ALGEBRAIC THINKING OF GRADE 8 STUDENTS IN SOLVING WORD PROBLEMS WITH A SPREADSHEET

Sandra Nobre, Escola E.B. 2, 3 Professor Paula Nogueira

Nélia Amado & Susana Carreira, FCT, University of Algarve & UIDEF

João Pedro da Ponte, Institute of Education, University of Lisbon

This paper describes and discusses the activity of grade 8 students on two word problems, using a spreadsheet. We look at particular uses of the spreadsheet, namely at the students' representations, as ways of eliciting forms of algebraic thinking involved in solving the problems. We aim to see how the spreadsheet allows the solution of formally impracticable problems at students' level of algebra knowledge, by making them treatable through the computational logic that is intrinsic to the operating modes of the spreadsheet. The protocols of the problem solving sessions provided ways to describe and interpret the relationships that students established between the variables in the problems and their representations in the spreadsheet.

Key-words: algebraic thinking, spreadsheet, representations.

INTRODUCTION

Representations have a dual role in learning and in mathematical communication. These resources serve the purpose of communicating with others about a problem or an idea but also constitute tools that help to achieve an understanding of a property, a concept or a problem (Dufour-Janvier, Bednarz & Belanger, 1987). This is one of the reasons why we consider students' use of representations as a lens from which we can grasp the meaning involved in the mathematical processes of solving a problem.

Spreadsheets have great potential for the construction of algebraic concepts, including the establishment of functional relationships, the representation of sequences or the use of recursive procedures in solving mathematical problems. The use of spreadsheets in problem solving has been deeply investigated by several authors (e.g., Ainley et al., 2004; Rojano, 2002) and revealed interesting processes in the development of algebraic thinking, particularly with regard to the transition from arithmetic to algebra. Within a spreadsheet environment, the symbolic representation of the relations present in a problem is initiated through the nomination of columns and writing of formulas. This is considered a stimulating environment that fosters an understanding of the relations of dependence between variables and encourages students to submit solutions gradually more algebraic and moving away from arithmetical methods (Rojano, 2002). These aspects encouraged us to carry out an analysis of how grade 8 students create their representations and how they conceive and display the problem conditions on the spreadsheet.

PROBLEM SOLVING AND THE LEARNING OF ALGEBRA

Problems are an important type of task that leads to algebraic activity. According to Kieran (2004) the work in algebra can be divided into three areas: generational, transformational and global/meta-level activities. Generational activities correspond to the construction and interpretation of algebraic objects. Transformational activities include simplifying algebraic expressions, solving equations and inequalities and manipulating expressions. Finally, global/meta-level activities involve problem solving and mathematical modelling, including pattern generalization and analysis of variation.

The nature of algebraic reasoning depends on the age and mathematical experience of the students. Students at a more advanced level may use in a natural way symbolic expressions and equations instead of numbers and operations. For students who have not yet learned the algebraic notation, the more general ways of thinking about numbers, operations and notations, may be effectively considered algebraic (Kieran, 2007). Contexts that involve numbers, functional relationships, regularities, and other properties, are an essential foundation for the understanding of algebraic structures. For instance, writing symbolic numerical relations may favour the use of letters. However, the use of technological tools allows other representations for such relations, as well as new forms of exploration, which may be seen as analogous to generational and transformational activities in algebra. Thus, it seems appropriate that such new representations, and the mathematical thinking associated with them, are included in the field of algebra (Kieran, 1996). Moreover, Lins & Kaput (2004) claim that algebra can be treated from the arithmetic field, since there are many properties, structures and relationships that are common to these two areas. Therefore, arithmetic and algebra may be developed as an integrated field of knowledge. In this study we adopt this perspective, considering algebraic thinking as a broad way of thinking that is not limited to the formal procedures of algebra. This entails to separate algebraic thinking from algebraic symbolism (Zazkis & Liljedhal, 2002).

SPREADSHEETS IN THE DEVELOPMENT OF ALGEBRAIC THINKING

A spreadsheet supports the connection between different registers (numerical, relational, and graphical). One feature that stands out in this tool is the possibility of dragging the handle of a cell containing a formula along a column. This action generates a “variable-column”. Using this tool in problem solving emphasizes the need to identify the relevant variables and encourages the search for relations of dependence between variables. The definition of intermediate relations between variables, that is, the breakdown of complex dependency relations in successive simpler relations is a process afforded by this tool, with decisive consequences in the process of problem solving (Carreira, 1992; Haspekian, 2005). As noted by Haspekian (2005) a spreadsheet also allows an algebraic organization of apparently arithmetical solutions and this kind of hybridism, where arithmetic and algebra naturally cohabit, becomes an educational option that may help students in moving from arithmetic to algebra (Kieran, 1996).

We want to see how this particular functioning of the spreadsheet is a valid route for solving problems where the formal algebraic approach is too heavy for the students’ level. More specifically, we aim to understand how far the spreadsheet, while being a means to promote algebraic thinking, can relieve the burden of formal algebraic procedures and as such can advance the possibility of solving certain types of problems. So far, research has shown the value of the spreadsheet in the transition from arithmetic thinking to algebraic thinking, but less is known about the utility of the spreadsheet to set up an alternative to formal and symbolic algebra and yet allowing the development of students' algebraic thinking (Carreira, 1992; Haspekian, 2005; Rojano, 2002)

METHODOLOGY

This study follows a qualitative and interpretative methodology. The participants are three grade 8 students (13-14 years). They had some previous opportunities to solve word problems with a spreadsheet in the classroom, from which they acquired some basics of the spreadsheet operation. Before the two tasks here presented, students had worked with the spreadsheet in solving other problems for six lessons. All problems involved relationships among variables (usually equations) and only one included a simple linear inequality. The detailed recording of student's processes was achieved with the use of *Camtasia Studio*. This software allows the simultaneous collecting of the dialogues of the students and the sequence of the computer screens that show all the actions that were performed on the computer. We were able to analyze the students' conversations while we observed their operations on a spreadsheet. This type of computer protocols is very powerful as it allows the description of the actions in real time on the computer (Weigand & Weller, 2001).

The two problems

King Edgar of Zirtuania decided to divide their treasure of a thousand gold bars by his four sons. The royal verdict is:

1- The 1st son gets twice the bars of the 2nd son.
2 - The 3rd son gets more bars than the first two together.
3 - The 4th son will receive less than the 2nd son.
What is the highest number of gold bars that the 4th son of the king may receive?

Figure 1: The treasure of King Edgar

From small equilateral triangles, rhombuses are formed as shown in the picture. We have 1000 triangles and we wish to make the biggest possible rhombus.
How many triangles will be used?

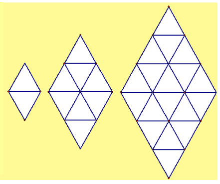


Figure 2: Rhombuses with triangles

A possible algebraic approach to the problems is presented in table 1. Solving these problems by a formal algebraic approach, namely using inequalities and systems such as these was beyond the reach of these students. Therefore, it is important to see which roads are opened by using the spreadsheet.

The treasure of King Edgar	Rhombuses with triangles
$\begin{cases} s_1 + s_2 + s_3 + s_4 = 1000 \\ s_1 = 2s_2 \\ s_3 >' s_1 + s_2 \\ s_4 < s_2 \end{cases} \quad (\max s_4)$ s_i – number of bars of $i\,(i = 1 \ldots 4)$ child	$2n^2 \leq 1000 \quad (\max n)$ n – figure number

Table 1: Algebraic approach to the problems

Problem 1 contains several conditions that relate to each other and the statements "gets more" and "receives less" involve an element of ambiguity and make the problem complex, for understanding it, for translating into algebraic language and for solving it. Problem 2 entails a pictorial sequence that can be translated algebraically into a single condition. However, this condition involves a quadratic function that does not arise immediately after reading the statement of the problem. These two problems represent instances of global/meta-level activities considered by Kieran (2004), insofar as they involve functional reasoning and pattern finding strategies. They both have in common the search for a maximum value, leading to some difficulties when a purely algebraic approach is envisioned. However, a spreadsheet provides alternative approaches to both problems that may make them clearer to students, facilitating their solution process and efficiently providing a solution. We examine how students approached these problems in the classroom, the strategies they used, how they connected the variables involved and expressed that on a spreadsheet. Excerpts of Excel computer protocols are offered to further clarify the description of students' activity.

In solving problem 1, Marcelo assigned and named a column to each of the four sons and a fifth column for the total of gold bars (table 2). Then, he started writing values in the cells corresponding to the sons in the following order: 2nd, 1st, 4th and 3rd, as follows: choosing a value for the 2nd son, then mentally doubling it for the 1st son; subtracting one unit to the 2nd son's number of bars to get the 4th son's; add the three values of the 2nd, 1st, and 4th sons and calculate the difference to 1000 to find the 3rd son's number. In another column, the student entered a formula that gives the total of gold bars and served as control for the total number of bars (1000).

1º filho	2º filho	3º filho	4º filho	Total
200	100	601	99	1000
300	150	451	99	1000
240	120	521	119	1000
260	130	481	129	1000
290	145	436	129	1000
284	142	433	141	1000
286	143	430	141	1000
288	144	433	135	1000

xls file

1º filho	2º filho	3º filho	4º filho	Total
200	100	601	99	=D2+E2+F2+G2
300	150	451	99	=D3+E3+F3+G3
240	120	521	119	=D4+E4+F4+G4
260	130	481	129	=D5+E5+F5+G5
280	140	441	139	=D6+E6+F6+G6
290	145	436	129	=D7+E7+F7+G7
284	142	433	141	=D8+E8+F8+G8
286	143	430	141	=D9+E9+F9+G9
288	144	433	135	=D10+E10+F10+G10

xls file (command "show formulas")

Table 2: Print screen of Marcelo's representation

Although Marcelo did not display the relations between the number of bars of the four brothers, using formulas or otherwise, he kept them always present in his thinking. The task required a greater effort for the student, since in each attempt he had to recall the relations, while carrying out the calculations mentally.

Marcelo: Teacher, I found the best! [the value 139 was obtained in cell G6]. If I choose 150 [for the 2nd son] it won't do. I've tried it.

Teacher: But this is not the maximum number of bars for the 4th son, is it?

Marcelo: I went from 100 to 150, and it turns out that 150 gets worse because the other gets over 450 and the last one falls to 99.

The teacher asked Marcelo to do more experiments to which he replied that he had already made some, for example 160 and 170. So she made another suggestion:

Teacher: Here you already got an excellent value and it increased significantly from 130 to 140 [referring to column E]. So, try around these values.

The student continued to experiment, always doing the calculations mentally. He found 141, confirming that it was the best. As an answer the student wrote: "I solved this problem taking into account the conditions of the problem, making four columns, one for each child, and trying to find a higher number".

In our view, Marcelo has developed algebraic thinking by focusing on dependence relationships between different variables to finding the optimal solution. As he stated, he took into account the five conditions of the problem and expressed them in the spreadsheet columns. From the standpoint of an algebraic approach, the student began by choosing an independent variable (the 2nd son's number of bars) and established relationships to express the number of bars for each remaining son.

1º filho	2º filho	3º filho	4º filho	Total
200	100	601	99	1000

$$\boxed{s_1 = 2s_2} \wedge \boxed{s_2} \wedge \boxed{s_3 = 1000 - (s_1 + s_2 + s_4)} \wedge \boxed{s_4 = s_2 - 1} \wedge \boxed{s_1 + s_2 + s_3 + s_4 = 1000}$$

The diagram above summarizes the translation of the student's algebraic thinking in solving the problem and shows how the Excel allowed dealing with simultaneous manipulation of several conditions, by means of numbers, rather then with letters and symbolic algebra. It is important to note that the condition set for the 4rd son demonstrates an understanding of looking for the highest possible value, given that the difference down to the 2nd was only one bar.

Maria and Jessica (table 3) started to solve the problem like Marcelo, with the allocation of columns to the number of gold bars for each son and another column for the total of bars. Then, they created a column of integers for the number of bars of the 2nd son; the number of bars of the 1st son was obtained by doubling the 2nd son's; the number of bars of the 3rd son was found by adding a unit to the sum of the 1st and 2nd sons' bars; the number of bars of the 4th son was obtained by subtracting one unit to 2nd son's; finally, the last column computed the sum of bars of the four sons.

2filho	1 filho	3 filho	4 filho	total de baras
1	2	4	0	7
2	4	7	1	14

141	282	424	140	987
142	284	427	141	994
143	286	430	142	1001
144	288	433	143	1008

xls file

2filho	1 filho	3 filho	4 filho	total de baras
1	2	4	0	=D2+E2+F2+G2
2	4	7	1	=D3+E3+F3+G3
3	6	10	2	=D4+E4+F4+G4
4	8	13	3	=D5+E5+F5+G5
5	10	16	4	=D6+E6+F6+G6
6	12	19	5	=D7+E7+F7+G7

xls file (command "show formulas")

Table 3: Print screen of Maria e Jessica's representation

At one point the students got a value higher than 1000 in the last column and concluded that it was necessary to remove a bar from one of the sons. Yet, it was necessary to realize that one bar could only be taken from the 4th in accordance with the terms of the problem.

Maria: It shows 1001, it is wrong!

Teacher: And now?

Maria: Take one out! Take one out from the 4th!... The largest number of bars that the 4th can receive is 141.

The diagram below shows the relations as they would be expressed symbolically in algebraic language:

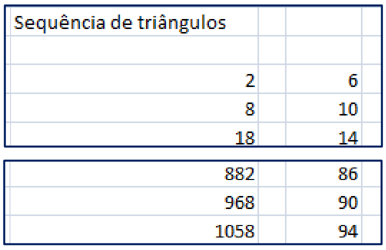
2filho	1 filho	3 filho	4 filho	total de baras
1	2	4	0	7

$$\boxed{s_1 = 2s_2} \wedge \boxed{s_2} \wedge \boxed{s_3 = s_2 + s_1 + 1} \wedge \boxed{s_4 = s_2 - 1} \wedge \boxed{s_1 + s_2 + s_3 + s_4 = 1000}$$

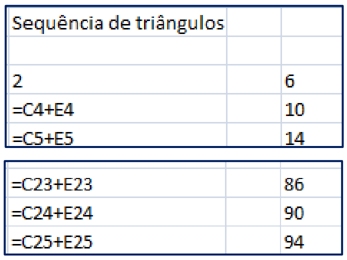
These students formulated two conditions intended to obtain the optimal solution, first, the difference between the number of bars of the 4th and the 2nd son must be a unit and, second, the difference between the number of bars of the 3rd and the sum of the 1st and 2nd must also be a unit.

In both solutions, data show that students use relationships between variables but they do it with numbers through the use of the spreadsheet. The fact that they are working with numbers does not deviate them from the mathematical structure of the problem. On the contrary, it helps them to better understand the problem and to deal with a set of simultaneous conditions of different nature: equations, inequalities and a free variable. We believe that the thinking involved in either approach is consistent with the perspective of Kieran (2007) and Lins & Kaput (2004) on genuine algebraic thinking development.

For the second problem, Marcelo (table 4) started to introduce the inputs 2, 8, 18. Then, he selected these three cells as a cluster and tried to drag them (figure 3), noticing that the numbers generated were not all integers.



Sequência de triângulos	
2	6
8	10
18	14
882	86
968	90
1058	94



Sequência de triângulos	
2	6
=C4+E4	10
=C5+E5	14
=C23+E23	86
=C24+E24	90
=C25+E25	94

xls file	*xls* file (command "show formulas")

Table 4: Print screen of Marcelo's representation

Marcelo enters 2, 8, 18, 32. He selects these three cells, and then tries to drag, without success. He abandoned the dragging and called the teacher:

Marcelo: I don't know if this works... How do I do this? Is there an easier way?

Teacher: To move from the 1st to the 2nd how much did you add?

The student writes in cell E4 the number 6.

Teacher: And from the 2nd to the 3rd how much do you add?

The student wrote in cell E5 the number 10, followed by 14 and 18.

Teacher: What are you going to do now?

Marcelo: If you pull it down [referring to column E] and then adding this column plus this one [referring to column C and column E]...

The student inserted the formula "=C4+E4" (below the first term of the sequence) and generated a variable-column:

Marcelo: 968! It's what we will use from 1000. We have 1000, so it can't be more than 1000 and 1058 already exceeds.

The student tried to find a pattern in the numbers of triangles. The construction of additional figures did not help the student to find a pattern based on the figure. One useful approach is to look at the differences between the consecutive terms.

From an algebraic point of view, this student is using a recursive method to generate the sequence of triangles with the help of the arithmetic progression which gives the difference between consecutive terms. Excel easily allows handling a recursive approach. Somehow it removed the requirement to find nth element to solve the inequality, although not hiding the mathematical structure of the problem.

Maria and Jessica (table 5) addressed the problem with a similar strategy, noticing that dragging the values 2, 8 and 18 did not produce the sequence of rhombuses presented in the problem. At one point they called the teacher:

nº de figuras	nº de triangulos	(+)4
1	2	6
2	8	10
3	18	14
4	32	18
5	50	22
6	72	26
7	98	30
8	128	34
21	882	86
22	968	90
23	1058	94

xls file

nº de figuras	nº de triangulos	(+)4
1	2	6
2	8	10
3	18	14
4	32	18
5	50	22
6	72	26
7	=G7+F7	30
8	=G8+F8	34
21	=G21+F21	86
22	=G22+F22	90
23	=G23+F23	94

xls file (command "show formulas")

Table 5: Print screen of Maria and Jessica's representation

Jessica: From this one to this one it goes 6 and from this one to this one it goes 10. [Referring to the differences between the terms].

Maria: From 2 to 8 it goes 6… From 8 to 18 it goes 10.

Jessica: Oh teacher, we don't know how to continue.

Teacher: Have you already drawn the next figure to see if there is any relation?

They drew it on paper, but only half of the picture.

Maria: 14, 15 e 16. 16 plus 16 is… 32

Teacher: And now?... How many will the next one have?

Maria: 50.

The students were still looking for a relation between the numbers.

Maria: I know what that is... Look... the link is…

Jessica: The number plus 4.

This was the decisive moment to build the column with the differences between consecutive rhombuses, and then a formula for the number of triangles.

CONCLUDING REMARKS

Our main aim was to understand the role of the spreadsheet in solving two word problems and examine how the solutions reflect students' algebraic thinking, regardless of the use of algebraic symbolism. It was not our intention to consider what students have done without the use of technology, since any of the problems demanded an algebraic knowledge that was beyond the level of students. In any case pencil and paper solutions could certainly come up with methods based on trial and error. We interpreted the students' processes based on the spreadsheet in light of what would be a possible use of symbolic algebra. Thus we intended to make clear students' algebraic thinking in establishing the relationships involved in the problems.

In the first problem, four columns corresponded to the four sons and the column for the 2nd son was reserved for the introduction of initial values (the *input*), serving as a column for the independent variable. The remaining columns were constructed through relations of dependence. For the second problem, the students were not able to express the general term of the sequence, but by counting the number of triangles in the sequence of rhombuses they used the differences between consecutive terms to generate the former sequence recursively. As reported in the research students when confronted with more demanding sequences tend to use the difference method.

We found that the spreadsheet helped the students to establish relations between variables, expressed through numerical sequences generated by the computer, and also with the use of formulas to produce variable-columns. We claim that algebraic thinking was fostered by the affordances of the Excel in generating the rules of the problems. This result resonates with other investigations such as Ainley et al. (2004) but it also highlights the structure of students' algebraic thinking expressed in a particular representation system. It provided a clear indicator of how students interpreted the problems in light of their mathematical knowledge and their knowledge of the tool. The analysis allows us to make inferences about what is gained in using Excel to solve algebraic problems, and helps to understand the relationship between the symbolic language of Excel and the algebraic language. The use of Excel can be seen as means to fill the gap between the algebraic thinking and the ability to use algebraic notation to express such thinking. The lack of algebraic notation and formal algebra methods does not prove the absence of algebraic thinking. The kind of algebraic thinking that emerges from the use of the spreadsheet is the kind that belongs to global algebraic activities (Kieran, 2004).

We highlight the following features of the spreadsheet in algebraic problem solving:

- *It was a way to anticipate complex algebraic problems*; our study shows how the spreadsheet was a tool that allowed 8th grade students to solve two problems that were impracticable from the point view of formal algebra. On the other hand it anticipated forms of algebraic reasoning involved in the problems that were elicited by the representation systems embedded in the spreadsheet.
- *It helped to understand the conditions in the problems*; students clearly understood the relations between the several variables involved and were able to express such conditions and restrictions appropriately. They were not given in algebraic notation but instead with the language of Excel.
- *It led to a numerical approach of an algebraic problem*; students found ways to represent the problem through numerical variable-columns without loosing the structure of the problems.

Our perspective of algebraic thinking stresses the distinction between algebraic notation and algebraic structures, separated by a gap that is often underestimated. We suggest that this gap can be gainfully filled with suitable spreadsheet activities.

> "Rather than insisting on any particular symbolic notation, this gap should be accepted and used as a venue for students to practice their algebraic thinking. They should have

the opportunity to engage in situations that promote such thinking without the constraints of formal symbolism" (Zazkis & Liljedhal, 2002, p. 400).

REFERENCES

Ainley, J., Bills, L., & Wilson, K. (2004). Construting meanings and utilities within algebraic tasks. In M. J. Høines & A. B. Fuglestad (Eds), *Proceedings of the 28th PME Conference* (Vol. 2, pp. 1-8). Bergen, Norway.

Carreira, S. (1992). *A aprendizagem da Trigonometria num contexto de aplicações e modelação com recurso à folha de cálculo* (Tese de Mestrado). Lisboa: APM.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 109-122). Hillsdale, NJ: Lawrence Erlbaum.

Haspekian, M. (2005). An 'instrumental approach' to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *International Journal of Computers for Mathematical Learning, 10*(2), 109-141.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde & A. Pérez (Eds.), *International Congress on Mathematical Education 8: Selected lectures* (pp. 271-290). Seville: SAEM Thales.

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey et al. (Eds). *The future of teaching and learning of algebra: The 12th ICMI Study* (pp. 21-34). Dordrecht: Kluwer.

Kieran, C. (2007). Developing algebraic reasoning: the role of sequenced tasks and teacher questions from the primary to the early secondary school levels. *Quadrante*, *16*(1), 5-26.

Lins, R., &. Kaput, J. (2004). The early development of algebraic reasoning: the current state of the field. In K. Stacey et al. (Eds). *The future of teaching and learning of algebra: The 12th ICMI Study*. (pp. 47-70). Dordrecht: Kluwer.

Rojano, T. (2002). Mathematics learning in the junior secondary school: Students' access to significant mathematical ideas. In L. English, M. B. Bussi, G. A. Jones, R. A. Lesh & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 143-161). Mahwah, NJ: Lawrence Erlbaum.

Weigand, H., & Weller, H. (2001). Changes in working styles in a computer algebra environment: The case of functions. *International Journal of Computers in Mathematical Learning*, *6*(1), 87-111.

Zazkis, R., & Liljedhal, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*, 379-402.