Mathematical investigations in the classroom: A collaborative project[1]

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Abstract. This paper presents some work developed through a collaborative project involving mathematics teachers and teacher educators, aimed at the development of classroom tasks involving students in mathematical investigations and the study of related teaching styles. We give a general overview of the project and present one task dealing with the concepts of powers and exponents. Then, we describe how it was used in the classroom (grade levels 5-7), and present our experience about the role and the activity of the teacher in the process of organizing, conducting and reflecting about this kind of activity.

Why mathematical investigations in the classroom?

Modern society requires from everyone a reasonable fluency in mathematics. It is specially important to be able to interpret information framed in mathematical language (numerical and graphical) and to think mathematically (seeking patterns and relationships and reasoning). However, mathematics is usually regarded as a most difficult subject. Students frequently view it just as doing computations and getting the correct answers. They tend to assume a dualist view in which things are either right or wrong. In many countries, including Portugal, the evaluations of students' learning, attitudes and views about mathematics are considered unsatisfactory — by all sorts of criteria (Neves e Serrazina, 1992; Ponte, 1994; Ramalho, 1994).

At the base of these difficulties it is the social filter role played by mathematics teaching and the subsequent emphasis on mastery of basic concepts and procedures. In general, mathematics teaching pays little attention to the more advanced aspects of mathematical activity such as the formulation and resolution of problems, the formulation and testing of conjectures, the pursuit of investigations and mathematical proofs, and the argumentation and critique of results. While these are fundamental and current

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themes of mathematics education expressed in many curriculum documents across the world (APM, 1988; Cockcroft, 1982; NCR, 1989; NCTM, 1989) they still find very little emphasis in classroom practice, both in our country and elsewhere (Lerman, 1989; Silver, 1993).

Mathematical investigations, based on open-ended problem solving tasks, are important from the educational point of view (Ernest, 1991; Mason, 1991). In fact, in our perspective, they:

- are indispensable to provide a complete view of mathematics, since they are an essential part of mathematical activity;
- stimulate the sort of student involvement required for significant learning;
- provide multiple entry points for students at different ability levels;
- stimulate a holistic mode of thought, relating many topics, an essential condition for significant mathematical reasoning.

Setting up a collaborative project

The common feeling that more attention should be paid to understanding what is involved in doing mathematical investigations in the classroom brought together a small group of teachers and teachers educators[2]. We decided to set up a project to experiment with tasks involving students exploring and investigating around mathematical ideas, concepts, and processes. Our aim was to engage students in the formulation of conjectures and in mathematical discussion and argumentation, aspects that we find essential in their mathematical experience. Our work includes producing, experimenting, and evaluating such tasks and studying the competencies needed for using them to be successful in the mathematics classroom.

This activity intends to contribute to furthering knowledge about innovative learning situations in mathematics classrooms. It is double sided, involving curriculum development (collecting information about the potential value of given types of activities in different school grades and acquiring experience in the preparation and evaluation of classroom tasks) and also research on teaching (studying the decisions, dilemmas, difficulties, etc. that teachers face in conducting these kinds of activities). Of

[2] One of us (João Pedro) has been a classroom teacher for 6 years and afterwards a teacher educator for 16 years. The other three are involved in a master's program. Two of them are classroom teachers who are now also working in teacher education. One (Hélia) has 4 years of experience in the classroom and worked for 3 years in teacher education and another (Helena) taught for 5 years in the classroom and has been in a school of education for 2 years. The other one (Irene) has been a classroom teacher for 16 years.

course, our ultimate purpose is to contribute to the dissemination of such activities in the Portuguese educational system.

The tasks that we are developing are drawn from the mathematics topics of the 2nd and 3rd cycles of basic education in Portugal[3], and are designed to fit within the existing curriculum, and be used in the regular classroom. They are not intended to suggest extra-curricular themes, but to present the topics currently in the curriculum, in a different way. The tasks do not just stand by themselves. They are organized in units, each focusing on a single topic. The full set of materials will include worksheets with a sequence of questions for students, teacher's guides with didactic suggestions and a supporting text for teachers and students, with relevant information and historical notes about the given topic. In this paper we present some work based on a worksheet dealing with the concepts of powers and exponents.

The development of tasks

This project hinges on the collaboration between classroom teachers and researchers/teacher educators[4]. We start discussing ideas for the tasks in project meetings. Then, someone from the group produces a first version of a possible worksheet with three or four questions which may be appropriate for students to follow for two or three successive lessons. This version is discussed by the whole group in a further meeting and is usually taken back by the same person to refine. One or two more iterations of this process usually make the task appropriate for experimentation in the classroom. Figure 1 presents a set of questions on the topic of powers of natural numbers.

One of our big questions is how much to structure the tasks. Too much structure implies that we are guiding the students and leaving not enough thinking for them to do. Too little may imply that the students move in different paths and perhaps miss the ideas most related to the mathematical concepts in the curriculum. Furthermore, different students may profit from being asked different kinds of questions. We finally decided to make something in between: some of the questions are more structured, others more open. Often we will begin one worksheet with one or two more structured questions (such as 1a, 2a, 2b, and 2c) and then provide one or two suggestions for open-ended and truly exploratory and investigative work (such as 1b and 3).

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[3] In the Portuguese educational system, this includes grades 5 to 9 (children 10 to 14 years old). In all these grades mathematics is taught as a separate subject by a specialist teacher.

[4] In fact, all four authors are or have been in varying degrees in the roles of classroom teacher and researcher/teacher educator.

Figure 1a - Questions on powers and patterns

POWERS AND PATTERNS

Questions

1.-The number 729 may be written as a power of 3. This can be seen making a table with the powers of 3:

$$3^2 = 9$$

$$3^3 = 27$$

$$3^4 = 81$$

$$3^5 = 243$$

$$3^6 = 729$$

a) Can you write as a power of 2?...

64 =

128 =

200 =

256 =

1000 =

b) What conjectures can you make regarding numbers that can be written as powers of 2? and as powers of 3?

2.- Look at the following table with powers of 5

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

a) The last digit of each power is always 5. Is that also true for the next powers of 5?
b) Investigate what happens for the powers of 6
c) Investigate also the powers of 9 and 7.

3.- The cubes of the first natural numbers satisfy the following relations:

$$1^3=1$$

$$2^3=3+5$$

$$3^3 = 7+9+11$$

• Note that in this example, 1^3 was written as a "sum" with just *one* odd number, 2^3 as a sum of *two* odd numbers and 3^3 as the sum of *three* odd numbers. Is it the case that the cube of any number can be written as the sum of odd numbers?

Experimenting and reflecting

Classroom experimentation is carried out by one or more of the teachers in the group. If possible, one member of the group observes the classes. Sometimes a video recording is also made. And in some cases we also invite other teachers not in the project to try out our tasks. For each experiment, we make a first reflection as soon as possible after the class and a more extended reflection, using the video recorder if available, sometime afterwards.

Experiment I. We had in mind that these tasks about powers and exponents could be used with students of a variety of grade levels. One of the teachers in our group started experimenting these tasks in a 6th grade class, in a 2-hour period. She regarded this as a difficult class, since the students had no motivation to study mathematics.

The teacher began by making a short introduction, explained the meaning of the word "conjecture" and provided the students with an orientation about what they were going to do. She gave the first page in the first hour (question 1) and students started working individually. However, they had the opportunity to interact informally with their peers as they wished. Then, in the second hour, the teacher gave them the second page (questions 2 and 3). She moved around in the groups, seeing the work of the students and discussing their results with them.

Surprisingly for the teacher, the students were very interested and even quite excited with this activity. This high involvement of the students is illustrated by the fact that some of them, having finished question 1, started by themselves trying some other possibilities, such as powers of 5, etc. even without knowing what was to come on the second sheet (question 2). The teacher felt the experience was a success but commented that she would like to know more about how to (a) observe the class and (b) evaluate if all students had understood the same points about the task.

Then we had a general reflection in our group about this activity. Focusing on the issues raised by the teacher, we concluded that observation and evaluation could be carried out during the students' activity and in a final discussion. Such discussion, in fact, may be a powerful means of providing all students with further insights about the work of their peers and giving the teacher a general overview of students' work. But how to start it? Previously, the teacher felt that after having discussed each task with all students, one at a time, it did not make much sense to make a final discussion. Also, she felt that there would be no time to have all the groups reporting their work. The first group to speak could refer most of the interesting things, leaving little to say to the other groups. She regarded the discussion as something valuable but difficult to conduct. In our reflection, a new suggestion come up. It sounds obvious, but the teacher had never considered it before. Instead of letting each group say all they wanted, she could go task after task, asking all groups for their results and explanations. Each group, in turn, would have the opportunity of saying just one thing about its work. The groups that have less chance of presenting their ideas now may be the first next time.

Other points discussed included how much should the teacher say to the students while they were working. There was an agreement that the teacher should ask more questions than provide answers or explanations. We also considered the advantages and disadvantages of encouraging the students to work in a more collaborative fashion.

Experiment II. A couple of days later this teacher presented the same task to a 5th grade class. When we were designing the questions several times she expressed doubts about how her 5th graders could handle these but she decided to try anyway. In this class she asked the students to register everything that they were doing, so that later she would be able to know what they were thinking, and what conjectures they had made. The students were working in pairs. In each pair, both students kept separate notes, but discussion between them was encouraged. The completion of this task took 3 hours (2+1).

The teacher tried a new different strategy regarding students' questions. She was saying less but asking more questions. This time there was a discussion at the end of the activity. For each task, one group gave their results and the others reported if they had something else to add.

Teacher: Let us see now what you have discovered regarding the powers of 5. It is the turn of AmŽlia's group.

Sílvia: It ends always in 5.

Daniel (a student from another group): And in 2.

Teacher: How is that? Let us hear Daniel. S'lvia says it always ends in 5, right? But Daniel's group added something else.

Daniel: After 25, it ends always in 25.

Teacher: Anything else about the powers of 5?

All the students were very attentive to their colleagues, verifying on the calculator the results that were given. Their enthusiasm was quite beyond the teacher's highest expectations.

In the ensuing reflection, we considered the power of these kind of activities to bring to the teacher new perspectives about her students' capacities. Of course, the younger kids needed more time to finish the work, but they were able to carry it through. We also concluded that the strategy used in this experience to make the students share results and ideas among themselves represented a big improvement over what had occurred in the first experiment.

Experiment III. A teacher from another school, not in the project, agreed to try this task with her 7th graders. This is a teacher who usually presents problem solving and exploratory activities in her class and often organizes the students in group work.

She began by distributing the full worksheet to the students asking them to set up groups of 4 or 5. Two hours were devoted to this work plus half an hour of discussion some days later. Being used to this kind of activity, the students could work productively by themselves, seldom calling on the teacher.

One interesting episode in this experience is that this teacher at first did not perceive question 3 as interesting. She commented that the question probably had an error somewhere. But the class was about to begin and there was no time to discuss the issue in detail. The project member present assured her that the students would be able to handle it. It was just when they were working on this question that she realized that the question makes sense, indeed. The students easily realized how they could write 4^3, 5^3, 6^3,... as the sum of odd numbers following the pattern presented in the question. Then, the teacher challenged them to search for a general law. Interestingly, some students had ideas that we had not thought about when we were designing the task...

The students had no major difficulties working with the questions. Their involvement was high and there were heated discussions within the groups. The students argued with each other and when they could not agree, they called on the teacher to be the referee. She avoided giving the answers, challenging the students to find them by themselves, using expressions such as "experiment", "did you verify?", "try it with another number", "read carefully what is asked"...

The discussion had to be a couple of days later. The teacher presented to the whole class the thoughts of two of the groups that she found particularly interesting. The initial involvement of the students was low, since they had forgotten what they had done. But some of the students became quite intrigued with the approach of one of the groups and asked if that was a mathematical proof. The teacher said that a real proof would be very difficult to understand, but they were so intrigued that she had a hard time convincing them of that. She highly enjoyed this since she thinks that the students should start exploring, justifying, convincing, and proving at an early level.

This teacher considered this experience rather positive and quite consistent with the activities that she often does in her classes. The students quickly got involved in the tasks and the fact that they were used to dealing with problematic situations and to make decisions on their own was probably an important factor. However, looking in retrospect at this set of lessons, we and the teacher felt that the time elapsed from the activity and the fact that she presented an initial summary of the student's work made the discussion much less successful than what it could have been otherwise.

General reflection. At the end of these three experiments, we felt quite strongly that:

- These tasks are suitable for all students and not just for the better students;
- Students can really get involved in significant mathematical activity and have a sense of doing something significant;
- It is possible that students develop a significant autonomy;
- There is great advantage in stimulating interaction between students, working in groups or in pairs in these tasks, since a lot of power emerges from their talking and arguing with each other.

We also concluded that four steps are essential in this kind of activity. In *planning and selecting the tasks* the teacher needs to be aware of the mathematical content, the curriculum goals and capacities, and the previous experience of the students. All of this must be taken into account and blended into a general perspective of how the work will proceed. In *presenting the activity to the students*, the teacher needs to make sure that they make sense out of it and get really involved. However, not too much must be said at this point, since the goal is to have the students working on their own. *During the students' activity*, the teacher needs to be attentive to successes and problems and be ready to help if necessary. But we also found that in this kind of activity one must leave the students to struggle for some time. The role of the teacher is to pose questions and not immediately tell them all the answers. And we realized that the *final discussions* are quite important. The system used with 5th graders worked well. For the students who have the chance to report first, the discussion may bring to the fore issues and strategies of which they were not aware. We also found that the discussion should be included in the activity, or right after if possible.

Planning classes together, and observing and discussing them made us aware of new aspects and issues that we had not consider before. Having some questions in mind, a process of recording, and an observer in the classroom, were important conditions to help us in this reflective process. As a group, we are learning about ways of framing tasks to initiate students' work, to interact with students in doing investigations and in reflecting about classroom activity. We began choosing the concepts of powers and exponents as a challenge. At first, this did not seem a very promising topic for our purposes. But, in the end, we are quite pleased with the students' work and with what we have been learning.

Future work

This work is to be continued in different directions. In terms of curriculum development we need to look at issues such as: what is the potential value of given types of activities in different school years? should the tasks have more or less structure? should we provide more or less suggestions for teachers? what kinds of suggestions? are investigations worthwhile in themselves or should they support the learning of mathematical "content"?

Another aspect of concern includes students' learning, where we need to look at points such as what is the influence of these activities in students' conceptions regarding school mathematics? What are the relationships between basic mathematical knowledge and skills and more advanced reasoning processes as conjecturing and proving? How should we evaluate mathematical processes, and therefore, this kind of activity?

We also need to consider the classroom dynamics: in doing this kind of activity what issues arise in doing investigations in groups, pairs, individually? what new patterns in classroom interactions are associated with doing mathematical investigations?

And finally, we will be looking further at other teachers, specially to the decision processes required in conducting these kind of activities: How to integrate these tasks in the curriculum? How to present them to students? What sort of support to offer students? How to promote classroom discussions? How to assess students?

For sure, it will not be possible to study all these points at the same level of depth. But, to conduct classroom investigations, we need to have a clear perspective of them. Reflecting in the classes where students are encouraged to think and behave mathematically is important. Considering the growing international literature on this topic, and discussing these issues with other teachers and researchers outside the group, we expect to develop and deepen our understandings regarding this kind of activity in mathematics teaching and learning and challenge others to try it as well in their classroom practice.

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