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Geometria e Topologia

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Tese Orientada pelo Prof. Doutor Orlando Manuel Bartolomeu Neto

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Resumo

Dada uma variedade X de dimensão $2n+1$, chama-se *forma de contacto* de X a uma forma diferencial ω de grau 1 tal que $\omega(d\omega)^n = \omega \wedge d\omega \wedge \cdots \wedge d\omega$ é não-nula em todos os pontos. Pelo teorema de Darboux existe localmente um sistema de coordenadas $(x_1, \ldots, x_n, p_1, \ldots, p_{n-1})$ tal que $\omega = dx_n - \sum_{i=1}^{n-1} p_i dx_i$. Seja $\mathcal{L}$ um $\mathcal{O}_X$-módulo do feixe das formas diferenciais de grau 1, Ω_X^1. O feixe $\mathcal{L}$ diz-se uma estructura de contacto sobre X se para todo o $o \in X$ existe uma forma de contacto ω definida numa vizinhança aberta U de o tal que $\mathcal{L}|_U = \mathcal{O}_X\omega$. O par $(X, \mathcal{L})$ diz-se uma *variedade de contacto.*

A geometria de contacto é o equivalente em dimensão ímpar da geometria simpléctica. Seja Γ um subconjunto analítico de uma variedadede de contacto $(X, \mathcal{L})$ de dimensão $2n-1$. O conjunto Γ diz-se uma *variedade Legendriana* se Γ tem dimensão $n-1$ e a restrição à parte regular de Γ de qualquer secção ω de $\mathcal{L}$ se anula identicamente. Uma variedade Legendriana é o equivalente em geometria de contacto a uma variedade Lagrangeana em Geometria Simplética.

Dada uma variedade complexa X de dimensão n, o fibrado cotangente T^*X de X está munido de uma forma diferencial θ de grau 1, a *forma canónica* de T^*X. Vamos denotar por π a projecção de T^*X sobre X. A forma $d\theta$ é uma forma simpléctica de T^*X. Na verdade $d\theta^n$ é não-nula em todos os pontos. O fibrado projective cotangente $\mathbb{P}^*X$ tem uma estructura canónica de variedade de contacto. Se $X = \mathbb{C}^n, T^*X = \mathbb{C}^n \times \mathbb{C}_n$ onde $\mathbb{C}_n$ representa o dual de $\mathbb{C}^n$. Se considerarmos em $\mathbb{C}^n$ as coordenadas $(x_1, \ldots, x_n)$ e em $\mathbb{C}_n$ as coordenadas duais $(\xi_1, \ldots, \xi_n)$, $\theta = \sum_{i=1}^n \xi_i dx_i$, e $d\theta = \sum_{i=1}^n d\xi_i dx_i$.

Então $\mathbb{P}^*\mathbb{C}^n = \mathbb{C}^n \times \mathbb{P}_n$, onde $\mathbb{P}_n$ denota o espaço projectivo de $\mathbb{C}^n$. Temos que $\mathbb{P}^*\mathbb{C}^n$ é a união dos abertos $U_i = \{\xi_i \neq 0\}, 1 \leq i \leq n$. Temos em U_i o sistema de coordenadas $(x_1, \ldots, x_n, \frac{\xi_1}{\xi_i}, \ldots, \frac{\xi_{i-1}}{\xi_i}, \frac{\xi_{i+1}}{\xi_i}, \ldots, \frac{\xi_n}{\xi_i})$. A forma de contacto $\omega_j = \dfrac{\theta}{\xi_j} = d\xi_j + \sum_{j \neq i} \frac{\xi_i}{\xi_j} dx_i$ é uma forma de contacto sobre U_i. As formas diferenciais $\omega_i, 1 \leq i \leq n$, determinam uma estructura de contacto $\mathcal{L}$ sobre $\mathbb{P}^*\mathbb{C}^n$.

Dada uma hipersuperfície $S = \{f = 0\}$ sobre um aberto de $\mathbb{C}^n$, temos uma aplicação

$$a \mapsto \left(\frac{\partial f}{\partial x_1}(a) : \cdots : \frac{\partial f}{\partial x_n}(a) \right)$$

definida sobre a parte regular de S com valores em $\mathbb{P}_n$. O fecho em $\mathbb{P}^*\mathbb{C}^n$ do gráfico desta aplicação diz-se o *conormal* de S. O conormal de S é uma variedade Legendriana de $\mathbb{P}^*\mathbb{C}^n$. Dado um ponto $a \in S$, o conjunto $\Sigma = \Gamma \cap \pi^{-1}(a)$ diz-se o *limite de tangentes de S no ponto a.*

Seja (S, o) um germe de hipersuperfície de uma variedade complexa X definido por um germe de função holomorfa $f \in \mathcal{O}_{X,o}$. Dizemos que (S, o) é uma hipersuperfície quasi-ordinária se existe um sistema de coordenadas locais $(x_1, \ldots, x_n)$ centrado em o tal que a imagem pela aplicação

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) \tag{0.0.1}$$

do conjunto

$$\left\{ f = \frac{\partial f}{\partial x_n} = 0 \right\} \tag{0.0.2}$$

é igual a

$$\{x_1 \cdots x_x = 0\}. \tag{0.0.3}$$

O conjunto (0.0.2) diz-se o *contorno aparente* de S relativamente à projecção (0.0.1) e o conjunto (0.0.3) diz-se o *discriminante* de S relativamente à projecção (0.0.1).

A singularidade quasi-ordinária caracteriza-se pelo facto de admitir parametrizações em séries de potências fracionárias do tipo

$$x_n = \varphi(x_1, \ldots, x_{n-1}). \tag{0.0.4}$$

Toda a curva (hipersuperfície de uma variedade de dimensão 2) é uma superfície quasi-ordinária. Newton foi o primeiro a descobrir que toda a curva complexa admite uma parametrização do tipo (0.0.4), normalmente chamada de *expansão de Puiseux.*

O objectivo central desta tese é o estudo das variedades Legendrianas que são conormais de hipersuperfícies quasi-ordinárias.

O primeiro capítulo dedica-se ao estudo das curvas Legendrianas. O resultado fundamental é um teorema de classificação de curvas Legendrianas. Trata-se de uma versão para curvas Legendrianas de um teorema de Delorme (ver [7]) para curvas planas. Mostra-se que o conjunto das curvas Legendrianas que verificam uma condição de genericidade associada ao semigrupo da curva formam um aberto de Zariski de um espaço projectivo pesado.

Um dos instrumentos fundamentais para a prova do teorema consiste num teorema que descreve todas as transformações de contacto de uma variedade de contacto de dimensão três. Consideramos em $(\mathbb{C}^3, 0)$ a estructura de contacto definida pela forma de contacto $dy - pdx$. toda a transformação de contacto cuja derivada deixe invariante a recta $\{y = p = 0\}$ é a composição de transformações do tipo

$$(x, y, p) \mapsto (\lambda x, \mu y, \frac{\mu}{\lambda} p) \tag{0.0.5}$$

e

$$(x, y, p) \mapsto (x + \alpha, y + \beta, p + \gamma) \tag{0.0.6}$$

onde α, β, γ pertencem ao ideal maximal do anel $\mathbb{C}\{x, y, p\}$. Dados $\alpha \in \mathbb{C}\{x, y, p\}$ e $\beta_0 \in \mathbb{C}\{x, y\}$, temos que β é solução do problema de Cauchy

$$\frac{\partial \beta}{\partial x} - (p + \gamma) \left(1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y}\right) + p \left(1 + \frac{\partial \beta}{\partial y}\right) = 0,$$

com $\beta - \beta_0 \in (p)$. Além disso,

$$\gamma = \left(1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y}\right)^{-1} \left(\frac{\partial \beta}{\partial x} + p \left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} - p \frac{\partial \alpha}{\partial y}\right)\right).$$

Temos que toda a transformação de contacto de $(\mathbb{C}^3, 0)$ em $(\mathbb{C}^3, 0)$ é a composição de transformações do tipo (0.0.5), (0.0.6) e uma transformação de contacto paraboloidal (ver [11])

$$(x, y, p) \mapsto (ax + bp, y - \frac{1}{2} acx^2 - \frac{1}{2} bdp^2 - bcxp, cx + dp), \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.$$

O teorema de classificação de transformações de contacto referido acima é talvez o mais importante resultado de [1], tendo já sido citado em [6]. É também citado em dois outros trabalhos actualmente em preparação.

Como consequência do teorema fundamental deste capítulo, é possível classificar explícitamente em muitas situações todas as curvas Legendrianas que são os conormais de uma curva plana com um único par de Puiseux (p, q).

O segundo capítulo desta tese dedica-se ao estudo dos limites de tangentes de uma hipersuperfície quasi-ordinária. Podemos encontrar a solução deste problema num caso muito particular em [2].

Uma das consequências fundamentais deste resultado é mostrar que, sempre que o cone tangente de uma hipersuperfície quasi-ordinária é um hiperplano, o limite de tangentes é um invariante topológico da hipersuperfície.
Este resultado leva-nos a perguntar se podemos esperar que, quando o cone tangente de uma hipersuperfície arbitrária é um hiperplano, o limite de tangentes é um invariante topológico da hipersuperfície.
No terceiro capítulo da tese aplica-se o resultado fundamental do segundo capítulo ao estudo do comportamento por explosão do conormal de uma hipersuperfície quasi-ordinária. Obtemos desta forma um teorema de resolução de singularidades para superfícies Legendrianas que são conormais de superfícies quasi-ordinárias.
Seja $\pi : \widetilde{X} \to X$ uma explosão de uma variedade de contacto X com um centro dado D. Dada uma estructura de contacto $\mathcal{L}$ em X não podemos esperar que exista em $\widetilde{X}$ uma estructura de contacto $\widetilde{\mathcal{L}}$ para a qual π é uma transformação de contacto. Na verdade toda a transformação de contacto é bijectiva, e π só é bijectiva se $D = \emptyset$. Neto mostrou em [18] que existe uma noção de variedade de contacto logaritmica que generaliza a noção de variedade de contacto. Dada uma variedade Legendriana lisa Λ de X, o blow up $\widetilde{X}$ de X com centro Λ tem uma estructura de variedade de contacto logaritmica com polos ao longo do divisor excepcional de π. As secções do fibrado cotangente T^*M são as formas diferenciais de grau 1 que são as secções do feixe Ω^1_M. Dado um divisor com cruzamentos normais N de M, vamos denotar por $\Omega^1_M\langle N\rangle$ o feixe das formas diferenciais logaritmicas de grau 1 com polos em N. Vamos chamar fibrado cotangente logaritmico ao fibrado $T^*\langle M/N\rangle$ cujo feixe de secções é $\Omega^1_M\langle N\rangle$. Vamos denotar por $\mathbb{P}^*\langle M/N\rangle$ a projectivização do fibrado $T^*\langle M/N\rangle$.
Seja L uma subvariedade lisa de M tal que para toda a componente irredutível N_i de N, L está contida em N_i ou L é transversal a N_i. Podemos definir $\mathbb{P}_L\langle M/N\rangle$ de forma semelhante à usada para definir $\mathbb{P}_L M$.
O resultado seguinte é um dos instrumentos essenciais na prova do teorema fundamental deste capítulo.

Theorem 0.0.1. (i) *Seja* $(X, \mathcal{L})$ *uma variedade de contacto logarítmica com polos ao longo de* Y. *Seja* Λ *uma subvariedade Legendriana bem comportada* *de X. Seja $\tau : \widetilde{X} \to X$ o blow up de X ao longo de Λ. Seja $E = \tau^{-1}(\Lambda)$. Então $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{O}_{\widetilde{X}}(E)\tau^*\mathcal{L}$ é uma estructura de contacto logarítmica em $\widetilde{X}$ com polos ao longo de $\tau^{-1}(Y)$.*

(ii) *Seja M uma variedade e N um divisor com cruzamentos normais de M. Seja L uma subvariedade bem comportada de M. O conormal $\Lambda = \mathbb{P}^*_L M$ de L é uma subvariedade Legendriana bem comportada de $P^*\langle M/N\rangle$. Seja $\rho : \widetilde{M} \to M$ o blow up de M ao longo de L. Seja $\widetilde{E} = \rho^{-1}(L)$. Seja $\widetilde{N} = \rho^{-1}(N)$. Então existe uma transformação de contacto injectiva φ de um subconjunto aberto denso Ω do blow up $\widetilde{X}$ de $P^*\langle M/N\rangle$ ao longo de Λ para $P^*\langle\widetilde{M}/\widetilde{N}\rangle$ tal que o diagrama (0.0.7) comuta.*

$$\begin{array}{ccccc} P^*\langle M/N\rangle & \stackrel{\tau}{\longleftarrow} & \widetilde{X} \hookleftarrow \Omega & \stackrel{\varphi}{\hookrightarrow} & P^*\langle\widetilde{M}/\widetilde{N}\rangle \\ \pi\downarrow & & & & \downarrow\pi \\ M & & \stackrel{\rho}{\longleftarrow} & & \widetilde{M} \end{array} \tag{0.0.7}$$

(iii) *Seja M uma variedade e N um divisor com cruzamentos normais de M. Seja L uma subvariedade bem comportada de (M,N). Seja σ a projecção canónica de $T_\Lambda\mathbb{P}^*\langle M/N\rangle$ sobre $T_L M$. Seja S um germe de um subconjunto analítico natural de (M,N) em $o \in N$. Seja $\Gamma = \mathbb{P}^*_S\langle M/N\rangle$. Se S tem limite de tangentes trivial em o, então $\Gamma \cap \pi^{-1}(o) = \{\lambda\}$ e $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \Lambda$, $\widetilde{\Gamma} \subset \Omega$ e $\varphi(\widetilde{\Gamma}) = \mathbb{P}^*_{\widetilde{S}}\langle\widetilde{M}/\widetilde{N}\rangle$.*

A prova do Teorema de resolução de singularidades depende de um argumento combinatório baseado no algoritmo de resolução de singularidades para superfícies quasi-ordinárias.

Palavras chave: Espaços de Moduli; Geometria Algébrica; Hipersuperfície quasi-ordinária; Limites de tangentes; Teoria das singularidades; Variedade de contacto; Variedade Legendriana.

Abstract

This thesis is a study of the Legendrian Varieties that are conormals of quasi-ordinary hypersurfaces.
In the first chapter we study the analytic classification of the Legendrian curves that are the conormal of a plane curve with a single Puiseux pair. Let $\chi_{m,n}$ be the set of Legendrian curves that are the conormal of a plane curve with a Puiseux pair (m, n), where $g.c.d.(m, n) = 1$ and $m > 2n$, with semigroup as generic as possible. We show that the quotient of $\chi_{m,n}$ by the group of contact transformations is a Zariski open set of a weighted projective space.
The main tool used in the proof of this theorem is a classification/construction theorem for contact transformation that has since proved useful in other instances.
In the second chapter we calculate the limits of tangents of a quasi-ordinary hypersurface. In particular, we show that the set of limits of tangents is, in general, a topological invariant of the hypersurface.
In the third chapter we prove a desingularization theorem for Legendrian hypersurfaces that are the conormal of a quasi-ordinary hypersurface. One of the main ingredients of the proof is the calculation of the limits of tangents achieved in chapter two.

Keywords: Algebraic Geometry; Contact Variety; Legendrian Variety; Limits of tangents; Moduli Spaces; Quasi-ordinary Hypersurface; Singularity theory.

Contents

Chapter 1

Moduli of Germs of Legendrian Curves

In this chapter We construct the generic component of the moduli space of the germs of Legendrian curves with generic plane projection topologicaly equivalent to a curve $y^n = x^m$.

1.1 Introduction

Zariski [23] initiated the construction of the moduli of plane curve singularities. Delorme [7] organized in a systematic way the ideas of Zariski, obtaining general results o the case of curves with one characteristic exponent in the generic case (see also [20]). Greuel, Laudal and Pfister (see the bibliography of [8]) stratified the space versal deformations of plane curves, constructing moduli spaces on each stratum.

In this chapter we initiate the study of the moduli of Legendrian curve singularities. We construct the moduli space of generic irreducible Legendrian singularities with equisingularity type equal to the topological type of the plane curve $y^n = x^m$, $(n, m) = 1$. Our method is based on the analysis of the action of the group of infinitesimal contact transformations on the set of Puiseux expansions of the germs of plane curves.

In section 2 we associate to each pair of positive integers n, m such that $(n, m) = 1$ a semigroup $\Gamma(n, m)$. We show that the semigroup of a generic element of this equisingularity class equals $\Gamma(n, m)$. In section 3 we classify the infinitesimal contact transformations on a contact threefold and study its action on the Puiseux expansion of a plane curve. In section 4 we discuss some simple examples of moduli of germs of Legendrian curves. In section 5 we show that the generic components of the moduli of germs of Legendrian curves with fixed equisingularity class are the points of a Zariski open subset of a weighted projective space.

1.2 Plane curves versus Legendrian curves

Let Λ be the germ at o of an irreducible space curve. A local parametrization $\imath : (\mathbb{C}, 0) \to (\Lambda, o)$ defines a morphism $\imath^*$ from the local ring $\mathcal{O}_{\Lambda,o}$ into its normalization $\mathbb{C}\{t\}$. Let $v : \mathcal{O}_{\Lambda,o} \to \mathbb{Z} \cup \{\infty\}$ be the map $g \mapsto \mathrm{order}(\imath^*(g))$. We call $v(g), g \in \mathcal{O}_{\Lambda,o}$, the *valuation* of g. We call $\Gamma = v(\mathcal{O}_{\Lambda,o})$ the semigroup of the curve Λ. There is an integer k such that $l \in \Gamma$ for all $l \geq k$. The smallest integer k with this property is denoted by c and called the *conductor* of Γ.

Let C be the germ at the origin of a singular irreducible plane curve C parametrized by

$$x = t^n, \qquad y = \sum_{i=m}^{\infty} a_i t^i, \tag{1.2.1}$$

with $a_m \neq 0$ and $(n, m) = 1$. The pair (n, m) determines the topological type of C (see for instance [5]).

Example 1.2.1. A *monomial space curve* is a curve defined by a parametrization of the type $t \mapsto (x, y, p) = (a_1 t^n, a_2 t^m, a_3 t^s)$, $a_i \in \mathbb{C}^*$. Let C be a monomial space curve. The semigroup of any space curve includes the valuations of all the monomials $x^i y^j p^k$, $i, j, k \in \mathbb{N}_0$, which are equal to $\text{order}(\iota^*(x^i y^j p^k)) = \text{order}(t^{in} t^{jm} t^{ks}) = in + jm + sk$. Hence $\Gamma \supseteq \{in + jm + ks, i, j, k \in \mathbb{N}_0\}$. Since C is a monomial curve, if u, v are monomials of $\mathcal{O}_{\Lambda,o}$, and $a, b \in \mathbb{C}$, then $\text{order}(\iota^*(au + bv)) = min(\text{order}(u), \text{order}(v))$, or $\iota^*(au+bv) = 0$. Hence, for a monomial curve $\Gamma = \{in+jm+ks, i, j, k \in \mathbb{Z}^+\}$. The same result applies to monomial plane curves as a particular case, with the obvious modifications.

Example 1.2.2. Let C be the germ of plane curve germ defined at the origin of $\mathbb{C}^2$ by $y^3 - x^{11} = 0$. Let ι be the parametrization of C defined by $t \mapsto (t^3, t^{11})$. Then $v(x^i y^j) = \text{order}(\iota^*(x^i y^j)) = \text{order}(t^{3i+11j}) = 3i + 11j$ and, since C is a monomial curve, the semigroup Γ of C is equal to the set of all such orders for $i, j \in \mathbb{N}_0$.

It is useful to represent the semigroup of a curve in a table with $v(x)$ columns, where each place (i, j) of the table represents the valuation $iv(x) + j$. In each place of the table we display a monomial that has the corresponding valuation. Once a monomial u is placed in the table we know that all places below that monomial along the same column are also in the semigroup, since moving down one line along a fixed column corresponds to multiplying u by powers of x. Hence we omit displaying all monomials in (x), except for x itself. In the current example, we obtain the table (1.1).

Hence, it is easy to see that in this case the semigroup equals

$$\Gamma = \{3, 6, 9, 11, 12, 14, 15, 17, 18, 20, 21, 22 \ldots\}$$

In particular, the conductor is $c = 20$. In general, $c = (n - 1)(m - 1)$ for a plane curve $t \mapsto (t^n, t^m + \sum_{i>m} a_i t^i)$ such that $(n, m) = 1$.

	0	1	2
0			
3	x		
6	.		
9	.		y
12	.		.
15	.		.
18	.		.
21	.	y^2	.

Table 1.1: Semigroup table of $\mathbb{C}\{x,y\}/(y^3 - x^{11})$

Example 1.2.3. Let Λ be the space curve defined in $\mathbb{C}^3_{x,y,p}$ by the ideal $(y^3 - x^{11}, y-(3/11)px)$. A parametrization of Γ is given by $\iota(t) = (t^3, t^{11}, (11/3)t^8)$. The semigroup is equal to the set of valuations $v(x^i y^j p^k), i, j, k \in \mathbb{N}_0$. The semigroup table is

	0	1	2
0			
3	x		
6	.		p
9	.		y
12	.		.
15	.	p^2	.
18	.	py	.
21	.	y^2	.

Table 1.2: Semigroup table of $\mathbb{C}\{x,y,p\}/(y^3 - x^{11}, y - (3/11)px)$

Hence the semigroup is the union of $\{3, 6, 8, 9\}$ with all the integers greater or equal to 11 except for 13, and the conductor is $c = 14$.

Example 1.2.4. Consider the family of plane curves defined by $t \mapsto (t^3, t^{11} + \sum_{i>11} a_i t^i)$, $a_i \in \mathbb{C}$. Since $(3, 11) = 1$, $c = (3-1)(11-1) = 20$, and for $k < c$, there is at most one monomial with valuation k (the smallest k

where two monomials coincide is $v(x^{11}) = v(y^3) = 33$). Hence we still have $\Gamma = \{v(x^i y^j), i, j \in \mathbb{N}_0\}$, and the semigroup table is the same as in example (1.2.2).

Example 1.2.5. Let Λ be the space curve defined in $\mathbb{C}^3_{x,y,p}$ by the parametrization $\iota(t) = (t^3, t^{11} + \sum_{i>11} a_i t^i, \frac{11}{3}t^8 + \sum_{i>11} \frac{i}{3} a_i t^{i-3})$, $a_i \in \mathbb{C}$. Notice that the projection of Λ through $(x, y, p) \mapsto (x, y)$ coincides with the curve C of the previous example. The semigroup of Λ contains all the valuations of the type $v(x^i y^j p^k), i, j, k \in \mathbb{N}_0$. In addition, we have

$$\iota^*(y - \frac{11}{3}px) = -a_{12}11t^{11} - \frac{2}{11}a_{13}t^{13} + O(t^{14}).$$

Hence, if $a_{12} \neq 0$, $v(y - \frac{11}{3}px) = v(x^4) = 12$. Suppose $a_{12} \neq 0$. Then

$$\omega = \iota^*(y - \frac{11}{3}px + \frac{a_{12}}{11}x^4) = -\frac{2}{11}a_{13}t^{13} + O(t^{14}).$$

Hence, if $a_{12} \neq 0, a_{13} \neq 0$, $13 \in \Gamma$, although 13 is not the valuation of a monomial. In this case the semigroup table is table (1.3). Therefore $\Gamma = \{3, 6, 8, 9\} \cup (11 + \mathbb{N}_0)$. Now suppose $a_{12} = 0, a_{13} \neq 0$. Then $v(y - \frac{11}{3}px) = 13$ and we get the same table again, so we see that the value of a_{12} is irrelevant. But if $a_{13} = 0$ then 13 no longer belongs to Γ and the semigroup is that of table (1.2).

	0	1	2
0			
3	x		
6	.		p
9	.		y
12	.	ω	.
15	.	p^2	.
18	.	py	.
21	.	y^2	.

Table 1.3: Semigroup table of Λ when $a_{13} \neq 0$.

Hence we see that the semigroup of a space curve depends on the values of at least some of the coefficients a_i.

Let M be a complex manifold of dimension n. The cotangent bundle $\pi_M : T^*M \to M$ of M is endowed of a canonical 1-form θ. The differential form $(d\theta)^{\wedge n}$ never vanishes on M. Hence $d\theta$ is a symplectic form on T^*M. Given a system of local coordinates $(x_1, \dots, x_n)$ on an open set U of X, there are holomorphic functions $\xi_1, \dots, \xi_n$ on $\pi_M^{-1}(U)$ such that $\theta \mid_{\pi_M^{-1}(U)} = \xi_1 dx_1 + \cdots + \xi_n dx_n$.

Let X be a complex threefold. Let Ω_X^k denote the sheaf of differential forms of degree k on X. A local section of Ω_X^1 is called a *contact form* if $\omega \wedge d\omega$ never vanishes. Let $\mathcal{L}$ be a subsheaf of the sheaf Ω_X^1. The sheaf $\mathcal{L}$ is called a *contact structure* on X if $\mathcal{L}$ is locally generated by a contact form. A pair $(X, \mathcal{L})$, where $\mathcal{L}$ is a contact structure on X, is called a *contact threefold.* Let $(X_i, \mathcal{L}_i)$, $i = 1, 2$, be two contact threefolds. A holomorphic map $\varphi : X_1 \to X_2$ is called a *contact transformation* if $\varphi^*\mathcal{L}_2 = \mathcal{L}_1$.

Let $\mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 = \{(x, y, (\xi : \eta)) : x, y, \xi, \eta \in \mathbb{C},\ (\xi, \eta) \neq (0, 0)\}$ be the projective cotangent bundle of $\mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 \to \mathbb{C}^2$ be the canonical projection. Let U and V be the open sets of $\mathbb{P}^*\mathbb{C}^2$ defined respectively by $\eta \neq 0$ and $\xi \neq 0$. Set $p = -\xi/\eta$, $q = -\eta/\xi$. The sheaf $\mathcal{L}$ defined by $\mathcal{L} \mid_U = \mathcal{O}_U(dy - pdx)$ and $\mathcal{L} \mid_V = \mathcal{O}_V(dx - qdy)$ is a contact structure on $\mathbb{P}^*\mathbb{C}^2$. By the Darboux theorem every contact threefold is locally isomorphic to $(U, \mathcal{O}_U(dy - pdx))$. We call *infinitesimal contact transformation* to a germ of a contact transformation $\Phi : (U, 0) \mapsto (U, 0)$.

A curve Λ on a contact manifold $(X, \mathcal{L})$ is called *Legendrian* if the restriction of ω to the regular part of Λ vanishes for each section ω of $\mathcal{L}$. Let $C = \{f = 0\}$ be a plane curve. Let Λ be the closure on $\mathbb{P}^*\mathbb{C}^2$ of the graph of the Gauss map $G : \{a \in C : df(a) \neq 0\} \to \mathbb{P}^1$ defined by $G(a) = \langle df(a) \rangle$. The set Λ is a Legendrian curve. We call Λ the *conormal* of the curve C. If C is irreducible and parametrized by (1.2.1) then Λ is parametrized by

$$x = t^n, \qquad y = \sum_{i=m}^{\infty} a_i t^i, \qquad p = \frac{dy}{dx} = \sum_{i=m}^{\infty} \frac{i}{n} a_i t^{i-n}. \tag{1.2.2}$$

Given a Legendrian curve Λ of $\mathbb{P}^*\mathbb{C}^2$ such that Λ does not contain any fibre of π, $\pi(\Lambda)$ is a plane curve. Moreover, Λ equals the conormal of $\pi(\Lambda)$ (see [21]).

Let $(X, \mathcal{L})$ be a contact threefold. A holomorphic map $\varphi : (X, o) \to (\mathbb{C}^2, 0)$

is called a *Legendrian map* if $D\varphi(o)$ is surjective and the fibers of φ are smooth Legendrian curves. The map φ is Legendrian if and only if there is a contact transformation $\psi : (X, o) \to (\mathbb{P}^*\mathbb{C}^2, (0, 0, (0 : 1))$ such that $\varphi = \pi\psi$. Let (Λ, o) be a Legendrian curve of X. Let $C_o(\Lambda)$ be the tangent cone of Λ at o. We say that a Legendrian map $\varphi : (X, o) \to (\mathbb{C}^2, 0)$ is *generic* relatively to (Λ, o) if it verifies the transversality condition $T_o\varphi^{-1}(0) \cap C_o(\Lambda) = \{0\}$. We say that a Legendrian curve (Λ, o) of $\mathbb{P}^*\mathbb{C}^2$ is in strong generic position if $\pi : (\mathbb{P}^*\mathbb{C}^2, o) \to (\mathbb{C}^2, \pi(o))$ is generic relatively to (Λ, o). The Legendrian curve Λ parametrized by (1.2.2) is in *strong generic position* if and only if $m \geq 2n + 1$. Given a Legendrian curve (Λ, o) of a contact threefold X there is a contact transformation $\psi : (X, o) \to (\mathbb{P}^*\mathbb{C}^2, (0, 0, (0 : 1))$ such that $(\psi(\Lambda), o)$ is in strong generic position (cf [10], section 1).

Example 1.2.6. Let C be the germ of plane curve $y^2 - x^3 = 0$. The tangent cone of C is obtained by considering the *d*eformation to the tangent cone *m*ap,

$$\lambda \mapsto \frac{(\lambda^2 y^2 - \lambda^3 x^3)}{\lambda^2} = y^2 - \lambda x^3,$$

and setting $\lambda = 0$. Hence the tangent cone of C is $\{y = 0\}$.
Let Λ be the conormal of C. Λ is the curve parametrized by

$$t \mapsto (x, y, p) = (t^2, t^3, (3/2)t),$$

hence Λ verifies the equations $y^2 - x^3 = 0$, $p^2 - (9/4)x = 0$. From the first equation, the tangent cone is contained in $\{y = 0\}$. from the second we get $\lambda p^2 - (9/4)x = 0$, hence $x = 0$. Hence the tangent cone of Λ is $\{x = y = 0\}$, therefore Λ is not in strong generic position.

We say that two germs of Legendrian curves are equisingular if their images by generic Legendrian maps have the same topological type.

1.3 Infinitesimal Contact Transformations

Let m be the maximal ideal of the ring $\mathbb{C}\{x, y, p\}$. Let $\mathcal{G}$ denote the group of infinitesimal contact transformations Φ such that the derivative of Φ leaves invariant the tangent space at the origin of the curve $\{y = p = 0\}$. Let $\mathcal{J}$

be the group of infinitesimal contact transformations

$$(x,y,p) \mapsto (x+\alpha, y+\beta, p+\gamma) \tag{1.3.1}$$

such that $\alpha, \beta, \gamma, \partial\alpha/\partial x, \partial\beta/\partial y, \partial\gamma/\partial p \in m$. Set $\mathcal{H} = \{\Psi_{\lambda,\mu} : \lambda, \mu \in \mathbb{C}^*\}$, where

$$\Psi_{\lambda,\mu}(x,y,p) = \left(\lambda x, \mu y, \frac{\mu}{\lambda} p\right). \tag{1.3.2}$$

Let $\mathcal{P}$ denote the group of *paraboloidal contact transformations* (see [11])

$$(x,y,p) \mapsto (ax+bp, y-\frac{1}{2}acx^2-\frac{1}{2}bdp^2-bcxp, cx+dp), \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1. \tag{1.3.3}$$

The contact transformation (1.3.3) belongs to $\mathcal{G}$ if and only if $c = 0$. The paraboloidal contact transformation

$$(x,y,p) \mapsto (-p, y-xp, x) \tag{1.3.4}$$

Is called the *Legendre transformation.*

Theorem 1.3.1. *The group $\mathcal{J}$ is an invariant subgroup of $\mathcal{G}$. Moreover, the quotient $\mathcal{G}/\mathcal{J}$ is isomorphic to $\mathcal{H}$.*

Proof. . If $H \in \mathcal{H}$ and $\Phi \in \mathcal{J}$, $H\Phi H^{-1} \in \mathcal{J}$. Hence it is enough to show that each element of $\mathcal{G}$ is a composition of elements of $\mathcal{H}$ and $\mathcal{J}$. Let $\Phi \in \mathcal{G}$ be the infinitesimal contact transformation $(x,y,p) \mapsto (x',y',p')$. There is $\varphi \in \mathbb{C}\{x,y,p\}$ such that $\varphi(0) \neq 0$ and

$$dy' - p'dx' = \varphi(dy - pdx). \tag{1.3.5}$$

Composing Φ with $H \in \mathcal{H}$ we can assume that $\varphi(0) = 1$. Let $\hat{\Phi}$ be the germ of the symplectic transformation $(x,y,p;\eta) \mapsto (x',y',-\eta p';\varphi^{-1}\eta)$. Notice that $\hat{\Phi}(0,0;0,1) = (0,0;0,1)$. Since $D\hat{\Phi}(0,0;0,1)$ leaves invariant the linear subspace μ generated by $(0,0;0,1)$, $D\hat{\Phi}(0,0;0,1)$ induces a linear symplectic transformation on the linear symplectic space $\mu^\perp/\mu$. There is a paraboloidal contact transformation P such that $D\hat{P}(0,0;0,1)$ equals $D\hat{\Phi}(0,0;0,1)$ on $\mu^\perp/\mu$. Since $D(\hat{P}^{-1}\hat{\Phi})(0,0;0,1)$ induces the identity map on $\mu^\perp/\mu$, $P^{-1}\Phi$ is an infinitesimal contact transformation of the type $(x,y,p) \mapsto (x+\alpha, y', p+\gamma)$, where

$$\frac{\partial\alpha}{\partial x}, \frac{\partial\alpha}{\partial p}, \frac{\partial\gamma}{\partial x}, \frac{\partial\gamma}{\partial p} \in m. \tag{1.3.6}$$

Set $\beta = y' - y$. It follows from (1.3.5) and (1.3.6) that $(\partial\beta/\partial y)(0) = 0$. Hence $P^{-1}\Phi \in \mathcal{J}$. Since Φ and $P^{-1}\Phi \in \mathcal{G}$, $P \in \mathcal{G}$. Therefore p is the composition of an element of $\mathcal{H}$ and an element of $\mathcal{J}$. $\square$

Theorem 1.3.2. *Let $\alpha \in \mathbb{C}\{x, y, p\}, \beta_0 \in \mathbb{C}\{x, y\}$ be power series such that*

$$\alpha, \beta_0, \frac{\partial\beta_0}{\partial y} \in m. \tag{1.3.7}$$

There are $\beta, \gamma \in \mathbb{C}\{x, y, p\}$ such that $\beta - \beta_0 \in (p)$, $\gamma \in m$ and α, β, γ define an infinitesimal contact transformation Φ_{α,β_0} of type (1.3.1). The power series β and γ are uniquely determined by these conditions. Moreover, (1.3.1) belongs to $\mathcal{J}$ if and only if

$$\frac{\partial\alpha}{\partial x}, \frac{\partial\beta_0}{\partial x}, \frac{\partial^2\beta_0}{\partial x\partial p} \in m. \tag{1.3.8}$$

The function β is the solution of the Cauchy problem

$$\left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)\frac{\partial\beta}{\partial p} - p\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x} = p\frac{\partial\alpha}{\partial p}. \tag{1.3.9}$$

with initial condition $\beta - \beta_0 \in (p)$.

Proof. . The map (1.3.1) is a contact transformation if and only if there is $\varphi \in \mathbb{C}\{x, y, p\}$ such that $\varphi(0) \neq 0$ and

$$d(y + \beta) - (p + \gamma)d(x + \alpha) = \varphi(dy - pdx). \tag{1.3.10}$$

The equation (1.3.10) is equivalent to the system

$$\frac{\partial\beta}{\partial p} = (p + \gamma)\frac{\partial\alpha}{\partial p} \tag{1.3.11}$$

$$\varphi = 1 + \frac{\partial\beta}{\partial y} - (p + \gamma)\frac{\partial\alpha}{\partial y} \tag{1.3.12}$$

$$-p\varphi = \frac{\partial\beta}{\partial x} - (p + \gamma)(1 + \frac{\partial\alpha}{\partial x}). \tag{1.3.13}$$

By (1.3.12) and (1.3.13),

$$\frac{\partial\beta}{\partial x} - (p + \gamma)\left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right) + p\left(1 + \frac{\partial\beta}{\partial y}\right) = 0, \tag{1.3.14}$$

By (1.3.11) and (1.3.14), (1.3.9) holds.

By the Cauchy-Kowalevsky theorem there is one and only one solution β of (1.3.9) such that $\beta - \beta_0 \in (p)$. It follows from (1.3.14) that

$$\gamma = \left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)^{-1}\left(\frac{\partial\beta}{\partial x} + p\left(\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x} - p\frac{\partial\alpha}{\partial y}\right)\right). \tag{1.3.15}$$

Since $\partial\beta_0/\partial y \in m$, $\partial\beta/\partial y \in m$. By (1.3.12), $\varphi(0) \neq 0$.
(ii) Since $\partial\beta_0/\partial x \in m$, $\partial\beta/\partial x \in m$. By (1.3.15), $\gamma \in m$. By (1.3.15),

$$\frac{\partial\gamma}{\partial p} \in \left(\frac{\partial^2\beta}{\partial x\partial p} + \frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x}, p\right).$$

By (1.3.7) and (1.3.8), $\partial\gamma/\partial p \in m$. □

Example 1.3.3. Setting $\alpha = \frac{k}{k-1}p^{k-1}$, $k \geq 2$, $a \in \mathbb{C}$ and $\beta_0 = 0$, we find that

$$\left\{\begin{array}{lcl} x' & = & x + \frac{k}{k-1}ap^{k-1} \\ y' & = & x + ap^k \\ p' & = & p \end{array}\right. \tag{1.3.16}$$

is a contact transformation.

Example 1.3.4. Setting $\alpha = \frac{k}{k-1}x^i y^j p^{k-1}$, such that $a \in \mathbb{C}$, and either $k \geq 2$ or $k \geq 1$ and $ij \neq 0$, there are $\varepsilon \in m$ and $\gamma \in \mathbb{C}\{x, y, p\}$, such that

$$\left\{\begin{array}{lcl} x' & = & x + \frac{k}{k-1}ax^i y^j p^{k-1} \\ y' & = & x + ax^i y^j p^k(1+\varepsilon) \\ p' & = & p + \gamma \end{array}\right. \tag{1.3.17}$$

is a contact transformation.

Corollary 1.3.5. *The elements of $\mathcal{J}$ are the infinitesimal contact transformations Φ_{α,β_0} such that α, β_0 verify (1.3.7) and (1.3.8).*

Lemma 1.3.6. *Given $\lambda \in \mathbb{C}$ and $w \in \Gamma(m, n)$ such that $w \geq m + n$, there are α, β_0 verifying the conditions of theorem 1.3.2 such that $\imath^*(\beta - p\alpha) = \lambda t^w + \cdots$.*

Proof. . By (1.5.1) there is $b \in \mathbb{C}\{x, y, p\}$ such that $\imath^* b = \lambda t^w + \cdots$, $b = \sum_{k\geq 0} b_k p^k$ and $v(b_k) \geq v(b) - v(x) - kv(p) + 1$. Set $\alpha = -\partial b/\partial p$, $\beta_0 = b_0$. Set $\alpha = \sum_{k\geq 0}\alpha_k p^k$, $\beta = \sum_{k\geq 0}\beta_k p^k$, where $\alpha_k, \beta_k \in \mathbb{C}\{x, y\}$. By (1.3.9),

$$k\beta_k \quad + \quad \sum_{j=1}^{k-1} j\beta_j\left(\frac{\partial\alpha_{k-j}}{\partial x} + \frac{\partial\alpha_{k-j-1}}{\partial y}\right) =$$

$$= \quad (k-1)\alpha_{k-1} + k\alpha_k \frac{\partial \beta_0}{\partial x} + \sum_{j=1}^{k-1} j\alpha_j \left(\frac{\partial \beta_{k-j}}{\partial x} + \frac{\partial \beta_{k-j-1}}{\partial y} \right),$$

for $k \geq 1$. Since $\alpha_l = -(l+1)b_{l+1}$ for $l \geq 1$, $v(\alpha_j p^k) \geq w+1$, if $j \leq k-2$. Moreover, $v(\alpha_{k-1}p^k) \geq w+1-n$ and $v(\alpha_k p^k) \geq w+1-m$. Therefore

$$k\beta_k p^k + \sum_{j=1}^{k-1} j\beta_j \left(\frac{\partial \alpha_{k-j}}{\partial x} + \frac{\partial \alpha_{k-j-1}}{\partial y} \right) p^k \equiv (k-1)\alpha_{k-1}p^k + (k-1)\alpha_{k-1}\frac{\partial \beta_1}{\partial x},$$

mod $\left(t^{w+1}\right)$ for $k \geq 1$. We show by induction in k that

$$k\beta_k p^k \equiv (k-1)\alpha_{k-1}p^k \quad \mod (t^{w+1}), \text{for } k \geq 1.$$

Hence $\beta - p\alpha \equiv b \bmod (t^{w+1})$. □

There is an action of $\mathcal{J}$ into the set of germs of plane curves C such that the tangent cone to the conormal of C equals $\{y = p = 0\}$. Given $\Phi \in \mathcal{J}$ we associate to C the image by $\pi\Phi$ of the conormal of C. Given integers n, m such that $(m,n) = 1$ and $m \geq 2n+1$, $\mathcal{J}$ acts on the series of type (1.2.1). Given an infinitesimal contact transformation (1.3.1) there is $s \in \mathbb{C}\{t\}$ such that $s^n = t^n + \alpha$ and for each $i \geq 1$

$$s^i = t^i \left(1 + \frac{i}{n}\frac{\alpha(t)}{t^n} + \frac{i}{n}\left(\frac{i}{n} - 1\right)\left(\frac{\alpha(t)}{t^n}\right)^2 + \cdots \right).$$

Lemma 1.3.7. *If $v(\beta_0) \geq v(\alpha) + v(p)$, the contact transformation* (1.3.1) *takes* (1.2.1) *into the plane curve parametrized by $x = s^n$, $y = y(s) + \beta(s) - p(s)\alpha(s) + \varepsilon$, where $v(\varepsilon) \geq 2v(\alpha) + m - 2n$.*

Proof. . Since $t^i = s^i - (i/n)t^{i-n}\alpha(t) + (i(i-n)/n^2)\alpha(t)^2 t^{i-2n} + \cdots$,

$$y(t) = \sum_{i \geq m} a_i s^i - \alpha(t) \sum_{i \geq m} \frac{i}{n} a_i t^{i-m} + \varepsilon' = y(s) - \alpha(t)p(t) + \varepsilon',$$

$$p(t)\alpha(t) = p(s)\alpha(t) - \alpha(t)^2 \sum_{i \geq m} (\frac{i}{n})^2 a_i t^{i-2m} + \varepsilon'' = p(s)\alpha(s) + \varepsilon''',$$

where $v(\varepsilon'), v(\varepsilon''), v(\varepsilon''') \geq 2v(\alpha) + m - 2n$. □

Example 1.3.8. Recall the family of contact transformations

$$\left\{ \begin{array}{lcl} x' & = & x + \frac{k}{k-1}ap^{k-1} \\ y' & = & x + ap^k \\ p' & = & p \end{array} \right. \qquad (1.3.18)$$

from example 1.3.3. A member of this family takes (1.2.1) into the plane curve parametrized by $x = s^n, y = y(s)+\beta(s)-p(s)\alpha(s)+O(t^{2v(\alpha)+m-2n}) = y(s) - \frac{a}{k-1}p^k + \varepsilon$, where $v(\varepsilon) > v(p^k)$. Hence these transformations allow us to eliminate the coeficients a_k, $k \in v(p^k)$ of the parametrization. In a similar fashion, the transformations of example 1.3.4 allows us to eliminate coefficients of the type a_i, $i = v(x^i y^j p^k)$, $k \geq 2$ or $k \geq 1$ and $ij \neq 0$.

1.4 Examples

Example 1.4.1. If m odd all plane curves topologicaly equivalent to $y^2 = x^m$ are analyticaly equivalent to $y^2 = x^m$ (cf. [23]). Hence all Legendrian curves with generical plane projection $y^2 = x^m$ are contact equivalent to the conormal of $y^2 = x^m$.

Example 1.4.2. Let m, s, ϵ be positive integers. Assume that $m = 3s + \epsilon$, $1 \leq \epsilon \leq 2$. Let $C_{3,m,\nu}$ be the plane curve parametrized by

$$x = t^3, \quad y = t^m + t^{m+3\nu+\epsilon-3}.$$

By [23] a plane curve topologically equivalent to $y^3 = x^m$ is analyticaly equivalent to $y^3 = x^m$ or to one of the curves $C_{3,m,\nu}$, $1 \leq \nu \leq s-1$. The infinitesimal contact transformation

$$(x, y, p) \mapsto (x - 2p, y + p^2, p)$$

takes the plane curve $C_{3,m,s-1}$ into the plane curve C' parametrized by

$$3x = 3t^3 - mt^{m-3} - \cdots, \quad y = t^m.$$

By Lemma 1.3.7, the curve C' admits a parametrization of the type $x = s^3$, $y = s^m + \delta$, where $v(\delta) \geq m + 3s + \epsilon - 6$. By [23], the curve C' is analyticaly equivalent to the plane curve $y^3 = x^m$.

The semigroup of the conormal of the plane curve $y^3 = x^m$ equals $\Gamma_{3,m,0} = \langle 3, m-3 \rangle$. The semigroup of the conormal of the curve $C_{3,m,\nu}$ equals $\Gamma_{3,m,\nu} = \langle 3, m-3, m+3\nu+\epsilon \rangle$, $1 \leq \nu \leq s-1$. The map from $\{0, 1, \ldots, s-2\}$ into $\mathcal{P}(\mathbf{N})$ that takes ν into $\Gamma_{3,m,\nu}$ is injective. Hence there are $s-1$ analytic equivalence classes of plane curves topologicaly equivalent to $y^3 = x^m$ and $s-2$ equivalence contact classes of Legendrian curves with generical plane projection $y^3 = x^m$. In this case the semigroup of a curve is an analytic invariant that classifies the contact equivalence classes of Legendrian curves. We will see that in the general case there are no discrete invariants that can classify the contact equivalence classes of Legendrian curves.

Given a plane curve

$$x = t^3, \quad y = t^m + \sum_{i \geq m+\epsilon} a_i t^i, \tag{1.4.1}$$

the semigroup of the conormal of (1.4.1) equals $\Gamma_{3,m,1}$ if and only if $a_{m+\epsilon} \neq 0$. It is therefore natural to call $\Gamma(3,m) := \Gamma_{3,m,1}$ the generic semigroup of the family of Legendrian curves with generic plane projection $y^3 = x^m$.

1.5 The generic semigroup of an equisingularity class of irreducible Legendrian curves

We will associate to a pair (n,m) such that $m \geq 2n+1$ and $(m,n) = 1$ a semigroup $\Gamma(n,m)$. Let $\langle k_1, \ldots, k_r \rangle$ be the submonoid of $(N,+)$ generated by $k_1, \ldots, k_r$. Let c be the conductor of the semigroup of the plane curve (1.2.1). Set $\Gamma_c = \langle n \rangle \cup \{c, c+1, ...\}$. We say that the trajectory of $k \geq c$ equals $\{k, k+1, ...\}$. Let us assume that we have defined Γ_j and the trajectory of j for some $j \in \langle n, m-n \rangle \setminus \Gamma_c$, $j \geq m$. Let i be the biggest element of $\langle n, m-n \rangle \setminus \Gamma_j$. Let $\sharp_i$ be the minimum of the cardinality of the set of monomials of $\mathbb{C}[x,y,p]$ of valuation i and the cardinality of $\{i, i+1, \ldots\} \setminus \Gamma_j$. Let ω_i be the $\sharp_i$-th element of $\{i, i+1, \ldots\} \setminus \Gamma_j$. We call *trajectory* of i to the set $\tau_i = \{i, i+1, \ldots, \omega_i\} \setminus \langle n \rangle$. Set $\Gamma_i = \tau_i \bigcup \Gamma_j$. Set $\Gamma(n,m) = \Gamma_{m-n}$. The main purpose of this section is to prove theorem 1.5.2. Let us show that

$$\omega_i \leq i + n - 2. \tag{1.5.1}$$

If $\omega_i \geq i+n-1$, $\Gamma_i \supset \{i, \ldots, i+n-1\}$. Hence $\Gamma_i \supset \{i, i+1, \ldots\}$ and $i \geq c$. Therefore (1.5.1) holds.

Let $X = t^n$, $Y = \sum_{i\geq 0} a_{m+i} t^{m+i}$, $P = \sum_{i \geq 0} (\mu + i) a_{m+i} t^{m-n+i}$ be power series with coefficients in the ring $\mathbb{Z}[a_m, \ldots, a_{c-1}, \mu]$. Given $J = (i,j,l) \in \mathbb{N}^3$, set $v(J) = v(x^i y^j p^l)$. Let $\mathcal{N} = \{J \in \mathbb{N}^3 : j + l \geq 1$ and $v(J) \leq c - 1\}$. Let $\Upsilon = (\Upsilon_{J,k})$, $J \in \mathcal{N}$, $m \leq k \leq c-1$ be the matrix such that

$$X^i Y^j P^l \equiv \sum_{k=m}^{c-1} \Upsilon_{J,k} t^k \qquad (mod\ (t^c)). \tag{1.5.2}$$

Since $\partial Y/\partial\mu = 0$ and $X\partial P/\partial\mu = Y$,

$$\frac{\partial X^i Y^j P^l}{\partial \mu} = l X^{i-1} Y^{j+1} P^{l-1} \qquad \text{and} \qquad \frac{\partial \Upsilon_{J,k}}{\partial \mu} = l \Upsilon_{\partial J,k}, \tag{1.5.3}$$

where $\partial(i+1, j, l+1) = (i, j+1, l)$. Moreover,

$$\Upsilon_{J,k} = \sum_{\alpha \in A(k)} \sum_{\gamma \in G(\alpha,l)} \frac{j!\ l!}{(\alpha-\gamma)!\gamma!}\, a^\alpha \mu^\gamma, \tag{1.5.4}$$

where $A(k) = \{\alpha = (\alpha_m, ..., \alpha_{c-1}) : |\alpha| = j+l$ and $\sum_{s=m}^{c-1} s\alpha_s = k-(i-l)n\}$, $G(\alpha, l) = \{\gamma : |\gamma| = l$ and $0 \leq \gamma \leq \alpha\}$ and $\mu^\gamma = \prod_{s=m}^{c-1}(\mu - m + s)^{\gamma_s}$. Let us prove (1.5.4). We can assume that $i = l$. Since $G(\alpha, N) = \{\alpha\}$ and $X^N P^N = \sum_{k\geq 0} t^k \sum_{\alpha\in A(k)} (N!/\alpha!)\mu^\alpha a^\alpha$, (1.5.4) holds for $J = (N, 0, N)$. Let us show by induction in j that (1.5.4) holds when $j + l = N$. Set $e_s = (\delta_{s,r})$, $0 \leq s, r \leq N$. Given $\gamma \in G(\alpha, l-1)$, set $\gamma_{(s)} = \gamma + e_s$. Set $\Delta_s^\gamma = 1$ if $\gamma_{(s)} \leq \alpha$. Otherwise, set $\Delta_s^\gamma = 0$. Since

$$\begin{aligned}
\frac{1}{l}\sum_{\gamma\in G(\alpha,l)} \frac{j!l!}{(\alpha-\gamma)!\gamma!}\frac{\partial\mu^\gamma}{\partial\mu} &= \sum_{\gamma\in G(\alpha,l-1)} \sum_{s=m}^{c-1} \frac{j!(l-1)!}{(\alpha-\gamma_{(s)})!\gamma_{(s)}!}(\gamma_s+1)\Delta_s^\gamma \mu^\gamma \\
&= \sum_{\gamma\in G(\alpha,l-1)} \frac{j!(l-1)!}{(\alpha-\gamma)!\gamma!}\mu^\gamma \sum_{s=m}^{c-1}(\alpha_s - \gamma_s) \\
&= \sum_{\gamma\in G(\alpha,l-1)} \frac{(j+1)!(l-1)!}{(\alpha-\gamma)!\gamma!}\mu^\gamma,
\end{aligned}$$

the induction step follows from (1.5.3). We will consider in the polynomial ring $\mathbb{C}[a_m, \ldots, a_{c-1}]$ the order $a^\alpha < a^\beta$ if there is an integer q such that $\alpha_q < \beta_q$ and $\alpha_i = \beta_i$ for $i \geq q+1$. Set $\omega(P) = \sup\{i : a_i$ occurs in $P\}$.

Lemma 1.5.1. *Let $M, N, q \in \mathbb{Z}$ such that $0 \leq M \leq N$ and $q + N \geq 0$. If $\lambda = (\lambda_{l,k})$, where $M \leq l \leq N$, $k \geq 0$, $\lambda_{l,k} = \Upsilon_{J,k}$ and $J = (q+l, N-l, l)$, the minors of λ with $N - M + 1$ columns different from zero do not vanish at $\mu = m$.*

Proof. . One can assume that $q = 0$. When we multiply the left-hand side of (1.5.2) by P the coefficients of Υ are shifted and multiplied by an invertible matrix. Hence one can assume that $M = 0$. Set $Z = (Z_{j,k})$, where $Z_{j,k} = \binom{j}{k}\mu^{j-k}$, $0 \leq j, k \leq N$. Notice that Z is lower diagonal, $\det(Z) = 1$ and

$$\frac{\partial Z_{j,k}}{\partial \mu} \;=\; jZ_{j-1,k} \;=\; (k+1)Z_{j,k+1}. \tag{1.5.5}$$

Let us show that

$$Z^{-1}\lambda = \lambda|_{\mu=0}. \tag{1.5.6}$$

Since $\lambda_{N,k}$ is a polynomial of degree N in the variable μ with coefficients in the ring $\mathbb{Z}[a_m, \ldots, a_{c-1}]$, there are polynomials $\mathcal{Z}_{i,k} \in \mathbb{Q}[a_m, \ldots, a_{c-1}]$ such that $\lambda_{N,k} = \sum_{i=0}^{N} \binom{N}{i} \mathcal{Z}_{i,k}\mu^{N-i}$. Set $\mathcal{Z} = (\mathcal{Z}_{i,k})$, $0 \leq i \leq N$, $0 \leq k \leq c-1$. Since $Z|_{\mu=0} = Id$, it is enough to show that $Z\mathcal{Z} = \lambda$. By construction,

$$\lambda_{j,k} = \sum_{i=0}^{N} Z_{j,i}\mathcal{Z}_{i,k} \tag{1.5.7}$$

when $j = N$. By (1.5.3) and (1.5.5) statement (1.5.7) holds for all j. Remark that

$$\lambda_{l,v(J)+k}|_{\mu=0} = 0 \qquad \text{if and only if} \qquad k < l. \tag{1.5.8}$$

Let $\theta_{l,k}$ be the leading monomial of $\lambda_{l,k}$. When $k \geq l$,

$$\theta_{l,v(J)+k} = a_m^{N-1}a_{m+k} \qquad \text{if} \qquad l = 0, \tag{1.5.9}$$

$$\theta_{l,v(J)+k} = a_m^{N-l}a_{m+1}^{l-1}a_{m+k-l+1} \qquad \text{if} \qquad l \geq 1. \tag{1.5.10}$$

Let us prove (1.5.10). Set $\alpha_0 = j$, $\alpha_1 = l - 1$, $\alpha_{k-l+1} = 1$ and $\alpha_s = 0$ otherwise. By (1.5.4), $\alpha \in A(k)$ and there is one and only one $\gamma \in G(\alpha, j)$ such that $\gamma_0 = 0$, the tuple $\overline{\alpha}$ given by $\overline{\alpha}_0 = 0$ and $\overline{\alpha}_i = \alpha_i$ if $i \neq 0$. Since

$$\sum_{\gamma \in G(\alpha,l)} \frac{j!l!\mu^{\gamma}}{(\alpha-\gamma)!\gamma!} \equiv \frac{j!l!\mu^{\overline{\alpha}}}{(\alpha-\overline{\alpha})!\overline{\alpha}!} \equiv l \prod_{s=0}^{c-m-1} s^{\overline{\alpha}_s} = (k-l+1)l \mod \mu,$$

the coefficient of $a_m^{N-l}a_{m+1}^{l-1}a_{k-l+1}$ does not vanish. By (1.5.4), $\alpha_{k-l+r} \neq 0$ for some $r > 1$ implies that $\gamma_0 > 0$ for all $\gamma \in G(\alpha, l)$. Hence (1.5.10) holds. Let λ' be the square submatrix of λ with columns $g(i) + Nm$, $0 \leq g(0) < \cdots < g(N)$. By (1.5.6), $\det(\lambda'|_{\mu=0}) = \det(Z^{-1}\lambda') = \det(Z)^{-1}\det\lambda' = \det\lambda'$.

Hence $\det\lambda'$ does not depend on μ and $\det(\lambda'|_{\mu=m}) = \det(\lambda'|_{\mu=0})$. Set $\det(\lambda') = \sum_\pi sgn(\pi)\lambda_\pi$, where $\lambda_\pi = \prod_{i=0}^N \lambda'_{i,\pi(i)}$. If $\lambda_\pi \neq 0$, let θ_π be the leading monomial of λ_π.

Let ε be the following permutation of $\{0,\dots,N\}$. Assume that ε is defined for $0 \leq i \leq l-1$. Let p_l and q_l be respectively the maximum and the minimum of $\{0,\dots,N\}\backslash\varepsilon(\{0,\dots,l-1\})$. If $\lambda_{l+1,q_l} = 0$, set $\varepsilon(l) = q_l$. Otherwise, set $\varepsilon(l) = p_l$. Let us show that (1.5.8) implies that $\lambda_\varepsilon \neq 0$. It is enough to show that $\lambda_{i,q_i} \neq 0$ for all i. Since $g(0) \geq 0$, $\lambda_{0,q_0} \neq 0$. Assume that $l \geq 1$ and $\lambda_{i,q_i} \neq 0$ for $0 \leq i \leq l-1$. Hence $g(q_{l-1}) \geq l-1$. If $\lambda_{l,q_{l-1}} \neq 0$ then $\lambda_{l,q_l} \neq 0$. If $\lambda_{l,q_{l-1}} = 0$ then $\varepsilon(l-1) = q_{l-1}$. Therefore $g(q_l) = g(q_{l-1}+1) \geq g(q_{l-1})+1 \geq l$ and $\lambda_{l,q_l} \neq 0$.

Let us show that θ_ε is the leading monomial of $\det(\lambda'|_{\mu=0})$. Let π be a permutation of $\{1,\dots,N\}$. Assume that $\pi(i) = \varepsilon(i)$ if $0 \leq i \leq l-1$ and $\pi(l) \neq \varepsilon(l)$. If $\lambda_{l,q_{l-1}} = 0$ then $\pi(l) \neq q_l$ and $\lambda_\pi = 0$. If $\lambda_{l,q_{l-1}} \neq 0$ then $\pi(l) \neq p_l$ and $\omega(\prod_{i=l}^N \lambda_{i,\pi(i)}) < \omega(\prod_{i=l}^N \lambda_{i,\varepsilon(i)})$. Therefore $\lambda_\pi < \lambda_\varepsilon$. □The semigroup of the legendrian curve (1.2.2) only depends on $(a_m,\dots,a_{c-1})$. We will denote it by $\Gamma_{(a_m,\dots,a_{c-1})}$.

Theorem 1.5.2. *There is a dense Zariski open subset U of $\mathbb{C}^{c-m}$ such that if $(a_m,\dots,a_{c-1}) \in U$, $\Gamma_{(a_m,\dots,a_{c-1})} = \Gamma(n,m)$.*

Proof. . Since U is defined by the non vanishing of several determinants, it is enough to show that $U \neq \emptyset$. Let $j \in \langle n, m-n\rangle$, $j \geq m$. Set $q = \sharp(\tau_j)$. Assume that we associate to j a family of triples $I_1,\dots,I_q \in \mathcal{N}$ such that $v(I_s) \geq j$, $1 \leq s \leq q$, and if E is the linear subspace of $\mathbb{C}[a_m,\dots,a_{c-1}]\{t\}$ spanned by $\Upsilon_{I_s,k}|_{\mu=m}$, $1 \leq s \leq q$, $v(E) = \tau_j \cup \{\infty\}$. Let i be the biggest element of $\langle n, m-n\rangle\backslash\Gamma_j$. Assume that $\tau_i \cap \tau_j \neq \emptyset$. Hence τ_i contains τ_j. Since $v(E) = \tau_j \cup \{\infty\}$ and $\sharp(\tau_j) = q$, the determinant D' of the matrix $(\Upsilon_{I_s,k})$, $1 \leq s \leq q$, $k \in \tau_j$, does not vanish at $\mu = m$. In order to prove the theorem it is enough to show that there are $I_{q+1},\dots,I_{q+\sharp_i} \in \mathcal{N}$ such that $v(I_s) = i$, $q+1 \leq s \leq q+\sharp_i$, and the determinant D of the matrix $(\Upsilon_{I_s,k})$, $1 \leq s \leq q+\sharp_i$, $k \in \tau_i$, does not vanish at $\mu = m$. Set $I_{q+s+1} = (M-s, s, N-s)$, $M \leq s \leq N$, where $i = v(x^M p^N)$. By (1.5.8), (1.5.9) and (1.5.10),

$$g(\Upsilon_{I_s,k}) < g(\Upsilon_{I_r,k}) \qquad \text{if } \; k \geq i \; \text{ and } \; s \leq q < r. \tag{1.5.11}$$

Set $\lambda' = (\Upsilon_{I_s,k})$, $q+1 \leq s \leq q+\sharp_i$, $k \in \tau_i \backslash \tau_j$. By lemma 1.5.1, $\det(\lambda'|_{\mu=m}) \neq 0$. Set $\Upsilon_\varepsilon = \prod_{s=1}^{q+\sharp_i} \Upsilon_{I_s,\varepsilon(i)}$ for each bijection $\varepsilon : \{1,\ldots,q+\sharp_i\} \to \tau_i$. By (1.5.11), $g(\Upsilon_\varepsilon) < g(D'\lambda'|_{\mu=m})$ if $\varepsilon(\{q+1,\ldots,q+\sharp_i\}) \neq \tau_i \setminus \tau_j$. Since

$$D'\lambda'|_{\mu=m} = \sum_{\varepsilon(\{q+1,\ldots,q+\sharp_i\})=\tau_i\backslash\tau_j} sign(\varepsilon)\Upsilon_\varepsilon,$$

the product of the leading monomials of $D'|_{\mu=m}$ and $\lambda'|_{\mu=m}$ is the leading monomial of $D|_{\mu=m}$. $\square$

1.6 The moduli

Set $s = s(n,m) = \inf(\Gamma(n,m)\backslash\langle n, m-n\rangle)$. We say that (1.2.1) is in *Legendrian short form* if $a_m = 1$ and if $a_i = 0$ for $i \in \Gamma(n,m)$, $i \notin \{m, s(n,m)\}$. If $n = 2$ or if $n = 3$ and $m \in \{7,8\}$, $\Gamma(n,m) = \langle n, m-n\rangle \supset \{m,\ldots\}$ and $x = t^n$, $y = t^m$ is the only curve in Legendrian normal form such that the semigroup of its conormal equals $\Gamma(n,m)$. If $n = 3$ and $m \geq 10$ or if $n \geq 4$, $\langle n, n-m\rangle \not\supset \{m,\ldots,m+n-1\}$ and $s(m,n) \in \{m,\ldots,m+n-1\}$.

Lemma 1.6.1. *If* (1.2.1) *is in Legendrian normal form,* $\Gamma(n,m) \neq \langle n, m-n\rangle$ *and the semigroup of the conormal of* (1.2.1) *equals* $\Gamma(n,m)$*,* $a_{s(n,m)} \neq 0$*.*

Proof. . Each $f \in \mathbb{C}\{x,y,p\}$ is congruent to a linear combination of the series

$$y,\ nxp - my,\ x^i,\ p^j, \qquad v(x^i), v(p^j) \leq s \tag{1.6.1}$$

modulo (t^s). Since the series (1.6.1) have different valuations, one of these series must have valuation s, $s \in \Gamma(n,m) \setminus \langle n, m-n\rangle$ and $nxp - my = sa_st^s + \cdots$, $a_s \neq 0$. $\square$

Let $\mathcal{X}_{n,m}$ denote the set of plane curves (1.2.1) such that (1.2.1) is in Legendrian normal form and the semigroup of the conormal of (1.2.1) equals $\Gamma(n,m)$. Let W_n be the group of n-roots of unity. There is an action of W_n on $\mathcal{X}_{n,m}$ that takes (1.2.1) into $x = t^n$, $y = \sum_{i\geq m} \theta^{i-m} a_i t^i$, for each $\theta \in W_n$. The quotient $\mathcal{X}_{n,m}/W_n$ is an orbifold of dimension equal to the cardinality of the set $\{m,...\}\backslash(\Gamma(n,m) \setminus \{s(n,m)\})$.

Theorem 1.6.2. *The set of isomorphism classes of generic Legendrian curves with equisingularity type* (n,m) *is isomorphic to* $\mathcal{X}_{n,m}/W_n$*.*

Proof. . Let Λ be a germ of an irreducible Legendrian curve. There is a Legendrian map π such that $\pi(\Lambda)$ has maximal contact with the curve $\{y = 0\}$ and the tangent cone of the conormal of Λ equals $\{y = p = 0\}$. Moreover, we can assume that $\pi(\Lambda)$ has a parametrization of type (1.2.1), with $a_m = 1$. Assume that there is $i \in \Gamma(m, n)$ such that $i \neq m, s(m, n)$ and $a_i \neq 0$. Let k be the smallest integer i verifying the previous condition. By lemmata 1.3.6 and 1.3.7 there are $a \in \mathbb{C}\{x, y, p\}$ and $\Phi \in \mathcal{J}$ such that $\imath^* a = a_k t^k + \cdots$ and Φ takes (1.2.1) into the plane curve $x = s^n$, $y = y(s) - a(s) + \delta$, where $v(\delta) \geq 2v(a) + m - 2n$. Hence we can assume that $a_i = 0$ if $i \in \Gamma(m, n)$, $i \neq m, s(m, n)$, and i is smaller then the conductor σ of the plane curve (1.2.1). There is a germ of diffeomorfism ϕ of the plane that takes the curve (1.2.1) into the curve $x = t^n$, $y = \sum_{i=m}^{\sigma-1} a_i t^i$ (cf. [23]). This curve is in Legendrian normal form. The diffeomorphism ϕ induces an element of $\mathcal{G}$.
Let Φ be a contact transformation such $\Phi(\mathcal{X}) = \mathcal{X}$. Since the tangent cone of the conormal of an element of $\mathcal{X}$ equals $\{y = p = 0\}$, $\Phi \in \mathcal{G}$. By theorem 1.3.1, $\Phi = \Psi\Psi_{\lambda,\mu}$, where $\Psi \in \mathcal{J}$ and $\lambda, \mu \in \mathbb{C}^*$. Moreover, $\lambda \in W_n$ and $\mu = \lambda^m$. By lemmata 1.3.6 and 1.3.7, $\Psi = Id$. $\square$

Chapter 2

Limits of tangents of quasi-ordinary hypersurfaces

We compute explicitly the limits of tangents of a quasi-ordinary singularity in terms of its special monomials. We show that the set of limits of tangents of Y is essentially a topological invariant of Y.

2.1 Introduction

The study of the limits of tangents of a complex hypersurface singularity was mainly developped by Le Dung Trang and Bernard Teissier (see [13] and its bibliography). Chunsheng Ban [2] computed the set of limits of tangents Λ of a quasi-ordinary singularity Y when Y has only one very special monomial (see Definition 2.1.3).

The main achievement of this chapter is the explicit computation of the limits of tangents of an arbitrary quasi-ordinary hypersurface singularity (see Theorems 2.2.17, 2.2.18 and 2.2.19). Corollaries 2.2.20, 2.2.21 and 2.2.22 show that the set of limits of tangents of Y comes quite close to being a topological invariant of Y. Corollary 2.2.21 shows that Λ is a topological invariant of Y when the tangent cone of Y is a hyperplane. Corollary 2.2.23 shows that the triviality of the set of limits of tangents of Y is a topological invariant of Y.

Let X be a complex analytic manifold. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Let Γ be a germ of a Lagrangean variety of T^*X at a point α. We say that Γ is in *generic position* if $\Gamma \cap \pi^{-1}(\pi(\alpha)) = \mathbb{C}\alpha$. Let Y be a hypersurface singularity of X. Let Γ be the conormal $T^*_Y X$ of Y. The Lagrangean variety Γ is in generic position if and only if Y is the germ of an hypersurface with trivial set of limits of tangents.

Let $\mathcal{M}$ be an holonomic $\mathcal{D}_X$-module. The characteristic variety of $\mathcal{M}$ is a Lagrangean variety of T^*X. The characteristic varieties in generic position have a central role in $\mathcal{D}$-module theory (cf. Corollary 1.6.4 and Theorem 5.11 of [10] and Corollary 3.12 of [16]). It would be quite interesting to have good characterizations of the hypersurface singularities with trivial set of limits of tangents. Corollary 2.2.23 is a first step in this direction.

After finishing this chapter, two questions arise naturally:

Let Y be an hypersurface singularity such that its tangent cone is an hyperplane. Is the set of limits of tangents of Y a topological invariant of Y?

Is the triviality of the set of limits of tangents of an hypersurface a topological invariant of the hypersurface?

Let $p : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection that takes $(x, y) = (x_1, \ldots, x_n, y)$ into x. Let Y be the germ of a hypersurface of $\mathbb{C}^{n+1}$ defined by $f \in \mathbb{C}\{x_1, \ldots, x_n, y\}$. Let W be the singular locus of Y. The set Z defined by the equations $f = \partial f/\partial y = 0$ is called the *apparent contour* of f relatively to the projection p. The set $\Delta = p(Z)$ is called the *discriminant* of f relatively to the projection p.

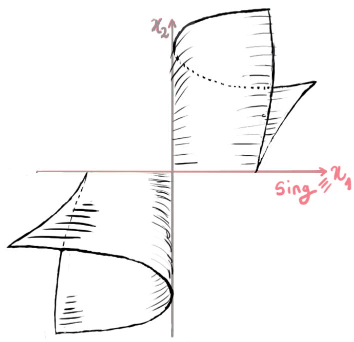
Example 2.1.1. The apparent contour consists of the singular points and of those points where the surface has a non-generic number of points with the same "shadow", or where the surface "turns" with regard to the projection axis. If $X = \{(x_1, x_2, y) : y^2 - x_1 x_2^3 = 0\}$, then

$$Sing(X) = \{(x_1, x_2, y) : f = \partial f/\partial x_1 = \partial f/\partial x_2 = \partial f/\partial y = 0\} = \{x_2 = y = 0\}.$$

Hence the apparent contour with regard to the projection $(x_1, x_2, y) \mapsto (x_1, x_2)$ is

$$\{(x_1, x_2, y) : f = \frac{\partial f}{\partial y} = 0\} = \{x_1 x_2 = y = 0\},$$

and the discriminant with regard to the projection is $\{(x_1, x_2) : x_1 x_2 = 0\}$.



Near $q \in Y \setminus Z$ there is one and only one function $\varphi \in \mathcal{O}_{\mathbb{C}^{n+1},q}$ such that $f(x, \varphi(x)) = 0$. The function f defines implicitly y as a function of x. Moreover,

$$\frac{\partial y}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} = -\frac{\partial f/\partial x_i}{\partial f/\partial y} \text{ on } Y \setminus Z. \tag{2.1.1}$$

Let $\theta = \xi_1 dx_1 + \ldots \xi_n dx_n + \eta dy$ be the canonical 1-form of the cotangent bundle $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{C}_{n+1}$. An element of the projective cotangent bundle $\mathbb{P}^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{P}_n$ i s represented by the coordinates

$$(x_1, \ldots, x_n, y; \xi_1 : \cdots : \xi_n : \eta).$$

We will consider in the open set $\{\eta \neq 0\}$ the chart

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n),$$

where $p_i = -\xi_i/\eta, 1 \leq i \leq n$. Let Γ_0 be the graph of the map from $Y \setminus W$ into $\mathbb{P}_n$ defined by

$$(x, y) \mapsto \left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y}\right).$$

Let Γ be the smallest closed analytic subset of $\mathbb{P}^*\mathbb{C}^{n+1}$ that contains Γ_0. The analytic set Γ is a Legendrian subvariety of the contact manifold $\mathbb{P}^*\mathbb{C}^{n+1}$. The projective algebraic set $\Lambda = \Gamma \cap \pi^{-1}(0)$ is called the *set of limits of tangents* of Y.

Remark 2.1.2. It follows from (2.1.1) that

$$\left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial y}{\partial x_1} : \cdots : -\frac{\partial y}{\partial x_n} : 1\right) \text{ on } Y \setminus Z.$$

Let $c_1, \ldots, c_n$ be positive integers. We will denote by $\mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$ the $\mathbb{C}\{x_1, \ldots, x_n\}$ algebra given by the immersion from $\mathbb{C}\{x_1, \ldots, x_n\}$ into $\mathbb{C}\{t_1, \ldots, t_n\}$ that takes x_i into $t_i^{c_i}, 1 \leq i \leq n$. We set $x_i^{1/c_i} = t_i$. Let $a_1, \ldots, a_n$ be positive rationals. Set $a_i = b_i/c_i, 1 \leq i \leq n$, where $(b_i, c_i) = 1$. Given a ramified monomial $M = x_1^{a_1} \cdots x_n^{a_n} = t_1^{b_1} \cdots t_n^{b_n}$ we set $\mathcal{O}(M) = \mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$.
Let Y be a germ at the origin of a complex hypersurface of $\mathbb{C}^{n+1}$. We say that Y is a quasi-ordinary singularity if Δ is a divisor with normal crossings. We will assume that there is $l \leq m$ such that $\Delta = \{x_1 \cdots x_l = 0\}$.
If Y is an irreducible quasi-ordinary singularity there are ramified monomials $N_0, N_1, \ldots, N_m, g_i \in \mathcal{O}(N_i), 0 \leq i \leq m$, such that $N_0 = 1$, N_{i-1} divides N_i in the ring $\mathcal{O}(N_i)$, g_i is a unit of $\mathcal{O}(N_i), 1 \leq i \leq m$, g_0 vanishes at the origin and the map $x \mapsto (x, \varphi(x))$ is a parametrization of Y near the origin, where

$$\varphi = g_0 + N_1 g_1 + \ldots + N_m g_m. \tag{2.1.2}$$

Replacing y by $y - g_0$, we can assume that $g_0 = 0$. The monomials $N_i, 1 \leq i \leq m$, are unique and determine the topology of Y (see [15]). They are called the *special monomials* of f. We set $\tilde{\mathcal{O}} = \mathcal{O}(N_m)$.

Definition 2.1.3. We say that a special monomial N_i, $1 \leq i \leq m$, is *very special* if $\{N_i = 0\} \neq \{N_{i-1} = 0\}$.

Let $M_1, \ldots, M_g$ be the very special monomials of f, where $M_k = N_{n_k}, 1 = n_1 < n_2 < \ldots < n_g, 1 \leq k \leq g$. Set $M_0 = 1, n_{g+1} = n_g + 1$. There are units f_i of $\mathcal{O}(N_{n_{i+1}-1})$, $1 \leq i \leq g$, such that

$$\varphi = M_1 f_1 + \ldots + M_g f_g. \tag{2.1.3}$$

Example 2.1.4. If $f(x_1, x_2, y) = y^2 - x_1 x_2^3$, the ramified series $y = x_1^{1/2} x_2^{3/2}$ is a root of f. The ramification order is 2 and $\varphi_1 = H(x_1^{1/2}, x_2^{1/2})$ with $H(x_1, x_2) = x_1 x_2^3$. The conjugates of φ_1 are the series

$$\varphi_{ij} = H(\varepsilon_i x_1^{1/2}, \varepsilon_j x_2^{1/2}), \varepsilon_i, \varepsilon_j \in \{-1, 1\}.$$

That is:

$$\begin{aligned}
&\varphi_{1,1} := \varphi_1,\\
&\varphi_{1,-1} = H(x_1^{1/2}, -x_2^{1/2}) = -x_1^{1/2} x_2^{3/2} := \varphi_2,\\
&\varphi_{-1,1} = H(-x_1^{1/2}, x_2^{1/2}) = -x_1^{1/2} x_2^{3/2} := \varphi_2,\\
&\varphi_{-1,-1} = H(-x_1^{1/2}, -x_2^{1/2}) = x_1^{1/2} x_2^{3/2} := \varphi_1.
\end{aligned}$$

Therefore $f(x_1, x_2, y) = (y - \varphi_1(x_1, x_2))(y - \varphi_2(x_1, x_2))$.

Example 2.1.5. Let X be defined by

$$y = x_1^{2/5} + x_1^{1/2} + x_1^{3/5} + x_1^{6/10} x_2^{1/2} + x_1^3 x_2^7.$$

The special monomials of X are

$$N_1 = x_1^{2/5}, N_2 = x_1^{1/2}, N_3 = x_1^{6/10} x_2^{1/2}.$$

The very special monomials of X are

$$M_1 = x_1^{2/5}, M_2 = x_1^{6/10} x_2^{1/2}.$$

Furthermore, we have

$$\mathcal{O}(N_1) = \mathbb{C}\{x_1^{1/5}\}, \mathcal{O}(N_2) = \mathbb{C}\{x_1^{1/10}\}$$

and

$$\tilde{\mathcal{O}} = \mathcal{O}(N_3) = \mathbb{C}\{x_1^{1/10}, x_2^{1/2}\}$$

2.2 Limits of tangents

After renaming the variables x_i there are integers $m_k, 1 \leq k \leq g+1$, and positive rational numbers $a_{kij}, 1 \leq k \leq g, 1 \leq i \leq k, 1 \leq j \leq m_k$ such that

$$M_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} x_{ij}^{a_{kij}}, \qquad 1 \leq k \leq g. \tag{2.2.1}$$

The canonical 1-form of $\mathbb{P}^*\mathbb{C}^{n+1}$ becomes

$$\theta = \sum_{i=1}^{g+1} \sum_{j=1}^{m_i} \xi_{ij} dx_{ij}. \tag{2.2.2}$$

We set $p_{ij} = -\xi_{ij}/\eta, 1 \leq i \leq g+1, 1 \leq j \leq m_i$. Remark that

$$\frac{\partial y}{\partial x_{ij}} = a_{iij} \frac{M_i}{x_{ij}} \sigma_{ij}, \tag{2.2.3}$$

where σ_{ij} is a unit of $\tilde{\mathcal{O}}$.

Example 2.2.1. In this notation,

$$y = x_1^{2/5} + x_1^{1/2} + x_1^{6/10} x_2^{1/2}$$

becomes

$$y = x_{11}^{2/5} + x_{11}^{1/2} + x_{11}^{6/10} x_{21}^{1/2}$$

and we have

$$\frac{\partial f}{\partial x_{11}} = \frac{M_1}{x_{11}} \sigma_{11}, \quad \frac{\partial f}{\partial x_{21}} = \frac{M_2}{x_{21}} \sigma_{21}.$$

The following examples motivate a strategy for constructing Λ, by establishing an "upper bound" that depends (almost) exclusively on the signal of the sums of the exponents of the very special monomials.

Example 2.2.2. Let $y = x_1^{1/2} x_2^{3/2}$. The conormal verifies the equations

$$\begin{aligned} p_1 &= \frac{\partial y}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{3}{2}}, \\ p_2 &= \frac{\partial y}{\partial x_2} = \frac{1}{2} x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}. \end{aligned}$$

Setting $x = 0$ we obtain from squaring both sides of the second equation that $\Lambda \subset \{\xi_2 = 0\}$. We notice that this happens because the x_2 is raised to a power greater than 1. We can't conclude anything from the first equation.

Example 2.2.3. For a slightly trickier case, let $y = x_1^{1/3} x_2^{4/5}$.
Now none of the powers are larger than 1, but their sum is. We have

$$\begin{aligned} p_1 &= \frac{\partial y}{\partial x_1} = -\frac{1}{3} x_1^{-2/3} x_2^{4/5}, \\ p_2 &= \frac{\partial y}{\partial x_2} = -\frac{4}{5} x_1^{1/3} x_2^{-1/5}. \end{aligned}$$

The product doesn't seem to work:

$$p_1 p_2 = \frac{1}{3}\frac{4}{5} x_1^{-1/3} x_2^{3/5}.$$

But raising p_1, p_2 to adequate powers c_1, c_2, maybe we can ensure only positive powers for x_1, x_2 (from now on we'll write the monomials modulo products by non-zero constants). We have

$$p_1^{c_1} p_2^{c_2} = x_1^{-2/3c_1 + 1/3c_2} x_2^{4/5c_1 - 1/5c_2}.$$

Then it is enough to find a solution of the system of inequalities

$$\begin{cases} -2/3c_1 + 1/3c_2 & > & 0, \\ 4/5c_1 - 1/5c_2 & > & 0. \end{cases}$$

Setting $c_1 = 1$ we get $2 < c_2 < 4$. Taking $c_2 = 3$ we get:

$$p_1 p_2^3 = \frac{1}{3}(\frac{4}{5})^3 x_1^{1/3} x_2^{1/5}$$

Then at $x = 0$ we get $p_1 p_2^3 = 0$. Therefore the limit of tangents verifies $p_1 p_2 = 0$. It remains to be shown if this procedure can always be made to work, even with more than one special monomial.

Example 2.2.4. Still trickier: Take

$$y = x_{11}^{1/2} x_{12}^{3/2} + x_{11}^{1/2} x_{12}^{3/2} x_{21}^{1/3} x_{22}^{4/5}.$$

We have combined the two previous examples into a case with two special monomials. Can we apply both the previous methods independently? We have

$$p_{12} = \frac{\partial y}{\partial x_{12}} = -\frac{1}{2} x_{11}^{\frac{1}{2}} x_{12}^{\frac{1}{2}} \phi, \quad \phi(0) \neq 0.$$

Then, setting $x = 0$, we conclude that $\Lambda \subset \{\xi_{12} = 0\}$.

Furthermore, when we take derivatives on variables x_{2i} we eliminate the first monomial, and the exponents of the variables of the first monomial present on those derivatives are always positive. Hence

$$p_{21}p_{22}^3 = \frac{1}{3}(\frac{4}{5})^3 x_{21}^{1/3} x_{22}^{1/5}\phi, \quad \phi(0) = 0.$$

Therefore $p_{21}p_{22} = 0$ (or $\xi_{21}\xi_{22} = 0$) in Λ. So the two monomials can be handled independently.

Example 2.2.5. Now suppose that $\sum_i a_{11i} < 1$. For example, consider the case

$$y = x_1^{2/3} x_2^{1/5}.$$

Then

$$\begin{aligned} p_1 &= \frac{\partial y}{\partial x_1} = x_1^{-1/3} x_2^{1/5}, \\ p_2 &= \frac{\partial y}{\partial x_2} = x_1^{2/3} x_2^{-4/5}. \end{aligned}$$

and

$$p_1 p_2 = x_1^{1/3} x_2^{-3/5}.$$

We notice that if we raise p_1 to a larger power we can make the exponent of x_1 positive in $p_1^{c_1} p_2^{c_2}$. But we cannot make it arbitrarily large otherwise x_2 will have a negative power, and we want both to be positive. We have

$$p_1^{c_1} p_2^{c_2} = x_1^{-1/3c_1 + 2/3c_2} x_2^{1/5c_1 - 4/5c_2}$$

In particular,

$$p_1^3 p_2 = x_1^{-1/3} x_2^{-1/5}$$

Then

$$\xi_1^3 \xi_2 x_1^{1/3} x_2^{1/5} = \eta^4$$

Setting $x_1 = x_2 = 0$, we get $\eta = 0$ in Λ. It remains to be shown that this works in general.

Example 2.2.6. Suppose $\sum_i a_{11i} = 1$. For example,

$$y = a x_1^{1/2} x_2^{1/2} + x_1^{1/2} x_2^{1/2} x_3^{1/2}, \quad a \in \mathbb{C}^*.$$

Then

$$\begin{aligned} p_1 &= (1/2)x_1^{-1/2}x_2^{1/2}(a + x_3^{1/2}), \\ p_2 &= (1/2)x_1^{1/2}x_2^{-1/2}(a + x_3^{1/2}) \end{aligned}$$

and

$$p_1p_2 = (1/4)(a^2 + 2ax_3^{1/2} + x_3).$$

Hence,

$$\xi_1\xi_2 = \eta^2(1/4)(a^2 + 2ax_3^{1/2} + x_3).$$

Therefore

$$\Lambda \subset \{\xi_1\xi_2 = (a^2/4)\eta^2\}.$$

One can always find powers c_i such that the product of the $p_i^{c_i}$ in the first monomial verifies a homogeneous relation with η. We note that the cone we obtained depends not only on the special exponents but also on the coefficient a. Hence the cone is not a topological invariant.
The following theorems show that the previous constructions will work in general.

Theorem 2.2.7. *If* $\sum_{i=1}^{m_1} a_{11i} < 1$, $\Lambda \subset \{\eta = 0\}$.

Proof. Set $m = m_1$, $x_i = x_{1i}$ and $a_i = a_{11i}$, $1 \leq i \leq m$. Given positive integers $c_1, \ldots, c_m$, it follows from (2.2.3) that

$$\prod_{i=1}^{m} p_i^{c_i} = \prod_{i=1}^{m} x_i^{a_i \sum_{j=1}^{m} c_j - c_i}\phi, \tag{2.2.4}$$

for some unit ϕ of $\tilde{\mathcal{O}}$. By (2.1.3) and (2.2.3),

$$\phi(0) = f_1(0)^{\sum_{j=1}^{m} c_j} \prod_{j=1}^{m} a_j^{c_j}. \tag{2.2.5}$$

Hence

$$\eta^{\sum_{i=1}^{m} c_i} = \psi \prod_{i=1}^{m} \xi_i^{c_i} x_i^{c_i - a_i \sum_{j=1}^{m} c_j}, \tag{2.2.6}$$

for some unit ψ. If there are integers $c_1, \ldots, c_m$ such that the inequalities

$$a_k \textstyle\sum_{j=1}^{m} c_j < c_k, \quad 1 \leq k \leq m, \tag{2.2.7}$$

hold, the result follows from (2.2.6). Hence it is enough to show that the set Ω of the m-tuples of rational numbers $(c_1, \ldots, c_m)$ that verify the inequalities (2.2.7) is non-empty. We will recursively define positive rational numbers l_j, c_j, u_j such that

$$l_j < c_j < u_j, \tag{2.2.8}$$

j=1,...,m. Let c_1, l_1, u_1 be arbitrary positive rationals verifying $(2.2.8)_1$. Let $1 < s \leq m$. If l_i, c_i, u_i are defined for $i \leq s-1$, set

$$l_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{m} a_j}, \quad u_s = (a_s/a_{s-1})c_{s-1}. \tag{2.2.9}$$

Since $\sum_{j \geq s} a_j < 1$ and

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{m} a_j)} \left((1 - \sum_{j=s-1}^{m} a_j)c_{s-1} - a_{s-1} \sum_{j<s-1} c_j \right) \\ &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{m} a_j)} \left((1 - \sum_{j=s-1}^{m} a_j)(c_{s-1} - l_{s-1}) \right), \end{aligned}$$

it follows from $(2.2.8)_{s-1}$ that $l_s < u_s$. Let c_s be a rational number such that $l_s < c_s < u_s$. Hence $(2.2.8)_s$ holds for $s \leq m$.
Let us show that $(c_1, \ldots, c_m) \in \Omega$. Since $c_k < u_k$, then

$$c_k < \frac{a_k}{a_{k-1}} c_{k-1}, \text{ for } k \geq 2.$$

Then, for $j < k$,

$$c_k < \frac{a_k}{a_{k-1}} \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_k}{a_j} c_j.$$

Hence,

$$a_k c_j < a_j c_k, \text{ for } j > k. \tag{2.2.10}$$

Since $l_k < c_k$,

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_j c_k.$$

Hence, by (2.2.10),

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_k c_j.$$

Therefore $a_k \sum_{j=1}^{m} c_j < c_k$. □

Theorem 2.2.8. *Let $1 \le k \le g$. Let $I \subset \{1, \ldots, m_k\}$. Assume that one of the following three hypothesis is verified:*

1. $\sum_{j \in I} a_{kkj} > 1$;
2. $k = 1$, $\sum_{j \in I} a_{11j} = 1$ and $\sum_{j=1}^{m_1} a_{11j} > 1$;
3. $k \ge 2$ *and* $\sum_{j \in I} a_{kkj} = 1$.

Then $\Lambda \subset \{\prod_{j \in I} \xi_{kj} = 0\}$.

Proof. Case 1: We can assume that $I = \{1, \ldots, n\}$, where $1 \le n \le m_k$. Set $a_i = a_{kki}$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.2.3) that

$$\prod_{i=1}^{n} \xi_{ki}^{c_i} = \prod_{i=1}^{n} x_{ki}^{a_i \sum_{j=1}^{n} c_j - c_i} \eta^{\sum_{i=1}^{n} c_i} \varepsilon, \tag{2.2.11}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$. Hence it is enough to show that there are positive rational numbers $c_1, \ldots, c_n$ such that

$$a_k(\sum_{j=1}^{n} c_j) - c_k > 0, \qquad 1 \le k \le n. \tag{2.2.12}$$

We will recursively define $l_j, c_j, u_j \in]0, +\infty]$ such that $c_j, l_j \in \mathbb{Q}$,

$$l_j < c_j < u_j, \tag{2.2.13}$$

j=1,...,n, and $u_j \in \mathbb{Q}$ if and only if $\sum_{i=j}^{n} a_i < 1$. Choose c_1, l_1, u_1 verifying (2.2.13). Let $1 < s \le n-1$. Suppose that l_i, c_i, u_i are defined for $1 \le i \le s-1$. If $\sum_{j=s}^{n} a_j < 1$, set

$$l_s = (a_s/a_{s-1})c_{s-1}, \quad u_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{n} a_j}. \tag{2.2.14}$$

Since

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{n} a_j)} \left(a_{s-1} \sum_{j=1}^{s-2} c_j - c_{s-1}(1 - \sum_{j=s-1}^{n} a_j) \right) \\ &\le \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{n} a_j)} \left((1 - \sum_{j=s-1}^{n} a_j)(u_{s-1} - c_{s-1}) \right), \end{aligned}$$

it follows from $(2.2.13)_{s-1}$ that $l_s < u_s$.

If $\sum_{j=s}^{n} a_j \geq 1$, set l_s as above and $u_s = +\infty$.
We choose a rational number c_s such that $l_s < c_s < u_s$. Hence $(2.2.13)_s$ holds for $1 \leq s \leq n$.
Let us show that $c_1, \ldots, c_n$ verify (2.2.12). We will proceed by induction. First we will show that $c_1, \ldots, c_n$ verify $(2.2.12)_n$. Suppose that $a_n < 1$. Since $c_n < u_n$, we have that

$$c_n < \frac{a_n \sum_{j=1}^{n-1} c_j}{1 - a_n}.$$

Hence $a_n \sum_{j=1}^{n} c_j > c_n$. If $a_n \geq 1$, then

$$a_n \sum_{j=1}^{n} c_j \geq \sum_{j=1}^{n} c_j > c_n.$$

Hence $(2.2.12)_n$ is verified. Assume that $c_1, \ldots, c_n$ verify $(2.2.12)_k$, $2 \leq k \leq n$. Since $c_k > l_k$,

$$a_k \sum_{j=1}^{n} c_j > c_k > \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence $a_{k-1} \sum_{j=1}^{n} c_j > c_{k-1}$. Therefore $(c_1, \ldots, c_n)$ verify $(2.2.12)_{k-1}$.
Case 2: Set $a_j = a_{11j}$ and $x_j = x_{1j}$. We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_1$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.1.2) that

$$\prod_{i=1}^{n} \xi_i^{c_i} = \prod_{i=1}^{n} x_i^{a_i \sum_{j=1}^{n} c_j - c_i} \eta^{\sum_{i=1}^{n} c_i} \varepsilon, \tag{2.2.15}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. Hence it is enough to show that there are positive rational numbers $c_1, \ldots, c_n$, such that

$$a_k \sum_{j=1}^{n} c_j = c_k, \quad 1 \leq k \leq n. \tag{2.2.16}$$

We choose an arbitrary positive integer c_1. Let $1 < s \leq n$. If the c_i are defined for $i < s$, set

$$c_s = \frac{a_s}{a_{s-1}} c_{s-1}. \tag{2.2.17}$$

Let us show that $c_1, \ldots, c_n$ verify (2.2.16). We will proceed by induction in k. First let us show that $(2.2.16)_n$ holds.

Let $j < n-1$. By (2.2.17),

$$c_{n-1} = \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_{n-1}}{a_j} c_j. \tag{2.2.18}$$

By (2.2.17), and since $\sum_{j=1}^{n} a_j = 1$,

$$c_n = \frac{a_n}{a_{n-1}} c_{n-1} = \frac{c_{n-1}}{a_{n-1}}(1 - \sum_{j=1}^{n-1} a_j) = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} \frac{a_j}{a_{n-1}} c_{n-1}.$$

Hence, by (2.2.18)

$$c_n = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} c_j.$$

Therefore, $\sum_{j=1}^{n} c_j = c_{n-1}/a_{n-1}$. Hence by (2.2.17),

$$a_n \sum_{j=1}^{n} c_j = a_n \frac{c_{n-1}}{a_{n-1}} = c_n.$$

Therefore $(2.2.16)_n$ holds.
Assume $(2.2.16)_k$ holds, for $2 \leq k \leq n$. Then

$$a_k \sum_{j=1}^{n} c_j = c_k = \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence, $a_{k-1} \sum_{j=1}^{n} c_j = c_{k-1}$.
Case 3: We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_k$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.2.3) that

$$\prod_{\imath=1}^{n} \xi_{ki}^{c_i} = \left(\prod_{i=1}^{n} x_{ki}^{a_{kki}(\sum_{j=1}^{n} c_j) - c_i} \right) \eta^{\sum_{i=1}^{n} c_i} \varepsilon,$$

where $\varepsilon \in \tilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. We have reduced the problem to the case 2. $\square$

Theorem 2.2.9. *If $\sum_{k=1}^{m_1} a_{11j} = 1$, Λ is contained in a cone.*

Proof. Set $a_i = a_{11i}, i = 1, \ldots m_1$. Given positive integers $c_1, \ldots, c_{m_1}$, there is a unit ϕ of $\tilde{\mathcal{O}}$ such that

$$\prod_{i=1}^{m_1} \xi_i^{c_i} = (-1)^{\sum_{j=1}^{m_1} c_j} \phi \prod_{i=1}^{m_1} x_i^{\sum_{j=1}^{m_1} c_j a_i - c_i} \eta^{\sum_{j=1}^{m_1} c_j}. \tag{2.2.19}$$

By the proof of case 2 of Theorem 2.2.8, there is one and only one m_1-tuple of integers $c_1, \ldots, c_{m_1}$ such that $(c_1, \ldots, c_{m_1}) = (1)$, $a_i \sum_{j=1}^{m_1} c_j = c_i, 1 \le i \le m_1$, and Λ is contained in the cone defined by the equation

$$\prod_{i=1}^{m_1} \xi_i^{c_i} - (-1)^{\sum_{j=1}^{m_1} c_j} \phi(0) \eta^{\sum_{j=1}^{m_1} c_j} = 0, \tag{2.2.20}$$

where $\phi(0)$ is given by (2.2.5). $\square$

Remark 2.2.10. Set $D_\varepsilon^* = \{x \in \mathbb{C} : 0 < |x| < \varepsilon\}$, where $0 < \varepsilon << 1$. Set $\mu = \sum_{k=1}^{g+1} m_k$. Let $\sigma : \mathbb{C} \to \mathbb{C}^\mu$ be a weighted homogeneous curve parametrized by

$$\sigma(t) = (\varepsilon_{ki} t^{\alpha_{ki}})_{1 \le k \le g+1, 1 \le i \le m_k}.$$

Notice that the image of σ is contained in $\mathbb{C}^\mu \setminus \Delta$. Set $\theta_0(t) = 1$ and

$$\theta_{ki}(t) = \frac{\partial \varphi}{\partial x_{ki}}(\sigma(t), \varphi(\sigma(t))), \quad 1 \le k \le g+1, 1 \le i \le m_k,$$

for $t \in D_\varepsilon^*$. The curve σ induces a map from D_ε^* into Γ defined by

$$t \mapsto (\sigma(t), \varphi(\sigma(t)); \theta_{11}(t) : \cdots : \theta_{g+1,m_g+1}(t) : \theta_0(t)).$$

Let $\vartheta : D_\varepsilon^* \to \mathbb{P}^\mu$ be the map defined by

$$t \mapsto (\theta_{11}(t) : \cdots : \theta_{g+1,m_g+1}(t) : \theta_0(t)). \tag{2.2.21}$$

The limit when $t \to 0$ of $\vartheta(t)$ belongs to Λ. The functions θ_{ki} are ramified Laurent series of finite type on the variable t. Let h a be ramified Laurent series of finite type. If $h = 0$, we set $v(h) = \infty$. If $h \neq 0$, we set $v(h) = \alpha$, where α is the only rational number such that $\lim_{t \to 0} t^{-\alpha} h(t) \in \mathbb{C} \setminus \{0\}$. We call α the *valuation* of h. Notice that the limit of ϑ only depends on the functions θ_{ki}, θ_0 of minimal valuation. Moreover, the limit of ϑ only depends on the coefficients of the term of minimal valuation of each θ_{ij}, θ_0. Hence the limit of ϑ only depends on the coefficients of the very special monomials of f. We can assume that $m_{g+1} = 0$ and that there are $\lambda_k \in \mathbb{C} \setminus \{0\}, 1 \le k \le g$, such that

$$\varphi = \sum_{k=1}^{g} \lambda_k M_k. \tag{2.2.22}$$

Remark 2.2.11. Let L be a finite set. Set $\mathbb{C}^L = \{(x_a)_{a\in L} : x_a \in \mathbb{C}\}$. Let $\sum_{a\in L} \xi_a dx_a$ be the canonical 1-form of $T^*\mathbb{C}^L$. Let Λ be the subset of $\mathbb{P}_L$ defined by the equations

$$\prod_{a\in I} \xi_a = 0, \quad I \in \mathcal{I}, \tag{2.2.23}$$

where $\mathcal{I} \subset \mathcal{P}(L)$. Set $\mathcal{I}' = \{J \subset L : J \cap I \neq \emptyset \text{ for all } I \in \mathcal{I}\}$, $\mathcal{I}^* = \{J \in \mathcal{I}'$ such that there is no $K \in \mathcal{I}' : K \subset J, K \neq J\}$. The irreducible components of Λ are the linear projective sets $\Lambda_J, J \in \mathcal{I}^*$, where Λ_J is defined by the equations

$$\xi_a = 0, \qquad a \in J.$$

Example 2.2.12. Suppose that

$$y = x_{11}^{a_{111}} x_{12}^{a_{112}} + x_{11}^{a_{211}} x_{12}^{a_{212}} x_{21}^{a_{221}} x_{22}^{a_{222}},$$

with $a_{111} + a_{112} > 1$, $a_{211} + a_{212} > 1$. By theorem 2.2.8, we have

$$\Lambda \subset \{\xi_{11}\xi_{12} = 0\} \cap \{\xi_{21}\xi_{22} = 0\}.$$

Call $\overline{\Lambda} := \{\xi_{11}\xi_{12} = 0\} \cap \{\xi_{21}\xi_{22} = 0\}$ the upper bound for Λ. Hence, with the notation $\xi_1 := \xi_{11}, \xi_2 := \xi_{12}, \xi_3 := \xi_{21}, \xi_4 := \xi_{22}$, we have that

$$\mathcal{I}' = \{\{1,2\}, \{1,3\}, \{1,4\}, (\ldots)\{1,2,3\}, \{1,2,4\}, (\ldots)\{1,2,3,4\}\}$$

and

$$\mathcal{I}^* = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}.$$

The irreducible components of $\overline{\Lambda}$ are:

$$\overline{\Lambda}_{\{1,3\}} = \{\xi_1 = 0 \wedge \xi_3 = 0\},$$

$$\overline{\Lambda}_{\{1,4\}} = \{\xi_1 = 0 \wedge \xi_4 = 0\},$$

$$\overline{\Lambda}_{\{2,3\}} = \{\xi_2 = 0 \wedge \xi_3 = 0\},$$

$$\overline{\Lambda}_{\{2,4\}} = \{\xi_2 = 0 \wedge \xi_4 = 0\}.$$

Let Y be a germ of hypersurface of $(\mathbb{C}^L, 0)$. Let Λ be the set of limits of tangents of Y. For each irreducible component Λ_J of Λ there is a cone V_J contained in the tangent cone of Y such that Λ_J is the dual of the projectivization of V_J. The union of the cones V_J is called the *halo* of Y. The halo of Y is called "la auréole" of Y in [13].

Remark 2.2.13. If Λ is defined by the equations (2.2.23), the halo of Y equals the union of the linear subsets $V_J, J \in \mathcal{I}^*$ of $\mathbb{C}^{\ L}$, where V_J is defined by the equations

$$x_a = 0, \qquad a \in L \setminus J.$$

Example 2.2.14. We have already established a method to find a set that constitutes an upper bound $\overline{\Lambda}$ for Λ. It remains to be seen if that set equals Λ. The following example sugests a method for "filling up" the upper bound of Λ.

Let $y = x_1^{1/2} x_2^{3/2}$. Then, by theorem 2.2.8, $\Lambda \subset \overline{\Lambda} := \{\xi_1 \xi_2 = 0\}$. The irreducible components of $\overline{\Lambda}$ are $\xi_1 = 0$ and $\xi_2 = 0$. We have

$$\theta = \left(\frac{\partial y}{\partial x_1} : \frac{\partial y}{\partial x_2} : -1 \right) = \left(\frac{1}{2} x_1^{-1/2} x_2^{3/2} : \frac{3}{2} x_1^{1/2} x_2^{1/2} : -1 \right).$$

Set

$$x_i = \varepsilon_i t^{\alpha_i}, \quad i \in \{1,2\}, \alpha_i \in \mathbb{Q}^+, \varepsilon_i \in \mathbb{C}^*.$$

Then

$$\theta = \left(\frac{1}{2} \varepsilon_1^{-1/2} \varepsilon_2^{3/2} t^{-\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2} : \varepsilon_1^{1/2} \varepsilon_2^{1/2} t^{\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2} : -1 \right).$$

This is valid modulo product by a non-zero constant, since we are working in $\mathbb{P}^2$. In particular we can multiply by powers of t, out of the origin. For this reason the valuation of the components of θ is defined modulo addition of a constant. Therefore we can set the valuation of the term of smallest valuation to zero and the other terms will be $O(t)$ and vanish as $t \to 0$. The vector of valuations is then

$$v(\theta) = \left(-\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2 : \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 : 0 \right).$$

What limits can we obtain? Suppose we want a limit with θ_1 and θ_2 non-zero. Then by equaling the valuations of both components we get:

$$-\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 \Leftrightarrow \alpha_1 = \alpha_2.$$

But then θ_y is the component with smallest valuation:

$$v(\theta_1) = v(\theta_2) = -\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_1 = \alpha_1 > 0 = v(\theta_y).$$

Therefore the only limit with $v(\theta_1) = v(\theta_2)$ is the trivial limt $(0 : 0 : 1)$. (as expected since the exponent of x_2 is larger than 1, therefore we know that $\Lambda \subset \{\xi_2 = 0\}$. Let's consider then the set irreducible component of Λ defined by $V_2 = \{\xi_2 = 0\}$. Can we get all the limits in V_2? All such limits are of the type $(\psi_1 : 0 : \psi_y)$. So we'd like to set $v(\theta_1) = v(\theta_y)$ and ensure that $v(\theta_2)$ is larger than both. We have

$$v(\theta_1) = v(\theta_y) = 0 \Leftrightarrow -\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2 = 0 \Leftrightarrow \alpha_1 = 3\alpha_2.$$

and with that choice,

$$v(\theta_2) = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 = 2\alpha_2 > 0 = v(\theta_y).$$

Hence with this choice of α_i we are restricted to the set $\{\xi_2 = 0\}$. Substituting into the expression of θ and passing to the limit $t \to 0$ we get

$$\psi_{\alpha,\varepsilon}(t) = \lim_{t\to 0}\theta = \lim_{t\to 0}\left(\frac{1}{2}\varepsilon_1^{-1/2}\varepsilon_2^{3/2}t^0 : \varepsilon_1^{1/2}\varepsilon_2^{1/2}t^{\alpha_2} : -1\right) = \left(\frac{1}{2}\varepsilon_1^{-1/2}\varepsilon_2^{3/2} : 0 : -1\right).$$

Choosing ε_i adequately we get all the limits in $\{\xi_2 = 0\}$.
This sugests the following strategy: Considering the map

$$(\alpha, \epsilon) \mapsto \psi_\alpha(\varepsilon) := \lim_{t\to 0}\vartheta(t).$$

we fix a certain $J \in \mathcal{I}^*$, that is, an irreducible component V_J of $\overline{\Lambda}$, by fixing the values of α, and then show that by varying the parameters ϵ for fixed α we can get all the limits in V_J (more precisely, that the image of the map restricted to the choice of α is dense in V_J).

Example 2.2.15. Consider the hypersurface defined by

$$y = x_{11}^{a_{111}}x_{12}^{a_{112}} + x_{11}^{a_{211}}x_{12}^{a_{212}}x_{21}^{a_{221}}x_{22}^{a_{222}}.$$

Suppose the two very special monomials are such that $a_{111} + a_{112} < 1$, $a_{221} + a_{222} < 1$. Then there is a single irreducible component V_J of $\overline{\Lambda}$ that can be identified with $\{\zeta = 0\}$ in $\mathbb{C}^5_{\xi_{11},\xi_{12},\xi_{21},\xi_{22},\eta}$. By fixing adequate values of α_{ij} for the parametrization $x_{ij} = \varepsilon_{ij}t^{\alpha_{ij}}$ we restrict ourselves to V_J. Set $M_i = \prod_{i=1}^{k}\prod_{j=1}^{m_k}\varepsilon_{ij}^{akij}$. Then

$$(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}) \mapsto \psi_\alpha(\varepsilon) = \left(a_{111}\frac{M_1}{\varepsilon_{12}} : a_{112}\frac{M_1}{\varepsilon_{12}} : a_{221}\frac{M_2}{\varepsilon_{21}} : a_{222}\frac{M_2}{\varepsilon_{22}}\right)$$

maps each choice of coeficients ε to the limit of tangents obtained through the corresponding curve. The Jacobian of ψ is

$$\begin{vmatrix} \frac{a_{111}(a_{111}-1)M_1}{\varepsilon_{11}^2}+m_{11} & \frac{a_{111}a_{112}M_1}{\varepsilon_{11}\varepsilon_{12}}+m_{12} & m_{13} & m_{14} \\ \frac{a_{112}a_{111}M_1}{\varepsilon_{12}\varepsilon_{11}}+m_{21} & \frac{a_{112}(a_{112}-1)M_1}{\varepsilon_{12}^2}+m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & \frac{a_{212}(a_{212}-1)M_2}{\varepsilon_{21}^2} & \frac{a_{212}a_{222}M_2}{\varepsilon_{21}\varepsilon_{22}} \\ m_{41} & m_{42} & \frac{a_{222}a_{221}M_2}{\varepsilon_{22}\varepsilon_{21}} & \frac{a_{222}(a_{222}-1)M_2}{\varepsilon_{22}^2} \end{vmatrix}$$

$= M_1^2M_2^2(c+\epsilon)$, where $\epsilon(0)=0, m_{ij}\in(M_2)$. The permutations that result in a minimum valuation monomial ($M_1^2M_2^2$) are the ones corresponding to the product of the determinants of the block diagonal (2×2 blocks, or, in the general case, $n_i\times n_i$, where n_i is the number of new variables in the i-th very special monomial). All other permutations result, as a consequence of the total ordering of special monomials, in monomials that are in the ideal generated by the first monomial. It is enough to show that the product of the diagonal blocks is not identically null in a neighbourhood of the origin. In each block we have something of the type

$$\begin{vmatrix} \frac{a_{111}(a_{111}-1)M_1}{\varepsilon_{11}^2} & \frac{a_{111}a_{112}M_1}{\varepsilon_{11}\varepsilon_{12}} \\ \frac{a_{112}a_{111}M_1}{\varepsilon_{12}\varepsilon_{11}} & \frac{a_{112}(a_{112}-1)M_1}{\varepsilon_{12}^2} \end{vmatrix}$$

$$= M_1^2\varepsilon_{11}\varepsilon_{12}a_{111}a_{112}\begin{vmatrix} a_{111}-1 & a_{112} \\ a_{111} & a_{112}-1 \end{vmatrix}$$

$$= M_1^2\varepsilon_{11}\varepsilon_{12}a_{111}a_{112}\begin{vmatrix} -1 & 0 \\ 0 & a_{111}+a_{112}-1 \end{vmatrix}$$

and this is non-zero since we suppose $a_{111}+a_{112}<1$. This Jacobian will be zero only in a closed set which is a divisor with normal crossings.

Lemma 2.2.16. *The determinant of the $n\times n$ matrix $(\lambda_i-\delta_{ij})$ equals*

$$(-1)^n(1-\sum_{i=1}^{n}\lambda_i).$$

Proof. Notice that $\det(\lambda_i - \delta_{ij}) =$

$$= \left|\begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n - 1 \end{array}\right| = \left|\begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & \sum_{i=1}^{n} \lambda_i - 1 \end{array}\right|.$$

□

Theorem 2.2.17. *Assume that* $\sum_{i=1}^{m_1} a_{11i} < 1$. *Set*

$$L = \cup_{k=2}^{g} \{k\} \times \{1, \ldots, m_k\}, \quad \mathcal{I} = \cup_{k=2}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ *is the union of the irreducible linear projective sets* $\Lambda_J, J \in \mathcal{I}^*$, *defined by the equations* $\eta = 0$ *and*

$$\xi_{kj} = 0, \quad (k, j) \in J. \tag{2.2.24}$$

The tangent cone of Y *equals* $\{x_{11} \cdots x_{1m_1} = 0\}$. *The halo of* Y *is the union of the cones* V_J, $J \in \mathcal{I}^*$, *where* V_J *is defined by the equations* $x_{1j} = 0$, $1 \leq j \leq m_1$, *and*

$$x_{kj} = 0, (k, j) \in L \setminus J. \tag{2.2.25}$$

Proof. Let us show that $\Lambda_J \subset \Lambda$. We can assume that there are integers $n_1, \ldots, n_g$, $1 \leq n_k \leq m_k$, $1 \leq k \leq g$, such that $J = \cup_{k=1}^{g} \{k\} \times \{n_k + 1, \ldots, m_k\}$. We will use the notations of Remark 2.2.10.
Set $m = \sum_{k=1}^{g} m_k, n = m - \#J$. Assume that there are positive rational numbers $\alpha_k, \beta_k, 1 \leq k \leq g$, such that $\alpha_{ki} = \alpha_k$ if $1 \leq i \leq n_k$, $\alpha_{ki} = \beta_k$ if $n_k + 1 \leq i \leq m_k$, and $\alpha_k > \beta_k$, $1 \leq k \leq g$. Since $v(\theta_{ki}) = v(M_k) - v(x_{ki}) = v(M_k) - \alpha_{ki}$,

$$\lim_{t \to 0} \vartheta(t) \in \Lambda_J.$$

Let $\psi : (\mathbb{C} \setminus \{0\})^n \to \Lambda_J$ be the map defined by

$$\psi(\varepsilon_{ij}) = \lim_{t \to 0} \vartheta(t). \tag{2.2.26}$$

The map ψ has components ψ_{ki}, $1 \leq i \leq n_k, 1 \leq k \leq g$. In order to prove the Theorem it is enough to show that we can choose the rational numbers

α_k, β_k in such a way that the Jacobian of ψ does not vanish identically. We will proceed by induction in k. Let $k = 1$. Since $\sum_{i=1}^{m_1} a_{11i} < 1$, $n_1 = m_1$. Choose positive rationals α_1, β_1, $\alpha_1 > \beta_1$. There is a rational number $v_0 < 0$ such that $v(\theta_{1i}) = v_0$, for all $1 \leq i \leq n_1$.

Assume that there are α_k, β_k such that $v(\theta_{ki}) = v_0$ for $1 \leq i \leq n_k$ and $v(\theta_{ki}) > v_0$ for $n_k + 1 \leq i \leq m_k$, $k = 1, \ldots, u$. Set

$$\underline{\alpha}_{u+1} = \frac{\alpha_u + \sum_{k=1}^{u} \sum_{i=1}^{m_k} (a_{u+1,k,i} - a_{uki})\alpha_{ki}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Since the special monomials are ordered by valuation and, by construction of Λ_J, $\sum_{i=1}^{n_k} a_{kki} < 1$ for all $1 \leq k \leq g$, $\underline{\alpha}_{u+1}$ is a positive rational number. Choose a rational number β_{u+1} such that $0 < \beta_{u+1} < \underline{\alpha}_{u+1}$. Set

$$\alpha_{u+1} = \underline{\alpha}_{u+1} + \frac{\sum_{i=n_{u+1}+1}^{m_{u+1}} a_{u+1,u+1,i}\beta_{u+1}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Then, $v(\theta_{u+1,i}) = v(M_{u+1}) - \alpha_{u+1} = v(M_u) - \alpha_u = v_0$ for $1 \leq i \leq n_{u+1}$. Set $\widehat{M}_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} \varepsilon_{ij}^{a_{kij}}, 1 \leq i \leq n_k, 1 \leq k \leq g$. With these choices of α_{ki}, we have that

$$\psi_{ki} = \frac{1}{\varepsilon_{ki}} \sum_{l=k}^{g} a_{kli}\widehat{M}_l, \quad 1 \leq i \leq m_k, 1 \leq k \leq g.$$

Let D be the jacobian matrix of ψ. The matrix D has $n_r \times\ n_s$ blocks D_{rs}, $1 \leq r, s \leq g$. If $r < s$, the entries of D_{rs} are

$$\frac{1}{\varepsilon_{ri}\varepsilon_{sj}} \sum_{l=s}^{g} a_{rli}a_{slj}\widehat{M}_l.$$

Moreover, D_{rr} has entries

$$\frac{\widehat{M}_r}{\varepsilon_{ri}\varepsilon_{rj}} \left(a_{rri}(a_{rrj} - \delta_{ij}) + \sum_{l=r+1}^{g} a_{rri}a_{rrj})\widehat{M}_l \right).$$

Let m be the maximal ideal of the ring $\mathcal{O}(\widehat{M}_g)$. If $r \leq s$ the entry (i, j) of D_{rs} belongs to the ideal generated by $\widehat{M}_s/(\varepsilon_{ri}\varepsilon_{rj})$. Hence $det(D_{rr})$ belongs to the ideal I_r generated by

$$\left(\widehat{M}_r^{m_r} \Big/ \prod_{i=1}^{m_r} \varepsilon_{ri} \right)^2, \quad 1 \leq r \leq g. \tag{2.2.27}$$

Moreover, $det(D)$ belongs to the ideal I generated by

$$\left(\prod_{l=1}^{g} \widehat{M}_l^{m_l} \Big/ \prod_{l=1}^{g} \prod_{i=1}^{m_l} \varepsilon_{li} \right)^2. \tag{2.2.28}$$

Let σ be a permutation of $\{(1,1),\dots,(1,m_1),\dots,(g,1),\dots,(g,m_g)\}$. If there are $(r,i),(s,j)$ such that $\sigma(r,i)=(s,j)$ and $r\neq s$, the product of the entries $(r,i),\sigma(r,i)$ of D belongs to the ideal Im. Therefore $det(D)$ is congruent modulo Im to the product of the determinants of the diagonal blocks $D_{rr}, 1\le r\le g$. Moreover, $det(D_{rr})$ is congruent modulo $I_r m$ to the determinant of the matrix D_r with entries

$$\frac{\widehat{M}_r}{\varepsilon_{ri}\varepsilon_{rj}}a_{rri}(a_{rri}-\delta_{ij}).$$

By Lemma 2.2.16 $det(D_r)$ equals the product of (2.2.27) by a nonvanishing complex number. Therefore there are $\lambda\in\mathbb{C}\setminus\{0\}$ and $\varepsilon\in m$ such that $det(D)$ equals the product of (2.2.28) by an unit of $\mathcal{O}(\widehat{M}_g)$. Hence $det(D)$ does not vanish identically and Λ contains an open set of Λ_J. Since Λ is a projective variety and Λ_J is irreducible, Λ contains Λ_J. $\square$

Theorem 2.2.18. *Assume that* $\sum_{i=1}^{m_1} a_{11i}>1$. *Set*

$$L=\cup_{k=1}^{g}\{k\}\times\{1,\dots,m_k\},\quad \mathcal{I}=\cup_{k=1}^{g}\{\{k\}\times I:\sum_{j\in I}a_{kkj}\ge 1\}.$$

The set Λ *is the union of the irreducible linear projective sets* $\Lambda_J, J\in\mathcal{I}^*$, *defined by the equations* (2.2.24).
The tangent cone of Y *equals* $\{y=0\}$. *The halo of* Y *is the union of the cones* V_J, $J\in\mathcal{I}^*$, *where* V_J *is defined by the equations* $y=0$ *and* (2.2.25).

Proof. The proof is analogous to the proof of Theorem 2.2.17. On the first induction step we choose

$$\beta_1=\left(\frac{1-\sum_{i=1}^{n_1}a_{11i}}{\sum_{i=n_1+1}^{m_1}a_{11i}}\right)\alpha_1.$$

Hence $\beta_1<\alpha_1$, $v(\theta_{1i})=v(\eta)=0$ for $1\le i\le n_1$ and $v(\theta_{1i})>0$ for $n_1+1\le i\le m_1$. The rest of the proof proceeds as in the previous case. $\square$

Theorem 2.2.19. *Assume that* $\sum_{i=1}^{m_1} a_{11i}=1$. *Set*

$$L=\cup_{k=2}^{g}\{k\}\times\{1,\dots,m_k\},\quad \mathcal{I}=\cup_{k=2}^{g}\{\{k\}\times I:\sum_{j\in I}a_{kkj}\ge 1\}.$$

The set Λ *is the union of the irreducible projective algebraic sets* $\Lambda_J, J\in\mathcal{I}^*$, *where* Λ_J *is defined by the equations* (2.2.20) *and* (2.2.24).

There are integers c, d_i such that $a_{11i} = d_i/c, 1 \leq i \leq m_1$ and c is the l.c.d. of $d_1, \ldots, d_{m_1}$. The tangent cone of Y equals

$$y^c - f(0)^c \prod_{i=1}^{m_1} x_{1i}^{d_i} = 0. \tag{2.2.29}$$

The halo of Y is the union of the cones V_J, $J \in \mathcal{I}^*$, where V_J is defined by the equations (2.2.25) *and* (2.2.29).

Proof. Following the arguments of Theorem 2.2.17, it is enough to show that $\Lambda_J \subset \Lambda$ for each $J \in \mathcal{I}^*$. Choose $J \in \mathcal{I}^*$. Let $\tilde{\Lambda}_J$ be the linear projective variety defined by the equations (2.2.24). We follow an argument analogous to the one used in Theorem 2.2.17. We have $n_1 = m_1$. We choose positive rational numbers α_1, β_1 such that $\beta_1 < \alpha_1$. Then $v(\theta_{1i}) = 0$ for all $i = 1, \ldots, m_1$. The remaining steps of the proof proceed as before. Hence

$$\lim_{t \to 0} \vartheta(t) \in \tilde{\Lambda}_J.$$

Let $\psi : (\mathbb{C} \setminus \{0\})^n \to \tilde{\Lambda}_J$ be the map defined by (2.2.26). By Theorem 2.2.9 the image of ψ is contained in Λ_J. By Lemma 2.2.16, $\det(D_1) = 0$. Let $D'{}_1$ be the matrix obtained from D_1 by eliminating the m_1-th line and column. The argument of the proof of Theorem 2.2.17 works when we replace D_1 by D_1'. Hence, $\Lambda_J \subset \Lambda$. $\square$

Let Y be a quasi-ordinary hypersurface singularity.

Corollary 2.2.20. *The set of limits of tangents of Y only depends on the tangent cone of Y and the topology of Y.*

Corollary 2.2.21. *If the tangent cone of Y is a hyperplane, the set of limits of tangents of Y only depends on the topology of Y.*

Corollary 2.2.22. *Let $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ be the first special monomial of Y. If $\alpha_1 + \cdots + \alpha_k \neq 1$, the set of limits of tangents of Y only depends on the topology of Y.*

Corollary 2.2.23. *The triviality of the set of limits of tangents of Y is a topological invariant of Y.*

Proof. The set of limits of tangents of Y is trivial if and only if all the exponents of all the special monomials of Y are greater or equal than 1. $\square$

Chapter 3

Desingularization of Legendrian Varieties

In this chapter we prove a desingularization theorem for Legendrian hypersurfaces that are the conormal of a quasi-ordinary hypersurface. One of the main ingredients of the proof is the logarithmic version of the results on limits of tangents proved in the previous chapter.

3.1 Introduction

Neto introduced in [18] the notion of logarithmic contact manifold and constructed the blow up of a contact manifold along a Legendrian variety. He proved in [17] a desingularization theorem for Legendrian varieties and applied it in [19] to prove a desingularization theorem for regular holonomic systems of partial differential equations with holomorphic coefficients.

The main idea of the proof of the desingularization theorem is that the blow up along the conormal of a point o of the conormal of a curve Y equals the conormal of the blow up of the curve along o. This means that we can use the algorithm of resolution of singularities of plane curves to desingularize Legendrian curves.

We cannot expect the same phenomena will always occur when we replace a curve by a surface S. We need at least to ask that the limit of tangents (or its logarithmic version) be trivial at each singular point of S. Moreover, we need to ask for a condition on the normal cone of the conormal of S along the conormal of each center.

The natural generalization of [17] would be a general theorem for Legendrian surfaces. We overcame in this chapter most of the problems that we can find on the way to reaching this goal. Unfortunately we could not find a good description of the limits of tangents in terms of topological invariants of a surface, if such a description exists.

The results we obtained in this direction for quasi-ordinary surfaces are already not completely trivial.

Hironaka [9] proved his celebrated theorem of resolution of singularities in 1964. Bierstone and Milman [3], and Villamayor [22] gave constructive versions of this result. Lipman [15] proved a desingularization thorem for quasi-ordinary surfaces and Ban and Mcewan [4] gave an ambedded version of this result using the invariants of [3].

We follow the algorithm of [3], which allows us to forget about the global problems and the "historical" invariants that dealt with them. The main result of this chapter relies on the commutation between the operations of blowing up and taking the conormal and the hereditarity of the conditions that guarantee it. Example 3.8.3 shows that there is at least a case where this hereditarity fails. This fact forces us to prove the theorem through a case by case combinatorial analysis, that the reader can find in Lemma 3.8.4. It is common to use the coordinates of $\mathbb{C}^n$ when dealing with the projective space $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$. We call these coordinates the homogeneous coordinates of $\mathbb{P}^{n-1}$. When dealing with contact manifolds it is common to use the coordinates of the associated symplectic manifolds within the same spirit. In particular we will often use the coordinates of $T^*\langle M/N\rangle$ when dealing with $\mathbb{P}^*\langle M/N\rangle$.

3.2 Logarithmic differential forms

Let X be a complex manifold. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Let Ω_X^* denote the sheaf of differential forms on X. A subset Y of X is called a divisor with normal crossings at $o \in X$ if there is an open neighborhood U of o, a system of local coordinates $(x_1, \ldots, x_n)$ and a nonnegative integer ν such that $x_i(o) = 0, i = 1, \ldots, n$, and

$$Y \cap U = \{x_1 \cdots x_\nu = 0\}. \tag{3.2.1}$$

We call ν the index of Y at o. We say that Y is a *divisor with normal crossings* if Y is a divisor with normal crossings at each point of X. We call index of Y to the maximum of the indexes of Y at $o, o \in X$. Notice that the index of Y is smaller or equal to the dimension of X.
A germ of a divisor with normal crossings (Y, o) defines a canonical stratification of (X, o). The k-strata are the connected components of the set of points of index k of X. A k-stratum is a locally closed submanifold of codimension k of X.
The closure Z of a k-stratum Z' of (X, Y) is a closed submanifold of X, the intersection of the irreducible components of Y that contain Z'. If o has index k, Z has codimension l and $l < k$, Y induces in Z the normal crossings divisor $Z - Z'$ of index $k - l$.

Example 3.2.1. Set $X = \mathbb{C}^3, o = (0, 0, 0)$. The strata induced in X by the divisor $Y = \{x_1x_2x_3 = 0\}$ are

$$\begin{aligned}
&\{x_1x_2x_3 \neq 0\}\\
&\{x_k = 0\} \setminus \{x_ix_j = 0\}, \quad k = 1,2,3, \{i,j,k\} = \{1,2,3\}.\\
&\{x_i = x_j = 0\} \setminus \{(0,0,0)\}, \quad i < j,\\
&\{(0,0,0)\}.
\end{aligned}$$

If $Z' = \{x_1 = 0, x_2x_3 \neq 0\}$, $Z = \{x_1 = 0\}$ and $Z - Z' = \{x_2x_3 = 0\} \cap Z$.
If $Z' = \{x_1 = x_2 = 0, x_3 \neq 0\}$, $Z = \{x_1 = x_2 = 0\}$ and $Z - Z' = \{x_3 = 0\} \cap Z$.

Let Y be a divisor with normal crossings of a complex manifold X. Let U be an open set of X. Let $j : U \setminus Y \hookrightarrow X$ be the open inclusion. Let $f \in \mathcal{O}_X(U)$. If $f^{-1}(0) \subset Y \cap U$ let δf denote the section df/f of $j_*\Omega^1_{U\setminus Y}$. Otherwise, set $\delta f = df$.
Let $\Omega^*_X\langle Y\rangle$ be the smallest complex of $j_*\Omega^*_{X\setminus Y}$ stable by exterior product that contains $\mathcal{O}_X$ and δf for each local section f of $\mathcal{O}_X$. The local sections of $\Omega^*_X\langle Y\rangle$ are called logarithmic differential forms with poles along Y.
Let Θ_X be the sheaf of vector fields of X. Let I_Y be the defining ideal of Y. We say that a vector field u of X is tangent to Y if $uI_Y \subset I_Y$. Let $\Theta_X\langle Y\rangle$ be the sheaf of vector fields tangent to Y.
The $\mathcal{O}_X$-modules $\Omega^1_X\langle Y\rangle$ and $\Theta_X\langle Y\rangle$ are locally free and dual of each other. Given a system of local coordinates verifying (3.2.1),

$$\begin{aligned}
\Omega^1_X\langle Y\rangle|_U &= \mathcal{O}_U\frac{dx_1}{x_1} \oplus \cdots \oplus \mathcal{O}_U\frac{dx_\nu}{x_\nu} \oplus \mathcal{O}_U dx_{\nu+1} \oplus \cdots \oplus \mathcal{O}_U dx_n,\\
\Theta_X\langle Y\rangle|_U &= \mathcal{O}_U x_1\frac{\partial}{\partial x_1} \oplus \cdots \oplus \mathcal{O}_U x_\nu\frac{\partial}{\partial x_\nu} \oplus \mathcal{O}_U\frac{\partial}{\partial x_{\nu+1}} \oplus \cdots \oplus \mathcal{O}_U\frac{\partial}{\partial x_n}.
\end{aligned}$$

Definition 3.2.2. Let W be a smooth irreducible component of Y. We can associate to $\alpha \in \Omega^1_X\langle Y\rangle$ an holomorphic function $Res_W\alpha \in \mathcal{O}_W$. We call $Res_W\alpha$ the *Poincaré residue of* α *along* W.
Assume that we are in the situation of (3.2.1),

$$\alpha_U = \sum_{i=1}^{\nu} \alpha_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n} \alpha_i dx_i$$

and $W \cap U = \{x_i = 0\}$, where $1 \leq i \leq \nu$. Then

$$Res_W\alpha|_{U\cap W} = \alpha_i|_{U\cap W}.$$

3.3 Logarithmic symplectic manifolds

Let us recall some definitions and some results introduced in [18].

Definition 3.3.1. Let X be a complex manifold and Y a divisor with normal crossings of X. Let

$$\pi : T^*\langle X/Y\rangle \to X \tag{3.3.1}$$

be the vector bundle with sheaf of sections $\Omega^1_X\langle Y\rangle$. We will call (3.3.1) the *logarithmic cotangent bundle of X along Y*. Let

$$\tau : T\langle X/Y\rangle \to X \tag{3.3.2}$$

be the vector bundle with sheaf of sections $\Theta_X\langle Y\rangle$. We call (3.3.2) the *logarithmic tangent bundle of X along Y*.

Remark 3.3.2. Given a section α of $\Omega^1_X(\Theta_X)$ we will represent its value at $x^0 \in X$ as a section of $\Omega^1_X(\Theta_X)$ by $\alpha_{(x^0)} \in T^*_{x^0}X(\in T_{x^0}X)$. Given a section α of $\Omega^1_X\langle Y\rangle(\Theta_X\langle Y\rangle)$ we will represent its value at $x^0 \in X$ as a section of $\Omega^1_X\langle Y\rangle(\Theta_X\langle Y\rangle)$ by $\alpha_{\langle x^0\rangle} \in T^*_{x^0}\langle X/Y\rangle(\in T_{x^0}\langle X/Y\rangle)$.

Definition 3.3.3. Let X be a complex manifold and Y a divisor with normal crossings of X. We say that a locally exact section σ of $\Omega^2_X\langle Y\rangle$ is a *logarithmic symplectic form with poles along Y* if $\sigma_{\langle x^0\rangle}$ is a symplectic form on $T_{x^0}\langle X/Y\rangle$ for any $x^0 \in X$.
We say that a complex manifold X endowed with a logarithmic symplectic form with poles along a divisor with normal crossings Y of X is a *logarithmic symplectic manifold with poles along Y*.
If X_1, X_2 are logarithmic symplectic manifolds with logarithmic symplectic forms σ_1, σ_2 and φ is a holomorphic map from X_1 to X_2 such that $\varphi^*\sigma_2 = \sigma_1$ then φ is called a *morphism of logarithmic symplectic manifolds.* If moreover φ is biholomorphic we say that φ is an *isomorphism of logarithmic symplectic manifolds* or a *canonical transformation.*

Remark 3.3.4. (i) If Y is the empty set we get the usual definition of symplectic manifold.
(ii) A logarithmic symplectic manifold has always even dimension.

(iii) Suppose that X has dimension $2n$. A locally exact section σ of $\Omega_X^2\langle Y\rangle$ is a logarithmic symplectic form with poles along Y if and only if σ^n is a generator of $\Omega_X^{2n}\langle Y\rangle$.

Definition 3.3.5. Given a complex manifold X we say that a $\mathbb{C}$-bilinear morphism

$$\{\star,\star\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

is a *Poisson bracket* if it verifies the following conditions:

(i) $\{f,g\} = -\{g,f\}$

(ii) $\{fg,h\} = f\{g,h\} + g\{f,h\}$

(iii) $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

We call $\{f,g\}$ the *Poisson bracket* of f and g. If f is a local section of $\mathcal{O}_X$, the derivation $g \mapsto \{f,g\}$ determines a vector field H_f, the *Hamiltonian vector field* of f.
We call a complex manifold X endowed with a Poisson bracket a *Poisson manifold.*
If $(X_1,\{\star,\star\}_1),(X_2,\{\star,\star\}_2)$ are Poisson manifolds and $\varphi : X_1 \to X_2$ is a complex map such that $\{\varphi^* f, \varphi^* g\}_1 = \varphi^*\{f,g\}_2$, for any holomorphic functions f,g defined in an open set of X_2 we call φ a morphism of Poisson manifolds.

Example 3.3.6. A logarithmic symplectic manifold has a canonical structure of Poisson manifold.

Definition 3.3.7. Let X be a Poisson manifold. An analytic subset V of X is called *involutive* if $\{I_V, I_V\} \subset I_V$.

Proposition 3.3.8. *Let σ be a logarithmic symplectic form on a symplectic manifold X. Then we can recover σ from the Poisson bracket it determines.*

Corollary 3.3.9. *Let X_1, X_2 be logarithmic complex manifolds and φ a biholomorphic map from X_1 onto X_2. The map φ is a canonical transformation if and only if it is a morphism of Poisson manifolds.*

Example 3.3.10. If X is a complex manifold and Y is a divisor with normal crossings of X then the vector bundle $\pi : T^*\langle X/Y\rangle \to X$ has a canonical structure of logarithmic symplectic manifold with poles along $\pi^{-1}(Y)$.

Actually, there is a canonical section θ of $\Omega^1_{T^*\langle X/Y\rangle}\langle \pi^{-1}(Y)\rangle$. We call θ the canonical *1-form* of $T^*\langle X/Y\rangle$. Given an integer ν and a system of local coordinates $(x_1, \ldots, x_n)$ on an open set U of X verifying (3.2.1) there is one and only one family of holomorphic functions $\xi_i, 1 \leq i \leq n$, defined on $\pi^{-1}(U)$ such that

$$\theta|_{\pi^{-1}(U)} = \sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n} \xi_i dx_i.$$

The functions

$$x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$$

define a system of local coordinates on $\pi^{-1}(U)$, called the *system of symplectic coordinates with poles along Y associated to the system of local coordinates* $(x_1, \ldots, x_n)$.
The 2-form $\sigma = d\theta$ is called the *canonical 2-form* of $T^*\langle X/Y\rangle$. The canonical 2-form is a symplectic form with poles along $\pi^{-1}(Y)$.
Given holomorphic functions f, g, defined on a open set V contained in $\pi^{-1}(U)$, we have that

$$\{f, g\} = \sum_{i=1}^{\nu} x_i \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} \right) + \sum_{i=\nu+1}^{n} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} \right).$$

Definition 3.3.11. Let (X, σ) be a logarithmic symplectic manifold with poles along a divisor with normal crossings Y. Let U be an open set of X and Y_0 a global smooth hypersurface contained in $Y \cap U$. A holomorphic function ξ defined on U is called a *residual function along* Y_0 if

$$d\xi|_{Y_0} = dRes_{Y_0}(\sigma|_U).$$

Let X be a complex manifold. A group action $\alpha : \mathbb{C}^* \times X \to X$ is called a free group action of $\mathbb{C}^*$ on X if, for each $x \in X$, the isotropy subgroup $\{t \in \mathbb{C}^* : \alpha(t, x) = x\}$ equals $\{1\}$. A manifold X with a free froup action α

of $\mathbb{C}^*$ is called a conical manifold. We associate to each free group action α of $\mathbb{C}^*$ on X a vector field ρ, the *radial vector field of* α, in the following way:

$$\rho f = \frac{\partial}{\partial t}\alpha_t^* f|_{t=1}, f \in \mathcal{O}_X.$$

Here $\alpha_t(x) = \alpha(t, x)$. We put

$$\mathcal{O}_X(\lambda) = \{f \in \mathcal{O}_X : \rho f = \lambda f\}$$

for any $\lambda \in \mathbb{C}$ and

$$\mathcal{O}_X^h = \oplus_{k \in \mathbb{Z}} \mathcal{O}_X(k).$$

A section f of $\mathcal{O}_X(\lambda)$ is called a homogeneous function of degree λ. Given conic complex manifolds (X_1, α_1) and (X_2, α_2), a holomorphic map $\varphi : X_1 \to X_2$ is called *homogeneous* if it commutes with the actions α_1, α_2, that is, if

$$\alpha_{2,t}\varphi = \varphi\alpha_{1,t},$$

for any $t \in \mathbb{C}^*$.

Definition 3.3.12. A logarithmic symplectic manifold (X, σ) with a free group action α is called a *homogeneous symplectic manifold* if

$$\alpha_t^*\sigma = t\sigma, t \in \mathbb{C}^*.$$

If $(X_1, \sigma_1), (X_2, \sigma_2)$ are homogeneous symplectic manifolds and $\varphi : X_1 \to X_2$ is a canonical transformation we say that φ is a *homogeneous canonical transformation* or a *contact transformation* if it is homogeneous.
Given a homogeneous logarithmic symplectic manifold (X, σ) we call the logarithmic differential form of degree 1

$$\theta = \iota(\rho)\sigma$$

the *canonical 1-form* of (X, σ), where $\iota(\rho)\sigma$ is the contraction of ρ and σ.
We notice that a canonical transformation $\varphi : (X_1, \sigma_1) \to (X_2, \sigma_2)$ is a homogeneous canonical transformation if and only if $\varphi^*\theta_2 = \theta_1$. Here $\theta_i = \iota(\rho)(\sigma_i)$, $i = 1, 2$.
A homogeneous logarithmic symplectic manifold is locally isomorphic to $\mathring{T}^*\langle X/Y\rangle$ in the category of homogeneous symplectic manifolds. Given a vector bundle E over X we denote by $\mathring{\mathrm{E}}$ the complex manifold $E \setminus X$, where we identify X with the image of the zero section of E.

Theorem 3.3.13. *Let σ be a homogeneous logarithmic symplectic form on a complex manifold X with poles along a divisor with normal crossings Y. Given $x^0 \in X$ let ν be the number of irreducible components of Y at x^0. Then there is a system of local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on U such that $Y \cap U = \{x_1 \cdots x_\nu = 0\}$, $x_1, \dots, x_n$ are homogeneous of degree 0, $\xi_1, \dots, \xi_n$ are homogeneous of degree 1 and*

$$\sigma|_U = \sum_{i=1}^{\nu} d\xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n} d\xi_i dx_i.$$

Remark 3.3.14. If (X, σ) is a homogeneous logarithmic symplectic manifold and x_j, $1 \leq j \leq n$, ξ_k, $1 \leq k \leq n$, is a system of homogeneous logarithmic symplectic coordinates for σ on an open set U of X then

$$\rho|_U = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} \text{ and } \theta|_U = \sum_{i=1}^{n} \xi_i \delta x_i.$$

Definition 3.3.15. Let (X, σ) be a homogeneous logarithmic symplectic manifold with poles along a divisor with normal crossing Y. Let W be the intersection of the smooth irreducible components $Y_1, \dots, Y_\mu$ of Y. We call *residual submanifold of X along W* to the set of points $o \in W$ such that the residual of θ along Y_i vanishes at o for $1 \leq i \leq \mu$. We will denote the residual submanifold of X along W by $R_W X$.

Proposition 3.3.16. *Let X be an homogeneous logarithmic symplectic manifold with poles along a smooth divisor Y. Let W be the intersection of the smooth irreducible components $Y_1, \dots, Y_\mu$ of Y. Then:*

(i) *X, $R_W X$ are involutive submanifolds of X.*

(ii) *The manifold $R_W X$ has a canonical structure of homogeneous symplectic manifold with poles along the divisor induced in W by Y.*

Proof. Let $o \in W$. There is a system of symplectic coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on a conic open set U that contains o such that

$$\theta\,|_U = \sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n} \xi_i dx_i$$

and $W = \{x_1 = \cdots = x_\mu = 0\}$. Hence

$$R_W X \cap U = \{x_1 = \cdots = x_\mu = \xi_1 = \cdots = \xi_\mu = 0\}.$$

The restriction to $R_W X \cap U$ of the Poisson bracket of X is given by

$$\{f, g\} = \sum_{i=\nu+1}^{\mu} x_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right) + \sum_{i=\mu+1}^{n} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right).$$

By proposition 3.3.8, $R_W X \cap U$ is endowed with a 1-form

$$\sum_{i=\nu+1}^{\mu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\mu+1}^{n} \xi_i dx_i$$

$\square$

Definition 3.3.17. Let X be a complex manifold of dimension $2n+1, n \geq 0$, and Y a divisor with normal crossings of X. A local secton ω of $\Omega_X^1\langle Y \rangle$ is called a *logarithmic contact form with poles along* Y if $\omega(d\omega)^n$ is a local generator of $\Omega_X^{2n+1}\langle Y \rangle$.
We say that a locally free sub $\mathcal{O}_X$-module $\mathcal{L}$ of $\Omega_X^1\langle Y \rangle$ is a *logarithmic contact structure on* X *with poles along* Y if it is locally generated by a logarithmic contact forms with poles along Y. We say that a complex manifold with a logarithmic contact structure with poles along a divisor with normal crossings Y is a *logarithmic contact manifold with poles along* Y. We call Y the *set of poles of the logarithmic contact manifold* $(X, \mathcal{L})$.
Let $(X_1, \mathcal{L}_1)$, $(X_2, \mathcal{L}_2)$ be logarithmic contact manifolds. We say that a holomorphic map $\varphi : X_1 \to X_2$ is a *contact transformation* if for any local generator of $\mathcal{L}_2$ its inverse by φ is a local generator of $\mathcal{L}_1$.

Let Y_0 be a smooth irreducible component of Y. We say that a point x^0 of Y is in the *residual set* of X along Y_0 if the residue along Y_0 of all the sections of $\mathcal{L}$ vanishes at x^0.

Proposition 3.3.18. *There is an equivalence of categories between the category of logarithmic contact manifolds and the category of homogeneous logarithmic symplectic manifolds.*

Let X be a homogeneous logarithmic symplectic manifold. Let θ be the canonical 1-form of X and let Y be the set of poles of X. Let X_* be the quotient of X by its $\mathbb{C}^*$ action. Then X_* is a complex manifold and the canonical epimorphism $\gamma : X \to X_*$ is a $\mathbb{C}^*$-bundle. Put $Y_* = \gamma(Y)$. Let $\mathcal{L}$ be the sub $\mathcal{O}_{X_*}$-module of $\Omega^1_{X_*}\langle Y\rangle$ generated by the logarithmic differential forms $s^*\theta$, where s is a holomorphic section of γ. Then $\mathcal{L}_*$ is a structure of logarithmic contact manifold with poles along Y_*.

Let $\mathbb{P}^*\langle X/Y\rangle$ be the projective bundle associated to $T^*\langle X/Y\rangle$. We call $\mathbb{P}^*\langle X/Y\rangle$ *the projective logarithmic cotangent bundle of X with poles along Y.*

The projective bundle $\mathbb{P}^*\langle X/Y\rangle$ has a canonical structure of logarithmic contact manifold. Moreover the associated homogeneous logarithmic symplectic manifold equals $\mathring{T}\langle X/Y\rangle$.

A logarithmic contact manifold of dimension $2n$ is locally isomorphic to $\mathbb{P}^*\langle \mathbb{C}^n/\{x_1 \cdots x_\nu = 0\}\rangle$, for some integer ν.

Theorem 3.3.19. *Let X be a complex manifold of dimension $2n+1$.*

(i) *Let ω be a logarithmic contact form of X. Given a point x^0 on the domain of ω there are holomorphic functions $x_1, \dots, x_{n+1}, \zeta_1, \dots, \zeta_{n+1}$ defined in an open neighbourhood U of X such that*

$$\omega|_U = \sum_{i=1}^{n+1} \zeta_i \delta x_i. \tag{3.3.3}$$

Moreover, there is an i such that $\zeta_i(x^0) \neq 0$. For any i_0 such that $\zeta_{i_0}(x^0) \neq 0$ the functions

$$x_i, \quad 1 \leq i \leq n+1, \quad \frac{\zeta_i}{\zeta_{i_0}}, \quad 1 \leq i \leq n+1, \quad i \neq i_0$$

are a local system of coordinates for X on U.

(ii) *Let $\mathcal{L}$ be a logarithmic contact structure on X with poles along a divisor with normal crossings Y. Given a point x^0 of X, suppose that the germ (Y, x_0) has irreducible components $Y_1, \dots, Y_\nu$ and that the residual values of x^0 along Y_i vanish for $1 \leq 1 \leq \nu$. Then there is a system of coordinates $(x_1, \dots, x_{n+1}, p_1, \dots, p_n)$ in a neighbourhood U of x^0 such that the logarithmic differential form*

$$dx_{n+1} - \sum_{i=1}^{\nu} p_i \frac{dx_i}{x_i} - \sum_{i=\nu+1}^{n} p_i dx_i \tag{3.3.4}$$

is a local generator of $\mathcal{L}$ and $Y_i \cap U = \{x_i = 0\}$, for $1 \leq i \leq \nu$.

3.4 Legendrian Varieties

Let $(X, \mathcal{L})$ be a contact manifold of dimension $2n+1$. An analytic subset Γ of X is a Legendrian variety of X if it verifies the following three conditions: Γ has dimension n, Γ is involutive and the restriction to the regular part of Γ of a local generator of $\mathcal{L}$ vanishes.
Each two of these three conditions imply the remaining one.
Given a manifold M and an irreducible analytic subset S of M there is one and only one Legendrian variety $\mathbb{P}^*_S M$ of $\mathbb{P}^* M$ such that $\pi(\mathbb{P}^*_S M) = S$. The analytic set $\mathbb{P}^*_S M$ is called the conormal of S (see for instance [12]). If S has irreducible components $S_i, i \in I$, the conormal $\mathbb{P}^*_S M$ of S equals $\cup_{i \in I} \mathbb{P}^*_{S_j} M$.
Let us introduce stratified versions of the definitions above.

Definition 3.4.1. Let X be a logaritmic contact manifold of dimension $2n+1$ with set of poles Y. An analytic subset Γ of X is called a Legendrian variety of X if Γ is involutive and :

1. The intersection of Γ with $X \setminus Y$ is a Legendrian variety of $X \setminus Y$.
2. If an irreducible component of Γ is contained in the closure Z of a codimension 1 stratum of Y, it is contained in the residual set $R_Z X$ of X along Z.
3. If Z is the closure of a codimension 1 stratum of (X, Y), the irreducible components of $\Gamma \cap Z$ that are not contained in the singular locus of Y are Legendrian varieties of the residual set $R_Z X$ of X along Z.

Remark 3.4.2. Let M be a manifold. Let N be a divisor with normal crossings of M. Let Γ be a Legendrian variety of $\mathbb{P}^*\langle M/N \rangle$. Let Q be a codimension 1 stratum of (M, N). Let R the divisor with normal crossings induced in Q by N. If Γ is contained in $\pi^{-1}(Q)$, it follows from condition 2) of definition 3.4.1 that Γ is contained in $\mathbb{P}^*\langle Q/R \rangle$.

Example 3.4.3. Let X be a logarithmic contact manifold of dimension $2n+1$ with poles along Y. If $n = 0$ the irreducible Legendrian varieties of

X are the points of $X \setminus Y$. If $n = 1$ the irreducible Legendrian varieties of X are the points of the residual set of X and the irreducible curves Γ of X such that $\Gamma \setminus Y$ is dense in Γ and $\Gamma \setminus Y$ is a Legendrian curve of $X \setminus Y$.

An analytic subset S of (X, Y) is *natural* if no germ of S is the germ of the closure of a stratum of (X, Y). A Legendrian variety of a logarithmic contact manifold X with poles along Y is a natural analytic subset of (X, Y).

Definition 3.4.4. Let S be a natural irreducible subset of (M, N). Let Q be the closure of the stratum Q' of (M, N) of biggest codimension such that S is contained in the closure of Q. Set $R = Q \cap N$. We call *conormal* of S to the closure $\mathbb{P}^*_S\langle Q/R\rangle$ of the conormal of the analytic subset $S \setminus R$ of $Q \setminus R$ in $\mathbb{P}^*\langle Q/R\rangle$.
Let S be a natural analytic subset of (M, N). We call *conormal* of S to the union $\mathbb{P}^*_S\langle M/N\rangle$ of the conormals of its irreducible components.

The two definitions above have even dimensional equivalents: A conic analytic subset Γ of a conic symplectic manifold is called a Lagrangian variety if $\gamma(\Gamma)$ is a Legendrian variety. The conic analytic subset $T^*_S\langle M/N\rangle = \gamma^{-1}(\mathbb{P}^*_S\langle M/N\rangle)$ of the conic symplectic manifold $T^*\langle M/N\rangle \setminus M$ is also called conormal of S.

Theorem 3.4.5. *The conormal of a natural analytic set is a Legendrian variety.*

Proof. Let S be a germ of a natural analytic subset of (M, N). We can assume that S is irreducible and that M is the closure of the stratum of (M, N) of biggest codimension that contains S. The intersection of Γ with $\pi^{-1}(M \setminus N)$ is the Legendrian variety $\mathbb{P}^*_{S \setminus N}(M \setminus N)$ of the contact manifold $\mathbb{P}^*(M \setminus N)$. Hence condition 1) is verified. Since Γ is the closure of $\mathbb{P}^*_{S \setminus M}(M \setminus N)$, Γ is involutive. Condition 2) follows from the definition of conormal variety.
Let us prove statement 3) by induction in the dimension of M. Statement 3) is trivial if $dim M = 1$. Let Z be the closure of a 1-stratum Z' of $\mathbb{P}^*\langle M/N\rangle, \pi^{-1}(N))$. Since Z' is invariant, Z is invariant. The set $Q = \pi(Z)$ is the closure of a 1-stratum of (M, N) and $Z = \pi^{-1}(Q)$. Let R be the

divisor induced in Q by N. Let Γ_0 be an irreducible component of $\Gamma \cap Z$ that is not contained in the singular locus of $\pi^{-1}(N)$. Let us show that

$$\Gamma_0 \subset \mathbb{P}^*\langle Q/R\rangle. \tag{3.4.1}$$

It is enough to show that $\gamma^{-1}(\Gamma_0)$ is contained in the residual set of $T^*\langle M/N\rangle$. Let $o \in \gamma^{-1}(\Gamma_0 \cap Z')$. There is an open conic neighborhood U of o and a system of local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on U such that

$$\theta\,|_U = \xi_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} \xi_i dx_i$$

and $\gamma^{-1}(Z') \cap U = \{x_1 = 0\}$.
There is a holomorphic map

$$\delta : \{t \in \mathbb{C} : |t| < 1\} \to \gamma^{-1}(\Gamma_0)$$

such that

$$\gamma(\delta(0)) = o \text{ and } \delta^{-1}(\gamma^{-1}(\pi^{-1}(N))) = \{0\}.$$

Set $\delta_i = x_i \circ \delta$, $1 \leq i \leq n$. Since θ vanishes on $\gamma^{-1}(\Gamma_0 \setminus Z')$,

$$\xi_1(\delta(t)) \frac{\delta_1'(t)}{\delta_1(t)} + \sum_{i=2}^{n} \xi_i(\delta(t))\delta_i'(t) = 0 \qquad \text{if } t \neq 0. \tag{3.4.2}$$

Hence

$$\xi_1(o) = 0, \tag{3.4.3}$$

and (3.4.1) holds.
Since Z is invariant, $\Gamma \cap Z$ is an involutive submanifold of $P^*\langle M/N\rangle$. Hence $\Gamma \cap Z$ is an involutive submanifold of $\mathbb{P}^*\langle Q/R\rangle$. Hence its irreducible components are involutive. Since the $\dim\Gamma_0 = \dim\Gamma - 1$, $\Gamma_0 \setminus \pi^{-1}(R)$ is a Legendrian subvariety of $\mathbb{P}^*(Q_0 \setminus R_0)$. Let S_0 be the closure in Q of the projection of $\Gamma_0 \setminus \pi^{-1}(R)$. Then Γ_0 is the conormal of S_0. By the induction hypothesis, Γ_0 is a Legendrian variety of $\mathbb{P}^*\langle Q/R\rangle$. □

Theorem 3.4.6. *An irreducible Legendrian subvariety of a projective logarithmic cotangent bundle is the conormal of its projection.*

Proof. The result is known for Legendrian subvarieties of a projective cotangent bundle (see for instance [21]). The theorem is an immediate consequence of this particular case. □

3.5 Blow up and deformation of the normal cone

We recall that the blow up $\widetilde{\mathbb{C}^n_D}$ of the set $D = \{x_1 = \cdots = x_k = 0\}$ of $\mathbb{C}^n$ is the glueing of k open affine sets U_{x_i}, $1 \le i \le k$, where U_{x_i} is a copy of $\mathbb{C}^n$ with coordinates

$$\left(\tfrac{x_1}{x_i}, \ldots, \tfrac{x_{i-1}}{x_i}, x_i, \tfrac{x_{i+1}}{x_i}, \ldots, \tfrac{x_k}{x_i}, x_{k+1}, \ldots, x_n\right)$$

and the restriction to U_{x_i} of the blow up map $\pi : \widetilde{\mathbb{C}^n_D} \to \mathbb{C}^n$ is given by

$$\pi\left(\tfrac{x_1}{x_i}, \ldots, x_i, \ldots, \tfrac{x_k}{x_i}, x_{k+1}, \ldots, x_n\right) = \left(x_i\tfrac{x_1}{x_i}, \ldots, x_i, \ldots, x_i\tfrac{x_k}{x_i}, x_{k+1}, \ldots, x_n\right)$$

The charts U_{x_j} and U_{x_k} are glued by the change of coordinates

$$\begin{aligned}
&\tfrac{x_i}{x_k} = \tfrac{x_i}{x_j}(\tfrac{x_k}{x_j})^{-1}, \quad 1 \le i \le k_1, \quad i \ne j, k,\\
&\tfrac{x_j}{x_k} = (\tfrac{x_k}{x_j})^{-1}, x_j = x_k(\tfrac{x_k}{x_j})^{-1},\\
&x_i = x_i, \quad k+1 \le i \le n.
\end{aligned}$$

Let M be a complex manifold of dimension n and D a closed submanifold of codimension k of M. We can cover M with open sets endowed with charts adapted to D and construct the blow up of M with center D,

$$\pi : \widetilde{M_D} \to M.$$

We call $E = \pi^{-1}(D)$ the *exceptional divisor* of the blow up.
Let us recall the construction of the normal cone of an analytic set S relatively to a submanifold D. See [12].
Consider in $\mathbb{C}^{n+1}$ the coordinates $(s, \widetilde{x}_1, \ldots, \widetilde{x}_k, x_{k+1}, \ldots, x_n)$. Let $\tau : \mathbb{C}^{n+1} \setminus \{\widetilde{x}_1 = \cdots = \widetilde{x}_k = 0\} \to \mathbb{C}^n$ be the map defined by

$$\tau(s, \widetilde{x}_1, \ldots, \widetilde{x}_k, x_{k+1}, \ldots, x_n) = (s\widetilde{x}_1, \ldots, s\widetilde{x}_k, x_{k+1}, \ldots, x_n).$$

Let $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ be the blow up of $\mathbb{C}^n$ with center $\{x_1 = \cdots = x_k = 0\}$. Let $U_i, 1 \le i \le k$, be the affine open set of $\widetilde{\mathbb{C}^n}$ with coordinates $\left(\frac{x_1}{x_i}, \ldots, x_i, \ldots, \frac{x_k}{x_i}, x_{k+1}, \ldots, x_n\right)$. Let $\Phi_i : \mathbb{C}^{n+1} \setminus \{x_i = 0\} \to U_i$ be the map defined by

$$\Phi_i(s, \widetilde{x}_1, \ldots, \widetilde{x}_k, x_{k+1}, \ldots, x_n) = \left(\frac{\widetilde{x}_1}{\widetilde{x}_i}, \ldots, s\widetilde{x}_i, \ldots, \frac{\widetilde{x}_k}{\widetilde{x}_i}, x_{k+1}, \ldots, x_n\right).$$

There is a map $\Phi : \mathbb{C}^{n+1} \setminus \{\widetilde{x}_1 = \cdots = \widetilde{x}_k = 0\} \to \widetilde{\mathbb{C}^n}$ such that

$$\Phi|_{\mathbb{C}^{n+1}\setminus\{\widetilde{x}_i=0\}} = \Phi_i$$

and

$$\pi \circ \Phi = \tau.$$

Let M be an open set of $\mathbb{C}^n$ that contains the origin. Set $D = \{x_1 = \cdots = x_k = 0\} \cap M$. We call $\widehat{M}_D = \tau^{-1}(M)$ *the deformation of the normal cone with center* D. There is a canonical map $\Phi : \widehat{M}_D \to \widetilde{M}_D$ such that $\pi \circ \Phi = \tau$. We can identify the subset $\{s = 0\}$ of $\widehat{M}_D$ with $\mathring{\mathrm{T}}_D M$. Here $T_D M$ is *the normal bundle* of M along D, defined by the exact sequence of vector bundles

$$0 \to TD \to D \times_M TM \to T_D M \to 0.$$

Notice that

$$\widehat{M}_D = \mathring{\mathrm{T}}_D M \sqcup M \setminus D.$$

Moreover, $\Phi(\mathring{\mathrm{T}}_D M)$ equals the exceptional divisor E of $\widetilde{M}_D$. Hence Φ induces an isomorphism of manifolds between the projective normal bundle $\mathbb{P}_D M$ and E.

Assume that M is the polydisc of $\mathbb{C}^n$. Let S be a hypersurface of M defined by $f \in \mathcal{O}(M)$. We can write $f = \sum_{l \geq m} f_l$, where $f_l \in \mathbb{C}\{x_1, \ldots, x_n\}$ and f_l is homogeneous of degree l in the variables $x_1, \ldots, x_k$. We assume that $f_m \neq 0$. Note that $f \circ \tau$ is divisible by s^m and

$$f(\tau(s, \widetilde{x}_1, \ldots, \widetilde{x}_k, x_{k+1}, \ldots, x_m))/s^m \equiv f_m(\widetilde{x}_1, \ldots, \widetilde{x}_k, x_{k+1}, \ldots, x_n) \mod (s).$$

Hence

$$\mathrm{closure}_{\widehat{M}_D}(\tau^{-1}(S) \setminus \{s = 0\}) \cap \{s = 0\} = \{f_m = 0\}.$$

Remark that when we fix $x^o = (x^o_{k+1}, \ldots, x^o_n) \in D$, $\{f_m = 0\} \cap \{x_i = x^o_i, k + 1 \leq i \leq n\}$ is a cone of the vector space $(T_D M)_{x^o}$.

Definition 3.5.1. We call

$$C_D(S) = \mathrm{closure}_{\widetilde{M}_D}(\tau^{-1}(S) \setminus \{s = 0\}) \cap \{s = 0\}$$

the *normal cone of* S along D. We call

$$C_D(S) = \text{closure}_{\widetilde{M}_D}(\tau^{-1}(S) \setminus \{s = 0\})$$

the *deformation of the normal cone of* S along D.

Remark that:

(i) The image by Φ of the deformation of the normal cone of S equals the proper inverse image $\widetilde{S}$ of S by π.

(ii) The image by Φ of $C_D(S)$ equals $\widetilde{S} \cap E$.

(iii) The map Φ induces an isomorphism between the analytic sets $C_D(S)/\mathbb{C}^*$ and $\widetilde{S} \cap E$.

Let M be a complex manifold and D a closed submanifold of M. We can generalize the construction of $\widehat{M_D}$ in the following way:

(i) We cover M with open sets M_i endowed with charts adapted to $D_i = M_i \cap D$.

(ii) We construct maps $\tau_i : \widehat{M}_{iD_i} \to M_i$.

(iii) We glue the manifolds $\widehat{M}_{D_i}$ and the maps τ_i.

This construction is quite similar to the construction of the blow up of a manifold M along a closed submanifold D.

Set $X = \mathbb{C}^{a+b+c}$ with coordinates

$$(x_1, \dots, x_a, y_1, \dots, y_b, z_1, \dots, z_c).$$

Set $x = (x_1, \dots, x_a), \widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_a)$ and so on.

Set $\Lambda = \{x = y = 0\}$,

$$L = \{(\tilde{x}, \tilde{y}, z) \in T_\Lambda X : \widetilde{x} = 0\}.$$

The blow up of X along Λ is the union of the affine open sets $U_{x_i}, 1 \leq i \leq a, U_{y_j}, 1 \leq j \leq b$.

Lemma 3.5.2. *Let* Γ *be the germ of a closed analytic subset of* X*. If* $C_\Lambda(\Gamma) \cap L \subset \{\tilde{x} = \tilde{y} = 0\}$,

$$\widetilde{\Gamma} \cap E \subset \cup_{i=1}^a U_{x_i}.$$

Proof. Let E be the exceptional divisor of the blow up of X along Λ. Notice that $E = \mathbb{P}^{a+b-1} \times \mathbb{C}^c$ and a point of E has coordinates

$$(\tilde{x}_1 : \cdots : \tilde{x}_a : \tilde{y}_1 : \cdots : \tilde{y}_b; z_1, \ldots, z_c).$$

Moreover,

$$\begin{aligned}(E \cap U_{x_i}) &= \tau(\{s = 0, \tilde{x}_i \neq 0\}),\\ E \setminus U_{x_i} &= \tau(\{s = 0, \tilde{x}_i = 0\}).\end{aligned}$$

Hence,

$$\begin{aligned}E \setminus \cup_{i=1}^a U_{x_i} &= \tau(\{s = 0, \widetilde{x} = 0\}),\\ (\widetilde{\Gamma} \cap E) \setminus \cup_{i=1}^a U_{x_i} &= \tau(C_\Lambda(\Gamma) \cap \{\tilde{x} = 0\}).\end{aligned}$$

Therefore the following statements are equivalent:

$$\begin{aligned}&\widetilde{\Gamma} \cap E \subset \cup_{i=1}^a U_{x_i},\\ &(\widetilde{\Gamma} \cap E) \setminus \cup_{i=1}^a U_{x_i} = \emptyset,\\ &C_\Lambda(\Gamma) \cap L \subset \{\tilde{x} = \tilde{y} = 0\}.\end{aligned}$$

$\square$

Lemma 3.5.3. *Let $f : X \to Y$ be a holomorphic map between complex manifolds. Let A $[B]$ be a submanifold of X $[Y]$. If $f(A) = B$ and f and $f|_A : A \to B$ are submersions, there is a canonical holomorphic map σ from $T_A X$ into $T_B Y$.*

Proof. Given $a \in X$, $Df(a)$ defines maps from $T_a X$ onto $T_{f(a)} Y$ and from $T_a A$ onto $T_{f(a)} B$. Hence $Df(a)$ induces a map from $T_a X / T_a A$ onto $T_{f(a)} X / T_{f(a)} B$. Therefore Df induces a map $\sigma : T_A X \to T_B Y$. Locally there are coordinates

$$(x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c, w_1, \ldots, w_d)$$

on X and $(u_1, \ldots, u_a, v_1, \ldots, v_c)$ on Y such that

$$A = \{z = w = 0\}, B = \{v = 0\}$$

and $f(x,y,z,w)=(x,z)$. Hence there are local coordinates

$$(x_1,\dots,x_a,y_1,\dots,y_b,\widetilde{z}_1,\dots,\widetilde{z}_c,\widetilde{w}_1,\dots,\widetilde{w}_d)$$

on T_AX and $(u_1,\dots,u_a,\widetilde{v}_1,\dots,\widetilde{v}_c)$ on T_BY such that A and B are respectively the zero sections $\{\widetilde{z}=\widetilde{w}=0\}$ of T_AX and $\{\widetilde{v}=0\}$ of T_BY and

$$\sigma(x,y,\widetilde{z},\widetilde{w})=(x,\widetilde{z}).$$

□

3.6 Blow ups

Theorem 3.6.1. *Let $(X,\mathcal{L})$ be a logarithmic contact manifold with poles along Y. Let Z be the closure of a stratum of Y contained in the singular locus of Y. Let $\tau:\widetilde{X}\to X$ be the blow up of X along Z. Then the $\mathcal{O}_{\widetilde{X}}$-module $\tau^*\mathcal{L}$ is a logarithmic contact structure on $\widetilde{X}$ with poles along $\tau^{-1}(Y)$. Let M be a manifold and let N be a divisor with normal crossings of M. Let Q be the closure of a stratum of N contained in the singular locus of N. The set $\pi^{-1}(Q)$ is the closure of a nowhere dense stratum of the set of poles of $\mathbb{P}^*\langle M/N\rangle$. Let $\rho:\widetilde{M}\to M$ be the blow up of M along Q. Set $\widetilde{N}=\rho^{-1}(N)$. Then the blow up of $\mathbb{P}^*\langle M/N\rangle$ along $\pi^{-1}(Q)$ is a logarithmic contact manifold isomorphic to $\mathbb{P}^*\langle\widetilde{M}/\widetilde{N}\rangle$ and diagram* (3.6.1) *commutes.*

$$\begin{array}{ccc} \mathbb{P}^*\langle M/N\rangle & \leftarrow & \mathbb{P}^*\langle\widetilde{M}/\widetilde{N}\rangle \\ \downarrow & & \downarrow \\ M & \leftarrow & \widetilde{M} \end{array} \tag{3.6.1}$$

If S is a natural analytic subset of M, the proper inverse image of the conormal of S equals the conormal of the proper inverse image of S.

Proof. Let θ be the logarithmic symplectic form of $\widehat{X}$. The blow up of $\widehat{X}$ along $\widehat{Z}$ is a conic manifold. Let us show that $\tau^*\theta$ is a homogeneous logarithmic symplectic form with poles along $\tau^{-1}(\widehat{Y})$. We can assume that $\widehat{X}$ is an open set of $\mathbb{C}^{2n}$ with coordinates $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ and $\widehat{Y}$ and $\widehat{Z}$ equal respectively

$$\{x_1\cdots x_\nu=0\} \text{ and } \{x_1=\cdots=x_k=0\}, \text{ where } 2\le k\le\nu. \tag{3.6.2}$$

The blow up of $\widehat{X}$ is the union of k open set $\widehat{X}_1, \ldots, \widehat{X}_k$. Let $1 \le j \le k$. There is a system of local coordinates $(x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_n)$ on $\widehat{X}_j$ such that $\tau^* x_j = x'_j$, $\tau^* x_i = x'_j x'_i$, $1 \le i \le k$, $i \neq j$, $\tau^* x_i = x'_i$, $k+1 \le i \le n$, and $\tau^* \xi_i = \xi'_i$, $1 \le i \le n$. Hence $\tau^* \theta$ equals

$$\left(\sum_{i=1, i\neq j}^{k} \xi'_i \right) \frac{dx'_j}{x'_j} + \sum_{i=1, i \neq j}^{\nu} \xi'_i \frac{dx'_i}{x'_i} + \sum_{i=\nu+1}^{n} \xi'_i dx'_i. \tag{3.6.3}$$

Set $\widehat{X} = T^*\langle M/N \rangle \setminus M$ and set $\widehat{Z} = \pi^{-1}(Q) \setminus Q$. Let E be the exceptional divisor of ρ. By the universal property of the blowing up there is a map $\widetilde{\pi}$ from the blow up of $\widehat{X}$ onto $\widetilde{M}$ such that the even dimensional version of diagram (3.6.1) commutes when we replace $T^*\langle \widetilde{M}/\widetilde{N} \rangle$ by the blow up of $\widehat{X}$. Moreover, $\tilde{\pi}|_{\widetilde{\pi}^{-1}(\widetilde{M} \setminus E)}$ equals $\pi|_{\widetilde{\pi}^{-1}(\widetilde{M} \setminus E)}$. By (3.6.3) the canonical 1-form of the blow up of $\widehat{X}$ along $\widehat{Z}$ equals the canonical 1-form of $T^*\langle \widetilde{M}/\widetilde{N} \rangle \setminus \widetilde{M}$. Set $\Gamma = \mathbb{P}^*_S\langle M/N \rangle$. By the definitions of proper inverse image and conormal, the proper inverse image $\widetilde{\Gamma}$ of Γ is the conormal of the proper inverse image of S. By theorem 3.4.5, Γ is a Legendrian variety of $\widetilde{X}$. $\square$

Let X be a manifold and let Y be a closed hypersurface of X. We will denote by $\mathcal{O}_{X(Y)}$ the sheaf of meromorphic functions f such that $f I_Y \subset \mathcal{O}_X$.

Theorem 3.6.2. *Let N be the normal crossings divisor of a complex manifold M. Let L be a well behaved submanifold of (M, N). Let τ be the blow up of X along $\Lambda = \mathbb{P}^*_L\langle M/N \rangle$. Set $E = \tau^{-1}(\Lambda)$. Let $\rho : \widetilde{M} \to M$ be the blow up of M along L. Set $\widetilde{N} = \rho^{-1}(N)$.*

(i) *If $\mathcal{L}$ is the canonical contact structure of $\mathbb{P}^*\langle M/N \rangle$, the $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{O}_{\widetilde{X}}(E)\tau^*\mathcal{L}$ is a structure of logarithmic contact manifold on $\widetilde{X}$ with poles along $\tau^{-1}(\pi^{-1}(M))$.*

(ii) *There is an injective contact transformation φ from a dense open subset Ω of $\widetilde{X}$ onto $P^*\langle \widetilde{M}/\widetilde{N} \rangle$ such that diagram* (3.6.4) *commutes.*

$$\begin{array}{ccccc} P^*\langle M/N \rangle & \xleftarrow{\tau} & \widetilde{X} \hookleftarrow \Omega & \xhookrightarrow{\varphi} & P^*\langle \widetilde{M}/\widetilde{N} \rangle \\ \pi \downarrow & & & & \downarrow \pi \\ M & & \xleftarrow{\rho} & & \widetilde{M} \end{array} \tag{3.6.4}$$

(iii) *Let S be a germ of a natural analytic subset of (M, N) at $o \in N$. Set $\Gamma = \mathbb{P}^*_S\langle M/N \rangle$. Let $\widetilde{S}$ be the proper inverse image of the blow up of M along*

L. If S has trivial limits of tangents at o and $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \Lambda$, then $\widetilde{\Gamma} \subset \Omega$ and $\varphi(\widetilde{\Gamma}) = \mathbb{P}^*_{\widetilde{S}}\langle \widetilde{M}/\widetilde{N} \rangle$, where σ denotes the canonical projection from $T_\Lambda \mathbb{P}^*\langle M/N \rangle$ onto $T_L M$ introduced in Lemma 3.5.3.

Proof. We can assume that there is an open neighbourhood U of o and a system of local coordinates

$$(x_1, \ldots, x_{n+1}, \xi_1, \ldots, \xi_{n+1})$$

on $\pi^{-1}(U)$ such that

$$\theta\,|_{\pi^{-1}(U)} = \sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n+1} \xi_i dx_i$$

and there is $\iota \in \{1, \ldots, \nu\}$ such that

$$L = \{x_\iota = \cdots = x_k = x_{n+1} = 0\}.$$

Hence,

$$\Lambda = \{x_\iota = \cdots = x_k = p_1 = \cdots = p_\nu = p_{k+1} = \cdots = p_n = x_{n+1} = 0\}.$$

Therefore Λ is contained in the open subset X of $\pi^{-1}(U)$ defined by the condition $\xi_{n+1} \neq 0$. Hence

$$\omega = dx_{n+1} - \sum_{i=1}^{\nu} p_i \frac{dx_i}{x_i} - \sum_{i=\nu+1}^{n} p_i dx_i$$

generates the logarithmic contact structure of $\mathbb{P}^*\langle M/N \rangle$ on X.
The blow up of X along Λ is the glueing of the open affine sets $U_j, j = \iota, \ldots, k, n+1$, and $V_j, j = 1, \ldots, \nu, k+1, \ldots, n$. The open sets U_i, V_j are associated to the generators x_i, p_j of the defining ideal of Λ.
If $\nu \leq j \leq k, \tau^*\omega/x_j$ equals

$$d\tfrac{x_{n+1}}{x_j} + \left(\tfrac{x_{n+1}}{x_j} - \sum_{i=\iota}^{\nu} \tfrac{p_i}{x_j} - \sum_{i=\nu+1}^{k} p_i \tfrac{x_i}{x_j} \right) \frac{dx_j}{x_j}$$
$$- \sum_{i=1}^{\iota-1} \tfrac{p_i}{x_j} \frac{dx_i}{x_i} - \sum_{i=\iota}^{\nu} \tfrac{p_i}{x_j} \frac{d\frac{x_i}{x_j}}{\frac{x_i}{x_j}} - \sum_{i=\nu+1}^{k} p_i \frac{dx_i}{x_j} - \sum_{i=k+1}^{n} \tfrac{p_i}{x_j} dx_i.$$

If $1 \le j \le \nu$, $\tau^*\omega/p_j$ equals

$$d\tfrac{x_{n+1}}{p_j} + \left(\tfrac{x_{n+1}}{p_j} - \sum_{i=\iota+1}^{\nu} \tfrac{p_i}{p_j}\right)\frac{dp_j}{p_j}$$
$$-\sum_{i=1}^{\nu} \tfrac{p_i}{p_j}\frac{dx_i}{x_i} - \sum_{\substack{i=\iota+1\\ i\neq j}}^{\nu} \tfrac{p_i}{p_j}\frac{d\frac{x_i}{p_j}}{\frac{x_i}{p_j}} - \sum_{i=k+1}^{n} \tfrac{p_i}{p_j} dx_i.$$

If $k+1 \le j \le n$, $\tau^*\omega/p_j$ equals

$$\frac{d\frac{x_i}{p_j} e^{\frac{x_{n+1}}{p_j}}}{\frac{x_i}{p_j} e^{\frac{x_{n+1}}{p_j}}} - \left(\sum_{i=\iota+1}^{\nu} \tfrac{p_i}{p_j} - \tfrac{x_{n+1}}{p_j}\right)\frac{dp_j}{p_j}$$
$$-\sum_{i=1}^{\nu} \tfrac{p_i}{p_j}\frac{dx_i}{x_i} - \sum_{\substack{i=\iota+1\\ i\neq j}}^{\nu} \tfrac{p_i}{p_j}\frac{d\frac{x_i}{p_j}}{\frac{x_i}{p_j}} - \sum_{i=\nu+1}^{n} \tfrac{p_i}{p_j} dx_i.$$

(ii) Assume that $M \subset \mathbb{C}^{n+1}$, $N = \{x_1 \cdots x_\nu = 0\}$ and $X = \mathbb{P}^*\langle M/N\rangle$. The canonical 1-form θ of $T^*\langle M/N\rangle$ equals

$$\sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n+1} \xi_i dx_i.$$

Let $\widehat{\widetilde{X}}$ be the homogeneous symplectic manifold associated to $\widetilde{X}$. Let $\widehat{\theta}$ be the canonical 1-form of $\widehat{\widetilde{X}}$. By the argument of (i) $\widehat{\widetilde{X}}$ is the union of open set $\widehat{U}_j, j = \iota, \cdots, k, n+1$ and $\widehat{V}_j, j = 1, \ldots, \nu, k+1, \cdots, n$.
Set $\widehat{\Omega} = \cup_j \widehat{U}_j$. Set $\widehat{\theta}_j = \widehat{\theta}\mid_{\widehat{U}_j}$. Endow $\mathbb{C}^{2n}$ with the coordinates

$$x_1, \ldots, x_{\iota-1}, \tfrac{x_\iota}{x_j}, x_{\iota+1}, \ldots, x_j, \ldots, x_{\nu-1}, \tfrac{x_\nu}{x_j}, x_{\nu+1}, \ldots, x_{n+1}, \eta_1, \ldots, \eta_{n+1}.$$

We can assume that

$$\widehat{U}_j = \{(\eta_1, \ldots, \eta_{n+1}) \neq (0, \cdots, 0)\}$$

and

$$\widehat{\theta}_j = \sum_{i=1}^{\iota-1} \eta_i \frac{dx_i}{x_i} + \sum_{i=\iota}^{\nu} \eta_i \frac{d\frac{x_i}{x_j}}{\frac{x_i}{x_j}} + \eta_j \frac{dx_i}{x_j} + \sum_{i=\nu+1}^{k} \eta_i d\tfrac{x_i}{x_j} + \sum_{i=k+1}^{n+1} \eta_i dx_i.$$

The blow up of M along L is the glueing of the open affine sets $M_j, j = \iota, \ldots, k, n+1$ where M_j is associated to the generator x_j of the defining ideal

L. Hence $T^*\langle M/N\rangle$ is the glueing of the open affine sets $T^*\langle M_j/N\cap M_j\rangle$, $j = \iota,\dots,k,n+1$.
Set $\widehat{W}_j = \mathring{\mathrm{T}}^*\langle M_j/N\cap M_j\rangle$. Let $\widetilde{\theta}$ be the canonical 1-form of $T^*\langle M/N\rangle$. Set $\widetilde{\theta}_j = \widetilde{\theta}\Big|_{\widehat{W}_j}$. Endow $\mathbb{C}^{2n+2}$ with the coordinates

$$x_1,\dots,x_{\iota-1},\tfrac{x_\iota}{x_j},x_{\iota+1},\dots,x_j,\dots,x_{\nu-1},\tfrac{x_\nu}{x_j},x_{\nu+1},\dots,x_{n+1},\zeta_1,\dots,\zeta_{n+1}.$$

We can assume that

$$\widehat{W}_j = \{(\zeta_1,\dots,\zeta_{n+1}) \neq (0,\cdots,0)\}$$

and

$$\widetilde{\theta}_j = \sum_{i=1}^{\iota-1}\zeta_i\frac{dx_i}{x_i} + \sum_{i=\iota}^{\nu}\zeta_i\frac{d\frac{x_i}{x_j}}{\frac{x_i}{x_j}} + \zeta_j\frac{dx_i}{x_j} + \sum_{i=\nu+1}^{k}\zeta_i d\frac{x_i}{x_j} + \sum_{i=k+1}^{n+1}\zeta_i dx_i.$$

Since

$$\widehat{X} \hookleftarrow T^*\langle M\setminus L/N\setminus L\rangle = T^*\langle M\setminus\rho^{-1}(L)/N\setminus\rho^{-1}(L)\rangle \hookrightarrow T^*\langle\widetilde{M}\setminus\widetilde{N}\rangle$$

There is a bimeromorphic contact transformation

$$\widehat{\varphi}^{-1} : \widehat{X} \to \mathring{\mathrm{T}}^*\langle\widetilde{M}/N\rangle.$$

It is enough to show that the domain of $\widehat{\varphi}$ contains $\widehat{\Omega}$ and its image equals $\mathring{\mathrm{T}}^*\langle\widetilde{M}/N\rangle$.
Since

$$U_j\setminus\pi^{-1}(L) = V_j\setminus\pi^{-1}(\rho^{-1}(L)),$$

$\eta_i = \zeta_i$ on a dense open set of their domain. Hence $\eta_i = \zeta_i$ everywhere and the domain of $\widehat{\varphi}$ contains U_j for $j = \iota,\dots,k,n+1$.
(iii) The result follows from the Lemma 3.5.2 and the arguments of the proof of theorem 3.6.1. $\square$

3.7 Resolution of quasi-ordinary surfaces

Quasi-ordinary surface singularities have a property that distinguishes them from other hypersurfaces singularities: they are stable by explosion of admissible centers. Lipman [14] used this fact to achieve the first algorithmic

proof of the existence of a desingularization procedure for this type of singularities. We will follow the procedure presented in [4]. Theorem 3.6.1 and **??** show that as long as its hypothesis are hereditary by explosion along admissible centers, we can reduce the algorithm of desingularization of the conormal of a quasi-ordinary surface to the algorithm of desingularization of the surface. Since the surface of the blow up in $\mathbb{P}^*\langle M/N\rangle$ is invariantly defined by the center of the blow up im (M/N), we only have to consider the local situation.
Let (M, o) be the germ of a complex manifold of dimension 3 and (S, o) the germ of a quasi-ordinary surface of M with characteristic pairs $(\lambda_i, \mu_i), i = 1, \cdots, s$. We will assume always that the characteristic pairs are labeled such that $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_s$ and $\mu_1 \leq \mu_2 \ldots \leq \mu_s$. Let (x, y, z) be a system of local coordinates such that $f(x, y, z) = z^m + a_{m-1}(x, y)z^{m-1} + \ldots + a_0(x, y)$, $(S, o) = \{f = 0\}$, and the discriminant of f relative to z is contained in $\{xy = 0\}$. If $\zeta_i, i = 1, \ldots, m$ are the roots of f, $f = \prod_{i=1}^m (z - \zeta_i)$. Set $\zeta = \zeta_1 = H(x^{1/n}, y^{1/n}), H \in \mathbb{C}\{x, y\}$. We call ζ a *parametrization* of (S, o). We say that a parametrization is *normalized* in (x, y, z) if

(i) $\lambda_1 \neq \mathbb{Z}$ or $\mu_1 \notin \mathbb{Z}$;

(ii) $\lambda_1 + \mu_1 < 1$ implies $\lambda_1, \mu_1 \neq 0$

We say that a normalized parametrization is *strongly normalized* if

$$(\lambda_1, \ldots, \lambda_s) \geq (\mu_1, \ldots, \mu_s) \tag{3.7.1}$$

for the lexicographic order.
Assume that we fix the hypersurface $Y' = \{z = 0\}$ and that the discriminant of f relatively to the projection $(x, y, z) \mapsto (x, y)$ is contained in $\{xy = 0\}$. Now we cannot perform changes of coordinates that take z into $z - h$. In this situation we say that the parametrization ζ is *normalized* if there is a polynomial p and a unit H of $\mathbb{C}\{x, y\}$ such that

$$\zeta = p(x, y) + x^{\lambda_1} y^{\mu_1} H(x^{1/n}, y^{1/n})$$

where $(\lambda_1, \mu_1) \notin \mathbb{Z}^2$ and $\lambda_1 + \mu_1$ is greater than the degree of p. If $p \neq 0$, let $x^\lambda y^\mu$ be the monomial of smallest degree of p.

We say that ζ is strongly normalized if $\lambda > \mu$ or $\lambda = \mu$ and (3.7.1) holds.
Assume that $\mu = 0$. If

$$\frac{\partial \zeta}{\partial y} \neq 0,$$

let $x^a y^b$ be the monomial of smallest degree of ζ that depends on y. Otherwise set $b = +\infty$.

Lipman presented a desingularization procedure for a quasi-ordinary hypersurface (c.f. [14]). Ban and McEwan have shown in [4] that the invariants of Bierstone and Milman constructive desingularization procedure depend only on the first characteristic pair and the history of the desingularization procedure. From now on, all sequences of blow-ups come from this constructive procedure. We will also give some information about the system of exceptional divisors. Let $\tilde{o}$ be a point of $\widetilde{S}$, the strict transform of (S, o) by a sequence of blow-ups. Let (x, y, z) be a system of local coordinates centered at $\tilde{o}$ such that $(\widetilde{S}, \tilde{o})$ is defined by a strongly normalized parametrization with characteristic pairs $(\widetilde{\lambda_i}, \widetilde{\mu_i}), i = 1, \ldots, \tilde{s}$. Assume that $\tilde{o}$ is a point where the maximum multiplicity has just dropped. Following [4] the exceptional divisors that pass through $\tilde{o}$ are contained in the set

$$\{\{x = 0\}, \{y - q(x, z - p(x, y)) = 0\}, \{z - q(x, y) = 0\}\},$$

where $y - q(x, z - p(x, y))|_{\{z=0\}} = y \cdot$unit or $x^{\lambda_1} y^{\mu_1}$ divides $q(x, y)$ or $q(x, y) = x^a y^b \cdot$ unit, for some a, b positive integers such that $x^a y^b$ divides $x^{\lambda_1} y^{\mu_1}$.

Let (S, o) be a quasi-ordinary surface. Let $\widetilde{S}$ be the strict transform of S with center L. Let E be the exceptional divisor. Let (x, y, z) be a system of local coordinates centered at o such that (S, o) admits a strongly normalized parametrization relatively to this system of local coordinates.

Assume $L = \{x = y = z = 0\}$. Notice that $U_x \cap E = \{(x, \frac{y}{x}, \frac{z}{x}) : x = 0\}$, $(U_x \cap E) \setminus (U_y \cup U_z) = \{(0, 0, 0)\}$.

We call $(0, 0, 0)$ the *non-generic* point of E in the affine open set $U_x \cap E$. We call the other points of $U \cap E$ the *generic points* of $U_x \cap E$.

Assume that $L = \{x = y = 0\}$. Notice that

$$U_x \cap E = \{(x, \tfrac{y}{x}, z) : x = 0\}$$

$$(U_x \cap E) \setminus U_y = \{(x, \tfrac{y}{x}, z) : x = 0, \tfrac{y}{x} = 0\} \tag{3.7.2}$$

We call the points of 3.7.2 the non-generic points of $U_x \cap E$.

Theorem 3.7.1. *[4] The special exponents of S are affected by the blow up with center at the origin according to the following table:*

	o non-generic	o generic	chart
$\lambda_1 + \mu_1 \geq 1,\ \mu_1 \neq 0$	$(\lambda_i + \mu_i - 1, \mu_i)$	$(\lambda_i + \mu_i - 1, 0)$	U_x
	$(\lambda_i, \lambda_i + \mu_i - 1)$	$(0, \lambda_i + \mu_i - 1)$	U_y
$\lambda_1 + \mu_1 < 1$	$(\lambda_i + \frac{(1+\mu_i)(1-\lambda_1)}{\mu_1} - 2, \frac{1+\mu_i}{\mu_1} - 1)$	$(\lambda_i + \frac{(1+\mu_i)(1-\lambda_1)}{\mu_1} - 2, 0)$	U_x
	$(\mu_i + \frac{(1+\lambda_i)(1-\mu_1)}{\lambda_1} - 2, \frac{1+\lambda_i}{\lambda_1} - 1)$	$(\mu_i + \frac{(1+\lambda_i)(1-\mu_1)}{\lambda_1} - 2, 0)$	U_y
	$(\frac{\lambda_i(1-\mu_1)+\mu_i\lambda_1}{1-(\lambda_1+\mu_1)}, \frac{\mu_i(1-\lambda_1)+\lambda_i\mu_1}{1-(\lambda_1+\mu_1)})$	-	U_z
$\lambda_1 > 2\ ,\mu_1 = 0$	$(\lambda_i + \mu_i - 1, \mu_i)$	$(\lambda_i + \mu_i - 1, 0)$	U_x
	$(\lambda_i, \lambda_i + \mu_i - 1)$	$(0, \lambda_i + \mu_i - 1)$	U_y
$\lambda_1 < 2\ ,\ \mu_1 = 0$	$(\frac{\lambda_i+\mu_i}{\lambda_1-1} - 1, \mu_i)$	$(\frac{\lambda_i+\mu_i}{\lambda_1-1} - 1, 0)$	U_x
	$(\lambda_i, \lambda_i + \mu_i - 1)$	$(0, \lambda_i + \mu_i - 1)$	U_y

Table 3.1:

The special characteristic exponents of S are affected by the blow up with center at a curve according to the following table:

center	conditions	o non-generic	o generic	chart
$\{x = z = 0\}$	$\lambda_1 \geq 1$ and $\mu_1 \neq 0$	$(\lambda_i - 1, \mu_i)$	$(\lambda_i - 1, 0)$	U_x
	$\lambda_1 > 2$ and $\mu_1 = 0$	$(\lambda_i - 1, \mu_i)$	$(\lambda_i - 1, 0)$	U_x
	$\lambda_1 < 2$ and $\mu_1 = 0$	$(\frac{\lambda_i}{\lambda_1-1} - 1, \mu_i)$	$(\frac{\lambda_i}{\lambda_1-1} - 1, 0)$	U_x
$\{y = z = 0\}$	$\mu_1 \geq 1$	$(\lambda_i, \mu_i - 1)$	$(0, \mu_i - 1)$	U_y

Table 3.2:

3.8 Resolution of Legendrian surfaces

Theorem 3.8.1. *Let N be a normal crossings divisor of a germ of complex manifold (M, o) of dimension 3, let (x, y, z) be a system of local coordinates of M centered at o. Let (S, o) be a germ of surface of M such that the discriminant of S relatively to the projection $(x, y, z) \mapsto (x, y)$ is contained in $\{xy = 0\} = 0$. Let Σ be the logarithmic limit of tangents of S relatively to N. Let (λ_i, μ_i) be the very special characteristic exponents of S. Assume that the parametrization of S is in strong normal form.*

(i) *If* $N = \emptyset$, Σ *is trivial if and only if*

(v1) $\mu_1 \geq 1$ *or*

(v2) $\mu_1 = 0$ *and* $\mu_2 \geq 1$.

(ii) *If* $N = \{x = 0\}$, Σ *is trivial if and only if*

(x1) $\mu_1 \geq 1$ *or*

(x2) $\mu_1 = 0$ *and* $\mu_2 \geq 1$.

(iii) *If* $N = \{y = 0\}$, Σ *is trivial if and only if*

(y1) $\lambda_1 \geq 1$.

(iv) *If* $N = \{xy = 0\}$,

Σ *is always trivial.*

(v) *If* $N = \{z = 0\}$, Σ *is trivial if and only if*

(z1) $\mu = 0$ *and* $b \geq 1$

(vi) *If* $N = \{xz = 0\}$, Σ *is trivial if and only if*

(xz1) $\mu \neq 0$ *or*

(xz2) $\mu = 0$ *and* $b \geq 1$.

(xz3) $\mu = 0$ *and* $b < 1, a = \lambda$.

(vii) *If* $N = \{yz = 0\}$,

Σ *is always trivial*

(viii) *If* $N = \{xyz = 0\}$,

Σ *is always trivial.*

Proof. (i) This case is treated in chapter 2.
(ii) Set $\theta = \xi \dfrac{dx}{x} + \eta dy + \zeta dz = \zeta(dz - p\dfrac{dx}{x} - qdy)$.
Assume that $0 < \mu_1 < 1$. There is an integer $m > 0$ and there are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}, 1 \leq i \leq 3$, such that $z = x^{\lambda_1} y^{\mu_1} \varepsilon_1$ is a parametrization of S and

$$z = x^{\lambda_1} y^{\mu_1} \varepsilon_1, \quad p = x^{\lambda_1} y^{\mu_1} \varepsilon_2, \quad q = x^{\lambda_1} y^{\mu_1 - 1} \varepsilon_3.$$

defines a parametrization of the regular part of Γ.
Set

$$\beta = \frac{\lambda_1 \alpha}{1 - \mu_1},$$

where α is a positive integer. There are $A, B \in \mathbb{C}^*$, and units δ_i of $\mathbb{C}\{t\}$, $1 \le i \le 3$, such that

$$t \mapsto \left(At^\alpha, Bt^\beta, A^{\lambda_1}B^{\mu_1}t^{\alpha\lambda_1+\beta\mu_1}\delta_1, A^{\lambda_1}B^{\mu_1}t^{\alpha\lambda_1+\beta\mu_1}\delta_2, A^{\lambda_1}B^{\mu_1-1}t^{\alpha\lambda_1+\beta(\mu_1-1)}\delta_3\right)$$

is a curve of Γ. Since $\alpha\lambda_1 + \beta\mu_1 > 0$ and $\alpha\lambda_1 + \beta(\mu_1 - 1) = 0$,

$$\left(A^{\lambda_1}B^{\mu_1}t^{\alpha\lambda_1+\beta\mu_1}\delta_2 : A^{\lambda_1}B^{\mu_1-1}t^{\alpha\lambda_1+\beta(\mu_1-1)}\delta_3 : 1\right)$$

converges to $(0 : A^{\lambda_1}B^{\mu_1-1}\delta_3(0) : 1)$. Hence Σ is not trivial.
Assume that $\mu_1 = 0$ and $\mu_2 < 1$. There are units ε_1 of $\mathbb{C}\{x^{\frac{1}{m}}\}$, ε_2, ε_3, ε_4 of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$z = x^{\lambda_1}\varepsilon_1 + x^{\lambda_2}y^{\mu_2}\varepsilon_2, \quad p = x^{\lambda_1}\delta_2, \quad q = x^{\lambda_2}y^{\mu_2-1}\delta_3$$

defines a parametrization of Γ_{reg}. Hence we can repeat the previous argument.
Assume that $\mu_1 \ge 1$. There are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \le i \le 2$, such that $z = x^{\lambda_1}y^{\mu_1}\varepsilon_1$ defines a parametrization of S and

$$q = x^{\lambda_1}y^{\mu_1-1}\varepsilon_2$$

defines a parametrization of a hypersurface that contains Γ.
Hence $\Sigma \subset \{\eta = 0\}$. By (3.4.3), $\Sigma \subset \{\xi = 0\}$.
If $\mu_1 = 0$ and $\mu_2 \ge 1$ we can obtain a proof of the triviality of Σ combining the arguments of the previous cases.
(iii) Set $\theta = \xi dx + \eta \dfrac{dy}{y} + \zeta dz = \zeta\left(dz - pdx - q\dfrac{dy}{y}\right)$.
By (3.4.3), $\Sigma \subset \{\eta = 0\}$.
If $\mu_1 = 0$, $\lambda_1 > 1$. By the arguments of case (ii), $\Sigma \subset \{\xi = 0\}$. The same arguments hold if $\lambda_1 \ge 1$ and $\mu_1 > 0$.
If $\lambda_1 < 1$, the argument of the first case considered in (ii) shows that Σ is not trivial.
(iv) By arguments very similar to the previous cases, Σ is always trivial.
(v) Set $\theta = \xi dx + \eta dy + \zeta\dfrac{dz}{z} = \zeta(\dfrac{dz}{z} - pdx - qdy)$.
Assume that $\mu \neq 0$. There are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \le i \le 3$, such that

$$z = x^\lambda y^\mu \varepsilon_1, \quad p = x^{\lambda-1}y^\mu\frac{\varepsilon_2}{z}, \quad q = x^\lambda y^{\mu-1}\frac{\varepsilon_3}{z}$$

defines a parametrization of Γ_{reg}. Hence there are units δ_i of $\mathbb{C}\{t\}$, $1 \leq i \leq 3$, such that

$$t \mapsto \left(At^\alpha, Bt^\alpha, A^\alpha B^\alpha t^{\alpha(\lambda+\mu)}\delta_1, \frac{\delta_2}{At^\alpha}, \frac{\delta_3}{Bt^\alpha} \right)$$

is a curve of Γ. Since

$$\left(\frac{\delta_2}{At^\alpha} : \frac{\delta_3}{Bt^\alpha} : 1 \right) = (B\delta_1 : A\delta_2 : ABt^\alpha)$$

converges to $(B\delta_1(0) : A\delta_2(0) : 0)$, $\Sigma = \{\zeta = 0\}$.
Assume that $\mu = 0$ and $b < 1$. Setting

$$\beta = \frac{\alpha + 1 - \lambda}{1 - b}\alpha$$

we can show that $\Sigma = \{\zeta = 0\}$.
Assume that $\mu = 0$ and $1 \leq b \leq +\infty$. There are units ε_1, of $\mathbb{C}\{x^{\frac{1}{m}}\}$, ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $2 \leq i \leq 5$, such that

$$z = x^\lambda \varepsilon_1 + x^a y^b \varepsilon_2 = x^\lambda \varepsilon_3$$

defines a parametrization of S and

$$z = x^\lambda \varepsilon_3, \quad p = \frac{\varepsilon_4}{x}, \quad q = x^{a-\lambda} y^{b-1} \varepsilon_5$$

defines a parametrization of Γ_{reg}. Moreover, Γ is contained in the hypersurfaces defined by the equations

$$x\xi + \varepsilon_4 \zeta = 0, \quad \eta + x^{a-\lambda} y^{b-1} \varepsilon_5 \zeta = 0.$$

Hence $\Sigma = \{\eta = \zeta = 0\}$.
If $\mu = 0$ and $b = +\infty$, $\Sigma \subset \{\eta = 0\}$. By the argument above, $\Sigma \subset \{\zeta = 0\}$.
(vi) Set $\theta = \xi \frac{dx}{x} + \eta dy + \zeta \frac{dz}{z} = \zeta(\frac{dz}{z} - p\frac{dx}{x} - q dy)$.
Assume that $\mu = 0$. Then $b \geq 1$, and this case is quite similar to the previous one.
Assume that $\mu \neq 0$. Then there are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \leq i \leq 3$, such that

$$z = x^\lambda y^\mu \varepsilon_1, \quad p = \varepsilon_2, \quad q = \frac{\varepsilon_3}{y}.$$

defines a parametrization of Γ_{reg}. Hence Γ is contained in the hypersurfaces

$$\xi + \varepsilon_2 \zeta = 0, \quad y\eta + \varepsilon_3 \zeta = 0.$$

Therefore $\Sigma = \{\xi = \zeta = 0\}$.
Assume that $\mu = 0, b < 1, a = \lambda$. There are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \le i \le 3$, such that

$$z = x^\lambda \varepsilon_1, \quad p = \varepsilon_2, \quad q = y^{b-1}\varepsilon_3$$

defines a parametrization of Γ_{reg}. Hence Γ is contained in the hypersurfaces determined by

$$\xi + \varepsilon_2 \zeta = 0, \quad y^{1-b}\eta + \varepsilon_3 \zeta = 0.$$

Therefore $\Sigma = \{\xi = \zeta = 0\}$.
Assume that $\mu = 0, b < 1, a > \lambda$. Then, setting

$$\beta = \alpha(a - \lambda)/(1 - b),$$

it can be shown by the previous methods that there is a $u \in \mathbb{C}^*$ such that

$$\Sigma \supset \{(u : v : 1) : v \in \mathbb{C}^*\}.$$

(vii) This case is symmetric with the previous one, except that, because we are assuming a parametrization in strong normal form, Σ is always trivial.
(viii) Set

$$\theta = \xi \frac{dx}{x} + \eta \frac{dy}{y} + \zeta \frac{dz}{z} = \zeta(\frac{dz}{z} - p\frac{dx}{x} - q\frac{dy}{y}).$$

Assume that $\mu \neq 0$. There are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \le i \le 3$, such that Γ is contained in the hypersurfaces with parametrizations defined by

$$p = \lambda x^\lambda y^\mu \frac{\varepsilon_2}{z} \text{ and } q = \mu x^\lambda y^\mu \frac{\varepsilon_3}{z},$$

where $\varepsilon_1(0) = \varepsilon_2(0) = \varepsilon_3(0)$. Hence

$$\Sigma = \{(\lambda : \mu : 1)\}.$$

A similar argument shows that we arrive to the same conclusion when $\mu = 0$. $\square$

Table 3.3 summarizes the results of the previous theorem and indicates what is in each case the admissible center chosen by the resolution algorithm. In some cases different centers can be chosen, depending on the previous history of the resolution procedure. We set

$$\sigma_0 = \{x = y = z = 0\}, \sigma_x = \{x = z = 0\} \text{ and } \sigma_y = \{y = z = 0\}.$$

Divisor	Conditions	Label	Center
$\emptyset$	$\mu_1 \geq 1$	v1	σ_0
	$\mu_1 = 0$ and $\mu_2 \geq 1$	v2	σ_x
$\{x = 0\}$	$\mu_1 \geq 1$	x1	σ_0, σ_y
	$\mu_1 = 0$ and $\mu_2 \geq 1$	x2	σ_x
$\{y = 0\}$	$\lambda_1 \geq 1$ and $\mu_1 \neq 0$	y1	σ_x, σ_0
$\{xy = 0\}$		xy1	σ_0 if $\lambda_1 < 1$ or $\mu_1 = 0$.
		xy2	σ_x if $\lambda_1 \geq 1$.
		xy3	σ_y if $\mu_1 \geq 1$.
$\{z = 0\}$	$\mu = 0$ and $b \geq 1$	z1	σ_x
$\{xz = 0\}$	$\mu \neq 0$	xz1	σ_0
	$\mu = 0$ and $b \geq 1$	xz2	σ_x
	$\mu = 0, b < 1$, and $a = \lambda$	xz3	σ_x
	$\mu \geq 1$	xz4	σ_y
$\{yz = 0\}$		yz1	σ_0 if $\lambda < 1$.
		yz2	σ_0, σ_x if $\lambda \geq 1$.
$\{xyz = 0\}$		xyz1	σ_0
		xyz2	σ_x if $\lambda \geq 1$.
		xyz3	σ_y if $\mu \geq 1$.

Table 3.3: List of conditions for generic position and admissible centers.

Table (3.3) is a compilation of tables (3.4) - (3.11), that describe the desingularization procedure considered in [4]. We do not transcribe here the notations that describe the history of the procedure since we make no use of them.

Center	Conditions
σ_0	$\mu_1 > 0$
σ_x	$\mu_1 = 0$

Table 3.4: Divisor $N = \emptyset$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $\lambda_1 \geq 1$ and $0 < \mu_1 < 1$ $\mu_1 > 0$ and $i = k(o_i), k(o_i) + 1$
σ_x	$\mu_1 = 0$
σ_y	$\mu_1 \geq 1$ and $i > k(o_i) + 1$

Table 3.5: Divisor $N = \{x = 0\}$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $i = k(o_i), k(o_i) + 1$
σ_x	$\lambda_1 \geq 1$ and $i > k(o_i) + 1$

Table 3.6: Divisor $N = \{y = 0\}$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $i = k(o_i), k(o_i) + 1$ $i > k(o_i) + 1, \mu_1 = 0$, and $\{y = 0\} \subset E_i(o_i)$
σ_x	$\lambda_1 \geq 1, i > k(o_i) + 1, \mu_1 > 0$, and $l = i$ $\lambda_1 \geq 1, i > k(o_i) + 1$, and $0 < \mu_1 < 1$ $\mu_1 = 0, i > k(o_i) + 1$, and $\{y = 0\} \not\subset E_i(o_i)$
σ_y	$\mu_1 \geq 1, i > k(o_i) + 1$, and $k = i$

Table 3.7: Divisor $N = \{xy = 0\}$.

Center	Conditions
σ_0	$\mu\lambda \neq 0$
σ_x	$\mu = 0$

Table 3.8: Divisor $N = \{z = 0\}$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $\lambda = \lambda_1 \geq 1$ and $0 < \mu = \mu_1 < 1$ $\mu > 0$, and $i = k(o_i), k(o_i) + 1$
σ_x	$\mu = 0$
σ_y	$\mu \geq 1$ and $i > k(o_i) + 1$

Table 3.9: Divisor $N = \{zx = 0\}$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $i = k(o_i), k(o_i) + 1$
σ_x	$\lambda \geq 1$ and $i > k(o_i) + 1$

Table 3.10: Divisor $N = \{zy = 0\}$.

Center	Conditions
σ_0	$\lambda_1 < 1$ $i = k(o_i), k(o_i) + 1$ $i > k(o_i) + 1, \mu = 0$ and $\{y = 0\} \subset E_i(o_i)$
σ_x	$\lambda \geq 1, i > k(o_i) + 1, \mu > 0$ and $l = i$ $\lambda = \lambda_1 \geq 1, i > k(o_i) + 1$ and $0 < \mu = \mu_1 < 1$ $\mu = 0, i > k(o_i) + 1$ and $\{y = 0\} \not\subset E_i(o_i)$
σ_y	$\mu \geq 1, i > k(o_i) + 1$ and $k = i$

Table 3.11: Divisor $N = \{zxy = 0\}$.

Theorem 3.8.2. *Let N be a normal crossings divisor of a germ of a complex manifold (M,o) of dimension 3. Let S be the germ of a quasi-ordinary surface at o with trivial limit of tangents. Set $\Gamma = \mathbb{P}^*_S\langle M/N\rangle$. Let L be one of the admissible centers for S considered in table (3.3). Set $\Lambda = \mathbb{P}^*_L\langle M/N\rangle$. Let $\widetilde{\Gamma}$ be the proper inverse image of Γ by the blow up of $\mathbb{P}^*\langle M/N\rangle$ with center Λ. Then*

$$\widetilde{\Gamma} \subset \Omega \text{ and } \widetilde{\Gamma} = \mathbb{P}^*_{\widetilde{S}}\langle M/N\rangle.$$

Proof. Let λ be the only limit of tangents of S at o. By Lemma 3.5.2 it is enough to prove that

$$C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \Lambda. \tag{3.8.1}$$

holds in order to prove that $\widetilde{\Gamma} \subset \Omega$.
(i) Set $\theta = \xi dx + \eta dy + \zeta dz = \zeta(dz - pdx - qdy)$.
(v1) Set $L = \{x = y = z = 0\}$. Hence $\Lambda = \{x = y = z = 0\}$. We identify L with the zero section $\{\widetilde{x} = \widetilde{y} = \widetilde{z} = 0\}$ of $\mathbb{P}_L M$. We identify Λ with the zero section $\{\widetilde{x} = \widetilde{y} = \widetilde{z} = 0\}$ of $\mathbb{P}_\Lambda\mathbb{P}^*\langle M/N\rangle$. Near λ,

$$\sigma : \mathbb{P}_\Lambda\mathbb{P}^*\langle M/N\rangle \to \mathbb{P}_L M$$

is given by

$$\sigma(\widetilde{x}, \widetilde{y}, \widetilde{z}, p, q) = (\widetilde{x}, \widetilde{y}, \widetilde{z}).$$

Hence, $\sigma^{-1}(L) = \Lambda$.
(v2) Set $L = \{x = z = 0\}$. Hence $\Lambda = \{x = z = q = 0\}$ and $\sigma(\widetilde{x}, y, \widetilde{z}, p, \widetilde{q}) = (\widetilde{x}, y, \widetilde{z})$. Since $\mu_1 = 0$,

$$\lambda_2 \geq \lambda_1 \geq 1. \tag{3.8.2}$$

Since

$$z = a_{\lambda_1 0}x^{\lambda_1} + \ldots + a_{\lambda_2\mu_2}x^{\lambda_2}y^{\lambda_2} + \cdots$$

there is a unit ε of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$q = \frac{\partial z}{\partial y} = x^{\lambda_2}y^{\mu_2 - 1}\varepsilon \tag{3.8.3}$$

for some integer m. It follows from (3.8.2) and (3.8.3) that Γ is contained in a hypersurface

$$q^n + \sum_{i=0}^{n-1} a_i q^i = 0$$

where $a_i \in \mathbb{C}\{x, y\}$ and $a_i \in (x)^{n-i}$. Hence there are $\widetilde{a}_i \in (\widetilde{x})$ such that $C_\Lambda(\Gamma)$ is contained in an hypersurface

$$\widetilde{q}^n + \sum_{i=0}^{n-1} \widetilde{a}_i \widetilde{q}^i = 0$$

and (3.8.1) holds.
(ii) Set $N = \{x = 0\}$, hence

$$\theta = \xi \frac{dx}{x} + \eta dy + \zeta dz = \zeta(dz - p\frac{dx}{x} - qdy).$$

(x1) Assume that $\mu_1 \geq 1$. If $L = \{x = y = z = 0\}$, $\Lambda = \{x = y = z = p = 0\}$ and $\sigma(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{p}, q) = (\widetilde{x}, \widetilde{y}, \widetilde{z})$.
Since there is a unit ε such that $z = x^{\lambda_1} y^{\mu_1} \varepsilon$, there is a unit δ such that $p = x^{\lambda_1} y^{\mu_1} \delta$. Since $\lambda_1 + \mu_1 > 2$,

$$C_\Lambda(\Gamma) \subset \{\widetilde{p} = 0\}.$$

Hence, (3.8.1) holds.
Assume $L = \{y = z = 0\}$. Hence $\Lambda = \{y = z = p = 0\}$. If $\mu_1 > 1$, the argument is similar to the previous one. Assume $\mu_1 = 1$.
There are units $\varepsilon_1, \varepsilon_2$ of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$z = x^{\lambda_1} y \varepsilon_1, \quad p = x^{\lambda_1} y \varepsilon_2.$$

Hence $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \{\widetilde{p} = 0\}$.
(x2) Since $L = \{x = z = 0\}$, $\Lambda = \{x = z = p = q = 0\}$. There are units ε of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $\varepsilon_1, \ldots, \varepsilon_4$ of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$z = x^{\lambda_1} \varepsilon_1 = x^{\lambda_1} \varepsilon + x^{\lambda_2} y^{\mu_2} \varepsilon_2, \quad p = x^{\lambda_1} \varepsilon_3, \quad q = x^{\lambda_2} y^{\mu_2 - 1} \varepsilon_4.$$

Since $\mu_1 = 0$, $\lambda_1 > 1$. Therefore $C_\Lambda(\Gamma) \subset \{\widetilde{p} = 0\}$.
Since $\lambda_2 \geq \lambda_1 > 1$, $\mu_2 \geq 1$. Therefore $C_\Lambda(\Gamma) \subset \{\widetilde{q} = 0\}$.
(iii) This case is similar to the previous one.
(iv) If $N = \{xy = 0\}$,

$$\theta = \xi \frac{dx}{x} + \eta \frac{dy}{y} + \zeta dz = \zeta(dz - p\frac{dx}{x} - q\frac{dy}{y}).$$

Set $L = \{x = y = z = 0\}$. Hence $\Lambda = \{x = y = z = p = q = 0\}$.

There are units ε_i of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $1 \leq i \leq 3$, such that

$$z = x^{\lambda_1} y^{\mu_1} \varepsilon_1, \quad p = z^{\lambda_1} y^{\mu_1} \varepsilon_2, \quad q = x^{\lambda_1} y^{\mu_1} \varepsilon_3$$

defines a parametrization of Γ_{reg}. Hence there are units $\varepsilon_4, \varepsilon_5$ of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$z = x^{\lambda_1} y^{\mu_1} \varepsilon_1, \quad p = z\varepsilon_4, \quad q = z\varepsilon_5$$

define a parametrization of Γ_{reg}. Therefore Γ is contained in the hypersurfaces with parametrizations given by $p = z\varepsilon_4$, $q = z\varepsilon_5$. Hence Γ is contained in hypersurfaces of the type

$$p^k + \sum_{i=0}^{k-1} a_i p^i = 0, \quad q^l + \sum_{i=0}^{l-1} b_i q^i = 0,$$

where $a_i \in (z^{k-i}), b_i \in (z^{l-i})$.
Assume that $\mu_1 = 0$, $L = \{x = z = 0\}$ or $\mu_1 \geq 1$, $L = \{y = z = 0\}$. In both cases $C_\Lambda(\Gamma) \subset \{\widetilde{p} = \widetilde{q} = 0\}$ by the standard arguments.
(v) If $N = \{z = 0\}$,

$$\theta = \xi dx + \eta dy + \zeta \frac{dz}{z} = \zeta \left(\frac{dz}{z} - pdx - qdy \right) = \xi \left(dx - rdy - s\frac{dz}{z} \right).$$

(z1) Assume that $\mu = 0$, $b \geq 1$ and $L = \{x = z = 0\}$. Hence $\Lambda = \{x = z = r = s = 0\}$. There are units $\varepsilon_1, \varepsilon_2$ of $\mathbb{C}\{x^{\frac{1}{m}}\}$, $\varepsilon_1, \ldots, \varepsilon_8$ of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ such that

$$z = x^\lambda \varepsilon_1 + x^a y^b \varepsilon_3 = x^\lambda \varepsilon_4 \tag{3.8.4}$$

$$p = \frac{x^{\lambda-1}}{z}\varepsilon_2 = \frac{\varepsilon_5}{x} = \frac{1}{x\varepsilon_6} \tag{3.8.5}$$

$$q = \frac{x^a y^{b-1}}{z}\varepsilon_7 = x^{a-\lambda} y^{b-1} \varepsilon_8. \tag{3.8.6}$$

Hence $x\varepsilon_6 \xi + \zeta = 0$, $\eta + x^{a-\lambda} y^{b-1} \varepsilon_8 \zeta = 0$.
Therefore Γ is contained in the hypersurfaces with parametrizations given by

$$s = \varepsilon_6 x, \quad r + x^{a-\lambda} y^{b-1} \varepsilon_8 s = 0.$$

Hence $C_\Lambda(\Gamma) \cap \{\widetilde{x} = \widetilde{z} = 0\} \subset \{\widetilde{r} = \widetilde{s} = 0\}$.
(vi) If $N = \{xz = 0\}$,

$$\theta = \xi \frac{dx}{x} + \eta dy + \zeta \frac{dz}{z} = \eta (dy - r\frac{dx}{x} - s\frac{dz}{z}).$$

Assume that $\mu \neq 0$. Set $L = \{x = y = z = 0\}$. Hence $\Lambda = \{x = y = z = r = s = 0\}$. Notice that

$$\begin{aligned}
&z = x^\lambda y^\mu \varepsilon_1, \quad p = \frac{x^\lambda y^\mu}{z}\varepsilon_2 = \varepsilon_3, \quad q = \frac{x^\lambda y^{\mu-1}}{z}\varepsilon_4 = \frac{1}{\varepsilon_5 y}.\\
&\xi + \varepsilon_3 \zeta = 0, \quad y\varepsilon_5 \eta + \zeta = 0.\\
&u = \varepsilon_3 s = 0, \quad s = y\varepsilon_5.
\end{aligned}$$

Hence $C_\Lambda(\Gamma) \cap \rho^{-1}(L) \subset (\widetilde{r} = \widetilde{s} = 0)$.
If $\mu \neq 0$ and $L = \{y = z = 0\}$, then $\Lambda = \{y = z = r = s = 0\}$, and this case is solved in a similar fashion.
Assume that $\mu = 0$. In this case $L = \Lambda = \{x = z = 0\}$. This situation is solved by theorem 3.6.1.
(vii) Set $N = \{yz = 0\}$.
If $\mu \neq 0$, we are in the situation of (xz1).
Assume that $\mu = 0$. Set $\theta = \xi dx + \eta \frac{dy}{y} + \zeta \frac{dz}{z} = \xi(dx - r\frac{dy}{y} - s\frac{dz}{z})$.
Following the scheme of the previous cases,

$$\begin{aligned}
z = x^\lambda \varepsilon_1 = x^\lambda \varepsilon_2 + x^a y^b \varepsilon_3, \text{ where } \frac{\partial \varepsilon_2}{\partial y} = 0\\
p = \frac{x^{\lambda-1}}{z}\varepsilon_4 = \frac{1}{x\varepsilon_5}, \quad q = \frac{x^a y^b}{z}\varepsilon_6 = x^{a-\lambda} y^b \varepsilon_7,\\
x\varepsilon_5 \xi + \zeta = 0, \quad \eta + a^{a-\lambda} y^b \varepsilon_7 \zeta = 0,\\
s = x\varepsilon_5, \quad r + x^{a-\lambda} y^b \varepsilon_7 s = 0.
\end{aligned} \tag{3.8.7}$$

It follows from 3.8.7 that $C_\Lambda(\Gamma) \cap \rho^{-1}(L) \subset \{\widetilde{r} = \widetilde{s} = 0\}$ if $L = \{x = y = z = 0\}$ or $L = \{x = z = 0\}$. □

Example 3.8.3. Given $\delta \in \mathbb{C}\{x^{\frac{1}{m}}\}$, $\varepsilon \in \mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$, $\lambda > 1$ and $0 < b < 1$, the surface S with parametrization

$$z = x^\lambda \delta + x^\lambda y^b \varepsilon$$

verifies the condition (xz3) of Theorem 3.8.1. Hence its logarithmic limits of tangents relatively to the divisor $\{xz = 0\}$ is trivial. The proper inverse image of S by the blow up with center $\{x = z = 0\}$ admits the parametrization

$$\tfrac{z}{x} = x^{\lambda-1}\delta_1 + x^{\lambda+b-1}y^b\varepsilon_1.$$

By theorem 3.8.1, the logarithmic limit of tangents of $\widetilde{S}$ relatively to the divisor $\{x\tfrac{z}{x} = 0\}$ is not trivial.

Example (3.8.3) shows that the triviality of limits of tangents is not hereditary by blowing up. Lemma 3.8.4 solves this problem.

Lemma 3.8.4. *Let N be the normal crossings divisor of a germ of complex manifold (M, o) of dimension three. Let S be a surface of M such that the logarithmic limit of tangents of S along N is trivial. Let $\pi : \widetilde{M} \to M$ be the blow up of M along an admissible center for S and N. Let E be the exceptional divisor of π. Let $p \in \widetilde{S} \cap E$. If S, N do not verify condition* (xz3) *of table (3.3) at o*
(i) *$\widetilde{S}$ has trivial logarithmic limit of tangents along $\widetilde{N}$ at p.*
(ii) *$\widetilde{S}, \widetilde{N}$ do not verify condition* (xz3) *at p.*

Proof. We will denote by ε_i a unit of $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$ and by δ_i a unit of $\mathbb{C}\{x^{\frac{1}{m}}\}$, for a convenient m. We will denote by (xy), (yz), (xyz) the situations (xyi), (yzi), (xyzi) for each i.
(v1) We can assume that $z = x^{\lambda_1}y^{\mu_1}\varepsilon_1$. On the chart $(x, \tfrac{y}{x}, \tfrac{z}{x})$,

$$\tfrac{z}{x} = x^{\lambda_1+\mu_1-1}\tfrac{y}{x}^{\mu_1}\varepsilon_2.$$

Since $\lambda_1 \geq \mu_1 \geq 1$ and $(\lambda_1, \lambda_2) \notin \mathbb{Z}^2$, we are in situation (x1) at each point of $\widetilde{N} = \{x = 0\}$. The same happens in the chart $(\tfrac{x}{y}, y, \tfrac{z}{y})$.
(v2) We can assume that $z = x^{\lambda_1}\delta_1 + x^{\lambda_2}y^{\mu_2}\varepsilon_1$. On the chart $(x, y, \tfrac{z}{x})$, $\widetilde{N} = \{x = 0\}$ and

$$\tfrac{z}{x} = x^{\lambda_1-1}\delta_2 + x^{\lambda_2-1}y^{\mu_2}\varepsilon_2.$$

If $\lambda_1 > 2$ we are in situation (x2). Assume $\lambda_1 < 2$. By table (3.2), $\widetilde{S}$ admits the parametrization

$$x = (\tfrac{z}{x})^{\frac{\lambda_1}{\lambda_1-1}} + (\tfrac{z}{x})^{\frac{\lambda_2}{\lambda_1-1}}y^{\mu_2}\varepsilon_4.$$

Hence we are in situation (z1).
The cases (x1),(x2),(y1) are similar to the previous cases.

(xy1) Assume that $\lambda_1 + \mu_1 \geq 1$. In the chart $(x, \frac{y}{x}, \frac{z}{x})$, $\widetilde{N} = \{x\frac{y}{x} = 0\}$ and $\widetilde{S}$ admits the parametrization

$$\tfrac{z}{x} = x^{\lambda_1+\mu_1-1}\tfrac{y}{x}^{\mu_1}\varepsilon_2.$$

Let o be a point of $\widetilde{N}$ where $x \neq 0$. If $\mu_1 = 0$, $\widetilde{S}$ is smooth at o. Otherwise

$$\tfrac{y}{x} = (\tfrac{z}{x})^{\frac{1}{\mu_1}}\varepsilon_3.$$

Hence we are in situation (z1). The same holds at a point of $\widetilde{N}$ where $\frac{y}{x} \neq 0$.
The situation is similar in the chart $(\frac{x}{y}, y, \frac{z}{y})$.
Assume that $\lambda_1 + \mu_1 < 1$. In the chart $(\frac{x}{z}, \frac{y}{z}, z)$, $\widetilde{N} = \{\frac{x}{z}\frac{y}{z}z = 0\}$ and $\widetilde{S}$ admits the parametrization

$$z = (\tfrac{x}{z})^{\frac{\lambda_1}{1-\lambda_1-\mu_1}}(\tfrac{y}{z})^{\frac{\mu_1}{1-\lambda_1-\mu_1}}\varepsilon_2.$$

We are in the situation (xyz) at $(0,0,0)$. Let o be a point of $\widetilde{N}$ where $\frac{x}{z} \neq 0$. If $\mu_1 = 0$ or $2\lambda_1 + \mu_1 = 1$, $\widetilde{S}$ is smooth at o. If $\mu_1 \neq 0$ and $2\lambda_1 + \mu_1 \neq 1$, we are in situation (x2) or in situation (z1). The situation is similar at a point of $\widetilde{N}$ where $\frac{y}{z} \neq 0$.
The cases (xy2) and (xy3) are quite similar.
(z1) The blow up produces situation (z1) if $\lambda \geq 2$ and (x1) if $\lambda < 2$.
(xz1) Assume that we blow up σ_0. Assume that $\lambda + \mu \geq 1$. In the chart $(x, \frac{y}{x}, \frac{z}{x})$, $\widetilde{N} = \{x\frac{z}{x} = 0\}$ and $\widetilde{S}$ admits the parametrization

$$\tfrac{z}{x} = x^{\lambda+\mu-1}\tfrac{y}{x}^{\mu}\varepsilon_2.$$

Assume that $\lambda + \mu > 1$. We are in situation (xz1) at $(0,0,0)$. Let o be a point of $\widetilde{N}$. If $x \neq 0$, $\widetilde{S}$ is in situation (yz) at o.
Assume that $\lambda + \mu = 1$. Setting $\widetilde{x} = \frac{z}{x}, \widetilde{y} = x, \widetilde{z} = \frac{y}{x}$, $\widetilde{S}$ admits the parametrization

$$\widetilde{z} = \widetilde{x}^{\frac{1}{\mu}}\varepsilon_4 \text{ and } \widetilde{N} = \{\widetilde{x}\widetilde{y} = 0\}.$$

We are in situation (xy) at $(0,0,0)$. Let o be a point of $\widetilde{N}$. If $\widetilde{y} \neq 0$ we are in situation (x2) at o.
The case $\lambda + \mu < 1$ is similar to the case $\lambda + \mu > 1$.
(xz2) We can assume that $z = x^{\lambda}\delta_1 + x^a y^b \varepsilon_1$. On the chart $(x, y\frac{z}{x})$, $\widetilde{N} = \{x\frac{z}{x} = 0\}$ and

$$\tfrac{z}{x} = x^{\lambda-1}\delta_2 + x^{a-1}y^b\varepsilon_2.$$

If $\lambda \geq 2$ we are in situation (xz2). Assume $\lambda < 2$. By table (3.2), $\widetilde{S}$ admits the parametrization

$$x = (\tfrac{z}{x})^{\frac{\lambda}{\lambda-1}}\delta_3 + (\tfrac{z}{x})^{\frac{a}{\lambda-1}}y^b\varepsilon_3$$

and we are in situation (xz2) at $(0,0,0)$.
(xz4) We can assume that $z = x^\lambda\delta_1 + x^a y^b\varepsilon$.
On the chart $(x, y, \frac{z}{y})$, $\widetilde{N} = \{xy\frac{z}{y} = 0\}$ hence we are in situation (xyz) at the origin.
The remaining cases are similar to those considered above. □

Theorem 3.8.5. *Let S be a quasi-ordinary surface of a germ of complex manifold of dimension 3, (M, o). Assume that the limit of tangents of S at o is trivial. Let $M_0 = M, \Gamma = \mathbb{P}^*_S M$. Let*

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_m$$

be the sequence of blow ups that desingularizes S. Let L_i be the center of the blow up $M_{i+1} \to M_i$ for $0 \leq i \leq m-1$. Let S_i be the proper inverse image of S by the map $M_i \to M_0$. Let N_i be the inverse image of $\{o\}$ by the map $M_i \to M_0$. Set $\Gamma_i = \mathbb{P}^*_{S_i}\langle M_i/N_i\rangle$, $\Lambda_i = \mathbb{P}^*_{L_i}\langle M_i/N_i\rangle$. Let X_i be the blow up of $\mathbb{P}^*\langle M_i/N_i\rangle$ along Λ_i. There are inclusion maps $\mathbb{P}^*\langle M_{i+1}/N_{i+1}\rangle \hookrightarrow X_i$ such that the diagram (3.8.8) commutes.

$$\begin{array}{ccccccc} \mathbb{P}^*M_0 & \leftarrow & \mathbb{P}^*\langle M_1/N_1\rangle & \leftarrow & \cdots & \leftarrow & \mathbb{P}^*\langle M_m/N_m\rangle \\ \downarrow & & \downarrow & & & & \downarrow \\ M_0 & \leftarrow & M_1 & \leftarrow & \cdots & \leftarrow & M_m \end{array} \qquad (3.8.8)$$

Moreover Γ_m is a regular Lagrangean variety transversal to the set of poles of $\mathbb{P}\langle M_m/N_m\rangle$ and Γ_m is the proper inverse image of Γ_0 by the map $\mathbb{P}^*\langle M_m/N_m\rangle \to \mathbb{P}^*M$.

Proof. This result is an immediate consequence of the Theorem of resolution of singularities for quasi-ordinary surface singularities, Theorems 3.6.1, 3.6.2, 3.8.1, 3.8.2 and Lemma 3.8.4. □

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