Investigating mathematics and learning to teach mathematics[1]

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Resumo. Este artigo baseia-se numa ideia que desempenha um papel crescente no ensino e na formação de professores – investigar constitui um paradigma poderoso de construção do conhecimento. Tanto podem ser realizadas investigações no ensino da Matemática como na formação inicial e contínua do professor de Matemática. Assim, analiso o papel das investigações em Matemática e no currículo de Matemática, apontando algumas questões que os professores enfrentam quando as propõem na sala de aula. De seguida, discuto a formação de professores e o desenvolvimento profissional, dando ênfase ao valor das investigações sobre a prática como meio de desenvolver novo conhecimento. Concluo com exemplos de trabalho realizado por professores em formação inicial e contínua e por equipas de professores e investigadores que se centram no trabalho investigativo dos alunos realizado nas aulas de Matemática, exemplos esses que ilustram o valor educacional desta actividade e permitem discutir os papéis do professor.

Abstract. This paper deals with an idea that plays an increasing role in teaching and in teacher education—investigating as a powerful paradigm of knowledge construction. Investigations may be carried out both in learning mathematics and in learning how to teach mathematics at preservice and inservice levels. I look into investigations in mathematics and in the mathematics curriculum, pointing out some issues that teachers face proposing them in the classroom. Then, I discuss teacher education and professional development, stressing the value of investigations about practice as a means of developing knowledge. I conclude with examples of work done by preservice and inservice teachers and by teams of teachers and researchers focusing on pupils' investigative work in mathematics classes that illustrate the educational value of this activity and discuss the roles of the teacher.

Palavras-chave. Formação de professores, Formação inicial, Formação contínua, Desenvolvimento profissional, Investigação matemática, Investigação sobre a prática, Ensino da Matemática.

Key words. Teacher education, Preservice, Inservice, Professional development, Mathematics investigation, Practitioner Research, Mathematics teaching.

1. PUPILS INVESTIGATING MATHEMATICS

Pupils learn mathematics doing investigations. Why is it so? Mathematics, on the one hand, is the logical and deductive subject depicted in the works of Euclid and Bourbaki. On the other hand, it involves features that come very close to the natural sciences: observation, experimentation, induction, analogy, and plausible reasoning

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(Kline, 1970; Mason, Burton, & Stacey, 1982; Poincaré, 1908; Pólya, 1945). Instead of establishing an irreducible conflict between these two sides of mathematics, one may ask how they can complement each other in the learning process. The inductive side is essential to creating new knowledge, and the deductive side is necessary for organizing it and deciding what is valid and what is not. Both the professional mathematician and the young pupil may get curious, raising questions to themselves about the properties of mathematical objects and seeking to validate their answers.

There is a parallel between the activity of the research mathematician and the activity of the pupil in the classroom. Of course, there are differences between the knowledge held by both, their degree of specialization, the time they spend, and their relation with the subject. However, their problem-solving activity is of a similar nature. Hadamard (1945), a well-known mathematician, refers to this, for example:

> Between the work of the pupil that tries to solve a problem in geometry or algebra and a work of invention [by the mathematician] one can say that there is only a difference of degree, a difference of level, both works being of a similar nature (p. 104).

In mathematics education, Ernest (1991) holds a similar view. He considers all mathematics learners as mathematics creators and argues that productive pupils' problem solving and problem posing parallels the mathematicians' activity: "They do not differ qualitatively" (p. 283). Pólya (1962-65/1981) also stresses that pupils must have the opportunity to experiment several aspects of the mathematical activity. Based on examples of research problems, he shows that the teacher can create conditions so that pupils develop creative and independent work.

Investigative work by pupils receives significant attention in the mathematics curriculum of several countries, either explicitly or implicitly. For example, the French curriculum for the first year of secondary school underlines the importance of making the pupils experience scientific activity, referring to the discovery process as follows:

> [The aim is] to accustom pupils to the scientific activity and to promote the acquisition of methods: the mathematics class is first of all a place of discovery, of exploring situations, of reflection and debate about the strategies employed and the results achieved, of synthesis that yield some clear ideas and essential methods, indicating the respective value. (Ministère de l'Education Nationale, de la Recherche et de la Technologie, 1997, p. 16)

The English curriculum includes aspects directly related to investigative work in one of its main areas ("using and applying mathematics"). It states, for example, that pupils at Key Stage 2 must

- understand and investigate general statements;
- search for patterns in their results;
- make general statements of their own, based on evidence they have produced;
- explain their reasoning (DFE, 1997).

The Portuguese curriculum speaks about exploration, research activities, and conjecturing by pupils. It includes specific suggestions for carrying out this kind of work. It states, for example, that the use of graphic calculators enables secondary school pupils "to carry out mathematical experiences, formulate and test conjectures" (Ministério da Educação, 1997, p. 11) and indicates that "in the study of families of functions pupils may undertake small investigations" (p. 20).

In the USA, there is no national curriculum. However, influential documents such as NCTM's (1991) *Professional Standards* state that "the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining and testing" (p. 95). The more recent *Standards 2000* draft (NCTM, 1998) emphasizes the role of conjecturing and indicates that pupils can investigate mathematics objects. This document says that "at all levels, all branches of mathematics provide opportunities of reasoning and conjecture..." (p. 83) and "students at all grade levels can engage—in age-level-appropriate ways—in the kind of systematic thinking, conjecturing, and marshaling of evidence that are precursors to formal mathematical argumentation" (p. 80).

As mathematical activities, investigations and problem solving are very close and many people use these terms interchangeably. Although both notions refer to complex mathematical processes and point towards problematic activity, I suggest that they also involve some differences. Problems may be more structured or more open. They may refer to purely mathematical situations or real-life contexts. In most cases, the conditions and the questions are clearly framed from the outset and are presented ready-made to the pupil. A mathematical investigation stresses mathematical processes such as searching regularities, formulating, testing, justifying and proving conjectures, reflecting, and generalizing. When one starts working on an investigation, the question and the conditions are usually not completely clear and making them more precise is the first part of the work. That is, investigations involve an essential phase of problem posing by the pupil—something that in problem solving is usually done by the teacher. However, investigations go much beyond simple problem posing and involve testing conjectures, proving, and generalizing. Here we have several examples[2]:

> Consider triangles with integer sides. There are 3 triangles with perimeter 12 units. Investigate.
>
> Investigate the properties of the powers $(1+i)^n$ where i is the imaginary unit. Contrast them with the properties of the powers a^n of real numbers.
>
> Study the properties of the function defined in **R** by
>
> $$y = ax^n+bx^{n-1}$$
>
> when n is odd and when n is even.

The terminology used by educators regarding investigations varies significantly from time to time, from country to country, and even from author to author (see Pehkonen, 1997; Ruthveen, this volume; Sullivan, this volume). Whatever the labelling used, this kind of task enables pupils to become involved in the activity from the very beginning. Designing strategies, generalizing results, establishing relations among concepts and areas of mathematics, systematizing ideas and outcomes, there are multiple opportunities for creative and significant work during an investigation. The challenge for current educational systems is to make such experiences accessible, not just to a few, but to all pupils who must have opportunities to engage in:

[2] The first task is from *SMILE Investigations*, in Lerman, 1989, p. 77.

- identifying and initiating their own problems for investigation;
- expressing their own ideas and developing them in solving problems;
- testing their ideas and hypotheses against relevant experiences;
- rationally defending their own ideas and conclusions and submitting the ideas of others to a reasoned criticism. (Love, 1988, p. 260)

2. THE TEACHER AND THE INVESTIGATION CLASS

Let us look at some of the issues that the teacher faces in planning and conducting investigative work in the classroom.

Selecting and designing investigative situations for the classroom is the first issue. Investigations, if they are to be taken seriously, must be part of the mathematics curriculum and relate to the objectives, topics, working methods, and assessment schemes. The possibilities of the teacher heavily depend on the emphasis that investigations are given in the official curriculum. However, teachers have a fundamental role in interpreting this curriculum and adapting it to their particular circumstances. Selecting tasks, establishing objectives, and defining class operation modes need to take into account pupils' characteristics and the class context. Critical factors to consider are the age range, pupils' mathematics development, and their previous experience in investigations.

A mathematics investigation may begin in different ways. However, some situations are more promising than others. A high concern with certain objectives can lead the teacher to construct tasks that are too structured, that pupils tend to face as recipes to follow through and not as investigations whose questions they must define. The following words of Ollerton (1994) express the attention one should put in the choice of tasks:

An important part of my planning is concerned in finding tasks that:

- are a suitable starter for everyone in the class to work on;
- provide rich opportunities for many developments;
- cause a variety of content skills to be worked on;
- create opportunities for students to explore ideas and ask questions;
- support different types of teacher interventions ranging from asking questions to explaining and telling;
- learners can take over more responsibility for developing;
- will have a variety of outcomes, some of which may be unexpected;
- enable content to be processed;
- draw upon 'real' cross-curricular type contexts, such as using information from a newspaper periodical, or problem-solving contexts;
- wherever possible have a practical beginning in order to provide concrete experiences from which abstractions can be made. (p. 64)

In a similar line of thinking, Lampert (1990) draws our attention to what she regards as the main criterion in the selection of a problem. For her, problems must encourage all pupils to make and test conjectures that the whole class will discuss. The situations that she presents to pupils intend to promote their progression towards more complex and abstract mathematics ideas, or, in her words, to create a "stage of zig-zag between inductive observation and deductive generalization, that Lakatos and Pólya see as characteristics of mathematical activity" (p. 39).

Investigations may arise in a natural way when working in many mathematics topics and allow pupils to use the concepts, representations, ideas, and procedures that they already know. The activity of pupils in an investigation, particular and unique, may originate new questions, follow uncommon paths, and end up relating to many mathematical topics. The teacher needs to find an equilibrium point between orderly following of the planned sequence of questions and valuing unforeseen spin-offs from pupils' investigations that may further promote their mathematical development.

After deciding on the task, there is still some planning to do. This includes making decisions regarding time, class organization and management, and assessment. How long shall the class work in this? Will the pupils work individually, in small groups or as a whole class? How will the pupils get feedback for the work done? Such decisions depend on the task presented, on the curriculum and context constraints, and on the objectives valued by the teacher. Creating, reformulating, and refining suitable tasks require time and an investigative attitude. In doing so, the teacher participates in the process of curriculum construction—formulating objectives, methodologies and strategies, and reformulating them by reflecting on practice.

Usually, carrying out investigative work involves three basic stages: starting the activity, developing it, and a final discussion and summing up (Chapman, 1997; Christiansen & Walther, 1986; Mason, 1991). At the start of the activity, the teacher's aim is to involve the pupils in the task. During the activity, the teacher supports them as they pose questions, represent the information given, formulate and test conjectures, and justify them. In the final phase, the teacher wants to know at which conclusions the pupils arrived, how they justify them, and what implications they suggest. The teacher needs to create a favorable environment for learning, stimulate the communication between pupils, and assume a variety of roles to favor their learning.

Starting the activity is a critical process. Behind a polished and neat question, there is much thinking that one cannot immediately grasp—just as the published work of the mathematician does not reveal the advances and the drawbacks it suffered. It is not reasonable to assume that pupils will view the questions necessarily in the same ways as those who generated them. As Mason (1978) says, "the pupil is not in the same state as the originator [of the question]" (p. 45). This author stresses that "a question is just words with a question mark" (1991, p. 16). That is, a question by itself cannot generate an investigation. The situation that the teacher creates needs to be recreated as a question by the pupil. Moreover, the teacher must consistently reveal an investigative attitude in class to have a positive influence on the curiosity of the pupils.

During the activity, the teacher has to pay attention to the development of pupils' work. The support that is to be granted, helping them overcome difficulties or make a richer investigation, is another complex aspect of the teacher's activity. Saying too much tends to kill the challenge and increases pupils' dependence. Saying too little may be highly frustrating and make pupils move off-task. In addition, the reflection of the pupils about the work done is extremely important in an investigation and the teacher needs to stimulate that. Dealing successfully with these problems requires a lot of experience and sensitivity. Teachers may use questions that stimulate pupils' thinking. However, the need for their orientation must decrease as pupils become more familiar with this type of activity.

Jaworski (1994) reports a study that considered the challenges that this approach raises to the teacher. One of these is the "didactic tension" described in these words of John Mason:

> The *more* explicit I am about the behavior I wish my pupils to display, the more likely it is that they will display the behavior without recourse to the understanding which the behavior is meant to indicate; that is the more they will take the *form* for the substance... The *less* explicit I am about my aims and expectations about the behavior I wish my pupils to display, the less likely they are to notice what is (or might be) going on, the less likely they are to see the point, to encounter what was intended, or to realize what it was all about (Mason, 1988, cited in Jaworski, 1994, p. 180)

Jaworski indicates that she observed this tension in some teachers who participated in her study. They "were reluctant to *tell* students the facts that they wanted them to know; yet were unhappy when those facts did not emerge through the investigation" (p. 207).

One of the great objectives of investigations is leading pupils up gradual levels of generalization and abstraction. In consequence, the justification of conjectures is an important aspect of the work and its degree of formalization should depend on the mathematics level of the pupils. The teacher must point out to them the need for becoming convinced and convincing the others of the value of their arguments (Mason, 1991).

Several authors emphasize that pupils must end the activity with a discussion, indicating that without it the value of the investigation may be lost (Christiansen & Walther, 1986; Crockroft, 1982). Usually, in this phase, the pupils (or groups of pupils) indicate their strategies, results, and justifications; the teacher assumes a function of moderator, stimulating them to question the assertions of their colleagues. Thus, the development of pupils' ability to communicate and argue mathematically is a main objective of this phase of the work.

If pupils are to feel authenticity in this process, it is necessary that the teacher demonstrates a high investigative spirit. Pupils will only be able to understand what it means to do mathematics if they have an opportunity to observe genuine mathematics behavior. In an investigation, it is not possible to foresee what will happen—and the teacher has to show how to deal with that in practice. Therefore, great flexibility is required in conducting this type of work.

The teacher needs significant knowledge and competencies to do investigative work in the classroom—regarding the nature of these activities, possible strategies and tools, the pupils' most common difficulties, and the way to lead the classroom dynamics. This takes us, naturally, to consider the role of teacher education.

3. INVESTIGATING IN TEACHER EDUCATION

Let me begin this section on teacher education and professional development with two comments. First, we must have a clear notion of the possibilities and limits of teacher education. As Perrenoud (1993) says: "teacher education can only influence teachers' practices within given conditions and within certain limits" (p. 93). Second, teacher education must support teachers' professional development. That is, teachers' professional learning is a process that involves multiple stages, is always incomplete, and where teachers are the main actors.

In a classic essay, Lesne (1984) contrasts several models for teacher education, according to their pedagogical nature and associated socialization processes. The first model, transmission teacher education, corresponds to a normative orientation. The teacher is considered an object of socialization, that is, a social product. The second, incitative teacher education, follows a personal orientation. The teacher is considered the subject of the socialization process, determining him/herself, and actively adapting to different social roles. The third, appropriative teacher education, is centered on the social insertion of the individual. The teacher is regarded as an agent of socialization, simultaneously being defined by the circumstances and defining them. Another author, Ferry (1987), also considers diverse theoretical models for teacher education. He is not so concerned with the intentions, educational devices, structure of objectives, or nature of the contents, but mostly with the type of processes and the educational dynamics. He distinguishes three main models of teacher education: centered on acquisitions, centered on experimentation, and centered on analysis.

In order to lead investigative lessons successfully, teachers need knowledge and competencies, which they must improve constantly, reflecting on the work done. Recently, the idea of the reflective teacher became a pervasive topic in discussions about teacher education. Schön (1983), one of the authors that has considered this question, underlines reflecting in action and on action as two distinctive aspects of competent professionals.

3.1 Professional Development

There is a close relationship between teachers' professional development and personal and organizational development. António Nóvoa (1991) underlines the importance of valuing the person (and his/her experience) and the profession (and its knowledge). He argues that personal development is necessary to stimulate a critical-reflexive perspective that provides the basis for autonomous thought. This author criticizes the practices of inservice education directed towards individual teachers, that favor the isolation and reinforce an image of teachers as transmitters of external knowledge. Nóvoa endorses the practices that take as reference the collective dimensions and indicates that inservice education must stimulate teachers to value what they know, working from theoretical and conceptual points of view. He considers that teachers need to solve problematic situations, emphasizing the role of investigative and collective networks, conjugating clinical with inquiry-oriented teacher education. Nóvoa also underlines the importance of organizational development. For him, it is necessary to change the contexts where teachers act. Teacher education becomes a permanent process, integrated in the daily life of teachers and schools. He underlines the importance of participation, indicating that teachers must be active protagonists in the conception, implementation, and evaluation of teacher education as well as the need for a new culture in teacher education, bringing together schools and teacher education institutions, in the framework of "positive partnerships" (p. 30).

We can draw several contrasts between teacher education and professional development (Ponte, 1998). First, teacher education is strongly associated with the idea of attending "courses," while professional development occurs through multiple forms, including courses, projects, sharing of experiences, readings, reflections, and so forth. Second, teacher education follows an outside-in movement, aiming for the teacher to assimilate ready-made knowledge and information, while professional development is inside out, leaving the basic decisions regarding the questions to consider, the projects to undertake, and the ways of carrying them out to the teacher. Third, teacher education mostly attends to teachers' deficiencies, and professional development stresses the teachers' potential. Fourth, teacher education is usually compartmentalized in subjects or disciplines, while professional development implies the teacher as a whole embracing cognitive, affective, and relational dimensions. Finally, teacher education usually starts from theory and frequently does not leave theory, whereas professional development may consider both theory and practice.

Professional development stresses the combination of formal and informal processes. The teacher is no longer an object but becomes the subject of the learning process. Attention is given to knowledge and cognitive aspects, but also to affective and relational issues. The aim is not "normalization" but promotion of the individuality of each teacher (Hargreaves, 1998).

Teacher education can be viewed in a logic different from the transmission of a body of knowledge or the training specific skills. In fact, there is no insuperable incompatibility between professional development and teacher education. Teacher education may favor the professional development of the teacher, as it can, through its "hidden curriculum," contribute to reduce his/her creativity, self-confidence, autonomy, and sense of professional responsibility. Teachers who want to develop professionally and personally need to take advantage of the educational opportunities that address their necessities and objectives.

3.2 The Difficult Relationship Between Didactics and Teacher Education

Teacher education at all levels (preservice, inservice, and specialist education) has to take into account the processes of professional development, their rhythms and dynamics. Professional development involves the gradual maturing of the potentialities of each teacher, the construction of new knowing, and bears the mark of the underlying social and collective contexts. On the other hand, didactics has essential contributions to make to the professional activity of the teacher. It suggests useful concepts for understanding educational situations and supplies resources for professional practice (in particular, to conduct pupils' investigative work).

Ignoring the contributions of didactics is to put aside a set of powerful perspectives for education and a set of basic concepts to analyze and intervene in situations of practice. It means wasting an important capital of experience and research, that one could use to the benefit of the pupil. Ignoring the nature of the teachers' processes of professional development leads to designing teacher education programs of the "transmission" type, imposing concepts, practices, and theories that teachers do not need or that they regard without interest. However, that is common in much teacher education (preservice and inservice) that is done all around the world.

The problem consists, then, in combining didactics with professional development. How can we use the perspectives and results of the first without opposing the nature of the processes of the second? This problem is impossible apparently. It concerns movements that go in opposing directions, one inside out and the other outside in.

This problem may have many solutions, if we take into account the appropriate time scales. In other terms, we need to identify the specific contribution of each formative moment to the development of the teacher—remembering that teacher education can not achieve everything one would like, much less in a short period.

People learn from their activity and from reflection on their activity (Bishop & Goffree, 1986; Christiansen & Walther, 1986). Pupils learn mathematics working on mathematics tasks that they consider important and worthwhile. They learn by speaking with their colleagues and reflecting on their reasoning and results. Pupils learn sciences, French, history, or geography in a similar way. In addition, teachers and teacher candidates learn from their activity and from their reflection on activity—carried out in contexts of practice within a well-defined professional culture. Teachers' professional knowledge emerges through their participation in educational practices (Crawford & Adler, 1996; Lerman, this volume). The object of the activity of the teacher is not mathematics, French, history, or geography but the activity of the pupils in educational tasks. That is, teachers (and preservice teachers) learn by processes similar to pupils'; what is different is the basic object of their activity.

Activity, reflection, and cultural participation are important in learning, but deep learning requires a strong involvement from the learner. We can achieve it through investigative activities, because:

- investigations develop the ability to deal with complex problem-solving issues, mobilizing our knowledge in flexible and integrated ways;
- only understanding our own learning, investigating it, can we understand the learning processes of our pupils; and
- investigations provide a working paradigm that constitutes a reference for the reflecting teacher. (Ponte, 1999)

This importance of practice and investigation on practice is also stressed by researchers such as Cooney and Krainer (1996) and Lampert and Ball (1998). However, practice and investigation, only by themselves, are not enough. It is necessary to know in what ways they can play an important role in teacher education.

In an investigation, we may deal with a theoretical problem. Some phenomenon or issue may intrigue us. Working on them may lead to an increase in our knowledge. Other investigations, usually termed as action-research, may emerge from concrete difficulties of practice. Working on them, we strive to change the situation and to obtain better results. To use Ferry's terms, problem-oriented investigations promote the development of analytical abilities and action-research activities promote the experimentation of new approaches. Both require us to start defining the objectives and designing a working plan to reach them. Just as the investigative work in mathematics is important for pupils' learning, the investigative work regarding professional questions is also necessary for the professional development of teachers.

In both cases, we start defining a problem or an objective. We elaborate a detailed working plan. That plan is carried out and its results are evaluated. From that evaluation, many ideas for new inquiries may appear. There is a strong parallel between the investigation process—in mathematics and in mathematics education—and the process of mathematics problem solving, as considered by Pólya (1945):

Stages	Resolution of a problem, according to Pólya	Investigation and action research
1	Understanding the problem	Characterizing the problem-situation
2	Designing a plan	Designing a plan
3	Carrying out the plan	Carrying out the plan
4	Looking back	Reflecting on the work carried out and identifying new questions for investigation

Closely related to the idea of investigation is the idea of project, a typical context for Lesne's appropriative teacher education. In a project, a teacher, or a group of teachers, identify an objective regarded as important and develop a set of activities to reach it. It can be greater or smaller and more or less ambitious. In any case, the actors are the essential protagonists, and they both are transformed by the process they live and contribute to the transformation of the involving situation.

As Jean-Pierre Boutinet (1996) says, in a project there is a strong demand of unity between conception and execution. It has to consider the singularity of the situation, living permanently between theory and practice, dealing with complexity and uncertainty. It stands between the individual logic and the collective logic, whose management is always difficult, and between success and failure. In a project, failure is always ahead, and success depends not on the ability of preventing failure but on the ability of overcoming it.

An investigation, action-research, or project requires us to start characterizing the problem or the problem-situation that we want to address well. This done, it is necessary to design a working plan, defining the activities to carry out, the instruments to use, the calendar to follow, the resources to mobilize, and the role of diverse people who will participate in the work. This is followed by a phase of execution of the plan, correcting the trajectory when necessary. Finally, one has to evaluate the reach of the work carried through, reflecting on the process and the product, and to identify new questions for further investigation.

4. PRESERVICE TEACHER EDUCATION: INVESTIGATING INVESTIGATIONS

This perspective informs preservice teacher education in my institution. Part of the preservice program is a yearlong *practicum* carried out in a school. Student teachers work in groups and often choose a topic for a project. A group of four student teachers[3] addressed the potential of mathematical investigations in mathematics teaching.

First, they reviewed the literature on mathematical investigations, in order to gather arguments for using these activities and to collect information about how to conduct them. They developed their own working approach and created instruments to assess pupils and to collect data to support their reflection on the activity. In a final report, they describe the experimentation of these activities as "case studies" and end up with conclusions about what they learned in this project (Esteves, Santos, Ramos, & Roque, 1996).

These student teachers considered it quite important that pupils reflect on their work and on their conclusions. Therefore, they asked pupils to answer a questionnaire. They collected pupils' most significant answers in terms of content, difficulties, and language. They presented these answers in overhead transparencies and discussed them with the whole class, aiming to clarify ideas, improve pupils' form of expression (mathematical or not), and improve their performance in future activities.

Here is an example of an analytic geometry task proposed to 10th grade pupils during the 3rd quarter. The pupils already knew the notion of vector and vector operations. They worked in groups of 4 to 5 and it was the seventh time that they were doing an investigation.

Collinear vectors

Definition: Two vectors are collinear if they have the same direction.

Investigate if there is any relation among the coordinates of collinear vectors.

(grid paper provided)

The following description concerns one class led by one of the student teachers. She initiated the activity handing out a worksheet. Then, she read the definition and asked a pupil to come to the blackboard and give an example and a counter-example of vectors satisfying the definition.

Then the pupils started working in groups. The following excerpt of a conversation that went on between the teacher and a group of pupils provides an idea of the nature of the interactions taking place:

> *Teacher* But these two are collinear... and these two also... and these... are not collinear with these.
>
> *Pupils* They all are!
>
> *Teacher* Then is there some relation between the coordinates of these two with the coordinates of these two?
>
> *Pupils* Yes, they are different!
>
> *Teacher* They are different?... For example?...
>
> *Pupil 2* Then the coordinates are different and then they [the vectors] are collinear.
>
> *Teacher* OK! So, when vectors have different coordinates, what happens?
>
> *Pupil 5* They are collinear!

[3] Ana Cristina Esteves, Cláudia Santos, Cristina Ramos and Cristina Roque—school year of 1995-96.

Teacher They are always? Vectors with different coordinates are always collinear? It is that what you find? If it is what you find, verify it!

(Esteves et al., p. 34)

In the end of the lesson, the teacher collected pupils' work and analyzed it. She found it useful to write some suggestions in their papers so that in the first 15 minutes of the next lesson the slower groups could arrive at more conclusions by themselves.

In the discussion, the teacher started asking the groups to show their conclusions and reasoning, registering them on the blackboard. Later, with the aid of the pupils, she systematized their answers in one sentence: "two vectors **u** and **v** are collinear when **u** = k**v**, with k a real number."

A group of pupils wrote in the questionnaire:

> We concluded that a vector is collinear to another vector when both coordinates are multiplied by the same number, either positive, negative, or zero. In this last case, the result is always the null vector that may take any direction, and therefore is always collinear to any vector. Thus, mathematically, **u** = k**v**. (Esteves et al., p. 39)

In the opinion of the pupils, the discussion is important because: "it helps to assemble the ideas of all pupils, clarifying them... and is also useful to dissipate [our] doubts..." and "we have an opportunity to argue, which helps us to understand things better" (p. 40).

These student teachers conclude their report saying that:

> The work that we carried out during this school year made us consider that investigation activities are as stimulating for the pupils as they are for us. This is because we think that this approach is a "true mathematical activity" and develop abilities, attitudes, and values that other pedagogical strategies do not develop so efficiently. (Esteves et al., p. 47)

5. INSERVICE TEACHER EDUCATION: TEACHER NARRATIVES

Next, I present an example of a narrative reflection of an inservice teacher—Irene Segurado—concerning an investigation class. It was carried out in an action-research project that went on from 1995 to 1997, aiming to study the professional knowledge needed to carry out these activities in the classroom. The members of the project team promoted investigative lessons with 10-14 year old pupils. Those lessons were the basis for discussions and the production of narratives on the lived experiences. In project meetings, the team also discussed theoretical issues regarding investigations, lesson dynamics, teachers' professional knowledge, and using narratives in educational research. A first version of a narrative written by Irene was subject to further discussion and analysis, from which more refined versions were developed.

An all-class investigation[4]

> The task that I presented to my 5th graders concerns the multiples of a number, the topic that I was teaching. I planned that they would work in small groups, as usual in this type of activity. The task is the following:

[4] Abridged from Ponte, Oliveira, Segurado, & Cunha (1988).

- write in columns the 20 first multiples of 5.
- look at the digits in the units and tens. Do you find some regularities?
- investigate what happens with the multiples of 4 and 6.
- investigate for other numbers.

When I entered the classroom, I noted that the pupils were rather agitated, perhaps due to the beautiful day outside or to the proximity of the holidays. The organization of the material in pupils' desks and the change in places necessary for group work could lead to an even bigger agitation. To not make this problem worse I decided to keep them in their places.

I decided to control the situation immediately. I placed myself in front of the blackboard, asked for the multiples of 5, and registered them on the board.

I questioned pupils if there was something interesting and curious with the units and tens. Tatiana, raising her arm, answered readily: *the number of the units is always 0 or 5,* which was accepted by her colleagues, echoing in the room: *it is always 0; 5, 0; 5...*

And...? —I stimulated them.

The tens digit repeats itself 0-0, 1-1, 2-2; 3-3... said Octávio, with a happy expression.

I was marking these two statements in the blackboard, with colored chalk, so that they could all verify its truth when Carlos, with a certain agitation, interrupted me. *I discovered another thing... Can I go to the blackboard to explain it?* I asked him to wait a little in order to finish what I was doing. He accepted, but communicated his discovery to the nearby colleagues ....

I was pleased, since my expectations had already been exceeded. I asked them to investigate what happens with the multiples of 4, that I placed in a column next to the multiples of 5 .... Quickly, almost all pupils answered in chorus: *they always finish in 0, 4, 8, 2, and 6.* They still discovered that: *they always finish in an even number, the tens happen again 2 times, 3 times, alternatively, the number of tens that repeat three times are always pair and those that repeat two times are always odd.*

The pupils, who were more passive in the beginning, were livening up with the discoveries of the colleagues and became more outspoken, showing great enthusiasm in the search of regularities.

We had discovered all the regularities that I had previously found in my planning and after some moments of fruitless search, I considered that we could move to investigate what happens with the multiples of 6. I wrote them next to the multiples of four—for no special reason, but simply not to lose time in erasing the blackboard.

0	0	0
5	4	6
10	8	12
15	12	18
20	16	24
25	20	30
30	24	36
35	28	42

40	32	48
45	36	54
50	40	60
55	44	66
60	48	72
65	52	78
70	56	84
75	60	90
80	64	96
85	68	102
90	72	108
95	76	114

The discoveries now appeared in batches and there was no pupil who would not pledge in giving a contribution. That made it difficult for me, sometimes, to record and to systematize:

The units are always 0, 6, 2, 8 and 4.

The units are always a number pair.

The tens do not repeat from 5 to 5.

I was breaking their enthusiasm: *Easy! Let us verify if what your colleague said is true; Attention! Look! Look at what an interesting thing your colleague discovered! ...*

Sónia suddenly affirmed: *They are the same numbers as in the multiples of 4.* Even before this statement made any sense to me, Vânia had already declared: *They are in another order.* I perceived that they were comparing the multiples of 4 and 6, and I explained that to the class.

Both start in zero, said Pedro who today was clearly awake.

The other numbers are in contrast, reported Ana.

There are multiples of 4 that are also multiples of 6.

The multiples of 6 from 12 on, are alternatively also multiple of 4….

The discoveries now came purring as cherries, one behind the other, exceeding all my expectations about the answers that the pupils would give. I had not foreseen comparing multiples of different numbers, because I never placed them next to each other. Therefore, I lived their discoveries with enormous enthusiasm. A more astute pupil observed: *The teacher is very happy with us, aren't you?* And I was!

The records made on the blackboard provided a new approach to the task. Moreover, working with the whole class enabled the contribution of each pupil to be grasped immediately by all colleagues, leading to more discoveries.

For me, group work is the best way to have pupils working on investigations. Small group work allows us to attain objectives that can hardly be reached with individual or all-class work: cooperation, teamwork, and organization. It also enables reflecting on the others' ideas, explaining and verifying their reasoning. However, carrying out an investigation with the whole class (an experience that I had for the first time), allowed for a widening of the discoveries. The strategy used by a pupil, for a given discovery, was used by a larger number of colleagues to generate new discoveries. This strategy also allowed the pupils to assume their interventions individually, which is quite important for the educational process. This task, perhaps because it dealt with the investigation of simple regularities, fully resulted in a whole class lesson far beyond my most optimistic expectations.

6. EDUCATIONAL RESEARCH: TEACHER'S ROLES IN INVESTIGATION CLASSES

A third example concerns a research study aiming to characterize the roles of the teacher when pupils carry out mathematical investigations (for detail see Ponte, Oliveira, Brunheira, Varandas, & Ferreira, 1998). This work was done in the assumption that to conduct classroom activities, teachers need special professional knowledge, know how, and experience. Moreover, it considered that doing new types of tasks in the classroom requires the personal construction of new principles and routines.

This study was carried out by a collaborative team of teachers and researchers using a qualitative-based methodology[5]. It analyzed selected episodes concerning the start, when the teacher presents a task to the class, the development of the investigation, with the pupils working in small groups, and the final discussion, where pupils present their results and all the class, together with the teacher, does a general evaluation of the work carried out.

Data collection involved audio and video records and field notes. The classes were conducted by three teachers from the project. Some of these classes were observed by other project members. In other cases, the teacher also collected images and sounds. The analysis of data involved several stages including (a) transcribing audio and video records; (b) selecting episodes to study, from the transcripts; (c) applying a system of categories to the transcripts (and sometimes re-viewing the video records of the episode); (d) refining the analysis, through discussion of a document produced in the previous stage; (e) making a cross-analysis of the several previous analyses; and (f) analyzing items of the previous step, to identify and characterize teachers' roles.

The study presents diverse roles of the teacher in conducting pupils' investigative work. They concern teachers' professional knowledge, including their mathematical knowledge (in particular, regarding the task), and their didactic knowledge (concerning the organization of the work and the conduction of pupils' activity):

1. *Challenge pupils*. The teacher, in the beginning and during the work, proposes questions that pupils can find challenging, for which they do not have an immediate response, arousing their mathematical curiosity.
2. *Support pupils.* During the development of an investigation, the teacher supports pupils' progress, considering the mathematical exploration of the task and the management of the didactic situation, promoting a balanced participation of the pupils in the activity.

[5] This team, that included J. P. Ponte, H. Oliveira, L. Brunheira, J. M. Varandas, and C. Ferreira, was part of the Project MPT-Mathematics for All (1995-99).

3. *Evaluate pupils' progress.* During the activity, the teacher collects information to know if pupils understand the task, if they are formulating questions and conjectures, if they are testing them, if they justify their results, or if they have difficulties and, if so, what their origin is.
4. *Think mathematically.* New mathematical questions can always arise, especially if the situation is truly open and the teacher may become involved in reasoning mathematically with pupils.
5. *Supply and recall information.* The teacher has to provide useful information to pupils, helping them to remember important ideas, to understand mathematical concepts, and important forms of representation.
6. *Promote pupils' reflection.* The teacher has to assure that pupils relate the work they are doing with known ideas and develop their understanding of mathematics.

These different roles can be systematized in the following diagram:

Mathematics strand

4. To think mathematically
(to investigate/to establish connections)

Didactic s strand	1. To challenge 2. To support 3. To evaluate	5. To provide informatio n	6. To encourage reflection

When selecting, adapting or elaborating the investigation to propose to pupils, the teacher needs to think mathematically. Therefore, the mathematical reasoning of the teacher (previous to the lesson) assumes a basic importance. However, questions, conjectures, and arguments considered by pupils can lead the teacher, during the lesson, to consider new aspects of the task, requiring additional mathematical reasoning. Continuing the investigation, the mathematical reasoning of the teacher develops in an analogous way to the mathematical reasoning of pupils—placing questions, formulating conjectures, making tests, and validating results, the typical processes of an investigation. Moreover, during the lesson, there are also frequent opportunities to establish relationships between the work in progress and other mathematics or extra-mathematics concepts—which require the teacher to hold a mathematics culture and ability to decide what connections to establish. By doing investigative activity in the classroom, the teacher constitutes a genuine mathematical model for his/her pupils (Lampert, 1990; Mason, 1991). This concerns teacher's role 4, to think mathematically.

The remaining roles assume a didactic nature. Before the lesson starts the teacher establishes the agenda, makes decisions concerning curricular priorities, the actual wording of the task, and the form of presenting it to pupils, as well as the type of class organization. During the lesson, the teacher moves between two poles: one concerns the curriculum, that marks the objectives (aims) to reach, while the other concerns the activity, witht respects to the actions (means), that are carried out to reach those objectives. The learning objectives involve two dimensions (Christiansen and Walther, 1986) that are always there, explicitly or implicitly. The first one concerns the mathematics contents, leading the teacher to explain a concept, to remember a notion, or to establish direct links with other ideas or mathematics or extra-mathematics representations—role 5, to give information. Here we find one of the "classic" roles of the teacher that, as Lampert (1990) indicates, can be carried out in a substantially different way, contextualized and integrated in the accomplishment of significant activity. Instead of assuming this work alone, the teacher can try to get the pupils to participate actively, helping to explain a concept to their colleagues, remembering ideas, representations, and procedures already studied. The second dimension concerns understanding what it is to learn, what mathematics is, and what it is to think mathematically. This dimension introduces another level of activity, evaluating, and commenting on the work done and the new ideas that emerge. We find another basic role of the teacher here—role 6, to promote the reflection—stressed, for example, by Bishop and Goffree (1986).

The actions that the teacher can use to reach the intended objectives are essentially expressed by the three basic roles: (1), to challenge, (2) to support, and (3) to evaluate. These roles are connected to the logic of the development of any activity. The teacher challenges the pupils with situations and questions in order to involve them in investigative work. The teacher supports them, asking questions, making comments, or providing suggestions. The teacher also tries to evaluate the progresses already done and possible difficulties, collecting information, and, based on that, decides to continue, to modify some aspects of the work, or to move to another phase of the activity.

The two strands, mathematics and didactics, are not independent of each other. On the contrary, they cross each other, as the previous picture suggests and as Shulman (1986) underlines. All the didactic work carried out by the teacher requires an understanding of the task and its mathematical connections. The most specific aspect of the activity of the mathematics teacher, as a teacher of a discipline, is supporting the development of mathematical thought, before, during, and after the lesson. However, the educational role of the teacher has other sides besides mathematics—it depends equally on the way the teacher faces education, curriculum, pupils' learning, and the profession. Five of the roles are placed in this interconnection. Thinking mathematically is not. It can be started off by pupils' questions, comments, or affirmations, but concerns the mathematics being of the teacher.

7. CONCLUSION

The investigation paradigm is a good framework to discuss pupils' work in the classroom as well as teachers' activity and professional development. Investigation processes are at the heart of the mathematical activity and, when experienced with authenticity, naturally enable pupils to have a stimulating relationship with mathematics. Investigations about teaching constitute a powerful framework for professional development, providing a bridge between theory and practice, bringing together what we are learning in mathematics education about mathematics learning and teaching and what we are learning in teacher education about professional development. We saw an example of an investigation in preservice teaching, when a group of student teachers collected information on carrying out this type of lesson, designed and made investigation lessons, gathered, compiled and analyzed data, and drew their conclusions about its value. We saw a rather different example in an action-research project, when an experienced teacher conducted a lesson changing her usual way of work and reflected on its implications for future activities. We saw yet another example of a more formal investigation on this type of work, aiming to identify different teacher's roles. Scholars such as Goffree and Oonk (this volume) and Lampert and Ball (1998) also favor the idea that investigations, in mathematics as well as in teacher education, supply a useful framework for fostering pupils' learning process and teachers' professional development.

Investigations may mean different things for different people. As personal experiences, they can be lived in rather different ways. In fact, investigatory work involves a variety of interpretations in different parts of the mathematics education community. In this paper, I stressed a view of investigations that emphasize its problematic nature and its openness, less structured than usual problem solving tasks. The role of the teacher regarding this activity also allow for a range of interpretations from the "sage on the stage" to the "guide on the side." However, whatever the case, learning is a central focus for pupils and teachers involved in this process.

An investigation involves several phases and is connected to the idea of project. However, it is essential that the form of investigations is not used to shadow what it is more important—its content. An activity is not an investigation just because it involves a revision of literature, the conception and application of instruments such as questionnaires, videos, or interviews, and a final report. The essential in an investigation is that it starts formulating genuine and interesting questions and that a careful process is designed to find some type of response. It is the value of the questions and the appropriateness of the answering process that are the basic marks of an investigative work.

Investigations about practice may be nurtured by collaborative activity involving educators and teachers within a research culture (Jaworski, this volume). Investigative activities can arise from initiative of the practicing teacher or in teacher education programs. They also can arise spontaneously during a lesson or in professional development. Finding devices, working ways and situations that favor this activity, and studying its conditions of success are important tasks for present day mathematics education.

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