EXPLORING FUNCTIONAL RELATIONSHIPS TO FOSTER ALGEBRAIC THINKING IN GRADE 8

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Abstract. *This article aims to identify the contributions of a teaching unit in fostering grade 8 students' algebraic thinking. The unit is based on exploratory and investigation tasks about functional relationships. Using a qualitative and interpretative methodology, data collection included participant observation of classes by the teacher, registered in her professional journal, and gathering written records from students. Results indicate that working in the proposed tasks and combining students' group work with classroom discussions promoted the development of meaning for the algebraic language. The work in this unit also encouraged the students to widen the number of strategies that they used to explore situations involving relations between variables, to reason in a more general way, and to express their generalizations using a formal language.*

1. INTRODUCTION

In Portugal, algebra is an important topic of the grade 8 mathematics curriculum, addressing the use of symbols with different purposes and in different situations such as representing functional relationships, solving problems, 1st and 2nd degree numerical equations and literal equations, and generalizing and proving numerical properties (ME, 1991). The students at grade 7 have former experiences with the algebraic language, particularly in the study of 1st degree equations. However, the use of algebraic language and the construction of the concept of variable usually create significant difficulties for many of them. This paper is drawn from a larger research study about the role of exploratory and investigation tasks, involving functional relationships in students' learning (Matos, 2007). Its main goal is to identify the contributions of a teaching unit based on this curriculum strategy in students' algebraic thinking.

2. ALGEBRAIC THINKING

Kaput (1999) argues that algebraic thinking appears when one establishes generalizations about data and mathematical relationships, through the processes of conjecturing and arguing, and expresses them in an increasingly formal language. This generalization process may occur from arithmetic, geometric, and mathematical modelling situations. The author identifies five faces of algebraic thinking, intimately interrelated: (i) generalizing and formalizing patterns and constraints; (ii) manipulating formalisms; (iii) studying abstract structures; (iv) studying functions, relations, and joint variation; and (v) using multiple languages in mathematical modelling and control of phenomena. In this way, Kaput stresses the need for a wider way of looking at algebra teaching and learning. The NCTM (2000) also underlines this idea, indicating that middle school students must learn algebra, both as a set of concepts and skills related to the representation of quantitative relationships and as a style of thinking that enables the formalization of patterns and generalizations.

Although algebra is not just a language, part of its power comes from the use of symbols that allows expressing mathematical ideas in a short and rigorous way (Sfard & Linchevski, 1994). Symbols also allow keeping a distance from the semantic elements they represent becoming powerful tools for problem solving (Rojano, 1996). They can be used to represent different mathematical ideas and can be interpreted by the students in different ways (Küchemann, 1981; Usiskin, 1988). Namely, symbols can be used for representing unknown numbers and for expressing generalizations or as variables. Students reveal several kinds of difficulties when they deal with the algebraic language (Matos & Ponte, 2008). For example, Booth (1984) indicates three main areas of difficulty: (i) interpreting letters; (ii) formalizing the methods used; and (iii) understanding notations and conventions. In fact, the multiple uses of algebraic symbols is a source of potential in algebra but also a source of conflicts and difficulties for the students.

The exploration of patterns, since the first years of school, is an activity that contributes towards the development of algebraic thinking (Driscoll, 1999; NCTM, 2000). The study of functions includes understanding the way how two variables are related, and this may include the identification of patterns. In fact, Smith (2003) identifies two distinct ways of analysing a function: (i) understanding the relationship between each value of the variable x and the associated value of y, which may enable to write an algebraic expression that represents it; and (ii) analysing the way how the variation of the values of a variable produces variation in the values of the other.

3. THE TEACHING UNIT

This teaching unit of this study addressed several mathematics curriculum topics – numerical sequences, functions, and 1st degree equations. Aiming to promote the development of algebraic thinking, based on exploring functional relationships, its specific objectives were the development of the students' ability to: (i) identify and describe patterns in situations involving variation and to formulate generalizations; (ii) to represent and analyse functional relationships through tables, graphs and algebraic expressions; and (iii) to ascribe meaning to algebraic expressions and to use the algebraic language in an efficient way. Students must use letters in different contexts and with different purposes – as instruments for generalizing, as variables in functions and as unknowns in solving problems and equations.

The teaching unit was carried out in 16 classes (90 minutes each). It included several types of learning experiences. In a first part, the beginning of each topic included exploratory and investigation tasks (Ponte, Brocardo & Oliveira, 2003) to foster the construction of new concepts. In the tasks about numerical sequences, students explored patterns, with or without pictorial representations and with different levels of difficulty. These tasks created opportunities to draw generalizations, which could be first expressed in natural language but should progressively be expressed more formally, using algebraic language. In task 3, the students worked with numerical sequences graphically represented. In this part of the unit letters were mostly used as generalized numbers and as unknowns in 1st degree equations.

In the second part of the teaching unit, the study of functions was introduced by two tasks involving relationships between variables. These tasks represented an extension from the discrete case to the continuous case. The first task involved a direct proportion. Both the first and the second task explored ways of representing functional relationships, moving from one representation to another. The third task had four distance-time graphs that students should interpret to design a situation adjusted to the information given. Although the letter is used both as a generalized number and as an unknown, in this part of the unit the focus was on its use as a variable and on the notion of joint variation.

The last two tasks continued the study of equations that had begun at grade 7 and was already revisited in previous topics, solving new kinds of problems and equations with denominators. Literal equations appeared through investigation activities involving generalizing relationships among more than two variables. After this initial moment, supported by the context of the situation, the students got involved in solving literal equations for one of the variables. Here, letters were mostly used as unknowns and as generalized numbers. All tasks allowed the students to use their own strategies. This approach stimulates their active participation giving them multiple entry points, adequate to their ability levels.

Identifying patterns and regularities, representing, generalizing and specializing are mathematical reasoning processes that play an important role in exploring functional relationships. Solving investigation tasks allows the students to learn through the mobilization of their own cognitive and affective resources, as they follow their own goals (Ponte, 2006). Proposing such tasks, complemented by a small set of instructions, at the beginning of each topic, seeks to empower the students' intuition, as they involve themselves on mathematical explorations. Throughout the teaching unit students also solved exercises and problems from the textbook.

These classes included individual work and work in pairs and in small groups. At the end of each task or group of tasks it took place a discussion involving all class. In these discussions, the students presented orally their strategies to their colleagues. While they were solving the tasks, they made written records of their work which allowed them to organize their reasoning and supported their participation in the general discussions. In two of the tasks, these records where analyzed by the teacher which addressed each group some new questions that stimulated the students to go beyond their first explorations. Students' assessment in the teaching unit included several modalities: an individual test, two worksheets solved in pairs, two written reports of exploratory and investigation tasks and the teacher's observations during classes.

4. RESEARCH METHODOLOGY

We used a qualitative methodology, descriptive and interpretative, based on participant observation (Bogdan & Biklen, 1994). The unit was taught to a grade 8 class from a school of Lisbon's suburban area. The first author of this article was the classroom teacher and she performed simultaneously the role of teacher and researcher. This was an investigation that she led on her own professional practice (Ponte, 2008). The class had 27 students (aged 13 to 16). Reflecting immigration trends in Portugal, 10 of these students were from Portuguese speaking countries (Angola, Cape Verde, S. Tomé and Príncipe, Guinea and Brazil) or from Eastern European countries (Romania). The students in this class had low achievement in almost all the subjects and several of them failed at least one year. It existed, however, a good relationship between them and their teachers.

Data for this paper is taken from (i) the teacher's journal; (ii) the audio records of all classes, that were transcribed and coded; and (iii) the written documents produced by the students, that were collected and organized. The data analysis was descriptive and interpretative and was done in an inductive and exploratory way (Bogdan & Biklen, 1994). This analysis considers some episodes occurred in the classroom during the unit, regarding the strategies that the students used to explore the tasks and the main difficulties observed.

5. RESULTS AND DISCUSSION

Numerical sequences. The students were puzzled with the first exploratory task. The teacher indicated that they should work in pairs, making notes from their conclusions. Since this was a new topic for them, they expressed difficulties in interpreting what was sought in some questions. When the teacher noticed this, she suggested them to look attentively at the pictures and share ideas with their colleagues. After this initial moment, the students drew their attention back to the task and detected some patterns, which encouraged them to continue. The first strategies that they used were very intuitive. Most of them analyzed the difference between the number of points on each picture, seeing that each picture had two more units than the previous picture. Other students represented all or just some of the pictures and counted directly the points they saw. Filipa and Carla chose a different strategy, based on the decomposition of the picture in two parts (Figure 1). The first one had as many points as the place it had in the sequence and the other part had one point more:

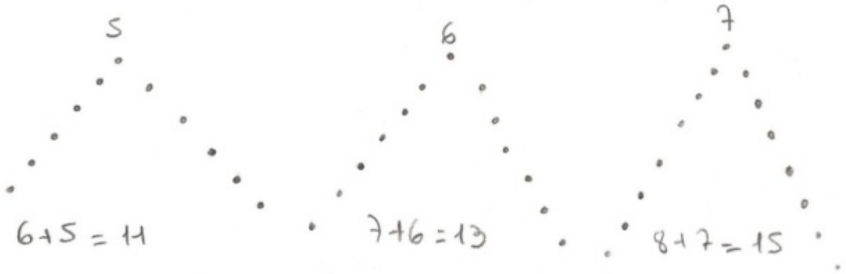

Figure 1 – Filipa and Carla – Task 1 – Numerical sequences

To describe a distant figure, the students used new strategies, considering the correspondence between variables. This relationship was expressed in two different ways. One was based on the decomposition of the picture in three parts: two main branches with the same number of points as the order of the picture and the point situated at the vertex. Adding each number of points, they obtained $100+100+1$ or $2\times 100+1$, totalizing 201 points. The other, as we saw, was done by Filipa and Carla, who just divided the picture in two parts. The students explained that there were 201 points: $101+100=201$. Supported by the pictorial representation, several pairs could actually imagine the 100th figure, opening the way for generalizing. However, this was also a point where some students showed some difficulties. For example, Jacinto and Carolina tried to use their knowledge about variation but they thought it was necessary to add 3 points instead of 2 and could not obtain the correct number of points.

There were also distinct answers concerning the generalization. It was quite apparent that the way students solved the first questions had a strong influence in how they described a general rule. Some students, such as Erica and Miguel, made general statements, directly related to the analysis of variation: “We always add two points to the existing number”. Other students based their generalization in the correspondence that they had previously detected. Therefore, students that in the 100th figure had multiplied the number 100 by 2 and added 1 answered as Pedro and Catarina: “It is twice a number plus the higher point”. Filipa and Carla, following their initial reasoning also stated: “When we have any number, we add one point more and then, with the result of the sum, we add that number again”. However, in this initial phase of the teaching unit some students in class still had problems in generalizing and in describing generalizations in an autonomous way.

The students were already familiar with this kind of classroom organization. They knew that, during the general discussions, the presentation of their strategies was valued, especially when they could add something new to what was already said. In the discussion of task 1, when Ricardo went to the board, he explained:

> **Ricardo:** So, we take this one [the figure’s order], we multiply it by two and we add one.

Teacher: But was that the first strategy you used?
Ricardo: No, we saw that we had to add two points.

At this point, the student recognized that his group started by observing the way how the number of points varied from one figure to the next. This strategy was very immediate for the most of the students. However, this task led them to go beyond this type of reasoning, looking for relations among different variables and describing such relations in natural language. Although students had already contacted with the algebraic language, they did not use it in this task to express the general rule they had found.

The possibility of representing that information in a more succinct way, using the algebraic language emerged during this general discussion. Some students suggested the use of the letter x to represent unknown values, as they had already done in the previous year. Taking this idea, the teacher asked the students if they could build a formula to represent the number of points of a general picture. Joaquim wrote immediately the expression $2 \times x + 1$ and said, with enthusiasm: "Teacher, this is our rule, isn't it?" Before the teacher confirmed this answer, she tried to know what the other students had in mind. She saw that a few other students had written the same formula. However, most of the students could not write suitable expressions. Influenced by the reasoning she made before about the 100th figure (to multiply the number of the figure by two and add one), Erica suggested: $x \times 2 = x + 1$. After this, Ricardo suggested $x \times 2 + 1$ completing his colleague's answer.

Filipa and Carla obtained different algebraic expressions, based on their previous decomposition of the figures: $x+1$ points to one side and x points to the other. Adding these two expressions, other student suggested that $x + x$ was equal to $2x$, which lead us again to the formula, so we got the expression $2x+1$. The students continued presenting their own algebraic expressions. Erica, for instance, suggested again: $x.2+1$. Miguel contributed with other possible expressions: $1+2x$ e $1+x.2$. This discussion about equivalent expressions was an important moment in the class. At the end, regarding the expression $2x+1$, Jacinto remembered something he learnt before: "Oh, we have to isolate…!" At that moment the teacher saw that the student was referring to the procedures for solving equations, without noticing that this was an algebraic expression. In this case, the aim was to generalize and not to determine the value of an unknown. His thinking was corrected by Joana who said: "No, that is only when we have an equal sign…" This dialogue was followed by a first reflection of the students about the difference between an equation and an algebraic expression.

The next task allowed students to contact with numerical sequences that were not related to pictorial representations. Most of the students started again to look for regularities between consecutive terms. The use of tables to represent the sequences was important because it made more visible the correspondence between the variables, favoring generalizations. The students had also the opportunity to work with the pictorial representations of the sequences and to solve small problems to find the order of some terms. Many students showed that they were able to invert their initial operations. This was a first step in formalizing their reasoning, that later became quite important, especially in solving equations.

Linear functions. The study of functions began with the analysis of two contextualized situations that could be modeled by linear functions. Both of them referred to shopping with or without discounts. The students started their work immediately building tables with some concrete cases. This situation was important because it created the opportunity to discuss about the values that made sense to use in that context. It was also interesting to verify that the students used the same strategies that they had already developed in the study of sequences. One of the main difficulties that they revealed was to analyze and describe the variation of the variables. After this initial moment, the students started to analyze equal increments on the independent variable: "They vary with each other. If you increase one litre you increase also 1.1 € in the price you pay" (Jacinto and Florbela). Other students also analyzed the variation and the correspondence between variables: "For each litre more, we add 1.1 € or we multiply the number of litres we buy for 1.1 €".

At this point, the students did not show difficulties in generalizing, suggesting $1{,}1n$ as a possible algebraic expression to represent the price of n litres of gas. When the students tried to express their generalization using a more formal language, they used the letter n as they did in the study of numerical sequences. That is, the work of some students in the continuous case revealed the influence of their previous work in the discrete case.

Numerical equations. The diversity concerning students' previous experiences in algebra was very visible in this class, especially concerning equations. Only some of the students that had studied this topic in grade 7 could effectively solve a 1st degree equation. The rest of the students, on the contrary, showed they had never learnt how to do it, because they had almost no interest about mathematics or because they had never studied equations at their previous schools in the former countries where they came from.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics generated the opportunity to solve simple equations, which was important to create a common understanding among students, allowing them to move on to study more complex algebraic concepts. In another class, the teacher asked the students to think about the sequence with general term $3n+5$. Then she asked if 300 was a term from this sequence. As the students tried to answer this question, in the general discussion, there was the following dialogue:

> **Teacher:** So, which was the order in which 300 was placed?
> **Erica:** Teacher, $3\times 100 \ldots$
> **Teacher:** Ok, but does that give 300?
> **Erica:** No, that is just with $3n$.
> **Teacher:** Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the n that makes this expression yield 300.
> **Sofia:** $300-5$? I don't know. [Students talk with each other.]
> **Erica:** So, we make $3n = 300-5$.

However, some students did not follow the Erica's reasoning, and went on designing their own strategies. One, Pedro, claimed with enthusiasm: "$3\times 98+5=299$; $3\times 99+5=302$. It will not pass on 300!" This discussion continued with the contributions of Isabel, who finished solving the equation in the board, according to her previous knowledge. The way the discussion developed allowed the confrontation between Erica's idea, the formal resolution proposed by Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and the advantages and weaknesses of each process.

Literal equations. The last task was one of the most challenging for the teacher in this unit. In the beginning she felt difficulties in dealing with the many paths students could choose. One of the situations involved studying the length of a wall built with yellow bricks. That length varied according to the number of bricks that the students could add to the wall in one of two directions: horizontal or vertical. There were three variables instead of only two, and most students chose to consider blocks of two or more bricks that they could repeat in order to build the wall. This strategy allowed them to know that the total length was a multiple of the length of each block. The first written records from Marisa and Helena showed this reasoning (Figure 2).

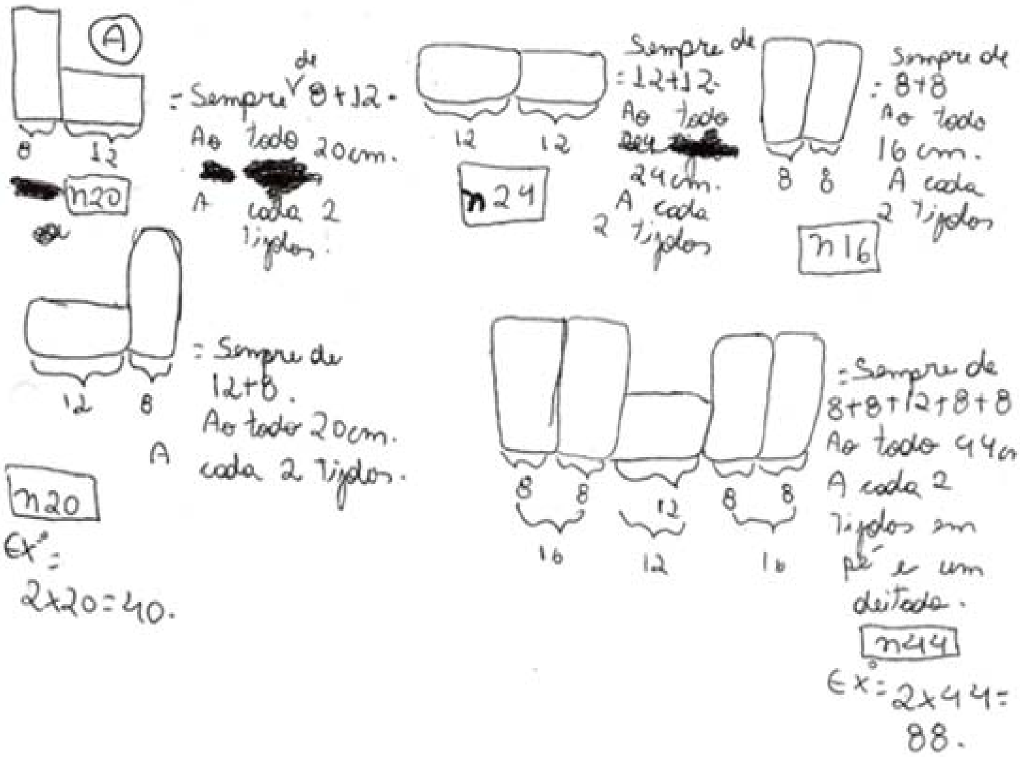

Figure 2 – Marisa and Helena – Task 8

To some groups, the analysis of walls built with two kinds of bricks brought the need to distinguish the length of the wall that was generated by each of them. The students began representing both lengths using the same letter, which led them to a situation of ambiguity. That happened to Sofia and Laura (Figure 3).

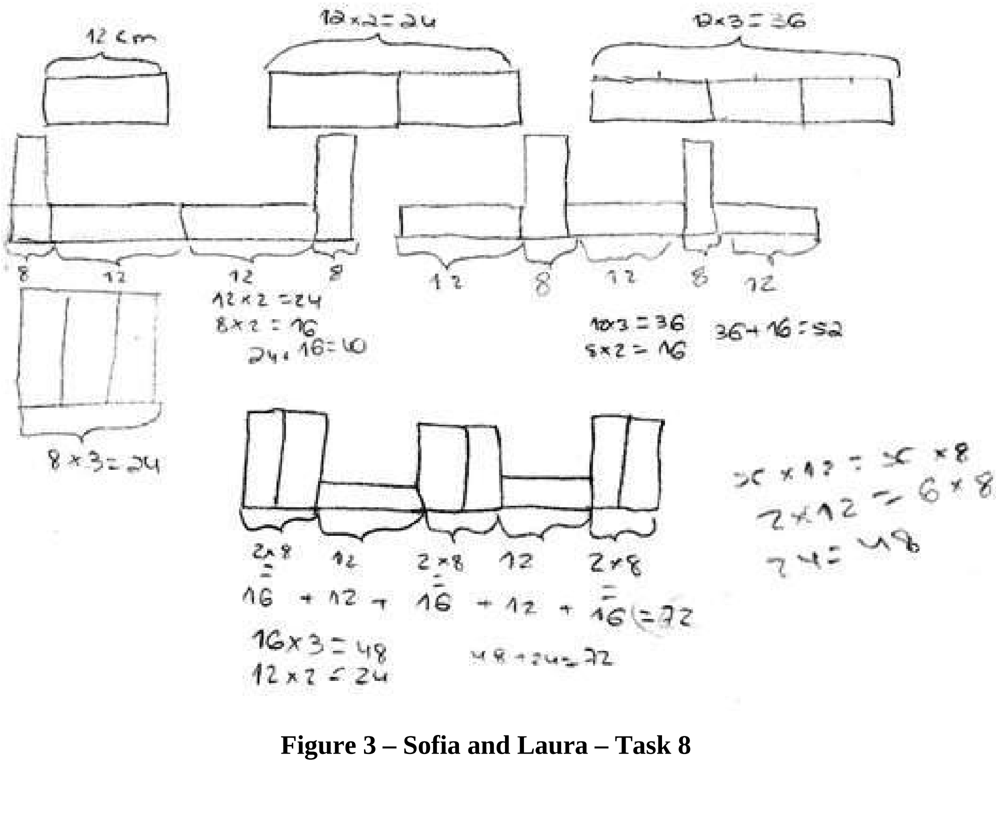

Figure 3 – Sofia and Laura – Task 8

Their initial reasoning had in consideration different possibilities. However, we can see that the value 72 cm came from the bricks in the horizontal direction and the value 48 cm came from the bricks in the vertical direction. Although the students used *x* as the number of bricks in the horizontal and *y* as the number of bricks in the vertical, they used the same letter "C" to describe the length of both sides and formulated the following equation to which they could not find any meaning: $x \times 12 = x \times 8$. As they were blocked, they asked for the teacher's intervention. She suggested them to think again about the meaning of all the expressions that they had used. After this they tried to explain those meanings in their notes and, considering C as the total length of the wall, they wrote the literal equation $C = 12x + 8y$. This was the first generalization that the students made with three different variables represented by three different letters. Other groups in the class reached similar conclusions, through the exploration of other possible walls. As the students explored this situation, first in concrete cases and then as a generalization, the meaning of those expressions became clear for them. For the teacher, the initial difficulty in dealing with so many paths of exploration gave place to the joy of seeing the different types of reasoning that the students were able to build and the strategies that they could share in the classroom discussion. Analyzing their written records before the general discussion allowed her to be better prepared for that moment.

6. CONCLUSION

The work in this teaching unit did not represent a reduction of algebra to just a formal language, although this aspect clearly maintained an important place. The situations proposed were based on a wider conception of algebra, involving the understanding of patterns and relationships through the exploration of sequences and functions. The initial part of the unit, with students' autonomous work and general discussions, was essential to activate the intuitive resources of the students and the knowledge that they developed in former school years. Presenting their own ideas, the students could clarify their questions about the meaning of algebraic expressions and become aware of the advantages that the use of this language could have for expressing generalizations and for solving problems. The first part of the unit was also important because it gave the students some contact with reasoning processes that they were not familiar with as generalizing and expressing generality. Sharing different strategies was an important feature that contributed to enrich the general discussions. These moments, most especially, the discussion of the first task, were very important to clarify difficulties and negotiate meanings.

All the students, even those who most feared mathematics, got immediately involved in solving the investigation tasks. At the beginning, their reasoning was supported by the pictorial representation of the sequences but the other tasks allowed them to go further on their reasoning. Some students began by making initial mistakes that were discussed and clarified in general discussions. However, most groups solved the tasks in an autonomous way, which gave them the self-confidence and motivation that they needed to solve the remaining tasks. This underscores the idea that these tasks yield multiple entry points to students with different ability levels in mathematics. The challenging nature of the tasks involved all students in developing their own strategies.

Working through exploratory and investigation tasks seems to have contributed towards: (i) developing a richer meaning for the algebraic language; (ii) widening the strategies to explore situations involving variables; (iii) using reasoning of an increasingly general nature; and (iv) expressing their reasoning using a more formal language. In this way, the work through the teaching unit contributed in an important way towards the development of students' symbol sense, yielding opportunities to strengthen their algebraic thinking. The exploratory and investigation tasks included in the unit, the work within the groups and the classroom discussions matched the initial expectations in planning the teaching unit, generating experimentation, autonomous work, lively discussions and most importantly, assuring that algebra was always a sense making activity. In this way, the students could construct new concepts and enlarge their algebraic knowledge and thinking.

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