Geometry learning: The role of tasks, working modes, and dynamic geometry software[1]

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Abstract: We present several learning experiences that illustrate how three aspects of the geometric competence, constructing and analyzing properties of figures, identifying patterns and investigating and geometric problem solving, were developed by pupils that participated in the implementation of an innovative geometry teaching unit in grade 8. The topics addressed were dealing with properties of two dimensional figures, Pythagoras theorem, loci, translations and similarity of triangles. The development of the geometric competence was clearly supported by the dynamic geometry environment but unfolded in different ways, depending on the way how pupils reacted to the different types of tasks.

Key words: Geometry learning, Dynamic geometry software, Investigation activities, Geometric problem solving.

Resumo. Apresentamos várias experiências de aprendizagem que ilustram como três aspectos da competência geométrica, construir e analisar propriedades de figuras, identificar regularidades e investigar e resolver foram desenvolvidos por alunos que participaram de uma experiência de ensino inovadora no campo da Geometria no 8.º ano de escolaridade. Os temas tratados incluem o trabalho com figuras bidimensionais, teorema de Pitágoras, lugares geométricos, translações e semelhança de triângulos. O desenvolvimento da competência geométrica foi claramente apoiado pelo ambiente de geometria dinâmica mas processou-se de formas diferentes, em resultado do modo como os alunos reagiram aos diferentes tipos de tarefa.

Palavras-chave: Aprendizagem da Geometria, Ambientes de Geometria Dinâmica, Actividades de Investigação, Resolução de problemas geométricos.

INTRODUCTION

In the last decades geometry and its teaching have regained importance within the mathematics curriculum. However Portuguese teachers of grades 7-9 (teaching pupils 12-15 years old), according to the study Matemática 2001 (APM, 1998), consider that geometry topics should be simplified or, in some cases, excluded from the curriculum. We wonder if

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the introduction of technologies in the teaching of geometry may help to modify such pessimistic vision concerning this topic.

In fact, recent curriculum proposals for geometry teaching give special emphasis to the use of technology. For instance, in Principles and standards 2000 (NCTM, 2000) technology, particularly computers, appear as one of the principles for mathematics teaching. Dynamic geometry software is highlighted because it allows to build the basic elements of Euclidean geometry (points, lines, segments and circles) and the relations among them. The constructions made with this kind of software are rigorous and the users can transform them by dragging their basic components. In addition, this software allows to measure lengths, angles, perimeters, areas and also to make calculations with these measurements.

In Portugal, dynamic geometry software have caught teachers' attention and led to a multiplication of workshops. At ProfMat2001, the Portuguese National Meeting of Mathematics Teachers, organized by the Association of Teachers of Mathematics (APM), the Geometry Working Group carried out a survey with 228 teachers. About 75% of the teachers had participated in at least a workshop to learn how to use dynamic geometry software. However only 34% of those who responded used it in their geometry classes (Veloso & Candeias, 2003). This shows that introducing this kind of software in the classroom has been a difficult process.

This paper concerns an investigation carried out with a group of grade 8 pupils when they learnt geometry topics with using a dynamic geometry software, The Geometer's Sketchpad (GSP). We intended to study how pupils developed their geometric competence when they used this kind of software to solve problems and carried out exploration and investigation tasks. We considered the following aspects of geometric competence: constructing and analyzing properties of figures, identifying patterns and investigating and geometric problem solving. Table 1 shows how these aspects were considered, based on the orientations of the Portuguese Curriculum (ME-DEB, 2001). Also following the curriculum orientations, the working mode gave especial attention to pupils having an opportunity to solve tasks on their own, work collaboratively in pairs, and present and discuss their results and strategies in all class discussions.

So, the main purpose of this study was to investigate how pupils developed their geometric competence when using GSP to solve geometric problems and undertake exploration and investigation tasks. More specifically, we aimed to answer to the following question: How does dynamic geometry software associated to these kinds of tasks and to a innovative working mode, contribute to the development of pupils' geometric competence?

Table 1. Aspects of geometric competence.

Constructing and analyzing properties of figures	The skill to make geometric constructions, namely polygons and locus, allowing the recognition and analysis of their properties.
Identifying patterns and investigating	The tendency to find invariants, to explore geometric patterns and to investigate properties and geometric relations.
Geometric problem solving	The skill to solve geometric problems through constructions, justifying the used process.

THE TEACHING UNIT

Tasks. The tasks used in the study were adapted or inspired in Bennett (1995, 1996), De Villiers (1999), Bully and Baldaque (2003), Key Curriculum Press (1995, 1997) and Lopes et al. (1996). Tasks 1 to 8 referred to the curriculum topic "From the space to the plan", tasks 9 to 17 dealt with "Decomposition of figures and Pythagoras theorem", tasks 18 to 21 were related to "Locus", tasks 22 to 24 concerned "Translations" and, finally, tasks 25 and 26 referred to "Similarity of triangles".

The tasks can be classified in two different ways. The first classification concerns their nature: explorations (tasks 1 to 6, 16, 18, 22 and 23, 25), investigations (7 to 10, 17, 19, 20 and 24) and problem solving (11 to 15, 21 and 26). We distinguish between exploration and investigation tasks, because:

> Many times we do not distinguish between investigation and exploration tasks, calling 'investigations' to all of them. That happens probably because it is complicated to know the difficulty degree that an open task will have for a certain group of pupils. However, once we attribute importance to the degree of difficulty of the tasks, it is preferable to have a designation to open tasks that are easier and for those that are more difficult. (Ponte, 2003, p. 28)

In this teaching unit exploration tasks were most prominent because one factor that we considered in this study was the use of computer software that pupils did not know well. There were also a considerable number of investigation tasks "to give pupils the responsibility of discovering and justifying their discoveries" (Ponte, 2003, p. 32). Problem solving activities were also present in this teaching unit, as well as in the textbook tasks that pupils had to do outside the classroom.

The second way of classifying the tasks is to consider the aspects of the geometric competence that each one contributes to develop. Constructing and analyzing properties of figures was present in 18 tasks since the geometric topics taught emphasized the construction of figures. Identifying patterns and investigating, included finding invariants, exploring geometric patterns, and investigating properties and geometric relations, was presented in 11 tasks. Finally, geometric problem solving, allowing pupils to develop skills to solve problems through constructions, was presented in 8 tasks.

Working mode. The first author was the teacher of this class. The class had 18 pupils and they worked in pairs, chosen by them, which corresponded to nine groups. The lessons had 90 minutes each and lasted approximately for four months. The classroom was recently equipped with 14 computers and a multimedia projector by the Portuguese Ministry of Education.

At the end of each task there was a small discussion about the pupils' results, problem solutions and processes. Sometimes there were also discussions on the difficulties related to the use of the software. This phase of the investigation process was very important, because it is in this "a reflection about the work done is carried out. Finishing an investigation task without reflecting about it, is somehow not to have concluded it" (Segurado, 2002, p. 58).

Each pair of pupils had only one worksheet where they had to write down the solutions for each question and, when asked, the processes that they used. Sketches made by pupils with GSP were saved in the computer used by each pair. Problems from the textbook related to the topics under study were suggested to pupils as homework or to be solved in "Supported study"[2]. The purpose of solving these problems was to clarify some doubts that pupils felt during the lessons.

INVESTIGATION METHODOLOGY

Design. Due to the nature of the problem and questions formulated, the study followed the interpretative paradigm (Bogdan & Biklen, 1994). The main instrument of data collection was one of the investigators, who was also the classroom teacher. The research focused then in his teaching practice. In this kind of research the teacher thinks and reflects about his own practice, inquiring about his pupils' learning (Alarcão, 2001). Sierpinska and Kilpatrick (1998) argue that the teacher who investigates his own

[2] "Supported study" is a component of the pupils' curriculum devoted to the study of subjects in which they have more difficulties such as Portuguese, English and mathematics.

practice is in a privileged position, because he can reflect about what his planning and teaching. This reflection can be done using existent theory or, sometimes, creating theory from practice. On the other hand, this research was based in the exchange of ideas between the two authors.

Data collection. In a qualitative investigation it is important to gather information from several sources because as that helps to answer the proposed problem. The use of different instruments allowed different approaches to the problem and helped to give a more complete answer (Bogdan & Biklen, 1994). In this study we used as instruments to gather data pupils' written records (tasks, investigation reports, problem solutions and homework), the teacher's journal (reflection notes and dialogues occurred between pupils and between pupils and teacher), two questionnaires (one at the beginning and other at the end of the study) and interviews carried out by the teacher to each pair of pupils at the end of the research.

Participants. The pupils who took part in the study belong to a grade 8 class of a school near Lisbon. The school had about 700 pupils from grades 5 to 9. The teaching unit was carried out with this class with 18 pupils, nine girls and nine boys. Fifteen of them were 13 years old, two were 15 years old, and one was 12 years old. The first author taught most of those pupils in the previous year and they were for the first time in grade 8. In general, these pupils had success in all subjects, having more difficulties in the Portuguese language.

In this paper we focus our attention on a single pair of pupils, André and José, who had a remarkable performance in this teaching experiment. André was 13 years old and was very reserved. He was born in Guinea, an African country, and came to Portugal when he was 8 years old. He lived with relatives, since his parents stayed in his homeland. His expressive look, last generation mobile phone and cap were his trade image. Inside the classroom, he had a good behavior but had some learning difficulties in mathematics. He was a quiet boy who did not intervene much in class. He was slow in doing the exercises and just copied what was on the board or in his classmate's notebook. During group work, he hid himself behind the work of his colleagues, because he felt some difficulties in speaking Portuguese. In spite of everything, he was successful in school.

André had a great respect and friendship regarding José. They started working together at grade 7, when André had negative marks in 8 subjects that he needed to improve. José started helping him in "supported study" and continued doing it in several other subjects. This partnership allowed André to improve considerably his school results and consequently to pass to grade 8 with no negative marks.

José was a brilliant pupil in all subjects, except physical education, in spite of practicing several sports. In the remaining subjects he usually had all the answers correct and was very concerned when that did not happen. In his interventions in class, always at a high level, he used a brilliant reasoning and a quite advanced vocabulary for his age. He did not turn down a challenge. José had great expectations about the kind of work that we asked them to do: take part in a study, in which pupils would use software to learn geometry for a considerable period of time.

DEVELOPING GEOMETRIC COMPETENCE

Constructing and analyzing properties of figures. In task 19 pupils studied the properties of the midpoint and the perpendicular bisector. The third question was the following:

> Construct a segment and its perpendicular bisector. Draw quadrilaterals with opposite vertices on the perpendicular bisector and on the endpoints of the initial segment as the other two vertices. Identify the quadrilaterals that you find and justify your answer.

The pupils drew a horizontal segment, so the perpendicular bisector was vertical. The first one they drew was a rhombus, but they did not make any attempt to justify it. Then, they observed and justified correctly both the square and the kite. They also observed the parallelogram. When dragging the points constructed in the perpendicular bisector they found a figure that they did not know. The teacher told them that they could name it "boomerang" because of its shape. However, they did not mention that it had two pairs of consecutive equal sides.

But the exploration of this task did not end here. As José and André finished earlier than their colleagues, the teacher asked them to help their classmates. Minutes later, while José was helping a group, one of his colleagues asked him how could he be sure that the figure on the screen (obtained when the points constructed on the perpendicular bisector were at the same distance as the endpoints of the initial segment) was a square. José accepted the challenge and a few moments later called the teacher to show him a sketch:

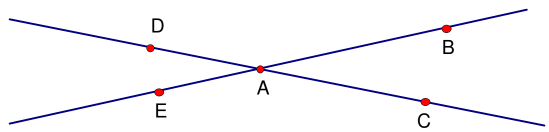
José: I constructed the square.

Teacher: How?

José: I used the Rotate menu to do a rotation with 90 degrees of this point [he pointed to one of the endpoints of the segment AB that had been drawn by a colleague].

José considered the midpoint of the initial segment as the rotation center. Afterwards, he drew the segments between that point and the endpoints of the initial segment. With an analogous process José obtained the other two sides. To confirm that the figure was really a square he measured the angles. He was excited when the 90 degrees for each measurement appeared on the screen: "Here it is! I managed to build it!"

Identifying patterns and investigating. We analyzed the way how pupils were evolving in finding invariants and exploring geometric patterns. In the second lesson, André and José felt some difficulty to understand what a conjecture about vertical opposite angles was. The difficulty emerged in question 1b) of task 2:

> Drag point B. The angles ∠BAC and ∠EAD are vertically opposed angles. The angles ∠EAB and ∠DAC are also vertically opposed angles.
>
> 
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>
> Write a conjecture about the measurement of these angles.

After reading the problem, José got up and addressed the teacher:

José:	What is a conjecture?
Teacher:	You must write what you observe regarding these angles. Have they any relation?
José:	Like an affirmation about the opposite angles?
Teacher:	Yes!
José:	So, vertically opposite angles are always equal.
Teacher:	That's it!

This first difficulty with the term conjecture was overcome, and the pupils wrote the following: "The measures of vertically opposite angles are always equal, regardless the position of the points".

The pupils went through other learning experiences that helped them to develop their skill to explore geometric patterns. That happened in task 9:

> Construct a quadrilateral and the midpoints of its sides. Then, construct the quadrilateral with vertices in the four midpoints. What

> quadrilateral did you obtain? Search for other quadrilaterals: square, rhombus, parallelogram, etc. Write your conjectures and try to justify them.

Based on their previous experiences, André and José started full of energy working on this task. They copied the quadrilaterals they already built in task 8 and started their investigation. They followed the following order: rhombus, square, rectangle, kite, parallelogram, isosceles trapezoid, rectangular trapezoid and scalene trapezoid. After, they extended their investigation to triangles: equilateral, rectangle, isosceles and scalene. At the end of this task they wrote the following conclusions:

> We conclude that, in spite of the several relations among them, there is a rule. When we connected the consecutive midpoints of any quadrilateral that does not have a formation rule, they form parallelograms. Because rhombus and kits belong to the same family regarding the diagonals, they form always rectangles. Because rectangles and isosceles trapezoids have opposite parallel sides and consecutive equal angles, they form rhombus. The square forms another square due to its unique properties. That happens in some regard with triangles, because when we connect their midpoints, they form triangles with the same properties as the original. In another class are the remaining trapezoids that form parallelograms. The parallelogram forms another parallelogram, since it does not have equal consecutive angles like the rectangle and the isosceles trapezoid.

The classification that they presented was different from the one the teacher expected. Nevertheless, this is a possible way to classify some quadrilaterals. They could have written more conclusions, but this was one of their first investigations. They already showed some progresses writing the conjectures that they arrived at.

At the end of the investigation it was possible to observe once again the pupils developing their capacity of searching invariants. In task 26 we asked them to write conjectures relating perimeters and areas of similar triangles. Regarding the first conjecture, José and André did not had difficulty relating the perimeters, because they suspected that it would be necessary to divide them as they did to find the ratio of similarity. So, they wrote that the ratio of the perimeters of similar triangles was equal to their ratio of similarity. Regarding the areas they carried out some attempts to find a relation, but they were having difficulties. So, they called the teacher:

José:	We discover that the ratio of the perimeters is equal to the ratio of the sides. The problem is to find a relation between areas!
Teacher:	Do you think that there is a relation with the ratio of similarity?
José:	I think so! There should be some relation, as it happens with perimeters.
Teacher:	Compare the ratio of similarity and the ratio of the areas.
José:	We tried to find a relation, but we could not.
Teacher:	Drag a vertex in the first triangle and compare the two values.

José followed the teacher's suggestion and, together with André, tried to find a relation. Some time later, he got up and talked with the teacher:

José:	We find it! The relation is the square of the ratio of similarity! I dragged the vertex and with the calculator did the square of the ratio of similarity. And it was equal!
Teacher:	Well done!

Their effort had been rewarded and the GSP tools gave an important help. The possibility to measure and to calculate in a dynamic way and the dragging of the triangles allowed the pupils to realize the relation and to formulate the right conjecture. We can see improvement in finding and writing conjectures.

Geometric problem solving. In this third aspect of the geometric competence, we tried to analyze how the pupils improved their geometrical problem solving ability with constructions, and how they justified the processes that they used.

In problem five of task 21 the pupils did also an interesting work. This problem was composed of two questions:

> Draw rectangle ABCD, in which A and C are opposite vertices, $\overline{AB}$ =10 cm and $\overline{BC}$ = 6 cm.
>
> a) What is the locus of the points that are less than 3 cm from vertex *B*?
>
> b) What is the locus of the points that are closer to point *C* than to point *A*?

When they finished and because they were not sure about their solution, José called the teacher to confirm it. While he was explaining the solution process he reread question b) and verified that in the sketch the points *A* and *C* were not opposite vertices. The solution that they had found was correct if *A* and *C* were adjacent vertices. André changed the vertices letters and both pupils continued looking at the screen, while another pair of pupils called the teacher. Some minutes later they called the teacher again. José had already drawn both diagonals and the intersection point, or rectangle midpoint as they called it. Then, they constructed the circle that passed by that point and with center at *C*. They thought that the solution were all the points inside the circle.

The teacher suggested them to place a "free" point on the figure, and measure the two distances between that point and the points *C* and *A*. That helped pupils to see that there were more points outside the circle that were also solutions to the problem. Then, they have drawn another circle with center in point A passing by the rectangle midpoint. Once again, there were points closer to *C* than *A* not included inside that circle. Later, the teacher asked them what they had found:

Teacher: Have you solved it?

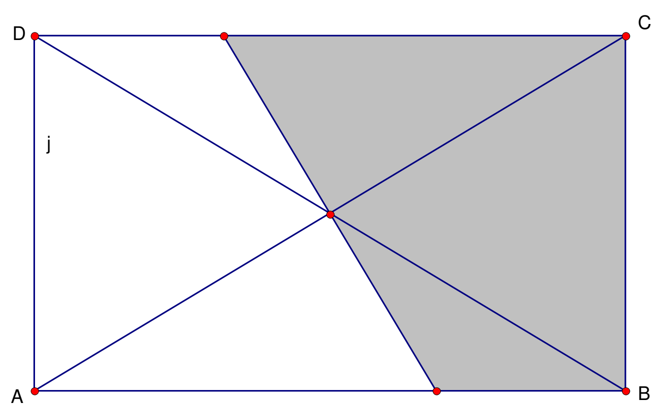
José: Yes! We draw a perpendicular to the diagonal at the midpoint. It is this segment. The points that are near to C are the ones in this zone. [See figure 1]

Teacher: And, what is the name of that figure?

José: It is a rectangular trapezoid! [He answered quickly.]

This episode shows how this problem was solved by these pupils: (i) they found a solution to a simplification of the initial problem; (ii) the solution did not allow them to answer the initial problem, because this was more complex; (iii) the experimental phase in which the pupils built both circles did not allow to answer the problem, but permitted to improve their knowledge about the points that were missing in the first try; (iv) they got involved in a new phase of construction, in which they drew the perpendicular bisector, since the construction of the circles did not result; and (v) they identified the figure that permitted to answer correctly to the problem.

5b) This locus is a rectangular trapezoid with a right angle in point C, and every point inside of it is closer to C than A.



Resolution process - We constructed the rectangle diagonal and its midpoint. Then we constructed the perpendicular bisector to diagonal AC.

Figure 1. Answer given by André and José to question 5 b) of task 21.

CONCLUSIONS

We presented several learning experiences that illustrate how the pupils involved in this study developed the three aspects of the geometric competence, constructing and analyzing properties of figures, identifying patterns and investigating, and geometric problem solving.

The teaching unit was based on three decisive elements: the dynamic geometry software, the tasks proposed, and the way how the pupils worked in the classroom. The use of GSP was an important support to the pupils' constructions and discoveries, and its functionalities allowed them to develop the recognition of properties and the analysis of figures. Dragging a geometric construction and verifying what stays invariant and the possibility of trying many cases allowed them to investigate and solve the proposed geometric problems.

All tasks were well accepted by the pupils, in particular the exploration and investigation ones. Their open nature allowed pupils to get involve actively in their learning and developed the skill to search for invariants. To solve geometric problems, they used constructions. The justifications that the pupils gave to conjectures that they elaborated, their solution processes and the way they improved strategies until they managed to solve a certain problem are aspects that reflect the challenge created by these kind of tasks.

Working in pairs promoted the learning of both pupils, although they were so different. The presentations that André and José made of the result that they found to the other classmates and the discussion of the solutions of the tasks permitted to evaluate the pupils' involvement and their development of geometric competence. Writing in paper or in GSP sketches permitted to evaluate pupils in a holistic[3] way and to give them a fast feedback about their work, as well as suggestions for possible improvements.

In the final interview the pupils expressed that they liked very much the teaching experiment and there had been a change in their perspective about geometry. This topic became more related to challenges and investigations and that happened largely due to the role of the dynamic geometry software.

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[3] Rubrics used to evaluate exploration and investigation tasks are available on-line in http://ia.fc.ul.pt/ (project evaluate geometric problem solving tasks were based on rubrics from Chicago Public Schools available on-line in: http://intranet.cps.k12.il.us/Assessments/Ideas_and_Rubrics/Rubric_Bank/ MathRubrics.pdf. Investigar e Aprender), and were based in the work of Cai and Jakabcsin (1996).

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