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THE HESTON MODEL UNDER STOCHASTIC INTEREST RATES

Hugo Miguel Fernandes Marques

Dissertação submetida como requisito parcial para obtenção do grau de

Mestre em Matemática Financeira

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Abstract

In this dissertation the Heston (1993) model is considered, but using, instead of a constant interest rate, stochastic interest rates according to Vasicek (1977) and to Cox, Ingersoll and Ross (1985) models. Under this framework, a closed-form solution is determined for the price of European standard calls, which, by using a manipulation implemented by Attari (2004), only require the evaluation of one characteristic function. For forward-start European calls, starting from the result for standard calls and using analytic characteristic functions, it is determined a closed-form solution that only requires one numerical integration. In the end, the results of these closed-form solutions are compared with the results presented by Monte Carlo simulations for the considered models.

Keywords: European Standard Call, Forward-Start European Call, Stochastic Volatility, Stochastic Interest Rate.

JEL Classification: G13.

Resumo

Nesta dissertação é considerado o modelo de Heston (1993), mas em vez de utilizar uma taxa de juro constante, considera-se taxas de juro estocásticas segundo os modelos de Vasicek (1977) e de Cox, Ingersoll e Ross (1985). Neste contexto, é determinada uma solução fechada para a avaliação de standard calls Europeias, que, por ter sido usada uma manipulação implementada por Attari (2004), apenas necessitará da avaliação de uma função característica. Para calls forward-start Europeias, partindo do resultado apresentado para standard calls e utilizando funções característica analíticas, é determinada uma solução fechada que também recorrerá a apenas uma integração numérica. No final, os resultados destas fórmulas fechadas são comparados com os resultantes de simulações de Monte Carlo para os modelos considerados.

Palavras Chave: Standard Call Europeia, Forward-Start Call Europeia, Volatilidade Estocástica, Taxa de Juro Estocástica.

Classificação JEL : G13.

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Chapter 1

Introduction

The Black-Scholes (1973) model brought a great development to option pricing theory, but it considers some unrealistic assumptions, namely that the stock's volatility and the interest rate are known and constant.

An alternative to Black-Scholes that considers stochastic volatility is the Heston model (1993). According to this model, the underlying price process S_t of a dividend paying asset and its instantaneous variance v_t follow, under the risk-neutral measure $\mathbb{Q}$ (the probability measure associated to the money market account), the stochastic differential equations:

$$dS_t = (r_t - q)\, S_t dt + \sqrt{v_t} S_t dW_1^{\mathbb{Q}}(t), \tag{1.1}$$

$$dv_t = \kappa\,(\theta - v_t)\, dt + \sigma \sqrt{v_t} dW_2^{\mathbb{Q}}(t), \tag{1.2}$$

and

$$d\left\langle W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}} \right\rangle_t = \rho dt \tag{1.3}$$

where, κ, θ and σ are positive constants, such that $2\kappa\theta > \sigma^2$, ensuring that v_t remains positive (Feller, 1951, p.180).

Seeking a more realistic framework, some models have also considered, besides equations (1.1) to (1.3), a stochastic interest rate. This is the case of Bakshi, Cao and Chen (1997), who have considered the short rate term structure model of Cox, Ingersoll and Ross (1985) (as well as random jumps), and the case of Hout, Bierkens, Ploeg and Panhuis (2007), who have used the Hull-White (1990) no-arbitrage extended Vasicek (1977) model.

I will consider the usual conditions of the Heston model - equations (1.1) to (1.3) - and two possibilities for the stochastic interest rate, namelly, the Vasiček (1977) and the Cox, Ingersoll and Ross (1985) models:

$$\text{(Vasicek)} \qquad dr_t = \kappa_r (\theta_r - r_t)\, dt + \rho_r dW_3^{\mathbb{Q}}(t), \tag{1.4}$$

$$\text{(Cox, Ingersoll, Ross(CIR))} \qquad dr_t = \kappa_r (\theta_r - r_t)\, dt + \sigma_r \sqrt{r_t} dW_3^{\mathbb{Q}}(t), \tag{1.5}$$

$$\text{with} \qquad d\left\langle W_1^{\mathbb{Q}}, W_3^{\mathbb{Q}} \right\rangle = 0, \tag{1.6}$$

$$d\left\langle W_2^{\mathbb{Q}}, W_3^{\mathbb{Q}} \right\rangle = 0, \tag{1.7}$$

and where κ_r, θ_r, ρ_r and σ_r are positive constants, such that $2\kappa_r\theta_r > \sigma_r^2$, ensuring that, in the CIR model, r_t remains positive (Feller, 1951, p.180). Unfortunately, in the Vasiček model r_t may become negative.

I will, when possible and practical, consider the generalization:

$$dr_t = \kappa_r (\theta_r - r_t)\, dt + \sqrt{\sigma_r^2 r_t + \rho_r^2} dW_3^{\mathbb{Q}}(t) \tag{1.8}$$

In the following Chapter, I start with the pricing of a standard European call, but instead of using two characteristic functions like Bakshi, Cao and Chen (1997), or Hout, Bierkens, Ploeg and Panhuis (2007), I will use a manipulation applied by Attari (2004) to the pricing of options under the Heston model (1993), that uses only one characteristic function. This approach will allow the computation of the option price with only one numerical integration. I will finish the Chapter by making some modifications to the obtained formulas so that discontinuities in the complex logarithms are avoided.

In Chapter 3, I will determine the price of an European forward-start call under the defined models. A closed-form solution has already been determined in the case of the Heston model (1993) with constant interest rates, by Kruse and Nögel (2005). I will start by following a similar path but, by using characteristic functions, I simplify the numerical implementation. Even without considering a stochastic interest rate, Kruse and Nögel's formula (2005) already requires the numerical calculation of four integrals. Despite considering a stochastic interest rate, because I will be using the formula determined in the previous Chapter along with characteristic functions, I will be able to narrow to only one integral to be computed numerically.

In the final chapter I will present the obtained results with the numerical implementation of the closed-form formulas and compare them with results attained through Monte Carlo simulations.

Chapter 2

Pricing of Standard European Options

2.1 Zero Coupon Bond

To price options under the described models, I will need $P(t,T)$, the time - t price of a default-free zero coupon bond which pays one monetary unit at the maturity date T ($\geq t$). Considering the interest rate process defined by equation (1.8), and following Björk (2003, p.320), $P(t,T)$ satisfies:

$$dP(t,T) = r_t P(t,T)dt + \sqrt{\sigma^2 r_t + \rho_r^2}\frac{\partial P}{\partial r_t}(t,T)dW_3^{\mathbb{Q}}(t). \tag{2.1}$$

Moreover and following again Björk (2003, p.328), in an arbitrage free bond market, $P(t,T)$ must satisfy the PDE:

$$-\frac{\partial P}{\partial \tau} + \kappa_r (\theta_r - r_t) \frac{\partial P}{\partial r_t} + \frac{1}{2}\left(\sigma_r^2 r_t + \rho_r^2\right)\frac{\partial^2 P}{\partial r_t^2} - r_t P = 0, \tag{2.2}$$

subject to $P(T,T) = 1$ and where $\tau = T - t$.

Equation (1.8) also ensures that the model has an affine structure (Björk, 2003, p.331), therefore admitting a solution of the type:

$$P\left(t,T\right) = \exp\left[a\left(\tau\right) - b\left(\tau\right)r_t\right], \tag{2.3}$$

where $a(\tau)$ and $b(\tau)$ are deterministic functions of time.

After incorporating equation (2.3) into equation (2.2) and solving the resulting ODE's (Appendix A), the bond's price is known:

Vasicek

$$b\left(\tau\right) = \frac{1-\exp\left(-\kappa_r\tau\right)}{\kappa_r} \tag{2.4}$$

$$a\left(\tau\right) = \left(b\left(\tau\right)-\tau\right)\left(\theta_r - \frac{\rho_r^2}{2\kappa_r^2}\right) - \left(b\left(\tau\right)\right)^2\frac{\rho_r^2}{4\kappa_r} \tag{2.5}$$

CIR

$$b\left(\tau\right) = \frac{2\left(\exp\left(\xi\tau\right)-1\right)}{2\xi+\left(\exp\left(\xi\tau\right)-1\right)\left(\xi+\kappa_r\right)} \tag{2.6}$$

$$a\left(\tau\right) = \frac{2\kappa_r\theta_r}{\sigma_r^2}\ln\left(\frac{2\xi\exp\left(\left(\xi+\kappa_r\right)\frac{\tau}{2}\right)}{2\xi+\left(\exp\left(\xi\tau\right)-1\right)\left(\xi+\kappa_r\right)}\right) \tag{2.7}$$

where $\xi = \sqrt{\kappa_r^2 + 2\sigma_r^2}$.

2.2 Change of Numeraire

Applying Itô's lemma to $\ln S_t$, a solution to equation (1.1) may be expressed as:

$$d\ln S_t = \left(r_t - q - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_1^{\mathbb{Q}}(t)$$

$$\Leftrightarrow \ln\left(\frac{S_T}{S_t}\right) = \int_t^T\left(r_u - q - \frac{1}{2}v_u\right)du + \int_t^T\sqrt{v_u}dW_1^{\mathbb{Q}}(u)$$

$$\Leftrightarrow S_T = S_t e^{\left(\int_t^T\left(r_u-q-\frac{1}{2}v_u\right)du+\int_t^T\sqrt{v_u}dW_1^{\mathbb{Q}}(u)\right)} \tag{2.8}$$

$$\Leftrightarrow S_T = S_t e^{-q\tau+x(t,T)} \tag{2.9}$$

where $x(t,T)$ is the random shock term in stock prices (which I will represent, to simplify notation, by x_T). That way, $x_t|\,\mathcal{F}_t = 0$ and the following equation may be considered:

$$dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_1^{\mathbb{Q}}(t) \tag{2.10}$$

To apply the multidimensional version of Girsanov's Theorem, the three Brownian motions can be made independent (Shreve, 2004, p.224). To restate this model with independent Brownian motions, consider the following three independent Brownian motions under measure $\mathbb{Q}$:

$$Z_1^{\mathbb{Q}} = W_1^{\mathbb{Q}}, \qquad Z_3^{\mathbb{Q}} = W_3^{\mathbb{Q}} \qquad \text{and} \qquad Z_2^{\mathbb{Q}} = \frac{W_2^{\mathbb{Q}} - \rho Z_1^{\mathbb{Q}}}{\sqrt{1-\rho^2}} \tag{2.11}$$

Consequently, the pricing model becomes:

$$dx_t = \left(r_t - \frac{1}{2}v_t\right) dt + \sqrt{v_t} dZ_1^{\mathbb{Q}}(t) \tag{2.12}$$

$$dv_t = \kappa\left(\theta - v_t\right) dt + \sigma\sqrt{v_t}\left(\rho dZ_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathbb{Q}}(t)\right) \tag{2.13}$$

$$dr_t = \kappa_r\left(\theta_r - r_t\right) dt + \sqrt{\sigma_r^2 r_t + \rho_r^2} dZ_3^{\mathbb{Q}}(t) \tag{2.14}$$

According to the general arbitrage theory, the option price can be computed as:

$$\begin{aligned} c_t\left(S_t, v_t, r_t, X, T\right) &= B_t \mathbb{E}_{\mathbb{Q}}\left[\left.\frac{\left(S_T - X\right)^+}{B_T}\right| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\left.\frac{B_t}{B_T} S_T 1_{\{S_T > X\}}\right| \mathcal{F}_t\right] - X\mathbb{E}_{\mathbb{Q}}\left[\left.\frac{B_t}{B_T} 1_{\{S_T > X\}}\right| \mathcal{F}_t\right], \end{aligned} \tag{2.15}$$

where

$$B_t := \exp\left(\int_0^t r_u du\right) \tag{2.16}$$

is the time - t value of the money market account.

At this point, in order to prepare the expression for the simplification introduced by Attari(2004), instead of making a numeraire change for each expected value, I will change them both from the risk-neutral measure $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}^T$, that uses $P\left(t, T\right)$ as numeraire. Attari (2004) did not consider a stochastic interest rate, but, as he refers, in that case, a forward measure should be used instead of the risk-neutral measure.

To determine the Radon-Nikodym derivative, another expression for $P(t,T)$ is convenient, which can be obtained by using equation (2.1) and applying Itô's lemma to $\ln P(t,T)$:

$$d\ln P(t,T) = \left[r_t - \frac{1}{2}\left(\frac{\sqrt{\sigma^2 r_t + \rho_r^2}}{P(t,T)} \frac{\partial P}{\partial r_t} \right)^2 \right] dt + \frac{\sqrt{\sigma^2 r_t + \rho_r^2}}{P(t,T)} \frac{\partial P}{\partial r_t} dZ_3^{\mathbb{Q}}(t)$$

$$\Leftrightarrow \frac{1}{P(t,T)} = e^{\int_t^T \left[r_u - \frac{1}{2} \frac{\sigma_r^2 r_u + \rho_r^2}{P^2(u,T)} \left(\frac{\partial P}{\partial r_u} \right)^2 \right] du + \int_t^T \left[\frac{\sqrt{\sigma_r^2 r_u + \rho_r^2}}{P(u,T)} \frac{\partial P}{\partial r_u} \right] dZ_3^{\mathbb{Q}}(u)} \tag{2.17}$$

Using equations (2.16) and (2.17), the Radon-Nikodym derivative can be determined as:

$$\begin{aligned} \left. \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \right| \mathcal{F}_t &= \frac{B_t}{B_T} \frac{P(T,T)}{P(t,T)} \\ &= e^{-\frac{1}{2} \int_t^T \left[\frac{\sigma_r^2 r_u + \rho_r^2}{P^2(u,T)} \left(\frac{\partial P}{\partial r_t} \right)^2 \right] du + \int_t^T \left[\frac{\sqrt{\sigma_r^2 r_u + \rho_r^2}}{P(u,T)} \frac{\partial P}{\partial r_t} \right] dZ_3^{\mathbb{Q}}(u)} \end{aligned} \tag{2.18}$$

But, to apply Girsanov′s theorem to make this numeraire change, it is necessary to verify that the right-hand side of equation (2.18) is defined and is a martingale. That can be made by verifying Novikov′s condition or, alternatively, according to Revuz and Yor (1999, p. 338), by verifying the existence of two constants a and c such that $\mathbb{E}_{\mathbb{Q}} \left[\exp(aY^2(s)) | \mathcal{F}_t \right] \leq c \; \forall s : t \leq s \leq T$, where $Y^2(t)$ is $\frac{\sigma_R^2 r_t + \rho_r^2}{P^2(t,T)} \left(\frac{\partial P}{\partial r_t} \right)^2$ (see Appendix B). Girsanov′s theorem then leads to:

$$dZ_1^{\mathbb{Q}^T}(t) = dZ_1^{\mathbb{Q}}(t) \tag{2.19}$$

$$dZ_2^{\mathbb{Q}^T}(t) = dZ_2^{\mathbb{Q}}(t) \tag{2.20}$$

$$dZ_3^{\mathbb{Q}^T}(t) = dZ_3^{\mathbb{Q}}(t) - \frac{\sqrt{\sigma_r^2 r_t + \rho_r^2}}{P(t,T)} \frac{\partial P}{\partial r_t} dt \tag{2.21}$$

With this numeraire change, and after returning to correlated Brownian motions ($W_1^{\mathbb{Q}^T} = Z_1^{\mathbb{Q}^T}$, $W_3^{\mathbb{Q}^T} = Z_3^{\mathbb{Q}^T}$ and $W_2^{\mathbb{Q}^T} = \rho Z_1^{\mathbb{Q}^T} + \sqrt{1-\rho^2} Z_2^{\mathbb{Q}^T}$), the pricing model becomes:

$$dx_t = \left(r_t - \frac{1}{2} v_t \right) dt + \sqrt{v_t} dW_1^{\mathbb{Q}^T}(t) \tag{2.22}$$

$$dv_t = \kappa \left(\theta - v_t \right) dt + \sigma \sqrt{v_t} dW_2^{\mathbb{Q}^T}(t) \tag{2.23}$$

$$dr_t = \left(\kappa_r \left(\theta_r - r_t \right) + \frac{\sigma_r^2 r_t + \rho_r^2}{P(t,T)} \frac{\partial P}{\partial r_t} \right) dt + \sqrt{\sigma_r^2 r_t + \rho_r^2} dW_3^{\mathbb{Q}^T}(t) \tag{2.24}$$

$$d \left\langle W_1^{\mathbb{Q}^T}, W_2^{\mathbb{Q}^T} \right\rangle_t = \rho dt \tag{2.25}$$

$$d \left\langle W_1^{\mathbb{Q}^T}, W_3^{\mathbb{Q}^T} \right\rangle_t = 0 \tag{2.26}$$

$$d \left\langle W_2^{\mathbb{Q}^T}, W_3^{\mathbb{Q}^T} \right\rangle_t = 0 \tag{2.27}$$

Proceeding with the numeraire change into equation (2.15):

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}^T} \left[\frac{P(t,T)}{P(T,T)} S_T 1_{\{S_T > X\}} \middle| \mathcal{F}_t \right] - X \mathbb{E}_{\mathbb{Q}^T} \left[\frac{P(t,T)}{P(T,T)} 1_{\{S_T > X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}^T} \left[P(t,T) S_T 1_{\{S_T > X\}} \middle| \mathcal{F}_t \right] - X \mathbb{E}_{\mathbb{Q}^T} \left[P(t,T) 1_{\{S_T > X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}^T} \left[P(t,T) S_t e^{-q\tau + x_T} 1_{\{x_T > l\}} \middle| \mathcal{F}_t \right] - X \mathbb{E}_{\mathbb{Q}^T} \left[P(t,T) 1_{\{x_T > l\}} \middle| \mathcal{F}_t \right] \\
&= S_t e^{-q\tau} \mathbb{E}_{\mathbb{Q}^T} \left[P(t,T) e^{x_T} 1_{\{x_T > l\}} \middle| \mathcal{F}_t \right] - X P(t,T) \mathbb{E}_{\mathbb{Q}^T} \left[1_{\{x_T > l\}} \middle| \mathcal{F}_t \right] \\
&= S_t e^{-q\tau} \int_l^\infty P(t,T) e^x f(x) dx - X P(t,T) \int_l^\infty f(x) dx \\
&= S_t e^{-q\tau} \Pi_1 \left(t, x_t, v_t, r_t, X, T \right) - X P(t,T) \Pi_2 \left(t, x_t, v_t, r_t, X, T \right)
\end{aligned} \tag{2.28}$$

where $l := \ln \left(\frac{X}{S_t} e^{q\tau} \right)$ and $f(x)$ is the density function associated to x_T under measure $\mathbb{Q}^T$.

Π_2 will be determined through the characteristic function as usual, but Π_1 will be determined through the same characteristic function as Π_2, using a modification applied by Attari(2004).

2.3 The Characteristic Function

2.3.1 A Single Integral Solution

According to the Fourier Inversion theorem, given the characteristic function of a random variable X: $\varphi(\phi) = \mathbb{E}\left[e^{i\phi X}\right]$, the density function $f(x)$ and the distribution function $F(x)$ may be written as:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x}\varphi(\phi)d\phi \tag{2.29}$$

$$F(x) = \frac{1}{2} + \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{i\phi x}\varphi(-\phi) - e^{-i\phi x}\varphi(\phi)}{i\phi}d\phi \tag{2.30}$$

Let $\varphi(x_t, v_t, r_t; T, \phi) = \mathbb{E}_{\mathbb{Q}^T}\left[\exp(i\phi x_T) | \mathcal{F}_t\right]$ be the characteristic function associated to x_T under the T-forward measure (for notational simplicity I will represent it by $\varphi(\phi)$).

From equation (2.30), and since the characteristic function is an hermitian function ($\varphi(-\phi) = \overline{\varphi(\phi)}$), Π_2 can be determined as follows:

$$\begin{aligned}
&\Pi_2(t, x_t, v_t, r_t, X, T) = \mathbb{Q}^T(x_T > l) = 1 - \mathbb{Q}^T(x_T \leq l) \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{i\phi l}\varphi(-\phi) - e^{-i\phi l}\varphi(\phi)}{i\phi}d\phi \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{i\phi l}\overline{\varphi(\phi)} - e^{-i\phi l}\varphi(\phi)}{i\phi}d\phi \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_{0}^{\infty} \frac{2i\sin(\phi l)\mathrm{Re}(\varphi(\phi)) - 2i\cos(\phi l)\mathrm{Im}(\varphi(\phi))}{i\phi}d\phi \\
&= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos(\phi l)\mathrm{Im}(\varphi(\phi)) - \sin(\phi l)\mathrm{Re}(\varphi(\phi))}{\phi}d\phi
\end{aligned} \tag{2.31}$$

To determine Π_1's expression, I follow Attari(2004), starting by using equation (2.29) in the expression that Π_1 had in equation (2.28), and then changing the order of integration:

$$\begin{aligned}\Pi_1 &= \int_l^\infty P(t,T)e^x \left(\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\phi x}\varphi(\phi)d\phi\right)dx \\ &= \frac{P(t,T)}{2\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_l^\infty e^{-i(\phi+i)x}dx\right)d\phi \end{aligned} \tag{2.32}$$

Because $\Pi_1|_{l=-\infty} = 1$ ($S_t e^{-q\tau}$ is the value of this option if the strike is zero), equation (2.32) implies that:

$$\begin{aligned}\Pi_1 =& 1 - \frac{P(t,T)}{2\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_{-\infty}^l e^{-i(\phi+i)x}dx\right)d\phi \\ =& \frac{1}{2} + \frac{\Pi_1|_{l=-\infty}}{2} - \frac{P(t,T)}{2\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_{-\infty}^l e^{-i(\phi+i)x}dx\right)d\phi \\ =& \frac{1}{2} + \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_l^\infty e^{-i(\phi+i)x}dx\right)d\phi \\ &+ \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_{-\infty}^l e^{-i(\phi+i)x}dx\right)d\phi \\ &- \frac{P(t,T)}{2\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_{-\infty}^l e^{-i(\phi+i)x}dx\right)d\phi \\ =& \frac{1}{2} + \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_l^\infty e^{-i(\phi+i)x}dx\right)d\phi \\ &- \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\int_{-\infty}^l e^{-i(\phi+i)x}dx\right)d\phi \\ =& \frac{1}{2} + \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\lim_{R\to\infty}\frac{e^{-i(\phi+i)R}}{-i(\phi+i)} - \frac{e^{-i(\phi+i)l}}{-i(\phi+i)}\right)d\phi \\ &- \frac{P(t,T)}{4\pi}\int_{-\infty}^\infty \varphi(\phi)\left(\frac{e^{-i(\phi+i)l}}{-i(\phi+i)} - \lim_{R\to-\infty}\frac{e^{-i(\phi+i)R}}{-i(\phi+i)}\right)d\phi \\ =& \frac{1}{2} + \frac{P(t,T)}{2\pi}\int_{-\infty}^\infty \varphi(\phi)\frac{e^{-i(\phi+i)l}}{i(\phi+i)}d\phi - \frac{P(t,T)}{4\pi}\lim_{R\to\infty}\int_{-\infty}^\infty \varphi(\phi)\frac{e^{R-i\phi R}}{i(\phi+i)}d\phi \end{aligned} \tag{2.33}$$

because $\lim_{R\to-\infty} e^{-i(\phi+i)R} = \lim_{R\to-\infty} e^R\left(\cos(-\phi R) + i\sin(-\phi R)\right) = 0$

The second integral may be evaluated by applying a result related to the Residue Theorem (see for instance Matos and Santos (2000, p. 236)). This result states that, being $f : U\backslash A \to \mathbb{C}$ an analytic function (where U is a open set containing the complex half-plane defined by $\text{Im}(z) \geq 0$, and A is a finite part of $U\backslash\mathbb{R}$), if $\lim_{z\to\infty} f(z) = 0$, then, for all positive a, $\int_{-\infty}^{+\infty} e^{iax} f(x)dx = 2\pi i \sum_{\xi\in\{z\in A:\text{Im}(z)>0\}} \text{res}(\xi, e^{iaz} f(z))$. After making the variable change $w = -\phi$, the integral is in the necessary conditions:

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(\phi)\frac{e^{R-i\phi R}}{i(\phi+i)}d\phi &= \int_{-\infty}^{\infty} \varphi(-w)\frac{e^{R+iwR}}{i(-w+i)}dw \\ &= 2\pi i \text{resid}\left(i, \varphi(-w)\frac{e^{R+iwR}}{i(-w+i)}\right) \\ &= -2\pi\varphi(-i) \end{aligned} \tag{2.34}$$

The value of $\varphi(-i)$ can be calculated by determining explicitly the value of $\Pi_1|_{l=-\infty}$ (proceeding like in the steps that led to equations (2.33) and (2.34)) and then using the fact that it has to be 1, like Attari(2004) did. Alternatively, knowing the expression of φ (which I will determine next), it is easy to conclude that $\varphi(-i) = P(t,T)^{-1}$.

Replacing equation (2.34) into (2.33):

$$\begin{aligned}\Pi_1 &= 1 + \frac{P(t,T)}{2\pi}\int_{-\infty}^{\infty} \varphi(\phi)\frac{e^{-i(\phi+i)l}}{i(\phi+i)}d\phi \\ &= 1 + \frac{e^l P(t,T)}{2\pi}\int_0^{\infty} \frac{\varphi(\phi)e^{-i\phi l}}{i(\phi+i)} + \frac{\varphi(-\phi)e^{i\phi l}}{i(-\phi+i)}d\phi \\ &= 1 + \frac{e^l P(t,T)}{2\pi}\int_0^{\infty} \frac{-\overline{\varphi(\phi)}e^{i\phi l}(i+\phi) - \varphi(\phi)e^{-i\phi l}(i-\phi)}{i(1+\phi^2)}d\phi \\ &= 1 - \frac{e^l P(t,T)}{2\pi}\int_0^{\infty} \frac{\varphi(\phi)e^{-i\phi l}(\phi-i) + \overline{\varphi(\phi)}e^{i\phi l}(i+\phi)}{i(1+\phi^2)}d\phi \\ &= 1 - \frac{e^l P(t,T)}{\pi}\int_0^{\infty} \frac{\text{Re}(\varphi)(\cos(\phi l) + \sin(\phi l)\phi) + \text{Im}(\varphi)(\sin(\phi l) - \cos(\phi l)\phi)}{1+\phi^2}d\phi \end{aligned} \tag{2.35}$$

Introducing equations (2.31) and (2.35) into equation (2.28) and rearranging the terms leads to the following option pricing formula:

$$
\begin{aligned}
&c_t(S_t, v_t, r_t, X, T) = S_t e^{-q\tau} - P(t,T)\frac{X}{2} \\
&-P(t,T)\frac{X}{\pi}\int_0^{\infty}\frac{\left(Re(\varphi(\phi)) + \frac{Im(\varphi(\phi))}{\phi}\right)\cos(\phi l) + \left(Im(\varphi(\phi)) - \frac{Re(\varphi(\phi))}{\phi}\right)\sin(\phi l)}{1+\phi^2}d\phi
\end{aligned}
\tag{2.36}
$$

2.3.2 The expression of the Characteristic Function

Due to the law of iterated expectations, the characteristic function φ must be a martingale:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^T}\left[\varphi\left(x_l, v_l, r_l; T, \phi\right) \middle| \mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}^T}\left[\mathbb{E}_{\mathbb{Q}^T}\left[\exp(i\phi x_T) \middle| \mathcal{F}_l\right] \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}^T}\left[\exp(i\phi x_T) \middle| \mathcal{F}_t\right] \\
&= \varphi(x_t, v_t, r_t; T, \phi)
\end{aligned}
$$

Replacing $\tau = T - t$ and applying Itô's lemma to $\varphi(x_t, v_t, r_t; T, \phi)$:

$$
\begin{aligned}
d\varphi(x_t, v_t, r_t; T, \phi) &= -\frac{\partial\varphi}{\partial\tau}dt + \frac{\partial\varphi}{\partial x_t}dx + \frac{\partial\varphi}{\partial v_t}dv + \frac{\partial\varphi}{\partial r_t}dr + \frac{1}{2}\frac{\partial^2\varphi}{\partial x_t^2}d\left\langle x_t, x_t\right\rangle \\
&+ \frac{1}{2}\frac{\partial^2\varphi}{\partial v_t^2}d\left\langle v_t, v_t\right\rangle + \frac{1}{2}\frac{\partial^2\varphi}{\partial r_t^2}d\left\langle r_t, r_t\right\rangle + \frac{\partial\varphi}{\partial x_t\partial v_t}d\left\langle x_t, v_t\right\rangle \\
&= \left[-\frac{\partial\varphi}{\partial\tau} + \left(r_t - \frac{1}{2}v_t\right)\frac{\partial\varphi}{\partial x_t} + \kappa\left(\theta - v_t\right)\frac{\partial\varphi}{\partial v_t}\right. \\
&+ \left(\kappa_r\left(\theta_r - r_t\right) + \frac{\sigma_r^2 r_t + \rho_r^2}{P(t,T)}\frac{\partial P}{\partial r_t}\right)\frac{\partial\varphi}{\partial r_t} + \frac{1}{2}v_t\frac{\partial^2\varphi}{\partial x_t^2} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2\varphi}{\partial v_t^2} \\
&\left.+\frac{1}{2}\left(\sigma_r^2 r_t + \rho_r^2\right)\frac{\partial^2\varphi}{\partial r_t^2} + \sigma\rho v_t\frac{\partial\varphi}{\partial x_t\partial v_t}\right]dt \\
&+ \sqrt{v_t}\frac{\partial\varphi}{\partial x_t}dW_1^{\mathbb{Q}^T} + \sigma\sqrt{v_t}\frac{\partial\varphi}{\partial v_t}dW_2^{\mathbb{Q}^T} + \sqrt{\sigma_r^2 r_t + \rho_r^2}\frac{\partial\varphi}{\partial r_t}dW_3^{\mathbb{Q}^T}
\end{aligned}
\tag{2.37}
$$

But, being a martingale, the drift of (2.37) must be zero so, the characteristic function must be the solution to:

$$-\frac{\partial\varphi}{\partial\tau}+\left(r_t-\frac{1}{2}v_t\right)\frac{\partial\varphi}{\partial x_t}+\kappa\left(\theta-v_t\right)\frac{\partial\varphi}{\partial v_t}+\left[\kappa_r\left(\theta_r-r_t\right)+\frac{\sigma_r^2r_t+\rho_r^2}{P(t,T)}\frac{\partial P}{\partial r_t}\right]\frac{\partial\varphi}{\partial r_t}$$
$$+\frac{1}{2}v_t\frac{\partial^2\varphi}{\partial x_t^2}+\frac{1}{2}\sigma^2v_t\frac{\partial^2\varphi}{\partial v_t^2}+\frac{1}{2}\left(\sigma_r^2r_t+\rho_r^2\right)\frac{\partial^2\varphi}{\partial r_t^2}+\sigma\rho v_t\frac{\partial\varphi}{\partial x_t\partial v_t}=0, \tag{2.38}$$

with terminal condition $\varphi\left(x_T,v_T,r_T;T;\phi\right)=\exp(i\phi x_T)$.

To solve this equation, solutions are considered to have the following structure:

$$\varphi\left(x_t,v_t,r_t;T,\phi\right)=\exp\left[F(\phi,\tau)+G(\phi,\tau)v_t+H(\phi,\tau)r_t+i\phi x_t-\ln\left(P(t,T)\right)\right] \tag{2.39}$$

Replacing φ into the partial differential equation (2.38):

$$-\frac{\partial F}{\partial\tau}-\frac{\partial G}{\partial\tau}v_t-\frac{\partial H}{\partial\tau}r_t+\frac{\partial P}{\partial\tau}\frac{1}{P(t,T)}+\left(r_t-\frac{1}{2}v_t\right)i\phi+\kappa\left(\theta-v_t\right)G$$
$$+\left[\kappa_r\left(\theta_r-r_t\right)+\frac{\sigma_r^2r_t+\rho_r^2}{P(t,T)}\frac{\partial P}{\partial r_t}\right]\left[H-\frac{1}{P(t,T)}\frac{\partial P}{\partial r_t}\right]-\frac{1}{2}v_t\phi^2+\frac{1}{2}\sigma^2v_tG^2$$
$$+\frac{1}{2}\left(\sigma_r^2r_t+\rho_r^2\right)\left[\left(H-\frac{1}{P(t,T)}\frac{\partial P}{\partial r_t}\right)^2-\frac{1}{P(t,T)}\frac{\partial^2P}{\partial r_t^2}+\left(\frac{1}{P(t,T)}\frac{\partial P}{\partial r_t}\right)^2\right]$$
$$+\sigma\rho v_ti\phi G=0$$
$$\Leftrightarrow-\frac{\partial F}{\partial\tau}-\frac{\partial G}{\partial\tau}v_t-\frac{\partial H}{\partial\tau}r_t-r_t+\left(r_t-\frac{1}{2}v_t\right)i\phi+\kappa\left(\theta-v_t\right)G+\kappa_r\left(\theta_r-r_t\right)H$$
$$-\frac{1}{2}v_t\phi^2+\frac{1}{2}\sigma^2v_tG^2+\frac{1}{2}\left(\sigma_r^2r_t+\rho_r^2\right)H^2+\sigma\rho v_ti\phi G=0 \tag{2.40}$$

The last partial differential equation can be decomposed into three ordinary differential equations subject to $F(\phi,0)=G(\phi,0)=H(\phi,0)=0$. After solving those equations (Appendix C):

Vasicek:

$$H = \frac{\left(1 - \exp\left(-\kappa_r \tau\right)\right)\left(i\phi - 1\right)}{\kappa_r} \tag{2.41}$$

$$G = \frac{\kappa - \rho\sigma i\phi + d_V}{\sigma^2} \frac{1 - \exp\left(d_V \tau\right)}{1 - q_V \exp\left(d_V \tau\right)} \tag{2.42}$$

$$\begin{aligned} F &= \theta_r i\phi\tau - \theta_r \tau + \frac{\kappa\theta}{\sigma^2}\left[\left(\kappa - \rho\sigma i\phi + d_V\right)\tau - 2\ln\left(\frac{1 - q_V \exp\left(d_V \tau\right)}{1 - q_V}\right)\right] \\ &+ \frac{\theta_r\left(i\phi - 1\right)}{\kappa_r}\left(\exp\left(-\kappa_r \tau\right) - 1\right) + \\ &+ \frac{\rho_r^2\left(i\phi - 1\right)^2}{\kappa_r^2}\left[\frac{-3 + 2\kappa_r\tau + 4\exp\left(-\kappa_r\tau\right) - \exp\left(-2\kappa_r\tau\right)}{4\kappa_r}\right] \end{aligned} \tag{2.43}$$

where

$$d_V = \sqrt{\left(\rho\sigma i\phi - \kappa\right)^2 + \left(i\phi + \phi^2\right)\sigma^2} \tag{2.44}$$

$$q_V = \frac{-\kappa + \rho\sigma\phi i - d_V}{-\kappa + \rho\sigma\phi i + d_V} \tag{2.45}$$

CIR:

$$H = \frac{\kappa_r + d_C}{\sigma_r^2} \frac{1 - \exp\left(d_C \tau\right)}{1 - q_C \exp\left(d_C \tau\right)} \tag{2.46}$$

$$\begin{aligned} F &= \frac{\kappa\theta}{\sigma^2}\left[\left(\kappa - \rho\sigma i\phi + d_V\right)\tau - 2\ln\left(\frac{1 - q_V \exp\left(d_V \tau\right)}{1 - q_V}\right)\right] \\ &+ \frac{\kappa_r\theta_r}{\sigma_r^2}\left[\left(\kappa_r + d_C\right)\tau - 2\ln\left(\frac{1 - q_C \exp\left(d_C \tau\right)}{1 - q_C}\right)\right] \end{aligned} \tag{2.47}$$

where

$$d_C = \sqrt{\kappa_r^2 - 2\left(i\phi - 1\right)\sigma_r^2} \tag{2.48}$$

$$q_C = \frac{\kappa_r + d_C}{\kappa_r - d_C}, \tag{2.49}$$

while G is still given by equation (2.42).

2.3.3 Continuity

For the numerical implementation of these formulas, some difficulties may arise from the discontinuity of the complex logarithm, consequence of always using the principal argument. In some cases, the curve of the numbers for which is calculated the logarithm will cross the negative part of the real axis, and, considering the principal argument, that will generate a discontinuity, as seen in Figures 2.1 and 2.2.

To avoid such problems, Jäckel and Kahl (2006) indicated an algorithm which, under certain conditions, should choose the argument so that it ensures continuity. Later, Kahl and Lord (2006) presented a demonstration that this algorithm chooses branches of the complex logarithm that ensures continuity but, they also mention that, under an alternative (and equivalent) formulation of the characteristic function, the principal branch ensures continuity.

This alternative formulation is easier to implement and numerically more stable than Heston's original formulation. Albrecher, Mayer, Schoutens and Tistaert (2006) also explore the alternative formulation, explaining that, under such formulation, stability is guaranteed. I will follow the alternative formulation, to which I must rearrange terms with $h = \frac{1}{q}$ in equations (2.41) to (2.49), so that the formulas depend on $-d$ instead of depending on d. Proceeding analogously to Albrecher, Mayer, Schoutens and Tistaert (2006), I reach the following formulas:

Vasicek:

$$G = \frac{\kappa - \rho\sigma i\phi - d_V}{\sigma^2} \frac{1 - \exp\left(-d_V \tau\right)}{1 - h_V \exp\left(-d_V \tau\right)} \tag{2.50}$$

$$\begin{aligned} F = {} & \theta_r i\phi\tau - \theta_r \tau + \frac{\kappa\theta}{\sigma^2}\left[\left(\kappa - \rho\sigma i\phi - d_V\right)\tau - 2\ln\left(\frac{1 - h_V \exp\left(-d_V\tau\right)}{1 - h_V}\right)\right] \\ & + \frac{\theta_r\left(i\phi - 1\right)}{\kappa_r}\left(\exp\left(-\kappa_r\tau\right) - 1\right) \\ & + \frac{\rho_r^2\left(i\phi - 1\right)^2}{\kappa_r^2}\left[\frac{-3 + 2\kappa_r\tau + 4\exp\left(-\kappa_r\tau\right) - \exp\left(-2\kappa_r\tau\right)}{4\kappa_r}\right] \end{aligned} \tag{2.51}$$

where

$$d_V = \sqrt{\left(\rho\sigma i\phi - \kappa\right)^2 + \left(i\phi + \phi^2\right)\sigma^2} \tag{2.52}$$

$$h_V = \frac{1}{q_V} = \frac{\kappa - \rho\sigma\phi i - d_V}{\kappa - \rho\sigma\phi i + d_V} \tag{2.53}$$

CIR:

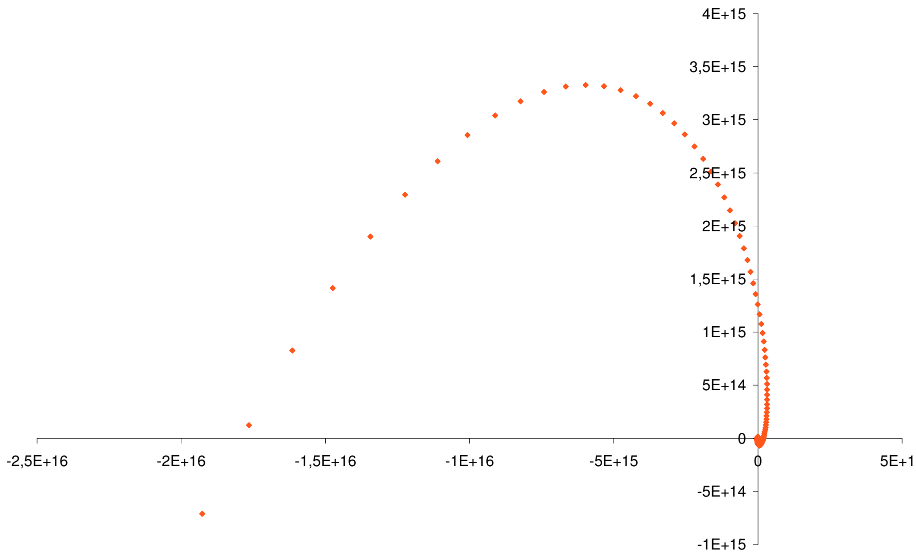
$$H = \frac{\kappa_r - d_C}{\sigma_r^2} \frac{1 - \exp\left(-d_C \tau\right)}{1 - h_C \exp\left(-d_C \tau\right)} \tag{2.54}$$

$$\begin{aligned} F = {} & \frac{\kappa\theta}{\sigma^2} \left[\left(\kappa - \rho\sigma i\phi - d_V\right) \tau - 2 \ln \left(\frac{1 - h_V \exp\left(-d_V \tau\right)}{1 - h_V} \right) \right] \\ & + \frac{\kappa_r \theta_r}{\sigma_r^2} \left[\left(\kappa_r - d_C\right) \tau - 2 \ln \left(\frac{1 - h_C \exp\left(-d_C \tau\right)}{1 - h_C} \right) \right] \end{aligned} \tag{2.55}$$

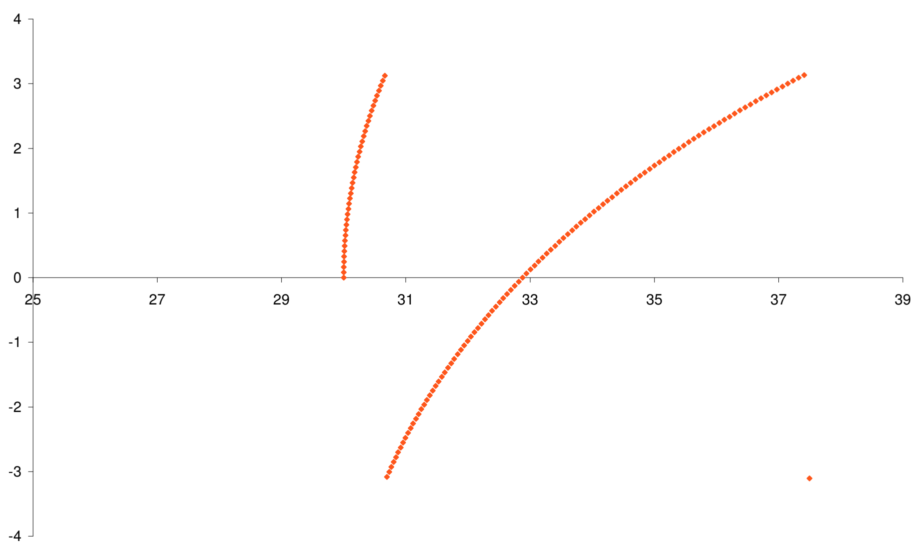
where

$$d_C = \sqrt{\kappa_r^2 - 2\left(i\phi - 1\right) \sigma_r^2} \tag{2.56}$$

$$h_C = \frac{1}{q_C} = \frac{\kappa_r - d_C}{\kappa_r + d_C} \tag{2.57}$$

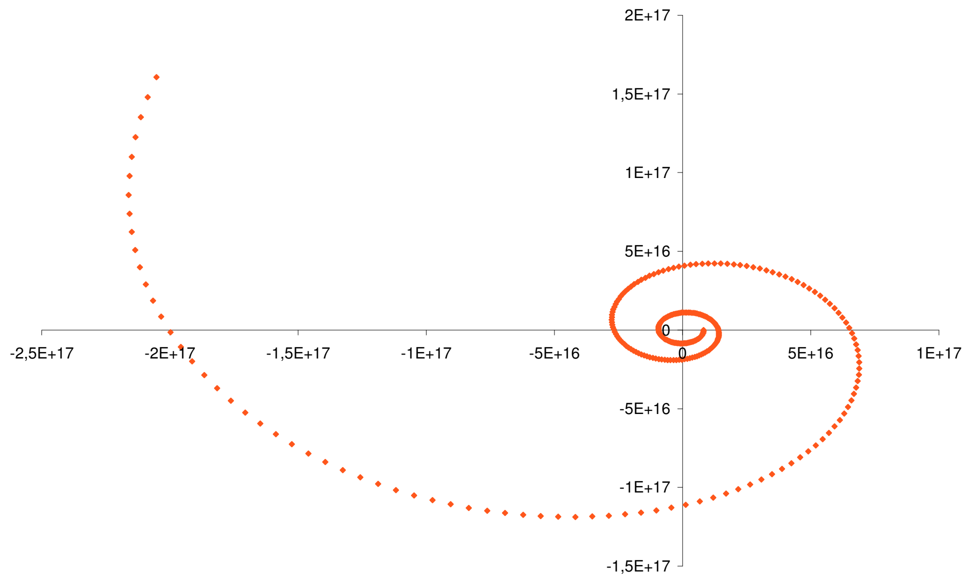


(a) $\frac{1-q_V e^{d_V \tau}}{1-q_V}$ with ϕ from 0.00001 to 1.44

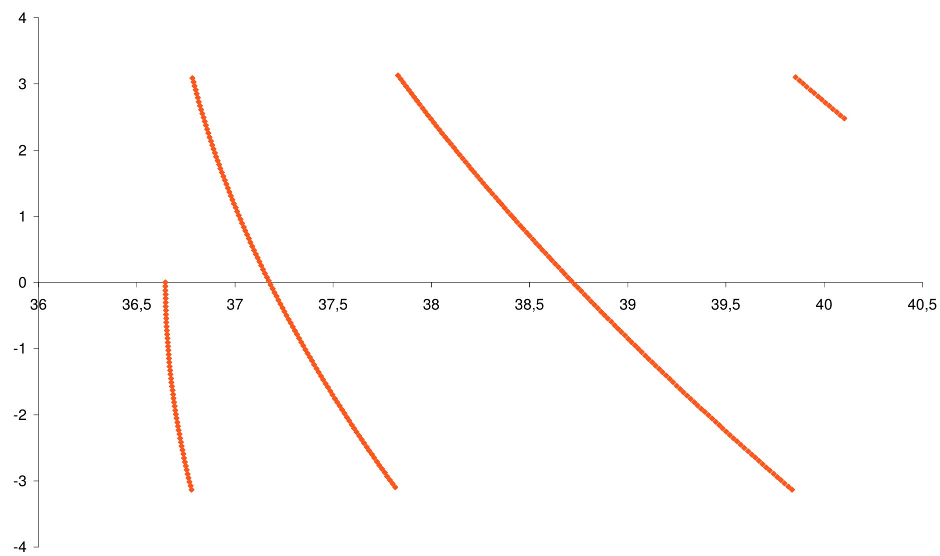


(b) $\ln\left(\frac{1-q_V e^{d_V \tau}}{1-q_V}\right)$ with ϕ from 0.00001 to 1.44

Figure 2.1: The logarithm in equation (2.43), considering $\tau = 30$, $\kappa = 1$, $\sigma = 0.5$ and $\rho = -0.3$

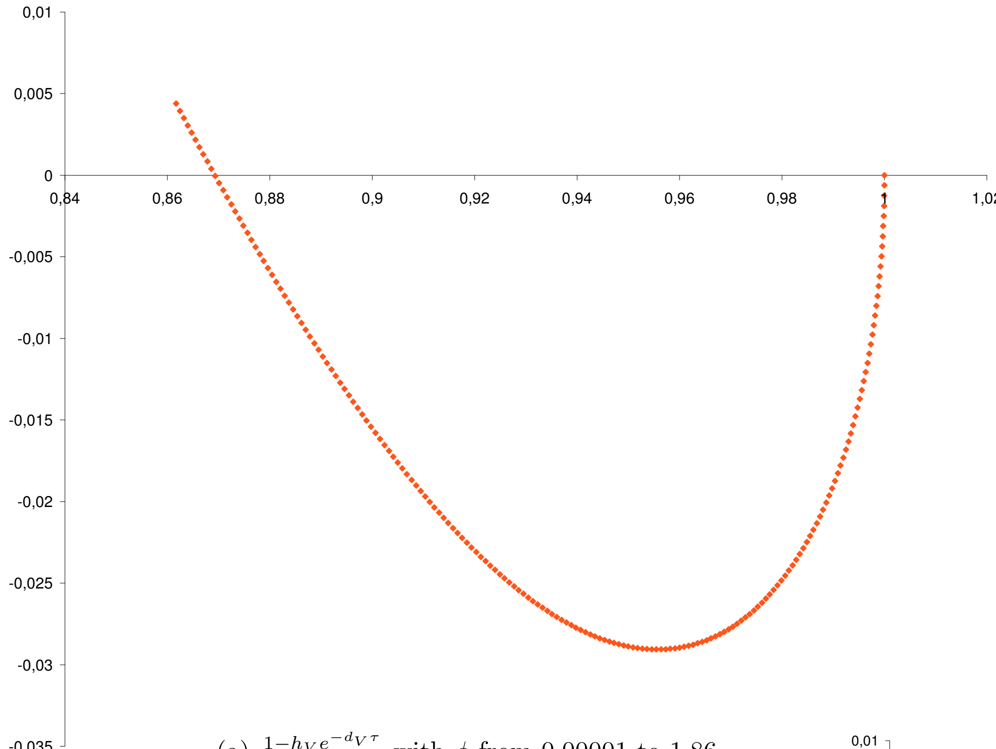


(a) $\frac{1-q_C e^{d_C \tau}}{1-q_C}$ with ϕ from 0.00001 to 2.96

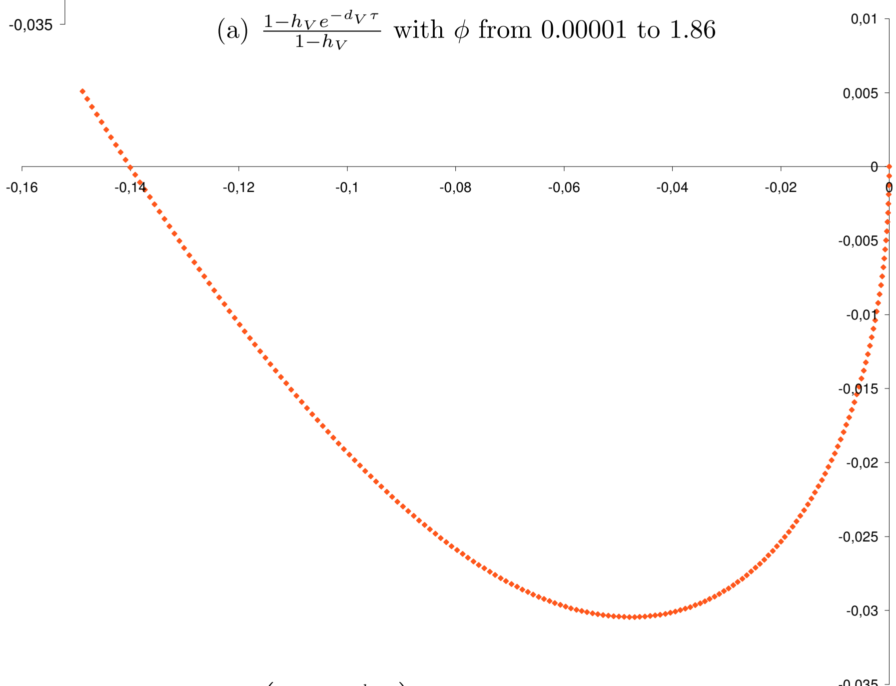


(b) $\ln\left(\frac{1-q_C e^{d_C \tau}}{1-q_C}\right)$ with ϕ from 0.00001 to 2.96

Figure 2.2: The other logarithm in equation (2.47), considering $\tau = 30$, $\kappa_r = 1$, $\sigma_r = 0.5$ and $\rho = -0.3$

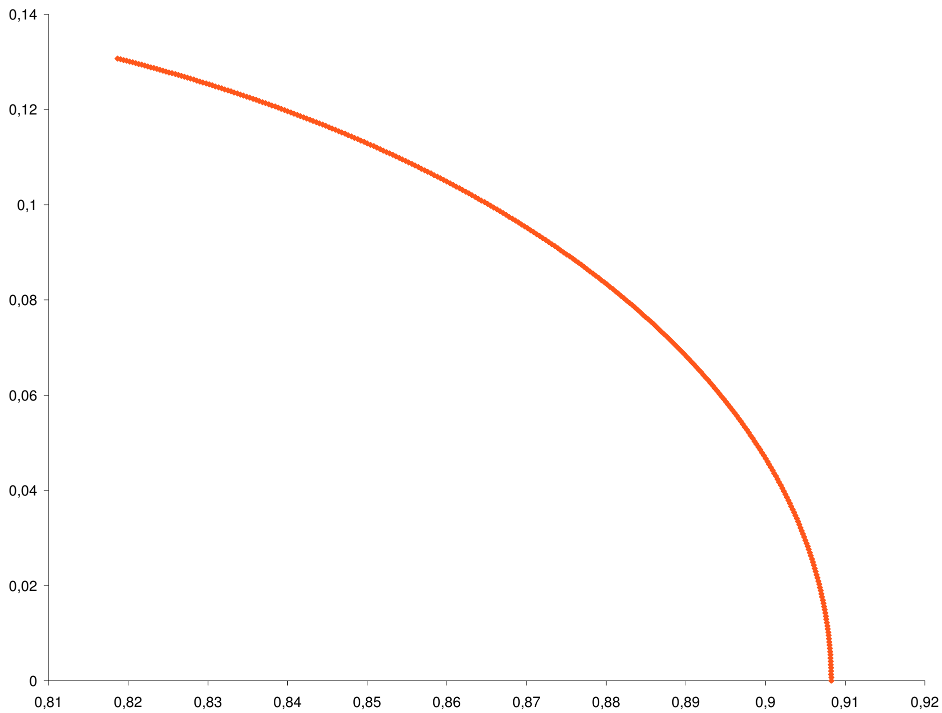


(a) $\frac{1-h_V e^{-d_V \tau}}{1-h_V}$ with ϕ from 0.00001 to 1.86

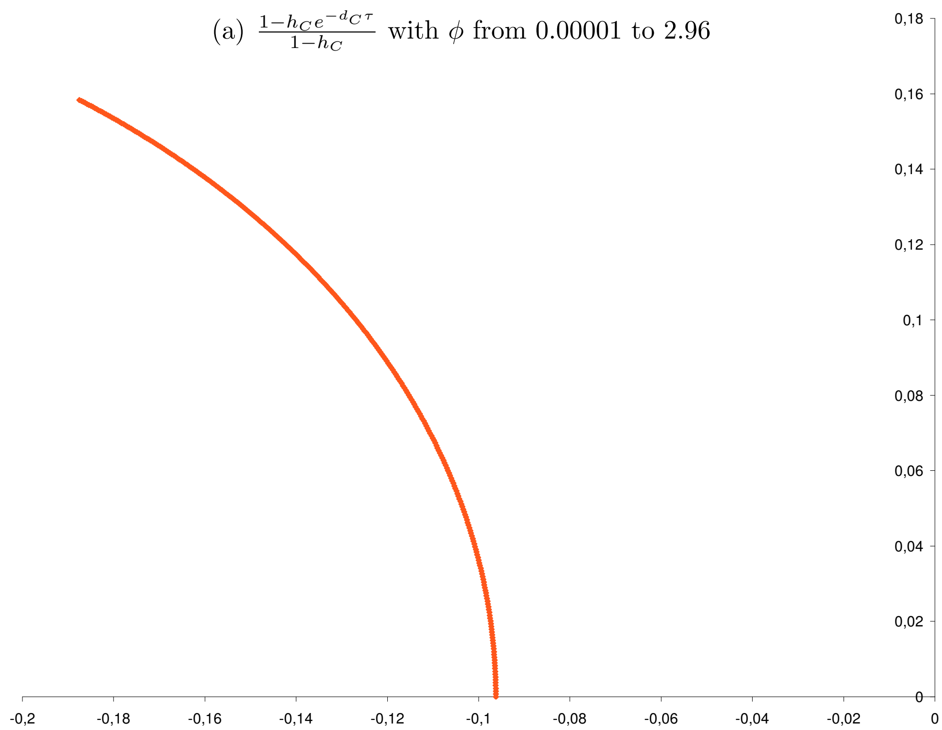


(b) $\ln\left(\frac{1-h_V e^{-d_V \tau}}{1-h_V}\right)$ with ϕ from 0.00001 to 1.86

Figure 2.3: The logarithm in equation (2.51), considering $\tau = 30$, $\kappa = 1$, $\sigma = 0.5$ and $\rho = -0.3$



(a) $\frac{1-h_C e^{-d_C \tau}}{1-h_C}$ with ϕ from 0.00001 to 2.96



(b) $\ln\left(\frac{1-h_C e^{-d_C \tau}}{1-h_C}\right)$ with ϕ from 0.00001 to 2.96

Figure 2.4: The other logarithm in equation (2.55), considering $\tau = 30$, $\kappa_r = 1$, $\sigma_r = 0.5$ and $\rho = -0.3$

Chapter 3

Pricing Forward Start Options

Forward-start options are exotic options which have its life starting on some future date t^*, although its premium is paid at the time of the purchase. It only makes sense to evaluate forward-start options with $t < t^*$, otherwise it would be a standard option with a determined strike since its inception. The strike will be considered to be a proportion (k) of the asset$'$s price at the time of its life start: kS_{t^*}.

Kruse and Nogel (2005) determined a solution for the pricing of this forward-start European option under the Heston (1993) model. I will follow their steps but considering a stochastic interest rate setup and using the expression I reached in the previous chapter for standard options. Also, unlike Kruse and Nogel (2005), I will seek to reduce the number of numerical procedures by using characteristic functions, avoiding the calculation of density functions and only computing one integral.

I will be changing from the risk-neutral measure $\mathbb{Q}$ to $\mathbb{Q}^S$, the measure that uses the stock price (compounded by the dividend yield) as numeraire. So, to apply Girsanov's theorem, I consider the independent Brownian motions defined in equations

(2.11) and the pricing model defined by equations (2.12) to (2.14). Using equations (2.8) and (2.16), the Radon-Nikodym derivative can be determined as:

$$\begin{aligned}\left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\right|\mathcal{F}_t &= \frac{B_t}{B_T}\frac{e^{qT}S_T}{e^{qt}S_t} \\ &= \exp\left(-\int_t^T r_u du\right)\frac{e^{qT}S_T}{e^{qt}S_t} = \\ &= \exp\left(-\frac{1}{2}\int_t^T v_u du + \int_t^T \sqrt{v_u} dZ_1^{\mathbb{Q}}(u)\right) \end{aligned} \tag{3.1}$$

To make the change of measure, it is necessary to verify that the right-hand side of equation (3.1) is defined and is a martingale. Once again I will do it by verifying the existence of two constants a and c such that $\mathbb{E}_{\mathbb{Q}}\left[\exp(aY^2(s)|\,\mathcal{F}_t\right] \leq c\ \forall s : t \leq s \leq T$, where $Y^2(t)$ is v_t (see Appendix B). Girsanov′s theorem then leads to:

$$dZ_1^{\mathbb{Q}^S}(t) = dZ_1^{\mathbb{Q}}(t) - \sqrt{v_t}dt \qquad dZ_2^{\mathbb{Q}^S}(t) = dZ_2^{\mathbb{Q}}(t) \qquad dZ_3^{\mathbb{Q}^S}(t) = dZ_3^{\mathbb{Q}}(t)$$

With this numeraire change, and after returning to correlated Brownian motions ($W_1^{\mathbb{Q}^S} = Z_1^{\mathbb{Q}^S}$, $W_3^{\mathbb{Q}^S} = Z_3^{\mathbb{Q}^S}$ and $W_2^{\mathbb{Q}^S} = \rho Z_1^{\mathbb{Q}^S} + \sqrt{1-\rho^2}Z_2^{\mathbb{Q}^S}$), the pricing model becomes:

$$dx_t = \left(r_t + \frac{v_t}{2}\right)dt + \sqrt{v_t}dW_1^{\mathbb{Q}^S}(t) \tag{3.2}$$

$$dv_t = (\kappa - \rho\sigma)\left(\frac{\kappa\theta}{\kappa - \rho\sigma} - v_t\right)dt + \sigma\sqrt{v_t}dW_2^{\mathbb{Q}^S}(t) \tag{3.3}$$

$$dr_t = \left(\kappa_r\left(\theta_r - r_t\right)\right)dt + \sqrt{\sigma_r^2 r_t + \rho_r^2}dW_3^{\mathbb{Q}^S}(t) \tag{3.4}$$

$$d\left\langle W_1^{\mathbb{Q}^S}, W_2^{\mathbb{Q}^S}\right\rangle_t = \rho dt \tag{3.5}$$

$$d\left\langle W_1^{\mathbb{Q}^S}, W_3^{\mathbb{Q}^S}\right\rangle_t = 0 \tag{3.6}$$

$$d\left\langle W_2^{\mathbb{Q}^S}, W_3^{\mathbb{Q}^S}\right\rangle_t = 0 \tag{3.7}$$

Although having different parameters, v_t still follows a square root process with parameters satisfying the stability condition:

$$2(\kappa - \rho\sigma)\frac{\kappa\theta}{\kappa - \rho\sigma} > \sigma^2 \Leftrightarrow 2\kappa\theta > \sigma^2$$

Consider $c_t^f\left(S_t, v_t, r_t, kS_{t^*}, t^*, T\right)$ to be the price of a forward-start European call in the mentioned conditions. Following Kruse and Nogel (2005), I begin with the numeraire change:

$$\begin{aligned} c_t^f\left(S_t, v_t, r_t, kS_{t^*}, t^*, T\right) &= \mathbb{E}_{\mathbb{Q}}\left[B_t \frac{(S_T - kS_{t^*})^+}{B_T}\middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^S}\left[S_t e^{qt} \frac{(S_T - kS_{t^*})^+}{S_T e^{qT}}\middle| \mathcal{F}_t\right] \end{aligned} \tag{3.8}$$

Because S_t is $\mathcal{F}_t$-measurable and applying the law of iterated expectation, then:

$$\begin{aligned} c_t^f\left(S_t, v_t, r_t, kS_{t^*}, t^*, T\right) &= S_t e^{qt} \mathbb{E}_{\mathbb{Q}^S}\left[\mathbb{E}_{\mathbb{Q}^S}\left[\frac{(S_T - kS_{t^*})^+}{S_T e^{qT}}\middle| \mathcal{F}_{t^*}\right]\middle| \mathcal{F}_t\right] \\ &= S_t e^{qt} \mathbb{E}_{\mathbb{Q}^S}\left[\mathbb{E}_{\mathbb{Q}^S}\left[S_{t^*} e^{qt^*} \frac{(S_T - kS_{t^*})^+}{S_T e^{qT}}\middle| \mathcal{F}_{t^*}\right] \frac{1}{S_{t^*} e^{qt^*}}\middle| \mathcal{F}_t\right] \end{aligned} \tag{3.9}$$

Knowing that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^S}\left[S_{t^*} e^{qt^*} \frac{(S_T - kS_{t^*})^+}{S_T e^{qT}}\middle| \mathcal{F}_{t^*}\right] &= c_{t^*}^f\left(S_{t^*}, v_{t^*}, r_{t^*}, kS_{t^*}, t^*, T\right) \\ &= c_{t^*}\left(S_{t^*}, v_{t^*}, r_{t^*}, kS_{t^*}, t^*, T\right) \end{aligned}$$

allows us to use equation (2.36) with $\tau^* = T - t^*$ and $\varphi = \varphi\left(x_{t^*}, v_{t^*}, r_{t^*}; T, \phi\right)$ (for notational simplicity I will sometimes write φ instead of $\varphi\left(x_{t^*}, v_{t^*}, r_{t^*}; T, \phi\right)$):

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^S}\left[S_{t^*} e^{qt^*} \frac{(S_T - kS_{t^*})^+}{S_T e^{qT}}\middle| \mathcal{F}_{t^*}\right] = \\ &= S_{t^*} e^{-q\tau^*} - P(t^*, T)\frac{kS_{t^*}}{2} \\ &- P(t^*, T)\frac{kS_{t^*}}{\pi} \int_0^{+\infty} \frac{(Re(\varphi) + \frac{Im(\varphi)}{\phi})\cos(\phi l) + (Im(\varphi) - \frac{Re(\varphi)}{\phi})\sin(\phi l)}{1 + \phi^2} d\phi \end{aligned} \tag{3.10}$$

Introducing this result into equation (3.9):

$$\begin{aligned}
&c_t^f\left(S_t, v_t, r_t, kS_{t^*}, t^*, T\right) = \\
&= S_t e^{qt}\mathbb{E}_{\mathbb{Q}^S}\left[e^{-qT} - e^{-qt^*}P(t^*,T)\frac{k}{2}\right. \\
&\left. -e^{-qt^*}P(t^*,T)\frac{k}{\pi}\int_0^{+\infty}\frac{(Re(\varphi)+\frac{Im(\varphi)}{\phi})\cos(\phi l)+(Im(\varphi)-\frac{Re(\varphi)}{\phi})\sin(\phi l)}{1+\phi^2}d\phi\right|\mathcal{F}_t\Bigg] \\
&= S_t e^{-q\tau} - S_t e^{-q(t^*-t)}\frac{k}{2}\mathbb{E}_{\mathbb{Q}^S}\left[P(t^*,T)|\,\mathcal{F}_t\right] - S_t e^{-q(t^*-t)}\frac{k}{\pi}\mathbb{E}_{\mathbb{Q}^S}\left[\Omega|\,\mathcal{F}_t\right] \qquad (3.11)
\end{aligned}$$

where

$$\Omega := P(t^*,T)\int_0^{+\infty}\frac{(Re(\varphi)+\frac{Im(\varphi)}{\phi})\cos(\phi l)+(Im(\varphi)-\frac{Re(\varphi)}{\phi})\sin(\phi l)}{1+\phi^2}d\phi \qquad (3.12)$$

Notice that neither $l = \ln\left(\frac{kS_t^*}{S_t^*}e^{q\tau^*}\right) = \ln ke^{q\tau^*}$ nor the conditional expectation depend on S_{t^*}. So, and because $x_{t^*} = x(t^*,t^*) = 0$, the variables involved are just v_{t^*} and r_{t^*}.

Because $P(t^*,T)$ depends only on r_{t^*}, it may be moved inside the integral. Because it has no imaginary part, it may be moved into $Re(\varphi)$ and $Im(\varphi)$, where, gathered with φ (given by equation (2.39)), may give place to [1]

$$\Upsilon\left(x_{t^*}, v_{t^*}, r_{t^*}; T, \phi\right) := P(t^*,T)\varphi = e^{F(\phi,\tau^*)+G(\phi,\tau^*)v_{t^*}+H(\phi,\tau^*)r_{t^*}+i\phi x_{t^*}}, \qquad (3.13)$$

allowing to define Ω as:

$$\Omega := \int_0^{+\infty}\frac{(Re(\Upsilon)+\frac{Im(\Upsilon)}{\phi})\cos(\phi l)+(Im(\Upsilon)-\frac{Re(\Upsilon)}{\phi})\sin(\phi l)}{1+\phi^2}d\phi \qquad (3.14)$$

In the second conditional expectation, v_{t^*} and r_{t^*} are envolved, so it is necessary to know its joint transition density. However, because v_{t^*} and r_{t^*} were assumed to be independent, the joint transition density may be obtained by multiplying the individual transition density from each one, represented by $f_v\left(v|\,v_t\right)$ and $f_r\left(r|\,r_t\right)$, respectively. The first conditional expectation, is simpler since it depends only on r_{t^*}:

$$\mathbb{E}_{\mathbb{Q}^S}\left[P(t^*,T)|\,\mathcal{F}_t\right] = \int_0^{+\infty}P(t^*,T)f_r\left(r|\,r_t\right)dr. \qquad (3.15)$$

[1] For notational simplicity I will refer to $\Upsilon\left(x_{t^*}, v_{t^*}, r_{t^*}; T, \phi\right)$ as simply Υ

However, [2]

$$\mathbb{E}_{\mathbb{Q}^S}\left[\Omega|\,\mathcal{F}_t\right] =$$
$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{(Re(\Upsilon) + \frac{Im(\Upsilon)}{\phi})\cos(\phi l) + (Im(\Upsilon) - \frac{Re(\Upsilon)}{\phi}\sin(\phi l))}{1+\phi^2}$$
$$d\phi f_v\left(v|\,v_t\right) f_r\left(r|\,r_t\right) dvdr. \tag{3.16}$$

The second conditional expectation led to a triple integral. However, assuming that the integrand is Lebesgue-integrable and that the characteristic function of both the random variables can handle a complex object, the number of integrations may be reduced.

According to Fubini's theorem, if the integrand is Lebesgue-integrable then the order of integration may be changed:

$$\int_0^{+\infty}\left[\frac{(Re\int_0^{+\infty}\int_0^{+\infty}\Upsilon f_v\left(v|\,v_t\right) f_r\left(r|\,r_t\right) dvdr + \frac{Im\int_0^{+\infty}\int_0^{+\infty}\Upsilon f_v(v|v_t) f_r(r|r_t)dvdr}{\phi})\cos(\phi l)}{1+\phi^2}\right.$$
$$\left. + \frac{(Im\int_0^{+\infty}\int_0^{+\infty}\Upsilon f_v\left(v|\,v_t\right) f_r\left(r|\,r_t\right) dvdr - \frac{Re\int_0^{+\infty}\int_0^{+\infty}\Upsilon f_v(v|v_t) f_r(r|r_t)dvdr}{\phi})\sin(\phi l)}{1+\phi^2}\right] d\phi \tag{3.17}$$

To simplify further, it is necessary to have under consideration that, for a random variable X:

$$\int_{-\infty}^{+\infty} \exp\left(iaX\right) f_X(a) dX = \Phi_X(a) \tag{3.18}$$

where Φ_X and f_X represent, respectively, X's characteristic and density functions. It is also necessary that a be allowed to be a complex number, which requires the characteristic function to be analytic.

The Normal Distribution has a characteristic function analytic in the entire complex plane. That conclusion can be reached by using a theorem presented by Kawata

[2] In the Vasicek case, the integral relative to r must be from $-\infty$ to $+\infty$ instead of 0 to $+\infty$, because while equation (1.5) defines r_t strictly positive, the same is not true for equation (1.4)

(1972, p.460), or by the expression of the characteristic function itself, which is known to be an analytic function. Concerning to the Non-central Chi-Squared Distribution, because its distribution function is null for negative objects, the characteristic function is analytic at least in the half-plane defined by $\mathrm{Im}(a) > 0$ - see, for instance, Kawata (1972, p.456).

Considering that (3.18) holds for complex a, an expression for the first integral may determined:

$$\begin{aligned}\int_0^{+\infty} P(t^*, T) f_r(r|\, r_t)\, dr &= \int_0^{+\infty} e^{a(\tau^*)-b(\tau^*)r} f_r(r|\, r_t)\, dr = \\ &= e^{a(\tau^*)} \int_0^{+\infty} e^{-b(\tau^*)r} f_r(r|\, r_t)\, dr = \\ &= e^{a(\tau^*)} \Phi_r(ib(\tau^*)) \end{aligned} \tag{3.19}$$

Similarly:[3]

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \Upsilon f_v(v|\, v_t) f_r(r|\, r_t)\, dvdr &= \int_0^{+\infty}\int_0^{+\infty} e^{F+Gv+Hr} f_v(v|\, v_t) f_r(r|\, r_t)\, dvdr \\ &= e^F \int_0^{+\infty} e^{Gv} f_v(v|\, v_t)\, dv \int_0^{+\infty} e^{Hr} f_r(r|\, r_t)\, dr \\ &= e^F \Phi_v(-iG)\Phi_r(-iH) \end{aligned} \tag{3.20}$$

The previous result allows equation (3.17) to be simplified into:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^S}[\Omega|\, \mathcal{F}_t] = \int_0^{+\infty} \Bigg[& \frac{(Re(e^F\Phi_v(-iG)\Phi_r(-iH)) + \frac{Im(e^F\Phi_v(-iG)\Phi_r(-iH))}{\phi})\cos(\phi l)}{1+\phi^2} \\ & + \frac{(Im(e^F\Phi_v(-iG)\Phi_r(-iH)) - \frac{Re(e^F\Phi_v(-iG)\Phi_r(-iH))}{\phi})\sin(\phi l)}{1+\phi^2}\Bigg] d\phi \end{aligned} \tag{3.21}$$

[3]For simplicity of notation I will write F, G or H instead of $F(\tau^*)$, $G(\tau^*)$ and $H(\tau^*)$

So, the fair-value of the forward-start option is given by:

$$
\begin{aligned}
&c_t^f\left(S_t, v_t, r_t, kS_{t^*}, t^*, T\right) = \\
&= S_t e^{-q\tau} - S_t e^{-q(t^*-t)} \frac{k}{2} e^{a(T-t^*)} \Phi_r\left(ib\left(\tau^*\right)\right) \\
&- S_t e^{-q(t^*-t)} \frac{k}{\pi} \int_0^{+\infty} \left[\frac{(Re(e^F \Phi_v(-iG)\Phi_r(-iH)) + \frac{Im(e^F \Phi_v(-iG)\Phi_r(-iH))}{\phi}) \cos(\phi l)}{1+\phi^2} \right. \\
&\left. + \frac{(Im(e^F \Phi_v(-iG)\Phi_r(-iH)) - \frac{Re(e^F \Phi_v(-iG)\Phi_r(-iH))}{\phi}) \sin(\phi l)}{1+\phi^2} \right] d\phi
\end{aligned}
\tag{3.22}
$$

The expression of the characteristic functions are known (see, for instance Kapadia, Owen and Patel (1976, p.46))):

Lemma 3.1. *Let X_1 be a random variable with non-central Chi-squared distribution with R degrees of freedom and non-centrality parameter Λ and X_2 a random variable with Normal distribution with mean μ and standard deviation σ. The characteristic function of X_1 is*

$$\Phi_1\left(a\right) = \left(1 - 2ia\right)^{-\frac{R}{2}} e^{\frac{i\Lambda a}{1-2ia}} \tag{3.23}$$

and the characteristic function of X_2 is:

$$\Phi_2\left(a\right) = \exp\left(\mu ia - \frac{1}{2}\sigma^2 a^2\right) \tag{3.24}$$

For r_t defined according to the Vasicek model, $r_{t^*}\left|\mathcal{F}_t\right.$ has a Normal distribution with mean $r_t e^{-\kappa_r(t^*-t)} + \theta_r\left(1 - e^{-\kappa_r(t^*-t)}\right)$ and variance $\frac{\rho_r^2}{2\kappa_r}\left(1 - e^{-2\kappa_r(t^*-t)}\right)$.

Considering $f_{\chi^2(\Lambda,R)}$ to be the density of a Non-Central Chi-Squared Distribution with R degrees of freedom and non-centrality parameter Λ, then, under measure $\mathbb{Q}^S$, the density function of $v_{t^*}\left|\mathcal{F}_t\right.$ and $r_{t^*}\left|\mathcal{F}_t\right.$ (according to the CIR model) are given by, respectively:

$$f_v\left(v\right|v_t) = B f_{\chi^2(\Lambda,R)}\left(Bv\right) \tag{3.25}$$

$$f_r\left(r\right|r_t) = B_r f_{\chi^2(\Lambda_r,R_r)}\left(B_r r\right) \tag{3.26}$$

$$\Lambda = Be^{-(\kappa-\rho\sigma)(t^*-t)}v_t \quad , \quad B = \frac{4(\kappa-\rho\sigma)}{\sigma^2}\frac{1}{1-e^{-(\kappa-\rho\sigma)(t^*-t)}} \quad , \quad R = \frac{4\kappa\theta}{\sigma^2}$$

$$\Lambda_r = B_r e^{-\kappa_r(t^*-t)}r_t \quad , \quad B_r = \frac{4\kappa_r}{\sigma_r^2}\frac{1}{1-e^{-\kappa_r(t^*-t)}} \quad , \quad R_r = \frac{4\kappa_r\theta_r}{\sigma_r^2}$$

So, according to Lemma 3.1, the characteristic functions are:

$$\Phi_v(a) = \left(1-2iaB^{-1}\right)^{-\frac{R}{2}} e^{\frac{\Lambda iaB^{-1}}{1-2iaB^{-1}}} \tag{3.27}$$

$$\text{(CIR)} \qquad \Phi_r(a) = \left(1-2iaB_r^{-1}\right)^{-\frac{R_r}{2}} e^{\frac{\Lambda_r iaB_r^{-1}}{1-2iaB_r^{-1}}} \tag{3.28}$$

$$\text{(Vasicek)} \qquad \Phi_r(a) = e^{\left(r_t e^{-\kappa_r(t^*-t)}+\theta_r\left(1-e^{-\kappa_r(t^*-t)}\right)\right)ia-\frac{\rho_r^2a^2}{4\kappa_r}\left(1-e^{-2\kappa_r(t^*-t)}\right)} \tag{3.29}$$

Chapter 4

Numerical Implementation

Due to the steps followed, the numerical implementation for standard and forward start options, only requires the computation of one integral, which was done with a Gauss-Laguerre Quadrature (see, for instance, Press, Flannery, Teukolsky and Vetterling (1997, p. 152)).

Kilin(2007) mentioned that the solution to option pricing in the Heston model presented by Attari(2004) had the advantages, relatively to the original formulation, of just using one characteristic function and of having a quadratic term in the integral's denominator, providing a faster rate of decay. This is also valid when comparing the closed-form solution to standard options presented by Bakshi, Cao and Chen(1997) with the one I reached in Chapter 2.

When comparing the solutions in Chapter 3 with the implementation for the forward-start options by Kruse and Nögel(2005), besides also having the advantage of only needing the numerical implementation of one integral and which also had a quadratic term in its denominator, there was the advantage of not needing the implementation of the modified Bessel function.

To verify the results given by the numerical implementation of the solutions, I used Monte Carlo Simulation through a standard Euler scheme. To generate the random deviates with Normal distribution required for the simulation I generated uniform deviates and then used the Box-Muller method (see, for instance, Press, Flannery, Teukolsky and Vetterling (1997, p. 275)).

For the comparison, I considered $S_0 = 100$, $r_0 = 0.04$, $q = 0.01$, $v_0 = 0.01$, $\kappa = 2$, $\theta = 0.01$, $\sigma = 0.1$, $\kappa_r = 0.3$, $\theta_r = 0.05$, $\sigma_r = 0.05$, and experimented different maturities, strikes and correlations ρ. I considered a 35-point Gauss-Laguerre Quadrature for the integral of the Closed-Form Solution, and, for the Monte Carlo solution, I considered 700 time-steps per year and one million simulations.

The following tables present, for each case, the Closed-Form Solution (CFS), Monte Carlo solution (MCS), relative error:=$\frac{CFS-MCS}{MCS}$ (MCE_{rel}) and standard error:=$\frac{MC_{StDev}}{\sqrt{N}}$($MCE_{std}$) (where MC_{StDev} is the standard deviation of the Monte Carlo simulation and N the number of simulations). It is possible to observe that most of the simulations returned values close to the ones presented by the closed-form solution. Almost only with higher strikes, which correspond to solutions near zero, there were MCE_{rel} superior to 1% and in many cases it was even below 0.1% .

Model	Strike	τ	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	50	1	-0.5	51.0148	51.008	0.01%	1.01%
Heston+CIR	50	1	-0.5	51.0302	51.0248	0.01%	0.98%
Heston+Vasicek	100	1	-0.5	5.72622	5.72293	0.06%	0.68%
Heston+CIR	100	1	-0.5	5.63308	5.63168	0.02%	0.66%
Heston+Vasicek	150	1	-0.5	0.000041	0.00003	37.42%	0.001%
Heston+CIR	150	1	-0.5	0.000015	0.000015	3.3%	0.001%

Table 4.1: Results for a Standard European Call with different strikes

Model	k	τ	$t^* - t$	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	0.5	2	1	-0.5	50.6134	50.5557	0.11%	1.22%
Heston+CIR	0.5	2	1	-0.5	50.6272	50.6098	0.03%	1.09%
Heston+Vasicek	1	2	1	-0.5	5.77113	5.94804	-2.97%	0.74%
Heston+CIR	1	2	1	-0.5	5.68765	5.67515	0.22%	0.66%
Heston+Vasicek	1.5	2	1	-0.5	0.0001585	0.000449	-64.7%	0.006%
Heston+CIR	1.5	2	1	-0.5	0.0001015	0.000097	4.59%	0.002%

Table 4.2: Results for a Forward-Start European Call with different strikes (kS_{t^*})

Model	Strike	τ	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	100	3	-0.5	12.7658	12.7509	0.12%	1.52%
Heston+CIR	100	3	-0.5	12.1794	12.1737	0.05%	1.30%
Heston+Vasicek	100	2	-0.5	9.40752	9.38098	0.28%	1.11%
Heston+CIR	100	2	-0.5	9.07573	9.05445	0.24%	1.01%
Heston+Vasicek	100	1	-0.5	5.72622	5.72293	0.06%	0.68%
Heston+CIR	100	1	-0.5	5.63308	5.63168	0.02%	0.66%
Heston+Vasicek	100	0.5	-0.5	3.63465	3.62971	0.14%	0.45%
Heston+CIR	100	0.5	-0.5	3.6136	3.60918	0.12%	0.45%

Table 4.3: Results for a Standard European Call with different maturities

Model	k	τ	$t^* - t$	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	0.5	3	1	-0.5	51.6525	51.5438	0.21%	1.76%
Heston+CIR	0.5	3	1	-0.5	51.7428	51.7358	0.01%	1.46%
Heston+Vasicek	0.5	2	1.5	-0.5	49.8357	49.8037	0.06%	1.01%
Heston+CIR	0.5	2	1.5	-0.5	49.8373	49.8238	0.03%	0.91%
Heston+Vasicek	0.5	2	1	-0.5	50.6134	50.5557	0.11%	1.22%
Heston+CIR	0.5	2	1	-0.5	50.6272	50.6098	0.03%	1.09%
Heston+Vasicek	0.5	2	0.5	-0.5	51.3343	51.2761	0.11%	1.38%
Heston+CIR	0.5	2	0.5	-0.5	51.3782	51.3647	0.03%	1.24%
Heston+Vasicek	0.5	1	0.5	-0.5	50.2861	50.2766	0.02%	0.80%
Heston+CIR	0.5	1	0.5	-0.5	50.2881	50.2855	0.01%	0.78%

Table 4.4: Results for a Forward-Start European Call with different maturities

Model	Strike	τ	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	100	1	-0.5	5.72622	5.72293	0.06%	0.68%
Heston+CIR	100	1	-0.5	5.63308	5.63168	0.02%	0.66%
Heston+Vasicek	100	1	-0.3	5.70613	5.7031	0.05%	0.71%
Heston+CIR	100	1	-0.3	5.60956	5.60846	0.02%	0.68%
Heston+Vasicek	100	1	0	5.67238	5.67003	0.04%	0.74%
Heston+CIR	100	1	0	5.57092	5.57049	0.01%	0.72%
Heston+Vasicek	100	1	0.3	5.63383	5.63249	0.02%	0.77%
Heston+CIR	100	1	0.3	5.52783	5.52835	-0.01%	0.75%
Heston+Vasicek	100	1	0.5	5.60516	5.60465	0.01%	0.79%
Heston+CIR	100	1	0.5	5.49638	5.49766	-0.02%	0.77%

Table 4.5: Results for a Standard European Call with different correlation ρ

Model	k	τ	$t^* - t$	ρ	CFS	MCS	MCE_{rel}	MCE_{std}
Heston+Vasicek	0.5	2	1	-0.5	50.6134	50.5557	0.11%	1.22%
Heston+CIR	0.5	2	1	-0.5	50.6272	50.6098	0.03%	1.09%
Heston+Vasicek	0.5	2	1	-0.3	50.6134	50.5546	0.12%	1.23%
Heston+CIR	0.5	2	1	-0.3	50.6272	50.6088	0.04%	1.10%
Heston+Vasicek	0.5	2	1	0	50.6134	50.5533	0.12%	1.24%
Heston+CIR	0.5	2	1	0	50.6272	50.6076	0.04%	1.12%
Heston+Vasicek	0.5	2	1	0.3	50.6134	50.5525	0.12%	1.25%
Heston+CIR	0.5	2	1	0.3	50.6272	50.6069	0.04%	1.13%
Heston+Vasicek	0.5	2	1	0.5	50.6134	50.5522	0.12%	1.26%
Heston+CIR	0.5	2	1	0.5	50.6272	50.6067	0.04%	1.14%

Table 4.6: Results for a Forward-Start European Call with different correlation ρ

Chapter 5

Conclusion

In this dissertation I found a closed-form solution for standard European calls under the Heston model but with stochastic interest rates. These closed-form solutions differ from the ones presented by Bakshi, Cao and Chen (1997) and Hout, Bierkens, Ploeg and Panhuis (2007) due to the use of a manipulation already used by Attari in the case of a constant interest rate, allowing the use of just one characteristic function, which facilitates the numerical implementation required for option pricing.

For the referred model I also determined a closed-form solution to European forward-start calls. Kruse and Nögel (2005) determined a closed-form for these options under the Heston model (with constant interest rates). Although having a stochastic interest rate, I was able to follow a similar path but, because I started from a closed-form solution for standard calls that only had one characteristic function, the formulas reached in this dissertation also require the calculation of just one characteristic function. By using properties of analytic characteristic functions I reduced the number of integrals and numerical procedures required, resulting in a formula which is more easily implemented than the formulas presented by Kruse and Nögel (2005), requiring just one numerical integration.

The Monte Carlo simulations made for the evaluation of these options under the considered mode presented results close to the ones returned by the closed-formula implementation.

Appendix A

Zero Coupon Bond's ODE

Incorporating equation (2.3) in equation (2.2):

$$-\left(\frac{\partial a}{\partial \tau}-\frac{\partial b}{\partial \tau}\right)-\kappa_r\left(\theta_r-r_t\right)b\left(\tau\right)+\frac{1}{2}\left(\sigma_r^2 r_t+\rho_r^2\right)b^2\left(\tau\right)-r_t=0$$

$$\Leftrightarrow r_t\left(\frac{\partial b}{\partial \tau}+\kappa_r b\left(\tau\right)+\frac{1}{2}\sigma_r^2 b^2\left(\tau\right)-1\right)+\frac{1}{2}\rho_r^2 b^2\left(\tau\right)-\kappa_r\theta_r b\left(\tau\right)-\frac{\partial a}{\partial \tau}=0$$

subject to $a\left(0\right)=b\left(0\right)=0$.

Since the equation must be equal to zero for all values of r_t, it can be decomposed in two ordinary differential equations:

$$\frac{\partial b}{\partial \tau}+\kappa_r b\left(\tau\right)+\frac{1}{2}\sigma_r^2 b^2\left(\tau\right)-1=0 \quad \text{(A.1)}$$

$$\frac{1}{2}\rho_r^2 b^2\left(\tau\right)-\kappa_r\theta_r b\left(\tau\right)-\frac{\partial a}{\partial \tau}=0 \quad \text{(A.2)}$$

A.1 Vasicek

In the Vasicek case, the equations are:

$$\frac{\partial b}{\partial \tau}+\kappa_r b\left(\tau\right)-1=0 \quad \text{(A.3)}$$

$$\frac{1}{2}\rho_r^2 b^2\left(\tau\right)-\kappa_r\theta_r b\left(\tau\right)-\frac{\partial a}{\partial \tau}=0 \quad \text{(A.4)}$$

Starting with equation $(A.3)$,

$$\frac{\partial b}{\partial \tau} = 1 - \kappa_r b(\tau) \Leftrightarrow \frac{1}{1-\kappa_r b(\tau)}\partial b = \partial \tau \Leftrightarrow -\frac{1}{\kappa_r}\ln(1-\kappa_r b(\tau)) = \tau + C,$$

where C is a constant of integration.

$$b(0) = 0 \Rightarrow C = 0 \Rightarrow -\frac{1}{\kappa_r}\ln(1-\kappa_r b(\tau)) = \tau \Leftrightarrow b(\tau) = \frac{1-e^{-\kappa_r \tau}}{\kappa_r} \tag{A.5}$$

Substituting equation $(A.5)$ into equation $(A.4)$:

$$\begin{aligned}
\frac{\partial a}{\partial \tau} &= -\kappa_r\theta_r\left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r}\right) + \frac{1}{2}\rho_r^2\left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r}\right)^2 \\
\Leftrightarrow a(\tau) &= -\kappa_r\theta_r\left(\frac{\tau + \frac{1}{\kappa_r}e^{-\kappa_r\tau}}{\kappa_r}\right) + \frac{1}{2}\rho_r^2\left(\frac{\tau + \frac{2}{\kappa_r}e^{-\kappa_r\tau} - \frac{1}{2\kappa_r}e^{-2\kappa_r\tau}}{\kappa_r^2}\right) + C \\
\Leftrightarrow a(\tau) &= -\theta_r\left(\tau + \frac{1}{\kappa_r}e^{-\kappa_r\tau}\right) + \frac{1}{2}\rho_r^2\left(\frac{\tau}{\kappa_r^2} + \frac{2}{\kappa_r^3}e^{-\kappa_r\tau} - \frac{1}{2\kappa_r^3}e^{-2\kappa_r\tau}\right) + C
\end{aligned}$$

$$a(0) = 0 \Rightarrow C = \frac{\theta_r}{\kappa_r} - \frac{1}{2}\rho_r^2\left(\frac{2}{\kappa_r^3} - \frac{1}{2\kappa_r^3}\right)$$

$$\begin{aligned}
a(\tau) &= \left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r} - \tau\right)\theta_r - \left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r} - \tau\right)\frac{\rho_r^2}{2\kappa_r^2} - \left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r} + \frac{e^{-2\kappa_r\tau}-1}{2\kappa_r}\right)\frac{\rho_r^2}{2\kappa_r^2} \\
&= \left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r} - \tau\right)\left(\theta_r - \frac{\rho_r^2}{2\kappa_r^2}\right) - \left(\frac{1-2e^{-\kappa_r\tau}+e^{-2\kappa_r\tau}}{\kappa_r^2}\right)\frac{\rho_r^2}{4\kappa_r} \\
&= (b(\tau) - \tau)\left(\theta_r - \frac{\rho_r^2}{2\kappa_r^2}\right) - (b(\tau))^2\frac{\rho_r^2}{4\kappa_r} \qquad \text{(A.6)}
\end{aligned}$$

A.2 CIR

In the CIR case, the ODE are:

$$\frac{\partial b}{\partial \tau} + \kappa_r b(\tau) + \frac{1}{2}\sigma_r^2 b^2(\tau) - 1 = 0 \tag{A.7}$$

$$-\kappa_r\theta_r b(\tau) - \frac{\partial a}{\partial \tau} = 0 \tag{A.8}$$

The following lemmas are useful:

Lemma A.1. *The solution of differential equations of the type* $\frac{\partial f}{\partial \tau} = af^2(\tau) + bf(\tau) + c$*, with* $a \neq 0$ *and subject to* $f(0) = 0$ *is:*

$$f(\tau) = \frac{-b+d}{2a} \frac{1-e^{d\tau}}{1-qe^{d\tau}} \tag{A.9}$$

where $d = \sqrt{b^2 - 4ac}$ *and* $q = \frac{b-d}{b+d}$

Proof.

$$af^2 + bf + c = 0 \Leftrightarrow f = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm d}{2a} \tag{A.10}$$

$$\frac{A}{\left(f+\frac{b-d}{2a}\right)} + \frac{B}{\left(f+\frac{b+d}{2a}\right)} = \frac{1}{a\left(f+\frac{b+d}{2a}\right)\left(f+\frac{b-d}{2a}\right)} \Leftrightarrow \begin{cases} -A = B \\ A = \frac{1}{d} \end{cases} \tag{A.11}$$

Due to equations $(A.10)$ and $(A.11)$:

$$\begin{aligned}
&\frac{\partial f}{\partial \tau} = af^2 + bf + c \\
\Leftrightarrow &\frac{\partial f}{\partial \tau} = a\left(f + \frac{b+d}{2a}\right)\left(f + \frac{b-d}{2a}\right) \\
\Leftrightarrow &\frac{1}{a\left(f+\frac{b+d}{2a}\right)\left(f+\frac{b-d}{2a}\right)}\partial f = \partial \tau \\
\Leftrightarrow &\left(\frac{1}{d\left(f+\frac{b-d}{2a}\right)} - \frac{1}{d\left(f+\frac{b+d}{2a}\right)}\right)\partial f = \partial \tau \\
\Leftrightarrow &\frac{1}{d}\ln\left(\frac{f2a+b-d}{f2a+b+d}\right) = \tau + C \\
\Leftrightarrow &\frac{f2a+b-d}{f2a+b+d} = e^{d(\tau+C)}
\end{aligned} \tag{A.12}$$

$$f(0) = 0 \Rightarrow \frac{b-d}{b+d} = e^{dC} \Leftrightarrow q = e^{dC} \tag{A.13}$$

$$\frac{f2a+b-d}{f2a+b+d} = qe^{d\tau} \Leftrightarrow \left(2a\left(1-qe^{d\tau}\right)\right) = qe^{d\tau}(b+d) - (b-d) \Leftrightarrow$$

$$\Leftrightarrow f\left(2a\left(1-qe^{d\tau}\right)\right) = \left(e^{d\tau}-1\right)(b-d) \Leftrightarrow f = \frac{-b+d}{2a} \times \frac{1-e^{d\tau}}{1-qe^{d\tau}}$$

□

Lemma A.2. *Defining* b *and* d *as in the previous Lemma, then*

$$\frac{\partial f}{\partial \tau} = \frac{1-e^{d\tau}}{1-qe^{d\tau}} \Leftrightarrow f(\tau) = \tau + \frac{2}{b-d}\ln\left(1-qe^{d\tau}\right) + C \tag{A.14}$$

Proof.

$$\frac{\partial f}{\partial \tau}=\frac{1-e^{d\tau}}{1-qe^{d\tau}}\Leftrightarrow\frac{\partial f}{\partial \tau}=1+\frac{(q-1)\,e^{d\tau}}{1-qe^{d\tau}}\Leftrightarrow f\left(\tau\right)=\tau+\frac{q-1}{-qd}\ln\left(1-qe^{d\tau}\right)+C$$

$$\frac{q-1}{-qd}=\frac{\frac{b-d}{b+d}-1}{-\frac{b-d}{b-d}d}=\frac{1-\frac{2d}{b+d}-1}{-\frac{b-d}{b-d}d}=\frac{-\frac{2d}{b+d}}{-\frac{b-d}{b-d}d}=\frac{2}{b-d}$$

So,

$$f\left(\tau\right)=\tau+\frac{2}{b-d}\ln\left(1-qe^{d\tau}\right)+C$$

□

Applying Lemma A.1 into equation $(A.7)$ and defining $\xi=\sqrt{\kappa_r^2+2\sigma_r^2}$:

$$b\left(\tau\right)=-\frac{\kappa_r+\xi}{\sigma_r^2}\frac{1-e^{\xi\tau}}{1+\left(\frac{\kappa_r+\xi}{\xi-\kappa_r}\right)e^{\xi\tau}} \tag{A.15}$$

Replacing equation $(A.15)$ in equation $(A.8)$ and using Lemma A.2:

$$\frac{\partial a}{\partial \tau}=\kappa_r\theta_r\frac{\kappa_r+\xi}{\sigma_r^2}\frac{1-e^{\xi\tau}}{1+\left(\frac{\kappa_r+\xi}{\xi-\kappa_r}\right)e^{\xi\tau}}\Leftrightarrow$$

$$\Leftrightarrow a\left(\tau\right)=\kappa_r\theta_r\frac{\kappa_r+\xi}{\sigma_r^2}\left(\tau-\frac{2}{\kappa_r+\xi}\ln\left(1+\frac{\kappa_r+\xi}{\xi-\kappa_r}e^{\xi\tau}\right)+C\right)$$

$$a(0)=0\Rightarrow C=\frac{2}{\kappa_r+\xi}\ln\left(1+\frac{\kappa_r+\xi}{\xi-\kappa_r}\right)$$

$$\Rightarrow a\left(\tau\right)=\kappa_r\theta_r\frac{\kappa_r+\xi}{\sigma_r^2}\left(\tau-\frac{2}{\kappa_r+\xi}\ln\left(\frac{1+\frac{\kappa_r+\xi}{\xi-\kappa_r}e^{\xi\tau}}{1+\frac{\kappa_r+\xi}{\xi-\kappa_r}}\right)\right)$$

Simplifying the previous expression:

$$\begin{aligned}a\left(\tau\right)&=\frac{2\kappa_r\theta_r}{\sigma_r^2}\left(\ln\left(e^{(\kappa_r+\xi)\frac{\tau}{2}}\right)-\ln\left(\frac{\xi-\kappa_r+(\kappa_r+\xi)\,e^{\xi\tau}}{\xi-\kappa_r+\kappa_r+\xi}\right)\right)\\&=\frac{2\kappa_r\theta_r}{\sigma_r^2}\ln\left(\frac{e^{(\kappa_r+\xi)\frac{\tau}{2}}}{\frac{\xi-\kappa_r+(\kappa_r+\xi)e^{\xi\tau}}{2\xi}}\right)=\frac{2\kappa_r\theta_r}{\sigma_r^2}\ln\left(\frac{2\xi e^{(\kappa_r+\xi)\frac{\tau}{2}}}{2\xi+(\kappa_r+\xi)\left(e^{\xi\tau}-1\right)}\right)\end{aligned} \tag{A.16}$$

Because $\xi=\sqrt{\kappa_r^2+2\sigma_r^2}\Leftrightarrow\sigma_r^2=\frac{\xi^2-\kappa_r^2}{2}\Leftrightarrow\sigma_r^2=\frac{(\xi-\kappa_r)(\xi+\kappa_r)}{2}$, equation $(A.15)$ can be transformed into:

$$b\left(\tau\right)=\frac{2(1-e^{\xi\tau})}{(\xi-\kappa_r)+(\kappa_r+\xi)\,e^{\xi\tau}}=\frac{2(e^{\xi\tau}-1)}{2\xi+\left(e^{\xi\tau}-1\right)(\kappa_r+\xi)} \tag{A.17}$$

Appendix B

Novikov's Condition

Revuz and Yor (1999, page 338, Exercise 1.40) present the following version of Novikov's condition:

Lemma B.1. *Let W_t be an $\mathcal{F}_t$-Brownian motion and Y_t an $\mathcal{F}_t$-adapted process such that*

$$\mathbb{E}\left[e^{aY^2(s)} \middle| \mathcal{F}_r \right] \leq c \tag{B.1}$$

for all $r < s \leq T$ and two positive constants a and c. Then the process

$$X_t = \exp\left(-\frac{1}{2} \int_0^t Y^2(s)\, ds + \int_o^t Y(s)\, dW_s \right), 0 \leq t \leq T \tag{B.2}$$

is a martingale.

The goal is to use Lemma B.1 with $Y^2(t)$ being v_t or $\frac{\sigma_r^2 r_t + \rho_r^2}{P^2(t,T)} \left(\frac{\partial P}{\partial r_t} \right)^2$. For the first case I proceed as Kruse and Nogel (2005, p.233). For the other case:

$$Vasicek \qquad \frac{\sigma_r^2 r_t + \rho_r^2}{P^2(t,T)} \left(\frac{\partial P}{\partial r_t} \right)^2 = \rho_r^2 b(\tau)^2 \tag{B.3}$$

$$CIR \qquad \frac{\sigma_r^2 r_t + \rho_r^2}{P^2(t,T)} \left(\frac{\partial P}{\partial r_t} \right)^2 = \sigma_r^2 r_t b(\tau)^2 \tag{B.4}$$

Note that in the Vasicek case $Y^2(t)$ is not a random variable. So, and because $b(\tau)$ is limited for the possible τ, the demanded condition in Lemma B.1 is satisfied, ensuring the Lemma's conclusion.

For the other two cases, it is necessary to use the following lemma (see, for instance, Kapadia, Owen and Patel (1976, p.46)):

Lemma B.2. *Let X be a random variable with non-central Chi-squared distribution with R degrees of freedom and non-centrality parameter Λ. The moment generating function of X is given by:*

$$\phi(a) = (1-2a)^{-\frac{R}{2}} e^{\frac{\Lambda a}{1-2a}} \tag{B.5}$$

Let $t < s$. Considering $f_{\chi^2(\Lambda,R)}$ to be the density of a non-central Chi-squared distribution with R degrees of freedom and non-centrality parameter Λ, the transition density of the variance process v_s and the interest rate r_s (according to the CIR model) are given by:

$$f\left(v_s \middle| v_t\right) = B f_{\chi^2(\Lambda,R)}\left(Bv\right) \tag{B.6}$$

$$CIR \quad f\left(r_s \middle| r_t\right) = B_r f_{\chi^2(\Lambda_r,R_r)}\left(B_r r\right) \tag{B.7}$$

where $\Lambda = Be^{-\kappa(s-t)}v_t$, $B = \frac{4\kappa}{\sigma^2}\frac{1}{1-e^{-\kappa(s-t)}}$ and $R = \frac{4\kappa\theta}{\sigma^2}$,
and $\Lambda_r = B_r e^{-\kappa_r(s-t)}r_t$, $B_r = \frac{4\kappa_r}{\sigma_r^2}\frac{1}{1-e^{-\kappa_r(s-t)}}$ and $R_r = \frac{4\kappa_r\theta_r}{\sigma_r^2}$.
So, using Lemma B.2:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{av_s}\middle| \mathcal{F}_t\right] = \left(1-2aB^{-1}\right)^{-\frac{R}{2}} e^{\frac{\Lambda a B^{-1}}{1-2aB^{-1}}} \tag{B.8}$$

$$\mathbb{E}_{\mathbb{Q}}\left[e^{a\sigma_r^2 b(\tau) r_s}\middle| \mathcal{F}_t\right] = \left(1-2a\sigma_r^2 b(\tau)B_r^{-1}\right)^{-\frac{R_r}{2}} e^{\frac{\Lambda_r a\sigma_r^2 b(\tau)B_r^{-1}}{1-2a\sigma_r^2 b(\tau)B_r^{-1}}} \tag{B.9}$$

As pointed by Kruse and Nogel for v_t, and also applicable to r_t (in the CIR case), due to the stability conditions $2\kappa\theta > \sigma^2$ and $2\kappa_r\theta_r > \sigma_r^2$, the processes v_t and r_t are strictly positive (Feller, 1951, p.180) and in addition $R, R_r > 2$, following from $\kappa, \kappa_r \geq 0$ that $\Lambda, \Lambda_r \geq 0$ and $B, B_r \geq 0$. This allows to choose two positive constants a and c as demanded in Lemma B.1, ensuring the Lemma's conclusion for each case.

Appendix C

The characteristic function's ODE

Equation (2.40) can be decomposed into three ordinary differential equations:

C.1 Vasicek

$$-\frac{\partial F}{\partial \tau} + \kappa\theta G + \kappa_r\theta_r H + \frac{1}{2}\rho_r^2 H^2 = 0 \tag{C.1}$$

$$-\frac{\partial G}{\partial \tau} - \frac{1}{2}i\phi - \kappa G - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 G^2 + \sigma\rho\phi i G = 0 \tag{C.2}$$

$$-\frac{\partial H}{\partial \tau} - 1 + i\phi - \kappa_r H = 0 \tag{C.3}$$

Starting by equation$(C.3)$:

$$\begin{aligned}
&\frac{\partial H}{\partial \tau} = -1 - \kappa_r H + i\phi \Leftrightarrow \frac{1}{-1 + i\phi - \kappa_r H}\partial H = \partial \tau \Leftrightarrow \\
&\Leftrightarrow -\frac{1}{\kappa_r}\ln\left(-1 + i\phi - \kappa_r H\right) = \tau + C \Leftrightarrow H = \frac{i\phi - e^{-\kappa_r(\tau + C)}}{\kappa_r} \\
&H(0) = 0 \Rightarrow -\frac{1}{\kappa_r}\ln\left(-1 + i\phi\right) = C \Rightarrow H = \left(i\phi - 1\right)\frac{1 - e^{-\kappa_r\tau}}{\kappa_r}
\end{aligned} \tag{C.4}$$

Equation $(C.2)$ can be solved with Lemma A.1:

$$\begin{aligned}
\frac{\partial G}{\partial \tau} &= -\frac{1}{2}i\phi - \frac{1}{2}\phi^2 + (-\kappa + \sigma\rho\phi i)\, G + \frac{1}{2}\sigma^2 G^2 \Leftrightarrow \\
\Leftrightarrow G &= \frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2}\frac{1 - e^{d_V \tau}}{1 - q_V e^{d_V \tau}}
\end{aligned} \tag{C.5}$$

where

$$d_V = \sqrt{\left(\rho\sigma i\phi - \kappa\right)^2 + \left(i\phi + \phi^2\right)\sigma^2} \tag{C.6}$$

$$q_V = \frac{-\kappa + \rho\sigma\phi i - d_V}{-\kappa + \rho\sigma\phi i + d_V} \tag{C.7}$$

To solve equation $(C.1)$, I start by replacing $(C.4)$ and $(C.5)$ into the equation, and then I use Lemma A.2:

$$\begin{aligned}
\frac{\partial F}{\partial \tau} &= \kappa\theta\left(\frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2}\frac{1 - e^{d_V \tau}}{1 - q_V e^{d_V \tau}}\right) \\
&\quad + \theta_r\left(\phi i - 1\right)\left(1 - e^{-\kappa_r \tau}\right) + \frac{1}{2}\rho_r^2\left(\left(i\phi - 1\right)\frac{1 - e^{-\kappa_r \tau}}{\kappa_r}\right)^2 \\
\Leftrightarrow \frac{\partial F}{\partial \tau} &= \kappa\theta\left(\frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2}\frac{1 - e^{d_V \tau}}{1 - q_V e^{d_V \tau}}\right) \\
&\quad + \theta_r\left(\phi i - 1\right)\left(1 - e^{-\kappa_r \tau}\right) + \frac{1}{2}\rho_r^2\left(i\phi - 1\right)^2\left(\frac{1 - 2e^{-\kappa_r \tau} + e^{-2\kappa_r \tau}}{\kappa_r^2}\right) \\
\Leftrightarrow F &= \kappa\theta\frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2}\left[\tau + \frac{2}{-\kappa + \sigma\rho\phi i - d_V}\ln\left(1 - q_V e^{d_V \tau}\right)\right] \\
&\quad + \theta_r\left(\phi i - 1\right)\left(\tau + \frac{e^{-\kappa_r \tau}}{\kappa_r}\right) \\
&\quad + \frac{\rho_r^2\left(i\phi - 1\right)^2}{2\kappa_r^2}\left(\tau + 2\frac{e^{-\kappa_r \tau}}{\kappa_r} - \frac{e^{-2\kappa_r \tau}}{2\kappa_r}\right) + C
\end{aligned}$$

$$F(0) = 0 \Rightarrow C = \kappa\theta\left[\frac{2}{\sigma^2}\ln\left(1 - q_V\right)\right] - \frac{\theta_r\left(\phi i - 1\right)}{\kappa_r} - \frac{3\rho_r^2\left(i\phi - 1\right)^2}{4\kappa_r^3}$$

$$
\begin{aligned}
F &= \kappa\theta\frac{\kappa-\sigma\rho\phi i+d_V}{\sigma^2}\left[\tau+\frac{2}{-\kappa+\sigma\rho\phi i-d_V}\ln\left(1-q_Ve^{d_V\tau}\right)\right] \\
&\quad+\theta_r(\phi i-1)\left(\tau+\frac{e^{-\kappa_r\tau}}{\kappa_r}\right)+\frac{\rho_r^2(i\phi-1)^2}{2\kappa_r^2}\left(\tau+2\frac{e^{-\kappa_r\tau}}{\kappa_r}-\frac{e^{-2\kappa_r\tau}}{2\kappa_r}\right) \\
&\quad+\kappa\theta\left[\frac{2}{\sigma^2}\ln(1-q_V)\right]-\frac{\theta_r(\phi i-1)}{\kappa_r}-\frac{3\rho_r^2(i\phi-1)^2}{4\kappa_r^3} \\
&= \theta_r i\phi\tau-\theta_r\tau+\frac{\kappa\theta}{\sigma^2}\left[(\kappa-\rho\sigma\phi i+d_V)\tau-2\ln\left(\frac{1-q_Ve^{d_V\tau}}{1-q_V}\right)\right] \\
&\quad+\frac{\theta_r(i\phi-1)}{\kappa_r}\left(e^{-\kappa_r\tau}-1\right) \\
&\quad+\frac{\rho_r^2(i\phi-1)^2}{2\kappa_r^2}\left(\frac{2\kappa_r\tau+4e^{-\kappa_r\tau}-e^{-2\kappa_r\tau}-3}{2\kappa_r}\right) \qquad \text{(C.8)}
\end{aligned}
$$

C.2 CIR

$$
\begin{aligned}
-\frac{\partial F}{\partial\tau}+\kappa\theta G+\kappa_r\theta_r H &= 0 \qquad \text{(C.9)} \\
-\frac{\partial G}{\partial\tau}-\frac{1}{2}i\phi-\kappa G-\frac{1}{2}\phi^2+\frac{1}{2}\sigma^2G^2+\sigma\rho\phi iG &= 0 \qquad \text{(C.10)} \\
-\frac{\partial H}{\partial\tau}-1+i\phi-\kappa_r H+\frac{1}{2}\sigma_r^2H^2 &= 0 \qquad \text{(C.11)}
\end{aligned}
$$

Equations $(C.2)$ and $(C.10)$ are the same, so, G has the same expression. To solve equation $(C.11)$, Lemma A.1 can be used:

$$
\frac{\partial H}{\partial\tau}=-1+i\phi-\kappa_rH+\frac{1}{2}\sigma_r^2H^2 \Leftrightarrow H=\frac{\kappa_r+d_C}{\sigma_r^2}\frac{1-e^{d_C\tau}}{1-q_Ced_C\tau} \qquad \text{(C.12)}
$$

where

$$
\begin{aligned}
d_C &= \sqrt{\kappa_r^2-2(i\phi-1)\sigma_r^2} \qquad \text{(C.13)} \\
q_C &= \frac{\kappa_r+d_C}{\kappa_r-d_C} \qquad \text{(C.14)}
\end{aligned}
$$

To solve $(C.9)$, I start by replacing $(C.5)$ and $(C.12)$ into the equation, and then I use Lemma A.2:

$$
\begin{aligned}
\frac{\partial F}{\partial \tau} &= \kappa\theta G + \kappa_r\theta_r H \\
\Leftrightarrow \frac{\partial F}{\partial \tau} &= \kappa\theta \left(\frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2} \frac{1 - e^{d_V \tau}}{1 - q_V e^{d_V \tau}} \right) + \kappa_r\theta_r \left(\frac{\kappa_r + d_C}{\sigma_r^2} \frac{1 - e^{d_C \tau}}{1 - q_C e^{d_C \tau}} \right) \\
\Leftrightarrow F &= \kappa\theta \frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2} \left[\tau + \frac{2}{-\kappa + \sigma\rho\phi i - d_V} \ln\left(1 - q_V e^{d_V \tau}\right) \right] \\
&\quad + \kappa_r\theta_r \frac{\kappa_r + d_C}{\sigma_r^2} \left(\tau - \frac{2}{\kappa_r + d_C} \ln\left(1 - q_C e^{d_C \tau}\right) \right) + C
\end{aligned}
$$

$$
F(0) = 0 \Rightarrow C = \frac{\kappa\theta}{\sigma^2} \left[2 \ln\left(1 - q_V\right) \right] + \frac{\kappa_r\theta_r}{\sigma_r^2} \left(2 \ln\left(1 - q_C\right) \right)
$$

$$
\begin{aligned}
F &= \kappa\theta \frac{\kappa - \sigma\rho\phi i + d_V}{\sigma^2} \left[\tau + \frac{2}{-\kappa + \sigma\rho\phi i - d_V} \ln\left(1 - q_V e^{d_V \tau}\right) \right] \\
&\quad + \kappa_r\theta_r \frac{\kappa_r + d_C}{\sigma_r^2} \left(\tau - \frac{2}{\kappa_r + d_C} \ln\left(1 - q_C e^{d_C \tau}\right) \right) \\
&\quad + \frac{\kappa\theta}{\sigma^2} \left[2 \ln\left(1 - q_V\right) \right] + \frac{\kappa_r\theta_r}{\sigma_r^2} \left(2 \ln\left(1 - q_C\right) \right) \\
&= \frac{\kappa\theta}{\sigma^2} \left[\left(\kappa - \rho\sigma\phi i + d_V\right) \tau - 2 \ln\left(\frac{1 - q_V e^{d_V \tau}}{1 - q_V} \right) \right] + \\
&\quad + \frac{\kappa_r\theta_r}{\sigma_r^2} \left[\left(\kappa_r + d_C\right) \tau - 2 \ln\left(\frac{1 - q_C e^{d_C \tau}}{1 - q_C} \right) \right] \qquad \text{(C.15)}
\end{aligned}
$$

Bibliography

Albrecher, H., P. Mayer, W. Schoutens, and J. Tistaert, 2007, The Little Heston Trap *Wilmott Magazine*, to appear.

Attari, M., 2004, Option pricing using Fourier Transforms: A Nummerically Efficient Simplification, Working Paper, (Charles River Associates Inc. , Boston)

Björk, T., 2003, *Arbitrage Theory in Continuous Time*, Second Edition (Oxford University Press, New York).

Bakshi,G., C. Cao, and Z. Chen, 1997, Empirical Performance of Alternative Pricing Models, *The Journal of Finance* 52, 2003–2049.

Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637-654

Feller, W., 1951, Two Singular Diffusion Problems, *Annals of Mathematics* 54, 173–182

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Hull, J. and A. White, 1990, Pricing Interest Rate Derivative Securities, *Review of Financial Studies* 3, 573–592.

Hout, K., J. Bierkens, A. Ploeg, and J. Panhuis, 2007, A semi closed-form analytic pricing formula for call options in a hybrid Heston-Hull-White model *Proceedings of the Fifty-Eighth European Study Group Mathematics with Industry*, University of Utrecht.

Kahl, C. and P. Jäckel, 2005, Not-so-complex Logarithms in the Heston Model, *Wilmott Magazine*, September 2005, 94–103.

Kahl, C. and R. Lord, 2006, Why the Rotation Algorithm Works, Working Paper, University of Wuppertal.

Kapadia, C., D. Owen and J. Patel, 1976, *Handbook of Statistical Distributions* (Marcel Dekker, New York).

Kawata, T., 1972, *Fourier Analysis in Probability Theory* (Academic Press, New York).

Kilin, F., 2007, Accelerating the Calibration of Stochastic Volatility Models, Working Paper, Frankfurt School of Finance Management and Quanteam AG.

Kruse, S., and U. Nögel, 2005, On the pricing of forward starting options in Heston's model on stochastic volatility, *Finance Stochast* 9, 233–250.

Matos, C., and J. Santos, 2000, *Curso de Análise Complexa*, (Escolar Editora, Lisboa).

Press, W., B. Flannery, S. Teukolsky, and W. Vetterling, 1997, *Numerical Recipes in C: The Art of Scientific Computing*, Second edition (Cambridge University Press, Cambridge).

Revuz, D. and M. Yor, 2004, *Continuous Martingales and Brownian Motion* (Springer, New York).

Shreve, S., 2004, *Stochastic Calculus for Finance II: Continuous-Time Models* (Springer, New York).

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.