositionality) at the categorial act. What this entails, concerning van Atten's claim, is that the apprehension of the categorial form 'and so on' does not concur for the constitution of the substrates of choice sequences. It is precisely the other way around: they are presupposed to be already constituted (or constituable) as substrates to make possible the intuition of a categorial object.

4.6 What should count as the noema for choice sequences?

In discussing the temporal modes of giveness of choice sequences we proposed *en passant* that Husserl would dismiss choice sequences as mathematical objects not because of their temporal mode of intratemporality but by the fact that they do not have a definite *noema* or, more precisely, a definite *mathematical noema*. (See p. 11 of this chapter, section 4: 'Omnitemporality *versus* Intratemporality'.) Let's now turn the attention to this question and analyze what it means and which consequences it entails.

The (in)adequate giveness of choice sequences. A claim advanced by van Atten concerning the intuition of choice sequences is that the form 'and so on' "adequately indicates the ever present possibility to construct additional elements of the sequence", meaning that "the absence of such further elements from an intuition of the sequence at a given moment does not render that intuition inadequate, because they do not yet even exist." (*CCM*, *ibidem*.) Choice sequences are given to consciousness necessarily as permanently growing objects and, as such, they cannot be given otherwise than the way they are. For van Atten, this means that they are adequately apprehended in virtue of they own unfinished nature. Moreover, the fact that in any moment of time there are parts of the sequence that are not necessarily apprehended because they do not exist yet is not a prejudice to their adequate apprehension since choice sequences are originarily given as incomplete objects.

For Husserl, an object is inadequately given when it is presented in such a way that:

> (...) the sense-correlate of «what properly appears» fashions a *non-selfsufficient* part which can only have unity and selfsufficiency of sense in a whole which *necessarily* includes in itself empty components and indeterminate components." (*Ideas I*, section 138, p. 331.)[17]

The apprehension of an individual object is inadequate if its particular sense is *per se* insufficient to guarantee a complete intuition of the object it refers to. And this is the case when the sense includes indeterminate or incomplete components. In other words, the apprehension of an individual object is inadequate if the respective sense leaves room to the «determination otherwise» of the intended substrate, that is, if it is left open by its sense that the substrate of determinations was capable of having other properties different from the ones it really has. (*Ideas I*, section 138, p. 332.) Another way Husserl describes the adequate/inadequate apprehension of an object is the follwing:

> Every such evidence – understanding the term in our broadend sense – is either *adequate* evidence, of essential necessity incapable of being further «strengthened» or «weakened», thus *without degrees of*

[17]in the original: (...) Es bildet sein SInneskorrelat im vollen Dingsinne einen unselbständingen Teil, welcher Sinneseinheit und -selbständigkeit nur haben kann in einem Ganzen, das notwendig Leerkomponenten und Unbestimmtheitskomponenten in sich birgt. Edmund Husserl, *Ideen zu einer reinen Phänomenologie, Erstes Buch: Allgemeine Einfürung in die reine Phänomenologie*, p. 319; Husserliana III/1 und V, 1992.

weight; or the evidence is *inadequate* and thus *capable of being increased and decreased*. (*Ideas I*, section 138, p. 333.)[18]

Given Husserl's description, the apprehension of an object is adequate if the evidence we have for it is not capable of degrees of weight, i.e., if its apprehension as an individual object is fully guaranteed or determinated by the *intentional content* of its respective noema; and thus, not able to be increased (or decreased) in its intuitive power by further possible determinations that are not already explicitly given in the noema's content. If the evidence for an object does not conform to these prescriptions, then it is inadequate. Let's see how is constituted the noema of a choice sequence, and then analyse if it fits Husserl's prescriptions for an adequate evidence.

The noema is the ideal correlate of the intentional acts or *noesis*. In the noema the intentional characteristics of the real intentional act, or multiplicity of acts, are crystallized in a permanent way. To each act or multiplicity of acts corresponds a noema, and each noema is correlated to an act or multiplicity of intentional acts. This mutual correlation between the noesis and the noema is the so-called *correlation* between intentional act and intentional sense. Moereover, the primitive intentionality of the consciousness acts, i.e., the fact that the consciousness is *consciousness of something* and, for that reason, *refers* to something in an intentive way, turns possible the distinction concerning the noesis between the intentional act itself and the 'something' the act is about. In other words, the distinction between the *intentive act* and the *intended object*. But this distinction also has a *parallel* within the noema. Which Husserl expresses as follows:

> Each noema has a «*content*», that is to say, its «sense», and is related through it to «its» *object*. (...) As soon as we go into it more precisely we are immediately cognitively aware that indeed the distinction between «content» and «object» is to be made not only for the «consciousness», for the intentive mental process, but also for the noema *taken in itself*. Thus the noema too is related to an object and possesses a «content» by «means» of which it relates to the object; in which case the object is the same as that of the noesis; as then the "parallelism" again completely confirms itself. (*Ideas I*, section 129, pp. 309 and 311, respectively.)[19]

Due to the paralelism between noesis and noema, the noema provides both the *sense* and, through this sense, the *reference*. A noema with a full sense can

[18]In the original: Eine jede solche Evidenz – das Wort in unseren erweiterten SInne verstanden – ist entweder adäquate, prinzipiell nicht mehr zu, "bekräftingende" oder zu "entkräftende", also ohne Gradualität eines Gewichts; oder sie ist inadäquate und damit steigerungs- und minderungssfähige. Edmund Husserl, *Ideen zu einer reinen Phänomenologie, Erstes Buch: Allgemeine Einfürung in die reine Phänomenologie*, p. 321.

[19]In the original: Jedes Noema hat einen "Inhalt", nämlich seinen "Sinn", und bezieht sich durch ihn auf "seinen" Gegenstand. (...) Sowie wir darauf genauer eingehen, werden wir dessen inne, daß in der Tat nicht nur für das "Bewuß tsein", für das intentionale Erlebnis, sondern auch für das Noema in sich genommen der Unterschied zwischen "Inhalt" und "Gegenstand" zu machen ist. Also auch das Noema bezieht sich auf einen Gegenstand und besitzt einen "Inhalt", "mittels" dessen es sich auf den Gegenstand bezieht: wobei der Gegenstand derselbe ist wie der der Noese; wie denn der "Parallelismus" wieder durchgängig sich bewährt. Edmund Husserl, *Ideen zu einer reinen Phänomenologie, Erstes Buch: Allgemeine Einfürung in die reine Phänomenologie*, p. 299.

be structured into several layers of senses. First of all, there are the properties of the object: its predicates, relations, etc. (*Ideas I*, section 129.) Then, there is the layer of sense referring to the object *in the how of its modes of giveness*. To this layer of sense corresponds *the way* the object is intended through its predicates: remembered, imagined, clearly or obscurely seen, adequately or inadequately given, etc. (*Ideas I*, section 130.) There is however a *central nucleous* around which all layers of sense are grouped:

> This nucleous or objective sense is the element in the noema in which the object of reference is meant as distinct from its various senses or predicates which are meant as belonging to this object of reference. This nucleous performs the role which was carried by the objectivating act in the *Investigations*. By virtue of it, an intention is referred to a definite object.[20]

In this central nucleous, the object of reference is meant as the *bearer* of the predicates, or as the pure substrate distinguishable from its predicates. Husserl also calls to the object in this sense the "the pure determinable X in abstraction from all predicates". (*Ideas I*, section 131, p. 313.) Moreover, the noema, the complete noema has no validity without its object of reference, without the central nucleous in which the object is intended as itself, through its own individual sense. In Husserl's own words:

> In no noema, however, can it or its necessary center, the point of unity, the pure determinable X, be missing. No «sense» without the «*something*» and, again, without «*determining content*». (*Ideas I*, section 131, p. 315.)[21]

Given Husserl's conception of the noema, we now ask: what should count as the noema of a particular choice sequence? Suppose we have a choice sequence α, then how is to be constituted the noema of α? In principle, we should have at least (1) the possible predicates of α: $A_1(\alpha), A_2(\alpha), ...$; (2) the possible modes of giveness of α: as clearly intuited, as remembered, as adequately given, etc.; and (3) the noematic nucleous of α: the intended object of reference in the core sense. Of course the problematic component of the noema of a choice sequence is the last one, its alleged object of reference or substrate given by the particular sense of the central nucleous. What should count as the noematic nucleous of α?

Towards the noema of a lawlike sequence. If a is a lawlike sequence, then what should count as the noematic nucleous is its particular *law*, which we express as 'f_a'. It is the determinate intensional generation process of its values that presents the object of reference. It is through f_a that a is presented as an object, and as a mathematical object since the law is a mathematical law. Therefore, for lawlike sequences, it is the sense expressed in the law that makes the reference to its substrate possible. Moreover, this law is presented

[20] Robert Sokolowski, *The Formation of Husserl's Concept of Constitution*, Martinus Nijhoff, The Hague, 1970, p. 144.

[21] In the original: In keinen Noema kann er fehlen und kann sein notwendiges Zentrum, der Einheitspunkt, das pure bestimmbare X fehlen. Kein "Sinn" ohne das "etwas" und wieder ohne " bestimmenden Inhalt". Edmund Husserl, *Ideen zu einer reinen Phänomenologie, Erstes Buch: Allgemeine Einfürung in die reine Phänomenologie*, p. 299.

as *continually* generating new values of the sequence. However, the sequence itself, albeit its growing character, is contained as a unity, as a single object, in the law. Again, it is through the law that the reference to the sequence in different contexts is made, no matter the predicates the sequence has. In one context we may present the sequence a as having the predicates: $A_1(a), A_2(a), A_3(a), ...$ and in another context as having the predicates: $B_1(a), B_2(a), B_3(a), ...$ But it is always the same object, namely a, that is presented in both contexts, no matter the predicates it has. This *univocal* reference to the object is only possible by virtue of its particular sense expressed in the law. Moreover, we do not have to stipulate in advance that two lawlike sequences are the same (or are different ones) in order to have identity (or non-identity). Their noemas will determine if they are the same sequence or not, i.e., if the laws attached to each of them are equivalent laws or not.

A consequence due to the type of sense attached to a lawlike sequence is that the initial segment of a lawlike sequence is not apprehended *separately* from the sequence itself. The law through which we apprehend a lawlike sequence turns the aprehension of its initial segment possible but un-necessary, meaning that we do not have to apprehend the initial segment as a detached part of the whole sequence because, in some sense, the sequence in its entirety is intuitively encapsulated in the law: any value of the sequence is determinated by the law, therefore we have access to any element of the sequence. Contrary to lawless sequences, the initial segment of a lawlike sequence is implicit in the law. Thus it is not necessary to make an essential differentiation between the sequence and its initial segment. This is the reason why WC-N need not hold for a restricted domain that contain the lawlike sequences. With respect to the class of lawlike sequences, because the initial segment is not necessary or sufficient to determine the predicates of the sequence, we may have extensional properties that do not depend on the extensional information of the sequence (viz. its initial segment) but on the intensional information of the sequence, viz. on the law that generate its values.

Towards the noema of a lawless sequence. For the noema of a lawless sequence α, however, it is not clear what should count as the noematic nucleous. By definition there is no law through which the values of α are generated, i.e., there is no f_α. Therefore, there is no determinate core sense that refers back to the substrate of α. In other words, there is not an expressed sense in the noema univocally attached to α and through which we are able to refer to α in each possible context. Notice that neither the inital segment nor the form 'and so on' can play this role. An initial segment can be presented as the initial segment of several distinct lawless sequences, therefore it should count as a determination of the object of reference (a fragment) and not as the object of reference itself.

The form 'and so on' is also an essential determination of any choice sequence, and by itself it does not determine univocally the object of reference. Instead, it should count as an essential characteristic pertaining to the layer of sense to which corresponds the property that α possesses of being a dynamic object. In other words, the form 'and so on' refers to the object of reference *as* permanently growing, it does not refers *simpliciter* to *the* object that is growing. Contrary to lawlike sequences, the form 'and so on' does not iterate a mathematically well defined process (it does not iterates the process '$n + 1$', or the process '$n \times 2$', etc.). Instead, for lawless sequences, the character 'and so on' iterates the more or less intuitive mental act of 'choosing a natural number', whose mathematical nature is debatable. Besides the uncertain mathematical character of the iterated process of 'choosing a natural number', it is not also clear how such a process univocally determines an individual object, and a mathematical individual object.

A possible candidate to stand as the noematic nucleous, in order to determine univocally the object of reference, would be the precise moment in which α was started (as van Atten proposes). The concrete moment in time in which we start to generate the values of α is not, however, part of the noema. Instead, it is a *real* moment of the noesis, and not an *ideal* component of the noema. The temporal moment in which α was started is a property of the noesis: it is a property of the *act* by which we generate the first value of α and then go on, it is not a property of the intentional correlate of such act, viz. the noema. Therefore, the particular moment in which α was started has no place in the noema and, in particular, it cannot stand as the noematic nucleous.

Another possible candidate (closely related to the last one) to stand as the noematic nucleous of a lawless sequence would be the ideal correlate of the *intuition of time*. This hypothesis seems to be explicitly made when van Atten says (concerning the noesis) that "The intuition obtained in these [objectifying] acts is closely related to the awareness of inner time." (CCM, p. 63.) But again, it is not clear how the correlate of the intuition of inner time would allow us to apprehend the object of reference of a lawless sequence (viz., the sequence) in itself, or to refer univocally to it. The flow of time (or 'form of time' as van Atten calls it, *ibidem*) and the substrate of a lawless sequence are different things, despite the fact that both are given as growing objects. It seems to be counter-intuitive to conceive that the substrate of a lawless sequence is the flow of time by the same reason it is counter-intuitive to conceive that the substrate of a lawless sequence is the form 'and so on', namely because the form 'and so on' (and the flow of time) cannot be a numerically different component in every noema of a lawless sequence. Moreover, it seems that there is no way to distinguish between the flow of time and the form 'and so on' since, as the form 'and so on', "the form of time constrains the purely categorial formations" (*ibidem*) and hence performs the same role that the form 'and so on' does. But if this identification is to be made, then the flow of time cannot stand as the noematic nucleous of a lawless noema precisely because we have already dismissed the form 'and so on' as a possible candidate.

Given the situation above, we think that we cannot assume the existence of a plausible noematic nucleous in the noema of a lawless sequence, since we are not able to determine what should count as a sense for the object of reference of a lawless sequence. Therefore, given Husserl's claim that there is no noema without the nuclear sense referring to the «something», the search for the proper noema of a lawless sequence is an open question. In particular, it is an open question to determine in which way an alleged noema of a lawless sequence would be a *mathematical* noema and in which sense is a lawless sequence a mathematical object. Another consequence of this state of affairs is that, without the core sense pointing to the object of reference, it is meaningless to ask whether such an object is adequately or inadequately given to consciousness because it is not given at all.

Chapter 5

On the elimination of lawless sequences

This chapter is the final one. We will advance a view on choice sequences of our own and present arguments in its favour. Such a view is not intended to be exhaustive or our last word on the matter, but rather a possible direction of work. In the first section, we discuss some aspects of the Elimination theorem, with particular emphasis on van Atten's arguments presented against the eliminitivist interpretation. In the second section, we develop a specific interpretation of Weyl's account of choice sequences, with the objective of bringing some clarification on the problematic claims he advocates. With this specific interpretation of Weyl's work on choice sequences we do not intend to defend the claims he proposed, but only to make a point: that they represent an unsurpassable difficulty posed by choice sequences taken as mathematical objects. In the third section, we develop and argue in favour of an interpretation of existential quantification over lawless sequences close to Weyl's view: there is in fact a sense in which existential quantification over lawless sequences can be interpreted as quantification over unproblematic mathematical objects and there is also a sense in which universal quantification can be interpreted as referring to choice sequences *en masse*.

5.1 Some aspects of the Elimination theorem

The question 'in what sense is the operation «sucessive free choices» a mathematical operation?' is not formulated nor answered by van Atten. And it would be interesting to show in what sense such operation is to be considered a mathematical operation. The way van Atten argues for the mathematical character of choice sequences is an *indirect* way: he argues for "the coherence of choice sequence as a mathematical concept." (*BmH*, p. 95.) The goal is to show that the mathematical speech about choice sequences is not meaningless; and, for him, to do this is equivalent to show that statements involving choice sequences have mathematical content, i.e., express genuine mathematical properties.

> Consider the lawless sequences, in fact the strongest case of time-dependent, subject dependent objects (...) For this class we have the

Kreisel-Troelstra translation, which shows that sentences quantifying over lawless sequences are equivalent to other sentences that do not, and whose mathematical nature goes unquestioned (...) But then the sentences that are translated must also be mathematical. Translations suffice to show that the concept of choice sequence, and specifically, the concept of lawless sequence, is mathematically coherent. (*BmH*, p. 96.)

For van Atten, the τ-translations guarantee the mathematical coherence and meaningfulness of speech about lawless sequences. Moreover the equivalences in the Elimination theorem hold in both directions, so the translations induces no *ontological preference*. The translations, after all, are equivalences between sentences A, with quantified lawless variables, and $\tau(A)$, not containing lawless variables at all. Suppose an opponent of choice sequences as mathematical objects argues that they do not have mathematical semantic content, i.e., that sentences involving lawless variables, albeit non-contradictory, do not express mathematical properties. Then the translations would stand for as counterexamples to that claim. In reality, semantical properties of lawless sequences, due to the translations, can be analyzed in terms of the semantical properties of unquestioned mathematical concepts (viz., the concepts of 'finite sequence' and 'constructive function').

The translations take the form of equivalences, i.e., the form $A \leftrightarrow \tau(A)$, where A has occurrences of lawless variables and $\tau(A)$ has not such occurrences. Van Atten's argument is that equivalence statements *go both ways*. There is one side of the equivalence $A \rightarrow \tau(A)$, meaning that sentences involving lawless sequences imply sentences not involving them. When we are talking about lawless sequences we are also talking about other mathematical objects (e.g., finite sequences, constructive functions) whose mathematical character is not questioned. There is the other side of the equivalence $\tau(A) \rightarrow A$, meaning that the speech about those mathematical unquestioned objects also imply speech about lawless sequences. Given this state of affairs, we cannot regard lawless sequences as eliminated from the mathematical discourse since they imply and are implied by unquestioned genuine mathematical statements. For this reason, and following a proposal of Kreisel, van Atten accepts that "such translations should not be understood (as they usually are) as 'elimination theorems', but as giving 'a complete analysis of lawless sequences'. (*BmH*, p. 41.)

Due to the translations, lawless sequences are coherent and meaningfull in the context of mathematical formulas. But now we ask: is coherence and meaningfulness of lawless sequences in mathematical formulas the only implication of the Elimination theorem? Notice that for any formula A involving lawless sequences we can substitute it and use its translation counterpart $\tau(A)$ with no lawless variables involved. Isn't this stronger than just coherence and meaningfulness? Is there any reason to take the elimination of lawless sequences literally? After all, what is the point of using lawless sequences if we can use unproblematic mathematical objects instead, viz. initial segments and lawlike sequences, with equivalent results? Should not the principle of *Occam's razor* prevail in this context, eliminating unnecessary elements in the process of constructing a theory? There is an argument which leads van Atten to answer 'No!' to these questions. In his words:

To speak of 'elimination' falsely suggests that the intuitive con-

> tinuum can be regarded as a discrete point set after all. Even if statements quantifying over lawless sequences are equivalent to statements that do not, the fact remains that we should think of the intuitive continuum as analysable into, among others, such sequences and not into elements of a set. (...) Only the notion of lawless sequence fully reflects both the inexhaustibility and non-discreteness of the intuitive continuum. (*BmH*, p. 42.)[1]

The strongest motivation for introducing lawless sequence into mathematics is that the *continuum* constructed on the basis of them has specific properties (e.g., inexhaustibility and non-discreteness) that the intuitionist takes as being essential to it. The intuitionist conception of the *continuum* is directed against the classic view of the *continuum* as a collection of discrete points and the notion of lawless sequence guarantees that. There are not, however, conclusive reasons to believe that the intuitionist conception of the *continuum* is the right one, or that it is better that the classical one. More relevant, even if we accept that the intuitionistic conception of the *continuum* is the right one, nothing ensures that the notion of lawless sequence is the best, or the only possible notion upon which it can be mathematically based.

The idea we want to stress here is that the motivation for lawless sequences does not force us to accept that they are legitimate mathematical objects. The *motivation* for considering that a determinate notion is mathematically useful, or advantageous and the *legitimacy* of applying it as a mathematical notion are not the same. They are independent of each other and, concerning discussions about the foundations of mathematics, the arguments for accepting such a notion as a legitimate mathematical notion are prior to the motivation for introducing them into mathematical theories. First we have to show that such notion is mathematically valid, then we have to show that we have good reasons to use them in our mathematical theories, i.e., that they have the intended mathematical properties we want to be inherited by the structures we use them to construct.

Does not such claim constitute a *petitio principii*? Van Atten assumes precisely what is in question: choice sequences are to be viewed as genuine mathematical objects and, hence, not eliminated since they are used to build up intuitionistic mathematical structures. Therefore, an eliminativist interpretation of the Elimination theorem is blocked in advance since they are necessary to build, e.g., the intuitionistic *continuum*. Notice, however, that at this point of the philosophical discussion the mathematical character of lawless sequences is not clearly established. Therefore, we think, such an argument is not yet available to be asserted. The Elimination theorem cannot be, in the first place, interpreted considering the (posterior) applications of lawless sequences. The significance of the Elimination theorem has to be settled before any application of lawless sequences.

In **LS**, lawless sequences are treated as objects *per se*, not *as* elements of the intuitonistic *continuum*. Any tension between the proof-theoretic implications of the Elimination theorem and posterior applications of the notion of lawless sequence is something we have to deal with by further investigation on both, the axiomatisation and on the applications of lawless sequences. It cannot be

[1] Dummett argues the same in *EI*, p. 222.

simply assumed that the applications are correct (or, cannot be wrong) and, therefore, that any proof-theoretic result which is incompatible with the applications is to be viewed with suspicion. In our understanding, the Elimination theorem is of interest precisely because of its intuitive 'incompatibility' with the need of lawless sequences in constructing the intuitionistic *continuum*. Can this tension the resolved? How to decide which of the readings (eliminativist or non-eliminativist) have conceptual predominance? These questions are very interesting ones, but they can only be properly answered in case we start to consider the Elimination theorem and similar (negative) proof-theoretic results with other eyes.[2] For this reason, we conclude that van Atten's argument, based on the motivation for lawless sequences, does not block in advance an eliminativist interpretation of the Elimination theorem.

Another argument is advanced by van Atten to support the non-eliminativist interpretation of the Elimination theorem. This argument is based on the negative view of the siginificance of formal theories, particularly advocated by Brouwer. For Brouwer, formal theories do not provide a complete grasp of mathematical objects precisely because mathematics is seen by him as a languageless activity whose objects (e.g., choice sequences) are freely generated by the mind. And because formal/axiomatic theories involve particular symbolisms, they are not faithful representations of mathematical knowledge. Inspired on Brouwer's view of mathematical resoning, van Atten states the following:

> [The] translations occur in the context of axiomatisations of choice sequences. These systems should not be confused with the sequences themselves. Axiomatisations are a way to present mathematical content, but they are not identical with it. Lawless sequences have been axiomatised in different, not always equivalent ways (e.g., Kreisel, Myhill, Troelstra); nevertheless something remains the same, namely, these axiomatisations are all about lawless sequences. (...) Clearly it is not the case that we were first in the dark about this and then learned it from finding mappings between the formalisms. These sequences (or any other mathematical object) cannot be identified with any particular axiomatisation (let alone formalisation). Moreover, doing so would force us to accept the implausible view that Brouwer's theorising before the introduction of axiomatic theories was, in fact, about nothing. (*BmH*, p. 42.)

Formal theories have limitations that are well known. It is a fact that, in general, formal theories are not *complete* in the sense that, assuming they are consistent, not all truths about the intuitive notion are provable within the particular formal theory in question; or that their consistency cannot be proved only by the means provided by the formalised theory in question (Gödel's Incompleteness theorems). However, the objective of formal/axiomatic theories is precisely to make as clear as possible the properties of intuitive notions. So, it is not clear what van Atten wants to stress in the quote above when he says that the axiomatic 'systems should not be confused with the sequences themselves'.

[2]See, for example, the fact that the universe of lawless sequences is not closed under any continuous operation except for the identity operation (*CinM II* vol, p. 650).

On one hand, this claim is trivially true (by Gödel's first Incompleteness theorem). On the other hand, if he wants to say that, somehow, what is proved in **LS** is not to be regarded as genuine knowledge about lawless sequences, then something is wrong with this claim. That **LS** possibly does not express *all* properties of lawless sequences, is acceptable. Another thing is to suggest that (some of) the properties expressed in **LS** are not genuine properties of lawless sequences, which is much harder to accept. And, if we regard the Elimination theorem with suspicion, why would we do not the same, e.g., in relation to WC-N; notice that both are proved in **LS**.

Of course, the Elimination theorem can be seen as a surprising (and unexpected?) formal result, but this fact does not mean that it should be regarded as being suspicious. In fact, it is the other way around. The Elimination theorem sould be regarded, we think, as a symptom of our lack of *intuitive* knowledge about lawless sequences since, in principle, they should not be eliminated precisely because they were introduced as original and fundamental objects. What is wrong after all with this notion? What we have just said does not force us to accept that 'Brouwer's theorising before the introduction of axiomatic theories was, in fact, about nothing', but leaves us to think that Brouwer's intuitive conception of lawless sequences was, of course, about *something*, but about something whose our (or his) undertanding was not completely accurate. For example, Brouwer's original notion of a lawless sequence, viz. the proto-lawless sequences, is a clear example of a notion with severe mathematical limitations. The axiomatisation of lawless sequences is a way to refine the notion of lawless sequence, a way of make it more accurate. And we have to deal with the negative results as well as we deal with the positive (or desired) ones. For this reason, we also conclude that van Atten's argument, based on the incompleteness of the axiomatisations of lawless sequences, does not block an eliminativist interpretation of the Elimination theorem. More arguments are needed to support the non-eliminativist interpretation of the Elimination theorem.

5.2 Weyl on mathematical existence

In chapter two (section 2.5) we advanced the hypothesis that the eliminativist interpretation of the τ-translations partially vindicates Weyl's account of choice sequences. But the proposed Weylian-type interpretation of the τ-translations is not easily accepted. For van Atten

> Weyl's effort to avail himself of the use of the concept of lawless sequence without allowing them into his mathematical ontology suffers from (...) a faulty premisse (after all, lawless sequences cannot be stipulated to follow lawlike sequences), (...) [which] requires an *unnatural* (asymmetrical) interpretation of the quantifiers." ('Brouwer and Weyl', p. 16.) At first glance it might seem that the justification for *Weyl's postulate* is to be founded in another principle advocated by him: that it is an essential property of all mathematical objects that they can be coded into natural numbers. But, as van Atten argues, "in phenomenology, one needs a constitution analysis to justify such claim [and], unless that is done, it is

only a pressuposition. ('Brouwer and Weyl', p. 16.)

We want now to present a new perspective on Weyl's account of choice sequences, based on his own conception of mathematical existence. With the purpose alone of introducing our personal view on choice sequences (which is inspired but not based on Weyl's account) we will here try to show the implicit reasons that lead Weyl to propose such a (untenable) principle on quantification over choice sequences. What we call here Weyl's postulate is essentially dependant on the way Weyl conceives existential statements of mathematics:

> An *existential statement* – say, "there is an even number" – is not at all a judgment *in the strict sense, which claims a state of affairs*. Existential states of affairs are empty inventions of logicians. "2 is an even number": This is an actual judgment expressing a state of affairs; "there is an even number" is merely a *judgment abstract* [*Ursteilabstrakt*] gained from this judgment. If knowledge is a precious treasure, then the judgment abstract is a piece of paper indicating the presence of a treasure, yet without revealing at which place. Its only value can be to drive me on to look for the treasure. The piece of paper is worthless as long as it is not realized by an underlying actual judgment like "2 is an even number." ('New Crisis', pp. 97-98.)

It is clear from the quote above that, for Weyl, statements of the form '$\exists x A(x)$' are not at the same epistemological level that statements of the form '$A(a)$'. We think that it would not be wrong to assert that, for Weyl, there are not mathematical states of affairs originally corresponding to judgments of the form '$\exists x A(x)$'. The genuine (*strict*) mathematical states of affairs correspond to judgments of the form '$A(a)$' and thus the genuine mathematical existence claims are of this form. It is an act of logical abstraction to go from $A(a)$ to $\exists x A(x)$. But we cannot confuse the two and assume that both have the same epistemological status: quantified existential claims are justified by their respective individual instances. We are allowed to assert $\exists x A(x)$ if, and only if, we have the construction of a proof of $A(a)$ in advance, not the reverse. Therefore, the statement $\exists x A(x)$ is worthless (i.e., express no genuine state of affairs) if it is not instantiated (*realized*) by the proper judgment $A(a)$, since from the mere assertion that $\exists x A(x)$ we are not able to indicate which individual object is in such conditions. In other words, from $\exists x A(x)$ we cannot extract a construction such that a proof of $A(a)$ is achieved for a particular a. ('New Crisis', p. 95.)

It seems that there is a strong resemblance between the way Weyl conceives existential mathematical judgments and the way Husserl does. Namely, that they are not directly about individual objects. Judgments of the form '$\exists x A(x)$' represent only (abstract) judgments about possible states of affairs about individual objects. And these possibilities are only realized when an actual or genuine mathematical judgment is achieved, i.e., when a judgment of the form '$A(a)$' is proved. Another way to bring closer this resemblance is by conceiving judgments of the form '$\exists x A(x)$' as *unfulfilled intentions*, whose fulfillment is only achieved when we are able to construct a particular instantiation of it (when we are able to find the treasure form whose it is only an indication), viz. a proof of $A(a)$. Notice that, within Weyl's conception of mathematical existence, is implicit the Husserlian distinction between *formal* and *material mathematics*[3]. Statements of the form '$\exists x A(x)$' belong to formal mathematics, while statements of the form '$A(a)$' belong to the material mathematics, i.e., belong to the traditional mathematical disciplines (Euclidean geometry, theory of numbers, etc.). Thus the judgments expressing genuine mathematical content or knowledge are the ones that are directly about the individual objects of the material branch of mathematics and whose reference is guaranteed by individuation criteria already belonging to that material disciplines (material ontology). For Weyl, statements of the form '$\exists x A(x)$' do not belong to the traditional material disciplines of mathematics, but to the formal ones (formal ontology) and, for that reason, they are not about determinate individual objects, but about possible objects. Therefore, they do not express actual states of affairs, but just possible states of affairs.

Applying this framework to lawless sequences, we have that a statement like $\exists \alpha A(\alpha)$, where α is a lawless variable, would only have a mathematical justification if, in advance, we had achieved a construction of the intended proof of $A(\alpha)$. But now a new element enter into the equation: the non-lawlike dynamic character of lawless sequences. Notice that now the difficulty does not lie in the level of the *judgments abstracts* but on the level of the genuine mathematical states of affairs. The difficulty is no longer to go from $A(\alpha)$ to $\exists \alpha A(\alpha)$ (or *vice-versa*), but lies on the ontological status of the judgments of the form $A(\alpha)$. The problem is that lawless sequences, because of their non-lawlike character, cannot be mathematically individuated. For lawlike sequences, individuation is guaranteed by the above enunciated criterion: they can be coded into natural numbers since any law can be replaced by a natural number. Given that lawlike sequences can be coded into natural numbers, judgments of the form $A(a)$, where a is a lawlike sequence, do express genuine mathematical states of affairs. Therefore, if we have a proof of a judgement $A(a)$, then we are able infer $\exists a A(a)$.

Lawless sequences, on the other hand, cannot be submitted to such a criterion. And, more generally, there is no a definite mathematical criterion to individuate them. For this reason judgments of the form $A(\alpha)$, where α is an individual lawless variable, seem to not express genuine mathematical states of affairs. As consequence, we don't know what is in general to be accepted as a (constructivist) proof of a statement as $\exists \alpha A(\alpha)$. How do we fulfill the intention of $\exists \alpha A(\alpha)$? Suppose $A(\alpha) := \forall n \alpha(n) > 100$. In Weyl's framework, in order to assert $\exists \alpha A(\alpha)$ we have to find first an individual α such that for any n, $\alpha(n) > 100$. But it is an essential character of lawless sequences that they are unfinished, free developing objects; therefore, we cannot have access to all values of α. Because A depends on the entire lawless sequence α and not just on its initial segment, the values of the initial segment are not sufficient to prove $A(\alpha)$ (but they can suffice to prove $\neg A(\alpha)$ if, at least, one value of the

[3]For a more detailed discusion about formal and material mathematics see Richard Tieszen's 'Husserl's Logic' in *Handbook of History of Logic vol. III*, North-holland, Amsterdam, 2008, pp. 285-291. Besides the resemblance with Husserl in the way Weyl conceives genuine/abstract mathematical statements, there is also, we think, a strong resemblance with Hilbert distinction between *finitistic* and *infinitary* mathematics. The crucial difference, of course, is that for both Hilbert and Husserl this fact have not the negative connotation Weyl attributes to it. For Hilbert and Husserl, both branchs of mathematics *are* mathematics, with the only difference being that they are in different 'levels' of generality. For Hilbert, infinitary mathematics needed to be reducted to finitary mathematics; for Husserl, it only needed to be explained.

initial segment is smaller than 100). The consequence is that we might be able to prove that $\neg A(\alpha)$, but (in the case we don't have evidence for $\neg A(\alpha)$) we are never in position to prove $A(\alpha)$ and, consequently, we are never in position to assert $\exists \alpha A(\alpha)$. In other words, we are in a position such that we are not able to fulfill the intention of $\exists \alpha A(\alpha)$. Generally, we cannot deal with propoerties that depend on the entire lawless sequence, we must restrict to the properties depending only on the initial segment (WC-N).

It is clear, we think, that the problem is not in Weyl's criterion of individuation for mathematical objects. The problem lies on the fact that, for lawless sequences, there is not a mathematical way of referring to them individually. Because lawlike sequences are never in the same position of lawless ones, i.e., because we virtually have acess to any value of a lawlike sequence a, we have the possibility to determine whether $A(a)$ is the case or not. In order to surpass this difficulty with lawless sequences Weyl proposes his postulate:

> It is one of the fundamental insights of Brouwer that number sequences, developing through free acts of choice, are possible objects of mathematical concept formation. The individual real number is represented by a law φ that determines a sequence *in infinitum*, while the *continuum* is represented by the choice sequence unrestricted by any law in the freedom of its development. ('New Crisis', p. 94.)

Notice that this principle is proposed for dealing with a problem arising within real analysis. In the context of real analysis, we are working within a regional branch of mathematics, i.e., we are working within a material discipline of mathematics. By this reason, the ontology pertaining to this particular branch of mathematics is well established: we are dealing with infinite sequences of natural (or rational) numbers. And because we need to deal, not just with the whole *continuum* of real numbers (represented by the collection of infinite sequences), but with individual real numbers given by individual infinite sequences we must have a way to individualize them in order to refer to them. However, it is a fact that lawless sequences cannot be individualized mathematically. Therefore, the only way we have to refer to them is to stipulate that individual choice sequences are given through a law, since "an individual determined sequence (...) can only be defined by a *law*." ('New Crisis', p. 94.) For Weyl, the individual real numbers are represented by lawlike sequences while the indifferentiated *continuum* of the reals is represented by the lawless ones. In particular, they allow us to correctly form the concept of the mathematical *continuum*. Lawless sequences, given that they cannot be defined by a law, can only be admitted as possible objects of mathematical concept formation, i.e., we can only deal with the whole collection of choice sequences at once. In other words, we can only deal with the *notion* of choice sequence, by interpreting universally quantified statements over lawless sequences as judgments about the notion of choice sequence.

Weyl's justification for this interpretation of universal quantification over choice sequences is to be found in one of his most important tenets about mathematical objects, which says that we "cannot get general [universal] judgments about [sequences of] numbers by looking at the individual [sequences of] numbers, but only looking at the essence of number [sequence]." ('New Crisis', p. 97.) For example, the justification of the statement $\forall \alpha A(\alpha)$, where α is a choice sequence variable, cannot be concluded by looking to all lawless sequences and confirm if they have the property A, since "a complete run through an infinite [number] of sequence[s] is nonsensical." (*Ibidem*) The only way we have to determine the truth of such a property for the domain of choice sequences is to investigate if the property A "is part of the *essence* of number [choice sequence]." (*Ibidem*) Therefore, universal statements about choice sequences are paraphrased into judgments about the notion of choice sequence. Here enters Weyl's asymmetrical interpretation of the quantifiers: we can universally quantify over lawless sequence, since when we do that we are not refferring to any individual choice sequence but making general statements about the *continuum*. But existential quantification is allowed only over lawlike sequences since only they can be individualized.

We can see Weyl's proposal as untenable, but now we understand better why he has proposed it. Indeed, as van Atten argues, we cannot stipulate lawless sequences to follow lawlike ones. But the failure of Weyl's postulate is quite instructive and it touches on a sensitive aspect of lawless sequences: we are not able to deal mathematically with individual lawless sequences. On the other hand, we cannot either substitue lawless sequences by lawlike ones if we want to idividualize them. So, how should we accomodate this situation?

In the next section we try to make plausible Weyl's insights on the individuation of choice sequences. Our proposal is the following: existential quantification over lawless sequences *is to be interpreted as* speech about initial segments only. Therefore, mathematical speech about lawless sequences is never about single individuals but about sequences *en masse*. This claim is based on a specific result obtained within **LS**, which is independent from the Elimination theorem. The proposed interpretation of existential quantification over lawless sequences is also based on the phenomenological discussion about lawless sequences presented in the previous chapter; but the emphasis on the phenomenological aspects of initial segments is developed further. We will see in what follows that Weyl was wrong in some technical aspects of his treatment of choice sequences, but that his overal insights about choice sequences point to the right direction.

5.3 Initial segments: a dilemma?

Both the concept and the categorical form 'and so on' were dismissed as the possible invariant substrate for choice sequences. The reason is the same: they are not the substrate that grows in the development of a choice sequence. What does grows in the process of dvelopment of a choice sequence, after all? The only remaining candidate to answer to this question is: the *initial segment*. However, it was regarded by van Atten as the wrong candidate. Let's see why. First, because different choice sequences may have the same initial segment, entailing that the same substrate would be the substrate of different growing objects. Second, in considering initial segments as the substrate invariant for choice sequences, the horizon 'and so on' is not thematized and, consequently, the unfinished character of choice sequences is not thematized as the essential feature it is.

In general, initial segments are the only *assured* information about choice sequences that we have, i.e., any choice sequence is given through a process of generation and an initial segment; in the case of lawlike sequences we have the additional information of the law by which their values are generated, but in the case of lawless sequences (the general case, with no law nor restriction assigned to them) initial segments are the only information we have. So, generally, initial segments are the only information we can count with when talking about choice sequences. This implies that all the information we can have and need to establish (mathematical) properties about them is given by initial segments. And, in fact, we don't need the 'entire' choice sequence nor the intensional information to establish relations among them, the initial segments suffice. This is precisely what the general principle of *open data* states.

Given the extensional nature of initial segments, we propose the following: the initial segment of lawless sequences is what remains identical through their process of development. We can always bring to the mind the same initial segment, the initial segment that is identical at any stage of growth of the choice sequence. Also, if we ask: what is the growing substrate in the development of a lawless sequence?, the natural answer is: its initial segment, for we can always extend an initial segment by making a further choice (in fact, this is a necessary condition). In this sense, initial segments can be seen as the invariant substrate for individual lawless sequences. In fact, it seems that there is no other possibility: to talk of an identical substrate for individual lawless sequences is to talk about their particular initial segments, because they are the only available answer to the question 'what grows?'.

However, if we accept initial segments as the invariant substrates for individual lawless sequences, unsurpassable problems will arise. How to deal with van Atten's criticism? In fact, due to the finite character of initial segments they do not encapsulate in themselves the potential or dynamic character of lawless sequences. This means that, being static and finished objects, initial segments lack the principal characteristic that makes lawless sequences be what they are. Notice that initial segments and lawless sequences are different objects. We could say, against this, that, albeit finite objects, initial segments are always extendable by making a further choice, or an infinity of further choices. Yes, but they remain always finite objects no matter how many further choices are made. And, besides, the extendable character of initial segments is a consequence of the 'and so on' character of choice sequences, which is not present in an initial segment as such. Another problem is that different choice sequences may share the same initial segment. In this case, it is the same substrate that grows in different lawless sequences? Can the same substrate originate different objects? How to accommodate this situation?

If we accept that the initial segment of a sequence and the sequence itself are different objects, we can argue that what grows is the 'sequence', not the initial segment. It is the sequence that is always being extended (by the creating subject), while the initial segments remain finite and the same. But what is 'the sequence'? In a determinate sense, the sequence is never 'there'; meaning that the sequence is never completed. What we have at any moment is a finite initial segment of the sequence. The sequence itself is not a finished object, in fact it is the outcome of the process, something never finished. Therefore, in the process of growth of the sequence what we have is a *sucession* of initial segments. At any stage what we have is a finite sequence.

At this point of the discussion, we have reached a dilemma: initial segments seem to be the only natural answer to the question 'what is the growing substrate of individual lawless sequences?', but accepting them as such would force the proponent of choice sequences to admit that the substrate of individual lawless sequences is something essentially classic. In conclusion, initial segments seem to be the only available candidates to stand as the substrate for individual lawless sequences, but they are inappropriate candidates. Surprisingly, there is, however, a technical aspect of the axiomatic theory of lawless sequences (**LS**) that seems to meet the perspective presented above and we will now turn to it. We will see that initial segments cannot stand as substrate for individual lawless sequence because they are not also able to individuate them. However, we will see also that existential quantification may be paraphrased as existential quantification over initil segments.

We present now a proof-theoretic result due to Troelstra which is not directly related to the Elimination theorem but encapsulates the general idea of elimination of existential quantifiers for lawless sequences. (*CinM II*, p. 654). In combining **LS1** and **LS3**, the following statement is true:

LS $\vdash \exists\alpha A(\alpha) \leftrightarrow \exists n\forall\alpha \in nA(\alpha)$.[4]

The sentence above states that existential quantification over a lawless sequence α is equivalent to existential quantification over an initial segment n of α. More precisely, to state that there is an individual lawless sequence α with property A, is equivalent to state that there is an initial segment n such that any lawless sequences with initial segment n have the property A. This result has the particular consequence of making possible the *factual* substitution of existential quantification over lawless sequences for existential quantification over their initial segments. In particular, says Troelstra, in formulas of elementary analysis all existential lawless quantifiers can be removed. (*Ibidem*.) Given this state of affairs, we ask: if we are allowed to substitute existential quantification over lawless sequences for existential quantification over their initial segments, over what are we quantifiying after all? The intuitive meaning of this proposition is that *particular instantiations* of lawless sequences are not mandatory to make the discourse about them intelligible. As consequence, we never have to deal directly with particular lawless sequences in theories involving speech about lawless sequences.

In fact, a stronger interpretation of the above result is possible. The lawless variables in the equivalence are not all eliminated, but they appear only as universal quantified variables. In this case, we are referring to a subclass of lawless sequences, namely to the subclass consisting of those lawless sequences sharing the initial segment n. Therefore, we advance that reference to individual lawless sequence is not necessary, but also the stronger claim that it is not possible. Initial segments themselves do not individuate lawless sequences at all, since several lawless sequences may share the same initial segment. This is the reason why the lawless variable appears in the range of a universal quantifier

[4]We shall present now a sketchy proof of this proposition. ($\rightarrow$) Assume (1) **LS3** and (2) $\exists\alpha A(\alpha)$ as premisses. By $\exists$-elimination from (2) we have (3) $A(\alpha)$; by $\forall$-elimination from (1) and $\rightarrow$-elimination from (2,3) we get (4) $\exists n(\alpha \in n \wedge \forall\beta(\beta \in n \rightarrow A(\beta)))$. By $\exists$-elimination and $\wedge$-elimination from (4) we get (5) $\forall\beta(\beta \in n \rightarrow A(\beta))$; and substituting the variable β by α in (5) we get (6) $\forall\alpha(\alpha \in n \rightarrow A(\alpha))$. Simplifying (6) we get (7) $\forall\alpha \in nA(\alpha)$. Then, from (6), and applying $\exists$-introduction we get the conclusion $\exists n\forall\alpha \in nA(\alpha)$. ($\leftarrow$) Assume (1') $\exists n\forall\alpha \in nA(\alpha)$ as premiss. Take n_0 in such conditions, then we get (2') $\forall\alpha \in n_0 A(\alpha)$. Applying **LS1** to (2'), exists α_0 such that $\alpha_0 \in n_0$; then we can infer (3') $A(\alpha_0)$. Finally, by $\exists$-introduction on (3') we get the conclusion $\exists\alpha A(\alpha)$.

in the right side of the equivalence. Initial segments do not individuate lawless sequences because any initial segment *n* determines, not just an individual, but a multiplicity (potentially infinite) of lawless sequences, viz. the collection (or species) of lawless sequences sharing the initial segment *n*. Moreover, the individual lawless sequences that are determined by an initial segment are indiscernible from each other on the basis solely of the initial segment.

This fact is the final argument in favour of the thesis that there is no mathematical way of individuating lawless sequences. It does not only show that initial segments do not individuate lawless sequences, but, more generally, that there is not a way of individuating them. We have reached the conclusion that the initial segment was the last possible candidate for a substrate which could individualize lawless sequences. The form 'and so on' does not individuate choice sequences by the same reason: it determines the full class of choice sequuences. Any choice sequence possesses the property 'and so on' as an essential property. The particular moment in time in which a choice sequence is started also does not individuate choice sequences: there is no mathematical property to which it corresponds. Moreover, it is conceivable that different choice sequences (e.g., with different initial segments) may start at the same specific moment (except, of course, if we stipulate that at any moment of time only a unique choice sequence is started by the creating subject, which is clearly an *ad hoc* stipuation). Therefore, there is no conceivable mathematical way to determine a unique individual lawless sequence. This state of affairs leaves us with the only possibility that we can only refer to lawless sequences *en masse*, i.e., as a whole.

If we take into consideration Weyl's interpretation of quantifiers for lawless sequences, the idea that they are only needed when we refer to them as a collection, i.e., as a notion and not as individuals is reinforced. And, vindicating Weyl's account of choice sequences, existential quantification over lawless sequences can be interpreted as existential quantification over unquestioned (lawlike) mathematical objects: over initial segments (and not over lawlike sequences as he has proposed). But also, contrary to Weyl, the individuation of lawless sequences is not just unnecessary, but impossible in mathematical grounds. The only choice sequences that can be individuated are the lawlike ones.

Conclusion

Given the material presented above, we conclude that the arguments for the legitimacy of introducing choice sequences into the mathematical ontology are far from being conclusive. When reading van Atten's attempt of constitution of choice sequences, there is a persistent tension. What is it to be constituted? Choice sequences as individual mathematical objects or choice sequence as a mathematical notion? It is obvious that van Atten's aim is to constitute them as individual mathematical objects. However, we can ask if this task was really achieved. In our understanding, the constitution of choice sequences as individual mathematical objects is not successfully achieved for the reasons presented in last two chapters. But, if we ask for a constitution of the general sense 'choice sequence' (by eidetic variation), i.e., as the way we can attribute a sense to what means to be a choice sequence, then there is a broad sense in which we can assume the notion as a valid one. Namely (as Weyl proposes) as a legitimate notion pertaining to the pre-theoretical sphere of constructing mathematical theories. Whether this notion is not an empty notion (i.e., is inhabited with individual objects) is dependent on evidence for considering choice sequences as individual mathematical objects. As we have seen, the phenomenological attempt to do this is not free from flaws. So, again, until a suitable account of choice sequences as mathematical objects is achieved, there is no reason to admit choice sequences as genuine, legitimate mathematical objects.

In order to achieve a suitable account of choice sequences as mathematical objects, one must justify the assumption that a 'free choice' is to be considered a legitimate mathematical operation. That a free choice is a possible mental operation and that it can constitute an acceptable mental process is not questioned. What is to be decided is if such a mental operation is also a genuine mathematical operation, like, for example, the mental act of *adding a unity to a number* is also a mathematical operation. In fact this is one of the main reasons for the general reluctance in accepting lawless sequences as genuine mathematical objects. How to make sense of this idea that arbitrarily choices of numbers constitute a mathematical operation to which corresponds genuine mathematical objects? Certainly, the fact that natural numbers are the chosen objects is not the required justification. Then, in what sense is *a succession of free choices* a mathematical operation?

The discussion presented until now gives rise to a tension between the intrinsic intensional properties that make choice sequences the intensional objects they are and the fact that such intensional properties have to be abstracted from the mathematical treatment they are subjected. This necessity is motivated by the impossibility of adequately incorporating such intensional properties into a mathematical framework. It is irrelevant for mathematical purposes which is the precise moment a choice sequence is started or the fact of having a bearer to individuate a choice sequence. Mathematically, choice sequences are not individuated nor characterized by such features. In fact, (lawless) choice sequences are not mathematically individualized at all. And this is so, because we cannot *define mathematically* the role that such features play in the way a choice sequence behaves within a mathematical theory. Troelstra, discussing the problem of individuation for lawless sequences (in the context of presenting the motivation for **LS2**), had to settle a new condition: that lawless sequences initially given as different sequences have to be mantained as different (even if they share the same initial segment or, possibly, have the same restrictions). The same goes for identity: lawless sequences initially given as identical, i.e., given by the same generation process, have to be treated as the same sequence, despite the fact that we have not access to the process of generation itself. Notice that no considerations about the relation to time nor about the subject carrying them plays a part in such 'criterion' of individuation. Troelstra's criterion had to be introduced in an *ad hoc* manner, for making intelligible the mathematical treatment of choice sequences.

Given the actual state of affairs, doesn't this tension constitutes an antinomy, i.e., an unsolvable philosophical problem? On one hand we have the fact that choice sequences are intrinsically intensional objects, in the sense that they can be individuated by the particular moment they start to grow or by considering the particular subject that started them, or both. On the other hand, we have the restriction that such features have to be abstracted from mathematical practice, in other words, they have to be considered irrelevant for mathematical treatment of choice sequences. Is this situation philosophically acceptable? A we have argued, however, in the actual state of affairs there is not a plausible solution to such a dichotomy.

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