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(LI)BOR: ONE MODEL AGAIN

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Resumo

A actual crise, conhecida por Crise do “Sub-Prime”, teve múltiplas consequências na forma como a teoria financeira se debruça sobre os mercados financeiros, uma das quais é a inconsistência que as actuais taxa oferecidas no mercado interbancario (BOR) colocam à teoria das taxas de juro.

A literatura e a comunidade de negociação ideintificaram diversos factores para o comportamento estranho tal como o prémio de risco e o prémio de liquidez (este último como residuo de regressões). A resposta desta última foi no sentido de tratar cada maturidade do mercado monetário de forma independente de outras maturidades e as diferenças entre si como uma classe de activos autónoma (Basis Swaps). A comunidade científica fcentrou-se no “fixing” de uma única maturidade, esquecedo inclusivé o mercado a prazo e providenciando uma resposta muito limitada sobre a natureza da dinâmica da BOR.

O que actualmente falta é um modelo consistente que capture a informação de todos estes “novos mercados” e que nos devolva a sensação de um mercado único.

Propomos uma moldura teórica para um modelo estocástico multifactor que incorpora cada um dos factores percebidos de uma forma consistente com a tomada de decisão dos agentes. Ela mantém a simplicidade dos modelos de taxa de juro standard e, mais importante, preserva a unicidade do mercado interbancário. Um modelo que trata todo o conjunto de curvas BOR.

As questões que tratamos são importantes não só para a comunidade de investimento mas também para reguladores e autoridades monetárias na resolução de problemas no mercado interbancário. É necessário identificar claramente o problema de forma a resolvelo e o modelo proposto constitui uma ferramente útil para a sua identificação.

Iremos desenvolver a referidade moldura teórica e ilustrar a sua dinâmica com alguns exemplos reais.

PALAVRAS CHAVE: MERCADO MONETÁRIO, LIBOR, CREDITO, LIQUIDEZ, FINAL DE ANO

Abstract

The current crisis, commonly named Sub-Prime crisis, had multiple implications on the way financial theory addresses financial markets, one of which is the curve inconsistency that current interBank Offered Rates (BOR) reality poses to interest rate theory.

Literature and trading community identified several factors for this strange behavior such as credit risk and liquidity premium (always the residual of regressions). The response of the later has been to treat each money market tenor independently from others and the difference between them as an autonomous asset class (Basis Swaps). Scientific community has been focused on one only tenor and also forgetting the market of forwards while providing a very limited answer about the nature of the BOR dynamics.

What currently lacks is a consistent model that captures the information in all these "new markets" and gives us back a sense of an unique market.

We propose the theoretical framework for multifactor stochastic model which incorporates each perceived factor in a way consistent with agent's decisions. It keeps the simplicity of standard interest rate models and, most importantly, it preserves the unity of the interbank market. One model to address the entire set of BOR curves.

The questions we approach are important not only to the derivatives community but also to regulators and monetary authorities in addressing the problems mirrored in the interbank market. One needs to clearly identify the problem in order to address it, and the proposed model in an useful tool to accomplish this goal.

We shall develop the referred theoretical framework and illustrate its dynamics with several real examples.

KEY WORDS: MONEY MARKET, LIBOR, CREDIT, LIQUIDITY, END OF YEAR

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Abreviations

ATS Affine Term Structure

ON, 1W, 2W, ... , 1M, 2M, ... ,12M Some Money Market standard maturities: OverNight, One Week, Two Weeks, ..., One Month, Two Months, ... , Twelve Months

BBA British Bankers Association

CC Continuously Compounded

DC Discreetly Compounded

EURIBOR EURo InterBank Offered Rate

FRA Forward Rate Agreement

IRS Interest Rate Swap

LIBOR London Interbank Offered Rate (Published by BBA)

OIS Overnight Interest Swap

RMV Recovery of Market Value (Recovery Model)

RP Recovery of Par (Recovery Model)

RT Recovery of Treasury (Recovery Model)

TAF Term Auction Facility

TIBOR Tokyo Interbank Offered Rate

ZC Zero Coupon

ZR Zero Recovery (Recovery Model)

Notation

u time variable

$\Delta_{t,T} = T - t$ number of days between t and T; Δ_t^i is the number of days for a i maturity money market in t

Statistics

$f(x)$ x density function

$P(A) = p^A$ probability of event A

$p_{t,T}^A$ probability of event A occurring from t to T

$E\ [A]$ expected value of A

1_A indicator function

$$1_A = \begin{cases} 1 & A = true \\ 0 & else \end{cases}$$

Stochastic Variables

Elementary

X_u^i i^{th} State Variable, $\mathbf{X_u}$ State Variables Vector

Y_u^j j^{th} Poisson Process, $\mathbf{Y_u}$ Poisson Process Vector

***X* Related**

$r_u = r(\mathbf{X_u})$ Instantaneous Risk Free Interest Rate

$\lambda_u^x = \lambda^x(\mathbf{X_u})$ event intensity, $\quad x = c, l$

$\phi_u^x = \phi^x(\mathbf{X_u})$ the recovery at the time of event, τ^x, $\quad x = c, l$

$\lambda *_u^x = \lambda *^x (\mathbf{X_u})$ recovery adjusted event intensity, $\quad x = c, l$

$\theta_u = \theta (\mathbf{X_u})$ ERP variable

Y Related

$\tau^x = \tau^x (\mathbf{Y_u})$ event time, $\quad x = c, l$

$\eta_u^x = \eta^x (\mathbf{Y_u})$ entities' class (with respect to superscript classification), $\quad x = c, l$

Subscripts and Superscripts

Type Superscripts

r Risk Free curve

c related to Credit Risk

l related to Liquidity Risk

z related to ERP Cost

b All previous Factors (BOR)

Time subscripts

$_{t,T}$ evaluated at t; maturity at T

$_{t,f,T}$ evaluated at t, maturity at T; to be initiated at f forward date

Math Fin

$\beta_{t,T}^{m+n+\dots}$ Stochastic Exponential

$$\beta_{t,T}^{m+n+\dots} = \exp\left(-\int_t^T m(\mathbf{X_u}) + n(\mathbf{X_u}) + \dots du\right)$$

where $m(\mathbf{X_u})$, $n(\mathbf{X_u})$, ... are linear functions of $\mathbf{X_u}$

$B_{t,f,T}^{m+n+\dots}$ Conditional Expected Value of Stochastic Exponential

$$B_{t,f,T}^{m+n} = E\left[\exp\left(-\int_f^T m(\mathbf{X_u}) + n(\mathbf{X_u}) + \dots du\right) |\mathscr{F}_t\right] = E\left[\beta_{f,T}^{m+n} |\mathscr{F}_t\right]$$

$$B_{t,T}^{m+n+\dots} = B_{t,t,T}^{m+n+\dots}$$

β_t Money Market Account

$$\beta_t = \exp\left(-\int_0^t r\left(\mathbf{X_u}\right) du\right)$$

$\beta_{t,T}$ Risk Free Sochastic Discount Factor

$$\beta_{t,T} = \frac{\beta_T}{\beta_t} = \exp\left(-\int_t^T r\left(\mathbf{X_u}\right) du\right) = \beta_{t,T}^r$$

Zero Coupon Bond

$B_{t,T} = B_{t,T}^r$ price of a Spot risk free ZC Bond

$B_{t,T}^{rx} = B_{t,T}^{r+\lambda^x}$ price of a Spot ZC Bond with x risk, $\quad x = c, l, z \quad \lambda^z = \theta$

$B_{t,T}^{x} = B_{t,T}^{\lambda^x}$ price of a single factor Spot ZC Bond with x risk, $\quad x = c, l, z$

$B_{t,T}^{b} = B_{t,T}^{r+\lambda^c+\lambda^l+\theta}$ price of a Spot InterBank ZC Bond

$B_{t,f,T}^{x}$ price of a Forward ZC Bond

$c_{t,T}^i = c_{t,T}$ day count factor of period $[t, T]$ applied to a reference DC yield , i=ACT/ACT, ACT/360,... when omitted is assumed to be the market standard for the reference DC yield

$y_{t,T}^x = \frac{\left(-\log B_{t,T}^x\right)}{c_{t,T}}$ implied CC spot yield

$y_{t,f,T}^x = \frac{\left(-\log B_{t,f,T}^x\right)}{c_{f,T}}$ implied CC forward yield

$v_{t,T}^x = \frac{\frac{1}{B_{t,T}^x} - 1}{c_{t,T}}$ DC spot yield, the

$v_{t,f,T}^x = \frac{\frac{1}{B_{t,f,T}^x} - 1}{c_{f,T}}$ DC forward yield

$f_{t,f}^x = -\frac{\partial \log B_{t,f}^x}{\partial f}$ instantaneous Forward yield

$y_t^x = f_{t,f}^x|_{f=t}$ instantaneous yield

When a time subscript t has a superscript ^{i} it refers to the class of the reference entity in moment t

Models

AFS model

$\varepsilon_A^i, \varepsilon_B^i, \varepsilon_C^i, \varepsilon_D^i$ Parameters from equation

$$dX_t^i = \varepsilon_A^i \left(\varepsilon_B^i - X_t\right) dt + \sqrt{\varepsilon_C^i + \varepsilon_D^i X_t} dW_t$$

Event Matrix

A^x Base Generator Matrix of some event, $\quad x = c, l$

ξ^x Eigenvalues vector of A^x, $\quad x = c, l$

B^x Eigenvectors from A^x, $\quad x = c, l$

$A_u^x = A^x \left(\mathbf{X_u}\right)$ Generator Matrix of some event at moment u, function of the State Vector, $x = c, l$

$\mu_u^x = \mu^x \left(\mathbf{X_u}\right)$ Stochastic Component of A_u^x, vector of K^x elements $\mu_i^x \left(X_u\right)$, $\quad x = c, l$

$\Pi_{t,T}^x$ Transition Matrix of some event, function of the State Vector, $\quad x = c, l$

$\mathsf{E}_{t,T}^x$ Auxiliary Diagonal Matrix with e_i^x in the diagonal, $\quad x = c, l$

$\alpha_{i,j}^x, b^x$ Generator Matrix Parameters, $\quad x = c, l$

K^x number of classes, the number of the event class, $K^x \cdot K^x$ matrix dimension $\quad x = c, l$

Discrete Time Environment

R Risk Free Interest Rate

Θ Recovery

Chapter 1

Introduction

The current crisis, commonly named Sub-Prime crisis, had multiple consequences on the way financial theory addresses financial markets. One of them is the curve inconsistency in current interBank Offered Rates (BOR), which challenges existing interest rate theory.

There are various examples of this inconsistency, the most striking one being the disconnection in the usual relationship between forward and spot rates. Before the current crisis we could, at t_0, synthetically build a FRA[1] from start date t_1 to maturity date t_2 by simply entering in a loan and a deposit starting at date t_0 and ending at dates t_1 and t_2. Table 1.1 shows that now, after the beginning of the crisis, market quoted FRAs are disconnected from synthetic FRAs.

Another example is the dynamic inconsistency between apparently interchangeable instruments at specific points in time: as we can see[2] in Figure1.1 some tenors curves (1M) may present an upward slope, others (6M) a downward slope and others a flat one (3M and 12M) for the same time horizon.

FIXING 3M	FIXING 6M	FRA 3Mx3M	SINTETHIC FRA 3Mx3M [3]
4.72	4.38	2.46	3.98

Table 1.1: 13Oct2008 USD LIBOR: Fixings and FRA

The immediate consequence of this disconnection is that each tenor has started being traded as an unique instrument with its own dynamics. Since the beginning of the current

[1] Forward Rate Agreement

[2] For help on graph interpretation see Appendix A

[3]

$$v^b_{0,3,6} = \frac{\frac{1+v^b_{0,6}\times c^{ACT/360}_{0,6}}{1+v^b_{0,3}\times c^{ACT/360}_{0,3}} - 1}{c^{ACT/360}_{3,6}} = \frac{\frac{1+0.0438\times\frac{182}{360}}{1+0.0472\times\frac{92}{360}} - 1}{\frac{90}{360}} = 3.98\%$$

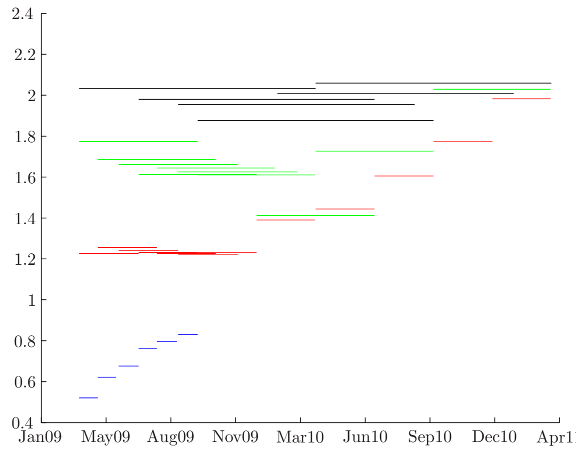


Figure 1.1: USD LIBOR 24Mar2009

crisis there are the ON, 1M, 3M and 6M curves, to name only the main ones. Also, when analyzing previous correlated instruments like Interbank Deposits, T-Bills and ON curve it is easy to see in Figure 1.2 that correlations from 3M intruments suddenly collapsed, making the hedging activity a nightmare.

Another example of the inconsistency mentioned above is the relationship between market prices inside each of the new curves: the humps and bumps in the short term curve, constructed using the quoted forwards, unsettle the most complex interest rate model. Of course we can always add n stochastic factors to fit a n-degree polynomial curve but that would not be useful as we would only be describing the curve in a meaningless way, without understanding it. The consequence of the phenomena described above is that market-making of derivative instruments on interbank rates started to be more of an art (not to mention witchcraft) than a technical job. Speaking of witchcraft, the analysts that trusted these instruments to "read the market" would be better described as mediums or psychics when they tried to interpret and decompose the factors impacting the interbank market.

At this moment, it is clear for every market participant that follows closely the interbank market that, besides interest rate risk, the current spot fixings and respective derivatives are influenced by at least three other factors: credit risk, liquidity risk and an "end of reporting period" risk. What currently lacks in the market is a consistent model that captures the information in all these "new markets" and gives us back a sense of an unique market again.

We will try to approach such problem by proposing a multifactor stochastic model that represents each perceived factor in a way consistent with agent's decisions while keeping

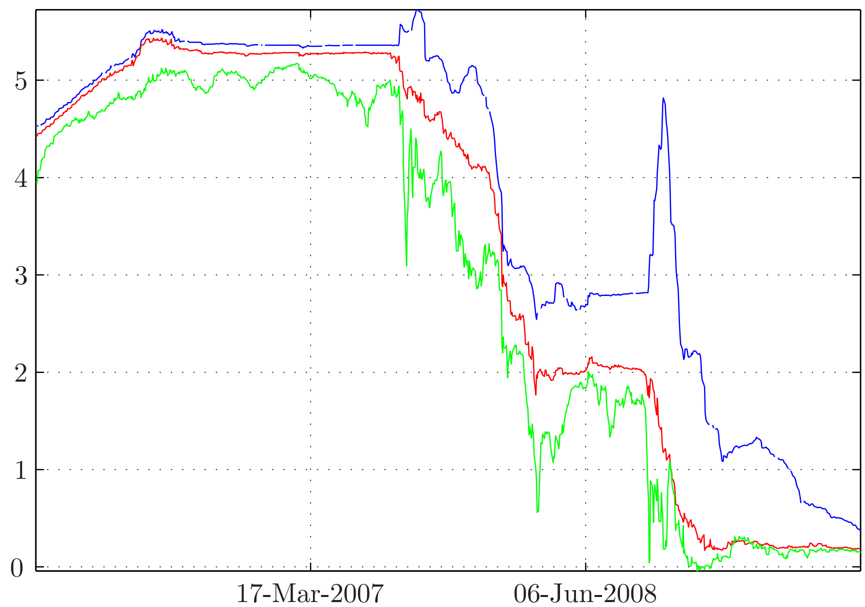


Figure 1.2: Evolution of the 3M Money Market instruments during the crisis

the simplicity of standard interest rate models and, most importantly, preserving the unity of the interbank market. One model to address the entire set of BOR curves.

The questions we approach are important not only to the derivatives community but also to regulators and monetary authorities that try to address the problems mirrored in the interbank market. One needs to identify the problem clearly in order to address it, and the proposed model is an useful tool to accomplish this goal.

We shall develop the referred theoretical framework and illustrate its dynamics with several real examples. We emphasize that we will not try to empirically estimate a model as the techniques and tools which are required to do so would duplicate the dimension of this work.

The structure of the work is as follows: In the current chapter we will give a brief description of some events and reactions in the interbank market, followed by literature that address some possible causes for such events. In the middle of the Chapter we will present some market terms that can be useful to understand this work. In Chapter 2 we will decompose our problem into several ones and treat each one separately. There will be a section for credit risk, liquidity risk and "End of Reporting Period" (ERP) factor. In Chapter 3 we will join all the factors in only one model dealing with the issues which are raised when we join variables. Chapter 4 will be dedicated to the analysis of some moments of the BOR curve with the technics developed in this work. In Chapter 5 we will briefly answer some questions raised during this work. Chapter 6 concludes the work.

1.1 Interbank Market

Before the crisis the interbank market had acquired a considerable importance for a significant number of agents mainly trough the different BOR fixings[4] and the associated derivatives market[5]. While BOR fixings and swap rates were widely used as a reference for bond pricing in the credit market, they also became the main vehicle for the transmission of the monetary policy[6]. For traders it was one of the most efficient ways of implementing trading strategies[7], and for fixed income investors they became the most efficient tool to hedge interest rate risk. Lastly BOR was viewed as an almost risk free rate by the derivatives industry[8].

When the crisis started the first market to feel it was the interbank market. Its rates started to fix in an unprecedented volatile way, going higher without any expected move by the monetary authorities and with strange features in the yield curve. These movements disturbed every market player. In the credit market, it became more costly to borrow money in the interbank market, the rate at which each player could borrow became very specific and many lenders started to fund themselves at higher rates than those at which they had lent. Then, as the interbank rates decoupled from central bank target rates and the velocity of money dropped, traditional monetary authorities' instruments became powerless to affect economic activity. For traders, the market became much more complex and illiquid as several curves, previously perceived as almost perfect substitutes, suddenly started to have their own dynamic[9]. For the investment community, the effectiveness of the hedge decreased enormously, as in many days investors would lose money both on the hedged asset and on the hedging instrument. Finally, for the derivatives industry, BOR was no longer the reflex of a risk free rate. To be clear, it was hard to tell what a risk free market was as the premium to pay for a credit riskless bond increased dramatically[10].

So it is clear that something changed in this market and that it would be useful to

[4] From which LIBOR is the most important example

[5] Swaps, FRAs and Futures.

[6] They could easily impact families's available income, and corporates P&L, by changing the short-term rates.

[7] Swaps are unfunded and highly liquid, have standard features but can be taylor made and have the same cost for being long or short.

[8] It was taken like that since its perceived credit risk was minimal, banks could actually borrow and lend at those rates, it had a liquid curve in every tradable currency along all maturities and did not have the same liquidity distortions as the treasury curve.

[9] You could had the correct view of the future path of interest rates but still loose money simply because you had bet on the wrong curve.

[10] Moreover, many derivatives desks were not able to borrow at BOR levels anymore.

have insights both on the causes and on the corresponding dynamics that produced such changes. Each player has it is own questions:

- Credit Market: at what rate will lenders be able to borrow in the future?
- Monetary Authorities: how can we start to be effective again? What problems to address in order to give traction to traditional tools?
- Trading communities: which factor impacts each curve?
- Investment community: how to hedge the interest rate risk?
- Derivatives industry: where is the risk free rate? What is the dynamic of BOR?

We shall now review some recent and past BOR dynamics in order to start understanding the phenomena. In the end of this work, in Chapter 5 we will return to these questions.

1.1.1 BOR: Pre Sub-Prime

Until recently the fixings of interbank market were driven by the expected path of interest rates for very short maturities (ON to 2W) which are manipulated/chosen by the monetary authorities. This expected path has a variable degree of uncertainty which was also reflected on its fixings (the so called "term premium").

As we are dealing with unsecured loans another "natural" contributor to BOR is credit risk. Despite this BORs were traditionally regarded as "risk-free" as they reflected the rate offered to high quality counterparts and the maturities were quite short (from 1W to 1Y). Moreover, given the squeeze of spreads in the last decade, the difference between risk-free rates and BOR was minimal as can be seen on Figure 1.3.

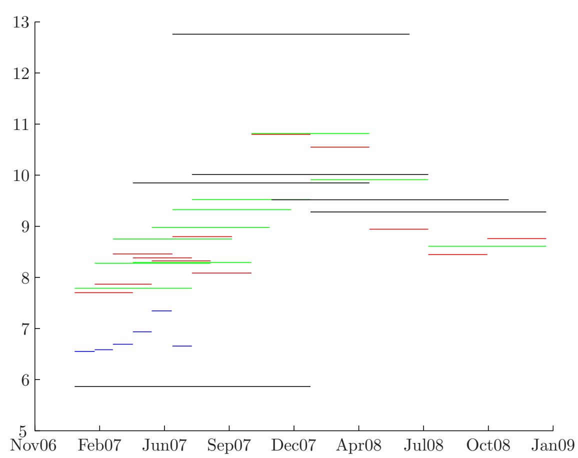


Figure 1.3: USD LIBOR-OIS 17Jan2007

Despite the historical dominance of the interest rate (risk-free) component, fixings sporadically moved away from interest rate expectations. Some reasons for such movements are:

- End of year: the eminence of presenting themselves to the market and regulators, makes banks clean their balance-sheets in order to decrease the amount of perceived risk and capital requirements.
- Disruptive events: surprise/shock events such as 9-11 create a high degree of uncertainty in the market and consequently in the confidence each bank has in its own cash flows projections. In these situations banks hoard money in order to secure their own liquidity position.
- Credit Problems on BOR Contributors: the Japanese inshore interbank market, mirrored by TIBOR[11], was disrupted when Japanese banks had credit problems in the 90s.

1.1.2 BOR since August 07

Since the current crisis began everything changed! The surge in write-downs and the increased difficulty in understanding the risks each counterpart was exposed to, drowned the interbank market and liquidity started to be exchanged only in very short terms (ON to 2W) and/or in a secured way (repo). If a bank had an excess of cash it would not know whom to lend and how its creditworthiness would be perceived in the market in the future. The safe bet would be to lend ON to a restricted number of counterparts.

With all these circumstances, BORs skyrocketed and became the center of the crisis just when they turned meaningless, since there were no transactions backing them levels and there were claims of underreporting.

At the same time, BORs became very relevant because of the pain they induced in the "real-economy", which was already facing difficulties, specially in a world were leverage turned from a virtue into a sin. As previously referred, the continuous increase in BOR also took power away from the traditional transmission channel of monetary policy: as monetary policy eased and BORs did not react as expected. Under these conditions, central banks expanded their liquidity provision to ensure the funding of the most illiquid assets for a considerable length of time, while governments tried to improve banks solvency and long term funding (equity injections and guaranteed debt).

Since then, conditions have improved but the dynamics of some BORs are still unsettling: EURIBOR, for instance, has fallen every day from 10 October 2008 until 19 May 2009 and surely the conditions in every factor that affects EURIBOR did not improved on every single day. Another issue were the claims that banks would be underreporting the contributed rate fearing a run on deposits if the contribution was too high.

[11] Tokyo Interbank Offered Rate

1.2 Some Market Terminology

In order to better understand the current work we will introduce and clarify some market terms that will be extensively used.

- LIBOR: is the BBA[12] fixing of the London Inter-Bank Offered Rate[13]. It is based on offered inter-bank deposit rates contributed by a panel of banks in accordance with the BBA instructions[14]. The main features of LIBOR are the following:
  - It is an offered rate: a rate at which the market offers funds to a specific contributor;
  - It is an average from which outliers are excluded: it captures the rate at which the average interquartile bank in the panel would borrow;
  - It is a rate offered in London, an offshore market for all the currencies except sterling;
  - It is not backed by effective transactions;
  - The credit quality is one important criterion for being in the panel, and therefore, LIBOR is the rate the market would offer to a high quality bank.
- Overnight Rate (ON): The rate at which banks exchange between each others ON deposits at the central bank. At the end of day the central banks make a weighted average of those exchanges and publish the number: EONIA for the ECB, Effective Fed Funds Rate for the FED, SONIA for the BOE. Contrary to BOR rates, these fixings are backed by real transactions. Among the factors affecting it are central bank rates and the rules to access it and the excess of funds supplied to the market by central banks. Is typically perceived as a risk free rate resulting merely from the supply and demand of funds since the banks which are expected to default the next day are "excluded" from the market. This happens because:
  - there are always rumors that precede the default (you would not lend to those counterparts) and some counterparts with difficulties would be paying an excess over the market rate (you don't lend at rates "out of the market" because it may mean that the borrower is desperate and that no one else lends him);
  - the supervision authorities [15] close a bank when they perceive insolvency risk;
  - the monetary authority can be expected to act as Lender of Last Resort, enabling the borrowers to fulfill their short term obligations when liquidity disturbances occur;

[12] British Bankers Association
[13] See Appendix B
[14] See Appendix B.1
[15] FDIC in US

  - in practice this feeling is reflected in the similarity of OIS and Repo GC[16] rates which trade at the same levels except for turbulent periods in the treasuries market (delivery failings, treasury interventions, FED interventions) and for bid-offer differences.

- Interest Rate Swap (IRS): an agreement by which two parties agree to exchange cash flows indexed to interest rate benchmarks. The most common one is the swap where one party periodically pays a fixed interest rate and receives a floating one (like LIBOR 3M for instance). In the fixed-float swap the fixed leg reflects (among other factors) the expected path for the floating leg and by this feature it is possible to build a curve of future BORs implied by the swap market.

- Overnight Interest Swap (OIS): an IRS where the floating rate is the official ON rate. As the central banks target the very short term of the curve it closely reflects the implementation of monetary policy. It can be thought as the basis for construction of a risk free curve as its underlying, the ON rate, is perceived as almost risk free.

- Treasury Curve: the set of yield rates at which sovereign bonds are traded plotted against its respective maturity. This curve reflects supply and demand specificness of each bond and the expected path of short term interest rates. For EUR, as many sovereigns share the same currency it also reflect different credit profiles.

- LIBOR-OIS spread: the excess yield of BOR over the OIS when both refer to the same maturity. If we assume that the OIS is the risk free rate (with the correspondent term premium to remunerate some uncertainty related with the path of monetary policy), the excess spread should contain all factors captured by in BOR besides interest rate risk.

1.3 Literature

There are a few published works on this issue, which have been produced recently and mostly motivated by central banks trying to justify or ascertain the effectiveness of their programs to address the problems described above. One of the conclusions is that the BOR curve no longer reflects only the future path of the central bank target rate, but also includes credit and liquidity spread among other possible factors. We review the contributions we find relevant in analyzing the recent behavior of the money market.

[16] Repo General Collateral (GC) refers to the repo rate applied to those US treasuries that are not trading with a premium.

1.3.1 Literature on Crisis

Here we shall refer to works analyzing the effects of the current environment on the interbank market. The first published piece we know of is an analysis box in BOE (2007). The authors analyze the LIBOR-OIS 12M spread and use the CDS market to extract the credit component and get the residual which they call non-credit premium. They conclude that in the beginning of the crisis increases in the LIBOR-OIS spread were due to other factors beyond the credit premium and suggest that banks were hoarding liquidity due to increased uncertainty about funding commitments with third parties. In their analysis they assume that all components (interest rate, credit premium, non credit premium) are independent which, as they acknowledge, may not be fully correct.

There was a surge in the production of papers by the staff at various regional FEDs with the purpose of evaluating the impact of FED's actions in the money market. The final balance is positive for FED's actions[17], with McAndrews, Sarkar & Wang (2008) and Wu (2008) finding statistical relevance in such actions, Taylor & Williams (2009) reaches the opposite conclusion. Although none of them uses a dynamic term structure model, they contain some interesting details that could be useful in this work:

- Wu (2008) tries to measure the impact of the TAF, which was designed to reduce liquidity stress, on the credit component; this tests the validity of the independence assumption (unfortunately the paper does not reach to a clear conclusion)

- McAndrews et al. (2008) uses a dummy variable to control for the quarter end effect and finds it statistically non-significant.

Another interesting paper is Eisenschmidt & Tapking (2009) where the authors construct a theoretical model to explain the behavior of a group of banks in an uncertain environment. The argument is that in a multiperiod horizon, where banks would be subject to random liquidity shocks, they will require a premium in their lending activity to compensate for their own credit risk, since they might have to refund themselves in the future in case of a liquidity shock. The authors also prove that it is conceivable that, in such situation, the bid and offer in the market for term liquidity may never settle so that the equilibrium in the money market would be one where every transaction would be ON. Evidence of this behavior is presented in Michaud & Upper (2008).

Still under the heading of "the current environment on the interbank market" we could easily include the papers on the TIBOR-LIBOR spread that emerged in the late 90s. At that time japanese banks were perceived to be undercapitalized and were charged an extra premium in the interbank market. The consequence was the emergence of a spread between TIBOR which included mostly japanese banks and LIBOR which had only a few ones. We found two papers, Melvin, Covrig & Low (2004) and Peek & Rosengren (2001), which tried

[17] The simple existence of liquidity programs trying to solve the problem means that there is a will to control such phenomena which advocates for the use of mean reverting processes for the liquidity component.

to regress the spread on the flow of news or risk indicators. Given their methodology these papers do not serve the purpose of this work, and therefore despite the easy parallel we could trace, our work will not have direct influence from this literature.

1.3.2 Credit

In what concerns credit risk, the present work is highly influenced by Lando's articles on intensity models with credit transition matrices and further developments. The first paper is Jarrow, Lando & Turnbull (1997), where the credit matrix applied to credit risk is developed, but our reference paper is Lando (1998) which introduces double stochastic Poisson Process, Cox Process, in the previous model. Other important reference for intensity models is Duffie & Singleton (1999) were the recovery process is modeled.

Two interesting papers which explore Cox Processes are Collin-Dufresne (2001) and Feldhütter & Lando (2008), the latter of which is a direct application of Lando (1998) as our work is.

1.3.3 Swap Spreads

Literature on both spreads and swap spreads is quite old[18]. Its relevance may seem obvious since we model the dynamics of a spread but it is even stronger when we look at the yield of a treasury as the average repo rate periodically incurred to finance the very same treasury until maturity. Under such interpretation the swap spread appears as the long term version of our work.

Among works on swap spreads we will focus on those that try to explain curves dynamics following Duffie & Singleton (1997). Two of those papers are of particular interest: He (2000) and Li (2004). He (2000) considers both government bonds and IRSs risk free instruments and so all the spread should be explained by liquidity concerns. The swap spread curve is modelled using a three factor Vazicek model[19]. Li (2004) goes one step further by trying to decompose the swap spread into a liquidity and a credit risk component. This work last is particularly interesting given the way the author identifies the liquidity factor from hardly any data available (Liu, Longstaff & Mandell (2006) for a similar model).

1.3.4 Liquidity

The literature concerning liquidity its quite ambiguous due to the vagueness of the concept and its widely usage. For this reason we shall state some definitions related to the liquidity concept.

Definition 1 (Liquidity)**.** Liquidity related definitions

[18]Swap Spread is the yield difference between Swaps and Government Debt for some maturity
[19]Grinblatt (2001) has a similar model.

Liquidity is cash, i.e., the asset that is generally accepted on exchanges.

Liquidity of an Asset is the ability that the asset has to be liquidated, i.e., transformed into cash. It would be very liquid if we lose little money on its conversion and it would qualify as being illiquid if we lose much of its assigned/perceived value in the conversion.

Liquidity Shock is an unexpected cash shortage.

Liquidity Risk is the risk of having a liquidity shock

Management of Liquidity Risk is about building a structure which would avoid the forced liquidation of illiquid assets when the balance-sheet is subject to a liquidity shock

Papers dealing exclusively with liquidity tend to analyze mostly bid-offer spreads and their impact on on the price of assets like Chollete, Naes & Skjeltorp (2008) for equity and Chen, Lesmond & Wei (2005) for corporate bonds. In the interest rate area some premium between apparently equal instruments are attributed to liquidity: the premium carried by on-the-run treasuries versus off-the-run treasuries, the premium the official bond issue program carries over other issues also guaranteed by the sovereign [20] or even the swap spread, as seen previously.

When associating BOR fluctuations with liquidity issues, we will try to connect them to the liquidity risk management of banks: the extra premium for lending funds would be a function of balance-sheet's capacity to absorb liquidity shocks and of the probability of such shocks occurring. By analyzing the problem in this fashion we would expect this component of the price to reflect precautions/sentiment of market participants due too the bad functioning of other markets. In proceeding this way we are in fact closer from Eisenschmidt & Tapking (2009) than from traditional liquidity literature. Despite this we will not adopt the discrete one period model from Eisenschmidt & Tapking (2009) since we need a continuum time approach which deals with all possible time horizons at each time.

Another contribution from other areas will be the way Li (2004) estimates the liquidity component. Likewise when estimating our model the approach would be simply to label the part of the spread that we are unable to explain using observable factors as the liquidity component.

1.3.5 End of Reporting Period

By "End of Reporting Period" we mean to refer to the effects that closing some period produces in asset prices. The most obvious effect is the year end, but the semester end can

[20] Longstaff (2002)

be another source of price changes[21].

Griffiths & Winters (2005) goes across several explanations for such behavior such as:

tax-effects where investors sell assets with taxable losses prior to the end of year to capture the loss in the current tax year

risk-shifting window dressing where "intermediaries alter portfolios at disclosure dates to underrepresent the riskiness of portfolios held on non-disclose dates"

preferred-habitat where price movements are driven by effective asset preferences and cash-flow restrictions of agents

The last two are the most interesting and Griffiths & Winters (2005) study them with reference to money market movements in year end, finding evidence in favor of the preferred-habitat hypothesis. In the current environment, where leverage is "forbidden" and capital is scarce and expensive we would privilege the risk-shifting hypothesis, but we will still employ part of the author's dummies methodology in our work.

[21] And, as the financial semesters for some financial agents differ in Europe and US, every quarter is a semester end for reporting purposes.

Chapter 2

Model

2.1 Introduction

We shall present a model that describes the current dynamics of interbank markets.

Our idea is that if a bank has excess cash it can lend it in the interbank market, hedge the interest rate risk[1] and then charge a spread to cover all the remaining risks and costs. Each lender shall have its own costs and they shall vary differently for each tenor across different lenders. Moreover, the credit risk is specific to the bank to whom the cash is lent, which makes the premium banks charge one another, for a specified time period, specific to that transaction.

Despite the specificity that each transaction carries we shall not target the individual transactions in our analysis. We will proceed in this way because there is no such data and also because if that data existed there would not exist a forward curve and a derivative market for each transaction which would make the exercise less interesting.

Our goal is to model benchmark interBank Offered Rates (BORs).

Definition 2 (BOR)**.** Is a interBank Offered Rate on a deposit to a relatively good credit quality bank by a counterpart with relatively sound liquidity position. The cash would be deposited in t and returned in T. Examples of BOR are LIBOR, EURIBOR and TIBOR.

We propose the following factors as determinants of BOR to complement the risk free dynamics :

- the potential loss of a credit event, of the borrower, which occurs at moment $\tau^c < T$
- the potential loss of a liquidity event, of the lender, which occurs at moment $\tau^l < T$
- the capital and reputation cost, of the lender, from holding a risky balance-sheet position in the i^{th} reporting moment. t_i will refer to the date of that deterministic moment.

[1] In a OIS for instance

Moreover we will not be very interested on contemporary BOR as most of the impact BOR have over the market is on the forward market. We shall define

Definition 3 (forward BOR)**.** Is a BOR where $t > t_{today}$. The biggest difference from a contemporary BOR is that credit and liquidity conditions are set to be valid only at the future start date and not "today". It can be defined as a defaultable claim conditional on the credit condition of the borrower and on the liquidity position of the lender at the forward start date. LIBOR futures, or any OTC equivalent, an examples of BOR forward.

In the following sections, after introducing the probability space and some auxiliary concepts, each factor will be analyzed *per se* while assuming the non existence of other factors. In each section the goal we will to capture the spread over the risk free rate that is due in order to compensate the lender for future costs related to the factor in analysis.

While proposing this model we are aware that the interbank rates should be the result of other factors, some of them common to all interbank market, others specific to one of them and that those factors could be correlated with the presented ones. There is also the possibility that factors which currently are irrelevant will be valued in the future and that the factors that currently drive interbank rates become less important.

Nevertheless, our aim is to state that, in today's markets, we should not model rates as blindly stochastic, just trying to discover the mathematical model that better fits the data: we should add more information about the conditions in which rates are set in order to better capture its dynamics.

2.2 Previous Remarks

2.2.1 Probability Space

We shall work on $(\Omega, \mathscr{F}, P)$ probability space large enough to support $\mathbb{R}^d$ - valued adapted stochastic process $\mathbf{X_t} = \{\mathbf{X_u} : 0 \leq u \leq t\}$ and two discrete stochastic variables, η_u^x (with $x = l, c$, where l and c stand for liquidity and credit), representing the $x-$class achieved by an asset at each moment and whose dynamics are given by a stochastic Markovian transition matrix.

Moreover, we assume that there is an equivalent risk neutral measure to $P : Q$. All the results will be computed on Q measure except when otherwise stated.

The following $\sigma - algebra$ will be used to distinguish different subsets of information:

$\mathscr{G}_t = \sigma\left\{\mathbf{X_u} : 0 \leqslant u \leqslant t\right\}$ represents the information about the state-space vector $\mathbf{X_u}$ up to moment t;

$\mathscr{H}_t^x = \sigma\left\{\eta_u^x : 0 \leqslant u \leqslant t\right\}$ holds the information about the path of η_u^x up to moment t;

$\mathscr{F}_t = \mathscr{G}_t \vee \mathscr{H}_t^l \vee \mathscr{H}_t^c$ all available information at t, ie, corresponds to knowing the evolution of $\mathbf{X_u}$ and the information about the path of η_u^x up to moment t.

Under this framework the credit and liquidity event moments are function of η_u^c and η_u^l:

$$\tau^x = \min\{u : \eta_u^x = K^x\}$$

where K^x is the class the asset achieves when the event x arrives.

In models where there are no class, just two states: "event x already occurred", $\tau^x < t$, and "event x did not occurred", $\tau^x > t$, η_s^x will be defined as:

$$\eta_t^x = \begin{cases} 1 & ,\tau^x > t \\ 0 & ,\tau^x \leqslant t \end{cases}$$

One last comment related to the nature of η^x variables: the trigger for η^x moves are the movements of an underlying hazard rate, the cumulated intensity and the threshold ξ which once touched triggers the event is a random variable which follows a standard exponential distribution. Quoting Brigo & Mercurio (2006) ξ *is independent of all default free market quantities and represents an external source of randomness that makes reduced form models incomplete.*

In this work we will pose a more restrict set of assumptions as we will impose independence between all ξ type variables which will exclude one source of dependency between η type variables.

2.2.2 Some Math Fin and Probability concepts and notation

Having $m(\mathbf{X_u})$, $n(\mathbf{X_u})$, ..., as linear functions of $\mathbf{X_u}$ we state some definitions:

Definition 4 (Stochastic Exponential)**.**

$$\beta_{t,T}^m = \exp\left(-\int_t^T m(\mathbf{X_u})\,du\right)$$

$$\beta_{t,T}^{m+n} = \exp\left(-\int_t^T m(\mathbf{X_u}) + n(\mathbf{X_u})\,du\right)$$

Definition 5 (Expected Value of Stochastic Exponential)**.**

$$B_{t,f,T}^m = E\left[\exp\left(-\int_f^T m(\mathbf{X_u})\,du\right)|\mathscr{F}_t\right] = E\left[\beta_{f,T}^m\,|\mathscr{F}_t\right]$$

$$B_{t,f,T}^{m+n} = E\left[\exp\left(-\int_f^T m(\mathbf{X_u}) + n(\mathbf{X_u})\,du\right)|\mathscr{F}_t\right] = E\left[\beta_{f,T}^{m+n}\,|\mathscr{F}_t\right]$$

Definition 6 (Money Market account)**.** We define β_t to be the value of a bank account at time $t > 0$. The bank account evolves according to the following differential equation:

$$d\beta_u = r_u \beta_u du \qquad \beta_0 = 1$$

were r_u is a stochastic variable function of the state-space vector $\mathbf{X_u}$. As a consequence,

$$\beta_t = \exp\left(-\int_0^t r\left(\mathbf{X_u}\right) du\right)$$

Definition 7 (Stochastic Discount Factor)**.** We define $\beta_{t,T}$ to be the value of a bank account at time $t > 0$ that is equivalent to one unit of currency payable at time $T > t$ and is given by

$$\beta_{t,T} = \frac{\beta_T}{\beta_t} = \exp\left(-\int_t^T r\left(\mathbf{X_u}\right) du\right) = \beta^r_{t,T}$$

Definition 8 (Risk Free Zero Coupon)**.**

$$B^r_{t,f,T} = B_{t,f,T} = E\left[\exp\left(-\int_f^T r\left(\mathbf{X_u}\right) du\right) | \mathscr{F}_t\right] = E\left[\beta_{f,T} \left| \mathscr{F}_t\right.\right]$$

Definition 9 (Inhomogeneous Poisson Process)**.** An Inhomogeneous Poisson Process Y_t with (non-negative) deterministic intensity function $l\left(.\right)$ satisfies:

$$P\left(Y_T - Y_t = k\right) = \frac{\left(\int_t^T l\left(u\right) du\right)^k}{k!} \exp\left(-\int_t^T l\left(u\right) du\right)$$

in particular, if $k = 0$:

$$P\left(Y_T - Y_t = 0\right) = \exp\left(-\int_t^T l\left(u\right) du\right)$$

Definition 10 (Cox Process)**.** A Cox Process is a generalization of a Poisson Process in which the intensity[2] is allowed to be random but in such way that if we condition on a particular realization $l\left(., \varpi\right)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $l\left(u, \varpi\right)$. The random intensity will be written on the form:

$$l\left(u, \varpi\right) = \lambda\left(\mathbf{X_u}\right) = \lambda_u$$

where $\mathbf{X_u}$ is a $\mathbb{R}^d$- valued stochastic process and $\lambda : \mathbb{R}^d \to [0, \infty[$.

$$P\left(Y_T - Y_t = 0\right) = \exp\left(-\int_t^T \lambda\left(\mathbf{X_u}\right) du\right) = \beta^\lambda_{t,T}$$

[2]Also known as hazard rate

2.3 Credit Risk

2.3.1 Introduction

In the current state of art it is quite strait forward to evaluate a defaultable claim of a known counterpart both in the spot and forward market. Despite such easiness the same does not happens when evaluating a defaultable claim of a counterpart conditional on its future rating, which is the concept behind forward BOR.

Looking to Def. 3 the crucial factor is the "credit condition of the borrower (...) at the forward start date". This nuance has been, quite often, ignored or minimized due to the low term of interbank deposits, low credit risk associated to high quality banks in the last two decades and also the cheapness of credit risk in the last decade.

Nevertheless such feature produces a yield curve different from others as there is the guarantee of credit quality being almost constant trough the time: the credit quality is periodically refreshed.

About credit risk modeling, one could argue that the credit risk associated to a specific credit class would be the same along time and that it would not make sense to use a stochastic model to capture its dynamics. In spite of that, in reality the premium demanded for bearing credit risk of a specific credit class do moves and there are good reasons explaining those movements.

One concerns the concept of credit class where credit agencies attribute the rating/class by valuing the ability of the company to survive in stress environments. So it is quite natural to observe some pro-ciclicity in the default probability of any rating class[3]. Other argument is that due to perceived assets correlations changes, the credit premium demanded by the market can evolve across time. Throughout this work we will forget this last argument as we will assume a constant price of credit risk.

The former argument also allows us to approach the credit factor as a two dimensions problem: the relative position of each counterpart, reflected on the rating, and the "underlying absolute level of risk", reflected on the procyclicality of defaults and ratings movements.

Given the described "refreshing character"[4] of BOR term structure and its natural dependence on ratings we propose the Jarrow et al. (1997) rating dependent default model with the stochasticity introduced by Lando (1998). Only with a model that incorporates credit quality we can price a forward defaultable claim conditional on the future credit quality. Moreover, by using a stochastic transition matrix we can model separately the the referred two dimensions of credit: the matrix parameters reflect the relative level of risk and the stochastic parameters the "underlying absolute level of risk".

[3] As well of downgrading probability

[4] The property of always standing a good rating at the beginning of the operation.

The impact, on pricing, of setting the future rating follows immediately since the price of a forward claim conditional on the present rating will depend on the rating path from now until maturity and the later only on that path from the forward start date until maturity.

In the present section we will study a zero coupon with T maturity only factoring for interest rate (the risk free curve) and credit risk, a characteristic which motivates a rc[5] superscript when appropriate. Most of the time the pricing will be conditional on the credit class/rating of the bond at a certain moment: that class will be visible as a superscript of the respective moment (t^i for instance). An hypothetical example would be a highly liquid corporate bond (so the liquidity premium would be small) whose sole buyers are non supervisioned entities (so no capital constraints in the reporting months).

2.3.2 Lando 1998

Here we shall present and comment Lando (1998) main results concerning the double stochastic process and the particular case of the stochastic Markovian rating transition matrix. Full details should be found in Appendix C.

We will try to provide an evaluation at t a Zero Coupon with N notional and T maturity:

$$B^{rc}_{t,T} = E\left[\exp\left(-\int_t^T r_u du\right) N 1_{\{\tau^c > T\}} + \exp\left(-\int_t^{\tau^c} r_u du\right) \phi^c_t 1_{\{\tau^c < T\}} \,|\mathscr{F}_t\right]$$

where ϕ_t is the recovery amount upon default event.

As in Lando (1998) we will work with the zero recovery assumption: $\phi_t = 0$ and later on[6] analyze the case where $\phi^c_t \neq 0$.

We will present Lando (1998) reasoning in four steps:

1. Lando (1998) start from Duffie & Singleton (1999) well known result – see Eq. C.3 in Appendix C – to value defaultable claims :

$$\begin{aligned} B^{rc}_{t,T} &= E\left[\exp\left(-\int_t^T r_u du\right) N 1_{\{\tau^c > T\}} \,|\mathscr{F}_t\right] \\ &= 1_{\{\tau^c > t\}} E\left[\exp\left(-\int_t^T r_u + \lambda^c_u du\right) N \,|\mathscr{G}_t\right] \end{aligned} \tag{2.1}$$

[5] Shorthand for $r + c$

[6] Section 2.5

which states that we can treat a risky claim in the same way we treat the risk free one[7]: it is just a mater of adding the continuous default intensity, λ_u^c, to the risk free interest rate.

2. Then, uses Jarrow et al. (1997) generator matrix, A^c, to model λ_u^c dynamics conditional on an initial credit class η_t^c. The difference from Jarrow et al. (1997) is that A^c is stochastic: $A_u^c = A^c(\mathbf{X_u})$. From all above we need to restate λ_u^c definition: $\lambda_u^c = \lambda^c(\eta_u^c, \mathbf{X_u})$.
   We will also enlarge the scope of λ^c based notation: $\lambda_{i,j}^c(\mathbf{X_u})$ will be the intensity of the transition from class i into class j and $\lambda_u^c = \lambda_{\eta_u^c,K^c}^c(\mathbf{X_u})$.

$$A_u^c = A^c(\mathbf{X_u}) = \begin{bmatrix} \lambda_1^c(\mathbf{X_u}) & \lambda_{1,2}^c(\mathbf{X_u}) & \cdots & \lambda_{1,K^c-1}^c(\mathbf{X_u}) & \lambda_{1,K^c}^c(\mathbf{X_u}) \\ \lambda_{2,1}^c(\mathbf{X_u}) & \lambda_2^c(\mathbf{X_u}) & \cdots & \lambda_{2,K^c-1}^c(\mathbf{X_u}) & \lambda_{2,K^c(\mathbf{X_u})}^c \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K^c-1,1}^c(\mathbf{X_u}) & \lambda_{K^c-1,2}^c(\mathbf{X_u}) & & \lambda_{K^c-1}^c(\mathbf{X_u}) & \lambda_{K^c-1,K^c}^c(\mathbf{X_u}) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

   The author states that such intensity can be stochastic in two different ways:

   - the intensity will change whenever the rating changes (which already happened in the standard Jarrow et al. (1997) model)
   - the intensity will change in accordance to a state vector: $\mathbf{X_u}$

   A full description of this step can be found on Appendix C.4.

3. Then Eq. 2.1 is restated – see eq. C.4 in Appendix – in a way that is function of A_u^c

$$B_{t^{\eta_t^c},T}^{rc} = E\left[\exp\left(-\int_t^T r_u du\right)\left(1 - \Pi_{t,T}^c|_{\eta_t^c,K}\right)|\mathscr{G}_t\right] \tag{2.2}$$

   where

$$\frac{\partial \Pi_{t,T}^c}{\partial t} = -A_t^c \Pi_{t,T}^c$$

   and $\Pi_{t,T}^c|_{l,c}$ is the element of $\Pi_{t,T}^c$ in l line and c column.

4. Finally, after some assumptions on the form of A_u^c, the author arrives to the following result – see eq. C.5 in Appendix –:

$$B_{t^{\eta_t^c},T}^{rc} = \sum_{j=1}^{K^c-1} -\alpha_{\eta_t^c,j}^c E\left[\exp\left(-\int_t^T r_u - \mu_j^c(\mathbf{X_u})\,du\right)|\mathscr{G}_t\right] \tag{2.3}$$

   where $\alpha_{i,j}^c$ and $\mu_j^c(\mathbf{X_u})$ are defined on Section C.5.

[7] In future references to this result, for economy of space, we will not use the $1_{\{\tau^c>t\}}$ part, omitting the obvious result: $P(\tau^c > T|\mathscr{G}_T \wedge \{\tau < t\}) = 0$

The last result main advantage is to enable the modeling of the credit risk factor in a way where we can perceive the credit dynamics as they are and not merely a binary (default/non-default) model while keeping the same framework.

The valuation of a forward start rating contingent claim would be as easy as a spot one. Following Eq. C.7,

$$B^{rc}_{t,f^{\eta^c_f},T} = \sum_{j=1}^{K-1} -\alpha^c_{\eta^c_f,j} E\left[\exp\left(-\int_f^T r_u - \mu^c_j\left(\mathbf{X_u}\right) du\right) |\mathscr{G}_t\right] \tag{2.4}$$

Furthermore, in case of independence of $r\left(\mathbf{X_u}\right)$ and $\mu^c_j\left(\mathbf{X_u}\right)$, following C.9 :

$$\begin{aligned} B^{rc}_{t,f^{\eta^c_f},T} &= B^r_{t,f,T} \sum_{j=1}^{K^c-1} -\alpha^c_{\eta^c_f,j} E\left[\exp\left(-\int_f^T -\mu^c_j\left(\mathbf{X_u}\right) du\right) |\mathscr{G}_t\right] \\ &= B^r_{t,f,T} B^{\lambda^c}_{t,f^{\eta^c_f},T} \end{aligned} \tag{2.5}$$

2.4 Liquidity Risk

2.4.1 Introduction

The current literature on liquidity does not offer much, if any, help if we want to implement a stochastic analysis[8]. Following the set of Definitions 1 the value of liquidity shall be defined as

Definition 11 (value of Liquidity)**.** the cost of not having enough cash in order to answer to an unexpected liquidity shock, and so the cost of having to find that money (either by selling assets or borrowing from someone) adjusted by the probability of such event. It can be expressed in a spread or in a present value format.

This definition is influenced by the recent paper from Eisenschmidt and Tapking (2009) who built a money market model in the context of uncertain liquidity shocks, where they argue that the demanded yield should compensate the lender for the possibility of him coming to the money market to fund a liquidity shock.

On the cost side, if the instrument that absorbed the liquidity is a tradable instrument one option is to sell the instrument and so the liquidity cost of holding such instrument would be, the maximum, the mid-offer spread (assuming that the "fair value" of such instrument is the mid price). Other option would be going to the money market and raise the required funds. The cost of raising funds in such fashion would be the excess that

[8]One should question the value of such approach if there is no previous research.

is paid over the risk free rate and such excess is dependent on the institution's perceived credit risk, market's overall liquidity among other factors (there is still the chance that the market stops to work or keeps not working).

When the instrument that absorbs liquidity is non tradable, like a deposit, the most natural option would be raising the money in the money market. The alternative option, the better one, would be to unwind the previous deposit such operation would be more costly than the money market since we only have one counterparty quoting and it may be unwilling to free such deposit (ultimately they could go to the market again for the remaining period). But despite being more costly the unwind of the operation reduces counterparty exposure and also frees up some capital.

In what concerns the probability of such event, it is not too strong to state that the marginal probability increases with time horizon since the factors that the institution is able to control to manage its liquidity decrease when the time horizon is expanded[9].

One last reference to correlations. There are several occasions where the liquidity shock and the credit risk of an entity may be correlated:

- when the liquidity shock is higher than bank's ability to fund in the short term
- when the credit quality of a counterparty turns doubtful there is a risk of a run on deposits
- when an asset suffers an credit event there is a loss of its ability to refinance via repo; moreover, when the credit event is the default the principal payment is not timely fulfilled

However, such situations will not be relevant as in the offered rate of a deposit we are evaluating the credit risk of the borrower and the liquidity risk of the lender. In order to this correlation to work we would have to make a stronger statement: that the liquidity of borrower and lender are also correlated.

2.4.2 BOR liquidity

As in the credit issue, BOR's liquidity component should reflect two different problems: "market sentiment" and individual decision. From this duality there may result an apparent contradiction: the market may signal an easing of liquidity stress in the medium term and individual agents will still demand a higher liquidity premium for longer time horizons.

The aversion of agents to lend long term liquidity will be modeled by creating liquidity profiles: using a transition matrix to model the event of a liquidity shock and also the transition between different states of liquidity shock probability. In order to deal with market

[9]This can be one reason for companies be willing to pay a big spread for issuing in the long term.

sentiment, as it is the market who lends to the panel, we will introduce stochasticity in the transition matrix, so that it reflects overall refinancing costs and uncertainty.

Turning to BOR's liquidity historic premium/importance, most of the time, when markets are functioning normally, there is widespread confidence and assets are liquid, the liquidity premium is quite small and stable. By contrast, when there are doubts about some market segment, when some agents start having difficulties in financing themselves, the lack of liquidity spreads into other markets as a pandemic and its premium increases in a vicious cycle: high liquidity premium generates fear which generates a higher demanded premium.

Meanwhile, as market discovers a price to liquidity, bank's regulators, the primary source of liquidity, also follow the conditions and consequences of such price. In some situations, when the overall uncertainty increases dramatically and there is a risk of system collapse, they intervene providing ample liquidity to the market even ensuring each bank unlimited liquidity for an extended period of time.

So, current experience tells us that the process of a liquidity premium is quite boring in stable times, violent in liquidity crisis and it reverts to a "stable" environment as soon as authorities eliminate all the funding risk[10].

Another important aspect is the meaning of forward liquidity and the difference between its price when is lent by the market, the BOR case, or by one particular agent. When an agent promises to lend in the future it is almost certain that it will be subject to various liquidity shocks and thereby its liquidity status will be different from the one at beginning. When dealing with the liquidity offered by the market, we will assume that market's liquidity price is formed by those who are in relatively good liquidity positions[11].

2.4.3 Liquidity Model

The purpose of this model is to provide a stochastic framework to an unexpected liquidity event and respective consequences. The event would be associated to the previous lock in of some amount of liquidity between t and T in a Zero Coupon Bond, a characteristic which motivates a rl[12] superscript when appropriate.

Moreover, in this section we will assume that there are no further risks or costs beyond the liquidity and risk free rate.

[10]Then the cost of refinancing in the authority is the penalty for accessing the extraordinary program designed by the authority. This cost should be adjusted to the market uncertainty: the largest the uncertainty the smaller the cost; the program should be unwound as the "market normality" is reestablished in order to motivate interbank lending.

[11]Those who are in a relatively bad liquidity position will demand a higher price for the liquidity, its offer will not met a demand and consequently their demanded price will not be reflected in market price.

[12]Shorthand for $r + l$.

To model the occurrence of a liquidity event we shall use a Cox Process in the same way[13] Lando (1998) did to model a credit event. We will start by presenting a model to evaluate a claim with no recovery upon liquidity event and only in section 2.5 we will address it.

In this section we shall discuss the hypothesis in which we incur, present the variables and the final results. All non original demonstrations can be found in Appendix C and be easily adapted to the current case[14] as they are formulated to a non specified event.

Basic Liquidity Model

Let $m^l(u,\omega)$ be a particular realization of the random intensity of a Y_u^l Cox Process defined on a $(\Omega, \mathscr{F}, Q)$ probability space. We will write $m^l(u,\omega) = \lambda^l(\mathbf{X_u}) = \lambda_u^l$ where $\mathbf{X_u}$ is a $\mathbb{R}^d$ - valued stochastic process and $\lambda^l : \mathbb{R}^d \to [0,\infty[$ defined on $(\Omega, \mathscr{F}, Q)$ probability space.

Definition 12. The liquidity event time τ^l can be thought as the time of the first jump after t of the Y^l Cox process with intensity process λ_t^l.

$$\tau^l = \inf\left\{u : Y_u^l - Y_t^l \geq 1\right\}$$

From the Definition 12 and according to eq. C.1 in Appendix C

$$P\left(\tau^l > T \left|\mathscr{G}_T \wedge \mathscr{H}_t^l\right.\right) = 1_{\{\tau^l > t\}} \exp\left(-\int_t^T \lambda^l(\mathbf{X_u})\, du\right) \tag{2.6}$$

We are able now to value a claim contingent on such event. Following eq. C.3 in Appendix C

$$\begin{aligned}
B_{t,T}^{rl} &= E\left[\exp\left(-\int_t^T r_u du\right) N 1_{\{\tau^l > T\}} \left|\mathscr{F}_t\right.\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) N P\left(\tau^l > T \left|\mathscr{G}_T \wedge \left\{\tau^l > t\right\}\right.\right) \left|\mathscr{F}_t\right.\right] && (2.7) \\
&= 1_{\{\tau^l > t\}} E\left[\exp\left(-\int_t^T r_u + \lambda_u^l du\right) N \left|\mathscr{G}_t\right.\right] && (2.8)
\end{aligned}$$

In future references to this result, for economy of space, we will not use the $1_{\{\tau^l > t\}}$ part, omitting the obvious result: $P\left(\tau^l > T \left|\mathscr{G}_T \wedge \{\tau < t\}\right.\right) = 0$.

[13] Using the same arguments and following the same line of thought.

[14] It is only need to change the x superscript into the l in the relevant variables to full convert the appendix results into the desired ones.

A Class Based Liquidity Model

As discussed previously BORs are rates offered by the market to the BOR panel. After going through a simple model of how each bank should value its liquidity risk our question is: how should this individual decision making be reflected on market offered rate?

One hypothesis would be that after each bank putting a premium on the liquidity component[15] the offers are ordered and only the best ones[16] are "hit". We would define then the market offered rate as some statistic of those hit offers: average, the highest/last, some quantil, etc. We would conclude then that

1. market offered rate reflects a state of liquidity better than markets average
2. changes in liquidity risk premium of offered rates reflect changes in overall liquidity[17].

Assuming that the costs from liquidity shocks have the same distribution for every bank, the only factor for difference would be the intensity. Then, taking the fact that market offered rate liquidity premium reflects a certain intensity and is associated with a good liquidity position we may build a structure where bank's would be classified accordingly to its liquidity state:

1. a fixed number of liquidity states to which we can associate different liquidity shock probabilities
2. intermediary intensities that attribute some probability of a bank in a given state to change to another state
3. an absorbing state which would be the liquidity shock state

The main problem with such structure is that every bank has the same expected/possible path. In reality it is quite possible that, given that different banks identify different weaknesses/strengths in different time horizons, a bank may have a liquidity advantage that enables it to lend at 1M and at the same time a disadvantage that make its offer out of the market for a 6M deposit. In the proposed framework such behavior would not be possible.

To be completely fair we would not be taking such strong statement since since the aggregate result of many individual decisions may be correctly described by it. In fact with such model we only pretend capture one effect: that sometime in the future a lender may be in a worse liquidity state than the one it currently faces and that at the same time this

[15]Function of individual balance-sheet risks and the perception of future developments of interbank market

[16]There is not an inelastic amount of demand of funds as each term competes with all the others and all compete with the secured lending and central bank facilities

[17]Changes on some of the best bidders would not affect much the market rate since they would be marginally replaced by others; only wide changes in overall liquidity sentiment would drive every bidder to change the offered rate

fact does not mean that the all market will be facing harder liquidity environment in the future. The transition matrix with stochastic intensities enables us to conciliate individual decisions with market sentiment in a way a simple n-factor linear model does not enable.

After laying the basis for a stochastic markovian transition matrix we shall describe its main features. Once more we will use the same approach Lando (1998) used.

Define the generator matrix:

$$A_u^l = A^l\left(\mathbf{X_u}\right) = \begin{bmatrix} \lambda_1^l\left(\mathbf{X_u}\right) & \lambda_{1,2}^l\left(\mathbf{X_u}\right) & \cdots & \lambda_{1,K^l-1}^l\left(\mathbf{X_u}\right) & \lambda_{1,K^l}^l\left(\mathbf{X_u}\right) \\ \lambda_{2,1}^l\left(\mathbf{X_u}\right) & \lambda_2^l\left(\mathbf{X_u}\right) & \cdots & \lambda_{2,K^l-1}^l\left(\mathbf{X_u}\right) & \lambda_{2,K^l}^l\left(\mathbf{X_u}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K^l-1,1}^l\left(\mathbf{X_u}\right) & \lambda_{K^l-1,2}^l\left(\mathbf{X_u}\right) & & \lambda_{K^l-1}^l\left(\mathbf{X_u}\right) & \lambda_{K^l-1,K^l}^l\left(\mathbf{X_u}\right) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where

- $1, ..., K^l-1$ are liquidity classes/ratings where 1 is the class with the lowest propensity to a liquidity shock, K^l-1 is where the propensity is higher and K^l is the liquidity shock state
- $\lambda_{i,j}^l\left(\mathbf{X_u}\right)$ is the intensity transition from state i to state j, function of stochastic state variables
- $\lambda_i^l = \lambda_{i,i}^l = \sum_{j=1,j\neq i}^{K^l} \lambda_{i,j}^l\left(\mathbf{X_u}\right), i = 1, ..., K^l-1$ by definition
- $\lambda_{i,K^l}^l = 0$, means no exit from liquidity shock state

With this construction[18] we obtain a continuous time process which conditionally on the evolution of the state variables is a non-homogeneous Markov chain.

Conditionally on the evolution of the state variables, the transition probabilities, $\Pi_X^l\left(t,T\right)$, of this Markov chain satisfy

$$\frac{\partial \Pi_{t,T}^l}{\partial t} = -A_t^l \Pi_{t,T}^l$$

Pricing in a Generalized Markovian Model

In the next paragraphs we shall deliver a pricing formula, based on previous framework, for a zero coupon bond with only liquidity risk conditional on the initial liquidity state: η_t^l

[18] See Lando (1998) pag 108 for details and further motivation

Recovering the 2.7 result and applying the previous results we obtain following equation (after Eq. C.4 in Appendix C):

$$\begin{aligned} B^{rl}_{t^{\eta^l_t},T} &= E\left[\exp\left(-\int_t^T r_u du\right) E\left[1_{\{\tau^l>T\}}\left|\mathscr{G}_T \vee \eta^l_t = i\right]\right|\mathscr{G}_t \vee \eta^l_t = i\right] \\ &= E\left[\exp\left(-\int_t^T r_u du\right)\left(1-\Pi^l_{t,T}|_{i,K^l}\right)|\mathscr{G}_t\right] \end{aligned} \tag{2.9}$$

where

- $\Pi^l_{t,T}|_{i,K^l} = P\left(t < \tau^l \leqslant T\left|\mathscr{G}_T \wedge \eta^l_t = i\right.\right)$ or the element from i^{th} line and K^l column from $\Pi^l_{t,T}$ matrix.
- the condition $\mathscr{H}^l_t \wedge \eta^l_t = i$ resumes to $\eta^l_t = i$. as there is more information in η^l_t than in $\mathscr{H}^l_t$

Finally, after many algebra manipulations to 2.9 and some assumptions, we arrive to the formula (after Eq. C.4 in Appendix C):

$$B^{rl}_{t^i,T} = \sum_{j=1}^{K^l-1} -\alpha^l_{i,j} E\left[\exp\left(-\int_f^T r_u - \mu^l_j\left(\mathbf{X_u}\right) du\right)|\mathscr{G}_t\right] \tag{2.10}$$

where $\alpha^l_{i,j}$ and $\mu^l_j\left(\mathbf{X_u}\right)$ are defined on Section C.5.

The valuation of a forward start rating contingent claim would be as easy as a spot one. Following Eq. C.7,

$$B^{rl}_{t,f^{\eta^l_f},T} = \sum_{j=1}^{K-1} -\alpha^l_{\eta^l_f,j} E\left[\exp\left(-\int_f^T r_u - \mu^l_j\left(\mathbf{X_u}\right) du\right)|\mathscr{G}_t\right] \tag{2.11}$$

Furthermore, in case of independence of $r\left(\mathbf{X_u}\right)$ and $\mu^l_j\left(\mathbf{X_u}\right)$, following C.9 :

$$\begin{aligned} B^{rl}_{t,f^{\eta^l_f},T} &= B^r_{t,f,T}\sum_{j=1}^{K^l-1} -\alpha^l_{\eta^l_f,j} E\left[\exp\left(-\int_f^T -\mu^l_j\left(\mathbf{X_u}\right) du\right)|\mathscr{G}_t\right] \qquad (2.12) \\ &= B^r_{t,f,T} B^{\lambda^l}_{t,f^{\eta^l_f},T} \end{aligned}$$

2.5 Recovery

2.5.1 Introduction

We decided to treat the recovery variable separately from the credit and liquidity sections, and not as a subsection in each section, since it applies to both in the exactly the same way and is a well studied subject.

Recovery can be roughly described as the value that the investor recovers upon an event after loosing all due future cash flows. Applying such concept to a credit event[19] is quite straightforward but to a liquidity one seams too violent. But in reality, in face of a liquidity shock we desire to anticipate all future cash flows and exchange them for a lump sum amount received at the liquidity shock moment, and that amount would be our recovery.

In what concerns our variable recovery, ϕ_u^x, there are several methods traditionally explored by the literature:

- Zero Recovery
- Recovery of Treasury
- Recovery of Market Value
- Recovery of Par
- Stochastic Recovery

The first is, in our context, an academic case, a starting point which enables us to progress to more elaborated models. The second, third and fourth are hypothesis that make the recovery as a fixed coefficient of other factor, and the last one is about making that fixed coefficient stochastic. All are analyzed separately on Appendix D.

2.5.2 Choosing a Method

One special feature of recovery is that it cannot be identified with a simple spread structure, as we would have one spread to identify both the probability of default and recovery rate. In such situation there are two options after choosing the recovery method:

- assume a given recovery rate and extract the event probability
- to join default probability and recovery rate in one single new unidimensional variable

Our option will be to privilege the easiness of modeling without taking too strong assumptions. The only method that enables us to do that is the Recovery of Market Value (RMV) which delivers a recovery adjusted event intensity[20]

$$\lambda^{xa}\left(\mathbf{X_u}\right) = q^x \lambda^x \left(\mathbf{X_u}\right)$$

[19] Historically it depends on numerous factors as the pre-default ratting or the business cycle evolution, among others. See Schonbucher (2003) for a discussion on the determinants of recovery and merits of each method dealing with it.

[20] As the 3.5 example in Lando (1998)

where q^x is the recovery parameter: $\phi^x_u = (1 - q^x) B^{rx}_{u-,T}$.

In case of stochastic recovery, Schonbucher (2003)[21] states that if the process that generates the recovery is independent from all other processes, we only need to use the expected value of q^x instead of itself to avoid incurring in errors:

$$q^x = E\left[q^x\left(\mathbf{X_u}\right)|\mathscr{G}_t\right]$$

And with such property we are able to join estimate event probability and recovery in a stochastic framework only taking the hypothesis that the recovery is independent from all other processes.

So if we want to apply any of the formulae in this chapter in a Recovery of Market Value perspective and not as Zero Recovery as was stated in the beginning we only have to adjust the interpretation of the intensity parameter from "event intensity" to "recovery adjusted event intensity".

Eq 2.1 and Eq 2.8 are now

$$\begin{aligned} B^{rxa}_{t,T} &= E\left[\exp\left(-\int_t^T r_u du\right) N 1_{\{\tau^x > T\}} + \exp\left(-\int_t^{\tau^x} r_u du\right) \phi^x_{\tau^x} 1_{\{\tau^x < T\}} |\mathscr{F}_t\right] \\ &= 1_{\{\tau > t\}} E\left[\exp\left(-\int_t^T r_u + \lambda^{xa}_u du\right) N |\mathscr{G}_t\right] \end{aligned} \tag{2.13}$$

where

$$\phi^x_u = (1 - q^x) B^{rx}_{u-,T} \qquad \lambda^{xa}_u = q^{xa}\lambda^x_u \qquad q^x = E\left[q^x\left(\mathbf{X_u}\right)|\mathscr{G}_t\right]$$

All remaining results are kept, only the interpretation of λ^x has to be adjusted into a recovery adjusted intensity under a a RMV model.

Furthermore, the generator matrix A^x_u is now A^{xa}_u after replacing $\lambda^x_{i,j}$ by $\lambda^{xa}_{i,j} = q^{xa}\lambda^x_{i,j}$.

2.6 End of Reporting Period - ERP

In the current crisis other recurrent phenomena gained unseen proportions: the premium banks charge each others to carry a position over a reporting date.

When leverage and balance-sheet size was seen as a strength and capital was cheap this phenomena was visible but kept under reasonable proportions. As we can see in Figure 2.1 the "instantaneous" impact on LIBOR-OIS of having to hold a credit in the 31-12-2006 is for the 3M LIBOR-OIS on the end of September about 2bp $\times$ 90days = 180bp and approximately 8bp $\times$ 30days = 240bp for the 1M LIBOR-OIS on the end of November.

During the current crisis with the delevereging pressure and scarcity of capital trading desks started to be pressed early on to avoid short term trades that would cross the End

[21] The main results can be founded in Appendix D

of Reporting Period (ERP) the premiums were significantly higher. In Figure 2.2 the "instantaneous" impact on LIBOR-OIS of having to hold a credit in the 31-12-2008 is for the 3M LIBOR-OIS on the end of September about 5bp $\times$ 90days = 450bp which is not much higher from the previous exemple. But it rose to approximately 50bp $\times$ 30days = 1500bp for the 1M LIBOR-OIS on the end of November.

As we saw in the literature review it is not easy to distinguish between ERP as we characterize here and liquidity risk since liquidity risk increases if lenders are not willing to lend. Eventhough we think that there is room in the current environment to study the ERP due to the big influence of the delevereging process and capital scarcity.

In the current environment, where deposits on the central bank do not consume capital and lending to banks does consume, the most natural behavior for banks approaching an ERP would be:

1. to cut all trading exposures that pass the ERP;

2. to borrow all needs in the last auctions form central bank (to avoid asking for liquidity in the ERP);

3. to lend excess liquidity in the interbank market while the term dos not pass the ERP;

4. in the last day of the current period deposit the excess on the central bank deposit facility or to lend ON at a higher than normal yield to other banks if the bank do not have capital constraints.

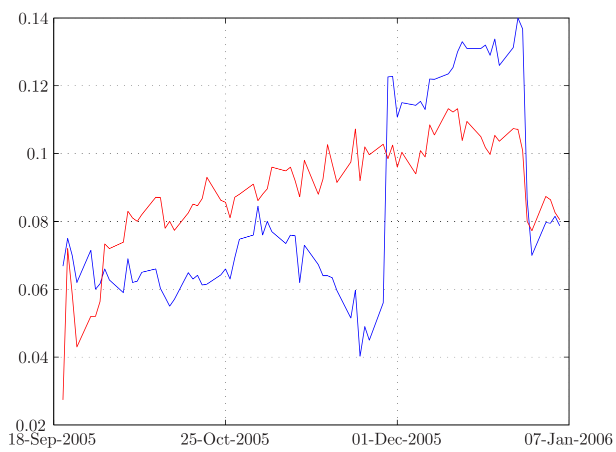


Figure 2.1: LIB-OIS Before Lehman

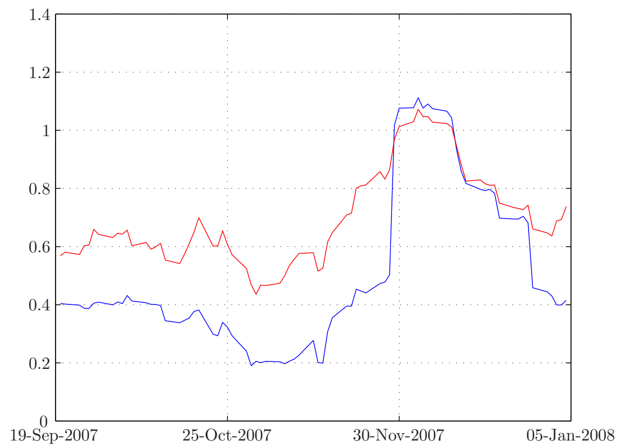


Figure 2.2: ERP After Lehman

The ON rate on the ERP usually do not follow the premium pattern of longer money market terms as the Central Banks flood the market with short term liquidity in this times biasing money market prices.

The way we see this phenomena impacting the BOR curve is the following:

- when the ERP is in the horizon banks start cutting risk positions that cross ERP to approach their desired balance-sheet state
- such behavior will diminish liquidity available from terms crossing the ERP and increase liquidity for the other terms
- the consequence would be changes in relative pricing of different tenors as we get closer of ERP
- Central Banks make liquidity injections in the eve of ERP to avoid market disruptions which makes the shorter end of the money market curve less affected by the phenomena
- return to normality as the ERP passes

Based on such diagnostic we model the ERP as a cost for which the lender has to be compensated:

$$B^{rz}_{t,T} \quad = \quad E\left[\beta_{t,T}N - \sum_i \beta_{t,t_i} z^i_{t_i} \,|\mathscr{F}_t\right] \tag{2.14}$$

where t_i is a ERP date and $z^i_{t_i}$ is the respective cost.

Once again, as in credit and liquidity risk, we will study a zero coupon with no other risk than the risk free curve and the factor which we are analyzing, a characteristic which motivates a rz[22] superscript when appropriate.

On the model of ERP premium we should add that we are not fully satisfied with 2.14 as we think ERP premium is more complex than that. For instance, there could be a process of discovering of the premium “fixing driven”: as the different money market terms pass the ERP the implied premium payed on other tenors is affected[23]. Other possibility is that the change of liquidity from one tenor to the other affects the price of both, and not only of the tenor that passes the ERP[24].

2.6.1 Single-Period Model

With such behavior in mind the price the banks attach to the extra room in balance-sheet would change as the circumstances and their positions change. We shall measure such price as the extra yield demanded for the last day of the period priced on interbank rates that cross the ERP. The yield would be charged over the market value of the asset as both capital consumption and perceived leverage are function of it.

This sort of pricing will produce the same framework as the “Recovery of Market Value” model described in Appendix D.

We will begin with a discrete time reasoning:

- time step from t to $t + \Delta_t$
- the period from t to $t + \Delta_t$ may or may not be an ERP
- The set A is the set of data periods that are classified as ERP periods, $D_t = 1_{\{]t,t+\Delta_t] \subset A\}}$
- if $D_t = 1$, Z_t will follow the RMV framework: $Z_{t+\Delta_t} = \Theta B^{rz}_{t,T} D_t$
- R as risk free interest rate from t to $t + \Delta_t$
- Θ is the RMV parameter

The price of an asset at time t must be the expected discounted value of its value at time $t + \Delta_t$:

[22] Shorthand for $r + z$

[23] See the jump of US LIBOR 3M when USD 1M LIBOR maturity crosses the ERP on Figure 2.2

[24] See the downward slope of USD 1M LIBOR in the two months following the day that the maturity of USD 3M LIBOR crosses ERP on Figure 2.1

$$\begin{aligned} B^{rz}_{t,T} &= \frac{1}{1+R\Delta_t}\left\{E\left[B^{rz}_{t+\Delta_t,T}\,|\mathscr{F}_t\right]-Z_{t+\Delta_t}\right\} \\ &= \frac{1}{1+R\Delta_t}\left\{E\left[B^{rz}_{t+\Delta_t,T}\,|\mathscr{F}_t\right]-D_t\Theta\Delta_t B^{rz}_{t,T}\right\} \end{aligned} \tag{2.15}$$

where $D_t = 1_{\{]t,t+\Delta_t]\subset A\}}$

Rearranging the terms, the price $B^{rz}_{t,T}$ should be

$$\frac{E\left[B^{rz}_{t+\Delta_t,T}\,|\mathscr{F}_\tau\right]}{B^{rz}_{t,T}} = 1+R\Delta_t - D_t\Theta\Delta_t$$

and then

$$\begin{aligned} E\left[\frac{B^{rz}_{t+\Delta_t,T}-B^{rz}_{t,T}}{B^{rz}_{t,T}}\,|\mathscr{F}_t\right] &= 1+R\Delta_t + D_t\Theta\Delta_t - 1 \\ &= R\Delta_t + D_t\Theta\Delta_t \end{aligned}$$

In the limit when $\Delta_t \to 0$, replacing interest and cost rates with their stochastic continuously compounded equivalents[25]

$$E\left[\frac{dB^{rz}_{u,T}}{B^{rz}_{u-,T}}\,|\mathscr{F}_u\right] = (r_u + d_u\theta_u)\,du$$

This leads to the conjecture that $B^{rz}_{t,T}$ is reached by discounting the expected price in survival $B^{rz}_{t+\Delta_t,T}$ with the $(r+d\theta)$ continuous rate.

The value of $B^{rz}_{t,T}$ from Eq 2.15 would be[26]

$$B^{rz}_{t,T} = E\left[\exp\left(-\int_t^T r\left(\mathbf{X_u}\right) + d_u\theta\left(\mathbf{X_u}\right)du\right)|\mathscr{F}_t\right] \tag{2.16}$$

where θ_u would be modeled through an Affine Term Structure (AFS) model and $d_u = 1_{\{u\in A\}}$.

The valuation of a forward start rating contingent claim would be as easy as a spot one. Eq. 2.16 can easily be modified into

$$B^{rz}_{t,f,T} = E\left[\exp\left(-\int_f^T r\left(\mathbf{X_u}\right) + d_u\theta\left(\mathbf{X_u}\right)du\right)|\mathscr{F}_t\right] \tag{2.17}$$

[25] $R \to r_u$, $\Theta \to \theta_u$ and $D_u \to d_u$

[26] proof on pag 139 of Schunbucher for a similar problem

Furthermore, in case of independence of $r(\mathbf{X_u})$ and $\theta(\mathbf{X_u})$:

$$B^{rz}_{t,f^{n^l_t},T} = B^{r}_{t,f,T} B^{z}_{t,f,T} \tag{2.18}$$

2.6.2 Multi-Period Model

At each moment there are multiple reporting periods in the horizon each of them with different degrees of importance but all affecting the BOR's curve[27]. How should we handle all this different periods?

1. Every ERP is a different phenomena and should be given a special treatment. We would need as many models (θ^i) as the number of ERP that will affect our variables. Naturally, correlations could be implemented when required. This is the most general way to express the Multi-Period Model.

$$B^{rz}_{t,T} = E\left[\exp\left(-\int_t^T r_u + \sum d^i_u \theta^i_u du\right)\right] \tag{2.19}$$

   where $d^i_u = 1_{\{u \in A^i\}}$

2. All ERP are the same phenomena and should be reflected in one single model.

$$B^{rz}_{t,T} = E\left[\exp\left(-\int_t^T r_u + d_u \theta_u du\right)\right] \tag{2.20}$$

   where $d_u = 1_{\{u \in A\}}$

3. There are different classes of ERP (EoY and EoS for instance) each class represent a single phenomena and should be modeled trough a different model. Naturally, correlations could be implemented when required.

$$B^{rz}_{t,T} = E\left[\exp\left(-\int_t^T r_u + d^{EoY}_u \theta^{EoY}_u + d^{EoS}_u \theta^{EoS}_u du\right)\right] \tag{2.21}$$

   where $d^{EoY}_u = 1_{\{u \in A^{EoY}\}}$ $\quad d^{EoS}_u = 1_{\{u \in A^{EoS}\}}$

4. There are different classes of ERP (EoY and EoS for instance) but all represent the same underlying phenomena, classes only reflect different intensities. There is only one model which is scaled with a coefficient (stochastic or deterministic) for each class of ERP.

$$B^{rz}_{t,T} = E\left[\exp\left(-\int_t^T r_u + d_u \theta_u du\right)\right] \tag{2.22}$$

[27] In some cases there are more than one affecting a Deposit.

where

$$d_u = \begin{cases} 1 & , u \in A^{EoY} \\ c_t^{EoS} & , u \in A^{EoS} \\ 0 & \text{else} \end{cases}$$

Our preference would go to the last option, Eq. 2.22, as it allows some degree of flexibility when compared with equation 2.20 without the increased complexity of equation 2.21. About equation 2.19, we think that the huge complexity due to a big number of models and a covar matrix behind them is not matched in gains for having one model for each ERP.

Chapter 3

Joint Model

3.1 Introduction

When trying to join all factors the first question should be: "How do those factors interact between themselves?" and not "What result are we expecting?". And, despite the unscientificness of the second question it will shape our path in a much stronger way than the first will do.

The first question will be extensively discussed but it can be easily modeled by inserting some extra state variable which connects two or more observed variables or other method for modeling co-movements.

Nevertheless it shall be noted that concerning the Poisson Processes implicit in some factors we will assume independence towards all continuous variables, which is a common assumption, but also independence between all Poisson Processes, which can be disputed. This assumption is mostly due to the added complexity[1] which the relaxing of the assumptions would introduce.

The second question, despite being much easier to answer since is based on intuition and some preconceived ideas, is much harder to deal with since the framework we design will affect the final result and the easiness of modeling it.

With such thoughts in mind we will start by displaying the various possible/acceptable frameworks and choose the one best fits our desired result and end the chapter discussing and modeling the correlations between factors.

[1] We would need to introduce copula models.

3.2 Framework

3.2.1 Choosing

When joining the models we have to take assumptions on the global consequences of an event, be it a credit or a liquidity one:

AD All risks (Liquidity and Credit) and costs (ERP) Disappear

- In a liquidity event is equivalent to assume that the deposit is unwound (so no more credit risk)
- In a credit event is equivalent to assume that the recovery amount is kept in ON deposits (so no more liquidity risk)

OK the Other risk and ERP costs are Kept but the event that occurred cannot happen again.

- In a liquidity event is equivalent to assume that the deposit is not unwound (so we keep credit risk)
- In a credit event is equivalent to assume that the recovery amount is only received at maturity (so we keep liquidity risk over the recovery amount)

AK All risks (liquidity and credit) and ERP costs are Kept

- In a liquidity event is equivalent to assume that the liquidity shock is partial, and so after a shock there would still be room for future shocks
- In a credit event is equivalent to assume that the recovery amount is reinvested again in a BOR deposit until maturity

Naturally, when a ERP cost is incurred nothing happens to the others risks and ERP costs

We shall now deduce an intermediary formulation which will be the starting point for the next section

3.2.2 Zero Recovery

By choosing a Zero Recovery assumption in credit and liquidity we are implicitly choosing the AD assumption, as there is no asset after the event. The value of such asset would be given by:

$$B_{t,T}^{b} = E\left[\beta_{t,T} N 1_{\{T>\tau^{cl}\}} + \sum_{i} \beta_{t,t_i} z_{t_i}^{i} 1_{t_i>\min\{T,\tau^{cl}\}} | \mathscr{F}_t\right] \tag{3.1}$$

where $\tau^{cl} = \min\left\{\tau^l, \tau^c\right\}$ is the time at which the first of the poisson processes jumps to 1.

The event time τ^{cl} can be thought as the time of the first jump after t of $Y_u^{cl} = Y_u^c + Y_u^l$. Given that Y_u^c and Y_u^l are independent from each other Y_u^{cl} is also a Cox process with intensity process $\lambda_t^{cl} = \lambda_t^c + \lambda_t^l$.

$$\tau^{cl} = \inf\left\{u : Y_u^{cl} - Y_t^{cl} \geq 1\right\}$$

Then, using Eq C.3 results:

$$\begin{aligned} B_{t,T}^b &= 1_{\{\tau^{cl}>t\}} E\left[\beta_{t,T}\beta_{t,T}^{\lambda^{cl}} + \sum_i \beta_{t,T}\beta_{t,t_i}^{\lambda^{cl}} z^i \,|\mathscr{G}_t\right] \\ &= 1_{\{\tau^{cl}>t\}} E\left[\beta_{t,T}^{r+\lambda^l+\lambda^c} + \sum_i \beta_{t,t_i}^{r+\lambda^l+\lambda^c} z^i \,|\mathscr{G}_t\right] \end{aligned} \tag{3.2}$$

And lastly, solving for the ERP factor under the simplest hypothesis, Eq 2.20:

$$B_{t,T}^b = E\left[\beta_{t,T}^{r+\lambda^l+\lambda^c+d_u\theta_u} \,|\mathscr{G}_t\right] \tag{3.3}$$

3.2.3 Recovery of Market Value

We will opt for the AD option for the following reasons:

- The AK assumptions imply slightly different models from the ones previously studied; the models are not much different but would add unnecessary complexity to our formulation
- The OK assumptions do not allow us to reach the desired final result: spreads additivity

Under the AD assumption, assuming RMV on credit, liquidity and ERP the value of the interbank zero coupon is:

$$\begin{aligned} B_{t,T}^b &= E\left[\beta_{t,T} N 1_{\{T>\tau^{cl})\}} + \beta_{t,\tau^l}\phi_{\tau^l}^l 1_{\{\tau^l<\min(T,\tau^c)\}}\right. \\ &+ \left.\beta_{t,\tau^c}\phi_{\tau^c}^c 1_{\{\tau^c<\min\left(T,\tau^l\right)\}} + z^i 1_{\{t_i<\min\left(T,\tau^{cl}\right)\}} \,|\mathscr{F}_t\right] \end{aligned} \tag{3.4}$$

We will now use the same kind of arguments presented on SECTION D.4, starting with a discrete model and extract the limit.

Discrete reasoning Assumptions:

- time step of Δ_t: $]t, t+\Delta_t]$
- R as risk free interest in $]t, t+\Delta_t]$
- the set A is the set of data periods that are classified as ERP periods, $D_t = 1_{\{]t,t+\Delta_t]\subset A\}}$; the moment $t+\Delta_t$ may or may not be an ERP
- if $D_t = 1$, z_t will follow the RMV framework: $z_{t+\Delta_t} = \Theta B^b_{t,T} D_t$
- Θ_t the ERP RMV parameter
- at the event moment, τ^x , the recovery of every event is $(1-q^x)$ times the pre-event value of the claim
$$\phi^x_{\tau^x} = (1-q^x)\, B^b_{\tau^x-,T} \quad x = l, c$$
if $\tau^x = t+\Delta_t$ then $\tau^x- = t$
- the event probability until $t+\Delta_t$ is $p^x_{t,t+\Delta_t} = p^x \quad x = l, c$
- the event x Hazard rate, H^x, is defined as $\Delta_t \cdot H^x = \frac{p^x}{1-p^x} \quad x = l, c$
- $p = p^c + p^l - p^c p^l$ is the probability of any event to happen until $t+\Delta_t$ and $\Delta_t \cdot H^x = \frac{p^c+p^l-p^cp^l}{1-p^c-p^l+p^cp^l}$ is the correspondent hazard rate
- Starting Point: The price of an asset at time t must be the expected discounted value of its value at time $t+\Delta_t$ as displayed on Figure 3.1:

$$B^b_{t,T} = \frac{(1-p)\left\{E\left[B^b_{t+\Delta_t,T}\,|\mathscr{F}_t\right] - z_{t+\Delta_t}\right\} + p^l\phi^l_{t+\Delta_t} + p^c\phi^c_{t+\Delta_t}}{1+R\Delta_t}$$

$$B^b_{t,T} = \frac{(1-p)\left\{E\left[B^b_{t+\Delta_t,T}\,|\mathscr{F}_t\right] - \Theta_t\Delta_t D_t B^b_{t,T}\right\} + p^l\left(1-q^l\right)B^b_{t,T} + p^c\left(1-q^c\right)B^b_{t,T}}{1+R\Delta_t}$$

$$B^b_{t,T}\left\{1+R\Delta_t - p^l\left(1-q^l\right) - p^c\left(1-q^c\right) + (1-p)\,\Theta_t\Delta_t D_t\right\} = (1-p)\,E\left[B^b_{t+\Delta_t,T}\,|\mathscr{F}_t\right]$$

In t	$\xrightarrow{R}$	*In* $t+\Delta_t$		
		Event	**Probability**	**Value**
$B^b_{t,T}$	$\to$	No Event	$1-p^l-p^c+p^cp^l$	$E\left[B^b_{t+\Delta_t,T}\,\middle\vert\,\mathscr{F}_t\right]-z_{t+\Delta_t}$
	$\to$	Credit	$p^c-p^cp^l$	$\phi^l_{t+\Delta_t}$
	$\to$	Liquidity	$p^l-p^cp^l$	$\phi^c_{t+\Delta_t}$
	$\to$	Credit & Liquidity	p^cp^l	$\phi^c_{t+\Delta_t}+\phi^c_{t+\Delta_t}$

Figure 3.1: Discrete Time Valuation

$$
\begin{aligned}
\frac{E\left[B^b_{t+\Delta_t,T}\,\middle\vert\,\mathscr{F}_t\right]}{B^b_{t,T}} &= \frac{1+R\Delta_t-p^l\left(1-q^l\right)-p^c\left(1-q^c\right)-(1-p)\,\Theta_t\Delta_tD_t}{1-p} \\
&= \frac{1+R\Delta_t-p^l+p^lq^l-p^c+p^cq^c+p^lp^c-p^lp^c+(1-p)\,\Theta_t\Delta_tD_t}{1-p} \\
&= \frac{R\Delta_t+p^lq^l+p^cq^c-p^lp^c}{1-p}\Delta_t\cdot H\frac{1-p}{p}+1+\Theta_t\Delta_tD_t \\
&= \frac{R\Delta_t+p^lq^l+p^cq^c-p^lp^c}{p}\Delta_t\cdot H+1+\Theta_t\Delta_tD_t \\
&= R\Delta_t\frac{\Delta_t\cdot H}{p}+\frac{p^lq^l+p^cq^c-p^lp^c}{p}\Delta_t\cdot H+1+\Theta_t\Delta_tD_t \\
&= R\Delta_t\left(1+\Delta_tH\right)+\left\{p^lq^l+p^cq^c-p^lp^c\right\}\frac{1}{1-p}+1+\Theta_t\Delta_tD_t \\
&= R\Delta_t+Rt\left(\Delta_t\right)^2H+\left\{p^cq^c\frac{1-p^c}{p^c}\Delta_tH^c+p^lq^l\frac{1-p^l}{p^l}\Delta_tH^l-p^lp^c\right\}\frac{1}{1-p} \\
&+ 1+\Theta_t\Delta_tD_t \\
&= \left(R+q^cH^c\frac{1-p^c}{1-p}+q^lH^l\frac{1-p^l}{1-p}+\Theta_tD_t\right)\Delta_t+Rt\left(\Delta_t\right)^2H-\frac{p^lp^c}{1-p}+1
\end{aligned}
$$

and then

$$\begin{aligned}E\left[\frac{B^b_{t+\Delta_t,T}-B^b_{t,T}}{B^b_{t,T}}|\mathscr{F}_t\right] &= \left(R+q^cH^c\frac{1-p^c}{1-p}+q^lH^l\frac{1-p^l}{1-p}+\Theta_tD_t\right)\Delta_t\\ &+ Rt\left(\Delta_t\right)^2H-\frac{p^lp^c}{1-p}+1-1\end{aligned}$$

In the limit when $\Delta_t \to 0$, $p^c_{t,t} = p^l_{t,t} = 0$, ignoring the $(\Delta_t)^2$ terms and replacing discrete variables with their continuous equivalents[2]

$$E\left[\frac{dB^b_{u,T}}{B^b_{u,T}}|\mathscr{F}_t\right] = \left(r_u+q^c_u\lambda^c_u+q^l_u\lambda^l_u+\theta_ud_u\right)du$$

This leads to the conjecture that $B^b_{t,T}$ is reached by discounting the expected price in survival $B^b_{t+\Delta_t,T}$ with the $\left(r_u+q^c\lambda^c+q^l\lambda^l+\theta_ud_u\right)$ continuous rate. After adjusting the notation of $q^c\lambda^c$ and $q^l\lambda^l$

$$B^b_{t,T} = E\left[\exp\left(-\int_t^T r_u+\lambda^{ca}_u+\lambda^{la}_u+\theta_ud_udu\right)|\mathscr{G}_t\right] \tag{3.5}$$

Then, applying the results obtained in Credit, Eq 2.3, and Liquidity section we obtain $B^b_{t,T}$ conditional on η^c_t and η^l_t:

$$\begin{aligned}B^b_{t,T} &= \sum_{j=1}^{K^c-1}-\alpha^c_{\eta^c_t,j}E\left[\exp\left(-\int_t^T r_u-\mu^c_j\left(\mathbf{X_u}\right)+\lambda^{la}_u+\theta_ud_udu\right)|\mathscr{G}_t\right]\\ &= \sum_{jj=1}^{K^l-1}\sum_{j=1}^{K^c-1}\alpha^c_{\eta^c_t,j}\alpha^l_{\eta^l_t,jj}E\left[\exp\left(-\int_t^T r_u-\mu^c_j\left(\mathbf{X_u}\right)-\mu^l_{jj}\left(\mathbf{X_u}\right)+\theta_ud_udu\right)|\mathscr{G}_t\right]\end{aligned} \tag{3.6}$$

As previously done the analogy into the forward start rating contingent claim follows. After Eq. 3.5,

$$B^b_{t,f,T} = E\left[\exp\left(-\int_f^T r_u+\lambda^{ca}_u+\lambda^{la}_u+\theta_ud_udu\right)|\mathscr{G}_t\right] \tag{3.7}$$

After Eq. 3.6, $B^b_{t,f,T} =$

$$\sum_{jj=1}^{K^l-1}\sum_{j=1}^{K^c-1}\alpha^c_{\eta^c_t,j}\alpha^l_{\eta^l_t,jj}E\left[\exp\left(-\int_f^T r_u-\mu^c_j\left(\mathbf{X_u}\right)-\mu^l_{jj}\left(\mathbf{X_u}\right)+\theta_ud_udu\right)|\mathscr{G}_t\right] \tag{3.8}$$

[2] $R \to r_u$, $H^x \to \lambda^x$, $\Theta \to \theta_u$ and $D_u \to d_u$

3.3 Independence

3.3.1 Discussion

We will analyze the six different relation that may be established between the factors.

	Interest Rate	Credit	Liquidity	ERP
Credit	X		X	X
Liquidity	X			X
ERP	X			

Table 3.1: Independence Analysis

Credit As already stated the correlation between business cycles and credit variables, both intensity and recovery, is quite obvious and if we add that central banks behave countercyclicy while setting interest rates the correlation between intensities and risk free rates follows immediately.

Against this argument:

- central banks do not respond to banking solvency issues with rate policy;
- we shall focus on short term high quality credit, less vulnerable to business cycles evolution
- the main movements we would like to capture are general collapses on banking industry confidence and those are rapid movements with low correlation to risk free rates (except for flight to quality and liquidity phenomena)
- for long lasting phenomena like Japan's TIBOR, its reversion is not function of the main interest rates and its impact on the business cycle, and so the rates, is quite differed

One other source of correlation is the ERP factor: a poor condition on the credit side means that banks do not have the appropriate capital which would push them to develereging in the End of the Reporting Period.

We do not find any particular reason for a correlation with liquidity factors.

Liquidity There seems to exist independence in what concerns the relation between factors behind risk free rates and liquidity variables. There are, however, two question that should be addressed.

Let's first consider central banks' reaction to generalized liquidity problems. While trying to mitigate agent's costs of an elevated liquidity cost one of the available instruments is the monetary policy and so a generalized liquidity uncertainty could trigger a lower policy rate.

Nevertheless, the policy rate is not the correct instrument to deal with a liquidity crisis and authorities should avoid using it[3].

The second issue is the reaction of BOR fixings after the central bank cutting rates: they may stand still for technical factors, making all the cut to be compensated by liquidity premium, as the former is estimated as a residual[4].

We do not find any particular reason for a correlation with ERP.

ERP In what concerns independence there is no apparent relation of the ERP phenomena with the risk free rate, however there exist chances for a big correlation which depends on the way we measure the risk free curve.

If we assume that the treasury curve is the benchmark for the risk free curve then the correlation may be huge, as banks substitute bank deposits for short term treasury bills, which consume zero capital, and respective yields completely diverge.

If however our risk free benchmark is the OIS curve the problem may be reduced, but it will still persist as one of the fixings, the last business day of the year for instance, will be affected by the same phenomena!

In the first case we have a negative correlation, in the second a positive one. And on the later we still be left thinking if the ERP effect should not be entirely reflected on the ON curve!

We will choose the ON curve as the treasury curve is hugely disruptive for its liquidity driven movements[5] and since the positive correlation does not occur in practice. The answer to this apparently incoherent pricing of liquidity on the last business day of a period is probably related to the liquidity flooding which most central banks do in this traditionally critic periods of the calendar. In proceeding this way the demand for funds in the interbank market drops making the transactions to settle at lower rates.

---

[3]However a high liquidity premium, signaling the freeze of money markets, can lead to the collapse of real economy which can later trigger the downward move of monetary policy.

[4]This argument can be applied to any illiquid instrument as the markets risk free curve moves almost continuously and the former instruments more exporadicly. As we are "artificially" splitting the yield of a risky instrument into risk-free component and a credit spread the different timings between prices can easily produce correlation in the extracted spread and risk free rates.

[5]Schonbucher (2003) on Section 3.7 refers some evidence from Houweling and Vorst (2001) that repo and swap rates outperform government rates as risk-free reference curve.

3.4 Modeling

3.4.1 Independence

Assuming independence makes estimation much easier than not doing so as it enables us to estimate each factor independently from others, which significantly decreases the complexity of the model. Starting from Eq. 3.5:

$$\begin{aligned} B^b_{t,T} &= E\left[\exp\left(-\int_t^T r\left(\mathbf{X_u}\right) + \lambda^{ca}\left(\mathbf{X_u}\right) + \lambda^{la}\left(\mathbf{X_u}\right) + \theta\left(\mathbf{X_u}\right) d_u du\right) | \mathscr{G}_t\right] \\ &= E\left[\beta^{r(\mathbf{X_u})}_{t,T}\beta^{\lambda^{ca}(\mathbf{X_u})}_{t,T}\beta^{\lambda^{la}(\mathbf{X_u})}_{t,T}\beta^{\theta(\mathbf{X_u})d_u}_{t,T} | \mathscr{G}_t\right] \end{aligned}$$

under independence

$$\mathbf{X_u} = \left[\mathbf{X^r_u}, \mathbf{X^c_u}, \mathbf{X^l_u}, \mathbf{X^z_u}\right]$$

$$\begin{aligned} B^b_{t,T} &= E\left[\beta^{r(\mathbf{X^r_u})}_{t,T} | \mathscr{G}_t\right] E\left[\beta^{\lambda^{ca}(\mathbf{X^c_u})}_{t,T} | \mathscr{G}_t\right] E\left[\beta^{\lambda^{la}(\mathbf{X^l_u})}_{t,T} | \mathscr{G}_t\right] E\left[\beta^{\theta(\mathbf{X^z_u})d_u}_{t,T} | \mathscr{G}_t\right] \\ &= B_{t,T}\left(\mathbf{X^r_t}\right) B^c_{t,T}\left(\mathbf{X^c_u}\right) B^l_{t,T}\left(\mathbf{X^l_u}\right) B^z_{t,T}\left(\mathbf{X^z_u}\right) \end{aligned} \quad (3.9)$$

followed by the simple spreads equation after using the relation $y^x_{t,T} = \frac{\left(-\log B^x_{t,T}\right)}{c_{t,T}}$

$$\begin{aligned} B^b_{t,T} &= \exp\left(-c_{t,T} y^r_{t,T}\right) \exp\left(-c_{t,T} y^c_{t,T}\right) \exp\left(-c_{t,T} y^l_{t,T}\right) \exp\left(-c_{t,T} y^z_{t,T}\right) \\ &= \exp\left(-c_{t,T}\left[y^r_{t,T} + y^c_{t,T} + y^l_{t,T} + y^z_{t,T}\right]\right) \end{aligned}$$

3.4.2 Dependence

About the dependence analysis we shall again stress that we will only treat dependence in vector $\mathbf{X_u}$, we will not study dependence of Y^x_u Cox Processes, and that thera are alternative ways of expressing dependence.

The $\mathbf{X_u}$ vector will be composed by the following elements

- $\mathbf{X^{i_1}_u}$, $\mathbf{X^{i_2}_u}$,..., $\mathbf{X^{i_{n_i}}_u}$ for modeling the i^{th} component, $i = r, c, l, z$
- $\mathbf{X^j_u}$ for modeling the covariation components: $j = 1, ..., J$

As in Eq. 3.6

$$B^b_{t,T} = \sum_{jj=1}^{K^l-1}\sum_{j=1}^{K^c-1} \alpha^c_{\eta^c_t,j}\alpha^l_{\eta^l_t,jj} E\left[\exp\left(-\int_t^T r_u - \mu^c_j\left(\mathbf{X_u}\right) - \mu^l_{jj}\left(\mathbf{X_u}\right) + \theta_u d_u du\right) | \mathscr{G}_t\right]$$

where

- $r_u = \sum_i^{n_r} X_u^{r_i} + \sum_j^{n_r} a_{r,j} X_u^j$
- $\mu_u^c(\mathbf{X_u}) = \sum_i^{n_c} X_u^{c_i} + \sum_j^J a_{c,j} X_u^j$
- $\mu_t^l(\mathbf{X_u}) = \sum_i^{n_l} X_u^{l_i} + \sum_j^J a_{l,j} X_u^j$
- $\theta_t = \sum_i^{n_z} X_u^{z_i} + \sum_j^{n_z} a_{z,j} X_u^j$

and $c_{x,j}$ is a parameter.

Chapter 4

Numeric Examples

4.1 Introduction

In this chapter we will present four examples of past structures of the money market, most of which the current theory in unable to deal with, and show how our proposal handles such reality. We aim to show how our model dynamics compare with market ones.

We shall also emphasize that, by default, model parameters will be chosen without any criteria beyond author's judgment.

We will use Eq. 3.7 as a starting point and evolve from there:

$$B^b_{t,f,T} = E\left[\exp\left(-\int_f^T r\left(\mathbf{X_u}\right) + \lambda^{ca}\left(\mathbf{X_u}\right) + \lambda^{la}\left(\mathbf{X_u}\right) + d_u\theta\left(\mathbf{X_u}\right)\,\mathrm{d}u\right)|\mathscr{G}_t\right] \quad (4.1)$$

Then we assume independence between the four factors in order to make the exemplification simpler. We remind that the assumption of Dependence or Independence may change the difficulty in optimizing the parameters of Eq. 4.1 but do not change the conceptual framework in which we are working. We will further assume that each factor is function of just one ATS state variable.

As such Eq. 4.1 is

$$\begin{aligned}
B^b_{t,f,T} &= E\left[\exp\left(-\int_f^T r\left(X^r_u\right) + \lambda^{ca}\left(X^c_u\right) + \lambda^{la}\left(X^l_u\right) + d_u\theta\left(X^z_u\right)\,\mathrm{d}u\right)|\mathscr{G}_t\right] \\
&= E\left[\exp\left(-\int_f^T r\left(X^r_u\right)\mathrm{d}u\right)|\mathscr{G}_t\right] E\left[\exp\left(-\int_f^T \lambda^{ca}\left(X^c_u\right)\mathrm{d}u\right)|\mathscr{G}_t\right] \\
&\quad E\left[\exp\left(-\int_f^T \lambda^{la}\left(X^l_u\right)\mathrm{d}u\right)|\mathscr{G}_t\right] E\left[\exp\left(-\int_f^T d_u\theta\left(X^z_u\right)\,\mathrm{d}u\right)|\mathscr{G}_t\right] \\
&= B^{r(X^r_u)}_{t,f,T} B^{\lambda^{ca}(X^c_u)}_{t,f,T} B^{\lambda^{la}\left(X^l_u\right)}_{t,f,T} B^{d_u\theta(X^z_u)}_{t,f,T} \\
&= B^r_{t,f,T} B^c_{t,f,T} B^l_{t,f,T} B^z_{t,f,T} \qquad (4.2)
\end{aligned}$$

where

$$dX^i_t = \varepsilon_{Ai}\left(\varepsilon_{Bi} - X^i_t\right) dt + \sqrt{\varepsilon_{Ci} + \varepsilon_{Di} X^i_t} dW^i_t$$

Then, developing Eq. 4.2 using the relation $B^x_{t,f,T} = \exp\left(-c_{t,f,T} \cdot y^x_{t,f,T}\right)$, CC spread additivity follows easily:

$$\begin{aligned}
B^b_{t,f,T} &= B^{r(X^r_u)}_{t,f,T} B^{\lambda^{ca}(X^c_u)}_{t,f,T} B^{\lambda^{la}\left(X^l_u\right)}_{t,f,T} B^{d_u\theta(X^z_u)}_{t,f,T} \\
\exp\left(-c_{t,f,T} \cdot y^b_{t,f,T}\right) &= \exp\left(-c_{t,f,T} \cdot y^r_{t,f,T}\right) \exp\left(-c_{t,f,T} \cdot y^c_{t,f,T}\right) \exp\left(-c_{t,f,T} \cdot y^l_{t,f,T}\right) \\
&\quad \exp\left(-c_{t,f,T} \cdot y^z_{t,f,T}\right) \\
\exp\left(-c_{t,f,T} \cdot y^b_{t,f,T}\right) &= \exp\left(-c_{t,f,T} \cdot y^r_{t,f,T} - c_{t,f,T} \cdot y^c_{t,f,T} - c_{t,f,T} \cdot y^l_{t,f,T} - c_{t,f,T} \cdot y^z_{t,f,T}\right) \\
y^b_{t,f,T} &= y^r_{t,f,T} + y^c_{t,f,T} + y^l_{t,f,T} + y^z_{t,f,T}
\end{aligned}$$

where $c_{t,f,T} = \frac{T-f}{360}$

4.2 Market Data

We have chosen to study the tenors 1M, 3M, 6M and 12M from the USD LIBOR curve. The USD LIBOR is fixed at 11:00, London Time, every day while there is no fixing neither for the LIBOR FRA OIS curve neither for the CDS one. For those reasons we will have to take extra care in the process of collecting data in order to avoid comparing data reflecting different market environments.

4.3 Method

4.3.1 Capture the Spot and Forward BOR curve

Forward Period	0	1	2	3	4	5	6	9	12	15	18	21
1M USD LIBOR	x	x	x	x	x							
3M USD LIBOR	x	x	x	x	x	x	x	x	x	x	x	x
6M USD LIBOR	x					x	x	x		x		
12M USD LIBOR	x		x	x			x	x				

Table 4.1: LIBOR Data

We will capture the DC yields[1], $v^b_{t,f,f+\Delta^i_f}$, displayed on Table 4.1 and from that data we will calculate the implied CC yields, $y^b_{t,f,f+\Delta^i_f}$ based on the following formula:

$$\exp\left(c^i_f \cdot y^b_{t,f,f+\Delta^i_f}\right) = \frac{1}{1 + c^i_f \cdot v^b_{t,f,f+\Delta^i_f}}$$

where

$c^i_t = \frac{\Delta^i_t}{360}, \quad i = 1\text{M}, 3\text{M}, 6\text{M}, 12\text{M}$

Δ^i_t is the number of days for a i maturity money market in t.

4.3.2 Extracting the Risk Free curve

We will use the OIS curve as our proxy for the risk free curve for the reasons previously outlined on section 3.3. The OIS curve will be captured trough the following spot contributions: 1M, 2M, 3M, 4M, 5M, 6M, 9M, 12M, 15M, 18M, 24M. Based on those contributions a risk free zero coupon curve B^r is build.

Model

We will not attempt to model the risk free dynamics as it is not the purpose of this work and consequently we will just calculate the $y^r_{t,f,f+\Delta^i_f}$ implied on the risk free zero coupon, $B^r_{t,f,f+\Delta^i_f}$, which is calculated using a linear interpolation on the yields.

$$\exp\left(c^i_f \cdot y^r_{t,f,f+\Delta^i_f}\right) = B^r_{t,f,f+\Delta^i_f}$$

[1] When the forward period is 0 it refers to the daily official fixing, whether the others refers to the market quoted FRAs.

Then we will show a graph with the spread LIBOR-OIS:

$$y^{b-r}_{t,f,f+\Delta^i_f} = y^{b}_{t,f,f+\Delta^i_f} - y^{r}_{t,f,f+\Delta^i_f}$$

4.3.3 Extracting the Credit curve

Model

Following Eq. C.9, with $K^c = 3$ and $\eta^c_f = 1$, we have

$$B^c_{t,f^1,T} = \sum_{j=1}^{2} -\alpha^c_{1,j} E\left[\exp\left(-\int_f^T -\mu^c_j\left(X^c_u\right)du\right)|\mathscr{G}_t\right] \tag{4.3}$$

The credit matrix we use follows the one presented in Lando (1998). From that matrix we extracted the classes 1 (Low Credit Event Probability), 2 (High Credit Event Probability) and 3 (Credit Event already occurred), and obtained the base generator matrix A^c.

$$A^c = \begin{bmatrix} -0.0165 & 0.0155 & 0.001 \\ 0.0128 & -0.0881 & 0.0753 \\ 0 & 0 & 0 \end{bmatrix}$$

Following Section C.6 we set $\mu^c_j\left(X^c_u\right) = \xi^c_j X^c_u$ where ξ^c_1, ξ^c_2 are A^c eigenvalues and Eq. 4.3 becomes

$$B^c_{t,f^1,T} = -\alpha^c_{1,1} E\left[\exp\left(-\int_f^T -\xi^c_1 X^c_u du\right)|\mathscr{G}_t\right] - \alpha^c_{1,2} E\left[\exp\left(-\int_f^T -\xi^c_2 X^c_u du\right)|\mathscr{G}_t\right] \tag{4.4}$$

where $\alpha^c_{1,1}$ and $\alpha^c_{1,2}$ are obtained from A^c eigenvectors following Annex C.

$$\alpha^c = \begin{bmatrix} -1.667 & 0.667 \\ -0.2011 & -0.7989 \end{bmatrix} \quad \xi^c = \begin{bmatrix} -0.0138 & -0.0908 \end{bmatrix}$$

Finally, applying Eq. E.3

$$\begin{aligned} B^c_{t,f^i,T} &= -\alpha^c_{1,1}\exp\left(\mathcal{A}^c_{t,T} - \mathcal{A}^c_{t,f} - \mathcal{B}^c_{t,T}X^{c1}_t + \mathcal{B}^c_{t,f}X^{c1}_t\right) + \\ &+ -\alpha^c_{1,2}\exp\left(\mathcal{A}^c_{t,T} - \mathcal{A}^c_{t,f} - \mathcal{B}^c_{t,T}X^{c2}_t + \mathcal{B}^c_{t,f}X^{c2}_t\right) \end{aligned} \tag{4.5}$$

In relation to the X^c_u ATS model we will only define three parameters

$$\varepsilon^c_A = 0.9 \quad \varepsilon^c_C = 0.05^2 \quad \varepsilon^c_D = 0$$

and use the ε^c_B to adjust to the data in each moment. Despite not being a correct procedure, since we would like to present a stable framework valid to all moments, we will use this simplification to avoid introducing more stochastic variables which would reach the same result preserving the framework.

After having calculated the parameter ε_B^c and the initial value of the state variable, X_t^c, we calculate $B^c_{t,f,f+\Delta_f^i}$ following Eq. 4.5 and then $y^c_{t,f,f+\Delta_f^i}$ for every LIBOR in our data:

$$\exp\left(c_f^i \cdot y^c_{t,f,f+\Delta_f^i}\right) = B^c_{t,f,f+\Delta_f^i}$$

Then we will show a graph with the spread LIBOR-OIS-CREDIT:

$$y^{b-r-c}_{t,f,f+\Delta_f^i} = y^b_{t,f,f+\Delta_f^i} - y^r_{t,f,f+\Delta_f^i} - y^c_{t,f,f+\Delta_f^i}$$

Market Data

The LIBOR credit curve is the most non-standard element of data we have to gather from market as it is, in our model, the only element specific to the LIBOR contributor. In order to do it properly we would have to compare each counterpart credit curve with its contribution. As the purpose of this chapter is simply to illustrate the mechanics of the model we will take a short cut based on two assumptions:

- the credit curve is flat for each tenor (from the observation that in all the considered period the credit curve for the generality of the contributor banks, namely the median, kept a flat profile): $X_t^c = \varepsilon_B^c$
- the credit spread of the 12M tenor is the LIBOR-OIS spread of the longest forward in the sample, $y^b_{t,t+\Delta_t^i} - y^r_{t,t+\Delta_t^i}$. We are assuming that both the ERP and the liquidity component are poorly valued by the market for longer maturities and that only the credit component appears: $B^{b-r}_{t,t+1,t+2} = B^c_{t,t+1,t+2}$. Following Eq. 4.5

$$\begin{aligned} B^c_{t,(t+1)^i,t+2} &= -\alpha^c_{1,1}\exp\left(\mathcal{A}^c_{t,t+2} - \mathcal{A}^c_{t,t+1} - \mathcal{B}^c_{t,t+2}X_t^{c1} + \mathcal{B}^c_{t,t+1}X_t^{c1}\right) \\ &+ -\alpha^c_{1,2}\exp\left(\mathcal{A}^c_{t,t+2} - \mathcal{A}^c_{t,t+1} - \mathcal{B}^c_{t,t+2}X_t^{c2} + \mathcal{B}^c_{t,t+1}X_t^{c2}\right) \end{aligned}$$

  And, given $B^{b-r}_{t,t+1,t+2}$ solve for X_t^c.

4.3.4 Extracting ERP costs

In this factor we will apply the Eq. 2.22 multi-period model:

$$B^z_{t,f,T} = E\left[\exp\left(-\int_f^T d_u X_u^z du\right)\right]$$

where

$$d_u = \begin{cases} 1 & , u \in A^{EoY} \\ \frac{1}{2} & , u \in A^{EoS} \\ \frac{1}{4} & , u \in A^{EoQ} \\ 0 & \text{else} \end{cases}$$

The parameters of the ATS model will be

$$\varepsilon^z_A = 0.9 \quad \varepsilon^z_B = 0.04 \quad \varepsilon^z_C = 0.05^2 \quad \varepsilon^z_D = 0.$$

and the initial value of the state variable, X^z_t, will be sloppily adjusted to the data.

We will then calculate $y^z_{t,f,f+\Delta^i_f}$ for every LIBOR in our data:

$$\exp\left(c^i_f \cdot y^z_{t,f,f+\Delta^i_f}\right) = B^z_{t,f,f+\Delta^i_f}$$

Then we will show a graph with the spread LIBOR-OIS-CREDIT-ERP:

$$y^{b-r-c-z}_{t,f,f+\Delta^i_f} = y^b_{t,f,f+\Delta^i_f} - y^r_{t,f,f+\Delta^i_f} - y^c_{t,f,f+\Delta^i_f} - y^z_{t,f,f+\Delta^i_f}$$

4.3.5 Liquidity

Model

Following Eq. C.9, with $K^l = 3$ and $\eta^l_f = 1$, we have

$$B^l_{t,f^1,T} = \sum_{j=1}^{2} -\alpha^l_{1,j} E\left[\exp\left(-\int_f^T -\mu^c_j\left(X^l_u\right)du\right)|\mathscr{G}_t\right] \tag{4.6}$$

The liquidity matrix, A^l, we be slightly different from the credit one, A^c, in order to increase the liquidity spread of the tenors with higher duration. We did it imposing a lower probability of transition from class 1 to class 2 and a higher probability of a Liquidity event given that the agent is on class 2.

$$A^l = \begin{bmatrix} -0.071 & 0.07 & 0.001 \\ 0.01 & -0.2 & 0.19 \\ 0 & 0 & 0 \end{bmatrix}$$

Following Section C.6 we set $\mu^l_j\left(X^l_u\right) = \xi^l_j X^l_u$ where ξ^l_1, ξ^l_2 are A^l eigenvalues and Eq. 4.6 becomes

$$B^c_{t,f^1,T} = -\alpha^l_{1,1} E\left[\exp\left(-\int_f^T -\xi^l_1 X^l_u du\right)|\mathscr{G}_t\right] - \alpha^c_{1,2} E\left[\exp\left(-\int_f^T -\xi^l_2 X^c_u du\right)|\mathscr{G}_t\right] \tag{4.7}$$

where $\alpha^l_{1,1}$ and $\alpha^l_{1,2}$ are obtained from A^l eigenvectors following Annex C.

$$\alpha^l = \begin{bmatrix} -1.4646 & 0.4646 \\ -0.1091 & -0.8909 \end{bmatrix} \quad \xi^l = \begin{bmatrix} -0.0329 & -0.1026 \end{bmatrix}$$

In relation to the ATS model we will only define two parameters and use the ε^l_B and ε^l_A to adjust to the data in each moment.

The argument is the same as in the credit shortcut. Despite not being a correct procedure, since we would like to present a stable framework valid to all moments, we will use this simplification to avoid introducing more stochastic variables which would reach the same result preserving the framework:

$$\varepsilon^l_C = 0.05^2 \quad \varepsilon^l_D = 0$$

After having calculated the parameters ε^l_A and ε^l_B and the initial value of the state variable, X^l_t, we calculate $B^l_{t,f,f+\Delta^i_f}$ following Eq. 4.7 and then $y^l_{t,f,f+\Delta^i_f}$ for every LIBOR in our data:

$$\exp\left(c^i_f \cdot y^l_{t,f,f+\Delta^i_f}\right) = B^l_{t,f,f+\Delta^i_f}$$

Then we will show a graph with the spread LIBOR-OIS-CREDIT-ERP-LIQ:

$$y^{b-r-c-l-z}_{t,f,f+\Delta^i_f} = y^b_{t,f,f+\Delta^i_f} - y^r_{t,f,f+\Delta^i_f} - y^c_{t,f,f+\Delta^i_f} - y^z_{t,f,f+\Delta^i_f} - y^l_{t,f,f+\Delta^i_f}$$

Market Data

The only factor from which we do not hold any evidence is the liquidity and as such we will use the usual procedure: it will be our residual. So we will start from a base model leaving some degrees of freedom, ε^l_A, ε^l_B and X^l_t to adjust the model to the residual LIBOR-OIS-CREDIT-ERP.

We will sloppily adjust ε^l_A, ε^l_B and X^l_t to the data in order to get a final $y^{b-c-l-z}_{t,f,f+\Delta^i_f}$ which is zero in average and where the difference between the points in the curve is minimal and seemingly random.

4.4 Examples

4.4.1 Before August 2007

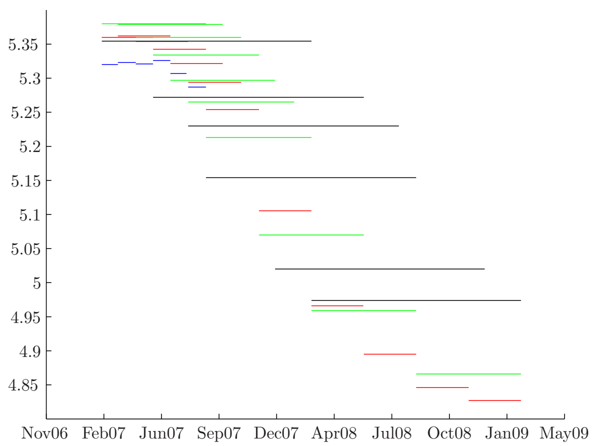
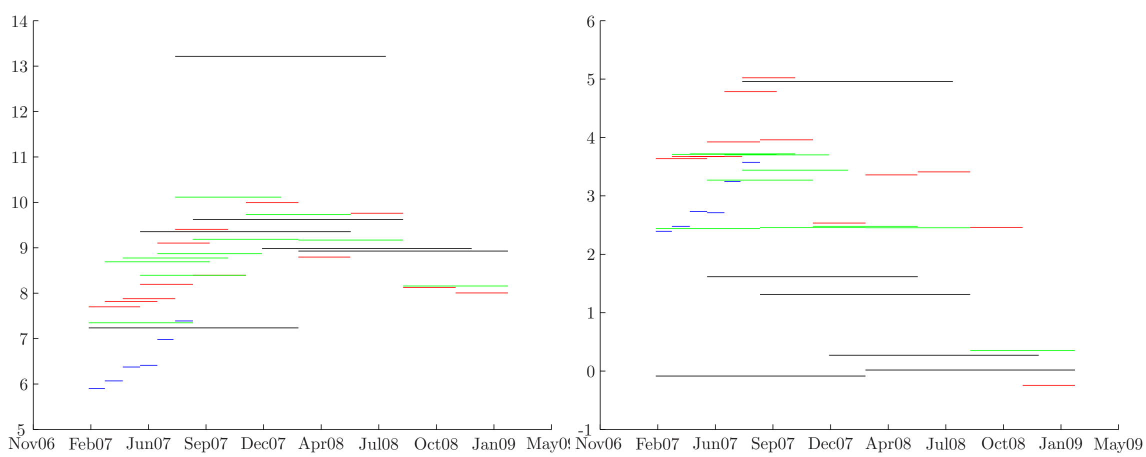


Figure 4.1: USD LIBOR 21Feb2007



(a) BOR-OIS (b) BOR-OIS-CREDIT-ERP-LIQ

Figure 4.2: USD LIBOR Decomposition 1 21Feb2007

	X_0^x	ε_A^x	ε_B^x
Credit	0.3	0.5	0.305
ERP	$\frac{200}{10000}$	0.5	$\frac{400}{10000}$
Liquidity	0.05	1	0.2

Table 4.2: AFS Parameters 21Feb2007

We have chosen to analyze the 21^{st} of February of 2007 with the structure displayed on Figure **??** using the method described on SubSection 4.3 and parameters from Table 4.2. As can be seen in Figure 4.2b final residual ranges between 0 and 5 which compares[2] with an initial residual between 6 and 13 as displayed on Figure 4.2a.

4.4.2 Right After August 2007

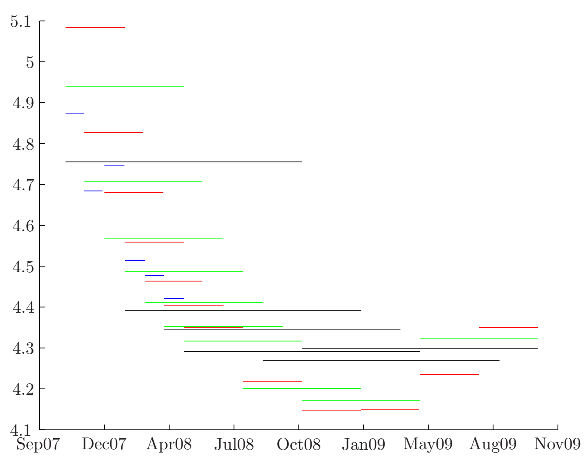
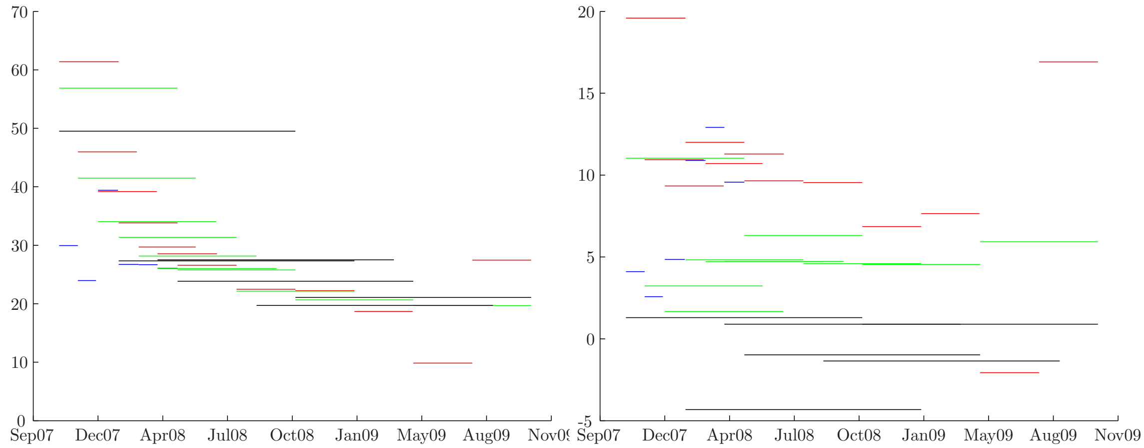


Figure 4.3: USD LIBOR 23Oct2007

[2]Intermediate Figures can be found on Section F.1



(a) BOR-OIS

(b) BOR-OIS-CREDIT-ERP-LIQ

Figure 4.4: USD LIBOR Decomposition 1 23Oct2007

	X_0^x	ε_A^x	ε_B^x
Credit	0.3	1	0.305
ERP	$\frac{510}{10000}$	0.5	$\frac{400}{10000}$
Liquidity	2.9	2.7	0.75

Table 4.3: AFS Parameters 23Oct2007

We have chosen to analyze the 23^{th} of October of 2007 with the structure displayed on Figure 4.3 using the method described on SubSection 4.3 and parameters from Table 4.3. As can be seen in Figure 4.4b final residual ranges between -5 and 20 which compares[3] with an initial residual between 60 and 10 as displayed on Figure 4.4a.

[3]Intermediate Figures can be found on Section F.2

4.4.3 Right After Lehman

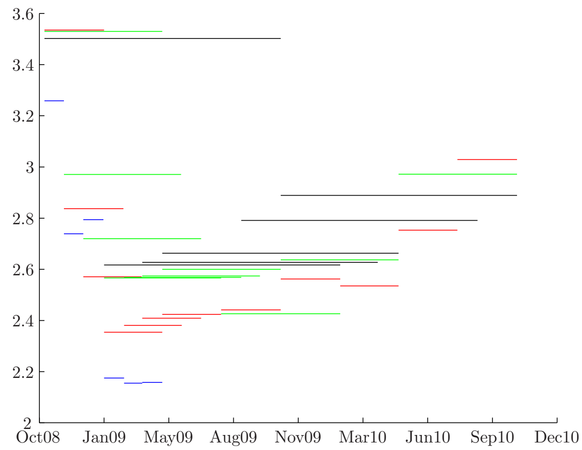
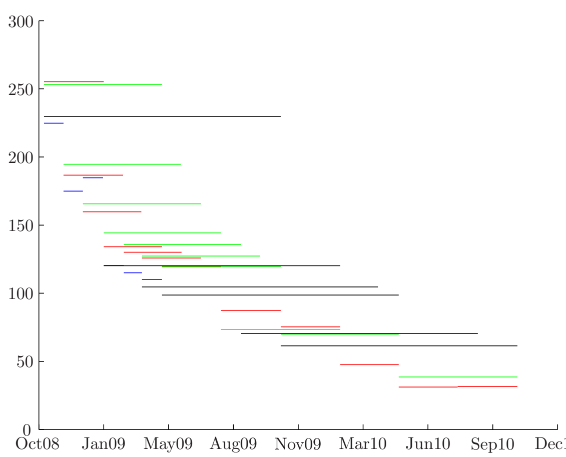
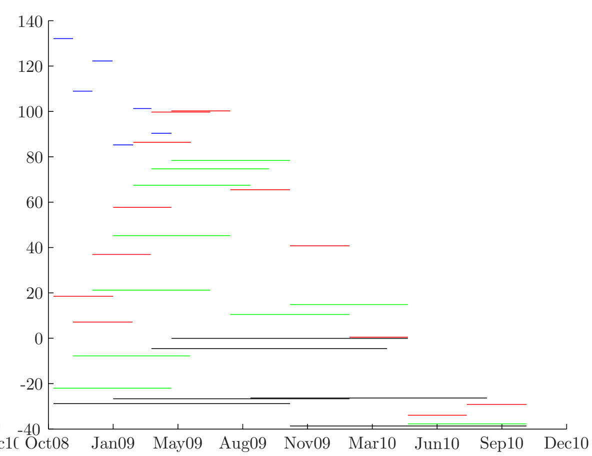


Figure 4.5: USD LIBOR 23Oct2008



(a) BOR-OIS



(b) BOR-OIS-CREDIT-ERP-LIQ

Figure 4.6: USD LIBOR Decomposition 1 23Oct2008

	X_0^x	ε_A^x	ε_B^x
Credit	1	0.5	1.005
ERP	$\frac{900}{10000}$	0.5	$\frac{400}{10000}$
Liquidity	9	3	0.75

Table 4.4: AFS Parameters 23Oct2008

We have chosen to analyze the 23^{th} of October of 2008 with the structure displayed on Figure 4.5 using the method described on SubSection 4.3 and parameters from Table 4.4. As can be seen in Figure 4.6b final residual ranges between -20 and 130 which compares[4] with an initial residual between 30 and 260 as displayed on Figure 4.6a.

4.4.4 Beginning 2009

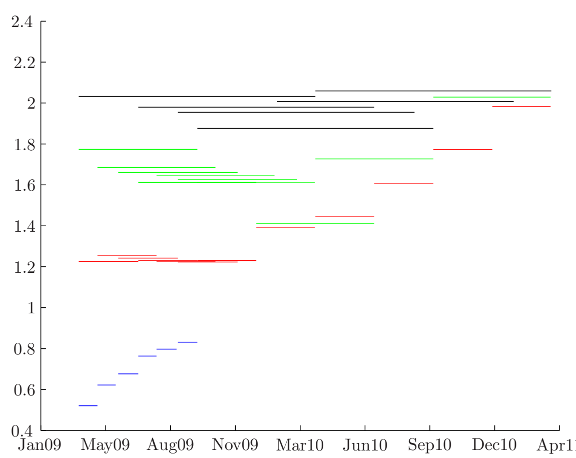
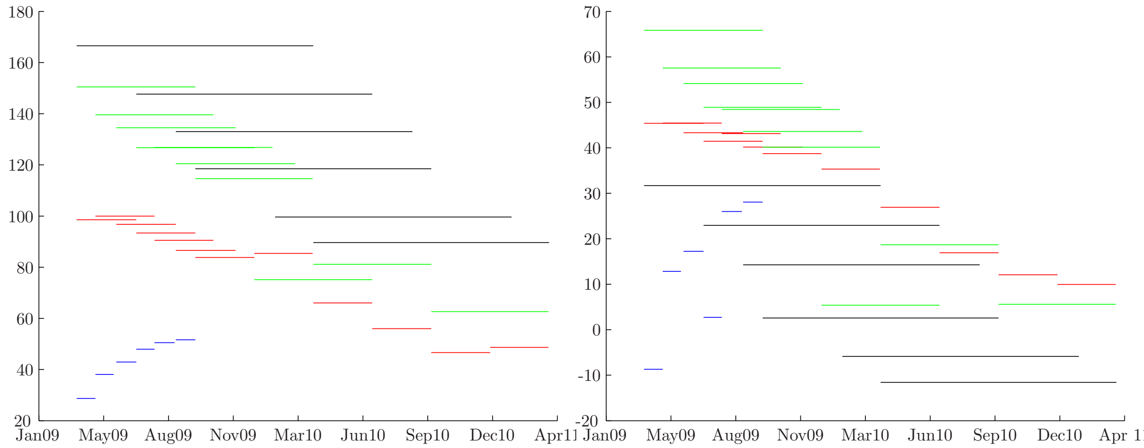


Figure 4.7: USD LIBOR 24Mar2009

[4]Intermediate Figures can be found on Section F.3



(a) BOR-OIS

(b) BOR-OIS-CREDIT-ERP-LIQ

Figure 4.8: USD LIBOR Decomposition 1 24Mar2009

	X_0^x	ε_A^x	ε_B^x
Credit	1.215	0.5	1.22
ERP	$\frac{1300}{10000}$	0.5	$\frac{400}{10000}$
Liquidity	2.5	0.3	0.15

Table 4.5: AFS Parameters 24Mar2009

We have chosen to analyze the 24^{th} of March of 2009 with the structure displayed on Figure 1.1 using the method described on SubSection 4.3 and parameters from Table 4.5. As can be seen in Figure 4.8b final residual ranges between -10 and 70 which compares[5] with an initial residual between 30 and 170 as displayed on Figure **??**.

4.4.5 Summer 2009

We have chosen to analyze the 23^{th} of July of 2009.

[5]Intermediate Figures can be found on Section F.4

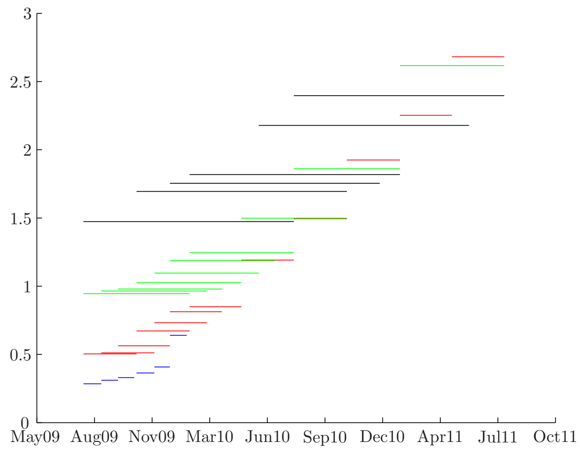
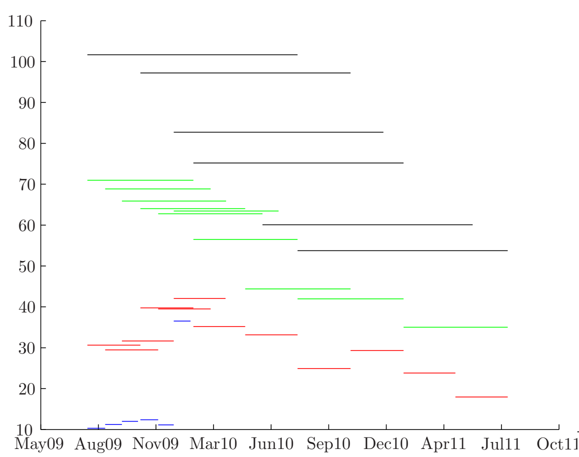
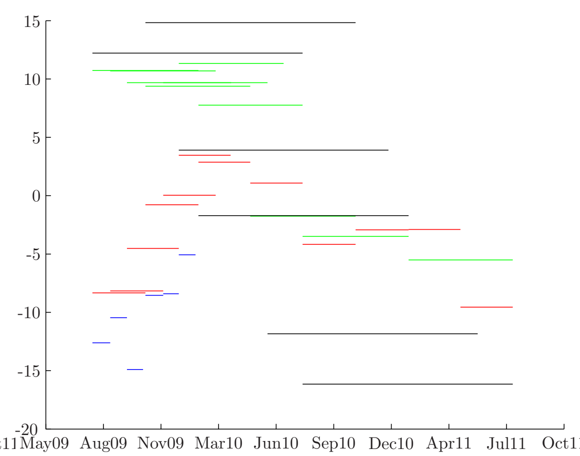


Figure 4.9: USD LIBOR 23Jul2009



(a) BOR-OIS



(b) BOR-OIS-CREDIT-ERP-LIQ

Figure 4.10: USD LIBOR Decomposition 1 23Jul2009

	X_0^x	ε_A^x	ε_B^x
Credit	0.3	1	0.305
ERP	$\frac{510}{10000}$	0.5	$\frac{400}{10000}$
Liquidity	2.9	2.7	0.75

Table 4.6: AFS Parameters 23Jul2009

We have chosen to analyze the 23^{th} of July of 2009 with the structure displayed on Figure 4.9 using the method described on SubSection 4.3 and parameters from Table 4.6. As can be seen in Figure 4.10b final residual ranges between 10 and 105 which compares[6] with an initial residual between -25 and 10 as displayed on Figure 4.10a.

4.5 Conclusion

In this small exercise, where we managed to deliver some results, we went through a lot of problems related to data collection.

In first place we had the difference in the hours at which each set of data refers to, as it is hard to control the capture of the OIS and FRA curve.

Then we had the illiquidity of some products as long maturity OIS and some less traded FRA like some from the 12 months curve. Some of examples of these problems were: understanding which bloomberg contributor quoted what and when, knowing where the market is l with high bid-offer spreads, believing in the actuality of some quotes which probably were not actualized at the same time the most liquid instruments were.

One other problem we had is related with the interpolation of the OIS curve which we had to produce under dramatic changes in the steepness along the yield curve. The result of this fact was that some y^{b-r} spreads were overstated and others understated.

All these problems added to the simplicity of the implementation, non-optimization and eventual structural problems in the model contributed to the final result of two major problems and one fine achievement.

The first problem was the apparent incompatibility between our residual spreads, $y^{b-r-c-z}$, and the liquidity model: in order to correct some differences we would exaggerate other differences in the other extremity of the curve. The second problem concerns some apparent autocorrelation of the residuals of the same tenor.

About the successes, we managed to reduce to almost zero the average residual (which is not a success as we target the zero average residual in the sloppily process of adjusting parameters to the data) and also, and this is an achievement, to reduce substantially the dispersion of the residuals.

[6] Intermediate Figures can be found on Section F.5

Chapter 5

Q & A

We we posed a couple of questions in Section 1.1. Based on the reflection and results of this work we tried to answer those questions:

- Q: at what rate will lenders be able to borrow in the future?

  A: in the Money Market, at a level consistent with its credit profile and market appetite for capital and liquidity

- Q: how can Monetary Authorities start to be effective again? What problems to address in order to give traction to traditional tools?

  A: Depending on the source of the problem the approach must be different:

  - liquidity problem: when impacting the BOR market is always a global problem and can be addressed by increasing the central bank liquidity available[1].
  - credit problem: central banks should not address individual credit problems as they are not prepared for that; cheap credit to finance high yielding assets is a highly opaque form of restoring bank balance-sheets. The generalized increase on banks spreads must be addressed with a framework which provides market participants information on banks risks and solvency[2], and a transparent way to deal with recapitalization and default.

- Q; which factor impacts each curve?

  A: Small tenors are most likely only affected in extreme scenarios by the own dynamics which the transition matrix generates and by the observed patterns. In a liquidity and/or credit crisis the most affected tenors are the longer ones.

[1] By acting in this way central banks will also decrease the risk of insolvency by illiquidity and thereby decrease the credit factor.

[2] So that only the impaired ones suffer higher spreads

- Q: how to hedge risks?

  A: Interest Rate risk can be best hedged with an OIS as is available for all maturities and it is not subject to liquidity issues as the government curve is neither interbank problems as a traditional BOR swap is. Global liquidity issues may be hedged with the BOR-OIS spread. Credit risk in the financial sector should be hedge with appropriate instruments and not with BOR-OIS spread, since the longer forwards almost do not react to credit crisis[3]

- Q: what is the dynamic of BOR?

  A: The dynamics of BOR are certainly connected to the dynamics of liquidity, credit and capital tensions on the financial sector. We believe that the proposed framework is a first step in order to understand them, but other factors may contribute to it.

[3] Due to the refreshing property of the BOR credit risk

Chapter 6

Conclusion

The Crisis We started our work with a dysfunctional BOR curve which reflected a new reality displaying old features.

The new problem was a frozen money market and the old features where the pricing of credit risk (of the borrower), liquidity risk (of the lender) and the pricing of capital requisites and leverage (of the borrower). All these factors importance changed throughout the time.

In the beginning of the crisis the main problem was liquidity. First, investors got scared with the Asset Backed CP from CIVs one of the biggest sources of money market funding. Then other agents, which usually trusted the money market to fund themselves, started to tap liquidity facilities of banks as the money markets started to be of more difficult access. This taps and difficulties on the money market affected severely the liquidity of the banks.

The liquidity problem was addressed by central banks which assured liquidity and BORs started to normalize from December 2007 onwards.

After Lehman collapse the problem was of solvency. Besides uncovering a balance-sheet worse than expected and an unwilling government to support the entire banking system the Lehman default trigged huge asset prices moves which impacted very negatively banks balance-sheets. This time money markets, as other markets, completely shut down: only the ON was traded. Naturally the liquidity problem appeared again on BOR pricing. It was almost impossible to find term liquidity outside the central banks.

The liquidity problem was again addressed by central banks which gave banks access to almost unlimited amounts of liquidity. This time the central banks where helped by governments which guaranteed banks newly issued debt which enabled liquidity flowing between banks again. Even the FED and ECB helped the issue of long term liquidity: the ECB with the Covered Bond program and the FED with the MBS program.

The solvency problem started to ease in March 2009, after the bottom on the equity market. The equity rally was followed by spreads tightening which came from improved economic outlook and risk appetite and the pressure of the wall of liquidity circulating in the market.

Throughout all this period the scarcity of capital and excess of leverage made the banks more reluctant in having credits on reporting dates which made the premium of such positions increase comparatively to previous periods.

All the aforementioned events impacted both BOR fixings and BOR forward curve in a specific way depending on the nature of the issues raised at each moment.

In our model we tried to provide a starting point for a further discussion on the contribution of each factor and the way they are modeled.

The Models We are aware that there are alternative approaches to the one we took, we will briefly expose the alternatives.

We could model the spot rate of each tenor as a stochastic variable. It provides an inconsistent model across tenors and also inside each tenor curve because of the ERP factor.

One advance from the first model would be to model the spot rate of each tenor, excluding the ERP cost, as a stochastic variable. It provides an inconsistent model across tenors but it would be still useful to trade each tenor curve as it provides a coherent curve.

One final option is to Model EONIA and also BOR-OIS spread, excluding the ERP cost, for each tenor as a transaction cost. This is a consistent model across tenors and a useful tool to trade the BOR curve. Despite being a good tool to trade and also to model past correlations it can say little about future evolution based on assumptions about future liquidity and credit conditions.

As opposed to the last option we started our model with one crucial assumption: the set of BOR curves reflects one reality. From this assumption we had legitimacy to analyze the entire set of BOR curves in an unified approach, which, under a disrupted market, was the first work doing so.

Our approach, while analyzing this suddenly complex set of curves, was to give an economic meaning to the factors underling the curves, attaching the agents decision process to the model. That is why we modeled BOR liquidity trough the valuation of cost and probabilities of a liquidity shock[1] and ERP phenomena as a undeterminable lump sum tax the lender has to pay in order to carry the asset on the end of the period.

In order to carry such analysis we had to introduce new tools and use old tools in a new environment. The most obvious was the application of Lando (1998) model to very short

[1] And an assumption about the lender initial liquidity status.

term tenors. But the “innovations” in our work, if any, were the treatment both of the liquidity event with a Markovian transition matrix and of the ERP as an ATS stochastic variable.

Joining the Models We analyzed carefully the hypothesis we had to incur in order to join the models in a way that it remained mathematical tractable, intuitive (spread additivity) and still having economic meaning.

We also showed that this model could handle easily correlations between factors as any sum of stochastic ATS variables does. In respect to correlations the only strong assumption we incurred was the independence of variables function of stopping times: η_t^c, η_t^l. The reason for keeping independence in moments of class changes (and event time as a particular case of class change) is quite simple: it would add too much complexity to our framework. We do not exclude it out as we think there can be important response by all market variables following a credit event, as it happened after Lehman.

The Experiences In the crude experiences we realized we managed to explain a considerable part of the BOR-OIS spread and also to decrease the difference between the highest and lowest BOR-OIS. Another result was that whenever a factor became more or less important during the crisis, the contribution to our rude calibration increased and decreased accordingly.

Nevertheless there were still big residuals and sometime an inconsistency in model parameters across time. Also important were the apparent residuals correlation for the same tenor, in opposition to the desired result of this model, which was is to left a random residual across tenors and position in the curve. We think this problems would be mitigated with a panel data calibration, by using more than one stochastic variable to each factor and by introducing correlations.

Problems Our work encompasses some problems related both to the concept of the object studied and to the way we study it.

First we are assuming that BOR is a meaningful thing, i.e., that it reflects market behaviors and market prices, but in fact it may be “artificial” as it is not backed by real transactions. There is also the risk that only part of the curve reflects market factors at each time, the tenors with more liquidity for instance, making false our assumption of unicity of the money market. We incur the risk of trying to model a meaningless/arbitrary/manipulated phenomena.

Second we may be missing some important factors, some that are current present and we do not value properly and others that are currently negligible and have potential to became important ones. Still related to the model, we may be modeling credit, liquidity and ERP in a way different from the one market players take the decisions and in that case the model would not be able to capture the market dynamics.

Third, related to the model itself, the correlations between intensities may provide an weak correlation between all factors and consequently the model may underestimate future volatility.

Goals The primary objective of this work was to provide a framework for future treatment of BOR curves more specifically, a framework which gives an economic meaning to liquidity and ERP factors, undoubtedly present in BOR curve, helping to forecast the curve and to intervene in the appropriate timing. Other primal goal was to achieve the former allowing for the intuitive spread additivity between the different factors. We think we hit this goals.

The secondary objective was to explain the dynamics of the BOR curve during the current crisis and that objective was poorly achieved as we did not have the space to make a proper estimation of the model. In face of such limitation we opted to show how the model works by assuming hypothetical parameters under the most simple stochastic and matricial framework. In such work we got already some results by reducing the unexplained factors but the final result still do not satisfy.

Future Work Behind the natural steps, which are to choose the appropriate stochastic model of the state vector and then to estimate the respective parameters, there are other areas which deserve a future study.

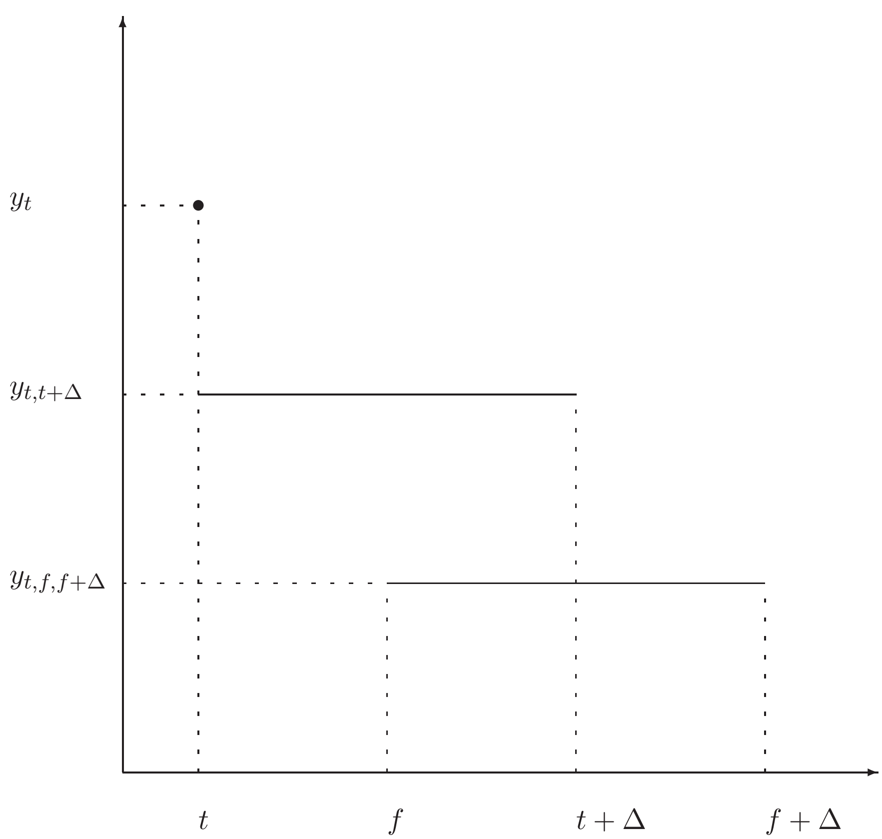
While estimating the model we should better understand the credit component of BOR as it is a statistic of a panel of banks. To target each individual contribution could be an option with enough data.

One component where there seems to exist a big opportunity for modeling is the ERP factor as there are obvious correlations/dynamics between the premiums of each tenor which were not taken into account in our model.

Other natural extension would be to study the Swap Spread curve, the long term version of this work focused on the short term conditions of the Money Market. Finally, we could use this framework to value the Bonds issued by a Bank or by a government.

Appendix A

Reading Graphs



Appendix B

The BBA LIBOR fixing & definition[1]

BBA LIBOR is the BBA fixing of the London Inter-Bank Offered Rate. It is based on offered interbank deposit rates contributed in accordance with the Instructions to BBA LIBOR Contributor Banks.

The BBA will fix BBA LIBOR and its decision shall be final. The BBA consults on the BBA LIBOR rate fixing process with the BBA LIBOR Steering Group. The BBA LIBOR Steering Group comprises leading market practitioners active in the interbank money markets in London.

BBA LIBOR is fixed on behalf of the BBA by the Designated Distributor and the rates made available simultaneously via a number of different information providers.

Contributor Panels shall comprise at least 8 Contributor Banks. Contributor Panels will broadly reflect the balance of activity in the interbank deposit market. Individual Contributor Banks are selected by the BBAs FX & Money Markets Advisory Panel after private nomination and discussions with the Steering Group, on the basis of reputation, scale of activity in the London market and perceived expertise in the currency concerned, and giving due consideration to credit standing.

The BBA, in consultation with the BBA LIBOR Steering Group, will review the composition of the Contributor Panels at least annually.

Contributed rates will be ranked in order and only the middle two quartiles averaged arithmetically. Such average rate will be the BBA LIBOR Fixing for that particular currency, maturity and fixing date. Individual Contributor Panel Bank rates will be released shortly after publication of the average rate.

The BBA, in consultation with the BBA LIBOR Steering Group, will review the BBA LIBOR Fixing process from time to time and may alter the calculation methodology after due consideration and proper notification of the planned changes.

[1]Extracted from http://www.bba.org.uk, according to last clarification in 18/12/2008

In the event that it is not possible to conduct the BBA LIBOR Fixing in the usual way, the BBA, in consultation with Contributor Banks, the BBA LIBOR Steering Group and other market practitioners, will use its best efforts to set a substitute rate. This will be the BBA LIBOR Fixing for the currency, maturity and fixing date in question. Such substitute fixing will be communicated to the market in a timely fashion.

If an individual Contributor Bank ceases to comply with the spirit of this Definition or the Instructions to BBA LIBOR Contributor Banks, the BBA, in consultation with the BBA LIBOR Steering Group, may issue a warning requiring the Contributor Bank to remedy the situation or, at its sole discretion, exclude the Bank from the Contributor Panel.

If an individual Contributor Bank ceases to qualify for Panel membership the BBA, in consultation with the BBA LIBOR Steering Group, will select a replacement as soon as possible and communicate the substitution to the market in a timely fashion.

B.1 Instructions To BBA Libor Contributor Banks

1. An individual BBA LIBOR Contributor Panel Bank will contribute the rate at which it could borrow funds, were it to do so by asking for and then accepting inter-bank offers in reasonable market size just prior to 1100.

2. Rates shall be contributed for currencies, maturities and fixing dates and according to agreed quotation conventions.

3. Contributor Banks shall input their rate without reference to rates contributed by other Contributor Banks.

4. Rates shall be for deposits:

   - made in the London market in reasonable market size;
   - that are simple and unsecured;
   - governed by the laws of England and Wales;
   - where the parties are subject to the jurisdiction of the courts of England and Wales.

5. Maturity dates for the deposits shall be subject to the ISDA Modified Following Business Day convention, which states that if the maturity date of a deposit falls on a day that is not a Business Day the maturity date shall be the first following day that is a Business Day, unless that day falls in the next calendar month, in which case the maturity date will be the first preceding day that is a Business Day.

6. Rates shall be contributed in decimal to at least two decimal places but no more than five.

7. Contributors Banks will input their rates to the Designated Distributor between 1100hrs and 1110hrs, London time.

8. The rate at which each bank submits must be formed from that banks perception of its cost of unsecured funds in the interbank market. This will be based on the cost of funds not covered by any governmental guarantee scheme.

9. Contributions must represent rates formed in London and not elsewhere.

10. Contributions must be for the currency concerned, not the cost of Producing one currency by borrowing in another currency and accessing the required currency via the foreign exchange markets.

11. The rates must be submitted by members of staff at a bank with primary responsibility for management of a banks cash, rather than a banks derivative book.

12. The definition of funds is: unsecured interbank cash or cash raised through primary issuance of interbank Certificates of Deposit.

The Designated Distributor will endeavour to identify and arrange for the correction of manifest errors in rates input by individual Contributor Banks prior to 1130.

The Designated Distributor will publish the average rate and individual Contributor Banks' rates at or around 1130hrs London time.

Remaining manifest errors may be corrected over the next 30 minutes. The Designated Distributor then will make any necessary adjustments to the average rate and publish it as the BBA LIBOR Fixing at 1200hrs.

Appendix C

Lando (1998)

C.1 Introduction

As far as I know the paper "On Cox Processes and Credit Risky Securities", was the first introducing Cox processes, the double stochastic Poisson processes, to the credit risk modeling. By doing this it enabled the stochascity of of credit spreads without the traditional requirement for rating changes and allowed the introduction of dependence between the risk free structure and credit spreads.

We will follow the paper with two slight differences on the approach. First, we will not attach the model to a credit event, we shall keep it general by studding the event x so that the explanation can fit any sort of events. Second, we will focus on an asset with specific cash flows, the zero coupon deposit with zero recovery, and ignore all other variants the author suggest throughout the paper.

In reality such approach does not pose any problem as the paper is written in such a way that we could easily replace credit by any other type of event without consequences and also because the author's main goal is to value the asset we choose.

The probability space and respective $\sigma - algebras$ will be the same to the presented in Section 2.2 and we will just adapt the notation to the present appendix.

C.2 Cox Process

Let $l^x(u,\omega)$ be a particular realization of the random intensity of a Y_u^x Cox Process defined on a $(\Omega, \mathscr{F}, Q)$ probability space. We will write $l^x(s,\omega) = \lambda(\mathbf{X_u}) = \lambda_u^x$ where X_t is a $\mathbb{R}^d$ - valued stochastic process and $\lambda_u^x : \mathbb{R}^d \to [0, \infty[$ defined on $(\Omega, \mathscr{F}, Q)$ probability space.

Definition 13. The event time τ^x can be thought as the time of the first jump after t of the Y_u^x Cox process with intensity process λ_u^x.

$$\tau^x = \inf\{u : Y_u^x - Y_t^x \geq 1\}$$

From the Definition 13 and the properties of the Poisson Process we obtain the following result

$$P\left(\tau^x > T \left| \mathscr{G}_T \wedge \mathscr{H}_t^x\right.\right) = 1_{\{\tau^x > t\}} \exp\left(-\int_t^T \lambda^x\left(\mathbf{X_u}\right) du\right) \tag{C.1}$$

Proof.

$$\begin{aligned}
P\left(\tau^x > T \left| \mathscr{G}_T \wedge \mathscr{H}_t^x\right.\right) &= \begin{cases} P\left(\tau^x > T \left| \mathscr{G}_T \vee \{\tau^x > t\}\right.\right) & ,\tau^x > t \\ 0 & ,\tau^x \leqslant t \end{cases} \\
&= 1_{\{\tau^x > t\}} P\left[\tau^x > T \left| \mathscr{G}_T \vee \{\tau^x > t\}\right.\right] \\
&= 1_{\{\tau^x > t\}} \frac{P\left[\tau^x > T \cap \{\tau^x > t\} \left| \mathscr{G}_T\right.\right]}{P\left[\{\tau^x > t\} \left| \mathscr{G}_T\right.\right]} \\
&= 1_{\{\tau^x > t\}} \frac{P\left[\tau^x > T \left| \mathscr{G}_T\right.\right]}{P\left[\{\tau^x > t\} \left| \mathscr{G}_T\right.\right]} \\
&= 1_{\{\tau^x > t\}} \frac{\exp\left(-\int_0^T \lambda_u^x du\right)}{\exp\left(-\int_0^t \lambda_u^x du\right)} \\
&= 1_{\{\tau^x > t\}} \exp\left(-\int_t^T \lambda_u^x du\right)
\end{aligned}$$

□

C.3 Valuing a Claim I

We will value a zero coupon claim contingent on the occurrence of an particular event x and subject to the interest risk free rates.

$$B_{t,T}^{rx} = E\left[\exp\left(-\int_t^T r_u du\right) N 1_{\{\tau > T\}} \left| \mathscr{F}_t\right.\right] \tag{C.2}$$

The superscript rx states the dependence on risk free curve and the credit risk.

Following Eq. C.1 our first approach while valuing $B_{t,T}^{rx}$ is the following:

$$B_{t,T}^{rx} = 1_{\{\tau > t\}} E\left[\exp\left(-\int_t^T r_u + \lambda_u du\right) N \left| \mathscr{G}_t\right.\right] \tag{C.3}$$

Proof.

$$\begin{aligned}
E\left[\exp\left(-\int_t^T r_u du\right) N1_{\{\tau^x>T\}}\left|\mathscr{F}_t\right.\right] &= E\left[E\left[\exp\left(-\int_t^T r_u du\right) N1_{\{\tau^x>T\}}\left|\mathscr{G}_T \vee \mathscr{H}_t^x\right.\right]\left|\mathscr{F}_t\right.\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) NE\left[1_{\{\tau^x>T\}}\left|\mathscr{G}_T \vee \mathscr{H}_t^x\right.\right]\left|\mathscr{F}_t\right.\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) N1_{\{\tau^x>t\}} \exp\left(-\int_t^T \lambda_u^x du\right)\left|\mathscr{F}_t\right.\right] \\
&= 1_{\{\tau^x>t\}} E\left[\exp\left(-\int_t^T r_u + \lambda_u^x du\right) N\left|\mathscr{F}_t\right.\right] \\
&= 1_{\{\tau^x>t\}} E\left[\exp\left(-\int_t^T r_u + \lambda_u^x du\right) N\left|\mathscr{G}_t \vee \mathscr{H}_t^x\right.\right] \\
&= 1_{\{\tau^x>t\}} E\left[\exp\left(-\int_t^T r_u + \lambda_u^x du\right) N\left|\mathscr{G}_t\right.\right]
\end{aligned}$$

□

In future references to this result we will not use the $1_{\{\tau^x>t\}}$ part. In fact, space will be a scarce resource and so we will omit the obvious result: $P\left(\tau^x > T\left|\mathscr{G}_T \wedge \{\tau < t\}\right.\right) = 0$

C.4 A Generalized Markovian Model

We shall now specify λ_u^x dynamics.

We start assuming that that there are different propensities to the materialization of the event. As consequence each deposit, in each moment, is classified accordingly to its propensity to the event: η_t^x is the stochastic $\mathscr{H}_t - measurable$ variable that reflects such classification. The transition between states throughout the time is governed by a markovian transition matrix which has A_u^x as a generator matrix.

The author goes a little further from this point stating that those propensities/intensities are themselves stochastic. This development is reflected in a stochastic generator matrix: $A_u^x = A^x\left(\mathbf{X_u}\right)$, function of the state vector X_u.

We may now define the generator matrix:

$$A^x\left(\mathbf{X_u}\right) = \begin{bmatrix}
\lambda_1^x\left(\mathbf{X_u}\right) & \lambda_{1,2}^x\left(\mathbf{X_u}\right) & \cdots & \lambda_{1,K^x-1}^x\left(\mathbf{X_u}\right) & \lambda_{1,K^x}^x\left(\mathbf{X_u}\right) \\
\lambda_{2,1}^x\left(\mathbf{X_u}\right) & \lambda_2^x\left(\mathbf{X_u}\right) & \cdots & \lambda_{2,K^x-1}^x\left(\mathbf{X_u}\right) & \lambda_{2,K^x}^x\left(\mathbf{X_u}\right) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{K^x-1,1}\left(\mathbf{X_u}\right) & \lambda_{K^x-1,2}^x\left(\mathbf{X_u}\right) & & \lambda_{K^x-1}^x\left(\mathbf{X_u}\right) & \lambda_{K^x-1,K^x}^x\left(\mathbf{X_u}\right) \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}$$

where

- $1, ..., K^x - 1$ are classes/states where 1 is the class with lowest propensity to event x, $K^x - 1$ is the worst and K^x represents the event itself
- $\lambda^x_{i,j}(\mathbf{X_u})$ is the intensity transition from state i to state j, function of stochastic state variables
- $\lambda^x_i(\mathbf{X_u}) = \lambda^x_{i,i}(\mathbf{X_u}) = \sum_{j=1, j\neq i}^{K} \lambda^x_{i,j}(\mathbf{X_u}), i = 1, ..., K-1$ by definition
- $\lambda^x_{i,K}(\mathbf{X_u}) = 0$, means no exit from event class once there

With this construction[1] we obtain a continuous time process which conditionally on the evolution of the state variables is a non-homogeneous Markov chain.

Conditionally on the evolution of the state variables, the transition probabilities, $\Pi^x_{t,T}$, of this Markov chain satisfy

$$\frac{\delta \Pi^x_{t,T}}{\delta t} = -A^x_t \Pi^x_{t,T}$$

despite this, the solution of this equation in the time-inhomogeneous case won't be always[2]

$$\Pi^x_{t,T} = \exp\left(\int_t^T A^x_u du\right)$$

Nevertheless it is possible to write $\Pi^x_{t,T}$ directly with recourse to product integral notation[3]:

$$S^x_{t,T} = \left(\int_t^T A^x_u du\right)$$

$$\Pi^x_{t,T} = \pi_{]t,T]}\left(\mathbf{1} + \mathbf{dS^x_{t,T}}\right)$$

where $\pi_{]t,T]}$ is the product integral from t to T

C.5 Valuing a Claim II

Now, having defined λ^x stochastics, we are able to change slightly eq. C.3 to incorporate the class information, reflected on t superscript, and then follow a different path of reasoning.

[1]See Lando (1998) pag 108 for details and further motivation

[2]It would only happen if $\int_t^a A^x_u du$ and $\int_a^T A^x_u du$ were commutative which typically will not occur.

[3]See Gill & Johansen (1990) section 4 for more detail on product integration of Markov Processes

The pricing formula for contingent bonds with zero recovery, conditional on its class is:

$$
\begin{aligned}
B^x_{t^i,T} &= E\left[\exp\left(-\int_t^T r_u du\right) N1_{\{\tau^x>T\}}\,|\mathscr{F}_t \wedge \eta^x_t = i\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) N1_{\{\tau^x>T\}}\,|\mathscr{G}_t \vee \mathscr{H}^x_t \wedge \eta^x_t = i\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) NE\left[1_{\{\tau^x>T\}}\,|\mathscr{G}_T \wedge \eta^x_t = i\right]|\mathscr{G}_t\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right)\left(1 - \Pi^x_{t,T}|_{i,K^x}\right)|\mathscr{G}_t\right] \qquad \text{(C.4)}
\end{aligned}
$$

where

- $\Pi^x_{t,T}|_{i,K^x} = P\left(t < \tau^x \leqslant T\,|\mathscr{G}_T \wedge \eta^x_t = i\right)$ or the element from i^{th} line and column K from $\Pi^x_{t,T}$ matrix.
- the condition $\mathscr{H}^x_t \wedge \eta^x_t = i$ resumes to $\eta^x_t = i$. as there is more information in η^x_t than in $\mathscr{H}^x_t$

In order to enable analytical solutions, the author impose a structure on the conditional intensity matrix, $A^x(\mathbf{X_u})$.

Assumption 1 (Structure of the conditional intensity matrix)**.** For each path of $\mathbf{X_u}$ assume that the time dependent generator has the representation

$$A^x_u = \mathsf{B}^x \mu^x_u \left[\mathsf{B}^x\right]^{-1}$$

where

$\mu^x(\mathbf{X_u}) = \mu^x_u$ denote de $K^x \times K^x$ diagonal matrix[4] configuration $diag\left(\mu^x_1(\mathbf{X_u}), \ldots, \mu^x_{K^x-1}(\mathbf{X_u}), 0\right)$;
B^x is a $K^x \times K^x$ matrix whose columns consist of K^x eigenvectors of A^x_u;

If we further define the diagonal matrix:

$$
\mathsf{E}^x_{t,T} = \begin{bmatrix}
\exp\left(\int_t^T \mu^x_1(\mathbf{X_u})\,du\right) & 0 & & \cdots & 0 \\
0 & \exp\left(\int_t^T \mu^x_2(\mathbf{X_u})\,du\right) & & \cdots & 0 \\
\vdots & & \ddots & \cdots & \\
\vdots & \cdots & & \exp\left(\int_t^T \mu^x_{K^x-1}(\mathbf{X_u})\,du\right) & 0 \\
0 & \cdots & & 0 & 1
\end{bmatrix}
$$

[4] See Assumption 2 for one hypothesis on μ

Then, by setting $\Pi^x_{t,T} = \mathsf{B}^x \mathsf{E}^x_{t,T} \left[\mathsf{B}^x\right]^{-1}$, $\Pi^x_{t,T}$ satisfies Kolmogorov's backward equation:

$$\begin{aligned}
\frac{\delta \Pi^x_{t,T}}{\delta t} &= \mathsf{B} \frac{\delta \mathsf{E}^x_{t,T}}{\delta t} \left[\mathsf{B}^x\right]^{-1} \\
&= \mathsf{B} \left[-\mu^x_u \mathsf{E}^x_{t,T}\right] \left[\mathsf{B}^x\right]^{-1} \\
&= -\mathsf{B}^x \mu^x_u \left[\mathsf{B}^x\right]^{-1} \mathsf{B}^x \mathsf{E}^x_{t,T} \left[\mathsf{B}^x\right]^{-1} \\
&= -A^x_t \Pi^x_{t,T}
\end{aligned}$$

It follows that:

$$\begin{aligned}
\Pi^x_{t,T} &= \begin{bmatrix} b^x_{1,1} & \cdots & b^x_{1,K^x} \\ \vdots & \ddots & \vdots \\ b^x_{K^x,1} & \cdots & b^x_{K^x,K^x} \end{bmatrix} \begin{bmatrix} e^x_1 & & \cdots & 0 \\ \vdots & \ddots & \cdots & \\ \vdots & & e^x_{K^x-1} & 0 \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} \left[b^x_{1,1}\right]^{-1} & \cdots & \left[b^x_{1,K}\right]^{-1} \\ \vdots & \ddots & \vdots \\ \left[b^x_{K^x,1}\right]^{-1} & \cdots & \left[b^x_{K^x,K^x}\right]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} b^x_{1,1}e^x_1 & b^x_{1,2}e^x_2 & \cdots & b^x_{1,K^x} \\ b^x_{2,1}e^x_1 & b^x_{2,2}e^x_2 & \cdots & b^x_{2,K^x} \\ \vdots & \vdots & \ddots & \vdots \\ b^x_{K,1}e^x_1 & b^x_{K,2}e^x_2 & \cdots & b^x_{K^x,K^x} \end{bmatrix} \begin{bmatrix} \left[b^{-1}_{1,1}\right]^{-1} & \cdots & \left[b^x_{1,K^x}\right]^{-1} \\ \vdots & \ddots & \vdots \\ \left[b^x_{K^x,1}\right]^{-1} & \cdots & \left[b^x_{K^x,K^x}\right]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{K^x} b^x_{1,i}e^x_i \left[b^x_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{1,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} b^x_{1,i}e^x_i \left[b^x_{i,K^x}\right]^{-1} \\ \sum_{i=1}^{K^x} b^x_{2,i}e^x_i \left[b^x_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{2,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} b^x_{2,i}e^x_i \left[b^x_{i,K^x}\right]^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{K^x} b^x_{K,i}e^x_i \left[b^x_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{K,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} b^x_{K,i}e^x_i \left[b^x_{i,K^x}\right]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{K^x} b^x_{1,i}e^x_i \left[b^x_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{1,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} \alpha^x_{1,i}e^x_i \\ \sum_{i=1}^{K^x} b^x_{2,i}e^x_i \left[b^x_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{2,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} \alpha^x_{2,i}e^x_i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{K^x} b^x_{K^x,i}e^x_i \left[b^{-1}_{i,1}\right]^{-1} & \sum_{i=1}^{K^x} b^x_{K,i}e^x_i \left[b^x_{i,2}\right]^{-1} & \cdots & \sum_{i=1}^{K^x} \alpha^x_{K^x,i}e^x_i \end{bmatrix}
\end{aligned}$$

where $\alpha^x_{i,j} = b^x_{i,j} \left[b^x_{j,K^x}\right]^{-1}$

Then,

$$\begin{aligned}
E\left[1_{\{t\leqslant\tau^x\leqslant T\}}\,|G_T \vee \eta_t^x = i\right] &= \Pi^x_{t,T}|_{j,K^x} \\
&= \sum_{j=1}^{K^x} \alpha^x_{i,j} e^x_j \\
&= \sum_{j=1}^{K^x-1} \alpha^x_{i,j} e^x_j + \alpha^x_{i,K^x} e^x_{K^x} \\
&= \sum_{j=1}^{K^x-1} \alpha^x_{i,j} e^x_j + 1 \\
&= \sum_{j=1}^{K^x-1} \alpha^x_{i,j} \exp\left(\int_t^s \mu^x_j\left(\mathbf{X_u}\right) du\right) + 1
\end{aligned}$$

given the result $\alpha^x_{i,K^x} = 1$
Finally, for valuing a risky bond,

$$\begin{aligned}
B^{rx}_{t^i,T} &= E\left[\exp\left(-\int_t^T r_u du\right)\left(1 - \Pi^x_{t,T}|_{i,K^x}\right)|\mathscr{G}_t\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right)\left(1 - \sum_{j=1}^{K-1} \alpha^x_{i,j}\exp\left(\int_t^T \mu^x_j\left(\mathbf{X_u}\right)du\right) - 1\right)|\mathscr{G}_t\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right)\left(\sum_{j=1}^{K-1} -\alpha^x_{i,j}\exp\left(\int_t^T \mu^x_j\left(\mathbf{X_u}\right)du\right)\right)|\mathscr{G}_t\right] \\
&= E\left[\left(\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(-\int_t^T r_u - \mu^x_j\left(\mathbf{X_u}\right)du\right)\right)|\mathscr{G}_t\right] \\
&= \sum_{j=1}^{K^x-1} -\alpha^x_{i,j} E\left[\exp\left(-\int_t^T r_u - \mu^x_j\left(\mathbf{X_u}\right)du\right)|\mathscr{G}_t\right] \qquad \text{(C.5)}
\end{aligned}$$

The valuation of a forward start rating contingent claim is as easy as a spot one. Eq. C.4 can easily be modified into

$$B^{rx}_{t,f^{\eta^x_f},T} = E\left[\exp\left(-\int_f^T r_u du\right)\left(1 - \Pi^x_{f,T}|_{\eta^x_f,K^x}\right)|\mathscr{G}_t\right] \qquad \text{(C.6)}$$

and the same applies to eq. C.5

$$B^{rx}_{t,f^{\eta^x_f},T} = \sum_{j=1}^{K-1} -\alpha^x_{\eta^x_f,j} E\left[\exp\left(-\int_f^T r_u - \mu^x_j\left(\mathbf{X_u}\right)du\right)|\mathscr{G}_t\right] \qquad \text{(C.7)}$$

Furthermore, in case of independence of $r\left(\mathbf{X_u}\right)$ and $\mu_j^x\left(\mathbf{X_u}\right)$:

$$\begin{aligned} B^{rx}_{t^{\eta^x_t},T} &= B^r_{t,T}\sum_{j=1}^{K^x-1} -\alpha^x_{\eta^x_t,j} E\left[\exp\left(-\int_t^T -\mu^x_j\left(\mathbf{X_u}\right)du\right)|\mathscr{G}_t\right] \\ &= B^r_{t,T} B^{\lambda^x}_{t^{\eta^x_t},T} \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} B^{rx}_{t,f^{\eta^x_f},T} &= B^r_{t,f,T}\sum_{j=1}^{K^x-1} -\alpha^x_{\eta^x_f,j} E\left[\exp\left(-\int_f^T -\mu^x_j\left(\mathbf{X_u}\right)du\right)|\mathscr{G}_t\right] \\ &= B^r_{t,f,T} B^{\lambda^x}_{t,f^{\eta^x_f},T} \end{aligned} \tag{C.9}$$

C.6 Extra Condition

In order to insure that we always have a coherent $A^x\left(\mathbf{X_u}\right)$ we will generally work with an extra extra Assumption to on top of Assumption 1.

Assumption 2. There exists and base generator matrix A^x from which all the others are produced such that

- *eigenvector* $\left(A^x\right) = \mathsf{B}^{\mathsf{x}} =$ *eigenvector* $\left(A^x\left(\mathbf{X_u}\right)\right)$
- *eigenvalue* $\left(A^x\right) = \xi^x = \begin{bmatrix} \xi^x_1 \\ \xi^x_2 \\ \xi^x_3 \\ \dots \\ \xi^x_{K-1} \\ 0 \end{bmatrix} = \frac{\mu^x(\mathbf{X_u})}{\frac{\sum g^i X^i_u}{\sum g^i}}$ which means $\mu^x_i\left(\mathbf{X_u}\right) = \xi^x_i \frac{\sum g^i X^i_u}{\sum g^i}$

This construction means that when all relevant $X^i_t = 1$ then $A^x\left(\mathbf{X_u}\right) = A^x$. | |

Appendix D

Recovery

D.1 Introduction

In this appendix we will work on the recovery of an zero coupon bond, $\phi^x_{\tau^x}$, upon event x. The purpose is to help on the decision of choosing the recovery model and to supply all needed results to pursue such application.

We will present some known results about recovery models applied to Zero Coupon Bonds following Schonbucher (2003). The results will be presented in a slightly different way from the book as the recovery will not be associated to a credit event.

The probability space and respective $\sigma - algebras$ will be the same to the presented in Section 2.2 and we will just adapt the notation to the present appendix.

We will explore several methods:

- Zero Recovery (ZR)
- Recovery of Treasury (RT)
- Recovery of Market Value (RMV)
- Recovery of Par (RP)
- Stochastic Recovery (all the above methods use a deterministic intensity)

D.2 Zero Recovery

Assumption:

- All claims have a zero recovery at event

$$\phi^x_u = 0 \qquad \forall u > 0$$

$$
\begin{aligned}
B_{t,T}^{x^{ZR}} &= E\left[\exp\left(-\int_t^T r_u du\right) 1_{\{\tau^x>T\}} \,|\mathscr{F}_t\right] \\
&= E\left[\exp\left(-\int_t^T r_u du\right) \exp\left(-\int_t^T \lambda_u^x du\right) |\mathscr{G}_t\right] \\
&= E\left[\beta_{t,T}\beta_{t,T}^{\lambda^x} \,|\mathscr{G}_t\right]
\end{aligned} \tag{D.1}
$$

assuming independence between λ^x and r

$$
\begin{aligned}
B_{t,T}^{x^{ZR}} &= E\left[\beta_{t,T}\,|\mathscr{G}_t\right] E\left[\beta_{t,T}^{\lambda^x} \,|\mathscr{G}_t\right] \\
&= B_{t,T}B_{t,T}^{\lambda^x}
\end{aligned} \tag{D.2}
$$

D.3 Recovery of treasury

Assumptions:

- there exists an equivalent default-free asset to every defaultable claim. This equivalent claim pays off for sure the payoffs that were promised in the defaultable asset and its price process is denoted by $B_{t,T}$
- at default, the recovery of every default claim is c times the value of the equivalent default-free asset
$$\phi_u^x = c^x B_{u,T}$$

If c^x is constant then one can decompose the problem into a zero recovery defaultable claim and a risk free claim:

$$
\begin{aligned}
B_{t,T}^{x^{RT}} &= (1-c^x)\, B_{t,T}^{x^{ZR}} + c^x B_{t,T} \\
&= (1-c^x)\, E\left[\beta_{t,T}\beta_{t,T}^{\lambda^x} \,|\mathscr{G}_t\right] + c^x B_{t,T}
\end{aligned} \tag{D.3}
$$

after assuming independence between λ^x and r.

$$
\begin{aligned}
B_{t,T}^{x^{RT}} &= (1-c^x)\, B_{t,T}B_{t,T}^{\lambda^x} + c^x B_{t,T} \\
&= B_{t,T}\left[(1-c^x)\, B_{t,T}^{\lambda^x} + c^x\right]
\end{aligned} \tag{D.4}
$$

D.4 Recovery of Market Value

Assumptions:

- at default, the recovery of every default claim is $(1-q^x)$ times the pre-default value of the defaultable claim

$$\phi^x_{\tau^x} = (1-q^x)\, B^{x^{RMV}}_{\tau^x-,T}$$

Like Schonbucher (2003) we will use a discrete time argument to start:

Discrete reasoning

- time step of Δ_t: $]t, t+\Delta_t]$
- R as risk free interest in $]t, t+\Delta_t]$
- the event probability until $t+\Delta_t$ is $p^x_{t,t+\Delta_t} = p^x$
- the event x Hazard rate, H^x, is defined as $\Delta_t \cdot H^x = \frac{p^x}{1-p^x}$
- at the event moment, τ^x , the recovery of every event is $(1-q^x)$ times the pre-event value of the claim

$$\phi^x_{\tau^x} = (1-q^x)\, B^b_{\tau^x-,T}$$

  if $\tau^x = t+\Delta_t$ then $\tau^x- = t$

The price of a zero coupon at time t must be the expected discounted value of its value at time $t+\Delta_t$:

$$B^{x^{RMV}}_{t,T} = \frac{1}{1+R\Delta_t}\left\{(1-p^x)\, E\left[B^{x^{RMV}}_{t+\Delta_t,T}\,|\mathscr{G}_t\right] + p^x\,(1-q^x)\, B^{x^{RMV}}_{t,T}\right\}$$

Rearranging the terms,

$$\begin{aligned} B^{x^{RMV}}_{t,T}\,(1+R\Delta_t - p^x(1-q^x)) &= (1-p^x)\, E\left[B^{x^{RMV}}_{t+\Delta_t,T}\,|\mathscr{G}_t\right] \\ \frac{E\left[B^{x^{RMV}}_{t+\Delta_t,T}\,|\mathscr{G}_t\right]}{B^{x^{RMV}}_{t,T}} &= \frac{1+R\Delta_t - p^x(1-q^x)}{1-p^x} \end{aligned}$$

and then

$$\begin{aligned}\frac{E\left[B^{x^{RMV}}_{t+\Delta_t,T}\,|\mathcal{G}_t\right]}{B^{x^{RMV}}_{t,T}} &= \frac{R\Delta_t + q^x\cdot p^x}{(1-p^x)}+1\\ &= \frac{R\Delta_t + q^x\cdot p^x}{(1-p^x)}H^x\Delta_t\frac{1-p^x}{p^x}+1\\ &= R\cdot\Delta_t\frac{\Delta_t H^x}{p^x}+q^x\Delta_t H^x+1\\ &= R\cdot\Delta_t\left(1+\Delta_t H^x\right)+q^x\Delta_t H^x+1\\ &= R\cdot\Delta_t + R\left(\Delta_t\right)^2 H^x+q^x\Delta_t H^x+1\\ &= \Delta_t\left(R+q^x H^x\right)+RH^x\left(\Delta_t\right)^2+1\end{aligned}$$

and then

$$E\left[\frac{B^{x^{RMV}}_{t+\Delta_t,T}-B^{x^{RMV}}_{t,T}}{B^{x^{RMV}}_{t,T}}\,|\mathcal{G}_t\right] = \Delta_t\left(R+q^x H^x\right)+RH^x\left(\Delta_t\right)^2+1-1$$

In the limit when $\Delta_t \to 0$,ignoring the $(\Delta_t)^2$ terms and replacing discrete variables with their continuous equivalents[1]

$$E\left[\frac{dB^{x^{RMV}}_{u,T}}{B^{x^{RMV}}_{u-,T}}\,|\mathcal{G}_t\right]=\left(r+q^x\lambda^x\right)du$$

This leads to the conjecture that $B^{x^{RMV}}_{t,T}$ is reached by discounting the expected price in survival $B^{x^{RMV}}_{t+\Delta_t,T}$ with the $(r+q^x\lambda^x)$ continuous rate.

The value of $B^{x^{RMV}}_{t,T}$ would be[2]

$$B^{x^{RMV}}_{t,T}=1_{\{\tau>t\}}E\left[\exp\left(-\int_t^T r\left(X_u\right)+q^x\lambda^x\left(X_u\right)du\right)|\mathcal{G}_t\right]+1_{\{\tau=t\}}\left(1-q^x\right)B^{x^{RMV}}_{t-,T} \tag{D.5}$$

D.5 Recovery of Par

Assumptions:

- at default, the recovery of every default claim is π^x times the notional, N_t, of the defaultable claim immediately before the default

$$\phi^x_{\tau^x}=\pi^x_{\tau^x}N_{\tau^x-}$$

[1] $R\to r_u$, $H^x\to\lambda^x$

[2] Proof on pag 139 of Schonbucher (2003); the result holds for a q^x predictable process.

The value the recovery this asset would be:

$$\begin{aligned}
E\left[\exp\left(-\int_t^{\tau^x} r_s ds\right)\pi_t^x N_t 1_{\{\tau^x<T\}}\,|\mathscr{F}_t\right] &= E\left[\int_t^T \exp\left(-\int_t^{\tau^x} r_u du\right)\pi_{\tau^x}^x N_{\tau^x} f\left(\tau^x\right) d\tau^x\,|\mathscr{G}_t\right] \\
&= E\left[\int_t^T \exp\left(-\int_t^{\tau^x} r_u du\right)\pi_{\tau^x}^x N_{\tau^x}\lambda_{\tau^x}^x \right.\\
&\quad \left.\exp\left(-\int_t^{\tau^x} \lambda_u^x du\right) d\tau^x\,|\mathscr{G}_t\right] \\
&= E\left[\int_t^T \pi_{\tau^x}^x N_{\tau^x}\lambda_{\tau^x}^x \exp\left(-\int_t^{\tau^x} r_u+\lambda_u^x du\right) d\tau^x\,|\mathscr{G}_t\right] \\
&= E\left[\int_t^T \pi_{\tau^x}^x N_{\tau^x}\lambda_{\tau^x}^x \beta_{t,\tau^x}^x d\tau^x\,|\mathscr{G}_t\right]
\end{aligned}$$

where $f\left(\tau^x\right)=\lambda_\tau \exp\left(-\int_t^{\tau^x}\lambda_u^x du\right)$

adding ${B_{t,T}^x}^{ZR}$

$$ {B_{t,T}^x}^{RP} = {B_{t,T}^x}^{ZR} + E\left[\int_t^T \pi_{\tau^x}^x N_{\tau^x}\lambda_{\tau^x}^x \beta_{t,\tau^x}^x d\tau^x\,|\mathscr{G}_t\right] \tag{D.6}$$

D.6 Stochastic Recovery

As long as there aren't correlations with other variables, which is commonly assumed, the stochastic recovery can be replaced by its expected value when pricing.

Appendix E

Dynamics Modeling

E.1 Affine Term Structure Model

We will present a general formulation that can be applied to any yield variable and some other environments. In fact in this work AFS models will be also applied to the modeling of a stochastic markovian generator matrix.

Yields can be represented by the sum of several X variables with the dynamics specified here and still preserve the desired mathematical tractability.

E.1.1 General Framework

Under an AFS model the X^x stochastic variable has the following dynamics:

$$dX_t^x = \varepsilon_A^x \left(\varepsilon_B^x - X_t^x\right) dt + \sqrt{\varepsilon_C^x + \varepsilon_D^x X_t^x} dW_t^x$$

where the parameters refer to:

ε_A^x mean rever. velocity ε_B^x mean rever. value ε_C^x const. vol. ε_D^x cond. vol.

- *Instantaneous Yield Rate*[1]

$$y_t^x = X_t^x$$

- *Value of a Spot ZC*

$$\begin{aligned} B_{t,T}^x &= E\left[\exp\left(-\int_t^T X_u^x du\right) |\mathscr{F}_t\right] \\ &= \exp\left(\mathcal{A}_{t,T}^x - \mathcal{B}_{t,T}^x X_t^x\right) \end{aligned}$$

[1]We could increase the complexity by making $y_t^x = \sum_i X_t^{x_i}$

where

$$\mathcal{A}^x_{t,T} = f^A\left(\varepsilon^x_A, \varepsilon^x_B, \varepsilon^x_C, \varepsilon^x_D\right)$$
$$\mathcal{B}^x_{t,T} = f^B\left(\varepsilon^x_A, \varepsilon^x_B, \varepsilon^x_C, \varepsilon^x_D\right)$$

- *Value of a Forward ZC*

$$\begin{aligned} B^x_{t,f,T} &= E\left[\exp\left(-\int_f^T X^x_u du\right)|\mathscr{F}_t\right] \\ &= \exp\left(\mathcal{A}^x_{f,T}\right) E\left[\exp\left(-\mathcal{B}^x_{f,T} X^x_f\right)|\mathscr{F}_t\right] \rightarrow fgm \\ &= \exp\left(\mathcal{A}^x_{f,T} - \mathcal{B}^x_{f,T} E\left[X^x_f\,|\mathscr{F}_t\right] + \frac{Var\left(X^x_f\,|\mathscr{F}_t\right)\left(\mathcal{B}^x_{f,T}\right)^2}{2}\right) \end{aligned}$$

$$\begin{aligned} B^x_{t,f,T} &= \frac{B^x_{t,T}}{B^x_{t,f}} \\ &= \frac{\exp\left(\mathcal{A}^x_{t,T} - \mathcal{B}^x_{t,T} X^x_t\right)}{\exp\left(\mathcal{A}^x_{t,f} - \mathcal{B}^x_{t,f} X^x_t\right)} \\ &= \exp\left(\mathcal{A}^x_{t,T} - \mathcal{A}^x_{t,f} - \mathcal{B}^x_{t,T} X^x_t + \mathcal{B}^x_{t,f} X^x_t\right) \end{aligned} \tag{E.1}$$

- *Continuously Compounded Yield Rate of a Spot ZC*

$$\begin{aligned} B^x_{t,T} &= \exp\left(-y^x_{t,T}\Delta_{t,T}\right) \\ y^x_{t,T} &= \frac{\left(-\log B^x_{t,T}\right)}{\Delta_{t,T}} \end{aligned}$$

- *Continuously Compounded Yield Rate of a Forward Start ZC*

$$\begin{aligned} B^x_{t,f,T} &= \exp\left(-y^x_{t,f,T}\Delta_{f,T}\right) \\ y^x_{t,f,T} &= \frac{\left(-\log B^x_{f,T}\right)}{\Delta_{f,T}} \\ &= \frac{-\mathcal{A}^x_{f,T} + \mathcal{B}^x_{f,T} E\left[X_f\,|\mathscr{F}_t\right]}{\Delta_{f,T}} \end{aligned}$$

- *Instantaneous Forward Interest Rate*

$$\begin{aligned} f^x_{t,f} &= E\left[X^x_f \,|\mathscr{F}_t\right] \\ &= -\frac{\partial \log B^x_{t,f}}{\partial f} \end{aligned}$$

- *Asymptotic Spot Yield Rate*

$$y^x_{t,\infty} = \lim_{t\to\infty} y^x_{t,\infty}$$

where $\Delta_{t,T} = T - t$

E.1.2 Vasicek

The Vasicek model is one of simplest examples of AFS models, which is the reason we have chosen it to illustrate our model.

Model

$$\varepsilon^x_D = 0 \quad \varepsilon^x_A, \varepsilon^x_B, \varepsilon^x_C \neq 0$$

$$dX^x_t = \varepsilon^x_A\left(\varepsilon^x_B - X^x_t\right)dt + \sqrt{\varepsilon^x_C}dW^x_t$$

$$\mathcal{B}^x_{t,T} = \frac{1-\exp\left(-\varepsilon^x_A\Delta_{t,T}\right)}{\varepsilon^x_A}$$

$$\mathcal{A}^x_{t,T} = \left(\varepsilon^x_B - \frac{\varepsilon^x_C}{2{\varepsilon^x_A}^2}\right)\left(\mathcal{B}^x_{t,T} - \Delta\right) - \frac{\varepsilon^x_C}{4\varepsilon^x_A}\left(\mathcal{B}^x_{t,T}\right)^2$$

and then,

- $E\left[y^x_f \,|\mathscr{F}_t\right] = f^x_{t,f} \sim \left(\mu, \sigma^2\right)$

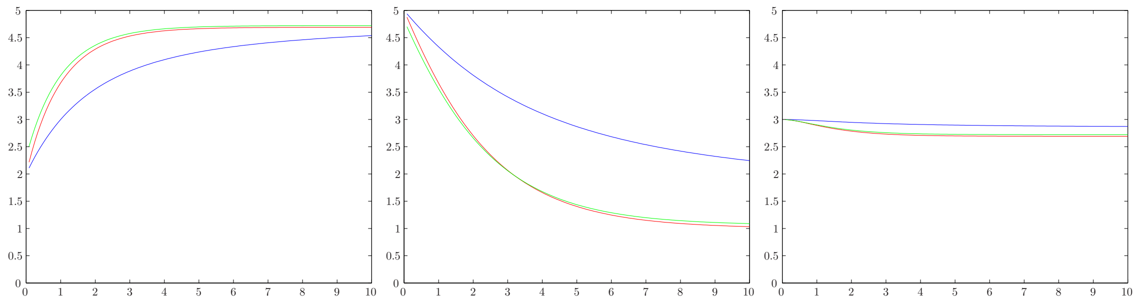
  $\mu = \exp\left(-\varepsilon^x_A\Delta_{t,f}\right)y^x_t + \varepsilon^x_B\left[1-\exp\left(-\varepsilon^x_A\Delta_{t,f}\right)\right]$
  $\sigma^2 = \frac{{\varepsilon^x_C}^2}{2\varepsilon^x_A}\left(1-\exp\left(-2\varepsilon^x_A\Delta\right)\right)$

- $y^x_{t,\infty} = \varepsilon^x_B - \frac{\varepsilon^x_C}{2{\varepsilon^x_A}^2}$

- $y^x_{t,T} = r_{t,\infty} + \left(y^x_t - y^x_{t,\infty}\right)\frac{1-\exp\left(-\varepsilon^x_A\Delta_{t,T}\right)}{\varepsilon^x_A} + \frac{\varepsilon^x_C}{4{\varepsilon^x_A}^2}\frac{\left(1-\exp\left(-\varepsilon^x_A\Delta_{t,T}\right)\right)^2}{{\varepsilon^x_A}^2\Delta_{t,T}}$

- $f^x_{t,f} = y^x_{t,\infty} + \left(y^x_t - y^x_{t,\infty}\right)\exp\left(-\varepsilon^x_A\Delta_{t,f}\right) + \frac{\varepsilon^x_C}{2{\varepsilon^x_A}^2}\left(\exp\left(-\varepsilon^x_A\Delta_{t,f}\right) - \exp\left(-2\varepsilon^x_A\Delta_{t,f}\right)\right)$

Curve Examples



(a) Negative Slope
$\varepsilon_A^x = 0.05 \quad \varepsilon_B^x = 0.9$
$\varepsilon_C^x = 0.9 \quad X_0^x = 0.05$

(b) Positive Slope
$\varepsilon_A^x = 0.05^2 \quad \varepsilon_B^x = 0.05^2$
$\varepsilon_C^x = 0.05^2 \quad X_0^x = 0.03$

(c) Flat
$\varepsilon_A^x = 0.02 \quad \varepsilon_B^x = 0.03$
$\varepsilon_C^x = 0.05^2 \quad X_0^x = 0.03$

Figure E.1: Vasicek Examples

This simple framework enables us to build a monotone yield curve with the slope we want which are depicted in Figure E.1.

In these graphics the blue line represents $y_{t,T}^x$, the red line represents $f_{t,f}^x$ and the green line represents $y_{t,f,f+3M}^x$.

E.2 JLT

The Jarrow et al. (1997) model is the starting point for Lando (1998) model described in appendix C. In order to illustrate the added value of the former we will start showing Jarrow et al. (1997) dynamics.

We will follow closely Appendix C, modeling the generic event x, with one small change: A_u^x will no longer be stochastic, and will equal the base generator matrix $A_u^x = A^x$.

Model

All Appendix C results apply after replacing the stochastic generator matrix A_u^x for a static one:

$$A_u^x = A^x = \begin{bmatrix} \lambda_1^x & \lambda_{1,2}^x & \cdots & \lambda_{1,k^x-1}^x & \lambda_{1,K^x}^x \\ \lambda_{2,1}^x & \lambda_2^x & \cdots & \lambda_{2,k^x-1}^x & \lambda_{2,k^x}^x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K^x-1,1}^x & \lambda_{K^x-1,2}^x & & \lambda_{k^x-1}^x & \lambda_{K^x-1,K^x}^x \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the intensities represent the transitions between a starting ratting (each line) and a terminal ratting (each column).

- *Value of a Forward Start ZC, conditional on* $\eta_t^x = i$
  Following Eq. C.7 with $\mu_j^x(\mathbf{X_u}) = \mu_j^x$

$$\begin{aligned}
B^{rx}_{t,f^i,T} &= E\left[\exp\left(-\int_f^T r(\mathbf{X_u})\,du\right)\left(\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\int_f^T \mu_j^x du\right)\right)|\mathscr{G}_t\right] \\
&= E\left[\exp\left(-\int_f^T r(\mathbf{X_u})\,du\right)|\mathscr{G}_t\right]\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right) \\
&= B_{t,f,T}\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right)
\end{aligned}$$

- *Value of a Spot ZC, conditional on* $\eta_t^x = i$

$$B^{rx}_{t^i,T} = B^{rx}_{t,t^i,T}$$

- *CC Spread of a Forward Start ZC, conditional on* $\eta_f^x = i$

$$B^{rx}_{t,f^i,T} = \exp\left(-y^{rx}_{t,f^i,T}\Delta_{f,T}\right) \quad y^{rx}_{t,f^i,T} = y^r_{t,f,T} + y^x_{t,f^i,T}$$

$$\begin{aligned}
B_{t,f,T}\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right) &= \exp\left(-\left(y^r_{t,f,T} + y^x_{t,f^i,T}\right)\Delta_{f,T}\right) \\
B_{t,f,T}\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right) &= B_{t,f,T}\exp\left(-y^x_{t,f^i,T}\Delta_{f,T}\right) \\
\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right) &= \exp\left(-y^x_{t,f^i,T}\Delta_{f,T}\right)
\end{aligned}$$

$$\begin{aligned}
-y^x_{t,f^i,T}\Delta_{f,T} &= \log\left(\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right)\right) \\
y^x_{t,f^i,T} &= -\frac{\log\left(\sum_{j=1}^{K^x-1} -\alpha^x_{i,j}\exp\left(\Delta_{f,T}\mu_j^x\right)\right)}{\Delta_{f,T}}
\end{aligned}$$

- *CC Spread of a Spot ZC, conditional on* $\eta_t^x = i$

$$y^x_{t^i,T} = y^x_{t,t^i,T}$$

- *Event Intensity at T of a Forward Start ZC, conditional on $\eta^x_f = i$*

$$\lambda^x_{t,f^i,T} = \frac{\partial \log \Pi^x_{t,T}|_{i,K^x}}{\partial T}$$

- *Event Intensity at T of a Spot ZC, conditional on $\eta^x_t = i$*

$$\lambda^x_{t^i,T} = \lambda^x_{t,t^i,T}$$

and since $\Pi^x_{t,t+\Delta_t}|_{i,K^x} = \Pi^x_{f,f+\Delta_t}|_{i,K^x}$ then $\lambda^x_{t,f^i,f+\Delta_t} = \lambda^x_{t^i,t+\Delta_t}$

Curve Examples

As in the previous section we can see on Figure E.2 that under such simple framework it is possible to built a monotonous spread structure with the desired slope (blue line represents $y^x_{t,T}$ and the red line represents $f^x_{t,f}$). We will use a three state framework (two pre-event states and the event state).

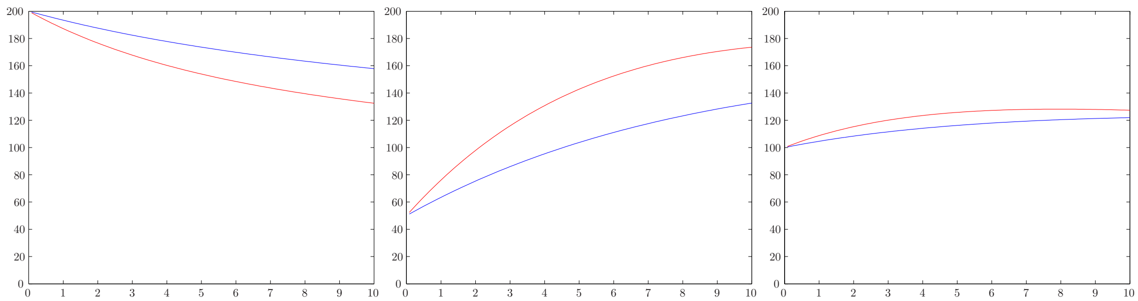
The examples are drawn from the generator matrixs, A^x, on Table E.1:

- The positive slope is obtained based on A^{x_p} and a initial state $\eta^x = 1$
- The negative slope is obtained based on A^{x_n} and a initial state $\eta^x = 2$
- The positive slope is obtained based on A^{x_n} and a initial state $\eta^x = 1$

$$A^{x_n} = \begin{bmatrix} -0.12 & 0.11 & 0.01 \\ 0.1 & -0.12 & 0.02 \\ 0 & 0 & 0 \end{bmatrix} \quad A^{x_p} = \begin{bmatrix} -0.12 & 0.115 & 0.005 \\ 0.02 & -0.05 & 0.03 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Negative Slope (b) Positive Slope

Table E.1: JLT Generator Matrix Examples



(a) Negative Slope (b) Positive Slope (c) Flat

Figure E.2: JLT Examples

E.3 Lando (1998) Model

Despite having already discussed exhaustively Lando (1998) model we did not gave an example with a concrete stochastic model and generator matrix. We shall do it now using the Vasicek model and the generator matrix described in the previous section under independence assumption between the stochastic factors behind interest rate and event:

$$\mathbf{X_u} = \begin{bmatrix} X_u^r \\ X_u^x \end{bmatrix}$$

$$\begin{aligned} y^r\left(\mathbf{X_u}\right) &= X_u^r \\ \mu_j^x\left(\mathbf{X_u}\right) &= \xi_j^x X_u^x \end{aligned}$$

where ξ_j^x is the i^{th} eigenvalue from base generator matrix A^x,.

Model

- *Value of a Forward Start ZC, conditional on* $\eta_f^x = i$
  Following eq. C.7

$$\begin{aligned}
B^{rx}_{t,f^i,T} &= \sum_{j=1}^{2} -\alpha^x_{i,j} E\left[\exp\left(-\int_f^T r\left(\mathbf{X_u}\right) - \mu^x_j\left(\mathbf{X_u}\right) du\right) | \mathscr{G}_t\right] \\
&= \sum_{j=1}^{2} -\alpha^x_{i,j} E\left[\exp\left(-\int_f^T X^r_u du\right) | \mathscr{G}_t\right] E\left[\exp\left(-\int_f^T \xi^x_j X^x_u du\right) | \mathscr{G}_t\right] \\
&= B_{t,f,T} \sum_{j=1}^{2} -\alpha^x_{i,j} E\left[\exp\left(-\int_f^T X^{xj}_u du\right) | \mathscr{G}_t\right] \\
&= B_{t,f,T} \sum_{j=1}^{2} -\alpha^x_{i,j} B^{X^{xj}}_{t,f,T} \qquad \text{(E.2)}
\end{aligned}$$

where $X^{xj} = \xi^x_j X^x_u$

$$\begin{aligned}
dX^x_t &= \varepsilon^2_A \left(\varepsilon^2_B - X^x_t\right) dt + \sqrt{\varepsilon^2_C + \varepsilon^2_D X^x_t} dW_t \\
dX^{xj} &= \xi^x_j dX^x \\
&= \xi^x_j \varepsilon^2_A \left(\varepsilon^2_B - X^x_t\right) dt + \xi^x_j \sqrt{\varepsilon^2_C + \varepsilon^2_D X^x_t} dW_t \\
&= \varepsilon^2_A \left(\xi^x_j \varepsilon^2_B - \xi^x_j \frac{X^{xj}_t}{\xi^x_j}\right) dt + \sqrt{\xi^{x2}_j \varepsilon^2_C + \xi^{x2}_j \varepsilon^2_D \frac{X^{xj}_t}{\xi^x_j}} dW_t \\
&= \varepsilon^2_A \left(\varepsilon^{2j}_B - X^{xj}_t\right) dt + \sqrt{\varepsilon^{2j}_C + \xi^{x2}_j \varepsilon^{2j}_D X^{xj}_t} dW_t
\end{aligned}$$

where $\varepsilon^{2j}_B = \xi^x_j \varepsilon^2_B \quad \varepsilon^{2j}_C = \xi^{x2}_j \varepsilon^2_C \quad \varepsilon^{2j}_D = \xi^x_j \varepsilon^2_D$
Then following Eq E.1

$$B^{rx}_{t,f^i,T} = B_{t,f,T} \sum_{j=1}^{2} -\alpha^x_{i,j} \exp\left(\mathcal{A}^x_{t,T} - \mathcal{A}^x_{t,f} - \mathcal{B}^x_{t,T} X^{xj}_t + \mathcal{B}^x_{t,f} X^{xj}_t\right) \qquad \text{(E.3)}$$

- *Value of a Spot ZC, conditional on* $\eta_t^x = i$

$$B^{rx}_{t^i,T} = B^{rx}_{t,t^i,T} \qquad \text{(E.4)}$$

- *CC Spread of a Forward Start ZC, conditional on* $\eta_f^x = i$

$$B^{rx}_{t,f^i,T} = \exp\left(-y^{rx}_{t,f^i,T}\Delta_{f,T}\right) \quad y^{rx}_{t,f^i,T} = r_{t,f,T} + y^x_{t,f^i,T}$$

$$\begin{aligned}
B_{t,f,T}\sum_{j=1}^{k^x-1} -\alpha^x_{i,j} B^{X^xj}_{t,f,T} &= \exp\left(-\left(r_{t,f,T} + y^x_{t,f^i,T}\right)\Delta_{f,T}\right)\\
B_{t,f,T}\sum_{j=1}^{k^x-1} -\alpha^x_{i,j} B^{X^xj}_{t,f,T} &= B_{t,f,T}\exp\left(-y^x_{t,f^i,T}\Delta_{f,T}\right)\\
\sum_{j=1}^{k^x-1} -\alpha^x_{i,j} B^{X^xj}_{t,f,T} &= \exp\left(-y^x_{t,f^i,T}\Delta_{f,T}\right)
\end{aligned}$$

$$\begin{aligned}
-\,y^x_{t,f^i,T}\Delta_{t,T} &= \log\left(\sum_{j=1}^{k^x-1} -\alpha^x_{i,j} B^{X^xj}_{t,f,T}\right)\\
y^x_{t,f^i,T} &= -\frac{\log\left(\sum_{j=1}^{k^x-1} -\alpha^x_{i,j} B^{X^xj}_{t,f,T}\right)}{\Delta_{f,T}}
\end{aligned} \tag{E.5}$$

- *CC Spread of a Spot ZC, conditional on* $\eta_t^x = i$

$$y^x_{t^i,T} = y^x_{t,t^i,T} \tag{E.6}$$

- *Event Intensity at* T *of a Spot ZC, conditional on* $\eta_t^x = i$

$$\begin{aligned}
\lambda^x_{t^i,T} &= \frac{\partial \log \Pi^x_{t,T}|_{i,k^x}}{\partial T}\\
&= \frac{\partial \log\left(1 - \sum_{j=1}^2 -\alpha^x_{i,j} B^{X^xj}_{t,T}\right)}{\partial T}
\end{aligned}$$

- *Event Intensity at* T *of a Forward Start ZC, conditional on* $\eta_f^x = i$

$$\begin{aligned}
\lambda^x_{t,f^i,T} &= \frac{\partial \log \Pi^x_{f,T}|_{i,k^x}}{\partial T}\\
&= \frac{\partial \log\left(1 - \sum_{j=1}^2 -\alpha^x_{i,j} B^{X^xj}_{t,f,T}\right)}{\partial T}\\
&\neq \lambda^x_{f^i,T}
\end{aligned}$$

Curve Examples

In the previous sections we shown how to build a monotone curve. It follows naturally that combining those simple models we are able to build a non monotone spread curve. Nevertheless our focus will not be the spot curve but the refreshed curve: a curve that periodically has its ratting reset so that its implied credit quality stays relatively constant. Using a matrix environment that produces positive slope spread curve, we will do some experiments to test the reaction of the forward of several tenors, to changes on the stochastic component.

Our base generator matrix, A^x, whose eigenvalues, ξ^x, and eigenvectors, B^{x}, produce the model parameters α^x.

$$A^x = \begin{bmatrix} -0.12 & 0.115 & 0.005 \\ 0.025 & -0.05 & 0.025 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathsf{B}^{\mathsf{x}} = \begin{bmatrix} -0.9696 & -0.7578 & 0.5774 \\ 0.2448 & -0.6525 & 0.5774 \\ 0 & 0 & 0.5774 \end{bmatrix} \quad \xi^x = \begin{bmatrix} -0.149 & 0 & 0 \\ 0 & -0.021 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha^x = \begin{bmatrix} 0.1247 & -1.1247 \\ -0.0315 & -0.9685 \end{bmatrix}$$

In order to show the impact of X_u^x in the generator matrix it is useful to show the generator matrix for different X_u^x inputs.

$$A^x(3) = \begin{bmatrix} -0.36 & 0.23 & 0.01 \\ 0.05 & -0.1 & 0.05 \\ 0 & 0 & 0 \end{bmatrix} \quad A^x(0.5) = \begin{bmatrix} -0.06 & 0.0575 & 0.0025 \\ 0.0125 & -0.025 & 0.0125 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^x(1) = \begin{bmatrix} -0.12 & 0.115 & 0.005 \\ 0.025 & -0.05 & 0.025 \\ 0 & 0 & 0 \end{bmatrix} = A^x$$

As we can see the probability of default and transition probabilities are multiplied by X_u^x, making it a determinant factor for the event intensity.

Then by manipulating the Vasicek parameters we can achieve the profiles depicted on Figure E.3 of the following four curves:

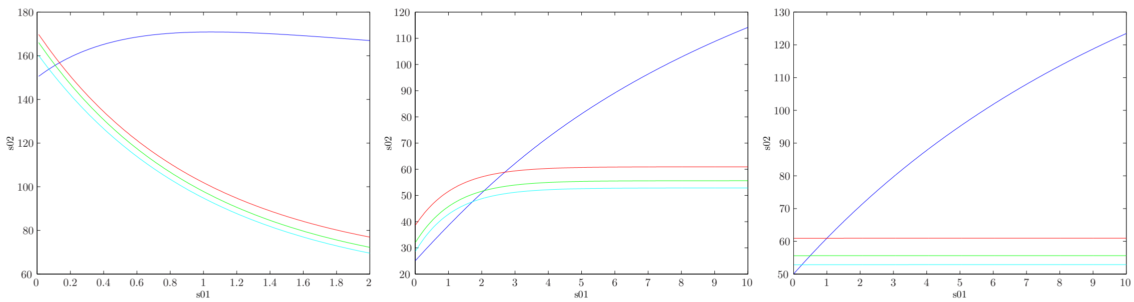
$y^x_{0^1,t}$ CC Spread of a Spot ZC, conditional on $\eta^x_0 = 1$

$y^x_{0,t^1,t+0.25}$ CC Spread of a Forward Start 3M ZC, conditional on $\eta^x_t = 1$

$y^x_{0,t^1,t+0.5}$ CC Spread of a Forward Start 6M ZC, conditional on $\eta^x_t = 1$

$y^x_{0,t^1,t+1}$ CC Spread of a Forward Start 12M ZC, conditional $\eta^x_t = 1$

As is clear the smaller the maturity of the forward the smaller the spread. This is related with the structural upward slope of the generator matrix: it would take a big downward movement of $X^x_u$to invert this situation[2] .



(a) Negative Slope
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 3$

(b) Positive Slope
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 0.5$

(c) Flat
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 1$

Figure E.3: Lando Spread Examples

In E.4 the evolution of the event intensity of some of the spot and forward instruments present in Figure E.3 are displayed:

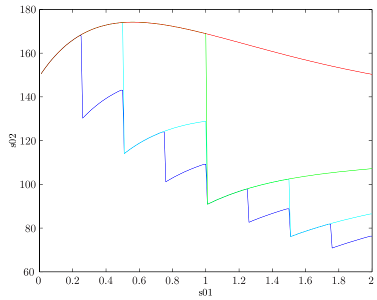
$\lambda^x_{0^1,t}$ Event Intensity at t of a Spot ZC, conditional on $\eta^x_0 = 1$

$\lambda^x_{0,f^1,t}$ Event Intensity at t of a Forward Start 3M ZC, conditional on $\eta^x_f = 1$ where f is the previous 3M multiple in t

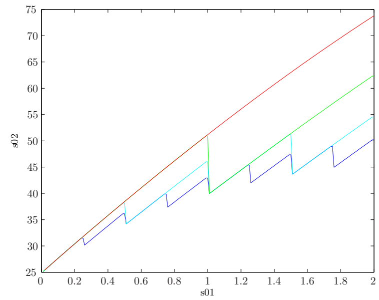
$y^x_{0,t^1,t+0.5}$ CC Spread of a Forward Start 6M ZC, conditional on $\eta^x_t = 1$

$y^x_{0,t^1,t+1}$ CC Spread of a Forward Start 12M ZC, conditional on $\eta^x_t = 1$

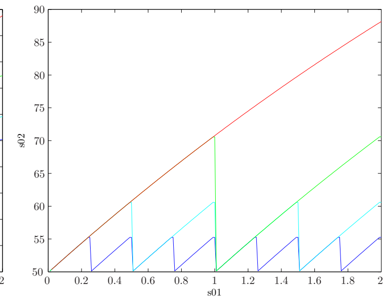
[2] It could be achieved setting the long term parameter much smaller than X^x_0



(a) Negative Slope
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 3$



(b) Positive Slope
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 0.5$



(c) Flat
$\varepsilon_A = 0.9 \quad \varepsilon_B = 1$
$\varepsilon_C = 0.05^2 \quad X_0 = 1$

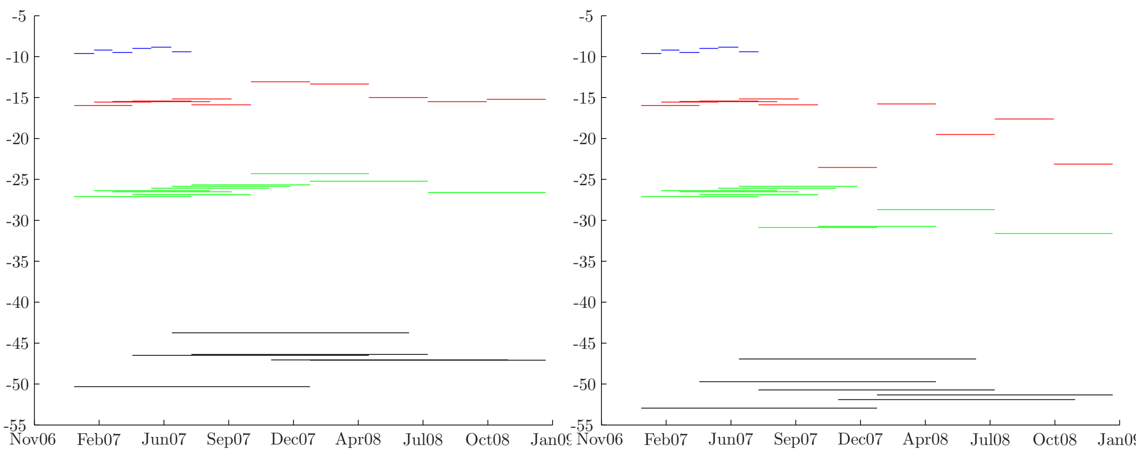
Figure E.4: Lando Intensity

As we can see a smaller intensity reflects a smaller "refresh" period and this result is obtained independently of the evolution of X_u^x as long as we have a structurally positive sloped matrix, which is always valid for the highest rating.

Appendix F

Graphs

F.1 Before August 2007

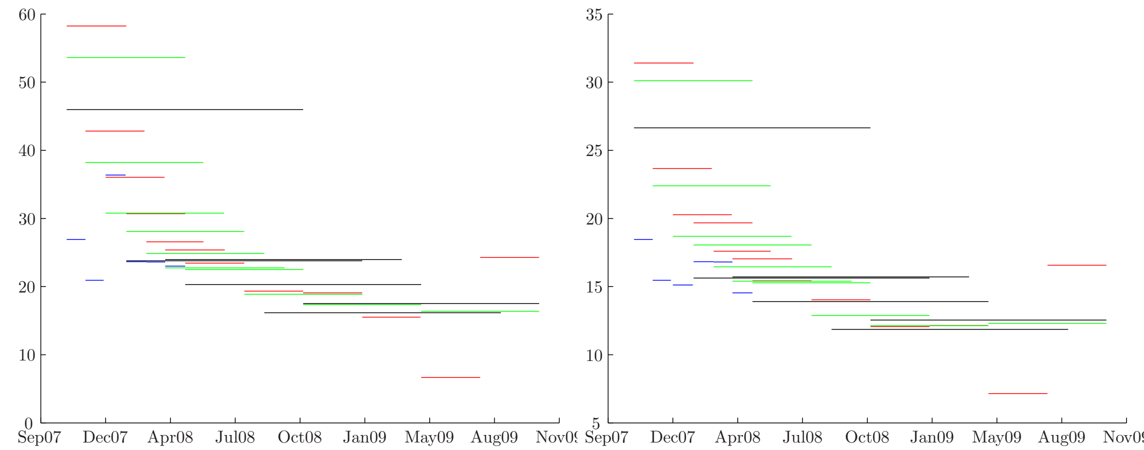


(a) BOR-OIS-CREDIT

(b) BOR-OIS-CREDIT-ERP

Figure F.1: USD LIBOR Decomposition 2 21Feb2007

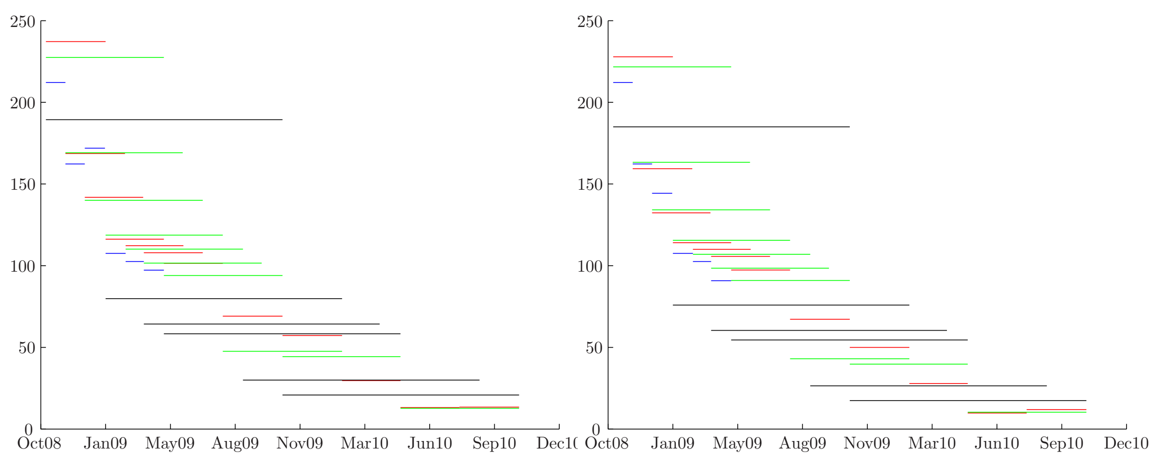
F.2 Right After August 2007



(a) BOR-OIS-CREDIT (b) BOR-OIS-CREDIT-ERP

Figure F.2: USD LIBOR Decomposition 2 23Oct2007

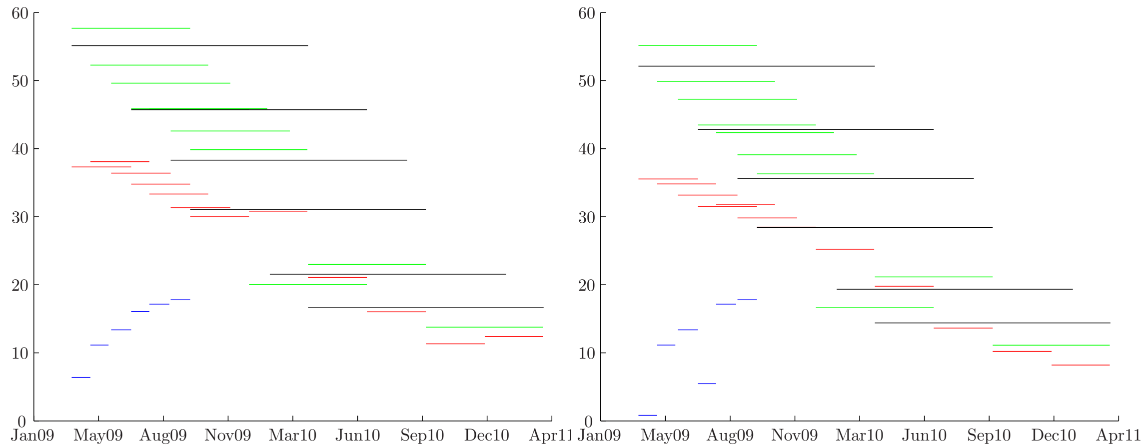
F.3 Right After Lehman



(a) BOR-OIS-CREDIT (b) BOR-OIS-CREDIT-ERP

Figure F.3: USD LIBOR Decomposition 2 23Oct2008

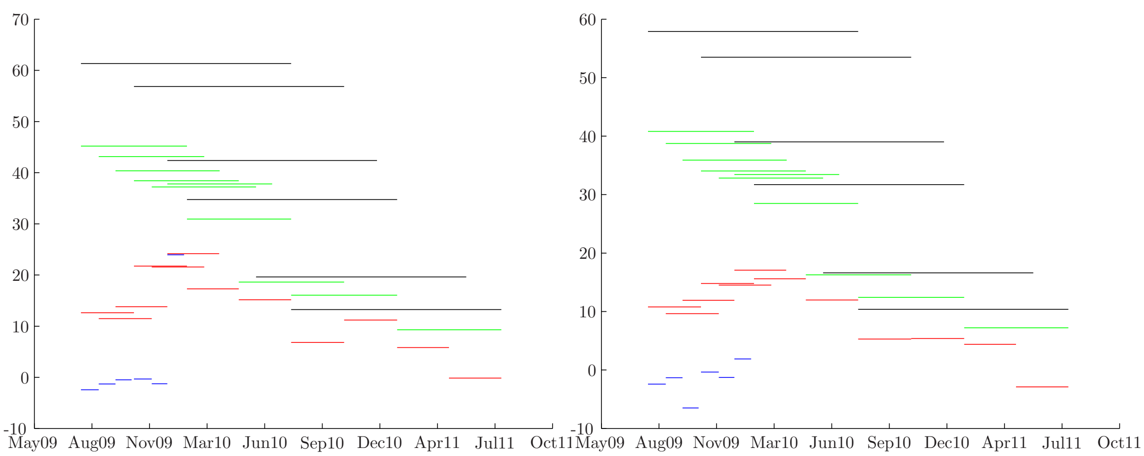
F.4 Beginning 2009



(a) BOR-OIS-CREDIT (b) BOR-OIS-CREDIT-ERP

Figure F.4: USD LIBOR Decomposition 2 24Mar2009

F.5 Summer 2009



(a) BOR-OIS-CREDIT (b) BOR-OIS-CREDIT-ERP

Figure F.5: USD LIBOR Decomposition 2 23Jul2009

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