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Hugo Ricardo Nabais Tavares

Doutoramento em Matemática

(Análise Matemática)

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Tese orientada pelos Prof. Doutores
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To Célia

"Chega de saudade..."
(Vinícius de Moraes)

Acknowledgements

"I am not an Athenian or a Greek, but a citizen of the world." (Socrates)

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Abstract

This thesis consists of three parts. In the first part we consider a system of partial differential equations arising in the theory of Bose-Einstein condensation, namely

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \omega_i u_i^3 - \beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{m} u_j^2, \\ u_i \in H_0^1(\Omega), \quad u_i > 0 \quad \text{in } \Omega, \end{cases} \qquad i = 1, \ldots, m,$$

where λ_i, ω_i are real parameters and Ω is a bounded domain of $\mathbb{R}^N$, $N = 2, 3$. We study the asymptotic behavior of the solutions of such system as $\beta \to +\infty$. We deduce that the limiting profiles are segregated, and study the regularity properties of their nodal sets. Moreover, in the particular case of $m = 2$ equations, we construct multiple solutions using a common minimax structure, proving convergence of both critical levels and optimal sets.

In the second part we deal with the strongly coupled system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = g(v), \\ -\varepsilon^2 \Delta v + V(x) v = f(u), \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega, \end{cases}$$

where Ω is a domain of $\mathbb{R}^N$, $N \geqslant 3$, not necessarily bounded, and $f(s) \sim s^{p-1}, g(s) \sim s^{q-1}$ with $1/p + 1/q > (N-2)/N$. We prove that there exist positive solutions $u_\varepsilon, v_\varepsilon$ such that the sum $u_\varepsilon + v_\varepsilon$ concentrates, as $\varepsilon \to 0^+$, at a prescribed finite number of local minimum points of V, possibly degenerate.

Finally, in the third and last part of the thesis, we deduce the existence of infinitely many sign-changing solutions for an elliptic problem of the type

$$-\Delta u = g(x, u) + f(x, u), \qquad u \in H_0^1(\Omega),$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, g has superlinear and subcritical growth and is odd in u, and f is a lower order term, not necessarily odd in u. Moreover, we also deal with the fourth order problem

$$\Delta^2 u = g(x, u) + f(x, u), \qquad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Keywords

Elliptic systems, variational methods, qualitative properties of solutions, free boundary problems, monotonicity formulae.

Resumo

Esta tese centra-se no estudo de sistemas de equações com derivadas parciais não lineares, de tipo elíptico. Todos os sistemas tratados possuem uma estrutura variacional, pelo que as suas soluções podem ser obtidas como pontos críticos de um determinado funcional (dito "funcional energia") definido num espaço de Hilbert.

Os problemas abordados são essencialmente de dois tipos. Por um lado, abordamos sistemas de tipo gradiente, onde os termos não lineares são o gradiente de um potencial,

$$-\Delta u_i + V_i(x)u_i = \partial_{u_i}F(u_1,\ldots,u_m), \qquad i=1,\ldots,m, \tag{1}$$

com funcional energia dado por

$$(u_1,\ldots,u_m)\mapsto \int\Big(\sum_{i=1}^m \frac{1}{2}(|\nabla u_i|^2+V_i(x)u_i)-F(u_1,\ldots,u_m)\Big).$$

Por outro lado, lidamos também com sistemas de tipo Hamiltoniano da forma

$$\begin{cases} -\Delta u + V(x)u = \partial_v F(u,v)\\ -\Delta v + V(x)v = \partial_u F(u,v), \end{cases} \tag{2}$$

com funcional energia

$$(u,v)\mapsto \int(\langle\nabla u,\nabla v\rangle + V(x)uv - F(u,v)).$$

Interessamo-nos nesta tese por diversas temáticas, tais como a existência de soluções positivas ou soluções que mudam de sinal, obtenção de estimativas *a priori*, e estudo do comportamento assimptótico de soluções. Num dos capítulos, esse estudo assimptótico leva-nos naturalmente a considerar um problema de fronteira livre, para o qual demonstramos resultados de regularidade. No tratamento destes assuntos recorremos a técnicas provenientes de diferentes áreas da matemática. Destacamos a utilização de resultados de Teoria da Medida e de Teoria Geométrica da Medida, o uso das Fórmulas de Monotonia de Alt-Caffarelli-Friedman e de Almgren, e a utilização de Métodos Variacionais (estes últimos na obtenção dos resultados de existência de soluções).

O presente trabalho encontra-se dividido em três partes, que passamos seguidamente a resumir.

Na primeira parte abordamos o sistema de competição-difusão

$$\begin{cases} -\Delta u_i + \lambda_i u_i = w_i u_i^3 - \beta u_i \sum_{\substack{j=1\\ j\neq i}}^m u_j^2, \\ u_i\in H^1_0(\Omega), \quad u_i>0 \text{ in } \Omega, \end{cases} \qquad i=1,\ldots,m, \tag{3}$$

(de tipo (1)) onde Ω é um domínio regular de $\mathbb{R}^N$ ($N = 2, 3$), e λ_i, w_i ($i = 1, \ldots, m$) são parâmetros reais. Impomos $\beta > 0$, o que significa que o termo de interacção $\beta u_i \sum_{j=1_{j\neq i}}^m u_j^2$ é de tipo repulsivo. O sistema em apreço surge associado ao estudo teórico do fenómeno físico designado por condensação de Bose–Einstein, o que possivelmente motiva a intensa actividade científica em redor de (3) existente na última década.

Através do uso de Métodos Variacionais, vários autores demonstram (para diferentes valores de λ_i, w_i e β) a existência de soluções de (3), principalmente no caso particular de um sistema com duas equações. Para além da questão da existência, interessamo-nos nesta tese pelo estudo do comportamento assimptótico das soluções quando o parâmetro de interacção β tende para mais infinito, bem como pela dedução de propriedades qualitativas das eventuais configurações limite. Neste campo, são de destacar os trabalhos seminais de Conti, Terracini, Verzini (2003) e de Caffarelli, Lin (2008), cujo estudo efectuado para soluções de energia mínima serviu de ponto de partida para o nosso trabalho.

Dada uma família de soluções $U_\beta = (u_{1,\beta}, \ldots, u_{m,\beta})$ de (3), limitada uniformemente em β na norma $L^\infty(\Omega)$, demonstramos estimativas uniformes na norma dos espaços de Hölder $C^{0,\alpha}(\Omega)$ para todo o $0 < \alpha < 1$. Como corolário, concluímos que, a menos de uma subsucessão, U_β converge para uma configuração limite $U = (u_1, \ldots, u_m)$ em $H_0^1(\Omega) \cap C^{0,\alpha}(\Omega)$ quando $\beta \to +\infty$. A demonstração deste facto baseia-se na dedução de novos teoremas de não existência de tipo Liouville (baseados por sua vez nas Fórmulas de Monotonia de Alt-Caffarelli-Friedman e de Almgren). Supondo que a conclusão não é verdadeira, isso permite-nos a construção de uma sucessão de tipo "blowup", que converge para uma solução de um dos problemas para os quais foram demonstrados os resultados de não existência.

No limite, deduzimos que as configurações $U = (u_1, \ldots, u_m)$ obtidas são Lipchitzianas, que os suportes de diferentes componentes têm interior disjunto ($u_i \cdot u_j \equiv 0$ para todo o $i \neq j$), e que

$$-\Delta u_i + \lambda_i u_i = w_i u_i^3 \qquad \text{no conjunto aberto } \{u_i > 0\}.$$

Na dedução da regularidade das configurações limite é de novo essencial o uso da Fórmula de Monotonia de Almgren. Posto isto, o novo objecto de estudo passa a ser o conjunto nodal do vector U, nomeadamente $\Gamma_U = \{x \in \Omega : U(x) = 0\}$, que corresponde à fronteira dos conjuntos $\{u_i > 0\}$. Demonstramos que Γ_U é, a menos de um conjunto singular com medida de Hausdorff menor ou igual a $N - 2$ (onde N é a dimensão do espaço), uma hipersuperfície de classe $C^{1,\alpha}$. A ideia essencial passa pela compreensão de que a validade da Fórmula de Monotonia de Almgren permite por um lado o uso do Princípio de Redução de Federer (usado para deduzir informações acerca das dimensões de Hausdorff dos conjuntos nodais e singulares), e por outro implica um princípio de reflexão fraco para o gradiente de U.

Por fim, para o caso particular de $m = 2$ equações e $w_1 = w_2 = -1$, procuram-se soluções de (3) como pontos críticos do funcional

$$J_\beta(u, v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) + \frac{1}{4}\int_\Omega (u^4 + v^4) + \frac{\beta}{2}\int_\Omega u^2 v^2$$

restringido ao conjunto

$$\mathcal{M} = \left\{(u, v) : H_0^1(\Omega) \times H_0^1(\Omega) : \int_\Omega u^2 = \int_\Omega v^2 = 1\right\}$$

(os parâmetros λ_i passam a depender de β, e são obtidos como corolário do Teorema dos Multiplicadores de Lagrange). Para cada $\beta > 0$ fixo, constroem-se níveis críticos de $J_\beta|_\mathcal{M}$ de tipo minimax, usando para tal uma noção de *genus* que toma em consideração a invariância do funcional J_β e do conjunto $\mathcal{M}$ para a involução $(u, v) \mapsto (v, u)$ (ideia já previamente utilizada num trabalho de Dancer, Wei, Weth (2010)). Demonstramos por um lado a existência de uma infinidade de pontos críticos de $J_\beta|_\mathcal{M}$, e por outro a convergência dos níveis críticos supramencionados para níveis críticos do funcional energia da equação

$$-\Delta w + \lambda_1 w^+ - \lambda_2 w^- = w^3. \tag{4}$$

Para além disso, demonstramos que existem soluções de (4) que são limite de soluções de um outro sistema, correspondente a uma perturbação L^2 de (3).

Na segunda parte deste trabalho abordamos o sistema Hamiltoniano (de tipo (2))

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(v), \\ -\varepsilon^2 \Delta v + V(x)v = f(u), \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega, \end{cases} \tag{5}$$

onde Ω é um domínio regular de $\mathbb{R}^N$ $(N \geqslant 3)$ possivelmente ilimitado, e f, g são não-linearidades do tipo potência, com crescimento sobrelinear e subcrítico no infinito. Sob estas hipóteses, demonstramos para ε pequeno a existência de soluções positivas $u_\varepsilon, v_\varepsilon$ tais que, quando $\varepsilon \to 0^+$, a soma $u_\varepsilon + v_\varepsilon$ se concentra num número finito prescrito de mínimos locais de V, possivelmente degenerados. A demonstração deste resultado resulta de extensões não triviais de técnicas usadas por del Pino e Felmer na segunda metade da década de noventa para lidar com o caso de uma única equação

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \qquad u \in H_0^1(\Omega). \tag{6}$$

Uma das dificuldades na extensão de resultados da equação (6) para o sistema (5) deve-se ao carácter fortemente indefinido da energia associada a (5) (observe-se, por exemplo, que a parte quadrática $\int_\Omega (\langle \nabla u, \nabla v \rangle + V(x)uv)$ não tem sinal definido). Isto leva a que, como demonstrado em trabalhos de Ramos et al., a variedade de Nehari “natural” apresente uma definição mais complexa, tendo tanto dimensão como co-dimensão infinita. A obtenção do nosso resultado passa por, fixados à partida k mínimos locais de V, minimizar a energia num conjunto que corresponde, aproximadamente, à união de k variedades de Nehari localizadas em regiões à volta dos mínimos de V.

Na terceira e última parte deste trabalho damos um contributo para o estudo de problemas conhecidos na literatura como “problemas de perturbação de simetria”. Mais concretamente, analisamos a equação

$$-\Delta u = g(x, u) + f(x, u), \qquad u \in H_0^1(\Omega),$$

onde Ω é um domínio regular de $\mathbb{R}^N$ $(N \geqslant 3)$, g tem crescimento sobrelinear e subcrítico no infinito e é impar como função de u, f é um termo de ordem inferior e não é necessariamente ímpar em u. Demonstramos para este problema a existência de uma infinidade de soluções que mudam de sinal (também conhecidas por soluções nodais), obtendo também para cada solução uma estimativa superior do número de regiões nodais. O argumento usado na demonstração passa primeiramente por um estudo detalhado do caso (simétrico) $f \equiv 0$. Através do uso de uma noção apropriada de enlace (*linking*, em inglês), definimos uma sucessão de níveis críticos de tipo minimax e deduzimos estimativas para o índice de Morse dos pontos críticos associados. Esta informação, aliada a um argumento de perturbação e a um resultado de extensão para aplicações ímpares, permite-nos obter a conclusão enunciada. Por fim, adaptamos a estratégia anterior de modo a lidar com o problema de quarta ordem

$$\Delta^2 u = g(x,u) + f(x,u), \qquad u \in H^2(\Omega) \cap H_0^1(\Omega)$$

(que, notamos, pode ser escrito como (2)). Sob uma hipótese adicional relativa à monotonia das funções f e g, demonstramos também neste caso a existência de uma infinidade de soluções que mudam de sinal. Observamos que, regra geral, é mais difícil determinar a existência de soluções nodais para problemas de quarta ordem; isto deve-se ao facto de, dada uma função de $H^2(\Omega)$, as suas partes positivas e negativas não pertencerem necessariamente ao espaço $H^2(\Omega)$. Contornamos esse problema usando as projecções nos cones das funções positivas e negativas, e nos seus cones duais.

Palavras-Chave

Sistemas elípticos, métodos variacionais, propriedades qualitativas de soluções, problemas de fronteira livre, fórmulas de monotonia.

Contents

Introduction

In this thesis we are concerned with the study of semilinear elliptic systems that possess a variational structure; with this, we mean that their solutions are obtained as critical points of some "energy functional" defined on a Hilbert space.

The problems we treat are essentially of one of the following two types. On the one hand, we deal with gradient-type systems, where the nonlinear terms are the gradient of a potential, such as in

$$-\Delta u_i + V_i(x)u_i = \partial_{u_i} F(u_1, \dots, u_m), \qquad i = 1, \dots, m,$$

which has as energy functional

$$(u_1, \dots, u_m) \mapsto \int \Big(\sum_{i=1}^{m} \frac{1}{2}(|\nabla u_i|^2 + V_i(x)u_i^2) - F(u_1, \dots, u_m) \Big).$$

On the other hand, we are also concerned with Hamiltonian systems of the form

$$\begin{cases} -\Delta u + V(x)u = \partial_v F(u, v) \\ -\Delta v + V(x)v = \partial_u F(u, v), \end{cases}$$

with energy functional given by

$$(u, v) \mapsto \int (\langle \nabla u, \nabla v \rangle + V(x)uv - F(u, v)).$$

We face several issues regarding these systems. At first, we are concerned with the existence of positive solutions (Chapters 4 and 5) as well as sign-changing solutions (Chapter 6). Afterwards, in two different situations, we deal with the asymptotic behavior of the solutions with respect to a parameter: while in Chapter 5 we are interested in the semiclassical limit for the case of the Hamiltonian systems, in Chapters 2 and 4 we deal with competition–diffusion systems with large interspecific interactions. In the latter case, as shown in Chapter 2, such study leads to the formulation of a free boundary problem, for which we develop a regularity theory in Chapter 3. In order to deal with these subjects we use techniques coming from different fields of mathematics. We emphasize the use of results from Measure Theory and Geometric Measure Theory, the use of the Monotonicity Formulae by Alt-Caffarelli-Friedman and Almgren, and the application of Variational Methods (for the deduction of the existence results).

This work is divided in three parts.

The first part is devoted to the analysis of the following class of competition-diffusion systems:

$$\begin{cases} -\Delta u_i + \lambda_i u_i = w_i u_i^3 - \beta u_i \sum\limits_{\substack{j=1 \\ j\neq i}}^{m} u_j^2, & i = 1, \dots, m, \\ u_i \in H_0^1(\Omega), \quad u_i > 0 \text{ in } \Omega, \end{cases}$$

arising in the theory of Bose-Einstein condensation in multiple spin-states. Here Ω is a regular bounded domain of $\mathbb{R}^N$ $(N = 2, 3)$, λ_i, w_i $(i = 1, \dots, m)$ are real parameters, and $\beta > 0$. Due to the sign restriction on β, the interaction term

$$\beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{m} u_j^2$$

is of competitive type. We are especially interested in the behavior of the solutions of the system when the interaction parameter β goes to infinity, as well in characterizing the limiting profiles.

In Chapter 1 we motivate and describe the problem, focusing our attention on the special case of minimal energy solutions.

In Chapter 2 we establish uniform bounds in the Hölder spaces $C^{0,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ for $L^\infty(\Omega)$–bounded solutions, and show that the associated limiting profiles are segregated and that actually are Lipschitz continuous. The regularity of the nodal sets of such profiles is studied in Chapter 3, where some matching conditions at the interfaces are deduced as well. It should be stressed that a common element in the proof of these results is the use of the Almgren's Monotonicity Formula, which is used at several points.

In Chapter 4, in the particular case of $m = 2$ equations and $w_1 = w_2 = -1$, the asymptotic study is carried out in connection with the associated single equation

$$-\Delta w + \lambda_1 w^+ - \lambda_2 w^- = w^3.$$

In the second part of this thesis (which corresponds to Chapter 5) we deal with the system in Hamiltonian form

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = g(v), \\ -\varepsilon^2 \Delta v + V(x) v = f(u), \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega, \end{cases}$$

where Ω is a domain of $\mathbb{R}^N$ $(N \geqslant 3)$, possibly unbounded, and f, g are power-type nonlinearities, having superlinear and subcritical growth at infinity. We prove the existence of positive solutions $u_\varepsilon, v_\varepsilon$ such that, as $\varepsilon \to 0^+$, $u_\varepsilon + v_\varepsilon$ concentrates at a prescribed finite number of local minimum points of V, possibly degenerate. Following some of the works by del Pino and Felmer, and Ramos *et al.*, the argument of the proof consists in considering a suitable variational framework which localizes the problem around the prescribed minima of V.

In the third and last part (which corresponds to Chapter 6) we prove the existence of infinitely many sign-changing solutions of the problem

$$-\Delta u = g(x,u) + f(x,u), \qquad u \in H_0^1(\Omega),$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $N \geqslant 3$, g is superlinear, subcritical and odd in u, and f is a lower order term not necessarily odd in u. The proof is based on a minimax-type argument that makes use of a suitable notion of linking. By using the same techniques, we are also able to draw similar conclusions for the fourth order problem

$$\Delta^2 u = g(x,u) + f(x,u), \qquad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

In an attempt to keep the exposition as self-contained as possible, at the end of the thesis we have included several appendices where we recall (and sometimes prove) some results that are used along the text.

In Appendix A we recall some of the notions of Measure Theory that are needed in Chapter 3, and in particular state and prove the Federer's Reduction Principle.

Afterwards, in Appendix B, we prove a version of the Ekeland's Variational Principle in the context of C^1 Hilbert manifolds, which is used in Chapter 5.

Finally, in Appendix C, we collect some results which concern pointwise estimates, as well as theorems involving the notion of Morse index.

This thesis is based on the work developed in the papers [93, 94, 107, 108, 125].

Some notations

Given $A, B \subseteq \mathbb{R}^N$, we will use the following notations.

∂A: the boundary of A;

$\bar{A}$: the closure of a set A;

$A \Subset B$: means that $\bar{A} \subseteq B$;

$|A|$: the Lebesgue measure of A;

χ_A: the characteristic funtion of A, namely $\chi(x) = 1$ if $x \in A$, $\chi(x) = 0$ if $x \notin A$.

The Euclidean inner product and norm in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. Namely, for $x, y \in \mathbb{R}^N$,

$$\langle x, y \rangle = x_1 y_1 + \ldots + x_N y_N, \qquad |x| = \langle x, x \rangle^{1/2}.$$

Given a normed space (V, $\|\cdot\|$) and a set $A \subseteq V$, we denote

$\mathrm{dist}(u, A) = \inf\{\|u - v\| : \ v \in A\}$;
$N_\delta(A) = \{x \in V : \mathrm{dist}(x, A) \leqslant \delta\}$.

Moreover, given a domain Ω, we will denote by $\|u\|_p = \left(\int_\Omega |u|^p\right)^{1/p}$ the usual norm in $L^p(\Omega)$ $(1 \leqslant p < \infty)$, and $\|u\|_\infty = \mathrm{ess\,sup}_\Omega\, |u|$.

Finally, given a function $u : \Omega \to \mathbb{R}$, we denote its positive and negative part respectively by

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}.$$

Part I

Asymptotic study of a class of gradient systems with competition terms

Chapter 1

Introducing the problem

Consider the following system of m elliptic partial differential equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = w_i u_i^3 - \beta u_i \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^m u_j^2, & i = 1, \ldots, m, \\ u_i \in H_0^1(\Omega), \quad u_i > 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain ($N = 2, 3$), λ_i, w_i are real parameters and $\beta > 0$. We observe that such system has a variational characterization, and that solutions of (1.1) can be obtained as critical points of the associated *energy functional* $J_\beta : (H_0^1(\Omega))^m \to \mathbb{R}$ defined by

$$J_\beta(U) = J_\beta(u_1, \ldots, u_m) = \frac{1}{2}\sum_{i=1}^m \int_\Omega (|\nabla u_i|^2 + \lambda_i u_i^2) - \frac{1}{4}\sum_{i=1}^m \int_\Omega w_i u_i^4 + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^m \int_\Omega \beta u_i^2 u_j^2.$$

We will be specially interested in the behavior of the solutions of the system (1.1) as $\beta \to +\infty$.

Although (1.1) can also be seen as a stationary case of a population model, the main motivation for its study comes from the field of Quantum Mechanics, namely from the study of phenomena arising from the Bose-Einstein condensation. To be more precise, (1.1) arises naturally when searching for solitary wave solutions of the so called system of Gross-Pitaevskii equations[1]

$$\begin{cases} \iota \partial_t \phi_i = \Delta \phi_i + w_i |\phi_i|^2 \phi_i - \beta \phi_i \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^m |\phi_j|^2, & i = 1, \ldots, m. \\ \phi_i \in H_0^1(\Omega, \mathbb{C}), \end{cases} \tag{1.2}$$

In fact, by putting $\phi_i := e^{-\iota\lambda_i t} u_i(x)$, we see that ϕ_i satisfies (1.2) if and only if u_i satisfies (1.1). The system (1.2) has been proposed as a mathematical model for multi-species Bose-Einstein condensates in m hyperfine spin-states (check for instance [96] and references therein), and ϕ_i denotes the wave function of each condensate. The parameters w_i

[1] Here ι denotes the (complex) square root of -1.

(called in this context the intraspecies scattering length) represent the interaction between particles of the same condensate; when $w_i > 0$ this is called the *focusing* case, in opposition to the *defocusing* one, when $w_i < 0$. As for β (the interspecies scattering length), it represents the interaction between unlike particles. Since we assume β positive, the interaction is of repulsive type.

Since the first experimental results concerning the observation of one single condensate [7], the subject of Bose-Einstein condensation has been attracting growing attention from both experimental and theoretical physicists, as well as from mathematicians. Experimental observations of multiple condensates have only been obtained very recently (see [88, 112]). In such experiments, it has been observed the occurrence of a phenomenon called *phase separation*, which means that the wave functions of different condensates have disjoint supports, that is, $\phi_i \cdot \phi_j \equiv 0$ for $i \neq j$. From theoretical physics considerations (check for instance [128]) such a phenomenon can be modelled in an efficient way by taking a large $\beta > 0$[2] in (1.2). This way, from a mathematical point of view, we are led to the asymptotic study of (1.1) as $\beta \to +\infty$.

Many interesting problems arise in this context. For each $\beta > 0$, let $U_\beta = (u_{1,\beta}, \dots, u_{m,\beta})$ be a solutions of (1.1).

1. Does U_β converge as $\beta \to +\infty$ to a limiting profile U in some normed space? If so, what is the expected regularity of the limiting profile?

2. Are the limiting profiles U segregated, that is, $u_i \cdot u_j \equiv 0$ whenever $i \neq j$? In such a case, what is the regularity of the free boundary $\Gamma_U := \{x \in \Omega : \ u_i = 0 \ \forall i\}$? Moreover, for $i \neq j$, how do u_i and u_j interact through the common boundary $\partial\{u_i > 0\} \cap \partial\{u_j > 0\}$?

Regarding these questions, the first mathematical results are due to Conti, Terracini and Verzini [45, 46] in the focusing case, and Chang *et al.* [40] in the defocusing one. These papers deal only with minimal energy solutions[3], and the phenomenon of phase separation as $\beta \to +\infty$ is rigorously proved. In the paper [46], the authors also deduce a strong characterization for the limiting profiles $U = (u_1, \dots, u_m)$. It is shown that they belong to the following class of functions

$$\mathcal{S}(\Omega) := \left\{ U \in (H^1(\Omega) \cap L^\infty(\Omega))^m : \begin{array}{c} u_i \geqslant 0, \ u_i \cdot u_j = 0 \text{ for } i \neq j, \ -\Delta u_i \leqslant f_i(u_i), \\ -\Delta\Big(u_i - \sum_{\substack{j=1 \\ j\neq i}}^m u_j\Big) \geqslant f_i(u_i) - \sum_{\substack{j=1 \\ j\neq i}}^m f_j(u_j) \end{array} \right\} \tag{1.3}$$

for $f_i(s) := w_i s^3 - \lambda_i s$. Afterwards, under different sets of assumptions, Conti, Terracini and Verzini [46, 47] and Caffarelli, Karakhanyan and Lin [29] develop a regularity theory

[2] Heuristically speaking, by making β large we are considering that the competition between different states becomes stronger, and hence it is somehow natural to expect that the components u_i concentrate in different regions.

[3] Along this introduction, the term minimal energy solution will be ambiguous, since its definition depends on the context. For us at this point a minimal energy solution will be a solution of (1.1) which minimizes the energy J_β over some set of functions.

for general elements belonging to the class $\mathcal{S}(\Omega)$. Namely, it is proved that each element $U \in \mathcal{S}(\Omega)$ is Lipschitz continuous, that each component u_i satisfies

$$-\Delta u_i + \lambda_i u_i = w_i u_i^3 \quad \text{in the open set } \{u_i > 0\} \tag{1.4}$$

(which can be seen directly from the definition of $\mathcal{S}(\Omega)$), and that the free boundary Γ_U is a $C^{1,\alpha}$–hypersurface everywhere except for a set with "small" Hausdorff dimension. The same regularity results are obtained Caffarelli and Lin [31] for other minimizing vector solutions of (1.1). We refer to Section 1.3 ahead for the precise statements.

At this point we stress that up to now no results had been obtained for general excited state solutions (*i.e.*, non minimal solutions). The recent literature shows that families of solutions with higher energy levels exist for large $\beta's$ [55, 92, 127, 133], but the only paper that dealt with the asymptotic behavior of such solutions was the one by Wei and Weth [132], where it is proved that, in the planar case, L^∞–bounded solutions of (1.1) converge uniformly to a segregated profile $U = (u_1, \ldots, u_m)$ where each component satisfies (1.4). It is the purpose of this part of the thesis to extend to more general families of solutions the regularity theory already established for the minimal ones.

In the following two subsections we start by considering two particular cases of the system (1.1), one focusing and the other defocusing, and deduce (following [45, 46]) that the limiting profiles coming from ground-state solutions belong to the class $\mathcal{S}(\Omega)$, hence allowing the application of the regularity results mentioned before. By doing so we want to point out why the minimality assumption makes it somehow easier to deduce regularity results, and why new tools had to be developed in order to deal with the general case of non-minimal solutions. This new approach will be developed in Chapters 2 and 3. More precisely, in Chapter 2 we prove that any given L^∞–bounded solution U_β of (1.1) converges, up to a subsequence, to a limiting profile U in the Hölder spaces $C^{0,\alpha}(\bar{\Omega})$, for every $0 < \alpha < 1$. Moreover we conclude that the limiting profiles are Lipschitz continuous, and that segregation occurs. Afterwards in Chapter 3 we study the regularity of the *nodal set* Γ_U defined before. We deduce regularity results which are similar to the ones obtained for the minimal cases, although the approach is quite different. In doing so, we actually develop a general theory which can be applied to the study of other free boundary problems arising for instance in optimization theory. After this, however, it remains unclear if, for non minimal solutions, there is some relation between (1.1) and the class $\mathcal{S}(\Omega)$. In Chapter 4 we provide some connections in the case of $m = 2$ equations. The original part of this work (namely Chapters 2, 3 and 4) corresponds to the papers [93, 94] (written in collaboration with B. Noris, S. Terracini and G. Verzini) and [125] (written in collaboration with S. Terracini).

As a concluding remark, we wish to mention that system (1.1) is also of great interest in the complementary case $\beta < 0$, which we do not address. Its study has applications in the theory of incoherent solitons in nonlinear optics. For results in this direction, we refer the reader to [6, 55, 85, 118] and references therein. Moreover, more recently a complex valued version of (1.1) has been treated in [83].

1.1 Minimal solutions: a focusing case

In this subsection we will study the ground-state solutions of (1.1) in a particular case of focusing type. The results we will present are collected from the works of Conti, Terracini and Verzini [46, 47] and from the work of Dancer, Wei and Weth [55]. Consider

$$\begin{cases} -\Delta u_i + u_i = u_i^3 - \beta u_i \sum_{\substack{j=1\\ j\neq i}}^m u_j^2 \quad \text{in } \Omega, \\ u_i \in H_0^1(\Omega),\ u_i > 0 \text{ in } \Omega, \end{cases} \qquad i = 1, \dots, m. \tag{1.5}$$

With this choice of parameters, the functional J_β becomes

$$J_\beta(U) = J_\beta(u_1, \dots, u_m) = \frac{1}{2}\sum_{i=1}^m \int_\Omega (|\nabla u_i|^2 + u_i^2) - \frac{1}{4}\sum_{i=1}^m \int_\Omega u_i^4 + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^m \int_\Omega \beta u_i^2 u_j^2.$$

Recall that critical points of J_β correspond to solutions of (1.5). In this subsection we will consider $H_0^1(\Omega)$ equipped with the inner product $\langle u, v\rangle = \int_\Omega(\langle \nabla u, \nabla v\rangle + uv)$ and will denote by $\|\cdot\|$ the associated norm. We define the Nehari's set to be:

$$\mathcal{N}_\beta = \left\{U = (u_1, \dots, u_m) \in (H_0^1(\Omega)\setminus\{0\})^m :\ \partial_{u_i} J_\beta(U) u_i = 0,\ i = 1, \dots, m\right\}$$
$$= \left\{U = (u_1, \dots, u_m) \in (H_0^1(\Omega)\setminus\{0\})^m :\ \|u_i\|^2 = \int_\Omega u_i^4 - \int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2,\ i = 1, \dots, m\right\}$$

and take the *least energy level*:

$$c_\beta = \inf_{\mathcal{N}_\beta} J_\beta.$$

In the following, first we prove that the level c_β is attained and that it is a critical level for J_β. Afterwards we study the behavior of the corresponding critical points as $\beta \to +\infty$.

Theorem 1.1. *There exists $U_\beta = (u_{1,\beta}, \dots, u_{m,\beta})$, with $u_{i,\beta} > 0$ in Ω for every $i = 1, \dots, m$, such that*

$$J_\beta(U_\beta) = c_\beta, \qquad \textit{and} \qquad J'_\beta(U_\beta) = 0.$$

We will denote by $J_\beta|_{\mathcal{N}_\beta}$ the restriction of the functional J_β to the set $\mathcal{N}_\beta$, and $\mathcal{H} := (H_0^1(\Omega))^m$. The arguments used in the proof of Theorem 1.1 are borrowed from [55], where the same result is proved in the case of $m = 2$ equations. We start with three auxiliary lemmas.

Lemma 1.2. *For every $\beta > 0$ it holds*

(a) $J_\beta(U) = \dfrac{1}{4}\sum_{i=1}^m \|u_i\|^2$ *whenever $U \in \mathcal{N}_\beta$;*

(b) there exists a constant $C > 0$ independent of β such that for every $i = 1, \dots, m$,

$$\|u_i\|, \|u_i\|_4 \geqslant C \qquad \textit{whenever } U = (u_1, \dots, u_m) \in \mathcal{N}_\beta.$$

Proof. (a) For each $U \in \mathcal{N}_\beta$ and $i = 1, \dots, m$,

$$-\frac{1}{4}\int_\Omega u_i^4 = -\frac{1}{4}\|u_i\|^2 - \frac{1}{4}\int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2.$$

By summing up in i the previous identities, one obtains

$$-\frac{1}{4}\sum_{i=1}^m \int_\Omega u_i^4 = -\frac{1}{4}\sum_{i=1}^m \|u_i\|^2 - \frac{1}{4}\sum_{\substack{i,j=1\\ i\neq j}}^m \int_\Omega \beta u_i^2 u_j^2 = -\frac{1}{4}\sum_{i=1}^m \|u_i\|^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^m \int_\Omega \beta u_i^2 u_j^2,$$

and hence

$$J_\beta(U) = \frac{1}{4}\sum_{i=1}^m \|u_i\|^2.$$

(b) Take $U \in \mathcal{N}_\beta$. For each i, since $\beta \geqslant 0$,

$$\|u_i\|^2 \leqslant \|u_i\|^2 + \int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2 = \int_\Omega u_i^4 \leqslant C_S^4 \|u_i\|^4 \tag{1.6}$$

where C_S denotes the Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Thus $\|u_i\| \geqslant 1/C_S^2$ and, again by (1.6), $\|u_i\|_4 \geqslant 1/C_S$. $\square$

Lemma 1.3. *The set $\mathcal{N}_\beta$ is a submanifold of $\mathcal{H}$ of codimension m for every $\beta > 0$. Moreover, if $J_\beta|'_{\mathcal{N}_\beta}(U) = 0$ then $J'_\beta(U) = 0$.*

Proof. The elements in $\mathcal{N}_\beta$ are zeros of the functional $F : \mathcal{H} \to \mathbb{R}^m$, $U = (u_1, \dots, u_m) \mapsto (F_1(U), \dots, F_m(U))$ where, for each $i = 1, \dots, m$, F_i is the $C^2(\mathcal{H}, \mathbb{R})$–functional defined by

$$F_i(U) = \|u_i\|^2 - \int_\Omega u_i^4 + \int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2.$$

Denote by $\mathbf{T_U}$ the $m \times m$ matrix whose i-th line is the vector

$$F'(U)(0, \dots, 0, u_i, 0, \dots, 0) = (\partial_{u_i} F_1(U)u_i, \dots, \partial_{u_i} F_m(U)u_i).$$

Given $U \in \mathcal{N}_\beta$, for each i we have

$$\partial_{u_i} F_i(U)u_i = 2\|u_i\|^2 - 4\int_\Omega u_i^4 + 2\int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2 = -2\int_\Omega u_i^4,$$

while for $j \neq i$

$$\partial_{u_j} F_i(U)u_j = 2\int_\Omega \beta u_i^2 u_j^2.$$

Thus,

$$\mathbf{T_U} = \begin{pmatrix} -2\int_\Omega u_1^4 & 2\int_\Omega \beta u_1^2 u_2^2 & \dots & 2\int_\Omega \beta u_1^2 u_m^2 \\ 2\int_\Omega \beta u_1^2 u_2^2 & -2\int_\Omega u_2^4 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 2\int_\Omega \beta u_1^2 u_m^2 & 2\int_\Omega \beta u_2^2 u_m^2 & \dots & -2\int_\Omega u_m^2 \end{pmatrix}$$

By the Gershgorin's theorem (see for instance [76, Appendix 7]), the eigenvalues of $\mathbf{T_U}$ lie in the set

$$\bigcup_{i=1}^m \Big\{\lambda :\ |\lambda + 2\int_\Omega u_i^4| \leqslant 2\int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2\Big\} \subseteq \bigcup_{i=1}^m \Big\{\lambda : \lambda \leqslant -2\int_\Omega u_i^4 + 2\int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2\Big\} \subseteq$$

$$\subseteq \bigcup_{i=1}^m \Big\{\lambda :\ \lambda \leqslant -2\|u_i\|^2\Big\}.$$

Hence all eigenvalues are strictly negative and the matrix $\mathbf{T_U}$ is negative definite. In particular its determinant is different from zero and the m vectors

$$F'(U)(u_1, 0, \dots, 0), \dots, F'(U)(0, \dots, 0, u_m)$$

are linearly independent. This implies that $F'(U) : \mathcal{H} \to \mathbb{R}^m$ is onto for every $U \in \mathcal{N}_\beta$, and hence $\mathcal{N}_\beta$ is indeed a submanifold of $\mathcal{H}$ of codimension m (*cf.* Theorem B.7).

As for the second part of the lemma, if $J_\beta|'_{\mathcal{N}_\beta}(U) = 0$ then there exist real numbers $\lambda_i,\ i = 1, \dots, m$, such that $J'_\beta(U) = \sum_{i=1}^m \lambda_i F'_i(U)$. By testing the previous equality with $(0, \dots, 0, u_j, 0, \dots, 0)$, one obtains

$$0 = \sum_{i=1}^m \lambda_i \partial_{u_j} F_i(U) u_j, \quad \forall j = 1, \dots, m,$$

which is equivalent to

$$\mathbf{T_U} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \mathbf{0}.$$

Hence $\lambda_i = 0$ for every i, and U is a critical point of the functional J_β. □

Lemma 1.4. *For every $\beta > 0$, the functional $J_\beta|_{\mathcal{N}_\beta}$ satisfies the Palais-Smale condition.*

Proof. Let $(U_k)_k \subseteq \mathcal{N}_\beta$ be a Palais-Smale sequence for $J_\beta|_{\mathcal{N}_\beta}$, that is,

$$J_\beta(U_k) \text{ is bounded in } k,$$

$$J'_\beta(U_k) - \sum_{i=1}^m \lambda_{i,k} F'_i(U_k) \to 0 \text{ in } \mathcal{H}', \text{ for some sequences } (\lambda_{1,k})_k, \dots, (\lambda_{m,k})_k. \tag{1.7}$$

By combining the fact that $J_\beta(U_\beta)$ is bounded with point (a) in Lemma 1.2 we conclude that the sequence $(U_k)_k$ is bounded in $\mathcal{H}$. Hence, up to a subsequence, we obtain the existence of $U \in \mathcal{H}$ such that

$$U_k \rightharpoonup U \text{ weakly in } H_0^1(\Omega), \qquad U_k \to U \text{ strongly in } L^4(\Omega).$$

By Lemma 1.2-(b), we deduce that $u_i \not\equiv 0$ for every i. Moreover, we have $\mathbf{T_{U_k}} \to \mathbf{T_U}$ in $\mathbb{R}^{2m}$ and, for each i,

$$\|u_i\|^2 \leqslant \liminf_k \|u_{i,k}\|^2 = \liminf_k \Big(\int_\Omega u_{i,k}^4 - \int_\Omega \beta u_{i,k}^2 \sum_{\substack{j=1\\ j\neq i}}^m u_{j,k}^2 \Big) = \int_\Omega u_i^4 - \int_\Omega \beta u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2.$$

Hence, by reasoning exactly as in the proof of Lemma 1.3, we obtain that $\mathbf{T_U}$ is a negative definite matrix. After testing (1.7) with $(0,\dots,0,u_{j,k},0,\dots,0)$ for every j, we obtain, as $k\to+\infty$,

$$\mathbf{o(1)} = \mathbf{T_{U_k}} \begin{pmatrix} \lambda_{1,k} \\ \vdots \\ \lambda_{m,k} \end{pmatrix} = \Big(\mathbf{T_U} + \mathbf{o(1)}\Big) \begin{pmatrix} \lambda_{1,k} \\ \vdots \\ \lambda_{m,k} \end{pmatrix}$$

and moreover

$$\begin{aligned} \mathrm{o}(|(\lambda_{1,k},\dots,\lambda_{m,k})|) &= \Big(\lambda_{1,k},\dots,\lambda_{m,k}\Big)\ \mathbf{T_U} \begin{pmatrix} \lambda_{1,k} \\ \vdots \\ \lambda_{m,k} \end{pmatrix} + \mathrm{o}(|(\lambda_{1,k},\dots,\lambda_{m,k})|^2) \\ &\leqslant -C|(\lambda_{1,k},\dots,\lambda_{m,k})|^2 + \mathrm{o}(|(\lambda_{1,k},\dots,\lambda_{m,k})|^2), \end{aligned}$$

for some $C>0$. Thus for every i we have $\lambda_{i,k}\to 0$ and $\lambda_{i,k}F_i'(U_k)\to 0$ in $\mathcal{H}'$ as $k\to\infty$, and therefore also $J_\beta'(U_k)\to 0$ in $\mathcal{H}'$ as $k\to\infty$. By taking $(0,\dots,0,u_{i,k}-u_i,0,\dots,0)$ as a test function we obtain

$$J_\beta'(U_k)(0,\dots,0,u_{i,k}-u_i,0,\dots,0) = \mathrm{o}(1) \qquad \text{as } k\to\infty,$$

which is equivalent to

$$\langle u_{i,k}, u_{i,k}-u_i\rangle - \int_\Omega u_{i,k}^3(u_{i,k}-u_i) = \int_\Omega \beta u_{i,k}(u_{i,k}-u_i) \sum_{\substack{j=1\\ j\neq i}}^m u_{j,k}^2 = \mathrm{o}(1) \qquad \text{as } k\to\infty.$$

Since

$$\int_\Omega u_{i,k}^3(u_{i,k}-u_i) + \int_\Omega \beta u_{i,k}(u_{i,k}-u_i) \sum_{\substack{j=1\\ j\neq i}}^m u_{j,k}^2 = \mathrm{o}(1) \qquad \text{as } k\to\infty,$$

it follows that $\|u_{i,k}\|\to\|u_i\|$, which provides the strong convergence $u_{i,k}\to u_i$ for every i. □

Proof of Theorem 1.1. By the point (a) in Lemma 1.2, we deduce that $c_\beta \geqslant 0$. Let U_k be a minimizing sequence for c_β, namely $U_k\in\mathcal{N}_\beta$ such that $J_\beta(U_k)\to c_\beta$ as $k\to\infty$. By the Ekeland's Variational Principle (*cf.* Theorem B.9) we can suppose, without loss of generality, that $(U_k)_k$ is a Palais-Smale sequence for the restricted functional $J_\beta|_{\mathcal{N}_\beta}$. Hence by Lemma 1.4 we have that, up to a subsequence, $U_k\to U$ strongly in $H_0^1(\Omega)$. In particular $U\in\mathcal{N}_\beta$ ($U\neq 0$ by Lemma 1.2-(b)) and $J_\beta(U)=c_\beta$. Moreover, U is a critical point of J_β. Now, by possibly replacing U with $(|u_1|,\dots,|u_m|)$ and by using the strong maximum principle the result follows. □

From now on let $U_\beta = (u_{1,\beta}, \ldots, u_{m,\beta})$, with $u_{i,\beta} > 0$ in Ω, denote a ground-state solution of (1.5) in the sense of Theorem 1.1. We are interested in the asymptotic behavior of U_β as $\beta \to +\infty$. With this in mind, define the auxiliary functionals $J_* : H^1_0(\Omega) \to \mathbb{R}$ and $J_\infty : \mathcal{H} \to \mathbb{R}$ by

$$J_*(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega u^4, \qquad J_\infty(U) = J_\infty(u_1, \ldots, u_m) = \sum_{i=1}^m J_*(u_i).$$

Define moreover the "limiting" Nehari's set

$$\begin{aligned}\mathcal{N}_\infty &= \Big\{U = (u_1, \ldots, u_m) \in \big(H^1_0(\Omega) \setminus \{0\}\big)^m : \;\; J_*'(u_i)u_i = 0 \;\text{ for } i = 1, \ldots, m\Big\} \\ &= \Big\{U = (u_1, \ldots, u_m) \in \big(H^1_0(\Omega) \setminus \{0\}\big)^m : \;\; \|u_i\|^2 = \int_\Omega u_i^4 \;\text{ for } i = 1, \ldots, m\Big\}\end{aligned}$$

and the level

$$c_\infty = \inf_{\substack{U \in \mathcal{N}_\infty \\ u_i \cdot u_j = 0, \forall j \neq i}} J_\infty(U),$$

Theorem 1.5. *There exist $U = (u_1, \ldots, u_m) \in \mathcal{N}_\infty$ with $u_i \geqslant 0$ ($i = 1, \ldots, m$), $u_i \cdot u_j = 0$ (whenever $i \neq j$) and $J_\infty(U) = c_\infty$ such that, up to a subsequence, as $\beta \to +\infty$,*

(i) $u_{i,\beta} \to u_i$ *in* $H^1_0(\Omega)$,

(ii) $\displaystyle\int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \to 0$ *for* $i \neq j$,

(iii) $c_\beta \to c_\infty$.

Proof. Observe first of all that for each $U = (u_1, \ldots, u_m) \in (H^1_0(\Omega) \setminus \{0\})^m$ with $u_i \cdot u_j = 0$ for $j \neq i$, a simple computation shows that $(\bar{t}_1 u_1, \ldots, \bar{t}_m u_m) \in \mathcal{N}_\infty$ for $\bar{t}_i = \|u_i\|/\|u_i\|_4^2$. Hence $\mathcal{N}_\infty \neq \emptyset$ and $c_\infty < +\infty$. Moreover, for each $U \in \mathcal{N}_\beta$ with $u_i \cdot u_j = 0$ ($j \neq i$) we see that $J_\beta(U) = J_\infty(U)$ for every $\beta > 0$, and thus $J_\beta(U_\beta) = c_\beta \leqslant c_\infty$. This immediately implies that $(U_\beta)_\beta$ is a bounded sequence in $\mathcal{H}$, and hence there exists $U \in \mathcal{H}$ such that, up to a subsequence

$$U_\beta \rightharpoonup U \quad \text{weakly in } H^1_0(\Omega), \qquad U_\beta \to U \quad \text{strongly in } L^4(\Omega).$$

From the strong convergence in $L^4(\Omega)$ and Lemma 1.2-(a) we deduce that $|u_i|_4 > 0$, and in particular $u_i \neq 0$. Moreover there exists a constant $C > 0$ such that

$$\frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^m \int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \leqslant J_\beta(U_\beta) + \frac{1}{4}\sum_{i=1}^m \int_\Omega u_{i,\beta}^4 \leqslant c_\infty + C$$

for every $\beta > 0$. Since $\beta \to +\infty$, we see that $\int_\Omega u_i^2 u_j^2 = 0$ and $u_i \cdot u_j = 0$ whenever $i \neq j$.

Take $\bar{t}_i = \|u_i\|/\|u_i\|_4^2$ so that $(\bar{t}_1 u_1, \ldots, \bar{t}_m u_m) \in \mathcal{N}_\infty$. Since $U_\beta \in \mathcal{N}_\beta$, we have

$$\|u_i\|^2 \leqslant \liminf_\beta \|u_{i,\beta}\|^2 \leqslant \liminf_\beta \Big(\|u_{i,\beta}\|^2 + \int_\Omega \beta u_{i,\beta}^2 \sum_{\substack{j=1 \\ j\neq i}}^m u_{j,\beta}^2\Big) = \liminf_\beta \int_\Omega u_{i,\beta}^4 = \int_\Omega u_i^4$$

and thus $\bar t_i \leqslant 1$. Therefore,

$$\begin{aligned}\frac{1}{4}\sum_{i=1}^m \|u_i\|^2 &\leqslant \liminf_\beta \frac{1}{4}\sum_{i=1}^m \|u_{i,\beta}\|^2 = \liminf_\beta c_\beta \leqslant \limsup_\beta c_\beta \leqslant c_\infty \leqslant \\ &\leqslant J_\infty(\bar t_1 u_1,\dots,\bar t_m u_m) = \frac{1}{4}\sum_{i=1}^m \|\bar t_i u_i\|^2 \leqslant \frac{1}{4}\sum_{i=1}^m \|u_i\|^2,\end{aligned}$$

and hence $u_{i,\beta} \to u_i$ in $H_0^1(\Omega)$ for every i, $U \in \mathcal{N}_\infty$ (because $\bar t_i = 1$) and $c_\beta \to c_\infty$. From this latter convergence we deduce that in particular $\int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \to 0$ for $i \neq j$. □

Now we prove that each limiting profile U belongs to the class S_m defined in (1.3). The following proof is taken from [46].

Theorem 1.6. *Let $U = (u_1,\dots,u_m)$ be a limiting profile as before. Then for every $i = 1,\dots,m$ we have*

(a) $-\Delta u_i + u_i \leqslant u_i^3$ *in* Ω*;*

(b) $-\Delta \hat u_i + \hat u_i \geqslant \hat u_i^3$ *in* Ω*, with* $\hat u_i := u_i - \sum_{\substack{j=1\\ j\neq i}}^m u_j$*;*

where both inequalities are understood in the distributional sense. In particular, $U \in \mathcal{S}(\Omega)$.

Proof. In order to simplify the notations, we present the proof for $i = 1$.
(a) Suppose, in view of a contradiction, the existence of a $C_c^\infty(\Omega)$ function $\varphi \geqslant 0$ such that

$$\langle u_1, \varphi\rangle - \int_\Omega u_1^3 \varphi > 0$$

and take $\bar\delta > 0$ such that

$$\langle \lambda u_1, \varphi\rangle - \int_\Omega (\lambda u_1)^3 \varphi > 0$$

for every $|\lambda - 1| \leqslant \bar\delta$. Due to the shape of the map $t \mapsto J_*(tu)$ for each $u \neq 0$, we have that

$$J_*'((1-\bar\delta)u_1)u_1 > 0 \quad \text{and} \quad J_*'((1+\bar\delta)u_1)u_1 < 0. \tag{1.8}$$

Define, for each $t \geqslant 0$ and $\lambda \in [1-\bar\delta, 1+\bar\delta]$, the following perturbation of λu_i:

$$\Phi(t,\lambda) = \lambda u_1 - t\varphi \in H_0^1(\Omega),$$

and the vector function $\tilde U = \tilde U(t,\lambda) = (\tilde u_1,\dots,\tilde u_m)$ by $\tilde u_1 = \Phi^+(t,\lambda)$ and $\tilde u_j = u_j$ for every $j \geqslant 2$. It is easy to check that $\tilde u_i \cdot \tilde u_j = 0$ whenever $j \neq i$ and that $\tilde u_1 \neq 0$ for every small t and every $\lambda \in [1-\bar\delta, 1+\bar\delta]$. We observe that $\tilde u_1(0,\lambda) = \lambda u_1$, and hence from (1.8) we deduce that

$$J_*'(\tilde u_1(t,1-\bar\delta))\tilde u_1(t,1-\bar\delta) > 0 \quad \text{and} \quad J_*'(\tilde u_1(t,1+\bar\delta))\tilde u_1(t,1+\bar\delta) < 0$$

for $0 < t < \bar t$ small enough. Therefore for each $0 < t < \bar t$ we can find $\lambda = \lambda(t) \in (1-\bar\delta, 1+\bar\delta)$ such that $J_*'(\tilde u_1(t,\lambda))\tilde u_1(t,\lambda) = 0$. Thus

$$\tilde U(t,\lambda) \in \mathcal{N}_\infty, \qquad \text{and hence } c_\infty \leqslant J_\infty(\tilde U).$$

On the other hand,

$$\begin{aligned}
J_\infty(\tilde U) &= \sum_{j=1}^m \left(\frac{1}{2}\|\tilde u_j\|^2 - \frac{1}{4}\int_\Omega \tilde u_j^4\right)\\
&= \sum_{j\geqslant 2} \left(\frac{1}{2}\|\tilde u_j\|^2 - \frac{1}{4}\int_\Omega \tilde u_j^4\right) + \frac{1}{2}\|(\lambda u_1 - t\varphi)^+\|^2 - \frac{1}{4}\int_\Omega [(\lambda u_1 - t\varphi)^+]^4\\
&\leqslant \sum_{j\geqslant 2} \left(\frac{1}{2}\|\tilde u_j\|^2 - \frac{1}{4}\int_\Omega \tilde u_i^4\right) + \frac{1}{2}\|\lambda u_1 - t\varphi\|^2 - \frac{1}{4}\int_\Omega (\lambda u_1)^3\varphi + o(t)\\
&= J_\infty(\lambda u_1, u_2, \ldots, u_m) - t\left(\langle \lambda u_1, \varphi\rangle - \int_\Omega (\lambda u_1)^3\varphi\right) + \mathrm{o}(t)\\
&< J_\infty(\lambda u_1, u_2, \ldots, u_m) \leqslant J_\infty(U) = c_\infty
\end{aligned}$$

for sufficiently small $t > 0$, which is a contradiction.
(b) We also prove this claim by means of a contradiction argument. Suppose there exists a $C_c^\infty(\Omega)$ function $\varphi \geqslant 0$ such that

$$\langle \hat u_1, \varphi\rangle - \int_\Omega \hat u_1^3 \varphi < 0. \tag{1.9}$$

Define $\Lambda u = (\lambda_1 u_1, \ldots, \lambda_m u_m)$, so that $\widehat{\Lambda u}_1 = \lambda_1 u_1 - \sum_{j\geqslant 2}\lambda_j u_j$. and take $\bar\delta > 0$ such that for $|\lambda_j - 1| \leqslant \bar\delta$ ($j = 1, \ldots, m$) it holds

$$\langle \widehat{\Lambda u}_1, \varphi\rangle - \int_\Omega \left(\widehat{\Lambda u}_1\right)^3 \varphi < 0.$$

We can reason exactly as before and conclude that for every j,

$$J_*'((1-\bar\delta)u_j)u_j > 0, \qquad \text{and} \qquad J_*'((1+\bar\delta)u_j)u_j < 0. \tag{1.10}$$

Define, for each $t > 0$ and $(\lambda_1, \ldots, \lambda_m) \in [1-\bar\delta, 1+\bar\delta]^m$ the following perturbation of $\widehat{\Lambda u}_1$

$$\Phi(t, \lambda_1, \ldots, \lambda_m) = \widehat{\Lambda u}_1 + t\varphi = \lambda_1 u_1 - \sum_{j\geqslant 2}\lambda_j u_j + t\varphi \in H_0^1(\Omega);$$

and the vector function $\tilde U(t, \lambda_1, \ldots, \lambda_m) = (\tilde u_1, \ldots, \tilde u_m)$ by

$$\tilde u_1(t, \lambda_1, \ldots, \lambda_m) = \Phi^+(t, \lambda_1, \ldots, \lambda_m)$$

and, for $j \geqslant 2$,

$$\tilde u_j(t, \lambda_1, \ldots, \lambda_m) = \begin{cases} \Phi^-(t, \lambda_1, \ldots, \lambda_m) & \text{if } u_j(x) > 0,\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $\tilde u_i \neq 0$, $\tilde u_i \cdot \tilde u_j = 0$ for every $j \neq i$ and $\sum_{j=1}^m = |\Phi|$. For each j, from (1.10) we obtain that

$$J_*'(\tilde u_j(0, \lambda_1, \ldots, \lambda_m))\tilde u_j(0, \lambda_1, \ldots, \lambda_m) = J_*'((1-\bar\delta)\tilde u_j)(1-\bar\delta)\tilde u_j > 0$$

for $\lambda_j = 1 - \bar\delta$ and $|\lambda_k - 1| \leqslant \bar\delta$ if $k \neq j$, and

$$J_*'(\tilde u_j(0,\lambda_1,\dots,\lambda_m))\tilde u_j(0,\lambda_1,\dots,\lambda_m) = J_*'((1+\bar\delta)\tilde u_j)(1+\bar\delta)\tilde u_j < 0$$

for $\lambda_j = 1+\bar\delta$ and $|\lambda_k - 1| \leqslant \bar\delta$ if $k \neq j$. Hence there exists $\bar t > 0$ such that for any $0 < t < \bar t$ the same inequalities hold, namely

$$J_*'(\tilde u_j(t,\lambda_1,\dots,\lambda_m))\tilde u_j(t,\lambda_1,\dots,\lambda_m) > 0 \quad \text{for } \lambda_j = 1-\bar\delta \text{ and } |\lambda_k - 1| \leqslant \bar\delta \text{ if } k \neq j$$

$$J_*'(\tilde u_j(t,\lambda_1,\dots,\lambda_m))\tilde u_j(t,\lambda_1,\dots,\lambda_m) < 0 \quad \text{for } \lambda_j = 1+\bar\delta \text{ and } |\lambda_k - 1| \leqslant \bar\delta \text{ if } k \neq j.$$

Therefore we can deduce, by Miranda's Theorem, that for every $0 < t < \bar t$ there exists an m-uple $(\lambda_1,\dots,\lambda_m) = (\lambda_1(t),\dots,\lambda_m(t))$ such that

$$\tilde U(t,\lambda_1,\dots,\lambda_m) \in \mathcal N_\infty, \qquad \text{and hence } c_\infty \leqslant J(\tilde U).$$

On the other hand,

$$\begin{aligned}
J_\infty(\tilde U) &\leqslant \sum_{i=1}^m \left(\frac12\|\tilde u_i\|^2 - \frac14\int_\Omega \tilde u_i^4\right) = \frac12\|\Phi\|^2 - \frac14\int_\Omega \Phi^4 \\
&= \frac12\|\widehat{\Lambda u_1}\|^2 + t\langle\widehat{\Lambda u_1},\varphi\rangle + \frac12 t^2\|\varphi\|^2 - \frac14\int_\Omega(\widehat{\Lambda u_1})^4 - t\int_\Omega(\widehat{\Lambda u_1})^3\varphi + \mathrm{o}(t) \\
&= J(\lambda_1 u_1,\dots,\lambda_m u_m) + t\left(\langle\widehat{\Lambda u_1},\varphi\rangle - \int_\Omega(\widehat{\Lambda u_1})^3\varphi\right) + \mathrm{o}(t) \\
&< J(\lambda_1 u_1,\dots,\lambda_m u_m) \leqslant J(U) = c_\infty,
\end{aligned}$$

for small t, where we have used the fact that $U \in \mathcal N_\infty$ as well as the contradiction hypothesis (1.9). This yields a contradiction and the result follows. □

The following observation is based on some remarks made in [131, Section 6].

Remark 1.7. The "hat"operation defined in Theorem 1.6 has a surprising geometric interpretation, as well as the deformation $\tilde U$ considered in the proof of part (b) of the mentioned theorem. Define the set

$$K = \{c \in \mathbb R^m : \ c_i \geqslant 0,\ c_i \cdot c_j = 0 \ \text{ for every } i \neq j\}.$$

We observe that $c \in K$ if and only if $c_i = 0$ for all $i = 1,\dots,m$ except for possibly one nonnegative component. Moreover, K is a metric space for the distance function

$$\mathrm{dist}_K(c,d) := \begin{cases} |d-c| & \text{if } \langle c,d\rangle \neq 0, \\ |c|+|d| & \text{if } \langle c,d\rangle = 0. \end{cases}$$

Given $c \in K$, we define the vector $\hat c$ by

$$\hat c = \begin{pmatrix} \hat c_1 \\ \vdots \\ \hat c_i \\ \vdots \\ \hat c_m \end{pmatrix} = \begin{pmatrix} c_1 - \sum_{j\geqslant 2} c_j \\ \vdots \\ c_i - \sum_{j\neq i} c_j \\ \vdots \\ \hat c_m - \sum_{j\leqslant m-1} c_j \end{pmatrix}.$$

Given $c, d \in K$, suppose that we want to find the geodesic homotopy $\gamma : [0,1] \to K$ between c and d, that is, $\gamma(t)$ is uniquely characterized by

$$\mathrm{dist}_K(c, \gamma(t)) = t \mathrm{dist}_K(c, d) \qquad \text{for } t \in [0,1].$$

A direct computation shows that

$$\gamma(t) = \Big((1-t)\hat{c} + t\hat{d}\Big)^+ = \begin{pmatrix} ((1-t)\hat{c}_1 + t\hat{d}_1)^+ \\ \vdots \\ ((1-t)\hat{c}_i + t\hat{d}_i)^+ \\ \vdots \\ ((1-t)\hat{c}_m + t\hat{d}_m)^+ \end{pmatrix}.$$

Thus, going back to the proof of Theorem 1.6 - (b) (consider $i = 1$ for simplicity), we observe that both $\Lambda u = (\lambda_1 u_1, \ldots, \lambda_m u_m)$ and $\boldsymbol{\varphi} := (\varphi, 0, \ldots, 0)$ belong to K for every $x \in \Omega$ and that, for every $(\lambda_1, \ldots, \lambda_m)$ fixed, we can rewrite $\tilde{U}$ as

$$\tilde{U}(t, \lambda) = \left(\widehat{\begin{pmatrix} \lambda_1 u_1 \\ \vdots \\ \lambda_i u_i \\ \vdots \\ \lambda_m u_m \end{pmatrix}} + t \widehat{\begin{pmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right)^+ = \begin{pmatrix} (\widehat{\Lambda u}_1 + t\varphi)^+ \\ (\widehat{\Lambda u}_2 - t\varphi)^+ \\ \vdots \\ (\widehat{\Lambda u}_m - t\varphi)^+ \end{pmatrix},$$

and thus $\tilde{U}$ corresponds to the geodesic homotopy in K between

$$(\lambda_1 u_1, \ldots, \lambda_m u_m) \qquad \text{and} \qquad ((\widehat{\Lambda u}_1 + \varphi)^+, (\widehat{\Lambda u}_2 - \varphi)^+, \ldots, (\widehat{\Lambda u}_m - \varphi)^+).$$

1.2 Minimal solutions: a defocusing case

In this subsection we turn our attention to the study of a particular case of system (1.1) of defocusing type, namely

$$\begin{cases} -\Delta u_i - \lambda_i u_i = -u_i^3 - \beta u_i \sum\limits_{\substack{j=1 \\ j \neq i}}^m u_j^2 \quad \text{in } \Omega, \\ u_i > 0 \text{ in } \Omega, \ u_i \in H_0^1(\Omega), \end{cases} \qquad i = 1, \ldots, m. \tag{1.11}$$

Due to the choice of signs, a natural way to obtain solution of (1.11) is (following [40]) to consider the energy functional

$$J_\beta(U) = J_\beta(u_1, \ldots, u_m) = \frac{1}{2} \sum_{i=1}^m \int_\Omega |\nabla u_i|^2 + \frac{1}{4} \sum_{i=1}^m \int_\Omega u_i^4 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^m \int_\Omega \beta u_i^2 u_j^2$$

restricted to the manifold

$$\mathcal{M} = \Big\{ U = (u_1, \ldots, u_m) \in \mathcal{H} : \int_\Omega u_i^2 = 1 \ \text{ for every } \ i = 1, \ldots, m \Big\}$$

(observe that J_β is a nonnegative functional and that the vector function $U = (0, \dots, 0)$ is a global minimum). The restriction $\mathcal{M}$ corresponds to an *a priori* constraint on the mass of the solution of system (1.11), and in this framework the parameters

$$\lambda_i = \lambda_{i,\beta}(U) = \int_\Omega (|\nabla u_i|^2 + u_i^4) + \beta \int_\Omega u_i^2 \sum_{\substack{j=1\\ j\neq i}}^m u_j^2 \geqslant 0$$

arise as Lagrange multipliers. Take the *least energy level*:

$$c_\beta = \inf_{\mathcal{M}} J_\beta.$$

We follow the same structure of the preceding section. First, we prove that the level c_β is attained, and afterwards we study the behavior of the least energy solutions, proving that the limiting profiles as $\beta \to +\infty$ belong to the class $\mathcal{S}(\Omega)$. Although these results are not stated in the literature, the proofs are simple adaptations of the arguments used in [46, 47].

Theorem 1.8. *There exists $U_\beta = (u_{1,\beta}, \dots, u_{m,\beta}) \in \mathcal{M}$, with $u_{i,\beta} > 0$ in Ω for every $i = 1, \dots, m$, such that*

$$J_\beta(U_\beta) = c_\beta \qquad \text{and} \qquad U_\beta \text{ solves } (1.11).$$

Proof. Since $J_\beta(U) \geqslant 0$ for every $U \in \mathcal{H}$, then $c_\beta \geqslant 0$. Now, take a minimizing sequence for c_β, namely $(U_k)_k \subseteq \mathcal{M}$ such that $J_\beta(U_k) \to c_\beta$. Then $(U_k)_k$ is bounded in $\mathcal{H}$, and there exists $U \in \mathcal{H}$ such that

$$U_k \rightharpoonup U \text{ weakly in } H_0^1(\Omega), \qquad U_k \to U \text{ strongly in } L^p(\Omega), \quad p = 2, 4.$$

In particular $\int_\Omega u_i^2 = \lim_k \int_\Omega u_{i,k}^2 = 1$ for every i (whence $U \in \mathcal{M}$), and

$$c_\beta \leqslant J_\beta(U) = \frac{1}{2}\sum_{i=1}^m \int_\Omega |\nabla u_i|^2 + \frac{1}{4}\sum_{i=1}^m \int_\Omega u_i^4 + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^m \int_\Omega \beta u_i^2 u_j^2 \leqslant \liminf_k J_\beta(U_k) = c_\beta.$$

Thus $J_\beta(U) = c_\beta$ and by the Lagrange's multiplier rule U is a solution of (1.11). By possibly replacing U with $(|u_1|, \dots, |u_m|)$ we obtain a positive solution of the problem. □

From now on we let $U_\beta = (u_{1,\beta}, \dots, u_{m,\beta}) \in \mathcal{M}$, with $u_{i,\beta} > 0$ in Ω, be such that $J_\beta(U_\beta) = c_\beta$, whence in particular a solution of (1.11). Once again we are interested in the asymptotic behavior of U_β as $\beta \to +\infty$. Define the auxiliary functionals $J_* : H_0^1(\Omega) \to \mathbb{R}$, $J_\infty : \mathcal{H} \to \mathbb{R}$ by

$$J_*(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega u^4, \qquad J_\infty(U) = J_\infty(u_1, \dots, u_m) = \sum_{i=1}^m J_*(u_i)$$

and consider the limiting level

$$c_\infty = \inf_{\substack{U \in \mathcal{M}\\ u_i \cdot u_j = 0, \forall j \neq i}} J_\infty(U).$$

Theorem 1.9. *There exists* $U = (u_1, \dots, u_m) \in \mathcal{M}$ *with* $u_i \geqslant 0$ *(*$i = 1, \dots, m$*),* $u_i \cdot u_j = 0$ *(whenever* $i \neq j$*) and* $J_\infty(U) = c_\infty$ *such that, up to a subsequence, as* $\beta \to +\infty$*,*

(i) $u_{i,\beta} \to u_i$ *in* $H_0^1(\Omega)$*;*

(ii) $\displaystyle\int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \to 0$ *for* $i \neq j$*,*

(iii) $c_\beta \to c_\infty$*.*

Proof. Notice that for every $U \in \mathcal{M}$ with $u_i \cdot u_j = 0$ ($i \neq j$) we have $c_\beta \leqslant J_\beta(U) = J_\infty(U)$. Hence $J_\beta(U_\beta) = c_\beta \leqslant c_\infty$ and $(U_\beta)_\beta$ is a bounded sequence in $\mathcal{H}$. Thus, up to a subsequence, there exists $U \in \mathcal{M}$ such that $U_\beta \rightharpoonup U$ weakly in $\mathcal{H}$, and in particular $U \in \mathcal{M}$. From the estimate $J_\beta(U_\beta) \leqslant c_\infty$ it follows also that

$$\frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{m} \int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \leqslant c_\infty < \infty,$$

and since $\beta \to +\infty$ we conclude that $u_i \cdot u_j = 0$ whenever $i \neq j$. Therefore, for every β,

$$\begin{aligned} c_\infty \leqslant J_\infty(U) \leqslant \liminf_\beta \sum_{i=1}^{m} \left(\frac{1}{2} \int_\Omega |\nabla u_{i,\beta}|^2 + \frac{1}{4} \int_\Omega u_{i,\beta}^4 \right) &\leqslant \liminf_\beta J_\beta(U_\beta) \\ &= \liminf_\beta c_\beta \leqslant \limsup_\beta c_\beta \leqslant c_\infty, \end{aligned}$$

and this provides all the conclusions of the theorem. □

Theorem 1.10. *Let* $U = (u_1, \dots, u_m)$ *be a limiting profile as before. Then for every* $i = 1, \dots, m$*, we have*

(i) $-\Delta u_i + u_i^3 \leqslant \lambda(u_i) u_i$ *in* Ω*,*

(ii) $\displaystyle -\Delta \hat{u}_i + \hat{u}_i^3 \geqslant \lambda(u_i) u_i - \sum_{\substack{j=1 \\ j \neq i}}^{m} \lambda(u_j) u_j$ *in* Ω*,*

where $\hat{u}_i = u_i - \sum_{j \neq i} u_j$ *and* $\lambda(w) := \int_\Omega (|\nabla w|^2 + w^4)$ *(again, both inequalities are understood in the distributional sense). In particular,* $U \in \mathcal{S}(\Omega)$*.*

Proof. In order to simplify notations, we present the proof for $i = 1$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a nonnegative function.
(i) Consider the deformation

$$\tilde{U}(t) = (\tilde{u}_1, \dots, \tilde{u}_m) = \left(\frac{(u_1 - t\varphi)^+}{\|(u_1 - t\varphi)^+\|_2}, u_2, \dots, u_m \right).$$

Obviously $\tilde{U}(t) \in \mathcal{M}$ and $\tilde{u}_i \cdot \tilde{u}_j \equiv 0$ whenever $i \neq j$, whence $c_\beta \leqslant J_\infty(\tilde{U}(t))$. Let us compute the energy of $\tilde{U}(t)$,

$$J_\infty(\tilde{U}(t)) = J_* \left(\frac{(u_1 - t\varphi)^+}{\|(u_1 - t\varphi)^+\|_2} \right) + \sum_{j \geqslant 2} J_*(u_j).$$

From the fact that

$$\frac{1}{\|(u_1-t\varphi)^+\|_2^2}=1+2t\int_\Omega u_1\varphi+\mathrm{o}(t),\qquad \frac{1}{\|(u_1-t\varphi)^+\|_2^4}=1+4t\int_\Omega u_1\varphi+\mathrm{o}(t) \quad (1.12)$$

and

$$\int_\Omega |\nabla(u_1-t\varphi)|^2\to\int_\Omega|\nabla u_1|^2,\qquad \int_\Omega|u_1-t\varphi|^4\to\int_\Omega u_1^4 \quad (1.13)$$

as $t\to 0$, we deduce that

$$\begin{aligned} J_*\left(\frac{(u_1-t\varphi)^+}{\|(u_1-t\varphi)^+\|_2}\right) &\leqslant \frac{1}{2}\frac{1}{\|(u_1-t\varphi)^+\|_2^2}\int_\Omega|\nabla(u_1-t\varphi)|^2+\frac{1}{4}\frac{1}{\|(u_1-t\varphi)^+\|_2^4}\int_\Omega(u_1-t\varphi)^4\\ &= J_*(u_1-t\varphi)+t\int_\Omega u_1\varphi\left(\int_\Omega|\nabla(u_1-t\varphi)|^2+\int_\Omega|u_1-t\varphi|^4\right)+\mathrm{o}(t)\\ &= J_*(u_1-t\varphi)+t\lambda(u_1)\int_\Omega u_1\varphi+\mathrm{o}(t). \end{aligned}$$

Moreover,

$$J_*(u_1-t\varphi)=J_*(u_1)+t\left(-\int_\Omega\langle\nabla u_1,\nabla\varphi\rangle-\int_\Omega u_1^3\varphi\right)+\mathrm{o}(t)$$

and hence

$$J_\infty(U)=c_\infty\leqslant J_\infty(\tilde U)\leqslant J_\infty(U)+t\left(-\int_\Omega\langle\nabla u_1,\nabla\varphi\rangle-\int_\Omega u_1^3\varphi+\lambda(u_1)\int_\Omega u_1\varphi\right)+\mathrm{o}(t)$$

as $t\to 0$, which implies that

$$-\int_\Omega\langle\nabla u_1,\nabla\varphi\rangle-\int_\Omega u_1^3\varphi+\lambda(u_1)\int_\Omega u_1\varphi\geqslant 0.$$

(ii) This time we consider the deformation:

$$\tilde U(t)=(\tilde u_1,\dots,\tilde u_m)=\left(\frac{(\hat u_1+t\varphi)^+}{\|(\hat u_1+t\varphi)^+\|_2},\frac{(\hat u_2-t\varphi)^+}{\|(\hat u_2-t\varphi)\|_2},\dots,\frac{(\hat u_m-t\varphi)^+}{\|\hat u_m-t\varphi\|_2}\right).$$

Recalling Remark 1.7, this corresponds to the normalization (in L^2) of the geodesic homotopy between $(u_1,\dots,u_m)$ and $((\hat u_1+\varphi)^+,(\hat u_2-\varphi)^+,\dots,(\hat u_m+\varphi)^+)$. Hence $\tilde U(t)\in\mathcal{M}$ for every $t\in[0,1]$, and $\tilde u_i\cdot\tilde u_j\equiv 0$ whenever $i\neq j$. Moreover, observe that $(\hat u_1+t\varphi)^+ + \sum_{j\geqslant 2}(\hat u_j-t\varphi)^+=\hat u_1+t\varphi$. We have

$$\begin{aligned} J_\infty(\tilde U(t))=&\frac{1}{2\|(\hat u_1+t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2+\frac{1}{4\|(\hat u_1+t\varphi)^+\|_2^4}\int_\Omega[(\hat u_1+t\varphi)^+]^4+\\ &+\sum_{j\geqslant 2}\left(\frac{1}{2\|(\hat u_j-t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_j-t\varphi)^+|^2+\frac{1}{4\|(\hat u_j-t\varphi)^+\|_2^4}\int_\Omega[(\hat u_j-t\varphi)^+]^4\right). \end{aligned}$$

By taking in consideration (1.12) and (1.13), it follows that

$$\begin{aligned} \frac{1}{2\|(\hat u_1+t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2 &= \frac{1}{2}\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2\left(1-2t\int_\Omega u_1\varphi+\mathrm{o}(t)\right)\\ &= \frac{1}{2}\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2-t\int_\Omega u_1\varphi\int_\Omega|\nabla u_1|^2+\mathrm{o}(t), \end{aligned}$$

$$
\begin{aligned}
\frac{1}{4\|(\hat u_1+t\varphi)^+\|_2^4}\int_\Omega[(\hat u_1+t\varphi)^+]^4 &= \frac14\int_\Omega[(\hat u_1+t\varphi)^+]^4\Big(1-4t\int_\Omega u_1\varphi+\mathrm{o}(t)\Big)\\
&= \frac14\int_\Omega[(\hat u_1+t\varphi)^+]^4-t\int_\Omega u_1\varphi\int_\Omega u_1^4+\mathrm{o}(t)
\end{aligned}
$$

and, for $j\geqslant 2$,

$$
\begin{aligned}
\frac{1}{2\|(\hat u_j-t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_j-t\varphi)^+|^2 &= \frac12\int_\Omega|\nabla(\hat u_j-t\varphi)^+|^2\Big(1+2t\int_\Omega u_j\varphi+\mathrm{o}(t)\Big)\\
&= \frac12\int_\Omega|\nabla(\hat u_j-t\varphi)^+|^2+t\int_\Omega u_j\varphi\int_\Omega|\nabla u_j|^2+\mathrm{o}(t),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4\|(\hat u_j-t\varphi)^+\|_2^4}\int_\Omega[(\hat u_j-t\varphi)^+]^4 &= \frac14\int_\Omega[(u_j-t\varphi)^+]^4\Big(1+4t\int_\Omega u_j\varphi+\mathrm{o}(t)\Big)\\
&= \frac14\int_\Omega[(\hat u_j-t\varphi)^+]^4+t\int_\Omega u_j\varphi\int_\Omega u_j^4+\mathrm{o}(t).
\end{aligned}
$$

We finally conclude that

$$
\begin{aligned}
J_\infty(U) &= c_\infty\leqslant J_\infty(\tilde U(t))=J_*(\hat u_1+t\varphi)-t\int_\Omega\lambda(u_1)u_1\varphi+t\sum_{j\geqslant2}\int_\Omega\lambda(u_j)u_j\varphi+\mathrm{o}(t)\\
&= J_\infty(U)+t\int_\Omega\Big(\langle\nabla\hat u_1,\nabla\varphi\rangle+\hat u_1^3\varphi-\Big(\lambda(u_1)u_1+\sum_{j\geqslant2}\lambda(u_j)u_j\Big)\varphi\Big)+\mathrm{o}(t)
\end{aligned}
$$

as $t\to0$, and hence

$$
\int_\Omega\Big(\langle\nabla\hat u_1,\nabla\varphi\rangle+\hat u_1^3\varphi-\Big(\lambda(u_1)u_1+\sum_{j\geqslant2}\lambda(u_j)u_j\Big)\varphi\Big)\geqslant 0.
$$

□

1.3 Regularity results for the class $\mathcal{S}(\Omega)$. Additional comments

In the two previous subsections we have studied two particular cases of solutions which satisfy a minimization principle. As illustrated, the minimality was essential to establish two points:

- up to a subsequence, there is strong H^1–convergence to some limiting profile,
- each limiting profile belongs to the following class of functions

$$
\mathcal{S}(\Omega)=\left\{U\in\big(H^1(\Omega)\cap L^\infty(\Omega)\big)^m:\ \begin{array}{c} u_i\geqslant0,\ u_i\cdot u_j\equiv0\text{ for }i\neq j,\ -\Delta u_i\leqslant f_i(u_i),\\ -\Delta\Big(u_i-\sum_{\substack{j=1\\ j\neq i}}^m u_j\Big)\geqslant f_i(u_i)-\sum_{\substack{j=1\\ j\neq i}}^m f_j(u_j)\end{array}\right\}.
$$

At this point, let us now explain in a detailed way what are the regularity results known to date for the elements in $\mathcal{S}(\Omega)$. The following two theorems were proved by Conti, Terracini and Verzini [46, 47].

Theorem 1.11. *Suppose that $f(s) = \mathrm{O}(s)$ as $s \to 0^+$ and take $U \in \mathcal{S}$. Then U is Lipschitz continuous in the interior of Ω. If moreover $\Omega \subseteq \mathbb{R}^N$ is a regular set of class C^2 and $U \in W^{1,\infty}(\partial\Omega)$, then U is Lipschitz continuous in $\bar{\Omega}$.*

In dimension $N = 2$, more is known.

Theorem 1.12. *Suppose $N = 2$, $f(s) = \mathrm{O}(s)$ as $s \to 0^+$, and let $U \in \mathcal{S}$. Define*

$$m(x) = \sharp\{i : \ |\{u_i > 0\} \cap B_r(x)| > 0, \ \forall r > 0\}.$$

If $m(x_0) \geqslant 3$ then $|\nabla U(x)| \to 0$ as $x \to x_0$.

Suppose moreover that $\{u_i > 0\}$ is connected for every $i = 1, \dots, m$. Then $\nabla U(x_0) \neq 0$ whenever x_0 is such that $m(x_0) = 2$, and hence the set

$$\{x \in \Omega : \ m(x) = 2\} \text{ is locally a } C^{1,\alpha}\text{–curve for every } 0 < \alpha < 1.$$

Furthermore,

- *The set $\{x \in \Omega : \ m(x) \geqslant 3\}$ is locally finite;*
- *for each $x_0 \in \Omega$ such that $m(x_0) = h \geqslant 3$ there exists $\theta_0 \in (-\pi, \pi]$ such that*

$$U(r, \theta) = r^{h/2} \left| \cos\left(\frac{h}{2}(\theta + \theta_0)\right)\right| + \mathrm{o}(r^{h/2}) \qquad \text{as } r \to 0,$$

  where (r, θ) denote the polar coordinates around x_0.

Next we state an improvement of the previous results which does not depend on the spacial dimension N, although is is assumed that $f_i \equiv 0$ for all i. These results are collected from [29, 31]. We believe that the same results should hold also in the case $f_i(s) = \mathrm{O}(s)$ as $s \to 0^+$, although this fact is not clear in the mentioned papers.

Theorem 1.13. *Suppose that $f_i = 0$ for every $i = 1, \dots, m$ and let $U \in \mathcal{S}$. Then*

- *$|\nabla U|$ is a continuous function in Ω,*
- *the nodal set $\Gamma_U = \{x \in \Omega : \ U(x) = 0\}$ can be spliced in $\Sigma_U \cup S_U$, where $\mathscr{H}_{\dim}(S_U) \leqslant N - 2$, and Σ_U is locally an $(N-1)$–dimensional manifold.*

The two last theorems are connected in the following way: for $N = 2$, under the assumptions of Theorem 1.12 we have that

$$\Sigma_U = \{x \in \Omega : \ m(x) = 2\} \qquad \text{and} \qquad S_U = \{x \in \Omega : \ m(x) \geqslant 3\}.$$

In the proof of these theorems, the key property is the fact that for each i it holds

$$-\Delta\Big(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{m} u_j\Big) \geqslant f_i(u_i) - \sum_{\substack{j=1 \\ j \neq i}}^{m} f_j(u_j), \tag{1.14}$$

which contains an information about the interaction between the different components u_i of each segregated configuration. The strength of this condition can be explained in a very simple way. If in some small ball $B_\delta(x_0)$ we only have two components, say u_1, u_2, then

$$-\Delta(u_1 - u_2) \geqslant f_1(u_1) - f_2(u_2), \qquad -\Delta(u_2 - u_1) \geqslant f_2(u_2) - f_1(u_1),$$

and therefore

$$-\Delta(u_1 - u_2) = f_1(u_1) - f_2(u_2) \qquad \text{in } B_\delta(x_0).$$

Unfortunately, condition (1.14) seems to be deeply connected with least energy solutions and it is very unclear whether it holds for other kind of solutions. Hence, in order to obtain regularity results for more general families of solutions, we must find a more general characterization between the components u_i. To do so, one of the first things we need to settle is the strong convergence of solutions of (1.1) to some limiting profile. This is not an immediate fact as it was when dealing with minimal energy solutions (recall Theorems 1.5 and 1.9). We will treat these subjects in the following two chapters.

We close this chapter with a final observation concerning the class $\mathcal{S}(\Omega)$.

Remark 1.14. Several papers show that the solutions to many other problems end up in the class $\mathcal{S}(\Omega)$. In fact, in [48] the authors show that the limits of a system with a Lotka-volterra type interaction (non variational) also belong to the class $\mathcal{S}(\Omega)$. Moreover, in [45, 46, 47, 48, 69] some optimal partition problems involving eigenvalues are considered, and a connection between their solutions and minimal solutions of (1.1) is shown. We will come back to this at the end of Chapter 3, where we will discuss with some detail these examples.

Chapter 2

Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition

2.1 Statement of the results

Consider the following system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_{i,\beta} u_i = \omega_i u_i^3 - \beta u_i \sum_{j\neq i} u_j^2 \\ u_i \in H_0^1(\Omega),\ u_i > 0 \text{ in } \Omega,\ i = 1, \dots, m \end{cases} \tag{2.1}$$

for the competition parameter $\beta \in (0, +\infty)$, with $\Omega \subseteq \mathbb{R}^N$ a smooth bounded domain in dimension $N = 2, 3$. The main results of this chapter will be the following.

Theorem 2.1. *Let $U_\beta = (u_{1,\beta}, \dots, u_{m,\beta})$ be a solution of (2.1) uniformly bounded in $L^\infty(\Omega)$–norm, where $\lambda_{i,\beta}$ are bounded in $\mathbb{R}$ and ω_i are fixed real constants for every i. Then for every $\alpha \in (0,1)$ there exists $C > 0$, independent of β, such that*

$$\|U_\beta\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant C \quad \textit{for every } \beta > 0.$$

Theorem 2.2. *Under the assumptions of the previous theorem, there exists an m–uple $U = (u_1, \dots, u_m)$ of Lipschitz continuous functions such that, up to a subsequence, it holds*

(i) $u_{i,\beta} \to u$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$, $\forall\ \alpha \in (0,1)$, *as* $\beta \to +\infty$;

(ii) $u_i \cdot u_j \equiv 0$ *in* Ω *and* $\displaystyle\int_\Omega \beta u_{i,\beta}^2 u_{j,\beta}^2 \to 0$ *as* $\beta \to +\infty$, *whenever* $i \neq j$;

(iii) the limiting functions $u_1, \dots, u_m$ *satisfy the system*

$$-\Delta u_i + \lambda_i u_i = \omega_i^3 \qquad \textit{in } \{u_i > 0\}, \quad i = 1, \dots, m, \tag{2.2}$$

with $\lambda_i := \lim_{\beta\to+\infty} \lambda_{i,\beta}$.

We would like to make two remarks about the assumptions of the previous theorems. The first is that it is natural to allow the dependence of the parameters λ_i on β, as we saw in the example considered in Section 1.2. Secondly, we observe that in Theorem 2.1 the assumption that the family $(U_\beta)_\beta$ should be uniformly bounded is essential, since there are in general no *a priori* bounds in L^∞ for the solutions of (2.1) for β large (not even for each fixed β). Indeed, in [55, Theorem 1.2] it is shown that, in the case of $m = 2$ equations and $\lambda_1 = \lambda_2 = 1 = \omega_1 = \omega_2$, for every $\beta \geqslant 1$ there exists a sequence $(u_1^k, u_2^k)_k$ of solutions of (2.1) such that $\|u_1^k\|_\infty + \|u_2^k\|_\infty \to +\infty$ as $k \to +\infty$.

The proofs of Theorems 2.1 and 2.2 will be carried out in the case of $m = 2$ equations. This will be done in order to simplify the presentation and to provide a better understanding of the techniques involved (some of these techniques will also be used in the subsequent chapter). The proof of the general case with an arbitrary number $m \geqslant 2$ of equations is almost analogous, and the few adaptations required will be pointed out along this chapter. Moreover, the study of system (2.1) for $m = 2$ will be carried out as a particular case of a more general one, where L^2–perturbations are allowed. The reason for this approach is that in Chapter 4 we intend to present a variational construction so as to obtain, for every fixed β, several solutions of (2.1); the present estimates, in their more general version, will then be used to study how, and in which sense, such a variational structure passes to the limit as $\beta \to +\infty$. To be more precise, let us consider the system

$$\begin{cases} -\Delta u + \lambda_\beta u = \omega_1 u^3 - \beta u v^2 + h_\beta & \text{in } \Omega \\ -\Delta v + \mu_\beta v = \omega_2 v^3 - \beta u^2 v + k_\beta & \text{in } \Omega \\ u, v \in H_0^1(\Omega), \quad u, v \geqslant 0 & \text{in } \Omega \end{cases} \tag{2.3}$$

under the assumptions that h_β, k_β are uniformly bounded in $L^2(\Omega)$, $\lambda_\beta, \mu_\beta \in \mathbb{R}$ are bounded in $\mathbb{R}$, and $\omega_1, \omega_2 \in \mathbb{R}$ are fixed constants. If we define

$$\alpha^* = \begin{cases} 1 & \text{if } N = 2 \\ 1/2 & \text{if } N = 3, \end{cases} \tag{2.4}$$

then by the Sobolev embeddings we have that any solution of (2.3) belongs to $C^{0,\alpha}(\overline{\Omega})$, for every $\alpha \in (0, \alpha^*)$ (and even $\alpha = \alpha^*$ if $N = 3$). As a consequence, in the general case of system (2.3) we can not expect boundedness for every Hölder exponent. In fact we have the following.

Theorem 2.3. *Let u_β, v_β be solutions of (2.3) uniformly bounded in $L^\infty(\Omega)$. Then for every $\alpha \in (0, \alpha^*)$ there exists $C > 0$, independent of β, such that*

$$\|(u_\beta, v_\beta)\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant C \quad \textit{for every } \beta > 0.$$

Theorem 2.4. *Let u_β, v_β be solutions of (2.3) uniformly bounded in $L^\infty(\Omega)$. Then there exists $(u, v) \in C^{0,\alpha}$, $\forall\, \alpha \in (0, \alpha^*)$, such that (up to a subsequence) there holds, as $\beta \to +\infty$,*

(i) $u_\beta \to u, v_\beta \to v$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$, $\forall\, \alpha \in (0, \alpha^*)$;

(ii) $u \cdot v \equiv 0$ *in* Ω *and* $\displaystyle\int_\Omega \beta u_\beta^2 v_\beta^2 \to 0$;

(iii) the limiting functions u, v satisfy the system

$$\begin{cases} -\Delta u + \lambda u &= \omega_1 u^3 + h \quad \text{in } \{u > 0\} \\ -\Delta v + \mu v &= \omega_2 v^3 + k \quad \text{in } \{v > 0\}, \end{cases} \tag{2.5}$$

where $\lambda := \lim \lambda_\beta$, $\mu := \lim \mu_\beta$, and h, k denote the L^2–weak limits of h_β, k_β as $\beta \to +\infty$.

Even though the result of Theorem 2.1 is stronger (no limitation on α), its proof is in fact a particular case of the one of Theorem 2.3, once one observes that, if $h_\beta \equiv k_\beta \equiv 0$, then u_β and v_β, at any fixed β, belong to $C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0, 1)$. For this reason, we will prove in detail all the results in the case of system (2.3), except for the Lipschitz continuity of the limiting state (Section 2.4), which requires $h_\beta \equiv k_\beta \equiv 0$. Our results rely upon a combination of blowup techniques (Section 2.3) together with some suitable Liouville–type theorems (Section 2.2). Such a strategy has been already adopted by Conti, Terracini and Verzini [48] in order to provide uniform Hölder estimates for competition–diffusion systems with Lotka–Volterra type of interactions. The arguments there, however, though helpful in the present situation, need to be complemented with some new ideas, including a proper use of the Almgren's frequency formula [1]. This requires the systems to have a gradient form, which is the case of (2.1). Let us mention that Hölder estimates for (non gradient) coupling arising in combustion theory have been obtained by Caffarelli and Roquejoffre [33].

2.2 Liouville–type results

In this section we prove some nonexistence results in $\mathbb{R}^N$. The main tools will be the monotonicity formula by Alt, Caffarelli and Friedman originally stated in [3], as well as some generalizations made by Conti, Terracini and Verzini [47, 48]. For a complete and self contained proof of the following lemma, see the work by Noris [91].

Lemma 2.5 (Alt-Caffarelli-Friedman's monotonicity formula)**.** *Let $u, v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be nonnegative functions such that $u \cdot v \equiv 0$. Assume moreover that $-\Delta u \leqslant 0, -\Delta v \leqslant 0$ in $\mathbb{R}^N$ and let $x_0 \in \mathbb{R}^N$ be such that $u(x_0) = v(x_0) = 0$. Then the function*

$$J(r) := \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \cdot \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x - x_0|^{N-2}}$$

only assumes finite values and is non decreasing for $r \in (0, +\infty)$.

Proposition 2.6. *Under the same assumptions of Lemma 2.5, assume moreover that for some $\alpha \in (0, 1)$ there holds*

$$\sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha} < \infty. \tag{2.6}$$

Then either $u \equiv 0$ or $v \equiv 0$.

Proof. Assume by contradiction that neither u nor v is zero, then none of them is constant since $u(x_0) = v(x_0) = 0$. Hence Lemma 2.5 ensures the existence of a constant $C > 0$ such that

$$\int_{B_r(x_0)} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \cdot \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x-x_0|^{N-2}} \geqslant C r^4 \tag{2.7}$$

for every r sufficiently large. Let $\eta_{a,b}$ $(0 < a < b)$ be any smooth, radial, cut-off function with the following properties: $0 \leqslant \eta_{a,b} \leqslant 1$, $\eta_{a,b} = 0$ in $\mathbb{R}^N \setminus B_b(x_0)$, $\eta_{a,b} = 1$ in $B_a(x_0)$ and $|\nabla \eta_{a,b}| \leqslant C/(b-a)$. Given $0 < \varepsilon \ll r$, let $A_\varepsilon := B_{2r}(x_0) \setminus B_\varepsilon(x_0)$ and $\eta := \eta_{r,2r}(1 - \eta_{\varepsilon,2\varepsilon})$. By testing the inequality $-\Delta u \leqslant 0$ with the function $\eta^2 u/|x-x_0|^{N-2}$, we obtain

$$\begin{aligned}\int_{A_\varepsilon} \frac{\eta^2 |\nabla u|^2}{|x-x_0|^{N-2}} &\leqslant -\int_{A_\varepsilon} \left(\frac{2\eta u}{|x-x_0|^{N-2}} \langle \nabla u, \nabla \eta \rangle + \eta^2 u \Big\langle \nabla u, \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle \right) \\ &\leqslant \int_{A_\varepsilon} \left(\frac{1}{2} \frac{\eta^2 |\nabla u|^2}{|x-x_0|^{N-2}} + 2 \frac{u^2 |\nabla \eta|^2}{|x-x_0|^{N-2}} - \eta^2 u \Big\langle \nabla u, \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle \right).\end{aligned}$$

We can rewrite the last term by using the fact that $1/|x-x_0|^{N-2}$ is harmonic in A_ε,

$$\begin{aligned} 0 &= \int_{A_\varepsilon} \Big\langle \nabla \Big(\frac{\eta^2 u^2}{2} \Big), \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle \\ &= \int_{A_\varepsilon} \left(\eta u^2 \Big\langle \nabla \eta, \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle + \eta^2 u \Big\langle \nabla u, \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle \right), \end{aligned}$$

obtaining

$$\frac{1}{2} \int_{A_\varepsilon} \frac{\eta^2 |\nabla u|^2}{|x-x_0|^{N-2}} \leqslant \int_{A_\varepsilon} \left(2 \frac{u^2 |\nabla \eta|^2}{|x-x_0|^{N-2}} + \eta u^2 \Big\langle \nabla \eta, \nabla \Big(\frac{1}{|x-x_0|^{N-2}} \Big) \Big\rangle \right).$$

By the definition of η, the last expression becomes

$$\begin{aligned} \frac{1}{2} \int_{B_r(x_0) \setminus B_{2\varepsilon}(x_0)} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} &\leqslant \int_{B_{2\varepsilon}(x_0) \setminus B_\varepsilon(x_0)} \left(\frac{C'}{\varepsilon^2} \frac{u^2}{|x-x_0|^{N-2}} + \frac{C''}{\varepsilon} \frac{u^2}{|x-x_0|^{N-1}} \right) + \\ &\quad + \int_{B_{2r}(x_0) \setminus B_r(x_0)} \left(\frac{C'}{r^2} \frac{u^2}{|x-x_0|^{N-2}} + \frac{C''}{r} \frac{u^2}{|x-x_0|^{N-1}} \right) \\ &\leqslant \frac{C}{\varepsilon^N} \int_{B_{2\varepsilon}(x_0)} u^2 + \frac{C}{r^N} \int_{B_{2r}(x_0)} u^2. \end{aligned}$$

Keeping in mind that $u(x_0) = 0$ as well as (2.6), we let now $\varepsilon \to 0$, obtaining

$$\int_{B_r(x_0)} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \leqslant \frac{C}{r^N} \int_{B_{2r}(x_0)} u^2 \leqslant \frac{C'}{r^N} \int_{B_{2r}(x_0)} |x-x_0|^{2\alpha} \leqslant C'' r^{2\alpha}$$

(in the same spirit, see also [28, Remark d), page 299]). Since the same result holds true for the function v, we finally obtain

$$\int_{B_r(x_0)} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \cdot \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x-x_0|^{N-2}} \leqslant C r^{4\alpha}.$$

By recalling that $\alpha < 1$, this contradicts (2.7) for r sufficiently large. $\square$

Corollary 2.7. *Let u be a harmonic function in $\mathbb{R}^N$ such that for some $\alpha \in (0,1)$ there holds*

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

Then u is a constant function.

Proof. If $u \geqslant 0$ or $u \leqslant 0$, then since u is harmonic it follows from the usual nonexistence Liouville theorem that u is a constant function. Otherwise, if u changes sign, then we can apply the previous result to its positive and negative parts obtaining that either $u^+ \equiv 0$ or $u^- \equiv 0$, and thus u is a constant function. □

Remark 2.8. The previous result does not hold for $\alpha = 1$: consider for instance the function $u(x) = x_1$ (analogously, by reasoning as in [49, Section 4], it is possible to see that also system (2.8) below admits non trivial solutions which are globally bounded in Lipschitz norm). These are the main reasons for which our strategy, as it is, cannot apply to prove uniform Lipschitz estimates.

We will need to prove a version of Proposition 2.6 suitable for functions u, v which do not have disjoint supports but are positive solutions in $H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of the system

$$\begin{cases} -\Delta u &= -uv^2 \quad \text{in } \mathbb{R}^N \\ -\Delta v &= -u^2 v \quad \text{in } \mathbb{R}^N. \end{cases} \tag{2.8}$$

Again, in order to obtain a Liouville–type result for the previous system, we will use a suitable generalization of the monotonicity formula (a similar idea, even though with slightly different equations, can be found in [47, 48]). To this aim we introduce the C^1 auxiliary function

$$f(r) = \begin{cases} \dfrac{2-N}{2} r^2 + \dfrac{N}{2} & r \leqslant 1 \\ \dfrac{1}{r^{N-2}} & r > 1 \end{cases}$$

and observe that $\Delta f(|x|) \leqslant 0$ for a.e. $x \in \mathbb{R}^N$.

Lemma 2.9. *Let u, v be positive solutions of (2.8) and let $\varepsilon > 0$ be fixed. Then there exists $\bar{r} = \bar{r}(u, v, \varepsilon) > 1$ such that the function*

$$J(r) := \frac{1}{r^{4-\varepsilon}} \int_{B_r(0)} f(|x|) \left(|\nabla u|^2 + u^2 v^2\right) \cdot \int_{B_r(0)} f(|x|) \left(|\nabla v|^2 + u^2 v^2\right)$$

is increasing for $r \in (\bar{r}, +\infty)$.

Proof. 1. Let us first evaluate the derivative of $J(r)$ for $r > 1$. In order to simplify notations we shall denote $J(r) = J_1(r) J_2(r)/r^{4-\varepsilon}$, with

$$J_1(r) = \int_{B_r(0)} f(|x|) \left(|\nabla u|^2 + u^2 v^2\right), \qquad J_2(r) = \int_{B_r(0)} f(|x|) \left(|\nabla v|^2 + u^2 v^2\right).$$

Then we have

$$\frac{J'(r)}{J(r)} = -\frac{4-\varepsilon}{r} + \frac{\int_{\partial B_r(0)} f(|x|)(|\nabla u|^2 + u^2 v^2)\, d\sigma}{J_1(r)} + \frac{\int_{\partial B_r(0)} f(|x|)(|\nabla v|^2 + u^2 v^2)\, d\sigma}{J_2(r)} \tag{2.9}$$

for a.e. $r > 0$. We can rewrite the term J_1 in a different way: by testing the equation for u with $f(|x|)u$ an integrating by parts in $B_r(0)$, we obtain

$$\begin{aligned}\int_{B_r(0)} f(|x|)(|\nabla u|^2 + u^2v^2) &= -\int_{B_r(0)} \langle \frac{1}{2}\nabla(u^2), \nabla f(|x|)\rangle + \int_{\partial B_r(0)} f(|x|)u\partial_\nu u\, d\sigma \\ &= \int_{B_r(0)} \frac{\Delta f(|x|)}{2}u^2 + \int_{\partial B_r(0)} \Big(f(|x|)u\partial_\nu u - \frac{u^2}{2}\partial_\nu(f(|x|))\Big)\, d\sigma \\ &\leqslant \int_{\partial B_r(0)} \Big(f(|x|)u\partial_\nu u - \frac{u^2}{2}\partial_\nu(f(|x|))\Big)\, d\sigma,\end{aligned}$$

which gives, recalling the definition of f,

$$J_1(r) \leqslant \frac{1}{r^{N-2}}\int_{\partial B_r(0)} u\partial_\nu u\, d\sigma + \frac{N-2}{2r^{N-1}}\int_{\partial B_r(0)} u^2\, d\sigma \tag{2.10}$$

for $r > 1$. In order to estimate this quantity we define

$$\Lambda_1(r) := \frac{r^2\int_{\partial B_r(0)}(|\nabla_\theta u|^2 + u^2v^2)\, d\sigma}{\int_{\partial B_r(0)} u^2\, d\sigma}, \qquad \Lambda_2(r) := \frac{r^2\int_{\partial B_r(0)}(|\nabla_\theta v|^2 + u^2v^2)\, d\sigma}{\int_{\partial B_r(0)} v^2\, d\sigma},$$

where $|\nabla_\theta u|^2 = |\nabla u|^2 - (\partial_\nu u)^2$. Then by Young's inequality we have that, for every $\delta \in \mathbb{R}$,

$$\begin{aligned}\Big|\int_{\partial B_r(0)} u\partial_\nu u\, d\sigma\Big| &\leqslant \Big(\int_{\partial B_r(0)} u^2\, d\sigma\Big)^{1/2}\Big(\int_{\partial B_r(0)} (\partial_\nu u)^2\, d\sigma\Big)^{1/2} \\ &\leqslant \frac{\sqrt{\Lambda_1(r)}}{2\delta^2 r}\int_{\partial B_r(0)} u^2\, d\sigma + \frac{\delta^2 r}{2\sqrt{\Lambda_1(r)}}\int_{\partial B_r(0)} (\partial_\nu u)^2\, d\sigma \\ &\leqslant \frac{r}{2\sqrt{\Lambda_1(r)}}\left(\frac{\Lambda_1(r)}{\delta^2 r^2}\int_{\partial B_r(0)} u^2\, d\sigma + \delta^2\int_{\partial B_r(0)} (\partial_\nu u)^2\, d\sigma\right) \\ &\leqslant \frac{r}{2\sqrt{\Lambda_1(r)}}\left(\frac{1}{\delta^2}\int_{\partial B_r(0)} \left(|\nabla_\theta u|^2 + u^2v^2\right)\, d\sigma + \delta^2\int_{\partial B_r(0)} (\partial_\nu u)^2\, d\sigma\right).\end{aligned} \tag{2.11}$$

Moreover,

$$\frac{N-2}{2r^{N-1}}\int_{\partial B_r(0)} u^2\, d\sigma = \frac{N-2}{2\Lambda_1(r)r^{N-3}}\int_{\partial B_r(0)}(|\nabla_\theta u|^2 + u^2v^2)\, d\sigma. \tag{2.12}$$

Thus, by combining (2.10)-(2.12) we obtain that

$$J_1(r) \leqslant \frac{1}{2r^{N-3}}\left[\left(\frac{1}{\delta^2\sqrt{\Lambda_1(r)}} + \frac{N-2}{\Lambda_1(r)}\right)\int_{\partial B_r(0)}(|\nabla_\theta u|^2 + u^2v^2) + \frac{\delta^2}{\sqrt{\Lambda_1(r)}}\int_{\partial B_r(0)}(\partial_\nu u)^2\right].$$

Now, we choose δ in such a way that $\dfrac{1}{\delta^2\sqrt{\Lambda_1(r)}} + \dfrac{N-2}{\Lambda_1(r)} = \dfrac{\delta^2}{\sqrt{\Lambda_1(r)}}$ or, equivalently,

$$\sqrt{\Lambda_1(r)}\delta^2 = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \Lambda_1(r)}.$$

Thus

$$\frac{\sqrt{\Lambda_1(r)}}{\delta^2} = \gamma(\Lambda_1(r)),$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\gamma(x) = \sqrt{\left(\frac{N-2}{2}\right)^2 + x} - \frac{N-2}{2}.$$

We remark that this function plays a crucial role in the proof of the Alt–Caffarelli–Friedman Monotonicity Formula (see [3]). Of particular importance is the following property: let E_1, E_2 be any couple of disjoint measurable subsets of the sphere S^{N-1} and denote by $\lambda_1(E_i)$ the first eigenvalue of the Dirichlet Laplacian[1] on S^{N-1}, then

$$\gamma(\lambda_1(E_1)) + \gamma(\lambda_1(E_2)) \geqslant 2. \tag{2.13}$$

With this choice of δ we have

$$\begin{aligned} J_1(r) &\leqslant \frac{1}{2r^{N-3}\gamma(\Lambda_1(r))} \int_{\partial B_r(0)} (|\nabla_\theta u|^2 + (\partial_\nu u)^2 + u^2v^2)\, d\sigma \\ &= \frac{r}{2\gamma(\Lambda_1(r))} \int_{\partial B_r(0)} f(|x|)(|\nabla u|^2 + u^2v^2)\, d\sigma \end{aligned}$$

(recall that $r > 1$ and consequently $f(r) = 1/r^{N-2}$) and a similar expression holds also for J_2. Coming back to (2.9), it follows that

$$\frac{J'(r)}{J(r)} \geqslant -\frac{4-\varepsilon}{r} + \frac{2\gamma(\Lambda_1(r))}{r} + \frac{2\gamma(\Lambda_2(r))}{r}.$$

2. At this point it only remains to prove that there exists $\bar{r} > 1$ such that for every $r \geqslant \bar{r}$ it holds

$$\gamma(\Lambda_1(r)) + \gamma(\Lambda_2(r)) > \frac{4-\varepsilon}{2}. \tag{2.14}$$

To this aim we define the functions $u_{(r)}, v_{(r)} : \partial B_1(0) \to \mathbb{R}$ by $u_{(r)}(\theta) := u(r\theta)$, $v_{(r)}(\theta) := v(r\theta)$. Then a change of variables yields

$$\Lambda_1(r) = \frac{\int_{\partial B_1(0)} (|\nabla_\theta u_{(r)}|^2 + r^2 u_{(r)}^2 v_{(r)}^2)\, d\sigma}{\int_{\partial B_1(0)} u_{(r)}^2\, d\sigma}, \qquad \Lambda_2(r) = \frac{\int_{\partial B_1(0)} (|\nabla_\theta v_{(r)}|^2 + r^2 u_{(r)}^2 v_{(r)}^2)\, d\sigma}{\int_{\partial B_1(0)} v_{(r)}^2\, d\sigma}.$$

The idea now is to show that the functions $u_{(r)}, v_{(r)}$ (after a normalization in $L^2(\partial B_1(0))$) converge as $r \to +\infty$ to some functions having disjoint supports, and then to take advantage of (2.13). Notice first of all that there exists a constant $C > 0$ such that $\int_{\partial B_1(0)} u_{(r)}^2 \geqslant C$ for r sufficiently large. Indeed, if we assume by contradiction this is not true, then $\frac{1}{|\partial B_r|} \int_{\partial B_r(0)} u \to 0$ as $r \to +\infty$, which implies (since u is subharmonic) that $u(0) = 0$, and this contradicts the assumption $u > 0$. The same result clearly holds also for $v_{(r)}$.

[1] The first eigenvalue is taken in a generalized sense: for any measurable set $E \subset S^1$, $\lambda_1(E) = \inf\{\int_{S^1} |\nabla u|^2 / \int_{S^1} u^2 : u \in H^1(S^1) \setminus \{0\} : \ u = 0 \text{ a.e. in } S^1 \setminus E\}$.

Now, assume that (2.14) does not hold. Then there exists $r_n \to +\infty$ such that

$$\gamma(\Lambda_1(r_n)) + \gamma(\Lambda_2(r_n)) \leqslant \frac{4-\varepsilon}{2} < 2.$$

In particular, $\Lambda_1(r_n)$ and $\Lambda_2(r_n)$ are bounded, and as a consequence the function

$$\tilde{u}_{(r_n)} := \frac{u_{(r_n)}}{\|u_{(r_n)}\|_{L^2(\partial B_1(0))}} \quad \text{satisfies} \quad C \geqslant \Lambda_1(r_n) \geqslant \int_{\partial B_1(0)} |\nabla \tilde{u}_{(r_n)}|^2$$

(and an analogous property holds for $\tilde{v}_{(r_n)} := v_{(r_n)}/\|v_{(r_n)}\|_{L^2(\partial B_1(0))}$). This ensures the existence of $\bar{u}, \bar{v} \neq 0$ such that, up to a subsequence, $\tilde{u}_{(r_n)} \rightharpoonup \bar{u}$, $\tilde{v}_{(r_n)} \rightharpoonup \bar{v}$ in $H^1(\partial B_1(0))$ and $\tilde{u}_{(r_n)} \to \bar{u}$, $\tilde{v}_{(r_n)} \to \bar{v}$ in $L^2(\partial B_1(0))$. Moreover, since

$$C \geqslant \Lambda_1(r_n) \geqslant r_n^2 \int_{\partial B_1(0)} \tilde{u}_{(r_n)}^2 \tilde{v}_{(r_n)}^2$$

we infer that $\bar{u} \cdot \bar{v} \equiv 0$. This immediately provides, by taking also in consideration (2.13), that

$$2 > \liminf_{n \to +\infty} [\gamma(\Lambda_1(r_n)) + \gamma(\Lambda_2(r_n))] \geqslant \gamma(\lambda_1(\{\bar{u} > 0\})) + \gamma(\lambda_1(\{\bar{v} > 0\})) \geqslant 2,$$

which is a contradiction. □

Now that we have a suitable monotonicity formula we are ready to prove a Liouville–type result for system (2.8).

Proposition 2.10. *Let u, v be nonnegative solutions of (2.8). Assume moreover that (2.6) holds for some $\alpha \in (0,1)$. Then one of the functions is identically zero and the other one is a constant.*

Proof. We start by noticing that, due to the form of system (2.8), if one of the functions is either identically zero or a positive constant, then the other one must be either a constant or identically zero, respectively. Hence we may assume by contradiction that neither u nor v is constant. Since $u \geqslant 0$ satisfies the equation $-\Delta u + c(x)u = 0$ with $c(x) = v^2$, then by the strong maximum principle we see that $u > 0$ in $\mathbb{R}^N$, and the same holds for the function v. Then Lemma 2.9 ensures that given $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\int_{B_r(0)} f(|x|) \left(|\nabla u|^2 + u^2 v^2\right) \cdot \int_{B_r(0)} f(|x|) \left(|\nabla v|^2 + u^2 v^2\right) \geqslant C r^{4-\varepsilon} \tag{2.15}$$

for r sufficiently large. Let $\eta = \eta_{r,2r}$ be the cut-off function defined in the proof of Proposition 2.6. By testing the equation for u with $\eta^2 f(|x|) u$ in $B_{2r}(0)$ and integrating by parts, we obtain

$$\int_{B_{2r}(0)} \eta^2 f(|x|)(|\nabla u|^2 + u^2 v^2) = -\int_{B_{2r}(0)} \left(2f(|x|)\eta u \langle \nabla u, \nabla \eta\rangle + \eta^2 \langle \nabla\Big(\frac{u^2}{2}\Big), \nabla f(|x|)\rangle\right) \leqslant$$
$$\leqslant \int_{B_{2r}(0)} \left(\frac{1}{2} f(|x|)\eta^2 |\nabla u|^2 + 2f(|x|)u^2 |\nabla \eta|^2 - \langle \nabla\Big(\frac{\eta^2 u^2}{2}\Big), \nabla f(|x|)\rangle + u^2 \eta \nabla \langle \eta, \nabla f(|x|)\rangle\right).$$

Since

$$-\int_{B_{2r}(0)} \langle \nabla\Big(\frac{\eta^2 u^2}{2}\Big), \nabla f(|x|)\rangle = \int_{B_{2r}(0)} \frac{\eta^2 u^2}{2} \Delta f(|x|) \leqslant 0,$$

then

$$\int_{B_{2r}(0)} \eta^2 f(|x|)(|\nabla u|^2 + u^2 v^2) \leqslant 2 \int_{B_{2r}(0)} \left[2 f(|x|) u^2 |\nabla \eta|^2 + u^2 \eta \langle \nabla \eta, \nabla f(|x|)\rangle\right].$$

Now, by recalling the definition of η and f and by using assumption (2.6), we finally obtain

$$\int_{B_r(0)} f(|x|)(|\nabla u|^2 + u^2 v^2) \leqslant \frac{C}{r^N} \int_{B_{2r}(0)\setminus B_r(0)} u^2 \leqslant \frac{C'}{r^N} \int_{B_{2r}(0)\setminus B_r(0)} |x|^{2\alpha} \leqslant C'' r^{2\alpha}$$

(in fact, from (2.6) we see that $|u(x)| \leqslant |x|^\alpha$ for large $|x|$). By reasoning in an analogous way with the equation for v, we obtain

$$\int_{B_r(0)} f(|x|)(|\nabla v|^2 + u^2 v^2) \leqslant C'' r^{2\alpha},$$

and hence

$$\int_{B_r(0)} f(|x|) \left(|\nabla u|^2 + u^2 v^2\right) \cdot \int_{B_r(0)} f(|x|) \left(|\nabla v|^2 + u^2 v^2\right) \leqslant C''' r^{4\alpha},$$

which contradicts (2.15) for r large enough, if we choose $\varepsilon < 4(1-\alpha)$. □

Arguing as above, one can easily prove the following Liouville–type theorem for systems with an arbitrary number of densities.

Proposition 2.11. *Let $m \geqslant 2$ and $u_1, \dots, u_m$ be nonnegative solutions of*

$$-\Delta u_i = -u_i \sum_{\substack{j=1 \\ j\neq i}}^{m} u_j^2 \quad \text{in } \mathbb{R}^N, \tag{2.16}$$

with the property that, for some $\alpha \in (0,1)$,

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u_i(x) - u_i(y)|}{|x-y|^\alpha} < \infty \quad \text{for every } i.$$

Then $k-1$ functions are identically zero and the remaining one is constant.

Proof. We claim that, for any $i \neq j$ fixed, (at least) one between u_i and u_j is identically zero (this, exploiting every possible choice of i and j, will readily complete the proof). Assuming that the claim is false, then by the maximum principle $u = u_i$ and $v = u_j$ are positive subsolutions of system (2.8). It is easy to see that Lemma 2.9 also holds for positive subsolutions of that system; as a consequence, (2.15) holds for $u = u_i$ and $v = u_j$. But this, by reasoning as in the proof of the previous proposition, is in contradiction with the global bound of the Hölder quotients. □

2.3 Uniform Hölder continuity

This section is mainly devoted to the proof of Theorem 2.3, which will provide, as a byproduct, also Theorem 2.4. As we said before we follow an argument by contradiction, combining a blowup analysis with the non existence results proved in the previous section (much in the spirit of the celebrated paper by Gidas and Spruck [66]). To start with, we need the following technical lemma, which refines the estimate in [49, Lemma 4.4].

Lemma 2.12. *Let $B_R \subset \mathbb{R}^N$ be any ball of radius R. Let M, A be positive constants, $h \in L^2(B_R)$, and let $u \in H^1(B_R)$ be a solution of*

$$\begin{cases} -\Delta u \leqslant -Mu + h & \text{in } B_R \\ u \geqslant 0 & \text{in } B_R \\ u \leqslant A & \text{on } \partial B_R. \end{cases} \tag{2.17}$$

Then for every $\varepsilon, \theta > 0$ such that $0 < \theta < \varepsilon < R$ there holds

$$\|u\|_{L^2(B_{R-\varepsilon})} \leqslant \frac{2AR^{N-1}|B_{R-\varepsilon}|^{1/2}}{(\varepsilon-\theta)^{N-1}} e^{-\theta\sqrt{M}} + \frac{1}{M}\|h\|_{L^2(B_R)},$$

where $B_{R-\varepsilon}$ is the ball of radius $R-\varepsilon$ which shares its center with B_R.

Proof. 1. To start with, take ψ to be a solution of

$$\begin{cases} \psi''(r) + \frac{N-1}{r}\psi'(r) = M\psi(r), & r > 0 \\ \psi(0) > 0, \ \psi'(0) = 0. \end{cases} \tag{2.18}$$

We claim that

$$\begin{cases} \psi(r) > 0, \quad \psi'(r) > 0 & \text{in } [0, +\infty) \\ \psi(r) \leqslant \psi_1(0) e^{r\sqrt{M}} & \text{in } [0, +\infty) \\ \psi(r) \geqslant \frac{\psi(0)\bar{r}^{N-1}}{2e^{\bar{r}\sqrt{M}}} \frac{e^{r\sqrt{M}}}{r^{N-1}} & \text{in } [\bar{r}, +\infty). \end{cases}$$

for every fixed $\bar{r} > 0$. Observe that $(r^{N-1}\psi')' = Mr^{N-1}\psi$ and hence ψ' is a strictly increasing function whenever $\psi > 0$. Since $\psi(0) > 0$ and $\psi'(0) = 0$, this yields that actually $\psi, \psi' > 0$ in $(0, +\infty)$. Now, since ψ' is positive, we see that $\psi'' \leqslant M\psi$ and hence, by using the initial conditions and comparison arguments, we conclude that $\psi(r) \leqslant \psi_1(0)e^{r\sqrt{M}}$. Finally, for the function $\bar{\psi}(r) = r^{N-1}\psi(r)$ we easily deduce that $\bar{\psi}'' \geqslant M\bar{\psi}$, $\bar{\psi}(\bar{r}) \geqslant \bar{r}^{N-1}\psi(0)$ and $\bar{\psi}'(\bar{r}) \geqslant 0$. Thus $\bar{\psi}(r) \geqslant \bar{r}^{N-1}\psi(0)\cosh((r-\bar{r})\sqrt{M}) \geqslant \bar{r}^{N-1}\psi(0)e^{(r-\bar{r})\sqrt{M}}/2$ for $r \geqslant \bar{r}$, as claimed.

Now we take a solution ψ of (2.18) such that $\psi(R) = A > 0$ (which is possible since the system in consideration is linear). By choosing $\bar{r} = \varepsilon - \theta$, we see that

$$\psi(R-\varepsilon) \leqslant \psi(0)e^{(R-\varepsilon)\sqrt{M}} \qquad \text{and} \qquad A = \psi(R) \geqslant \frac{\psi(0)(\varepsilon-\theta)^{N-1}}{2e^{(\varepsilon-\theta)\sqrt{M}}} \frac{e^{R\sqrt{M}}}{R^{N-1}},$$

which imply that

$$\psi(r) \leqslant \psi(R-\varepsilon) \leqslant \frac{2AR^{N-1}}{(\varepsilon-\theta)^{N-1}} e^{-\theta\sqrt{M}}, \qquad \text{for all } r \in [0, R-\varepsilon].$$

2. Coming back to (2.17), we can estimate u as $|u| \leqslant |u_1| + |u_2|$, where u_1, u_2 are solution of

$$\begin{cases} -\Delta u_1 &= -Mu_1 & \text{in } B_R \\ u_1 &\geqslant 0 & \text{in } B_R \\ u_1 &= u & \text{on } \partial B_R \end{cases} \qquad \begin{cases} -\Delta u_2 &= -Mu_2 + h & \text{in } B_R \\ u_2 &= 0 & \text{on } \partial B_R. \end{cases}$$

Now, by defining $v(x) = \psi(|x - x_0|)$ (where x_0 is the center of the ball B_R) and by using the maximum principle we infer that

$$0 \leqslant u_1(x) \leqslant v(x) \leqslant \frac{2AR^{N-1}}{(\varepsilon - \theta)^{N-1}} e^{-\theta\sqrt{M}} \qquad \text{for all } x \in B_{R-\varepsilon}$$

and thus

$$\|u_1\|_{L^2(B_{R-\varepsilon})} \leqslant \frac{2AR^{N-1}|B_{R-\varepsilon}|^{1/2}}{(\varepsilon - \theta)^{N-1}} e^{-\theta\sqrt{M}}.$$

In order to obtain an upper estimate for u_2, let us multiply the equation for u_2 by u_2 itself and integrate; by the zero boundary conditions we have

$$\int_{B_R} Mu_2^2 \leqslant \int_{B_R} |\nabla u_2|^2 + Mu_2^2 = \int_{B_R} hu_2 \leqslant \left(\int_{B_R} h^2\right)^{1/2} \left(\int_{B_R} u_2^2\right)^{1/2},$$

and therefore $\|u_2\|_{L^2(B_R)} \leqslant \frac{1}{M}\|h\|_{L^2(B_R)}$. In conclusion we see that

$$\|u\|_{L^2(B_{R-\varepsilon})} \leqslant \|u_1\|_{L^2(B_{R-\varepsilon})} + \|u_2\|_{L^2(B_R)},$$

which gives the desired estimates. □

2.3.1 Normalization and blowup

To start with, we recall the standard regularity properties for solutions of system (2.3).

Remark 2.13. Let u_β, v_β be solutions of (2.3). Then, since h_β, k_β belong to $L^2(\Omega)$ and Ω is bounded and regular, by elliptic regularity theory we have that

$$u_\beta, v_\beta \in H^2(\Omega), \quad \text{which implies} \quad u_\beta, v_\beta \in C^{0,\alpha}(\overline{\Omega})$$

for every $\alpha \in (0, \alpha^*)$, where α^* is defined as in (2.4). Let us mention that, if $h_\beta \equiv k_\beta \equiv 0$, then, by a bootstrap argument, we can choose $\alpha^* = 1$ also in dimension $N = 3$.

Coming back to the proof of Theorem 2.3, let us assume by contradiction the existence of $\alpha \in (0, \alpha^*)$ such that, up to a subsequence,

$$L_\beta := \max\left\{ \max_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|u_\beta(x) - u_\beta(y)|}{|x - y|^\alpha}, \max_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|v_\beta(x) - v_\beta(y)|}{|x - y|^\alpha} \right\} \longrightarrow +\infty$$

as $\beta \to +\infty$. We can assume that L_β is achieved, say, by u_β at the pair (x_β, y_β). We observe that

$$|x_\beta - y_\beta| \to 0 \quad \text{as} \quad \beta \to +\infty,$$

since we have $|x_\beta - y_\beta|^\alpha = |u_\beta(x_\beta) - u_\beta(y_\beta)|/L_\beta \leqslant 2\|u_\beta\|_\infty/L_\beta \leqslant 2C/L_\beta \to 0$.

The idea now is to consider a uniformly α–Hölder continuous blowup centered at x_β. Keeping this in mind, let us define the rescaled functions

$$\bar{u}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} u_\beta(x_\beta + r_\beta x),\ \bar{v}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} v_\beta(x_\beta + r_\beta x), \quad \text{for } x \in \Omega_\beta := \frac{\Omega - x_\beta}{r_\beta},$$

where $r_\beta \to 0$ will be chosen later. Depending on the asymptotic behavior of the distance $d(x_\beta, \partial\Omega)$ and on r_β, we have that Ω_β approaches Ω_∞, where Ω_∞ is either $\mathbb{R}^N$ or a half-space (when either $d(x_\beta, \partial\Omega)/r_\beta \to \infty$ or the limit is finite, respectively).

First of all we observe that the $\bar{u}_\beta, \bar{v}_\beta$'s are uniformly α–Hölder continuous for every choice of r_β, with Hölder constant equal to one:

$$\max\left\{ \max_{\substack{x,y\in\overline{\Omega}_\beta \\ x\neq y}} \frac{|\bar{u}_\beta(x) - \bar{u}_\beta(y)|}{|x-y|^\alpha},\ \max_{\substack{x,y\in\overline{\Omega}_\beta \\ x\neq y}} \frac{|\bar{v}_\beta(x) - \bar{v}_\beta(y)|}{|x-y|^\alpha} \right\} = \\ = \frac{\left|\bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} = 1. \quad (2.19)$$

Moreover the rescaled functions satisfy the following system in Ω_β:

$$\begin{cases} -\Delta\bar{u}_\beta + \lambda_\beta r_\beta^2 \bar{u}_\beta &= \omega_1 M_\beta \bar{u}_\beta^3 - \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \bar{h}_\beta(x) \\ -\Delta\bar{v}_\beta + \mu_\beta r_\beta^2 \bar{v}_\beta &= \omega_2 M_\beta \bar{v}_\beta^3 - \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta + \bar{k}_\beta(x) \\ \bar{u}_\beta,\ \bar{v}_\beta \in H_0^1(\Omega_\beta), \end{cases} \quad (2.20)$$

where

$$M_\beta := L_\beta^2 r_\beta^{2\alpha+2}$$

and

$$\bar{h}_\beta(x) := \frac{r_\beta^{2-\alpha}}{L_\beta} h_\beta(x_\beta + r_\beta x), \quad \bar{k}_\beta(x) := \frac{r_\beta^{2-\alpha}}{L_\beta} k_\beta(x_\beta + r_\beta x).$$

Remark 2.14. Since u_β, v_β are $L^\infty(\Omega)$–bounded, h_β, k_β are $L^2(\Omega)$–bounded, λ_β, μ_β are bounded in $\mathbb{R}$, and $r_\beta \to 0$, $L_\beta \to +\infty$, by direct calculations it is easy to see that

$$\begin{aligned} &\lambda_\beta r_\beta^2 \bar{u}_\beta,\ \mu_\beta r_\beta^2 \bar{v}_\beta \to 0 \quad \text{in } L^\infty(\Omega_\beta) \\ &\omega_1 M_\beta \bar{u}_\beta^3,\ \omega_2 M_\beta \bar{v}_\beta^3 \to 0 \quad \text{in } L^\infty(\Omega_\beta) \\ &\bar{h}_\beta,\ \bar{k}_\beta \to 0 \quad \text{in } L^2(\Omega_\beta). \end{aligned}$$

In fact, we have for instance that

$$\|\lambda_\beta r_\beta^2 \bar{u}_\beta\|_{L^\infty(\Omega_\beta)} \leqslant |\lambda_\beta| \|u_\beta\|_{L^\infty(\Omega)} r_\beta^{2-\alpha}/L_\beta \to 0,\ \|\omega_1 M_\beta \bar{u}_\beta^3\|_{L^\infty(\Omega_\beta)} \leqslant \omega_1 \|u_\beta\|^3_{L^\infty(\Omega)} r_\beta^{2-\alpha}/L_\beta \to 0,$$

and $\|\bar{h}_\beta\|_{L^2(\Omega_\beta)} \leqslant \|h_\beta\|_{L^2(\Omega)} r_\beta^{2-\alpha-N/2}/L_\beta \to 0$ as $\beta \to +\infty$.

In order to manage the different parts of the proof, we will need to make different choices of the sequence r_β. Once r_β is chosen, we wish to pass to the limit (on compact sets), and to this aim we will use Ascoli–Arzelà's Theorem. To apply such theorem, since the $\bar u_\beta, \bar v_\beta$'s are uniformly α–Hölder continuous, it suffices to show that the sequences $(\bar u_\beta(0))_\beta$, $(\bar v_\beta(0))_\beta$ are bounded in β. The following lemma provides a sufficient condition on r_β for such a bound to hold.

Lemma 2.15. *Under the previous notations, let $r_\beta \to 0$ as $\beta \to +\infty$ be such that*

(i) $\dfrac{|y_\beta - x_\beta|}{r_\beta} \leqslant R'$ *for some $R' > 0$,*

(ii) $\beta M_\beta \not\to 0$.

Then the sequences $(\bar u_\beta(0))_\beta, (\bar v_\beta(0))_\beta$ are uniformly bounded in β.

Proof. Assume by contradiction that $(\bar u_\beta(0))_\beta$ is unbounded, and let $R \geqslant R'$. Since the $\bar u_\beta$'s are uniformly Hölder continuous and vanish on $\partial\Omega_\beta$, we can consider β sufficiently large such that $B_{2R}(0) \subset \Omega_\beta$. Moreover since $\beta M_\beta \not\to 0$, we have that (up to a subsequence)

$$I_\beta := \inf_{B_{2R}(0)} \beta M_\beta \bar u_\beta^2 \longrightarrow +\infty.$$

As in Remark 2.14, we see that $\|\omega_2 M_\beta \bar v_\beta^2\|_{L^\infty(B_{2R}(0))} \to 0$ as $\beta \to +\infty$, and hence

$$\begin{aligned} -\Delta \bar v_\beta &= -\mu_\beta r_\beta^2 \bar v_\beta + \omega_2 M_\beta \bar v_\beta^3 - \beta M_\beta \bar u_\beta^2 \bar v_\beta + \bar k_\beta \\ &\leqslant -\frac{I_\beta}{2}\bar v_\beta + \bar k_\beta \end{aligned}$$

in $B_{2r}(0)$ for large β. In order to use Lemma 2.12, we need to show that $\bar v_\beta$ is bounded on $\partial B_{2R}(0)$. With this in mind, let us choose a smooth cut-off function η that vanishes outside $B_{2R}(0)$. Then by testing the second equation in (2.20) with $\eta^2 \bar v_\beta$ and integrating in $B_{2R}(0)$, we obtain

$$\begin{aligned} &\int_{B_{2R}(0)} \left(\eta^2 |\nabla \bar v_\beta|^2 + 2\eta \bar v_\beta \langle \nabla\eta, \nabla \bar v_\beta\rangle + \mu_\beta r_\beta^2 \bar v_\beta^2 \eta^2\right) \\ &\qquad\qquad \leqslant \int_{B_{2R}(0)} \left(\omega_2 M_\beta \bar v_\beta^4 \eta^2 - I_\beta \eta^2 \bar v_\beta^2 + \bar k_\beta \eta^2 \bar v_\beta\right), \end{aligned}$$

and thus

$$\begin{aligned} &\int_{B_{2R}(0)} \left(\frac12 \eta^2 |\nabla \bar v_\beta|^2 + I_\beta \eta^2 \bar v_\beta^2\right) \\ &\qquad \leqslant 2\int_{B_{2R}(0)} \left(\bar v_\beta^2 |\nabla\eta|^2 + |\mu_\beta| r_\beta^2 \bar v_\beta^2 \eta^2 + |\omega_2| M_\beta \bar v_\beta^4 \eta^2 + \bar k_\beta \eta^2 \bar v_\beta\right) \\ &\qquad\qquad \leqslant C(R)\left(\sup_{B_{2R}(0)} \bar v_\beta^2 + 1\right). \end{aligned}$$

On the other hand, since $\bar{v}_\beta$ is uniformly Hölder continuous,

$$\int_{B_{2R}(0)} I_\beta \eta^2 \bar{v}_\beta^2 \geqslant C'(R) I_\beta \inf_{B_{2R}(0)} \bar{v}_\beta^2 \geqslant C'(R) I_\beta \sup_{B_{2R}(0)} \bar{v}_\beta^2 - C''(R) I_\beta.$$

Therefore, by putting together the two previous inequalities, we obtain

$$C'(R) I_\beta \sup_{B_{2R}(0)} \bar{v}_\beta^2 \leqslant C(R)(\sup_{B_{2R}(0)} \bar{v}_\beta^2 + 1) + C''(R) I_\beta$$

which, since $I_\beta \to +\infty$, implies the boundedness of $\bar{v}_\beta$ in $B_{2R}(0)$ (and in particular on $\partial B_{2R}(0)$).

Thus we can apply Lemma 2.12, which gives

$$\|\bar{v}_\beta\|_{L^2(B_R(0))} \leqslant C e^{-C'\sqrt{I_\beta}} + \frac{2}{I_\beta} \|\bar{k}_\beta\|_{L^2(B_{2R}(0))}.$$

Hence, from the fact that in $B_{2R}(0)$ we have $\bar{u}_\beta \leqslant \bar{u}_\beta^2$ for β large,

$$\begin{aligned} \|\beta M_\beta \bar{u}_\beta \bar{v}_\beta\|_{L^2(B_R(0))} \leqslant \|\beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta\|_{L^2(B_R(0))} &\leqslant (I_\beta + \beta M_\beta (2R)^{2\alpha}) \|\bar{v}_\beta\|_{L^2(B_R(0))} \\ &\leqslant 2 I_\beta \left(C e^{-C'\sqrt{I_\beta}} + \frac{2}{I_\beta} \|\bar{k}_\beta\|_{L^2(B_{2R}(0))} \right) \to 0 \end{aligned}$$

as $\beta \to +\infty$. This, together with Remark 2.14 and the boundedness of $\bar{v}_\beta$, gives

$$\|\Delta \bar{u}_\beta\|_{L^2(B_R(0))} \to 0 \qquad \text{as } \beta \to +\infty, \tag{2.21}$$

for every $R \geqslant R'$.

Consider now the auxiliary function $\tilde{u}_\beta(x) := \bar{u}_\beta(x) - \bar{u}_\beta(0)$. By the uniform Hölder continuity and Ascoli–Arzelà's Theorem we know that $\tilde{u}_\beta \to \tilde{u}_\infty$ on compact sets. Moreover by (2.21) we have that $\tilde{u}_\beta$ is bounded in $C^{0,\gamma}_{\mathrm{loc}}$ for every $\gamma \in (0, \alpha^*)$ (in fact, Theorem 8.12 of [67] gives us boundedness in $W^{2,2}$, and the result follows by the Sobolev imbeddings). We now claim that

$$\max_{\substack{x,y \in \overline{\Omega}_\infty \\ x \neq y}} \frac{|\tilde{u}_\infty(x) - \tilde{u}_\infty(y)|}{|x-y|^\alpha} = 1 \tag{2.22}$$

holds. Indeed, notice that by assumption (i), $(y_\beta - x_\beta)/r_\beta$ must converge, up to a subsequence, to some point. But it cannot be $(y_\beta - x_\beta)/r_\beta \to 0$, otherwise we would have (considering $\varepsilon > 0$ sufficiently small)

$$\frac{\left|\bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} = \left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\varepsilon \frac{\left|\tilde{u}_\beta(0) - \tilde{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^{\alpha+\varepsilon}} \leqslant C \left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\varepsilon \to 0 \tag{2.23}$$

a $\beta \to +\infty$, which contradicts (2.19). Therefore, there exists $a \in \mathbb{R}^N \setminus \{0\}$ such that $(y_\beta - x_\beta)/r_\beta \to a$, and hence the left hand side of (2.19) also passes to the limit in β, providing (2.22).

Finally, we have that $\Delta \tilde{u}_\infty = 0$ in Ω_∞. If $\Omega_\infty = \mathbb{R}^N$, then by Corollary 2.7 $\tilde{u}_\infty$ must be a constant function, which is in contradiction with (2.22). On the other hand, if Ω_∞ is a half-space, then we have that $\tilde{u}_\infty = 0$ on $\partial\Omega_\infty$. If we extend it to the whole $\mathbb{R}^N$ by an odd symmetry (that is, $\tilde{u}_\infty(x) = -\tilde{u}_\infty(-\sigma(x))$, were σ denotes the reflection in $\mathbb{R}^N$ with respect to $\partial\Omega_\infty$) then $\tilde{u}_\infty$ satisfies once more the assumptions of Corollary 2.7, which is again a contradiction.

We have shown that $(\bar{u}_\beta(0))_\beta$ is bounded. Let us now check that the same happens with $(\bar{v}_\beta(0))_\beta$. Assume then that $(\bar{v}_\beta(0))_\beta$ is unbounded, and consider the quantity (for $R \geqslant R'$ fixed)

$$\tilde{I}_\beta := \inf_{B_{2R}(0)} \beta M_\beta \bar{v}_\beta^2 \to +\infty.$$

We have

$$-\Delta \bar{u}_\beta \leqslant -\frac{\tilde{I}_\beta}{2}\bar{u}_\beta + \bar{h}_\beta$$

and $\bar{u}_\beta$ is bounded on $\partial B_{2R}(0)$. Therefore by Lemma 2.12

$$\|\bar{u}_\beta\|_{L^2(B_R(0))} \leqslant Ce^{-C'\sqrt{\tilde{I}_\beta}} + \frac{2}{\tilde{I}_\beta}\|\bar{h}_\beta\|_{L^2(B_{2R}(0))}$$

and hence

$$\|\beta M_\beta \bar{u}_\beta \bar{v}_\beta^2\|_{L^2(B_R(0))} \to 0$$

as $\beta \to +\infty$. Once again this gives $\|\Delta \bar{u}_\beta\|_{L^2(B_R)(0)} \to 0$ and the proof follows exactly as before. □

With the help of the previous lemma we can now quantify the asymptotic relation between β, L_β and $|x_\beta - y_\beta|$.

Lemma 2.16. *Under the previous notations, we have (up to a subsequence)*

$$\beta L_\beta^2 |x_\beta - y_\beta|^{2\alpha+2} \to +\infty.$$

Proof. Assume by contradiction that $\beta L_\beta^2 |x_\beta - y_\beta|^{2\alpha+2}$ is a bounded sequence. Then we can choose

$$r_\beta = (\beta L_\beta^2)^{-\frac{1}{2\alpha+2}} \qquad \text{(and thus } \beta M_\beta = 1\text{)},$$

so that the assumptions of Lemma 2.15 are satisfied and thus $(\bar{u}_\beta(0))_\beta, (\bar{v}_\beta(0))_\beta$ are bounded. By uniform Hölder continuity and Ascoli–Arzelà's Theorem we have that, up to a subsequence, there exist u_∞, v_∞ such that $\bar{u}_\beta \to u_\infty$, $\bar{v}_\beta \to v_\infty$ uniformly in the compact subsets of $\overline{\Omega}_\infty$. From the fact that $\beta M_\beta = 1$ and by Remark 2.14, we have that $\Delta \bar{u}_\beta$, $\Delta \bar{v}_\beta$ are bounded in $L^2_{\text{loc}}(\Omega_\infty)$ and therefore the same happens to $\bar{u}_\beta, \bar{v}_\beta$ in $C^{0,\gamma}_{\text{loc}}(\overline{\Omega}_\infty)$, for all $\gamma \in (0, \alpha^*)$. We are now going to show that, as a consequence, u_∞, v_∞ are α–Hölder continuous and that the maximum of the Hölder quotients is given by:

$$\max_{\substack{x,y\in\overline{\Omega}_\infty \\ x\neq y}} \frac{|u_\infty(x) - u_\infty(y)|}{|x-y|^\alpha} = 1. \tag{2.24}$$

Indeed notice that we cannot have $(y_\beta - x_\beta)/r_\beta \to 0$, otherwise we would obtain the same contradiction as in (2.23). Therefore, there is an $a \in \mathbb{R}^N \backslash \{0\}$ such that $(y_\beta - x_\beta)/r_\beta \to a$, and hence the left hand side of (2.19) also passes to the limit in β, providing (2.24). Moreover, at the limit we have

$$\begin{cases} -\Delta u_\infty &= -u_\infty v_\infty^2 \quad \text{in } \Omega_\infty \\ -\Delta v_\infty &= -u_\infty^2 v_\infty \quad \text{in } \Omega_\infty. \end{cases}$$

If $\Omega_\infty = \mathbb{R}^N$, then u_∞, v_∞ are constant functions by Proposition 2.10, which contradicts (2.24).

On the other hand, let Ω_∞ be equal to a half-space. Since $u_\infty = 0$ on $\partial\Omega_\infty$, we can extend u_∞ to the whole space by even symmetry and obtain a pair of functions satisfying the hypotheses of Proposition 2.6 (apply it to the pair $(u_\infty|_{\Omega_\infty}, u_\infty|_{\mathbb{R}^N\setminus\overline{\Omega}_\infty})$ – where we consider both functions extended by 0 – and choose for x_0 any point on $\partial\Omega_\infty$). Therefore $u_\infty \equiv 0$, which contradicts (2.24). □

Now we are in a position to define our choice of r_β and to deduce the convergence of the blowup sequences.

Lemma 2.17. *Let*

$$r_\beta = |x_\beta - y_\beta|.$$

Then there exist $u_\infty, v_\infty \in C^{0,\alpha}(\mathbb{R}^N)$ such that, as $\beta \to +\infty$ (up to subsequences), the following holds.

(i) $\bar{u}_\beta \to u_\infty$ and $\bar{v}_\beta \to v_\infty$ uniformly in compact subsets of $\Omega_\infty = \mathbb{R}^N$;

(ii) $\beta M_\beta \to +\infty$, and moreover for any fixed $r > 0$ and $x_0 \in \mathbb{R}^N$ there holds

$$\int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta^2 \to 0;$$

(iii) $\|\bar{u}_\beta - u_\infty\|_{H^1(B_r(x_0))} \to 0$ and $\|\bar{v}_\beta - v_\infty\|_{H^1(B_r(x_0))} \to 0$.

Proof. With this choice of r_β, we obtain $\beta M_\beta = \beta L_\beta^2 |x_\beta - y_\beta|^{2\alpha+2} \to +\infty$ by Lemma 2.16. Once again the assumptions of Lemma 2.15 are satisfied and hence, by reasoning as in the initial part of the proof of Lemma 2.16, we deduce that the rescaled functions $\bar{u}_\beta, \bar{v}_\beta$ must converge uniformly to some u_∞, v_∞, in every compact set of $\overline{\Omega}_\infty$. In this situation (2.19) reads

$$\left| \bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right| = 1$$

and hence, by $L^\infty_{\mathrm{loc}}(\overline{\Omega}_\beta)$ convergence, u_∞, v_∞ are globally α–Hölder continuous and in particular

$$\max_{x\in\partial B_1(0)\cap\overline{\Omega}_\infty} |u_\infty(0) - u_\infty(x)| = 1. \tag{2.25}$$

Now if Ω_∞ is a half-space we can proceed exactly as in the last part of the proof of Lemma 2.16, obtaining a contradiction. Therefore $\Omega_\infty = \mathbb{R}^N$, and (i) is proved.

In order to prove the second part of the lemma, let us fix any ball $B_r(x_0)$ of $\mathbb{R}^N$, and let β be large so that $B_r(x_0) \subset \Omega_\beta$. Let us consider a smooth cut-off function $0 \leqslant \eta \leqslant 1$

such that $\eta = 1$ in $B_r(x_0)$, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2r}(x_0)$. By testing the equation for $\bar{u}_\beta$ with η, we obtain (since the $\bar{u}_\beta$'s are uniformly bounded in $B_{2r}(x_0)$)

$$\int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 \leqslant \int_{B_{2r}(x_0)} \beta M_\beta \bar{u}_\beta \eta \bar{v}_\beta^2 \leqslant \int_{B_{2r}(x_0)} |\bar{u}_\beta \Delta\eta - \lambda_\beta r_\beta^2 \eta \bar{u}_\beta + \omega_1 M_\beta \eta \bar{u}_\beta^3 + \eta \bar{h}_\beta| \leqslant C \tag{2.26}$$

and analogously $\int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta \leqslant C$. This immediately implies, recalling that $\beta M_\beta \to +\infty$,

$$u_\infty \cdot v_\infty \equiv 0 \quad \text{ in } \mathbb{R}^N. \tag{2.27}$$

Thus, since $B_r(x_0) \subseteq (B_r(x_0) \cap \{u_\infty = 0\}) \cup (B_r(x_0) \cap \{v_\infty = 0\})$,

$$\begin{aligned} &\int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta^2 \leqslant \\ &\leqslant \|\bar{u}_\beta\|_{L^\infty(B_r(x_0)\cap\{u_\infty=0\})} \int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \|\bar{v}_\beta\|_{L^\infty(B_r(x_0)\cap\{v_\infty=0\})} \int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta^2 \bar{v} \\ &\qquad \leqslant C \left(\|\bar{u}_\beta\|_{L^\infty(B_r(x_0)\cap\{u_\infty=0\})} + \|\bar{v}_\beta\|_{L^\infty(B_r(x_0)\cap\{v_\infty=0\})} \right) \to 0, \end{aligned} \tag{2.28}$$

which is (ii).

As for (iii), we test the equation for $\bar{u}_\beta$ with $\bar{u}_\beta \eta^2$ and integrate by parts in $B_{2r}(x_0)$. By reasoning as in Remark 2.14 and taking into account (2.28), we have

$$\begin{aligned} &\int_{B_{2r}(x_0)} |\nabla \bar{u}_\beta|^2 \eta^2 \leqslant \\ &\leqslant \int_{B_{2r}(x_0)} |-\lambda_\beta r_\beta^2 \bar{u}_\beta^2 + \omega_1 M_\beta \bar{u}_\beta^4 - \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta^2 + \bar{h}_\beta \bar{u}_\beta| + 2 \int_{B_{2r}(x_0)} |\nabla \bar{u}_\beta| |\nabla \eta| \bar{u}_\beta \eta \\ &\qquad \leqslant C + \frac{1}{2} \int_{B_{2r}(x_0)} |\nabla \bar{u}_\beta|^2 \eta^2 + 2 \int_{B_{2r}(x_0)} |\nabla \eta|^2 \bar{u}_\beta^2 \end{aligned}$$

which implies that $\int_{B_r(x_0)} |\nabla \bar{u}_\beta|^2 \leqslant C$. Doing the same with $\bar{v}_\beta$, we obtain the weak H^1–convergences $\bar{u}_\beta \rightharpoonup u_\infty$, $\bar{v}_\beta \rightharpoonup v_\infty$. Now, let η_ε be a smooth cut-off function such that $\eta_\varepsilon = 1$ in $B_r(x_0)$, $\eta_\varepsilon = 0$ in $\mathbb{R}^N \setminus B_{r+\varepsilon}(x_0)$. By testing the equation for $\bar{u}_\beta$ with $(\bar{u}_\beta - u_\infty)\eta_\varepsilon$ and integrating by parts in $B_{r+\varepsilon}(x_0)$, we finally obtain

$$\begin{aligned} &\int_{B_{r+\varepsilon}(x_0)} \eta_\varepsilon \langle \nabla \bar{u}_\beta, \nabla(\bar{u}_\beta - u_\infty)\rangle = \\ &= \int_{B_{r+\varepsilon}(x_0)} (-\lambda_\beta r_\beta^2 \bar{u}_\beta + \omega_1 M_\beta \bar{u}_\beta^3 - \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \bar{h}_\beta) \eta_\varepsilon (\bar{u}_\beta - u_\infty) - \\ &\qquad - \int_{B_{r+\varepsilon}(x_0)} \langle \nabla \bar{u}_\beta, \nabla \eta_\varepsilon \rangle (\bar{u}_\beta - u_\infty) = \mathrm{o}_\beta(1) \end{aligned}$$

as $\beta \to +\infty$, for every $\varepsilon > 0$ fixed. From this we deduce that

$$\begin{aligned}\limsup_{\beta\to+\infty} \int_{B_r(x_0)} |\nabla(\bar u_\beta - u_\infty)|^2 &\leqslant \limsup_{\beta\to+\infty} \int_{B_{r+\varepsilon}(x_0)} \eta_\varepsilon |\nabla(\bar u_\beta - u_\infty)|^2 \\ &= \limsup_{\beta\to+\infty} - \int_{B_{r+\varepsilon}(x_0)} \eta_\varepsilon \langle \nabla u_\infty, \nabla(\bar u_\beta - u_\infty)\rangle \\ &\leqslant C\left(\int_{B_{r+\varepsilon}(x_0)\setminus B_r(x_0)} |\nabla u_\infty|^2\right)^{1/2}\end{aligned}$$

for every $\varepsilon > 0$, which proves the H^1_{loc}–convergence. □

In the following we collect the properties satisfied by the limiting states u_∞, v_∞.

Lemma 2.18. *Let u_∞, v_∞ be defined as in Lemma 2.17. Then the following holds.*

(i) $u_\infty \cdot v_\infty \equiv 0$ *in* $\mathbb{R}^N$*;*

(ii) We have

$$\max\left\{\sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u_\infty(x)-u_\infty(y)|}{|x-y|}, \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u_\infty(x)-u_\infty(y)|}{|x-y|}\right\} = \max_{x\in\partial B_1(0)} |u_\infty(0)-u_\infty(x)| = 1$$

and, in particular, u_∞ is not a constant function;

(iii) $\begin{cases} -\Delta u_\infty = 0 & \text{in } \{u_\infty > 0\}, \\ -\Delta v_\infty = 0 & \text{in } \{v_\infty > 0\}. \end{cases}$

Proof. Properties (i) and (ii) are merely (2.27) and (2.25), respectively. Let us check that u_∞ is harmonic in the (open) set $\{x \in \mathbb{R}^N : u_\infty(x) > 0\}$ (the same is true for v_∞ in the set $\{x \in \mathbb{R}^N : v_\infty(x) > 0\}$). Given any point x_0 such that $u_\infty(x_0) > 0$, we have to find a neighborhood of it where u_∞ is harmonic. By continuity we can consider a ball $B_\delta(x_0)$ where $u_\infty \geqslant 2\gamma > 0$, and hence by local L^∞–convergence $\bar u_\beta \geqslant \gamma > 0$ in $B_\delta(x_0)$ for large β. Therefore we have

$$-\Delta \bar v_\beta \leqslant -\beta M_\beta \frac{\gamma^2}{2} \bar v_\beta + \bar k_\beta \qquad \text{in } B_\delta(x_0)$$

and thus, by using Lemma 2.12, we obtain

$$\|\bar v_\beta\|_{L^2(B_{\delta/2}(x_0))} \leqslant C e^{-C'\sqrt{\beta M_\beta}} + \frac{1}{\beta M_\beta} \|\bar k_\beta\|_{L^2(B_\delta(x_0))}.$$

Hence

$$\|\beta M_\beta \bar u_\beta \bar v_\beta^2\|_{L^2(B_{\delta/2}(x_0))} \to 0$$

and, using also Remark 2.14, we conclude that $\|\Delta \bar u_\beta\|_{L^2(B_{\delta/2}(x_0))} \to 0$, which implies the harmonicity of u_∞ in $B_{\delta/2}(x_0)$. □

Remark 2.19. By the previous lemma we obtain that u_∞ must vanish somewhere in $\mathbb{R}^N$ (indeed, if not, u_∞ would be a positive non–constant harmonic function in $\mathbb{R}^N$, a contradiction), and also v_∞ must vanish somewhere (otherwise we would have $u_\infty \equiv 0$ in $\mathbb{R}^N$, again a contradiction). This, by continuity, implies that u_∞ and v_∞ must have a common zero, and thus they satisfy all the assumptions of Proposition 2.6. Since u_∞ is not constant, we deduce that

$$v_\infty \equiv 0 \qquad \text{in } \mathbb{R}^N.$$

Moreover, we have

$$\{x \in \Omega : u_\infty(x) = 0\} \neq \emptyset, \qquad \text{and} \qquad \{x \in \Omega : u_\infty(x) > 0\} \text{ is a connected set.}$$

This last claim is due to the fact that, were $\{u_\infty > 0\}$ non trivially decomposed into $\Omega_1 \cup \Omega_2$, then again $u = u_\infty|_{\Omega_1}$ and $v = u_\infty|_{\Omega_2}$ - extended by 0 to the whole $\mathbb{R}^N$ - would be non–zero and would satisfy the assumptions of Proposition 2.6, a contradiction.

2.3.2 Almgren's Formula

In order to conclude the proof of Theorem 2.3 we will show that u_∞ is radially homogeneous; this crucial information will come from a generalization of the Almgren's Monotonicity Formula. This formula was first introduced in [1] and used for instance in [31, 64, 81] to prove some regularity issues related to free boundary problems. The aim is to study the monotonicity properties of the functions

$$E(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2\right),$$

$$H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} \left(u_\infty^2 + v_\infty^2\right) d\sigma,$$

and of the Almgren's quotient (whenever it is defined)

$$N(x_0, r) = \frac{E(x_0, r)}{H(x_0, r)},$$

where u_∞, v_∞ are defined in Lemma 2.17, $x_0 \in \mathbb{R}^N$ and $r > 0$.

Remark 2.20. Given a function u, in the literature the quotient $N(r) = N(x_0, r)$ is usually written as $N(r) = rE(r)/H(r)$, with $E(r) = \int_{B_r(x_0)} |\nabla u|^2$ and $H(r) = \int_{\partial B_r(x_0)} u^2 \, d\sigma$. We prefer to write it in a different way, stressing the fact that it is the quotient of two averages. The benefits of this point of view will become clear along this text.

It is worthwhile noticing that the result we prove for u_∞, v_∞ holds in fact for any non trivial strong H^1_{loc}–limit of solutions of variational systems; indeed, we will perform the proof without using all the other properties we collected about u_∞, v_∞. The reason for doing so is that we will need a similar result, for different functions, in Section 2.4.

Proposition 2.21. *Under the previous notations, for every $x_0 \in \mathbb{R}^N$ we have that $H(x_0, r) \neq 0$ for every $r > 0$, and*

$$r \mapsto N(x_0, r) \textit{ is an absolutely continuous, non decreasing function}$$

such that

$$\frac{d}{dr}\log(H(x_0,r)) = \frac{2}{r}N(x_0,r). \tag{2.29}$$

Moreover if $N(x_0,r)=\gamma$ *for all* $r>0$*, then* $u_\infty(x)=r^\gamma g_1(\theta)$*,* $v_\infty(x)=r^\gamma g_2(\theta)$ *in* $\mathbb{R}^N$*, for some functions* g_1,g_2 *(where* (r,θ) *denote the polar coordinates centered at* x_0*).*

Proof. We divide the proof into several steps.

1. Approximated quotients. Fix $x_0\in\mathbb{R}^N$ and let $0<r_1<r_2$ be such that $H(x_0,r)\neq 0$ in $[r_1,r_2]$ (such an interval exists for sure, since $u_\infty\not\equiv 0$ and it is a continuous function). Let us check that the conclusions of the proposition are true in this interval. To evaluate the derivatives of $E(r)$, $H(r)$ and $N(r)$ we have to face two main problems: first, it is not clear how regular these functions are; second, we have no global equation for u_∞,v_∞. To overcome these difficulties, the idea is to consider analogous functions for the approximated problem (2.20) which will result to be C^1, and then pass to the limit as $\beta\to+\infty$. In order to simplify notations we will denote for the moment $u:=\bar u_\beta$ and use similar notations for $\bar v_\beta,\bar h_\beta,\bar k_\beta$. We then define the approximated Almgren's quotient

$$N_\beta(x_0,r)=\frac{E_\beta(x_0,r)}{H_\beta(x_0,r)},$$

where

$$E_\beta(x_0,r)=\frac{1}{r^{N-2}}\int_{B_r(x_0)}\big(|\nabla u|^2+|\nabla v|^2+r_\beta^2(\lambda_\beta u^2+\mu_\beta v^2)- \\ -M_\beta(\omega_1u^4+\omega_2v^4)+2\beta M_\beta u^2v^2\big),$$

$$H_\beta(x_0,r)=\frac{1}{r^{N-1}}\int_{\partial B_r(x_0)}\left(u^2+v^2\right)\,d\sigma.$$

We also observe that, by multiplying system (2.20) by (u,v) and integrating in $B_r(x_0)$, we obtain

$$E_\beta(x_0,r)=\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}(u\,\partial_\nu u+v\,\partial_\nu v)\,d\sigma+\frac{1}{r^{N-2}}\int_{B_r(x_0)}(h(x)u+k(x)v) \tag{2.30}$$

(the boundary integrals above, and all the following ones, are well defined, for β fixed and for every $0<r<\operatorname{dist}(B_r(x_0),\partial\Omega_\beta)$, by Remark 2.13 and by the continuous immersion of $H^2(B_r)$ in $H^1(\partial B_r)$).

2. Derivatives of $\mathbf{E}_\beta$, $\mathbf{H}_\beta$. In order to compute the derivatives of these expressions, we consider the rescaled function $u_r(x):=u(x_0+rx)$ and similar expressions for v,h,k. System (2.20) now becomes

$$\begin{cases} -\Delta u_r+r^2\lambda_\beta r_\beta^2u_r &= r^2\omega_1M_\beta u_r^3-r^2\beta M_\beta u_rv_r^2+r^2h_r\\ -\Delta v_r+r^2\mu_\beta r_\beta^2v_r &= r^2\omega_2M_\beta v_r^3-r^2\beta M_\beta u_r^2v_r+r^2k_r.\end{cases} \tag{2.31}$$

Performing the change of variables $x=x_0+ry$ in $E_\beta(x_0,r)$ we obtain

$$E_\beta(x_0,r)=\int_{B_1(0)}\big(|\nabla u_r|^2+|\nabla v_r|^2+r^2r_\beta^2(\lambda_\beta u_r^2+\mu_\beta v_r^2)- \\ -r^2M_\beta(\omega_1u_r^4+\omega_2v_r^4)+2r^2\beta M_\beta u_r^2v_r^2\big),$$

and hence (Remark 2.13 implies that E_β is in fact C^1 in r)

$$\begin{aligned}\tfrac{d}{dr}E_\beta(x_0,r) \; = \; & 2\int_{B_1(0)} (\langle\nabla u_r, \nabla\langle\nabla u(x_0+rx),x\rangle\rangle + \langle\nabla v_r, \nabla\langle\nabla v(x_0+rx),x\rangle\rangle)+\\ & +2r^2\int_{B_1(0)} (r_\beta^2\lambda_\beta u_r - 2M_\beta\omega_1 u_r^3 + 2\beta M_\beta u_r v_r^2)\langle\nabla u(x_0+rx),x\rangle+\\ & +2r^2\int_{B_1(0)} (r_\beta^2\mu_\beta v_r - 2M_\beta\omega_2 v_r^3 + 2\beta M_\beta u_r^2 v_r)\langle\nabla v(x_0+rx),x\rangle+\\ & +2r\int_{B_1(0)} \left(r_\beta^2(\lambda_\beta u_r^2+\mu_\beta v_r^2) - M_\beta(\omega_1 u_r^4+\omega_2 v_r^4) + 2\beta M_\beta u_r^2 v_r^2\right).\end{aligned}$$

Multiplying the first equation in (2.31) by $\langle\nabla u(x_0+rx),x\rangle$, the second one by $\langle\nabla v(x_0+rx),x\rangle$, and integrating by parts in $B_1(0)$ yields

$$\begin{aligned}\int_{B_1(0)}\langle\nabla u_r,\nabla\langle\nabla u(x_0{+}rx),x\rangle\rangle = r^2\int_{B_1(0)} & (-\lambda_\beta r_\beta^2 u_r{+}\omega_1 M_\beta u_r^3{-}\beta M_\beta u_r v_r^2)\langle\nabla u(x_0{+}rx),x)\rangle+\\ + r^2\int_{B_1(0)} & h_r\langle\nabla u(x_0+rx),x\rangle + r\int_{\partial B_1(0)}\langle\nabla u(x_0+rx),x\rangle^2\,d\sigma\end{aligned}$$

and

$$\begin{aligned}\int_{B_1(0)}\langle\nabla v_r,\nabla\langle\nabla v(x_0{+}rx),x\rangle\rangle = r^2\int_{B_1(0)} & (-\mu_\beta r_\beta^2 v_r{+}\omega_2 M_\beta v_r^3{-}\beta M_\beta u_r^2 v_r)\langle\nabla v(x_0{+}rx),x)\rangle+\\ + r^2\int_{B_1(0)} & k_r\langle\nabla v(x_0+rx),x\rangle + r\int_{\partial B_1(0)}\langle\nabla v(x_0+rx),x\rangle^2\,d\sigma,\end{aligned}$$

which yields that

$$\begin{aligned}\frac{d}{dr}E_\beta(x_0,r) =& 2r\int_{\partial B_1(0)}(\langle\nabla u(x_0+rx),x\rangle^2+\langle\nabla v(x_0+rx),x\rangle^2)+\\ &+2r^2\int_{B_1(0)} h_r\langle\nabla u(x_0+rx),x\rangle + k_r\langle\nabla v(x_0+rx),x\rangle+\\ &+2r\int_{B_1(0)}(r_\beta^2(\lambda_\beta u_r^2+\mu_\beta v_r^2) - M_\beta(\omega_1 u_r^4+\omega_2 v_r^4)+2\beta M_\beta u_r^2 v_r^2)\\ &-2r^2\int_{B_1(0)}(\omega_1 M_\beta u_r^3\langle\nabla u(x_0+rx),x\rangle + \omega_2 M_\beta v_r^3\langle\nabla v(x_0+rx),x\rangle)+\\ &+2r^2\int_{B_1(0)}(\beta M_\beta u_r v_r^2\langle\nabla u(x_0+rx),x\rangle + \beta M_\beta u_r^2 v_r\langle\nabla v(x_0+rx),x\rangle)\\ =&\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}\Big((\partial_\nu u)^2+(\partial_\nu v)^2\Big)\,d\sigma+\\ &+\frac{2}{r^{N-1}}\int_{B_r(x_0)}(h(x)\langle\nabla u,x-x_0\rangle + k(x)\langle\nabla v,x-x_0\rangle)+\\ &+\frac{2}{r^{N-1}}\int_{B_r(x_0)}\left(r_\beta^2(\lambda_\beta u^2+\mu_\beta v^2) - M_\beta(\omega_1 u^4+\omega_2 v^4)+2\beta M_\beta u^2v^2\right)-\\ &-\frac{2}{r^{N-1}}\int_{B_r(x_0)}\left(\omega_1 M_\beta u^3\langle\nabla u,x-x_0\rangle) + \omega_2 M_\beta v^3\langle\nabla v,x-x_0\rangle\right)+\\ &+\frac{2}{r^{N-1}}\int_{B_r(x_0)}\left(\beta M_\beta u\langle\nabla u,x-x_0\rangle v^2 + \beta M_\beta u^2 v\langle\nabla v,x-x_0\rangle\right).\end{aligned}$$

Let us rewrite some terms by using the divergence theorem. We have

$$\frac{2}{r^{N-1}}\int_{B_r(x_0)}\big(\beta M_\beta u\langle\nabla u, x-x_0\rangle v^2+\beta M_\beta u^2 v\langle\nabla v, x-x_0\rangle\big)=$$
$$=\frac{\beta M_\beta}{r^{N-1}}\int_{B_r(x_0)}\langle\nabla(u^2v^2), x-x_0\rangle=-\frac{N}{r^{N-1}}\int_{B_r(x_0)}\beta M_\beta u^2v^2+\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}\beta M_\beta u^2v^2\,dx,$$

and moreover

$$-\frac{2}{r^{N-1}}\int_{B_r(x_0)}\big(\omega_1 M_\beta u^3\langle\nabla u, x-x_0\rangle+\omega_2 M_\beta v^3\langle\nabla v, x-x_0\rangle\big)=$$
$$=-\frac{1}{2r^{N-1}}\int_{B_r(x_0)}\big(\omega_1 M_\beta\langle\nabla(u^4), x-x_0\rangle+\omega_2 M_\beta\langle\nabla(v^4), x-x_0\rangle\big)=$$
$$=\frac{N}{2r^{N-1}}\int_{B_r(x_0)}\big(\omega_1 M_\beta u^4+\omega_2 M_\beta v^4\big)-\frac{1}{2r^{N-2}}\int_{\partial B_r(x_0)}\big(\omega_1 M_\beta u^4+\omega_2 M_\beta v^4\big)\,d\sigma.$$

At the end we obtain

$$\frac{d}{dr}E_\beta(x_0,r)=\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}\Big((\partial_\nu u)^2+(\partial_\nu v)^2\Big)\,d\sigma+$$
$$+\frac{2}{r^{N-1}}\int_{B_r(x_0)}\big(h(x)\langle\nabla u, x-x_0\rangle+k(x)\langle\nabla v, x-x_0\rangle\big)+$$
$$+\frac{1}{r^{N-1}}\int_{B_r(x_0)}\left(2r_\beta^2(\lambda_\beta u^2+\mu_\beta v^2)+\frac{N-4}{2}M_\beta(\omega_1 u^4+\omega_2 v^4)+(4-N)\beta M_\beta u^2v^2\right)+$$
$$+\frac{1}{2r^{N-2}}\int_{\partial B_r(x_0)}\big(2\beta M_\beta u^2v^2-\omega_1 M_\beta u^4-\omega_2 M_\beta v^4\big)\,d\sigma.$$

As for H_β, we have

$$H_\beta(x_0,r)=\int_{\partial B_1(0)}(u_r^2+v_r^2)\,d\sigma$$

and hence

$$\begin{aligned}\frac{d}{dr}H_\beta(x_0,r)&=2\int_{\partial B_1(0)}(u_r\langle\nabla u(x_0+rx),x\rangle+v_r\langle\nabla v(x_0+rx),x\rangle)\,d\sigma\\&=\frac{2}{r^{N-1}}\int_{\partial B_r(x_0)}(u\,\partial_\nu u+v\,\partial_\nu v)\,d\sigma.\end{aligned}$$

3. Estimate of $\mathbf{N}_\beta(\mathbf{r}+\delta)-\mathbf{N}_\beta(\mathbf{r})$. At this point, let us recover the original notations $\bar u_\beta,\bar v_\beta,\bar h_\beta,\bar k_\beta$. Recalling equation (2.30) we can compute the derivative of N_β in (r_1,r_2) as

$$\frac{d}{dr}N_\beta(x_0,r)=\frac{1}{H_\beta^2(x_0,r)}\left(\frac{d}{dr}E_\beta(x_0,r)\cdot H_\beta(x_0,r)-E_\beta(x_0,r)\cdot\frac{d}{dr}H_\beta(x_0r)\right)=$$
$$=\frac{2}{r^{2N-3}H_\beta^2(x_0,r)}\left[\int_{\partial B_r(x_0)}\Big((\partial_\nu\bar u_\beta)^2+(\partial_\nu\bar v_\beta)^2\Big)\,d\sigma\cdot\int_{\partial B_r(x_0)}(\bar u_\beta^2+\bar v_\beta^2)\,d\sigma-\right.$$
$$\left.-\left(\int_{\partial B_r(x_0)}(\bar u_\beta\partial_\nu\bar u_\beta+\bar v_\beta\partial_\nu\bar v_\beta)\,d\sigma\right)^2\right]+R_\beta(x_0,r),$$

where

$$
\begin{aligned}
R_\beta(x_0,r) =& \frac{2}{r^{N-1}H_\beta(x_0,r)}\int_{B_r(x_0)}\left(\bar h_\beta(x)\langle\nabla\bar u_\beta, x-x_0\rangle+\bar k_\beta(x)\langle\nabla\bar v_\beta, x-x_0\rangle+r_\beta^2(\lambda_\beta\bar u_\beta^2+\mu_\beta\bar v_\beta^2)\right)+\\
&+\frac{1}{r^{N-1}H_\beta(x_0,r)}\int_{B_r(x_0)}\left(\frac{N-4}{2}M_\beta(\omega_1\bar u_\beta^4+\omega_2\bar v_\beta^4)+(4-N)\beta M_\beta\bar u_\beta^2\bar v_\beta^2\right)+\\
&+\frac{1}{2r^{N-2}H_\beta(x_0,r)}\int_{\partial B_r(x_0)}\left(2\beta M_\beta\bar u_\beta^2\bar v_\beta^2-\omega_1 M_\beta\bar u_\beta^4-\omega_2M_\beta\bar v_\beta^4\right)\,d\sigma-\\
&-\frac{2}{r^{2N-3}H_\beta^2(x_0,r)}\int_{\partial B_r(x_0)}\left(\bar u_\beta\,\partial_\nu\bar u_\beta+\bar v_\beta\,\partial_\nu\bar v_\beta\right)\,d\sigma\int_{B_r(x_0)}\left(\bar h_\beta(x)\bar u_\beta+\bar k_\beta(x)\bar v_\beta\right).
\end{aligned}
$$

Notice that, since $H(x_0,r)\neq 0$ in $[r_1,r_2]$, for every $\delta>0$ such that $r,r+\delta\in(r_1,r_2)$, there exists a constant $C>0$ depending only on r_1,r_2 and δ such that

$$
\begin{aligned}
&\int_r^{r+\delta}|R_\beta(x_0,s)|ds\leqslant\\
&\quad\leqslant C\int_{B_{r_2}(x_0)}(|\bar h_\beta||\nabla\bar u_\beta|+|\bar k_\beta||\nabla\bar v_\beta|+r_\beta^2(\bar u_\beta^2+\bar v_\beta^2)+M_\beta(\bar u_\beta^4+\bar v_\beta^4)+\beta M_\beta\bar u_\beta^2\bar v_\beta^2)+\\
&\qquad\qquad+\int_{B_{r_2}(x_0)}(|\bar u_\beta||\nabla\bar u_\beta|+|\bar v_\beta||\nabla\bar v_\beta|)\int_{B_{r_2}(x_0)}\left(|\bar h_\beta||\bar u_\beta|+|\bar k_\beta||\bar v_\beta|\right)\longrightarrow 0
\end{aligned}
$$

as $\beta\to+\infty$, where we used Remark 2.14 and Lemma 2.17, $(ii)-(iii)$. Therefore

$$
\begin{aligned}
&N_\beta(x_0,r+\delta)-N_\beta(x_0,r)=\\
&\quad=\int_r^{r+\delta}\frac{2}{s^{2N-3}H_\beta^2(x_0,s)}\Bigg[\int_{\partial B_s(x_0)}\left((\partial_\nu\bar u_\beta)^2+(\partial_\nu\bar v_\beta)^2\right)d\sigma\cdot\int_{\partial B_s(x_0)}(\bar u_\beta^2+\bar v_\beta^2)\,d\sigma-\\
&\qquad\qquad-\left(\int_{\partial B_s(x_0)}(\bar u_\beta\partial_\nu\bar u_\beta+\bar v_\beta\partial_\nu\bar v_\beta)\,d\sigma\right)^2\Bigg]+\mathrm{o}_\beta(1),
\end{aligned}
$$

where $\mathrm{o}_\beta(1)\to0$ as $\beta\to+\infty$ for each r,δ fixed such that $r,r+\delta\in(r_1,r_2)$.

4. Derivatives of N, H, E, $\log$ H. Now we are in a position to pass to the limit in β. Indeed, Lemma 2.17, (iii) (that is, strong convergence) ensures that $N_\beta(x_0,r)\to N(x_0,r)$ for every r. Moreover it implies the existence of a function $f(\rho)\in L^1(r_1,r_2)$ such that, up to a subsequence, $\int_{\partial B_\rho(x_0)}|\nabla\bar u_\beta|^2\leqslant f(\rho)$ and $\int_{\partial B_\rho(x_0)}|\nabla\bar u_\beta|^2\to\int_{\partial B_\rho(x_0)}|\nabla u_\infty|^2$ for a.e. $\rho\in(r_1,r_2)$ (and analogously for $\bar v_\beta$). Hence, we let $\beta\to+\infty$ in the previous equation and readily obtain that N is absolutely continuous and that (for almost every r)

$$
\begin{aligned}
\frac{d}{dr}N(x_0,r)=\frac{2}{r^{2N-3}H^2(x_0,r)}\Bigg[&\int_{\partial B_r(x_0)}\left((\partial_\nu u_\infty)^2+(\partial_\nu v_\infty)^2\right)\,d\sigma\cdot\int_{\partial B_r(x_0)}(u_\infty^2+v_\infty^2)\,d\sigma-\\
&-\left(\int_{\partial B_r(x_0)}(u_\infty\partial_\nu u_\infty+v_\infty\partial_\nu v_\infty)\,d\sigma\right)^2\Bigg]\geqslant 0, \qquad (2.32)
\end{aligned}
$$

by Hölder inequality. This implies that $N(x_0,r)$ is a non decreasing function in $[r_1,r_2]$ and in addition gives an explicit expression for its derivative. Reasoning as above, we can conclude moreover that

$$\begin{aligned}
\frac{d}{dr}H(x_0,r) &= \frac{2}{r^{N-1}}\int_{\partial B_r(x_0)} (u_\infty\,\partial_\nu u_\infty + v_\infty \partial_\nu v_\infty)\,d\sigma,\\
H(x_0,r) &= \lim_{\beta\to+\infty} H_\beta(x_0,r) = \frac{1}{r^{N-1}}\int_{\partial B_r(x_0)} \left(u_\infty^2+v_\infty^2\right)\,d\sigma,\\
E(x_0,r) &= \lim_{\beta\to+\infty} E_\beta(r) = \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)} (u_\infty\,\partial_\nu u_\infty + v_\infty \partial_\nu v_\infty)\,d\sigma
\end{aligned}$$

(where we used (2.30) to obtain the last limit). Therefore a direct computation shows that

$$\begin{aligned}
\frac{d}{dr}\log(H(x_0,r)) &= \frac{\frac{d}{dr}H(x_0,r)}{H(x_0,r)} = \frac{2}{r^{N-1}}\frac{\int_{\partial B_r(x_0)}(u_\infty\partial_\nu u_\infty + u_\infty\partial_\nu v_\infty)\,d\sigma}{H(x_0,r)}\\
&= \frac{2}{r}\frac{E(x_0,r)}{H(x_0,r)} = \frac{2}{r}N(x_0,r),
\end{aligned}$$

which yields (2.29) for $r\in(r_1,r_2)$. Observe moreover that the latter equation also implies that $\log H$, and hence H, are C^1–functions in $(0,+\infty)$.

5. Validity for every $\mathbf{r>0}$. Equality (2.29) also implies that $\frac{d}{dr}H(x_0,r)\geqslant 0$ whenever $H(x_0,r)>0$, and therefore there exists $r_0:=\inf\{r>0:\ H(x_0,r)\neq 0\}$ such that $H(x_0,r)\neq 0$ for every $r>r_0$. Assume by contradiction that $r_0>0$; by continuity, we have $H(x_0,r)\equiv 0$ on $[0,r_0]$. For every $r\in(r_0,r_0+1)$ we obtain

$$\frac{d}{dr}\log H(x_0,r) = \frac{2}{r}N(x_0,r) \leqslant \frac{2}{r}N(x_0,r_0+1) = \frac{C}{r} = \frac{d}{dr}\log r^C,$$

with $C:=2N(x_0,r_0+1)$. We can now integrate the previous inequality between $r_1<r_2$ (with $r_1,r_2\in(r_0,r_0+1)$), obtaining

$$H(x_0,r_2)\leqslant H(x_0,r_1)\left(\frac{r_2}{r_1}\right)^C.$$

By letting $r_1\to r_0$ we obtain $H(x_0,r_2)=0$, which contradicts the definition of r_0. Hence $r_0=0$, and the conclusions obtained from Step 1 to 4 hold true for $r\in(0,+\infty)$.

6. Case $\mathbf{N(r)}$ constant. Let us now analyze what happens when $N(x_0,r)=\gamma$ for every $r>0$. In such situation, the Hölder inequality in (2.32) must in fact be an equality. This implies that there exists $c(r)$ such that

$$\partial_\nu u_\infty = c(r)u_\infty, \qquad \partial_\nu v_\infty = c(r)v_\infty.$$

Multiplying each equation by $e^{-C(r)}$, where $C(r):=\int_0^r c(\xi)\,d\xi$, yields

$$\frac{\partial}{\partial r}\left(u_\infty e^{-C(r)}\right)=0, \qquad \frac{\partial}{\partial r}\left(v_\infty e^{-C(r)}\right)=0$$

and hence

$$u_\infty = f(r)g_1(\theta), \qquad v_\infty = f(r)g_2(\theta),$$

for some $f(r) > 0$. Now, from the fact that

$$\gamma = N(x_0, r) = \frac{\frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} (u_\infty \partial_\nu u_\infty + v_\infty \partial_\nu v_\infty)\, d\sigma}{\frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} (u_\infty^2 + v_\infty^2)\, d\sigma} = r\frac{f'(r)}{f(r)}$$

we see that $f(r) \equiv r^\gamma$ and finally conclude that $u_\infty(x) = r^\gamma g_1(\theta)$, $v_\infty(x) = r^\gamma g_2(\theta)$ in $\mathbb{R}^N$. □

Remark 2.22. Starting from system (2.1), one can perform the same blow–up argument than above, obtaining in particular that, for the limiting states $(u_{1,\infty}, \ldots, u_{m,\infty})$, a result analogous to Proposition 2.21 holds with the choice

$$E(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \sum_{i=1}^m |\nabla u_{i,\infty}|^2, \qquad H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} \sum_{i=1}^m u_{i,\infty}^2.$$

2.3.3 Proof of the main results

End of the proof of Theorem 2.3. By Lemma 2.18 we know that the blowup limiting profiles u_∞ and v_∞ are globally α–Hölder continuous. Moreover, by Remark 2.19, $v_\infty \equiv 0$ and $u_\infty \not\equiv 0$. Take any

$$x_0 \in \mathbb{R}^N \text{ such that } u_\infty(x_0) = 0.$$

For $N(x_0, r)$ defined as before, we claim that $N(x_0, r) = \alpha$ for all $r > 0$. Assume that there exists $\bar{r} > 0$ such that $N(x_0, \bar{r}) \leqslant \alpha - \varepsilon$ for some $\varepsilon > 0$. From Proposition 2.21 we deduce that $N(x_0, r) \leqslant \alpha - \varepsilon$ for all $0 < r < \bar{r}$ and

$$\frac{d}{dr}\log(H(x_0, r)) = \frac{2}{r}N(x_0, r) \leqslant \frac{2}{r}(\alpha - \varepsilon) = \frac{d}{dr}\log r^{2(\alpha-\varepsilon)}.$$

Hence (after and integration between r and $\bar{r}$) we have

$$Cr^{2\alpha-2\varepsilon} \leqslant H(r, x_0) \qquad \text{for all } 0 < r < \bar{r}.$$

Moreover, from the α–Hölder continuity and the fact that $u_\infty(x_0) = v_\infty(x_0) = 0$, we also have

$$H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} \left((u_\infty(x) - u_\infty(x_0))^2 + (v_\infty(x) - v_\infty(x_0))^2\right) d\sigma \leqslant C' r^{2\alpha},$$

a contradiction for small $r > 0$. On the other hand let us assume now that $N(x_0, \bar{r}) \geqslant \alpha + \varepsilon$ for some $\varepsilon > 0$. Again by Proposition 2.21 we see that $N(x_0, r) \geqslant \alpha + \varepsilon$ for all $r > \bar{r}$, and thus

$$\frac{d}{dr}\log H(x_0, r) \geqslant \frac{2}{r}(\alpha + \varepsilon),$$

which implies (integrating between $\bar{r}$ and r and again by the α–Hölder continuity) that

$$Cr^{2\alpha+2\varepsilon} \leqslant H(x_0, r) \leqslant C' r^{2\alpha} \qquad \text{for large } r > 0,$$

a contradiction.

Therefore $N(x_0, r) \equiv \alpha$ for all $r > 0$, and by Proposition 2.21 we know that $u_\infty(x) = |x - x_0|^\alpha g_1(\theta)$ for $\theta = (x - x_0)/|x - x_0|$. This implies that the nodal set $\Gamma = \{u_\infty = 0\}$ is a cone with respect to x_0. Since this can be done for any $x_0 \in \Gamma$, we obtain that Γ is in fact a cone with respect to each of its points, and thus it is an affine subspace of $\mathbb{R}^N$. Moreover, again by Remark 2.19, Γ has dimension strictly smaller than $N - 1$ (otherwise $\{u_\infty > 0\}$ would be disconnected). Thus we have deduced that u_∞ is a nonnegative, non constant function in $H^1_{\rm loc}(\mathbb{R}^N)$, which is harmonic on the complement of an affine subspace Γ having at most dimension $N - 2$. We claim that then u_∞ is harmonic on the whole $\mathbb{R}^N$, a contradiction. With this in mind, take a tubular neighborhood of Γ, $N_\varepsilon(\Gamma) = \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \Gamma) \leqslant \varepsilon\}$, and consider a cut-off function $0 \leqslant \varphi_\varepsilon \leqslant 1$, $|\nabla\varphi_\varepsilon| \leqslant C/\varepsilon$ such that $\varphi_\varepsilon = 1$ in $\mathbb{R}^N \setminus N_{2\varepsilon}(\Gamma)$ and $\varphi_\varepsilon = 0$ in $N_\varepsilon(\Gamma)$. Then, for every $\eta \in C^\infty_{\rm c}(\mathbb{R}^N)$, we have

$$0 = \int_{\mathbb{R}^N} \langle \nabla u_\infty, \nabla(\eta\varphi_\varepsilon)\rangle = \int_{\mathbb{R}^N} \langle \nabla u_\infty, \nabla\eta\rangle\varphi_\varepsilon + \int_{N_{2\varepsilon}(\Gamma)\setminus N_\varepsilon(\Gamma)} \langle \nabla u_\infty, \nabla\varphi_\varepsilon\rangle\eta.$$

Since $|N_{2\varepsilon}(\Gamma) \setminus N_\varepsilon(\Gamma) \cap \mathrm{supp}\eta| \leqslant C\varepsilon^2$ (recall that Γ has at most dimension $N - 2$), we obtain from the previous identity that, as $\varepsilon \to 0$,

$$0 = \int_{\mathbb{R}^N} \langle \nabla u_\infty, \nabla\eta\rangle.$$

Hence u_∞ is harmonic in $\mathbb{R}^N$. □

Remark 2.23. After a careful examination of the proof of Theorem 2.3 we actually deduce that we have proved the following Liouville–type result. Let $u, v \in H^1_{\rm loc} \cap C(\mathbb{R}^N)$ be such that $u, v \geqslant 0$, $u \cdot v \equiv 0$,

$$\Delta u = 0 \quad \text{in } \{u > 0\}, \qquad \Delta v = 0 \quad \text{in } \{v > 0\},$$

and suppose that

(i) $\displaystyle\sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \ \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha} < \infty.$

(ii) the conclusions of Proposition 2.21 hold.

Then either $u \equiv 0$ and v is constant, or u is constant and $v \equiv 0$.

Proof of Theorem 2.4. By Theorem 2.3 we know that for every $\alpha < \alpha' < \alpha^*$ there exists a constant $C > 0$ such that $\|(u_\beta, v_\beta)\|_{C^{0,\alpha'}} \leqslant C$, for every $\beta > 0$. By the compact embedding $C^{0,\alpha'}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ (see [67, Theorem 7.26]) we obtain the existence of $(u, v) \in C^{0,\alpha'}$ which is, up to a subsequence, a strong $C^{0,\alpha}$–limit of (u_β, v_β). By uniqueness of the limit, this proves that

$$(u_\beta, v_\beta) \to (u, v) \quad \text{in } C^{0,\alpha}(\overline{\Omega}) \text{ for every } \alpha < \alpha^*.$$

Now we reason as in the proof of Lemma 2.17 in order to obtain the other claims of the theorem. By testing system (2.3) with (u_β, v_β) we get

$$\int_\Omega |\nabla u_\beta|^2 \leqslant \int_\Omega \left(-\lambda_\beta u_\beta + \omega_1 u_\beta^3 + h_\beta\right) u_\beta,$$

and an analogous inequality also holds for v_β. By uniform convergence, the right-hand side is bounded and therefore (u_β, v_β) is bounded in the $H_0^1(\Omega)$–norm. Thus, again up to a subsequence, we have

$$(u_\beta, v_\beta) \rightharpoonup (u, v) \quad \text{weakly in } H_0^1(\Omega).$$

On the other hand, integrating system (2.3) we have

$$-\int_{\partial\Omega} \partial_\nu u_\beta + \beta \int_\Omega u_\beta v_\beta^2 = \int_\Omega \left(-\lambda_\beta u_\beta + \omega_1 u_\beta^3 + h_\beta\right).$$

Again, the right-hand side is bounded and, since $u_\beta \geqslant 0$ in Ω and $u_\beta = 0$ on $\partial\Omega$, we deduce that $\partial_\nu u_\beta \leqslant 0$ on $\partial\Omega$ (see Lemma C.1). We infer that

$$\beta \int_\Omega u_\beta v_\beta^2 \leqslant C \qquad \text{and} \qquad \beta \int_\Omega u_\beta^2 v_\beta \leqslant C,$$

where C does not depend on β. This immediately provides $u \cdot v \equiv 0$ almost everywhere in Ω (recall that $\beta \to +\infty$) and, in turn, reasoning as (2.28),

$$\beta \int_\Omega u_\beta^2 v_\beta^2 \to 0 \quad \text{as } \beta \to +\infty,$$

which completes the proof of (*ii*). Now we can test system (2.3) with $(u_\beta - u, v_\beta - v)$, obtaining

$$\left|\int_\Omega \langle \nabla u_\beta, \nabla(u_\beta - u)\rangle\right| \leqslant \|u_\beta - u\|_{L^\infty(\Omega)} \int_\Omega \left|-\lambda_\beta u_\beta + \omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 + h_\beta\right|,$$

and the same for v. By uniform convergence we infer convergence in norm, and hence strong H_0^1–convergence of (u_β, v_β) to (u, v), and so also (*i*) is proved. Finally, in order to prove (*iii*), we observe that by the continuity of the limiting profile we know that $\{u > 0\}$ is an open set. Therefore for any given $x_0 \in \{u > 0\}$ there exists $B_\delta(x_0)$ such that $u \geqslant 2\gamma > 0$ in $B_\delta(x_0)$, for some positive constant γ. Let us show that the equation is satisfied in this open neighborhood. By (*i*) we see that $u_\beta \geqslant \gamma$ in $B_\delta(x_0)$ for large β, therefore

$$\int_{B_\delta(x_0)} \beta u_\beta v_\beta^2 \leqslant \frac{1}{\gamma} \int_{B_\delta(x_0)} \beta u_\beta^2 v_\beta^2 \to 0$$

because of (*ii*). By testing the equation for u_β with a test function $\varphi \in C_0^1(B_\delta(x_0))$ we obtain

$$\int_{B_\delta(x_0)} (\langle \nabla u_\beta, \nabla\varphi\rangle + \lambda_\beta u_\beta \varphi) = \int_{B_\delta(x_0)} (\omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 + h_\beta)\varphi,$$

and the previous estimate together with the H^1–convergence ends the proof. □

Proof of Theorems 2.1 and 2.2 ($m = 2$). As we have just noticed, with a small change in the previous arguments one can prove also these two theorems, except for the Lipschitz continuity of the limiting profile (u, v) (this will be the object of the following section). More precisely, in dimension $N = 2$, since $\alpha^* = 1$, then the theorems follow directly from Theorems 2.3 and 2.4. In dimension $N = 3$, according to Remark 2.13, if $h_\beta \equiv k_\beta \equiv 0$ then we can choose $\alpha^* = 1$ and repeat, word by word, all the arguments in this section. Then Theorem 2.1 straightly follows, while the proof of Theorem 2.2 will be completed by Proposition 2.25 and Remark 2.37 below. □

Remark 2.24. With exactly the same strategy it is also possible to prove analogous results for L^∞–bounded, positive solutions of system (2.1). The only differences are pointed out in Proposition 2.11 and in Remark 2.22.

2.4 Lipschitz continuity of the limiting profile

Throughout all this section, let $(u,v) \in C^{0,\alpha}(\bar{\Omega}) \cap H^1_0(\Omega)$ denote the limiting profile introduced in Theorem 2.4, and $h_\beta, k_\beta \equiv 0$ (that is, we are dealing with system (2.1)). As we noticed, in this case the uniform Hölder continuity result holds for every $\alpha \in (0,1)$ also if $N = 3$. In such a situation, although we are not able to prove uniform Lipschitz continuity of the solutions with respect to β (recall Remark 2.8), one can prove that the limiting profile is in fact Lipschitz continuous. To be more precise, we will first give the details of the proof of the local Lipschitz continuity for (u,v), an then we will advise (in Remarks 2.36 and 2.37) how this proof can be modified in order to obtain the Lipschitz regularity up to the boundary of Ω. Also, after Remarks 2.22 and 2.24, it will be easy to see how this result holds true for m–tuples of densities that are solutions of system (2.1). Let us fix a regular domain $\tilde{\Omega} \Subset \Omega$, and let us define the nodal set

$$\Gamma = \{x \in \tilde{\Omega} : u(x) = v(x) = 0\}.$$

Without loss of generality, we can suppose that $\Gamma \neq \tilde{\Omega}$.

Proposition 2.25. *Let (u,v) be the limiting profile introduced in Theorem 2.4, $h_\beta \equiv k_\beta \equiv 0$ and $\tilde{\Omega} \Subset \Omega$. Then $(u,v) \in W^{1,\infty}(\tilde{\Omega})$.*

Again, in order to prove the proposition, the main tool will be the Almgren's Monotonicity Formula introduced in Subsection 2.3.2, with some small change in its definition. Indeed, due to the fact that the limiting profiles satisfy system (2.5), the natural definition for $E(x_0,r)$ is

$$E(x_0,r) \quad = \quad \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2 - \omega_1 u^4 - \omega_2 v^4\right).$$

Now, as before, we define

$$H(x_0,r) \quad = \quad \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} (u^2+v^2)\, d\sigma,$$

and the Almgren's quotient by

$$N(x_0,r) = \frac{E(x_0,r)}{H(x_0,r)}$$

(whenever $H(x_0,r) \neq 0$). We observe that the quantities E and H are well defined for $x \in \tilde{\Omega}$ and $0 < r < \operatorname{dist}(\tilde{\Omega}, \partial\Omega)$.

Lemma 2.26. *There exists $\bar{r}_1 < \operatorname{dist}(\tilde{\Omega}, \partial\Omega)$ such that for every $0 < r \leqslant \bar{r}_1$ and for every $x_0 \in \tilde{\Omega}$ we have*

$$E(x_0,r) + H(x_0,r) \geqslant \frac{1}{2}\Big(\frac{1}{r^{N-2}} \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) + \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} (u^2+v^2)\Big) \geqslant 0.$$

Proof. We will use the following formulation of Poincaré's inequality: for every $w \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, $x_0 \in \mathbb{R}^N$ and $r > 0$ the following estimate holds[2]

$$\frac{1}{r^N}\int_{B_r(x_0)} w^2 \leqslant \frac{1}{N-1}\Big(\frac{1}{r^{N-2}}\int_{B_r(x_0)} |\nabla w|^2 + \frac{1}{r^{N-1}}\int_{\partial B_r(x_0)} w^2\Big).$$

By recalling that $u, v \in L^\infty(\Omega)$, let now $C > 0$ be such that

$$\Big|\frac{1}{r^{N-2}}\int_{B_r(x_0)} \lambda(u^2+\mu v^2-\omega_1 u^4-\omega_2 v^4)\Big| \leqslant \frac{C}{r^{N-2}}\int_{B_r(x_0)} (u^2+v^2)$$

(observe that C depends on u, v, ω_i, λ, μ, but not on x_0 and r). Then, by choosing $\bar r_1$ so that $C\bar r_1^2 \leqslant (N-1)/2$ we have, for $0 < r < \bar r_1$,

$$\begin{aligned}\frac{1}{r^{N-2}}\int_{B_r(x_0)} (|\nabla u|^2+|\nabla v|^2) &+ \frac{1}{r^{N-1}}\int_{\partial B_r(x_0)} (u^2+v^2)\,d\sigma \leqslant \\ &\leqslant E(x_0,r)+H(x_0,r)+\frac{Cr^2}{r^N}\int_{\partial B_r(x_0)}(u^2+v^2)\,d\sigma \leqslant \\ &\leqslant E(x_0,r)+H(x_0,r)+\frac{Cr^2}{N-1}\Big(\frac{1}{r^{N-2}}\int_{B_r(x_0)}(|\nabla u|^2+|\nabla v|^2)+H(x_0,r)\Big)\end{aligned}$$

and the result follows. □

Now, with the new notations of this section, let us present a result which corresponds to Proposition 2.21 in this context.

Proposition 2.27. *There exist $\bar r \leqslant \bar r_1$ and $C > 0$ such that, for every $x_0 \in \tilde\Omega$ and $0 < r \leqslant \bar r$, we have $H(x_0,r) \neq 0$,*

$$\frac{d}{dr}N(x_0,r) \geqslant -2Cr(N(x_0,r)+1),$$

and in particular $r \mapsto e^{Cr^2}(N(x_0,r)+1)$ is a non decreasing function for $r \in (0,\bar r]$. Moreover,

$$\frac{d}{dr}\log(H(x_0,r)) = \frac{2}{r}N(x_0,r). \tag{2.33}$$

Proof. We will follow very closely the ideas of the proof of Proposition 2.21. Fix $x_0 \in \tilde\Omega$.
1. Proof when $\mathbf{H(x_0,r)\neq 0}$. Let us first suppose that there is an interval $[r_1,r_2]$, with $r_2 < \bar r_1$ (defined in the previous lemma), such that $H(x_0,r) > 0$ in $[r_1,r_2]$. Again, we first consider the approximated problem

$$\begin{aligned}E_\beta(x_0,r) &= \frac{1}{r^{N-2}}\int_{B_r(x_0)}\left(|\nabla u_\beta|^2+|\nabla v_\beta|^2+\lambda_\beta u_\beta^2+\mu_\beta v_\beta^2-\omega_1 u_\beta^4-\omega_2 v_\beta^4+2\beta u_\beta^2 v_\beta^2\right),\\ H_\beta(x_0,r) &= \frac{1}{r^{N-1}}\int_{\partial B_r(x_0)}(u_\beta^2+v_\beta^2)\,d\sigma,\\ N_\beta(x_0,r) &= \frac{E_\beta(x_0,r)}{H_\beta(x_0,r)}.\end{aligned}$$

[2] This inequality is a direct consequence of the following one: $\int_{B_1(0)} w^2 \leqslant \frac{1}{N-1}\int_{B_1(0)}|\nabla w|^2 + \frac{1}{N-1}\int_{\partial B_1(0)} w^2\,d\sigma$ (for $w \in H^1(\Omega)$). This one can in turn be proved by applying the divergence theorem to the vector function $w^2 x$ in the domain $B_1(0)$.

We proceed exactly as in Proposition 2.21. In fact, for $u_r(x) = u_\beta(x_0 + rx)$, $v_r(x) = v_\beta(x_0 + rx)$, we have

$$\begin{aligned}\frac{d}{dr}E_\beta(x_0, r) &= 2\int_{B_1(0)}\Big(\langle \nabla u_r, \nabla\langle \nabla u(x_0+rx), x\rangle\rangle + \langle \nabla v_r, \nabla\langle \nabla v(x_0+rx), x\rangle\rangle\Big) + \\ &\quad +2r^2\int_{B_1(0)}(\lambda_\beta u_r - 2\omega_1 u_r^3 + 2\beta u_r v_r^2)\langle \nabla u(x_0+rx), x\rangle + \\ &\quad +2r^2\int_{B_1(0)}(\mu_\beta v_r - 2\omega_2 v_r^3 + 2\beta u_r^2 v_r)\langle \nabla v(x_0+rx), x\rangle + \\ &\quad +2r\int_{B_1(0)}(\lambda_\beta u_r^2 + \mu_\beta v_r^2 - \omega_1 u_r^4 - \omega_2 v_r^4 + 2\beta u_r^2 v_r^2).\end{aligned}$$

From the fact that (u_β, v_β) solves (2.1) ($m = 2$), we get

$$\begin{aligned}\int_{B_1(0)}&\Big(\langle \nabla u_r, \nabla\langle \nabla u(x_0+rx), x\rangle\rangle + \langle \nabla v_r, \nabla\langle \nabla v(x_0+rx), x\rangle\rangle\Big) = \\ &= r\int_{\partial B_1(0)}(\langle \nabla u(x_0+rx), x\rangle^2 + \langle \nabla v(x_0+rx), x\rangle^2)\,d\sigma \\ &\quad + r^2\int_{B_1(0)}(-\lambda_\beta u_r + \omega_1 u_r^3 - \beta u_r v_r^2)\langle \nabla u(x_0+rx), x\rangle + \\ &\qquad + r^2\int_{B_1(0)}(-\mu_\beta v_r + \omega_2 v_r^3 - \beta u_r^2 v_r)\langle \nabla v(x_0+rx), x\rangle. \end{aligned} \tag{2.34}$$

and hence

$$\frac{d}{dr}E_\beta(x_0, r) = \frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}((\partial_\nu u_\beta)^2 + (\partial_\nu v_\beta)^2)\,d\sigma + R_\beta(x_0, r),$$

where

$$\begin{aligned}R_\beta(x_0, r) &= \frac{1}{r^{N-1}}\int_{B_r(x_0)}\left(2\lambda_\beta u_\beta^2 + 2\mu_\beta v_\beta^2 + \frac{N-4}{2}(\omega_1 u_\beta^4 + \omega_2 v_\beta^4) + (4-N)\beta u_\beta^2 v_\beta^2\right) - \\ &\quad - \frac{1}{2r^{N-2}}\int_{\partial B_r(x_0)}(\omega_1 u_\beta^4 + \omega_2 v_\beta^4 - 2\beta u_\beta^2 v_\beta^2)\,d\sigma.\end{aligned}$$

On the other hand, we have

$$\frac{d}{dr}H_\beta(x_0, r) = \frac{2}{r^{N-1}}\int_{\partial B_r(x_0)}(u_\beta \partial_\nu u_\beta + v_\beta \partial_\nu v_\beta)\,d\sigma$$

and, by using the fact that (u_β, v_β) solves (2.1), we can rewrite E as

$$E_\beta(x_0, r) = \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}(u_\beta \partial_\nu u_\beta + v_\beta \partial_\nu v_\beta)\,d\sigma.$$

Thus, as $\beta \to +\infty$, it follows that

$$E(x_0, r) = \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}(u\partial_\nu u + v\partial_\nu v)\,d\sigma,$$

$$\frac{d}{dr}H(x_0,r) = \frac{2}{r^{N-1}}\int_{\partial B_r(x_0)} (u\partial_\nu u + v\partial_\nu v)\,d\sigma$$

and

$$\frac{d}{dr}E(x_0,r) = \frac{2}{r^{N-2}}\int_{\partial B_r(x_0)} ((\partial_\nu u)^2 + (\partial_\nu v)^2)\,d\sigma + R(x_0,r),$$

with

$$R(x_0,r) = \frac{1}{r^{N-1}}\int_{B_r(x_0)} \left(2\lambda u^2 + 2\mu v^2 + \frac{N-4}{2}(\omega_1 u^4 + \omega_2 v^4)\right) - \\ - \frac{1}{2r^{N-2}}\int_{\partial B_r(x_0)} (\omega_1 u^4 + \omega_2 v^4)\,d\sigma.$$

Hence in the interval (r_1, r_2) we obtain, by using Hölder's inequality,

$$\frac{d}{dr}N(x_0,r) = \frac{1}{H^2(x_0,u)}\left(\frac{d}{dr}E(x_0,r)H(x_0,r) - E(x_0,r)\frac{d}{dr}H(x_0,r)\right) \geqslant \frac{R(x_0,r)}{H(x_0,r)}.$$

Now, from Lemma 2.26 and the Poincaré's inequality we deduce the existence of two constants $C, \tilde{C} > 0$ (depending only of $\bar{r}_1$, independent of x_0) such that

$$\begin{aligned} |R(x_0,r)| &\leqslant \frac{\tilde{C}}{r^{N-1}}\int_{B_r(x_0)} (u^2+v^2) + \frac{\tilde{C}}{r^{N-2}}\int_{\partial B_r(x_0)} (u^2+v^2)\,d\sigma \\ &\leqslant 2\tilde{C}r(E(x_0,r) + H(x_0,r)) + \tilde{C}rH(x_0,r) \\ &\leqslant 2Cr(E(x_0,r) + H(x_0,r)) \end{aligned}$$

and thus

$$\frac{d}{dr}N(x_0,r) \geqslant -2Cr(N(x_0,r)+1).$$

Finally, (2.33) comes from a direct calculation as in Proposition 2.21.

Therefore, at this point, we have proved the proposition for every interval $[r_1, r_2]$ with $r_2 < \bar{r}_1$ such that $H(x_0,r) > 0$. Next we will show that in fact $H(x_0,r) \neq 0$ for r small, after which the proof will be complete. This will be done in two more steps.

2. Γ has empty interior. Assume the contrary, and let $x_1 \in \Gamma$ be such that $d_1 := \mathrm{dist}(x_1, \partial\Gamma) \in (0, \bar{r}_1)$ (recall that we are assuming that $u^2 + v^2$ is not identically zero in $\tilde{\Omega}$). We have $H(x_1, r) > 0$ for $r \in (d_1, d_1+\varepsilon)$ for some small $\varepsilon > 0$. By what we have done so far the function $H(r) = H(x_1, r)$ verifies, in $(d_1, d_1+\varepsilon)$, the initial value problem

$$\begin{cases} H'(r) = a(r)H(r) & r \in (d_1, d_1+\varepsilon) \\ H(d_1) = 0, \end{cases}$$

with $a(r) = 2N(r)/r$, which is continuous also at $d_1 > 0$ by the monotonicity of the function $e^{Cr^2}(N(r)+1)$. Then $H(x_0,r) \equiv 0$ for $r > d_1$, a contradiction with the definition of d_1.

3. Definition of $\bar{\mathbf{r}}$. Finally we observe that, by (2.2), we have

$$-\Delta u \leqslant (\omega_1 u^2 - \lambda)u < \lambda_1(B_r(x_0))u \qquad \text{in } \Omega$$

for small r, let us say $0 < r < \bar{r}_2$, independent of x_0 (indeed $\lambda_1(B_r(x_0)) \to +\infty$ as $r \to 0$); an analogous inequality holds for v. Fixing now $\bar{r} < \min\{\bar{r}_1, \bar{r}_2\}$, for $0 < r \leqslant \bar{r}$ we must have $H(x_0, r) \neq 0$ for every $x_0 \in \tilde{\Omega}$. In fact, if for some $x_1 \in \tilde{\Omega}$ and some $0 < r < \bar{r}$ we had $u, v = 0$ on $\partial B_r(x_1)$, then this fact together with the previous inequality would give $u, v \equiv 0$ in $B_r(x_1)$, a contradiction with the fact that Γ has empty interior. □

Remark 2.28. For future reference, we mention that in the previous proposition it was shown that Γ has empty interior.

Remark 2.29. We observe that the proof of the previous proposition does not rely on the non negativity of u, v, but only on the fact that $(u_\beta, v_\beta) \to (u, v)$ in $L^\infty(\Omega) \cap H^1_0(\Omega)$, where (u_β, v_β) is a solution of (2.1). In the following we will prove that the conclusions of Proposition 2.27 together with the fact that $(u, v) \in C^{0,\alpha}(\bar{\Omega})$ for every $0 < \alpha < 1$ are sufficient conditions to provide the Lipschitz continuity of (u, v).

Remark 2.30. In Proposition 2.27, the fact that (u, v) is the limit of a sequence of solutions of (2.1) enters essentially at two points. First of all, it was used in order to obtain (2.33) (actually, the validity of this identity justifies *a posteriori* the choice of the quantity $E(x_0, r)$). The second point is that system (2.1) provided an expression for the derivative of the quantity $E(x_0, r)$. Recalling its definition, we observe that only the expression of the gradient part

$$\tilde{E}(x_0, r) := \frac{1}{r^{N-2}} \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2)$$

depends on the equations that u and v solve. In fact, from (2.1) we have obtained (2.34) which yields, as $\beta \to +\infty$,

$$\begin{aligned} \frac{d}{dr}\tilde{E}(x_0, r) &= \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} ((\partial_\nu u)^2 + (\partial_\nu v)^2)\, d\sigma + \\ &+ \frac{2}{r^{N-1}} \int_{B_r(x_0)} (\omega_1 u^3 - \lambda u)\langle \nabla u, x - x_0\rangle + \frac{2}{r^{N-1}} \int_{B_r(x_0)} (\omega_2 v^3 - \mu v)\langle \nabla v, x - x_0\rangle. \end{aligned}$$

This, as we will see in the next chapter, is a rather strong characterization for the limiting profiles (u, v).

Remark 2.31. Going back to the general case of system (2.1) with $m \geqslant 2$ equations, the results of Proposition 2.27 still hold true with the choice

$$E(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \sum_{i=1}^m \Big(|\nabla u_i|^2 - (\omega_i u_i^4 - \lambda_i u_i^2)\Big) \qquad H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} \sum_{i=1}^m u_i^2\, d\sigma.$$

Once again the key estimate is the derivative of

$$\tilde{E}(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \sum_{i=1}^m |\nabla u_i|^2,$$

which in this case can be proved (by using (2.1)) to be given by

$$\frac{d}{dr}\tilde{E}(x_0, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} \sum_{i=1}^m (\partial_\nu u_i)^2\, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m (\omega_i u_i^3 - \lambda_i u_i)\langle \nabla u_i, x - x_0\rangle.$$

The previous lemma immediately provides lower estimates for N in the special case when x_0 is a zero point of the pair (u,v).

Lemma 2.32. *Under the previous notations, for every $x_0 \in \Gamma$ it follows that*

$$N(x_0, 0^+) \geqslant 1.$$

Proof. First of all, the limit exists and is finite because of the monotonicity of the function $e^{Cr^2}(N(x_0,r)+1)$. Let us assume by contradiction that, for some $x_0 \in \Gamma$, $N(x_0,0^+) < 1$. As a consequence by continuity there exist $r^* < \bar{r}$ and $\varepsilon > 0$ such that, for $0 < r < r^*$, we have $N(x_0,r) \leqslant 1-\varepsilon$. Then

$$\frac{d}{dr}\log(H(x_0,r)) = \frac{2}{r}N(x_0,r) \leqslant \frac{2}{r}(1-\varepsilon)$$

which, after an integration between r and r^*, yields

$$\frac{H(x_0,r^*)}{H(x_0,r)} \leqslant \left(\frac{r^*}{r}\right)^{2(1-\varepsilon)}$$

and in particular $Cr^{2(1-\varepsilon)} \leqslant H(x_0,r)$ for some $C = C(x_0,r^*)$. On the other hand, from the fact that u,v are α–Hölder continuous for every $\alpha \in (0,1)$ and $u(x_0) = v(x_0) = 0$ we obtain $H(x_0,r) \leqslant C' r^{2\alpha}$, a contradiction for $\alpha > 1-\varepsilon$ and r small. $\square$

Remark 2.33. Let $\bar{r}$ be as in Proposition 2.27. Since the map $x_0 \mapsto N(x_0,\bar{r})$ is continuous, there exists $C_1 > 0$ such that $N(x_0,\bar{r}) \leqslant C_1$ for every $x_0 \in \tilde{\Omega}$.

Lemma 2.34. *Under the previous notations there exists a constant $C > 0$ such that*

$$\frac{1}{r^{N-1}}\int_{\partial B_r(x_0)} (u^2+v^2)\,d\sigma \leqslant Cr^2 \qquad \text{and} \qquad \frac{1}{r^N}\int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) \leqslant C$$

for every $x_0 \in \bar{\Gamma}$, $0 < r \leqslant \bar{r}$.

Proof. By combining Proposition 2.27 with Lemma 2.32 we see that

$$\begin{aligned}\frac{d}{dr}\log\left(\frac{H(x_0,r)}{r^2}\right) &= \frac{2}{r}(N(x_0,r)-1) = \frac{2}{r}\left((N(x_0,r)+1)e^{Cr^2}e^{-Cr^2} - 2\right)\\ &\geqslant \frac{2}{r}\left((N(x_0,0^+)+1)e^{-Cr^2} - 2\right) \geqslant \frac{4}{r}(e^{-Cr^2}-1).\end{aligned}$$

For every $0 < r < \bar{r}$, we integrate the previous inequality between r and $\bar{r}$, obtaining

$$\log\left(\frac{H(x_0,\bar{r})}{\bar{r}^2}\frac{r^2}{H(x_0,r)}\right) \geqslant \int_r^{\bar{r}} \frac{4}{s}(e^{-Cs^2}-1)\,ds,$$

which yields that

$$\frac{H(x_0,r)}{r^2} \leqslant \frac{H(x_0,\bar{r})}{\bar{r}}\exp\left(\int_0^{\bar{r}}\frac{4}{s}(1-e^{-Cs^2})\,ds\right) \leqslant C'\|(u,v)\|_{L^\infty(\Omega)} \leqslant C,$$

for some constants $C, C' > 0$ not depending on r and x_0. Furthermore, by Lemma 2.26 and Remark 2.33,

$$\begin{aligned}
\frac{1}{r^N}\int_{B_r(x_0)}(|\nabla u|^2+|\nabla v|^2) &\leqslant \frac{2}{r^2}(E(x_0,r)+H(x_0,r)) = \frac{2}{r^2}(N(x_0,r)+1)H(x_0,r)\\
&= \frac{2}{r^2}(N(x_0,r)+1)e^{Cr^2}e^{-Cr^2}H(x_0,r)\\
&\leqslant \frac{2}{r^2}(N(x_0,\bar r)+1)e^{C\bar r^2}H(x_0,r)\\
&\leqslant C''\frac{H(x_0,r)}{r^2}\leqslant C.
\end{aligned}$$

□

Before we pass to the proof of the Lipschitz continuity of u, v, we present one simple lemma. We postpone its proof to Appendix C (see Lemma C.2).

Lemma 2.35. *Let $u \in C^2(\Omega)$ satisfy $-\Delta u \leqslant au$ for some $a > 0$. Then for any ball $B_R(x_0) \Subset \Omega$ we have*

$$u(x_0) \leqslant \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u + \frac{a}{2(N+2)}R^2\|u\|_{L^\infty(B_R(x_0))}.$$

Proof of Proposition 2.25. Before we start, recall that since $\tilde\Omega$ is a regular domain, the Lipschitz continuity of (u, v) is equivalent to having $|\nabla u|, |\nabla v| \in L^\infty(\tilde\Omega)$. Moreover, we know that

$$\lim_{r\to 0}\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}(|\nabla u|^2+|\nabla v|^2) = |\nabla u(x_0)|^2+|\nabla v(x_0)|^2$$

for almost every $x_0 \in \tilde\Omega$ (because $|\nabla u|^2, |\nabla v|^2 \in L^1(\tilde\Omega)$). Hence, if we suppose by contradiction that (u, v) is not Lipschitz continuous in $\tilde\Omega$, then we deduce the existence of $x_n \in \tilde\Omega$ and $r_n \to 0$ such that

$$\lim_{n\to+\infty}\frac{1}{r_n^N}\int_{B_{r_n}(x_n)}(|\nabla u|^2+|\nabla v|^2) = +\infty. \tag{2.35}$$

Up to subsequences, there are three possibilities.

Case 1. $d(x_n, \Gamma)/r_n \leqslant K$ for some constant $K > 0$.

In this case, let $y_n \in \overline{\Gamma}$ be such that $|y_n - x_n| = d(x_n, \Gamma)$, and let us define $s_n = r_n + d(x_n, \Gamma)$, in such a way that $B_{r_n}(x_n) \subseteq B_{s_n}(y_n)$. Moreover, since $d(x_n, \Gamma) \leqslant Kr_n$, we have that $s_n \leqslant (K+1)r_n$. It follows from (2.35) that

$$\frac{1}{s_n^N}\int_{B_{s_n}(y_n)}(|\nabla u|^2+|\nabla v|^2) \geqslant \frac{1}{(K+1)^N r_n^N}\int_{B_{r_n}(x_n)}(|\nabla u|^2+|\nabla v|^2) \to +\infty,$$

which is in contradiction with Lemma 2.34.

Case 2. $d(x_n, \Gamma) \geqslant \gamma$ for some $\gamma > 0$. We observe that in $B_\gamma(x_n)$ only one density (say u) is non trivial and

$$-\Delta u = w_1 u^3 - \lambda u \qquad \text{in } B_\gamma(x_n), \text{ for every } n.$$

Fix $q > N$. The Calderón-Zygmund inequality together with the Sobolev embedding $W^{2,q} \hookrightarrow C^{0,1}$ provide the existence of a constant $C > 0$ independent of n such that

$$\begin{aligned} [u]_{C^{0,1}(\bar{B}_{\gamma/2}(x_n))} &\leqslant C\left(\|u\|_{L^q(B_\gamma(x_n))} + \|w_1 u^3 - \lambda u\|_{L^q(B_\gamma(x_n))}\right) \\ &\leqslant C'\gamma^{N/q}(\|u\|_{L^\infty(\Omega)} + \|u\|^3_{L^\infty(\Omega)}) \leqslant C'', \end{aligned}$$

which is in contradiction with (2.35) for n large so that $B_{r_n}(x_n) \subseteq B_{\gamma/2}(x_n)$.
Case 3. $d(x_n, \Gamma) \to 0$ and $d(x_n, \Gamma)/r_n \to +\infty$.

Once again we denote $R_n = d(x_n, \Gamma)/2 \to 0$ and observe that in $B_{R_n}(x_n)$ only one density (say u) is non trivial and

$$-\Delta u = w_1 u^3 - \lambda u \qquad \text{in } B_{R_n}(x_n), \text{ for every } n.$$

Proceeding exactly as in Case 2, for each $q > N$ we get a constant $C > 0$ independent of n such that

$$\begin{aligned} [u]_{C^{0,1}(\bar{B}_{R_n/2}(x_n))} &\leqslant CR_n^{-1}\left(R_n^{-N/q}\|u\|_{L^q(B_{R_n}(x_n))} + R_n^{2-N/q}\|w_1 u^3 - \lambda u\|_{L^q(B_{R_n}(x_n))}\right) \\ &\leqslant C' R_n^{-1}\left(\|u\|_{L^\infty(B_{R_n}(x_n))} + R_n^2\right). \end{aligned}$$

We claim the existence of $C > 0$ such that $\|u\|_{L^\infty(B_{R_n}(x_n))} \leqslant CR_n$ for every n, which ends the proof of this case since $B_{r_n}(x_n) \subseteq B_{R_n/2}(x_n)$ for large n. First of all observe that in Lemma 2.34 we showed the existence of $C, \bar{r} > 0$ such that

$$H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} (u^2 + v^2)\, d\sigma \leqslant Cr^2 \qquad \text{for every } x_0 \in \bar{\Gamma} \text{ and } 0 < r < \bar{r},$$

which implies in particular that

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (u^2 + v^2) \leqslant C' r^2 \qquad \text{for every } x_0 \in \bar{\Gamma} \text{ and } 0 < r < \bar{r}.$$

Now, take an arbitrary sequence $y_n \in B_{R_n}(x_n)$ and denote $s_n = d(y_n, \Gamma)/2$. Take $w_n \in \Gamma$ such that $d(y_n, \Gamma) = |y_n - w_n|$. Observing that $s_n \leqslant d(y_n, x_n)/2 + d(x_n, \Gamma)/2 \leqslant 3R_n/2$ and using Lemmas 2.34 and 2.35 we have, for n large enough so that $9R_n/2 \leqslant \bar{r}$,

$$\begin{aligned} u^2(y_n) &\leqslant \frac{1}{|B_{s_n}|} \int_{B_{s_n}(y_n)} u^2 + Cs_n^2 \\ &\leqslant \frac{C}{|B_{3s_n}|} \int_{B_{3s_n}(w_n)} (u^2 + v^2) + cs_n^2 \leqslant C' s_n^2 \leqslant C'' R_n^2 \end{aligned}$$

for some constant C'' that does not depend on y_n. Hence $\|u\|^2_{L^\infty(B_{R_n}(x_n))} \leqslant C'' R_n^2$, which proves the claim and ends the proof. □

Remark 2.36. Following [64], one can see that all the Almgren–type formulae can in fact be proved in a more general setting, that is when the Laplace operator is replaced with uniformly elliptic operators of the type

$$-Lu = -\operatorname{div}(A(x)\nabla u),$$

where A is smooth (at least C^1). The key ingredient is to replace the usual polar coordinates with coordinates which are polar with respect to the geodesic distance associated to A. Of course the energies in the Almgren's quotient must be defined in a suitable way. We refer to [64] for further details.

Remark 2.37. Once suitable Almgren's formulae are settled as in the previous remark, one can treat the Lipschitz continuity of u and v up to $\partial\Omega$ in the following way: with a local change of coordinates, and hence changing the differential operator, it is possible to assume that $\partial\Omega$ is locally a hyperplane, and reflect u and v with respect to this hyperplane. It turns out that we find new functions $\tilde{u}$, $\tilde{v}$ which satisfy a new system of equations, with different differential operators, in a larger domain $\Omega' \Supset \Omega$. We can then prove Lipschitz regularity of $(\tilde{u}, \tilde{v})$, locally in Ω', and deduce Lipschitz regularity of (u, v) in $\overline{\Omega}$.

2.5 Additional comments

We have concluded that every sequence of solution $(U_\beta)_\beta$ of (2.1) that is uniformly bounded in $L^\infty(\Omega)$ converges, up to a subsequence, to some limiting profile $U = (u_1, \dots, u_m)$. We have deduced that U is Lipschitz continuous, segregation occurs ($u_i \cdot u_j \equiv 0$ for $i \neq j$) and each component u_i satisfies

$$-\Delta u_i + \lambda_i u_i = \omega_i u_i^3 \qquad \text{in } \{u_i > 0\}, \quad i = 1, \dots, m.$$

The next natural question to study is the regularity of the free boundary

$$\Gamma_U = \{x \in \Omega : \ u_i(x) = 0 \ \forall i\}.$$

What is the regularity of such set, and how do the different components u_i interact through it? We answer these questions in the next chapter. To do so, we will use the following additional property (recall Remark 2.30): for

$$\tilde{E}(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \sum_{i=1}^m |\nabla u_i|^2,$$

we have

$$\frac{d}{dr}\tilde{E}(x_0, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} \sum_{i=1}^m (\partial_\nu u_i)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m (\omega_i u_i^3 - \lambda_i u_i) \langle \nabla u_i, x - x_0 \rangle.$$

These assumptions are verified not only by the limiting profiles of (2.1), but appear as well in many other situations (*cf.* Section 3.7 ahead). It turns out that these properties alone provide good regularity results for Γ_U. For this reason, in the next chapter we will approach these problems from a general point of view.

Chapter 3

Regularity of the nodal set of segregated critical configurations under a weak reflection law

3.1 Introduction

3.1.1 Statement of the results

Let Ω be an open bounded subset of $\mathbb{R}^N$, with $N \geqslant 2$. Our main interest in this chapter is the study of the regularity of the nodal set $\Gamma_U = \{x \in \Omega : U(x) = 0\}$ of segregated configurations $U = (u_1, \ldots, u_m) \in (H^1(\Omega))^m$ associated with systems of semilinear elliptic equations. The main result of this chapter is the following.

Theorem 3.1. *Let $U = (u_1, \ldots, u_m) \in (H^1(\Omega))^m$ be a vector of nonnegative Lipschitz functions in Ω, having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$. Assume that $U \not\equiv 0$ and*

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i > 0\ , \quad i = 1, \ldots, m,$$

where $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are C^1 functions such that $f_i(x, s) = O(s)$ as $s \to 0$, uniformly in x. Moreover, defining for every $x_0 \in \Omega$ and $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$ the energy

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2\ ,$$

then $\tilde{E}(x_0, U, \cdot)$ is an absolutely continuous function of r, and we assume that it satisfies the following differential equation

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2\, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^{m} f_i(x, u_i)\langle \nabla u_i, x - x_0\rangle.$$

Let us consider the nodal set $\Gamma_U = \{x \in \Omega : U(x) = 0\}$. Then we have[1] $\mathscr{H}_{dim}(\Gamma_U) \leqslant N - 1$. Moreover there exists a set $\Sigma_U \subseteq \Gamma_U$, relatively open in Γ_U, such that

[1] Here $\mathscr{H}_{\dim}(\cdot)$ denotes the Hausdorff dimension of a set, whose meaning can be found in Definition A.24. We refer the reader to Appendix A for other definitions and results concerning measures.

- *$\mathscr{H}_{\dim}(\Gamma_U \setminus \Sigma_U) \leqslant N-2$, and if $N=2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- *Σ_U is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0<\alpha<1$). Furthermore for every $x_0 \in \Sigma_U$*
$$\lim_{x\to x_0^+} |\nabla U(x)| = \lim_{x\to x_0^-} |\nabla U(x)| \neq 0, \tag{3.1}$$

where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface. Finally,
$$\lim_{x\to x_0} |\nabla U(x)| = 0 \qquad \text{for every } x_0 \in \Gamma_U \setminus \Sigma_U. \tag{3.2}$$

From the fact that in dimension $N=2$ we know that the singular set $\Gamma_U \setminus \Sigma_U$ is locally finite, in the planar case we obtain the following additional result.

Theorem 3.2. *Under the previous assumptions, let $N=2$. Then for every $x_0 \in \Gamma_U \setminus \Sigma_U$ there exists $h \in \mathbb{N}$ and $\theta_0 \in (-\pi,\pi]$ such that*
$$\sum_{i=1}^m u_i = r^{h/2} \left| \cos\left(\frac{h}{2}(\theta+\theta_0)\right)\right| + \mathrm{o}(r^{h/2}) \qquad \text{as } r \to 0,$$
where (r,θ) denotes the polar coordinates centered at x_0. Thus Σ_U consists of a locally finite collection of curves meeting with equal angles at singular points.

To proceed with, it is convenient to group the vector functions satisfying the assumptions of Theorem 3.1 in the following class.

Definition 3.3. *We define the class $\mathcal{G}(\Omega)$ as the set of functions $U=(u_1,\dots,u_m) \in (H^1(\Omega))^m$, whose components are all nonnegative and Lipschitz continuous in the interior of Ω, and such that $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$. Moreover, $U \not\equiv 0$ and it solves a system of the type*
$$-\Delta u_i = f_i(x,u_i) - \mu_i \qquad \text{in } \mathscr{D}'(\Omega) = (C_c^\infty(\Omega))', \quad i=1,\dots,m, \tag{3.3}$$
where

(G1) $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are C^1 functions such that $f_i(x,s) = O(s)$ when $s \to 0$, uniformly in x;

(G2) $\mu_i \in \mathcal{M}(\Omega) = (C_0(\Omega))'$ are some nonnegative Radon measures, each supported on the nodal set $\Gamma_U = \{x \in \Omega : \ U(x) = 0\}$,

and moreover

(G3) associated to system (3.3), if we define for every $x_0 \in \Omega$ and $r \in (0, \mathrm{dist}(x_0,\partial\Omega))$ the quantity
$$\tilde{E}(r) = \tilde{E}(x_0,U,r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2,$$
then $\tilde{E}(x_0,U,\cdot)$ is an absolutely continuous function of r and
$$\frac{d}{dr}\tilde{E}(x_0,U,r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m f_i(x,u_i)\langle \nabla u_i, x-x_0\rangle. \tag{3.4}$$

To check the equivalence between the two sets of assumptions, we observe that equation (3.3) together with (G2) yields that $-\Delta u_i = f_i(x, u_i)$ over the set $\{u_i > 0\}$. Reciprocally, if such equation holds in $\{u_i > 0\}$ then (3.3) holds in the whole Ω for some measure μ_i concentrated in Γ_U (a proof of this fact will be provided in Lemma 3.39 in a similar situation). We will work from now on with this second formulation of the assumptions.

Notations. For any vector function $U = (u_1, \ldots, u_m)$, we define $\nabla U = (\nabla u_1, \ldots, \nabla u_m)$, $|\nabla U|^2 = |\nabla u_1|^2 + \ldots + |\nabla u_m|^2$, $(\partial_\nu U)^2 = (\partial_\nu u_1)^2 + \ldots + (\partial_\nu u_m)^2$ and $U^2 = u_1^2 + \ldots + u_m^2$. Moreover, $F(x, U) = (f_1(x, u_1), \ldots, f_m(x, u_m))$. We will denote by $\{U > 0\}$ the set $\{x \in \Omega : \ u_i(x) > 0 \text{ for some } i\}$. Hence, with these notations, we have for instance that

$$\langle F(x, U), U\rangle = \sum_{i=1}^{m} f_i(x, u_i)u_i \qquad \text{and} \qquad \langle U, \partial_\nu U\rangle = \sum_{i=1}^{m} u_i(\partial_\nu u_i).$$

Remark 3.4. It is easily checked that equation (3.4) always holds for balls lying entirely inside one of the supports of the components, as a consequence of the elliptic equation (3.3) (see also Subsection 3.1.2). Hence, for our class of systems, (G3) represents the only interaction between the different components u_i through the common boundary of their supports; as we are going to discuss in Subsection 3.1.2 this can be seen as a weak form of a reflection property through the interfaces. Although this hypothesis may look weird and may seem hard to check in applications, as we already seen in the previous chapter, it occurs naturally when U is the limit configuration of solutions of system (2.1). It is our belief that, in general, it occurs in many other situations where the vector U appears as a limit configuration in problems of spatial segregation.

Remark 3.5. Theorem 3.1 applies to the nodal components of solutions to a single semilinear elliptic equation of the form $-\Delta u = f(u)$. Hence, in a sense, our work generalizes [68, 81]. In the paper [31], Caffarelli and Lin proved that the same conclusion of Theorem 3.1 holds for vector functions U minimizing the Lagrangian functional associated with the system. They also proved that equation (3.4) holds for such energy minimizing configurations. At the end of this chapter we show that (3.4) is fullfilled also for strong limits to competition–diffusion systems, both those possessing a variational structure and those with Lotka-Volterra type interactions (see Section 3.7 for some applications of Theorem 3.1). Thus we find that property (G3) is a suitable substitute for the minimization property, and that the class $\mathcal{G}(\Omega)$ is a good replacement for the class $\mathcal{S}(\Omega)$ defined in Chapter 1 (more precisely, equation (3.4) is a good replacement for property (1.14)).

Remark 3.6. Our theorem applies also to sign changing functions u_i, by considering their positive and negative parts. Moreover, we observe that the conclusions of Theorem 3.1 are all of local type. Hence, the conclusions are still valid for the case of an unbounded domain Ω by applying our main theorem to each bounded subset $\Omega' \subset \Omega$.

The approach here differs from the viscosity one proposed by Caffarelli in [27] (which we think does not apply to elements in $\mathcal{G}(\Omega)$) and follows rather the mainstream of [31, 81], based upon a classical dimension reduction principle by Federer. It has the main advantage of avoiding the *a priori* assumption of non degeneracy of the free boundary (which is considered for instance in [2, Section 4]): in contrast, non degeneracy will turn out to hold true on the non singular part of the nodal set as a consequence of the weak reflection principle. Compared with [31], a major difficulty here arises from the fact that we lack the essential information of the minimality of the solution. It should be stressed that the techniques we present here are not mere generalizations of the ones used in [31]: we will use a different approach when proving compactness of the blowup sequences as well as when classifying the conic functions (blowup limits); finally we will exploit an inductive argument on the dimension. This will allow us to extend the results of [31] concerning the asymptotic limits of solutions of systems arising in Bose-Einstein condensation (*cf.* Subsection 3.7.1) to the case of excited state solutions.

3.1.2 Motivations and heuristic considerations

We have already seen in the previous chapter (recall Section 2.5) that the limiting profiles of (2.1) as $\beta \to +\infty$ belong to the class $\mathcal{G}(\Omega)$. In $\mathbb{R}^2$, the functions of the form $r^{m/2}cos(m\theta/2)$ (in polar coordinates) for any integer $m \geqslant 2$ serve also as good prototypes of elements in $\mathcal{G}$. The nodal sets of the latter functions can be divided in two parts: the regular part is a union of curves where a reflection principle holds (the absolute value of the gradient is the same when we approach each curve from opposite sides); the remaining part has small Hausdorff measure (it is a single point). Our aim is to show that this is a general fact, in any space dimension.

More generally, let u be a locally Lipschitz H^1–solution of $-\Delta u = f(x,u)$ in Ω for $f \in C^1(\Omega \times \mathbb{R} \setminus \{0\})$ with $f(x,s) = O(s)$ as $s \to 0$, uniformly in x. For

$$\tilde{E}(r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u|^2 \, dx$$

it holds

$$\tilde{E}'(r) = \frac{2-N}{r^{N-1}} \int_{B_r(x_0)} |\nabla u|^2 + \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} |\nabla u|^2 \, d\sigma \tag{3.5}$$

for almost every r. If we integrate the following Pohǒzaev–type (Rellich) identity in $B_r(x_0)$

$$\operatorname{div}\big((x-x_0)|\nabla u|^2 - 2\langle x-x_0, \nabla u\rangle \nabla u\big) = (N-2)|\nabla u|^2 - 2\langle x-x_0, \nabla u\rangle \Delta u \tag{3.6}$$

then we obtain

$$r\int_{\partial B_r(x_0)} |\nabla u|^2 \, d\sigma = 2r\int_{\partial B_r(x_0)} (\partial_\nu u)^2 \, d\sigma + (N-2)\int_{B_r(x_0)} |\nabla u|^2 + \int_{B_r(x_0)} 2f(x,u)\langle \nabla u, x-x_0\rangle.$$

This, together with (3.5), readily implies that equation (3.4) is verified by $U = (u)$. Hence, if we define $u_1 = u^+$ and $u_2 = u^-$ we deduce that $(u_1, u_2) \in \mathcal{G}(\Omega)$.

In order to better motivate property (G3) and to better understand the information that it contains about the interaction between the different components u_i, let us show what happens in the presence of exactly two components, each satisfying an equation on its support. Suppose $m = 2$ and take $U = (u_1, u_2) \in \mathcal{G}(\Omega)$ such that $\Omega \cap \partial\{u_1 > 0\} = \Omega \cap \partial\{u_2 > 0\} = \Gamma_U$. Assume sufficient regularity in order to perform the following computations (see the proof of Lemma 3.40 and Subsection 3.7.2 for related discussions).

For every point x_0 and radius $r > 0$, take identity (3.6) with $u = u_i$ ($i = 1, 2$) and integrate it in $\{u_i > 0\} \cap B_r(x_0)$. We obtain, for each i,

$$r \int_{\partial B_r(x_0)\cap\{u_i>0\}} |\nabla u_i|^2 \, d\sigma = 2r \int_{\partial B_r(x_0)\cap\{u_i>0\}} (\partial_\nu u_i)^2 \, d\sigma + (N-2) \int_{B_r(x_0)\cap\{u_i>0\}} |\nabla u_i|^2 + \\ + 2 \int_{B_r(x_0)\cap\{u_i>0\}} f_i(x, u_i)\langle \nabla u_i, x - x_0\rangle + \int_{B_r(x_0)\cap\partial\{u_i>0\}} |\nabla u_i|^2 \langle x - x_0, \nu\rangle \, d\sigma.$$

This implies, by summing up the equalities for $i = 1, 2$ and dividing the result by r^{N-1},

$$\frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} |\nabla U|^2 \, d\sigma = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 + \frac{N-2}{r^{N-1}} \int_{B_r(x_0)} |\nabla U|^2 + \\ + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^{2} f_i(x, u_i)\langle \nabla u_i, x - x_0\rangle + \frac{1}{r^{N-1}} \int_{B_r(x_0)\cap\partial\{u_1>0\}} |\nabla u_1|^2 \langle x - x_0, \nu\rangle \, d\sigma + \\ + \frac{1}{r^{N-1}} \int_{B_r(x_0)\cap\partial\{u_2>0\}} |\nabla u_2|^2 \langle x - x_0, \nu\rangle \, d\sigma$$

and

$$\tilde{E}'(r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^{2} f_i(x, u_i)\langle \nabla u_i, x - x_0\rangle + \\ + \frac{1}{r^{N-1}} \int_{B_r(x_0)\cap\partial\{u_1>0\}} |\nabla u_1|^2 \langle x - x_0, \nu\rangle \, d\sigma + \frac{1}{r^{N-1}} \int_{B_r(x_0)\cap\partial\{u_2>0\}} |\nabla u_2|^2 \langle x - x_0, \nu\rangle \, d\sigma. \tag{3.7}$$

for every point x_0 and radius $r > 0$. Hence in this case (G3) holds if and only if the sum of the last two integrals in (3.7) is equal to zero for every x_0, r, that is, $|\nabla u_1| = |\nabla u_2|$ on Γ_U. Thus, in some sense, (G3) is a weak formulation of a reflection principle. Observe moreover that the equality of the gradients over the common boundary yields in particular that

$$-\Delta(u_1 - u_2) = f_1(x, u_1) - f_2(x, u_2),$$

which should be compared with the considerations made for (1.14) in the presence of only two components.

This chapter is organized as follows: in the next section we prove that elements in $\mathcal{G}(\Omega)$ satisfy a modified version of the Almgren's Monotonicity Formula; by exploiting this fact, in Section 3.3 we prove convergence of blowup sequences as well as some closure properties of the class $\mathcal{G}(\Omega)$. In Section 3.4 we use the Federer's Reduction Principle in order to prove some Hausdorff estimates for the nodal sets, define the set Σ_U (recall Theorem 3.1) and prove part of Theorem 3.1 in dimension $N = 2$. In Section 3.5 we prove that, under an appropriate assumption, Σ_U is a hyper-surface satisfying the reflection principle (3.1) and in Section 3.6 we prove by induction in the dimension N that such assumption is indeed satisfied for every $N \geqslant 2$. In Section 3.7 we present some applications of our theory and solve two different problems by showing that their solutions belong to the class $\mathcal{G}(\Omega)$. Finally, in Section 3.8 we make some considerations regarding the connection between the classes $\mathcal{S}(\Omega)$ and $\mathcal{G}(\Omega)$ and present an open problem.

3.2 Preliminaries

The functions belonging to $\mathcal{G}(\Omega)$ have a very rich structure, mainly due to property (G3), which will enable us to prove the validity of the Almgren's Monotonicity Formula (Theorem 3.9 below). With this purpose, just like in Chapter 2, it is more convenient to use a slightly modified version of (G3), including in the definition of the energy also a potential term which takes into consideration system (3.3). We will use this second version from now on.

(G3) Define for every $x_0 \in \Omega$ and $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ the quantity

$$E(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U|^2 - \langle F(x,U), U\rangle\right);$$

then $E(x_0, U, \cdot)$ is an absolutely continuous function on r and

$$\frac{d}{dr} E(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + R(x_0, U, r), \tag{3.8}$$

with

$$\begin{aligned} R(x_0, U, r) = & \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m f_i(x, u_i) \langle \nabla u_i, x - x_0\rangle + \\ & + \frac{1}{r^{N-1}} \int_{B_r(x_0)} (N-2) \langle F(x,U), U\rangle - \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F(x,U), U\rangle \, d\sigma. \end{aligned} \tag{3.9}$$

The two versions of (G3) are clearly equivalent, as the following lemma shows.

Lemma 3.7. *We have*

$$\begin{aligned} & \frac{d}{dr}\Big(\frac{1}{r^{N-2}} \int_{B_r(x_0)} \langle F(x,U), U\rangle\Big) = \\ & \qquad = \frac{1}{r^{N-1}} \int_{B_r(x_0)} (2-N) \langle F(x,U), U\rangle + \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F(x,U), U\rangle \, d\sigma \end{aligned}$$

and hence the two formulations of (G3) - equations (3.4) *and* (3.8) *- are equivalent.*

Proof. For each i, we have

$$\begin{aligned} & \frac{d}{dr}\Big(\frac{1}{r^{N-2}} \int_{B_r(x_0)} f_i(x, u_i) u_i\Big) = \frac{d}{dr}\Big(r^2 \int_{B_1(0)} f_i(x_0 + rx, u_i(x_0+rx)) u_i(x_0+rx)\Big) \\ & = 2r \int_{B_1(0)} f_i(x_0+rx, u_i(x_0+rx)) u_i(x_0+rx) + r^2 \int_{B_1(0)} \langle \nabla \Big(f_i(\cdot, u_i(\cdot)) u_i(\cdot)\Big)(x_0+rx), x\rangle \\ & \qquad = \frac{2}{r^{N-1}} \int_{B_r(x_0)} f_i(x,u_i) u_i + \frac{1}{r^{N-1}} \int_{B_r(x_0)} \langle \nabla (f_i(x,u_i)u_i), x - x_0\rangle \\ & \quad = \frac{2}{r^{N-1}} \int_{B_r(x_0)} f_i(x,u_i)u_i - \frac{N}{r^{N-1}} \int_{B_r(x_0)} f_i(x,u_i)u_i + \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} f_i(x,u_i) u_i \, d\sigma. \end{aligned}$$

By summing up in i the previous identities, we obtain the desired result. $\square$

Furthermore define for every $x_0 \in \Omega$ and $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ the average

$$H(x_0, U, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} U^2 \, d\sigma$$

and, whenever $H(x_0, U, r) \neq 0$, the generalized Almgren's quotient by

$$N(x_0, U, r) = \frac{E(x_0, U, r)}{H(x_0, U, r)}.$$

Remark 3.8. With respect to Chapter 2, in this more general setting we take E, H and N also as functions of U. This is done because ahead it will be important to understand how these quantities change with respect to rescaling, translation and normalization. We would like to stress that these notations are to be used with some caution. In fact, each quantity also depends on the function F that is associated (through system (3.3)) to each $U \in \mathcal{G}(\Omega)$. Although this function is not uniquely determined for any given U, we prefer to omit its reference in the definitions, with some abuse of notations.

The following statements are extensions of some of the results that have been shown in the previous chapter.

Theorem 3.9. *Given $U \in \mathcal{G}(\Omega)$ and $\tilde{\Omega} \Subset \Omega$, there exist $\tilde{C} = \tilde{C}(d, N, \tilde{\Omega}) > 0$ and $\tilde{r} = \tilde{r}(d, N, \tilde{\Omega}) > 0$*[2] *such that for every $x_0 \in \tilde{\Omega}$ and $r \in (0, \tilde{r}]$ we have $H(x_0, U, r) \neq 0$, $N(x_0, U, \cdot)$ is an absolutely continuous function and*

$$\frac{d}{dr} N(x_0, U, r) \geqslant -\tilde{C}(N(x_0, U, r) + 1). \tag{3.10}$$

In particular $e^{\tilde{C}r}(N(x_0, U, r) + 1)$ is a non decreasing function for $r \in (0, \tilde{r}]$ and the limit $N(x_0, U, 0^+) := \lim_{r \to 0^+} N(x_0, U, r)$ exists and is finite. Moreover,

$$\frac{d}{dr} \log(H(x_0, U, r)) = \frac{2}{r} N(x_0, U, r). \tag{3.11}$$

Proof. The proof follows very closely the one of Proposition 2.27. For this reason we only present a sketch of it, stressing however the dependence of $\tilde{C}$ and $\tilde{r}$ on d. Fix $U \in \mathcal{G}(\Omega)$ and take $\tilde{\Omega} \Subset \Omega$. Since $U \not\equiv 0$ in Ω, we can suppose without loss of generality that $U \not\equiv 0$ in $\tilde{\Omega}$.

Observe that since Ω is bounded and U is Lipschitz continuous in Ω, $\|U\|_{L^\infty(\Omega)} < +\infty$. Hence property (G1) provides the upper bound $|f_i(x, u_i)| \leqslant d u_i$ for all $x \in \Omega$ and $i = 1, \ldots, m$, and therefore there exists $C = C(d, N, \tilde{\Omega})$ such that for every $x_0 \in \tilde{\Omega}$ and $0 < r < \mathrm{dist}(\tilde{\Omega}, \partial\Omega)$,

$$\begin{aligned} |R(x_0, U, r)| &\leqslant \frac{2d}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m u_i |\nabla u_i| r + \frac{(N-2)d}{r^{N-1}} \int_{B_r(x_0)} U^2 + \frac{d}{r^{N-2}} \int_{\partial B_r(x_0)} U^2 \, d\sigma \\ &\leqslant C \Big(\frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 + \frac{1}{r^N} \int_{B_r(x_0)} U^2 + \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} U^2 \, d\sigma \Big). \end{aligned}$$

[2] With $d = \max_i \sup_{\substack{0 < s \leqslant \|U\|_{L^\infty(\Omega)} \\ x \in \Omega}} |f_i(x, s)/s|$

Moreover, we have

$$\frac{1}{r^{N-2}}\int_{B_r(x_0)}|\nabla U|^2 = E(x_0,U,r)+\frac{1}{r^{N-2}}\int_{B_r(x_0)}\langle F(x,U),U\rangle \leqslant E(x_0,U,r)+\frac{dr^2}{r^N}\int_{B_r(x_0)}U^2 \tag{3.12}$$

and, by using Poincaré's inequality,

$$\begin{aligned}\frac{1}{r^N}\int_{B_r(x_0)}U^2 &\leqslant \frac{1}{N-1}\Big(\frac{1}{r^{N-2}}\int_{B_r(x_0)}|\nabla U|^2+\frac{1}{r^{N-1}}\int_{\partial B_r(x_0)}U^2\,d\sigma\Big)\\ &\leqslant \frac{1}{N-1}\left(E(x_0,U,r)+H(x_0,U,r)\right)+\frac{r^2C'}{r^N}\int_{B_r(x_0)}U^2.\end{aligned}$$

Thus we obtain the existence of $\bar r < \mathrm{dist}(\tilde\Omega,\partial\Omega)$ such that

$$\frac{1}{r^N}\int_{B_r(x_0)}U^2 \leqslant 2\left(E(x_0,U,r)+H(x_0,U,r)\right) \qquad \text{for every } x_0\in\tilde\Omega,\ 0<r<\bar r, \tag{3.13}$$

which, together with (3.12), yields $|R(x_0,U,r)| \leqslant \tilde C\,(E(x_0,U,r)+H(x_0,U,r))$ for some $\tilde C = \tilde C(d,N,\tilde\Omega) > 0$ and for every $x_0\in\tilde\Omega$, $0<r<\bar r$. The function $r\mapsto H(x_0,U,r)$ is absolutely continuous and for almost every $r>0$

$$\frac{d}{dr}H(x_0,U,r) = \frac{2}{r^{N-1}}\int_{\partial B_r(x_0)}\langle U,\partial_\nu U\rangle\,d\sigma$$

(to check it, use a sequence of smooth functions approximating U). Moreover if we multiply system (3.3) by U, integrate by parts in $B_r(x_0)$ and take into account property (G2) we can rewrite E as

$$E(x_0,U,r) = \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}\langle U,\partial_\nu U\rangle\,d\sigma.$$

Thus, by performing a direct computation, identity (3.11) holds true whenever $H(x_0,U,r) > 0$ for $r<\bar r$, as well as

$$\frac{d}{dr}N(x_0,U,r) \geqslant \frac{R(x_0,U,r)}{H(x_0,U,r)} \geqslant -\tilde C\frac{E(x_0,U,r)+H(x_0,U,r)}{H(x_0,U,r)},$$

which provides (3.10).

The only thing left to prove is that $H(x_0,U,r)>0$ for every $x\in\tilde\Omega$ and small $r>0$. Now, since $H(r) = H(x_0,U,r)$ solves the equation $H'(r) = a(r)H(r)$ with $a(r) = 2N(r)/r$, one can prove that Γ_U has an empty interior. Next, take $\tilde r<\bar r$ such that

$$-\Delta u_i \leqslant f_i(x,u_i) \leqslant du_i \leqslant \lambda_1(B_{\tilde r})u_i \tag{3.14}$$

for all i (where λ_1 denotes the first eigenvalue of $-\Delta$ in H^1_0). If there were $x_0\in\tilde\Omega$ and $0<r<\tilde r$ such that $H(x_0,U,r)=0$, then by multiplying inequality (3.14) by u_i and integrating by parts in $B_r(x_0)$ we would obtain $U\equiv 0$ in $B_r(x_0)$, a contradiction. Hence $H(x_0,U,r)>0$ whenever $x_0\in\tilde\Omega$, $0<r<\tilde r$. □

Remark 3.10. At this point we would like to stress that the hypotheses in $\mathcal{G}(\Omega)$ can be weakened. In Proposition 2.25, by making use of the previous Almgren's Monotonicity Formula, it is shown that if in $\mathcal{G}(\Omega)$ we replace the Lipschitz continuity assumption with α–Hölder continuity for every $\alpha \in (0,1)$, then actually each element $U \in \mathcal{G}(\Omega)$ is Lipschitz continuous (recall moreover Remark 2.29).

Remark 3.11. If $U \in \mathcal{G}(\Omega)$ has as associated function $F \equiv 0$, then $R(x_0, U, r) \equiv 0$ and by repeating the previous procedure we conclude that in this case $N(x_0, U, r)$ is actually a non decreasing function.

Remark 3.12. As observed in the above proof, Γ_U has an empty interior whenever $U \in \mathcal{G}(\Omega)$.

Another simple consequence of the monotonicity result is the following comparison property (which, with $r_2 = 2r_1$, is the so called *doubling property*).

Corollary 3.13. *Given $U \in \mathcal{G}(\Omega)$ and $\tilde{\Omega} \Subset \Omega$, there exist $\tilde{C} > 0$ and $\tilde{r} > 0$ such that*

$$H(x_0, U, r_2) \leqslant H(x_0, U, r_1) \left(\frac{r_2}{r_1}\right)^{2\tilde{C}}$$

for every $x_0 \in \tilde{\Omega}$, $0 < r_1 < r_2 \leqslant \tilde{r}$.

Proof. For each U and $\tilde{\Omega}$ fixed, let $\tilde{C}$ and $\tilde{r}$ be the associated constants according to the previous theorem. Let also $C := \sup_{x_0 \in \tilde{\Omega}} |N(x_0, U, \tilde{r})| < \infty$. Then

$$\begin{aligned}
\frac{d}{dr} \log\left(H(x_0, U, r)\right) &= \frac{2}{r} N(x_0, U, r) = \frac{2}{r}\left((N(x_0, U, r) + 1)e^{\tilde{C}r} e^{-\tilde{C}r} - 1\right) \\
&\leqslant \frac{2}{r}\left((N(x_0, U, \tilde{r}) + 1)e^{\tilde{C}\tilde{r}} e^{-\tilde{C}r} - 1\right) \\
&\leqslant \frac{2}{r}\left((C + 1)e^{\tilde{C}\tilde{r}} - 1\right) =: \frac{2\bar{C}}{r}
\end{aligned}$$

for every $0 < r < \tilde{r}$. Now we integrate between r_1 and r_2, $0 < r_1 < r_2 \leqslant \tilde{r}$, obtaining

$$\frac{H(x_0, U, r_2)}{H(x_0, U, r_1)} \leqslant \left(\frac{r_2}{r_1}\right)^{2\bar{C}},$$

as desired. □

Corollary 3.14. *For any $U \in \mathcal{G}(\Omega)$ and $x_0 \in \Gamma_U$, we have $N(x_0, U, 0^+) \geqslant 1$.*

Proof. Suppose that the conclusion is false. Since the limit $N(x_0, U, 0^+)$ exists, we obtain the existence of $\bar{r}$ and ε such that $N(x_0, U, r) \leqslant 1 - \varepsilon$ for all $0 \leqslant r \leqslant \bar{r}$. By Theorem 3.9 we have that in this interval (by possibly replacing $\bar{r}$ with a smaller radius)

$$\frac{d}{dr} \log(H(x_0, U, r)) \leqslant \frac{2}{r}(1 - \varepsilon).$$

Integrating this inequality between r and $\bar{r}$ ($r < \bar{r}$) yields

$$\frac{H(x_0, U, \bar{r})}{H(x_0, U, r)} \leqslant \left(\frac{\bar{r}}{r}\right)^{2(1-\varepsilon)}$$

which, together with the fact that U is a Lipschitz continuous function at x_0 and that $U(x_0) = 0$, implies

$$Cr^{2(1-\varepsilon)} \leqslant H(x_0, U, r) \leqslant C' r^2,$$

a contradiction for small r. □

Corollary 3.15. *The map $\Omega \to [1, +\infty)$, $x_0 \mapsto N(x_0, U, 0^+)$ is upper semi-continuous.*

Proof. Take a sequence $x_n \to x$ in Ω. By Theorem 3.9 there exists a constant $C > 0$ such that for small $r > 0$

$$N(x_n, U, r) = (N(x_n, U, r) + 1)e^{Cr}e^{-Cr} - 1 \geqslant (N(x_n, U, 0^+) + 1)e^{-Cr} - 1.$$

By taking the limit superior in n and afterwards the limit as $r \to 0^+$ we obtain

$$N(x, U, 0^+) \geqslant \limsup_n N(x_n, U, 0^+).$$

□

3.3 Compactness of blowup sequences

All techniques presented in this chapter involve a local analysis of the solutions, which will be performed via a blowup procedure for each fixed $U \in \mathcal{G}(\Omega)$. Therefore in this section we start with the study of the behavior of the class $G(\Omega)$ under rescaling, which will be followed by a convergence result for blowup sequences. This will be a key tool in the subsequent arguments.

Fix $U \in \mathcal{G}(\Omega)$ and let f_i, μ_i be associated functions and measures (respectively) in the sense of Definition 3.3 (*i.e.*, such that (3.3) holds). For every fixed $\rho, t > 0$ and $x_0 \in \Omega$ define the rescaled function

$$V(x) = \frac{1}{\rho} U_{x_0,t}(x) = \frac{U(x_0 + tx)}{\rho}, \qquad \text{for } x \in \Lambda := \frac{\Omega - x_0}{t}.$$

It is straightforward to check that V solves the system

$$-\Delta v_i = g_i(x, v_i) - \lambda_i, \qquad \text{in } \mathscr{D}'(\Lambda), \quad i = 1, \ldots, m, \tag{3.15}$$

where

$$g_i(x, s) = \frac{t^2}{\rho} f_i(x_0 + tx, \rho s) \quad \text{and} \quad \lambda_i(E) := \frac{1}{\rho t^{N-2}} \mu_i(x_0 + tE) \text{ for every Borel set } E \text{ of } \Lambda.$$

Indeed, for any given $\varphi \in \mathscr{D}(\Lambda)$,

$$\begin{aligned}
&\int_\Lambda (\langle \nabla v_i, \nabla\varphi\rangle - g_i(x,v_i)\varphi) + \int_\Lambda \varphi\, d\lambda_i = \\
&= \int_\Lambda \left(\frac{t}{\rho}\langle \nabla u_i(x_0+tx), \nabla\varphi\rangle - \frac{t^2}{\rho} f_i(x_0+tx, u_i(x_0+tx))\varphi\right) dx + \frac{1}{\rho t^{N-2}}\int_\Lambda \varphi(x)\, d\mu_i(x_0+t\cdot) \\
&\qquad = \frac{1}{\rho t^{N-2}}\int_\Omega \Big[\langle \nabla u_i, \nabla\Big(\varphi\Big(\frac{x-x_0}{t}\Big)\Big)\rangle - f_i(x,u_i)\varphi\Big(\frac{x-x_0}{t}\Big)\Big]\, dx + \\
&\qquad\qquad + \frac{1}{\rho t^{N-2}}\int_\Omega \varphi\Big(\frac{x-x_0}{t}\Big)\, d\mu_i(x) = 0.
\end{aligned}$$

In this setting, for any $y_0 \in \Lambda$ and $r \in (0, \operatorname{dist}(y_0, \partial\Lambda))$, by definition we have that

$$E(y_0, V, r) = \frac{1}{r^{N-2}}\int_{B_r(y_0)} \left(|\nabla V|^2 - \langle G(x,V), V\rangle\right),$$

and the following identities hold:

$$E(y_0,V,r) = \frac{1}{\rho^2}E(x_0+ty_0, U, tr), \qquad H(y_0,V,r) = \frac{1}{\rho^2}H(x_0+ty_0,U,tr), \tag{3.16}$$

and hence

$$N(y_0,V,r) = N(x_0+ty_0,U,tr). \tag{3.17}$$

Moreover,

Proposition 3.16. *With the previous notations, $V \in \mathcal{G}(\Lambda)$.*

Proof. At this point the only thing left to prove is property (G3). In order to check its validity, just observe that by using (3.16) and by performing a change of variables of the form $x = x_0 + ty$, we obtain

$$\begin{aligned}
\frac{d}{dr}E(y_0,V,r) &= \frac{d}{dr}\left(\frac{1}{\rho^2}E(x_0+ty_0,U,tr)\right) = \frac{t}{\rho^2}\frac{dE}{dr}(x_0+ty_0,U,tr) \\
&= \frac{2t}{\rho^2(tr)^{N-2}}\int_{\partial B_{tr}(x_0+ty_0)} (\partial_\nu U)^2\, d\sigma + \frac{t}{\rho^2}R(x_0+ty_0,U,tr) \\
&= \frac{2}{r^{N-2}}\int_{\partial B_r(y_0)} (\partial_\nu V)^2 + \frac{t}{\rho^2}R(x_0+ty_0,U,tr),
\end{aligned}$$

and

$$\begin{aligned}
\frac{t}{\rho^2}R(x_0+ty_0,U,tr) &= \frac{2t}{\rho^2(tr)^{N-1}}\int_{B_{tr}(x_0+ty_0)} \sum_{i=1}^m f_i(x,u_i)\langle \nabla u_i, x-(x_0+ty_0)\rangle + \\
&+ \frac{t}{\rho^2(tr)^{N-1}}\int_{B_{tr}(x_0+ty_0)} (N-2)\langle F(x,U),U\rangle - \frac{t}{\rho^2(tr)^{N-2}}\int_{\partial B_{tr}(x_0+ty_0)} \langle F(x,U),U\rangle\, d\sigma \\
&= \frac{2}{r^{N-1}}\int_{B_r(y_0)} \sum_{i=1}^m g_i(x,v_i)\langle \nabla v_i, x-y_0\rangle + \frac{1}{r^{N-1}}\int_{B_r(y_0)} (N-2)\langle G(x,V),V\rangle - \\
&- \frac{1}{r^{N-2}}\int_{\partial B_r(y_0)} \langle G(x,V),V\rangle\, d\sigma \\
&= R(y_0,V,r).
\end{aligned}$$

□

Next we turn our attention to the convergence of blowup sequences. Let $\tilde{\Omega} \Subset \Omega$ and take some sequences $x_k \in \tilde{\Omega}$, $t_k \downarrow 0$. We define a blowup sequence by

$$U_k(x) = \frac{U(x_k + t_k x)}{\rho_k}, \qquad \text{for } x \in \frac{\Omega - x_k}{t_k},$$

with

$$\rho_k^2 = \|U(x_k + t_k \cdot)\|^2_{L^2(\partial B_1(0))} = \frac{1}{t_k^{N-1}} \int_{\partial B_{t_k}(x_k)} U^2 \, d\sigma = H(x_k, U, t_k).$$

We observe that $\|U_k\|_{L^2(\partial B_1(0))} = 1$ and, by the previous computations,

$$U_k \in \mathcal{G}((\Omega - x_k)/t_k)$$

and

$$-\Delta u_{i,k} = f_{i,k}(x, u_{i,k}) - \mu_{i,k}, \tag{3.18}$$

with

$$f_{i,k}(s) = \frac{t_k^2}{\rho_k} f_i(x_k + t_k x, \rho_k s), \qquad \mu_{i,k}(E) = \frac{1}{\rho_k t_k^{N-2}} \mu_i(x_k + t_k E).$$

We observe moreover that $(\Omega - x_k)/t_k$ approaches $\mathbb{R}^N$ as $k \to +\infty$ because $\operatorname{dist}(x_k, \partial\Omega) \geqslant \operatorname{dist}(\tilde{\Omega}, \partial\Omega) > 0$ for every k. In order to simplify the upcoming statements, we introduce the following auxiliary class of functions.

Definition 3.17. *We say that $U \in \mathcal{G}_{\text{loc}}(\mathbb{R}^N)$ if $U \in \mathcal{G}(B_R(0))$ for every $R > 0$.*

In the remaining part of this section we will prove the following convergence result and present some of its main consequences.

Theorem 3.18. *Under the previous notations there exists a function $\bar{U} \in \mathcal{G}_{\text{loc}}(\mathbb{R}^N)$ such that, up to a subsequence, $U_k \to \bar{U}$ in $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for every $0 < \alpha < 1$ and strongly in $H^1_{\text{loc}}(\mathbb{R}^N)$. More precisely there exist $\bar{\mu}_i \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$, concentrated on $\Gamma_{\bar{U}}$, such that $\mu_{i,k} \rightharpoonup \bar{\mu}_i$ weakly in $\mathcal{M}_{\text{loc}}(\mathbb{R}^N)$, $\bar{U}$ solves*

$$-\Delta \bar{u}_i = -\bar{\mu}_i \qquad \text{in } \mathscr{D}'(\mathbb{R}^N) \tag{3.19}$$

and, for

$$E(x_0, \bar{U}, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla \bar{U}|^2 \qquad (\text{the energy associated with } (3.19)),$$

we have that

$$\frac{d}{dr} E(x_0, \bar{U}, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu \bar{U})^2 \, d\sigma \qquad \text{for a.e. } r > 0 \text{ and every } x_0 \in \mathbb{R}^N. \tag{3.20}$$

The proof will be presented in a series of lemmas.

Lemma 3.19. *There exists $\tilde{r} > 0$ such that for every $0 < r < \tilde{r}$ and $x_0 \in \tilde{\Omega}$ we have*

$$\frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 + \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} U^2 \, d\sigma \leqslant 2(E(x_0, U, r) + H(x_0, U, r)).$$

Proof. This result is a direct consequence of inequalities (3.12) and (3.13). □

Lemma 3.20. *For any given $R>0$ we have $\|U_k\|_{H^1(B_R(0))} \leqslant C$, independently of k.*

Proof. Let $\tilde{C}$ and $\tilde{r}$ be constants such that Theorem 3.9, Corollary 3.13 and Lemma 3.19 hold for the previously fixed domain $\tilde{\Omega}$. We have, after taking k so large that $t_k, t_k R \leqslant \tilde{r}$,

$$\begin{aligned}\int_{\partial B_R(0)} U_k^2\, d\sigma &= \frac{1}{\rho_k^2}\int_{\partial B_R(0)} U^2(x_k+t_kx)\, d\sigma = \frac{1}{\rho_k^2 t_k^{N-1}}\int_{\partial B_{t_kR}(x_k)} U^2\, d\sigma \\ &= R^{N-1}\frac{H(x_k,U,t_kR)}{H(x_k,U,t_k)} \leqslant R^{N-1}\left(\frac{t_kR}{t_k}\right)^{2\tilde{C}} =: C(R)R^{N-1}\end{aligned}$$

(by Corollary 3.13). Moreover,

$$\begin{aligned}\frac{1}{R^{N-2}}\int_{B_R(0)}|\nabla U_k|^2 &= \frac{H(0,U_k,R)}{H(0,U_k,R)}\frac{1}{R^{N-2}}\int_{B_R(0)}|\nabla U_k|^2 \\ &\leqslant \frac{C(R)}{H(0,U_k,R)}\Big(\frac{1}{R^{N-2}}\int_{B_R(0)}|\nabla U_k|^2 + \frac{1}{R^{N-1}}\int_{\partial B_R(0)} U_k^2\, d\sigma\Big) - C(R) \\ &= \frac{C(R)}{H(x_k,U,t_kR)}\Big(\frac{1}{(t_kR)^{N-2}}\int_{B_{t_kR}(x_k)}|\nabla U|^2 + \frac{1}{(t_kR)^{N-1}}\int_{\partial B_{t_kR}(x_k)} U^2\, d\sigma\Big) - C(R) \\ &\leqslant \frac{2C(R)}{H(x_k,U,t_kR)}\left(E(x_k,U,t_kR)+H(x_k,U,t_kR)\right) - C(R) \\ &= 2C(R)N(x_k,U,t_kR)+C(R) \leqslant 2C(R)(N(x_k,U,\tilde{r})+1)e^{\tilde{C}\tilde{r}} - C(R) \leqslant C'(R),\end{aligned}$$

where we have used identities (3.16), the continuity of the function $x \mapsto N(x,U,\tilde{r})$, as well as Theorem 3.9 and Lemma 3.19. □

Remark 3.21. Since $-\Delta u_{i,k} \leqslant f_{i,k}(x,u_{i,k}) = \frac{t_k^2}{\rho_k} f_i(x_k+t_kx, u_i(x_k+t_kx)) \leqslant d t_k^2 u_{i,k}$ (by property (G1)), then a standard Brezis-Kato type argument together with the H^1_{loc}–boundedness provided by the previous lemma yield that $\|U_k\|_{L^\infty(B_R(0))} \leqslant C(R)$ for every k.

Lemma 3.22. *For any given $R>0$ there exists $C>0$ such that $\|\mu_{i,k}\|_{\mathcal{M}(B_R(0))} = \mu_{i,k}(B_R(0)) \leqslant C$ for every $k\in\mathbb{N}$ and $i=1,\dots,m$.*

Proof. We multiply equation (3.18) by φ, a smooth cut-off function such that $0\leqslant\varphi\leqslant 1$, $\varphi=1$ in $B_R(0)$ and $\varphi=0$ in $\mathbb{R}^N\setminus B_{2R}(0)$. It holds

$$\begin{aligned}\mu_{i,k}(B_R(0)) &\leqslant \int_{B_{2R}(0)}\varphi\, d\mu_{i,k} = -\int_{B_{2R}(0)}\langle\nabla u_{i,k},\nabla\varphi\rangle + \int_{B_{2R}(0)} f_{i,k}(x,u_{i,k})\varphi \\ &\leqslant C(R)\|\nabla u_{i,k}\|_{L^2(B_{2R}(0))} + C(R)\|u_{i,k}\|_{L^\infty(B_{2R}(0))} \leqslant \tilde{C}(R),\end{aligned}$$

by Lemma 3.20 and Remark 3.21. □

So far we have proved the existence of a non trivial function $\bar{U}\in H^1_{\mathrm{loc}}(\mathbb{R}^N)\cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\bar{\mu}_i\in\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$ such that (up to subsequences)

$$U_k \rightharpoonup \bar{U} \qquad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^N),$$

$$\mu_{i,k} \rightharpoonup \bar\mu_i \qquad \text{in } \mathcal{M}_{\rm loc}(\mathbb{R}^N).$$

Moreover since

$$-\Delta u_{i,k} = f_{i,k}(x, u_{i,k}) - \mu_{i,k} \quad \text{ and } \quad \|f_{i,k}(x,u_{i,k})\|_{L^\infty(B_R(0))} \leqslant dt_k^2 \|u_{i,k}\|_{L^\infty(B_R(0))} \to 0,$$

then

$$-\Delta \bar u_i = -\bar\mu_i \qquad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

The next step is to prove that the convergence $U_k \to \bar U$ is indeed strong in $H^1_{\rm loc}$ and in $C^{0,\alpha}_{\rm loc}$ (see Lemmas 3.25 and 3.26 ahead). These facts will come out as a byproduct of some uniform Lipschitz estimates.

Lemma 3.23. *Fix $R > 0$. Then there exist constants $C, \bar r, \bar k > 0$ such that for $k \geqslant \bar k$ we have*

$$H(x, U_k, r) \leqslant Cr^2$$

for $0 < r < \bar r$ and $x \in B_{2R}(0) \cap \Gamma_{U_k}$.

Proof. We recall that $U_k \in \mathcal{G}(B_{3R}(0))$ for k large and apply Theorem 3.9 to the subset $B_{2R}(0) \Subset B_{3R}(0)$. First of all observe that for $0 < s \leqslant \|U_k\|_{L^\infty(B_{3R}(0))}$ it holds $\rho_k s \leqslant \|U(x_k + t_k\cdot)\|_{L^\infty(B_{3R}(0))} \leqslant C'(R)$ (*cf.* Remark 3.21) and hence by taking into account property (G1) we obtain the existence of $\bar k > 0$ such that

$$\max_i \sup_{\substack{0<s\leqslant \|U_k\|_{L^\infty(B_{3R}(0))} \\ x\in B_{3R}(0)}} |f_{i,k}(x,s)/s| \leqslant \max_i \sup_{\substack{0<\rho_k s\leqslant C'(R) \\ x\in B_{3R}(0)}} t_k^2 |f_i(x_k + t_k x, \rho_k s)/(\rho_k s)| \leqslant 1$$

for $k \geqslant \bar k$. Therefore there exist $\bar C, \bar r > 0$ independent of k such that the function $r \mapsto (N(x, U_k, r) + 1)e^{\bar C r}$ is non decreasing for $x \in B_{2R}(0)$ and $0 < r < \bar r$. If we suppose moreover that $x \in \Gamma_{U_k}$ then Corollary 3.14 yields

$$\frac{d}{dr}\log\left(\frac{H(x,U_k,r)}{r^2}\right) = \frac{2}{r}(N(x,U_k,r) - 1) = \frac{2}{r}((N(x,U_k,r)+1)e^{\bar C r}e^{-\bar C r} - 2) \geqslant \frac{4}{r}(e^{-\bar C r} - 1),$$

which implies (after an integration)

$$\frac{H(x,U_k,r)}{r^2} \leqslant \frac{H(x,U_k,\bar r)}{\bar r^2}\exp\left(\int_0^{\bar r}\frac{4}{s}(1 - e^{-\bar C s})\,ds\right) \leqslant C'\|U_k\|^2_{L^\infty(B_{2R+\bar r}(0))} \leqslant C.$$

□

Next we state a technical and general lemma which was already used in the previous chapter (its proof can be found in Appendix C, Lemma C.2).

Lemma 3.24. *Let $u \in C^2(\Omega)$ satisfy $-\Delta u \leqslant au$ for some $a > 0$. Then for any ball $B_R(x_0) \Subset \Omega$ we have*

$$u(x_0) \leqslant \frac{1}{|B_R|}\int_{B_R(x_0)} u + \frac{a}{2(N+2)}R^2\|u\|_{L^\infty(B_R(x_0))}.$$

Now we are in position to prove the $C^{0,1}_{\rm loc}$–boundedness of $(U_k)_k$. This proof contains some of the ideas used to prove Proposition 2.25.

Lemma 3.25. *For every $R > 0$ there exists $C > 0$ (independent of k) such that*

$$\|U_k\|_{C^{0,1}(\bar{B}_R(0))} \leqslant C \qquad \text{for every } k.$$

Proof. Suppose, without loss of generality, that

$$[U_k]_{C^{0,1}(\bar{B}_R(0))} := \max_{i=1,\dots,m} \sup_{\substack{x,y\in\bar{B}_R(0)\\ x\neq y}} \frac{|u_{i,k}(x)-u_{i,k}(y)|}{|x-y|} = \frac{|u_{1,k}(y_k)-u_{1,k}(z_k)|}{|y_k-z_k|}.$$

Define $r_k = |y_k - z_k|$ and suppose that

$$2R_k := \max\{\mathrm{dist}(y_k,\Gamma_{u_{1,k}}), \mathrm{dist}(z_k,\Gamma_{u_{1,k}})\} = \mathrm{dist}(z_k,\Gamma_{u_{1,k}}).$$

We can assume that $\mathrm{dist}(z_k,\Gamma_{u_{1,k}}) > 0$, otherwise $[U_k]_{C^{0,1}(\bar{B}_R(0))} = 0$ and the lemma trivially holds. Moreover, in such a case we obtain that $\mathrm{dist}(z_k,\Gamma_{u_{1,k}}) = \mathrm{dist}(z_k,\Gamma_{U_k})$ because $u_{i,k}\cdot u_{j,k} = 0$ for $i \neq j$.

We divide the proof in several cases. The idea is to treat the problem according to the interaction between y_k, z_k and Γ_{U_k}.

Case 1. $r_k \geqslant \gamma$ for some $\gamma > 0$.

From the L^∞–boundedness of U_k, we see that

$$\frac{|u_{1,k}(y_k)-u_{1,k}(z_k)|}{|y_k-z_k|} \leqslant \frac{2\|U_k\|_{L^\infty(B_R(0))}}{\gamma} \leqslant C.$$

Case 2. $r_k \to 0$ and $R_k \geqslant \gamma$ for some $\gamma > 0$.

Observe that in $B_{R_k}(z_k)$ the function $u_{1,k}$ solves the equation $-\Delta u_{1,k} = f_{1,k}(x,u_{1,k})$. By taking $q > N$ we obtain the existence of $C > 0$ independent of k such that

$$\begin{aligned}[u_{1,k}]_{C^{0,1}(B_{\gamma/2}(z_k))} &\leqslant C\left(\|u_{1,k}\|_{L^q(B_\gamma(z_k))} + \|f_{1,k}(x,u_{1,k})\|_{L^q(B_\gamma(z_k))}\right)\\ &\leqslant C'\gamma^{N/q}\|u_{1,k}\|_{L^\infty(B_\gamma(z_k))} \leqslant C''.\end{aligned}$$

Since $y_k \in B_{\gamma/2}(z_k)$ for large k, then $[u_{1,k}]_{C^{0,1}(\bar{B}_R(0))} \leqslant C$ in this case.

Case 3. $R_k,\ r_k \to 0$ and $R_k/r_k \leqslant C$.

Notice first of all that we can apply Lemma 3.24 to $u_{1,k}^2$ in $B_{R_k}(z_k)$, obtaining

$$u_{1,k}^2(z_k) \leqslant \frac{1}{|B_{R_k}|}\int_{B_{R_k}(z_k)} u_{1,k}^2 + CR_k^2.$$

On the other hand, let $w_k \in \Gamma_{U_k} \cap B_{2R}(0)$ be such that $\mathrm{dist}(z_k,\Gamma_{U_k}) = |z_k - w_k|$. Lemma 3.23 then yields the existence of $C > 0$ and $\bar{r} > 0$ such that for k large

$$H(w_k,U_k,r) \leqslant Cr^2, \qquad \text{which implies} \qquad \frac{1}{|B_r|}\int_{B_r(w_k)} U_k^2 \leqslant C'r^2 \qquad \text{for } r \leqslant \bar{r}.$$

By taking k sufficiently large in such a way that $3R_k \leqslant \bar{r}$, we have

$$u_{1,k}^2(z_k) \leqslant \frac{1}{|B_{R_k}|}\int_{B_{R_k}(z_k)} u_{1,k}^2 + CR_k^2 \leqslant \frac{C_1}{|B_{3R_k}|}\int_{B_{3R_k}(w_k)} U_k^2 + C_1R_k^2 \leqslant C_2R_k^2 \leqslant C_3r_k^2.$$

As for y_k, either $\mathrm{dist}(y_k, \Gamma_{u_{1,k}}) = 0$ (and $u_{1,k}(y_k) = 0$) or $\mathrm{dist}(y_k, \Gamma_{u_{1,k}}) = \mathrm{dist}(y_k, \Gamma_{U_k}) > 0$ and we can apply the same procedure as before (with R_k replaced by $\mathrm{dist}(y_k, \Gamma_{U_k})/2$ - observe that $\mathrm{dist}(y_k, \Gamma_{U_k}) \leqslant 2R_k \to 0$), obtaining $u_{1,k}^2(y_k) \leqslant C\mathrm{dist}^2(y_k, \Gamma_{U_k}) \leqslant C'R_k^2 \leqslant C''r_k^2$. Hence

$$|u_{1,k}(y_k) - u_{1,k}(z_k)|^2 \leqslant Cr_k^2 = C|z_k - y_k|^2.$$

Case 4. $R_k,\ r_k \to 0$ and $R_k/r_k \to +\infty$.

In this case observe that once again if we fix $q > N$ there exists $C > 0$ such that

$$\begin{aligned}
[u_{1,k}]_{C^{0,1}(\bar B_{R_k/2}(z_k))} &\leqslant CR_k^{-1}\left(R_k^{-N/q}\|u_{1,k}\|_{L^q(B_{R_k}(z_k))} + R_k^{2-N/q}\|f_{1,k}(x, u_{1,k})\|_{L^q(B_{R_k}(z_k))}\right)\\
&\leqslant C\left(R_k^{-1}\|u_{1,k}\|_{L^\infty(B_{R_k}(z_k))} + R_k\right).
\end{aligned}$$

Arguing as in Case 3, we prove the existence of $C > 0$ such that for large k and for every $x \in B_{R_k}(z_k)$ it holds $u_{1,k}^2(x) \leqslant C\mathrm{dist}^2(x, \Gamma_{U_k}) \leqslant C'R_k^2$, and thus $[u_{1,k}]_{C^{0,1}(\bar B_{R_k/2}(z_k))} \leqslant C$. Since $y_k \in B_{R_k/2}(z_k)$ for large k, the proof is complete. □

By the compact embeddings $C^{0,1}(B_R(0)) \hookrightarrow C^{0,\alpha}(B_R(0))$ for $0 < \alpha < 1$ we deduce the existence of a converging subsequence $U_k \to \bar U$ in $C^{0,\alpha}_{\mathrm{loc}}$. Now we pass to the proof of the H^1- strong convergence, after which we finish the proof of Theorem 3.18.

Lemma 3.26. *For every $R > 0$ we have (up to a subsequence) that $U_k \to \bar U$ strongly in $H^1(B_R(0))$.*

Proof. We already know that the following equations are satisfied in $\mathscr{D}'(B_{2R}(0))$ (for every $i = 1, \dots, m$):

$$-\Delta u_{i,k} = f_{i,k}(x, u_{i,k}) - \mu_{i,k}, \qquad\qquad -\Delta \bar u_i = -\bar\mu_i.$$

If we subtract the second equation from the first one and multiply the result by $(u_{i,k} - \bar u_i)\varphi$ (where φ is a smooth cut-off function such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $B_R(0)$ and $\varphi = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$), we obtain

$$\begin{aligned}
&\int_{B_{2R}(0)} |\nabla(u_{i,k} - \bar u_i)|^2\varphi + \int_{B_{2R}(0)} \langle\nabla(u_{i,k} - \bar u_i), \nabla\varphi\rangle\,(u_{i,k} - \bar u_i) =\\
&= \int_{B_{2R}(0)} f_{i,k}(x, u_{i,k})(u_{i,k} - \bar u_i)\varphi - \int_{B_{2R}(0)} (u_{i,k} - \bar u_i)\varphi\, d\mu_{i,k} + \int_{B_{2R}(0)} (u_{i,k} - \bar u_i)\varphi\, d\bar\mu_i.
\end{aligned}$$

Now we can conclude by observing that

$$\left|\int_{B_{2R}(0)} \langle\nabla(u_{i,k} - \bar u_i), \nabla\varphi\rangle\,(u_{i,k} - \bar u_i)\right| \leqslant C\|u_{i,k} - \bar u_i\|_{L^\infty(B_{2R}(0))}\|\nabla u_{i,k}\|_{L^2(B_{2R}(0))} \to 0,$$

$$\left|\int_{B_{2R}(0)} f_{i,k}(x, u_{i,k})(u_{i,k} - \bar u_i)\varphi\right| \leqslant C\|u_{i,k}\|_{L^\infty(B_{2R}(0))}\|u_{i,k} - \bar u_i\|_{L^\infty(B_{2R}(0))} \to 0,$$

and

$$\begin{aligned}
&\left|\int_{B_{2R}(0)} -(u_{i,k} - \bar u_i)\varphi\, d\mu_{i,k} + (u_{i,k} - \bar u_i)\varphi\, d\bar\mu_i\right| \leqslant\\
&\qquad\qquad \leqslant \|u_{i,k} - \bar u_i\|_{L^\infty(B_{2R}(0))}\left(\mu_{i,k}(B_{2R}(0)) + \bar\mu_i(B_{2R}(0))\right) \to 0.
\end{aligned}$$

□

End of the proof of Theorem 3.18. After Lemmas 3.19–3.26 the only thing left to prove are the claims that the measures μ_i are concentrated on $\Gamma_{\bar U}$ (for $i = 1, \dots, m$) and that property (G3) holds with $F \equiv 0$.

As for the first statement is concerned we start by fixing $R > 0$ and by considering a smooth cut-off function φ equal to one in $B_R(0)$, zero outside $B_{2R}(0)$. Since

$$\int_{B_{2R}(0)} U_k \varphi \, d\mu_{i,k} = \int_{B_{2R}(0)\cap\Gamma_{U_k}} U_k \varphi \, d\mu_{i,k} = 0,$$

then

$$\begin{aligned} 0 = \lim_k \int_{B_{2R}(0)} U_k \varphi \, d\mu_{i,k} &= \lim_k \int_{B_{2R}(0)} (U_k - \bar U)\varphi \, d\mu_{i,k} + \lim_k \int_{B_{2R}(0)} \bar U \varphi \, d\mu_{i,k} \\ &= \int_{B_{2R}(0)} \bar U \varphi \, d\mu_i. \end{aligned}$$

Thus $\int_{B_R(0)} \bar U \, d\mu_i = 0$ for every $R > 0$ and in particular $\bar\mu_i(K) = 0$ for every compact set $K \subset \mathbb{R}^N \setminus \Gamma_{\bar U}$, which proves the first claim.

As for the proof of the second claim, we recall that $U_k \in \mathcal{G}((\Omega - x_k)/t_k)$ and hence for any given $0 < r_1 < r_2$ the following equality holds

$$E(x_0, U_k, r_2) - E(x_0, U_k, r_1) = \int_{r_1}^{r_2} \Big(\frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U_k)^2 \, d\sigma \Big) \, dr + \int_{r_1}^{r_2} R(x_0, U_k, r) \, dr. \tag{3.21}$$

Since $|\langle F_k(U_k), U_k \rangle| \leqslant d t_k^2 |U_k|^2 \to 0$, we obtain

$$E(x_0, U_k, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \big(|\nabla U_k|^2 - \langle F_k(U_k), U_k \rangle \big) \xrightarrow[k]{} \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla \bar U|^2 = E(x_0, \bar U, r)$$

for each fixed $r > 0$. Moreover,

$$\begin{aligned} \Big| \int_{r_1}^{r_2} R(x_0, U_k, r) \, dr \Big| &\leqslant \Big| \int_{r_1}^{r_2} \Big(\frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m f_i(x, u_{i,k}) \langle \nabla u_{i,k}, x - x_0 \rangle \Big) \, dr \Big| + \\ + \Big| \int_{r_1}^{r_2} \Big(\frac{1}{r^{N-1}} \int_{B_r(x_0)} (N-2) \langle F_k(U_k), U_k \rangle \Big) \, dr \Big| &+ \Big| \int_{r_1}^{r_2} \Big(\frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F_k(U_k), U_k \rangle \, d\sigma \Big) \, dr \Big| \\ &\leqslant C(r_1, r_2) t_k^2 \int_{B_{r_2}(x_0)} \sum_{i=1}^m u_{i,k} |\nabla u_{i,k}| + C(r_1, r_2) t_k^2 \int_{B_{r_2}(x_0)} U_k^2 \to 0. \end{aligned}$$

Finally, the fact that $U_k \to U$ strongly in H^1_{loc} implies that, up to a subsequence of $(U_k)_k$, there exists a function $h(r) \in L^1(r_1, r_2)$ such that $\int_{\partial B_r(x_0)} |\nabla (U_k - \bar U)|^2 \, d\sigma \leqslant h(r)$, and moreover $\int_{\partial B_r(x_0)} |\nabla (U_k - \bar U)|^2 \, d\sigma \to 0$ for a.e. $r \in (r_1, r_2)$. Thus

$$\int_{r_1}^{r_2} \Big(\frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U_k)^2 \, d\sigma \Big) \, dr \to \int_{r_1}^{r_2} \Big(\frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu \bar U)^2 \, d\sigma \Big) \, dr.$$

We can now pass to the limit in (3.21) as $k \to +\infty$, obtaining

$$E(x_0, \bar U, r_2) - E(x_0, \bar U, r_1) = \int_{r_1}^{r_2} \left(\frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu \bar U)^2 \, d\sigma \right) dr,$$

i.e., (G3) holds for $\bar U$ with $F \equiv 0$. □

Up to now we have dealt with blowup sequences with arbitrary moving centers x_k. Next we observe that some particular choices of $(x_k)_k$ provide additional informational on the limit $\bar{U}$. More precisely, we have

Corollary 3.27. *Under the previous notations, suppose that one of these situations occurs:*

1. $x_k = x_0$ *for every* k,
2. $x_k \in \Gamma_U$ *and* $x_k \to x_0 \in \Gamma_U$ *with* $N(x_0, U, 0^+) = 1$.

Then $N(0, \bar{U}, r) = N(x_0, U, 0^+) =: \alpha$ *for every* $r > 0$, *and* $\bar{U} = r^\alpha G(\theta)$, *where* (r, θ) *are the generalized polar coordinates centered at the origin.*

Proof. We divide the proof in three steps.
Step 1. $N(0, \bar{U}, r)$ is constant.

First observe that

$$N(0, U_k, r) = N(x_k, U, t_k r)$$

and that Theorem 3.18 yields

$$\lim_k N(0, U_k, r) = N(0, \bar{U}, r).$$

As for the right hand side, if $x_k = x_0$ for some x_0, then

$$\lim_k N(x_0, U, t_k r) = N(x_0, U, 0^+)$$

for every $r > 0$ by Theorem 3.9. In the second situation ($x_k \in \Gamma_U$ and $x_k \to x_0$ with $N(x_0, U, 0^+) = 1$) we claim that

$$\lim_k N(x_k, U, t_k r) = 1.$$

Indeed, denoting by $\tilde{r}$ the radius associated to $\tilde{\Omega}$ in the context of Theorem 3.9, for any given $\varepsilon > 0$ take $0 < \bar{r} = \bar{r}(\varepsilon) \leqslant \tilde{r}$ such that

$$N(x_0, U, r) \leqslant 1 + \frac{\varepsilon}{2} \qquad \text{for every } 0 < r \leqslant \bar{r}, \qquad \text{and} \qquad e^{\tilde{C}\bar{r}} \leqslant \frac{2 + 2\varepsilon}{2 + \varepsilon}.$$

Moreover there exists $\delta_0 > 0$ such that

$$N(x, U, \bar{r}) \leqslant 1 + \varepsilon \qquad \text{for } x \in B_{\delta_0}(x_0) \subseteq \tilde{\Omega}.$$

Thus, again by Theorem 3.9, we obtain

$$N(x, U, r) \leqslant (2 + \varepsilon) e^{\tilde{C}\bar{r}} - 1 \leqslant 1 + 2\varepsilon, \qquad \text{for every } x \in B_{\delta_0}(x_0),\ 0 < r \leqslant \bar{r},$$

and hence $\limsup_k N(x_k, U, t_k r) \leqslant 1$. On the other hand, by taking into account Corollary 3.14,

$$N(x_k, U, t_k r) = (N(x_k, U, t_k r) + 1) e^{\tilde{C} t_k r} e^{-\tilde{C} t_k r} - 1 \geqslant 2 e^{-\tilde{C} t_k r} - 1,$$

which implies that

$$\liminf_k N(x_k, U, t_k r) \geqslant 1.$$

Step 2. The derivative of N.

An easy computation gives

$$\frac{d}{dr}H(0,\bar{U},r) = \frac{2}{r^{N-1}}\int_{\partial B_r(x_0)} \langle \bar{U}, \partial_\nu \bar{U}\rangle \, d\sigma \qquad \text{for a.e. } r>0$$

which together with identity (3.20) - for $y_0 = 0$ - readily implies

$$0 = \frac{d}{dr}N(0,\bar{U},r) = \frac{2}{r^{2N-3}H^2(0,\bar{U},r)}\left(\int_{\partial B_r(0)} \bar{U}^2 \, d\sigma \int_{\partial B_r(0)} (\partial_\nu \bar{U})^2 \, d\sigma - \right.$$
$$\left. -\left(\int_{\partial B_r(0)} \langle \bar{U}, \partial_\nu \bar{U}\rangle \, d\sigma\right)^2\right)$$

for a.e. $r > 0$.

Step 3. U is homogeneous.

The previous equality yields the existence of $C(r) > 0$ such that $\partial_\nu \bar{U} = C(r)\bar{U}$ for a.e. $r > 0$. By using this information in (3.11) we get

$$2C(r) = \frac{2\int_{\partial B_r(0)} \langle \bar{U}, \partial_\nu \bar{U}\rangle \, d\sigma}{\int_{\partial B_r(0)} \bar{U}^2 \, d\sigma} = \frac{d}{dr}\log(H(0,\bar{U},r)) = \frac{2}{r}N(0,\bar{U},r) = \frac{2}{r}\alpha$$

(by Step 1), and thus $C(r) = \alpha/r$ and $\bar{U}(x) = r^\alpha G(\theta)$. □

3.4 Hausdorff dimension estimates for nodal and singular sets

As we mentioned before, our main interest is the study of the free boundary $\Gamma_U = \{x \in \Omega : U(x) = 0\}$ for every $U \in \mathcal{G}(\Omega)$. As a first step towards its characterization we will provide an estimate of its Hausdorff dimension. Regarding its regularity, we shall decompose Γ_U in two parts:

- the first one - which will be denoted by S_U - where we are not able to prove any kind of regularity result, but which has a "small" Hausdorff dimension,
- the second one - Σ_U - where we are able to prove regularity results (*cf.* Theorem 3.1).

Definition 3.28. *Given $U \in \mathcal{G}(\Omega)$ we define its regular and singular sets respectively by*

$$\Sigma_U = \{x \in \Gamma_U : \ N(x,U,0^+) = 1\}, \quad \textit{and} \quad S_U = \Gamma_U \backslash \Sigma_U = \{x \in \Gamma_U : \ N(x,U,0^+) > 1\}.$$

In the same spirit of [31, Lemma 4.1] we prove that there exists a jump in the possible values of $N(x_0, U, 0^+)$ for $x_0 \in \Gamma_U$ (recall that $N(x_0, U, 0^+) \geqslant 1$ by Corollary 3.14). In [31], the authors deal with solutions of minimal energy, proving directly the existence of a jump in any dimension. In our general framework their strategy does not work; instead, we will obtain the same results via an iteration procedure. In the following proposition we start to prove the existence of a jump in dimension $N = 2$. The extension to higher dimensions will be treated in the subsequent sections.

Proposition 3.29. *Let $N = 2$. Given $U \in \mathcal{G}(\Omega)$ and $x_0 \in \Gamma_U$, then either*

$$N(x_0, U, 0^+) = 1 \qquad \text{or} \qquad N(x_0, U, 0^+) \geqslant 3/2.$$

Proof. We perform a blowup at x_0 by considering $U_k(x) = U(x_0 + t_k x)/\rho_k$, where $\rho_k = \|U(x_0 + t_k \cdot)\|_{L^2(\partial B_1(0))}$ and $t_k \downarrow 0$ is an arbitrary sequence. Theorem 3.18 together with Corollary 3.27 (case 1) yield the existence of $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\alpha = N(x_0, U, 0^+)$ such that (up to a subsequence) $U_k \to \bar{U}$ strongly in $H^1_{\mathrm{loc}} \cap C^{0,\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \beta < 1$. Moreover, each component $\bar{u}_i$ is harmonic in the open set $\{\bar{u}_i > 0\}$, which implies that on every given connected component $A \subseteq \{g_i > 0\} \subseteq \partial B_1(0)$ it holds

$$-g_i''(\theta) = \lambda g_i(\theta), \qquad \text{with } \lambda = \alpha^2.$$

In particular $\lambda = \lambda_1(A)$ (the first eigenvalue) because $g_i \geqslant 0$ and $g_i \not\equiv 0$, and moreover $\lambda_1(\cdot)$ has the same value on every connected component of $\{G > 0\}$.

Suppose that $\{G > 0\}$ has at least three connected components. Then one of them, denote it by C, must satisfy $\mathscr{H}^1(C) \leqslant \mathscr{H}^1(\partial B_1(0))/3$. By using spherical symmetrization (Sperner's Theorem) and the monotonicity of the first eigenvalue with respect to the domain, we obtain

$$\lambda = \lambda_1(C) \geqslant \lambda_1\left(E\left(\pi/3\right)\right), \quad \text{where } E\left(\pi/3\right) = \{x \in \partial B_1(0) : \ arcos(\langle x, e_3 \rangle) < \pi/3\}$$

($e_3 = (0, 0, 1)$). Since $\lambda_1(E(\pi/3)) = (3/2)^2$ with eigenfunction $\cos(3\theta/2)$ - in polar coordinates - we deduce that $\alpha \geqslant 3/2$.

Suppose now that $\{G > 0\}$ has at most two connected components. Since $N = 2$ and $\{\bar{U} = 0\}$ has an empty interior (Remark 3.12), then the number of components is equal to the number of zeros of G on $\partial B_1(0)$. Moreover G must have at least one zero, because otherwise $G > 0$ on $\partial B_1(0)$, $\bar{U}$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ and hence $\bar{U} \equiv 0$ (recall that $\bar{U}(0) = 0$), a contradiction. If G has one single zero then $\lambda = \lambda_1(E(\pi)) = 1/4$ and $\alpha = 1/2$, contradicting Corollary 3.14. Hence we have concluded that G must have exactly two zeros. Denote by Ω_1 and Ω_2 the two connected components of $\{G > 0\}$. Since $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$, Ω_1 and Ω_2 must cut the sphere in two equal parts and thus $\lambda = \lambda_1(E(\pi/2)) = 1$, $\alpha = 1$. □

Corollary 3.30. *For $N = 2$ the set S_U is closed in Ω, whenever $U \in \mathcal{G}(\Omega)$.*

Proof. This is a direct consequence of Proposition 3.29 together with the upper semicontinuity of the map $x \mapsto N(x, U, 0^+)$ stated in Corollary 3.15. □

Moreover a careful examination of the proof of Proposition 3.29 provides a more detailed description of the blowup limits:

Remark 3.31. Let $N = 2$ and let $\bar{U}$ be a blowup limit under the hypotheses of Corollary 3.27. Then $\{\bar{U} > 0\}$ has at least three connected components if and only if $\alpha = N(x_0, U, 0^+) > 1$. If on the other hand $\alpha = N(x_0, U, 0^+) = 1$ then $\{\bar{U} > 0\}$ is made of exactly two connected components and $\Gamma_{\bar{U}}$ is a hyper-plane (more precisely, denoting by ν a normal vector of $\Gamma_{\bar{U}}$, then on one side of $\Gamma_{\bar{U}}$ the non trivial component of $\bar{U}$ is equal to $a_1(x \cdot \nu)^+$, and on the other equals $a_2(x \cdot \nu)^-$, for some $a_1, a_2 > 0$).

Next we state and prove some estimates regarding the Hausdorff dimensions of the sets under study. The following result implies part of Theorem 3.1.

Theorem 3.32. *Let $U \in \mathcal{G}(\Omega)$. Then*

1. $\mathscr{H}_{\text{dim}}(\Gamma_U) \leqslant N-1$ *for any* $N \geqslant 2$.

2. $\mathscr{H}_{\text{dim}}(S_U) = 0$ *for* $N=2$, *and moreover for any given compact set* $\tilde{\Omega} \Subset \Omega$ *we have that* $S_U \cap \tilde{\Omega}$ *is a finite set.*

For the moment the second statement holds only for $N=2$ because of the dimension restriction in Proposition 3.29 (which provides the closedness of S_U). As we said before we shall extend ahead these results to any dimension greater than or equal to two.

The rest of this section is devoted to the proof of this result. The idea is to apply a version of the so called Federer's Reduction Principle, which we now state.

Theorem 3.33. *Let $\mathcal{F} \subseteq (L^\infty_{\text{loc}}(\mathbb{R}^N))^m$, and define, for any given $U \in \mathcal{F}$, $x_0 \in \mathbb{R}^N$ and $t>0$, the rescaled and translated function*

$$U_{x_0,t}(\cdot) := U(x_0 + t\cdot).$$

We say that $U_n \to U$ in $\mathcal{F}$ if and only if $U_n \to U$ uniformly on every compact set of $\mathbb{R}^N$. Assume that $\mathcal{F}$ satisfies the following conditions:

(A1) (Closure under rescaling, translation and normalization) Given $|x_0| \leqslant 1-t, 0<t<1$, $\rho>0$ and $U \in \mathcal{F}$, we have that also $\rho \cdot U_{x_0,t} \in \mathcal{F}$.

(A2) (Existence of a homogeneous "blowup") Given $|x_0|<1, t_k \downarrow 0$ and $U \in \mathcal{F}$, there exist a sequence $\rho_k \in (0,+\infty)$, a real number $\alpha \geqslant 0$ and a function $\bar{U} \in \mathcal{F}$ homogeneous of degree[3] α such that, if we define $U_k(x) = U(x_0+t_kx)/\rho_k$, then

$$U_k \to \bar{U} \qquad \text{in } \mathcal{F}, \qquad\qquad \textit{up to a subsequence.}$$

(A3) (Singular set hypotheses) There exists a map $\mathscr{S} : \mathcal{F} \to \mathcal{C}$ (where $\mathcal{C} := \{A \subset \mathbb{R}^N : A \cap B_1(0)$ is relatively closed in $B_1(0)\}$) such that

(i) Given $|x_0| \leqslant 1-t$, $0<t<1$, $\rho>0$ and $U \in \mathcal{F}$, it holds

$$\mathscr{S}(\rho \cdot U_{x_0,t}) = (\mathscr{S}(U))_{x_0,t} := \frac{\mathscr{S}(U)-x_0}{t}.$$

(ii) Given $|x_0|<1$, $t_k \downarrow 0$ and $U, \bar{U} \in \mathcal{F}$ such that there exists $\rho_k>0$ satisfying $U_k := \rho_k U_{x_0,t_k} \to \bar{U}$ in $\mathcal{F}$, the following "continuity" property holds:

$$\forall \varepsilon > 0\ \exists k(\epsilon)>0 :\ k \geqslant k(\varepsilon) \Rightarrow \mathscr{S}(U_k) \cap B_1(0) \subseteq \{x \in \mathbb{R}^N :\ \text{dist}(x, \mathscr{S}(\bar{U})) < \varepsilon\}.$$

[3] That is, $\bar{U}(tx) = t^\alpha U(x)$ for every $t>0$.

Then, if we define

$$d = \max\Big\{\dim L : \ L \text{ is a subspace of } \mathbb{R}^N \text{ and there exist } U \in \mathcal{F} \text{ and } \alpha \geqslant 0 \\ \text{such that } \mathscr{S}(U) \neq \emptyset \text{ and } U_{y,t} = t^\alpha U \ \forall y \in L,\ t > 0\Big\}, \quad (3.22)$$

either $\mathscr{S}(U) \cap B_1(0) = \emptyset$ *for every* $U \in \mathcal{F}$, *or else* $\mathscr{H}_{\dim}(\mathscr{S}(U) \cap B_1(0)) \leqslant d$ *for every* $U \in \mathcal{F}$. *Moreover in the latter case there exist a function* $V \in \mathcal{F}$, *a* d*-dimensional subspace* $L \leqslant \mathbb{R}^N$ *and a real number* $\alpha \geqslant 0$ *such that*

$$V_{y,t} = t^\alpha V \qquad \forall y \in L,\ t > 0, \qquad \text{and} \qquad \mathscr{S}(V) \cap B_1(0) = L \cap B_1(0).$$

If $d = 0$ *then* $\mathscr{S}(U) \cap B_\rho(0)$ *is a finite set for each* $U \in \mathcal{F}$ *and* $0 < \rho < 1$.

Up to our knowledge, this principle (due to Federer) appeared in this form for the first time in the book by Simon [116, Appendix A]. The version we present here can be seen as a particular case of a generalization made by Chen (see [41, Theorem 8.5] and [42, Proposition 4.5]). A detailed proof of this result can be found in Appendix A (*cf.* Theorem A.26).

Proof of Theorem 3.32. A first observation is that we only need to prove that the Hausdorff dimension estimates of the theorem hold true for the sets $\Gamma_U \cap B_1(0)$ and $S_U \cap B_1(0)$ whenever $U \in \mathcal{G}(\Omega)$ with $B_2(0) \Subset \Omega$. In fact, if we prove so, then we obtain that for any given Ω and $U \in \mathcal{G}(\Omega)$ it holds $\mathscr{H}_{\dim}(\Gamma_U \cap K) \leqslant N-1$, $\mathscr{H}_{\dim}(S_U \cap K) \leqslant N-2$ for every $K \Subset \Omega$ (because rescaling a function does not change the Hausdorff dimension of its nodal and singular sets). Being this true the theorem follows because a countable union of sets with Hausdorff dimension less than or equal to some $n \in \mathbb{R}_0^+$ also has Hausdorff dimension less than or equal to n.

Thus we apply the Federer's Reduction Principle to the following class of functions

$$\mathcal{F} = \left\{ U \in \left(L^\infty_{\text{loc}}(\mathbb{R}^N)\right)^m : \begin{array}{c} \text{there exists some domain } \Omega \text{ such that} \\ B_2(0) \Subset \Omega \text{ and } U|_\Omega \in \mathcal{G}(\Omega) \end{array} \right\}.$$

Let us start by checking (A1) and (A2). Proposition 3.16 immediately implies that condition (A1) is satisfied. Moreover, let $|x_0| < 1$, $t_k \downarrow 0$ and $U \in \mathcal{F}$, and choose $\rho_k = \|U(x_0 + t_k x)\|_{L^2(\partial B_1(0))}$. Theorem 3.18 and Corollary 3.27 (case 1) yield the existence of $\bar{U} \in \mathcal{F}$ such that (up to a subsequence) $U_k \to \bar{U}$ in $\mathcal{F}$ and $\bar{U}$ is a homogeneous function of degree $\alpha = N(x_0, U, 0^+) \geqslant 0$. Hence also (A2) holds. Next we choose the map $\mathscr{S}$ according to our needs.

1. (dimension estimate of the nodal sets in arbitrary dimensions) We want to prove that $\mathscr{H}_{\dim}(\Gamma_U \cap B_1(0)) \leqslant N-1$ whenever $U \in \mathcal{F}$. Define $\mathscr{S} : \mathcal{F} \to \mathcal{C}$ by $\mathscr{S}(U) = \Gamma_U$ ($\Gamma_U \cap B_1(0)$ is obviously closed in $B_1(0)$ by the continuity of U). It is quite straightforward to check hypothesis (A3)-(i), and the local uniform convergence considered in $\mathcal{F}$ clearly yields (A3)-(ii). Therefore, in order to end the proof in this case the only thing left to prove is that the integer d associated to $\mathscr{S}$ (defined in (3.22)) is less than or equal to $N-1$. Suppose by contradiction that $d = N$; then this would imply the existence of $V \in \mathcal{F}$ with $\mathscr{S}(V) = \mathbb{R}^N$, *i.e.*, $V \equiv 0$, which contradicts the definition of $\mathcal{G}$. Thus[4] $d \leqslant N-1$.

[4] Actually $d = N-1$, as it can be seen by taking $L = \mathbb{R}^{N-1} \times \{0\}$ and $U = (x_N^+, x_N^-, 0, \dots, 0)$.

2. (dimension estimate of the singular sets in the case $N = 2$) This is the most delicate case. As we said before, the restriction on the dimension N is only due to Proposition 3.29. As we shall see, the rest of the argument does not depend on the chosen dimension; for this reason, and since moreover we will prove the closedness of S_U for any dimension $N \geqslant 2$ in Section 3.6, we decide to keep N in the notations. We define $\mathscr{S} : \mathcal{F} \to \mathcal{C}$ by $\mathscr{S}(U) = S_U$ (which belongs to $\mathcal{C}$ by Corollary 3.30). The map satisfies (A3)-(i) thanks to identity (3.17), more precisely

$$x \in \mathscr{S}(U_{x_0,t}/\rho) \Leftrightarrow N(x, U_{x_0,t}/\rho, 0^+) > 1 \Leftrightarrow N(x_0 + tx, U, 0^+) > 1 \Leftrightarrow x_0 + tx \in \mathscr{S}(U).$$

As for (A3)-(ii), take $U_k, U \in \mathcal{F}$ as stated. Then in particular $U_k \to U$ uniformly in $B_2(0)$ and by arguing as in the proof of Lemma 3.26 it is easy to obtain strong convergence in $H^1(B_{3/2}(0))$. Suppose now that (A3)-(ii) does not hold; then there exists a sequence $x_k \in B_1(0)$ ($x_k \to x$, up to a subsequence, for some x) and $\bar{\varepsilon} > 0$ such that $N(x_k, U_k, 0^+) \geqslant 1 + \delta$ and $\mathrm{dist}(x_k, \mathscr{S}(U)) \geqslant \bar{\varepsilon}$. But then for small r we obtain (as in the proof of Corollary 3.15)

$$N(x_k, U_k, r) \geqslant (2 + \delta)e^{-Cr} - 1,$$

and hence (since $N(x_k, U_k, r) \to N(x, U, r)$ in k for small r) $N(x, U, 0^+) \geqslant 1 + \delta$, a contradiction.

Finally let us prove that $d \leqslant N - 2$. If $d = N - 1$ then we would have the existence of a function V, homogeneous with respect to every point in[5] $\mathbb{R}^{N-1} \times \{0\}$ such that $S_V \cap B_1(0) = (\mathbb{R}^{N-1} \times \{0\}) \cap B_1(0)$. Now, if we take the usual blowup sequence centered at $x_0 = 0$ (namely $V(t_k x)/\rho_k$), we obtain at the limit a function $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\alpha = N(x_0, V, 0^+) > 1$, harmonic in $\mathbb{R}^N \setminus \Gamma_{\bar{U}}$ such that $\bar{U}(y + \lambda x) = \lambda^\alpha \bar{U}(x)$ whenever $y \in \mathbb{R}^{N-1} \times \{0\}$, $x \in \mathbb{R}^N$. We prove that $\Gamma_{\bar{U}} = \mathbb{R}^{N-1} \times \{0\}$, which leads to a contradiction since Hopf's Lemma implies $\alpha = 1$. Since $\bar{U}(x) = \lim V(t_k x)/\rho_k$ and $\mathbb{R}^{N-1} \times \{0\} \subseteq \Gamma_V$, it is obvious that $\mathbb{R}^{N-1} \times \{0\} \subseteq \Gamma_{\bar{U}}$. If there were $y \in \Gamma_{\bar{U}} \setminus (\mathbb{R}^{N-1} \times \{0\})$, then since $\bar{U}$ is homogeneous with respect to every point in $\mathbb{R}^{N-1} \times \{0\}$, we would have that either $\mathbb{R}^{N-1} \times [0, +\infty)$ or $\mathbb{R}^{N-1} \times (-\infty, 0]$ would be contained in $\Gamma_{\bar{U}}$, contradicting Remark 3.12. □

Remark 3.34. The proof of Theorem 3.32-2 would hold in an arbitrary dimension provided that for every $N \geqslant 2$ there exists a universal constant $\delta_N > 1$ such that either $N(x_0, U, 0^+) = 1$ or $N(x_0, U, 0^+) \geqslant \delta_N$, whenever $U \in \mathcal{G}(\Omega)$ and $x_0 \in \Gamma_U$. A careful examination of the proof of Proposition 3.29 shows that the latter statement is equivalent to the following one:

- for every $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\Delta \bar{U} = 0$ in $\{\bar{U} > 0\}$, either $\alpha = 1$ or $\alpha \geqslant 1 + \delta_N$.

3.5 Regularity results under a flatness-type assumption

This section is devoted to the proof of the following auxiliary result.

[5]For some $\alpha > 0$ we have $V(y + \lambda x) = \lambda^\alpha V(x)$ for every $y \in \mathbb{R}^{N-1} \times \{0\}$, $x \in \mathbb{R}^N$.

Theorem 3.35. *Let Ω be a domain in $\mathbb{R}^N$ with $N \geqslant 2$. Fix $U \in \mathcal{G}(\Omega)$ and let $\Gamma^\star$ be a relatively open subset of Γ_U such that the following property holds:*

$$(P)\quad \begin{array}{l}\text{For any } x_0 \in \Gamma^\star \text{ take } x_k \in \Gamma^\star \text{ with } x_k \to x_0,\ t_k \downarrow 0, \text{ and let } \bar{U} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)\\ \text{be such that } \bar{U} = \lim_k U(x_k + t_k x)/\rho_k, \text{ with } \rho_k = \|U(x_k + t_k \cdot)\|_{L^2(\partial B_1(0))}.\\ \text{Then } \Gamma_{\bar{U}} \text{ is a hyper-plane passing through the origin.}\end{array}$$

Then $\Gamma^\star$ is a hyper-surface of class $C^{1,\alpha}$ for every $0 < \alpha < 1$ and, for every $x_0 \in \Gamma^\star$,

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0, \tag{3.23}$$

where the limits represent an approximation to x_0 coming from opposite sides of the hyper-surface.

Remark 3.36. In dimension $N = 2$, for every $U \in \mathcal{G}(\Omega)$, property (P) holds for $\Gamma^\star := \Sigma_U$, as previously observed in Remark 3.31.

In general, Theorem 3.18 yields that every blowup limit $\bar{U}$ belongs to $\mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ and that $\Delta \bar{u}_i = \bar{\mu}_i$, with $\bar{\mu}_i \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$ nonnegative and concentrated on $\Gamma_{\bar{U}}$. Property (P) says that such nodal sets are "flat", whenever the blowup limit is taken at points of Γ^*. Hence Theorem 3.35 states that "locally flat" points of the free boundary Γ_U (for $U \in \mathcal{G}(\Omega)$) are regular and that a reflection law holds. The previous theorem will be an important tool in the proof of Theorem 3.1 (this will become clear in Section 3.6 ahead): indeed, we will be able to apply this result to Σ_U in any dimension $N \geqslant 2$.

The strategy of the proof of Theorem 3.35 is as follows: property (P) will provide a local separation property (Proposition 3.38). This, together with the fact that $\bar{U} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ will allow us the use of a reflection principle (Lemma 3.40), which will in turn imply that in a small neighborhood of each point in $\Gamma^\star$ a certain equation can be solved and has a $C^{1,\alpha}$ solution (Theorem 3.41). The nodal set of this solution will be equal to Γ_U, and the final step will be to establish that its gradient is non zero on Γ_U.

From now on we fix $U \in \mathcal{G}(\Omega)$ with $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ and let $\Gamma^\star$ be a relatively open subset of Γ_U satisfying assumption (P). Take an open set $\tilde{\Omega} \Subset \Omega$ such that $\Gamma_U \cap \overline{\tilde{\Omega}} = \Gamma^\star \cap \overline{\tilde{\Omega}}$, that is, all the nodal points of U in the closure of $\tilde{\Omega}$ belong to $\Gamma^\star$. In the following lemma we prove that $\Gamma_U \cap \tilde{\Omega}$ verifies the so called $(N-1)$*–dimensional* (δ, R)*–Reifenberg flat condition* for every $0 < \delta < 1$ and some $R = R(\delta) > 0$.

Lemma 3.37. *Within the previous framework, for any given $0 < \delta < 1$ there exists $R > 0$ such that for every $x \in \Gamma^\star \cap \tilde{\Omega} = \Gamma_U \cap \tilde{\Omega}$ and $0 < r < R$ there exists a hyper-plane $H = H_{x,r}$ containing x such that* [6]

$$d_{\mathscr{H}}(\Gamma_U \cap B_r(x), H \cap B_r(x)) \leqslant \delta r. \tag{3.24}$$

Proof. Arguing by contradiction, suppose there exist $\bar{\delta} > 0$ and subsequences $x_k \in \Gamma^\star \cap \tilde{\Omega}$, $r_k \to 0$ such that

$$d_{\mathscr{H}}(\Gamma_U \cap B_{r_k}(x_k), H \cap B_{r_k}(x_k)) > \bar{\delta} r_k.$$

[6] Here $d_{\mathscr{H}}(A, B) := \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(A, b)\}$ denotes the Hausdorff distance. Notice that $d_{\mathscr{H}}(A, B) \leqslant \delta$ if and only if $A \subseteq N_\delta(B)$ and $B \subseteq N_\delta(A)$, where $N_\delta(\cdot)$ is the closed δ–neighborhood of a set.

whenever H is a hyper-plane passing through x_k. If we take a blowup sequence of type $U_k(x) = U(x_k + r_k x)/\rho_k$ (here we use the notations of Section 3.3), then the contradiction statement is equivalent to have

$$d_{\mathscr{H}}(\Gamma_{U_k} \cap B_1(0), H \cap B_1(0)) > \bar{\delta}$$

whenever H is a hyper-plane that passes through the origin. Since, up to a subsequence, $x_k \to \bar{x} \in \Gamma_U \cap \bar{\tilde{\Omega}} = \Gamma^\star \cap \bar{\tilde{\Omega}}$, Theorem 3.18 together with property (P) implies the existence of a blowup limit $\bar{U}$ whose nodal set $\Gamma_{\bar{U}}$ is a hyper-plane containing the origin. Hence we obtain a contradiction once we are able to prove that

$$d_{\mathscr{H}}(\Gamma_{U_k} \cap B_1(0), \Gamma_{\bar{U}} \cap B_1(0)) \to 0.$$

i) For every $\varepsilon > 0$ there exists $\bar{k} > 0$ such that

$$\Gamma_{U_k} \cap B_1(0) \subseteq N_\varepsilon(\Gamma_{\bar{U}} \cap B_1(0)) \qquad \text{for every } k \geqslant \bar{k}.$$

Were the previous inclusion not true and we would obtain the existence of $\bar{\varepsilon} > 0$ and of a sequence $y_k \in \Gamma_{U_k} \cap B_1(0)$ such that $\operatorname{dist}(y_k, \Gamma_{\bar{U}} \cap B_1(0)) > \bar{\varepsilon}$. Up to a subsequence, $y_k \to y \in \Gamma_{\bar{U}} \cap \bar{B}_1(0)$ by the L^∞_{loc} convergence $U_k \to \bar{U}$; moreover, since $\Gamma_{\bar{U}}$ is a hyper-plane passing through the origin, we deduce that $\operatorname{dist}(y, \Gamma_{\bar{U}} \cap B_1(0)) = 0$, which provides a contradiction.

ii) For every $\varepsilon > 0$ there exists $\bar{k} > 0$ such that

$$\Gamma_{\bar{U}} \cap B_1(0) \subseteq N_\varepsilon(\Gamma_{U_k} \cap B_1(0)) \qquad \text{for every } k \geqslant \bar{k}. \tag{3.25}$$

First of all we prove that given $x \in \Gamma_{\bar{U}}$ and $\delta > 0$, U_k must have a zero in $B_\delta(x)$ for large k. If not, by recalling that $u_{i,k} \cdot u_{j,k} \equiv 0$ whenever $i \neq j$, we would have $u_{i,k} > 0$ in $B_\delta(x)$ for some i and moreover $-\Delta u_{i,k} = f_{i,k}(u_{i,k})$ and $u_{j,k} \equiv 0$ (for $j \neq i$) in such ball. This would imply $\bar{u}_j \equiv 0$, $\Delta \bar{u}_i = 0$ in $B_\delta(x)$ with $x \in \Gamma_{\bar{U}}$, and therefore $\bar{U} \equiv 0$ in $B_\delta(x)$, a contradiction by Remark 3.12.

Now we are in condition to prove (3.25). We use once again a contradiction argument: suppose the existence of $\bar{\varepsilon} > 0$ and $y_k \in \Gamma_{\bar{U}} \cap B_1(0)$ such that $y_k \to y \in \Gamma_{\bar{U}} \cap \bar{B}_1(0)$, and $\operatorname{dist}(y_k, \Gamma_{U_k} \cap B_1(0)) > \bar{\varepsilon}$. Since $\Gamma_{\bar{U}}$ is a hyper-plane passing trough the origin, we can take $\bar{y} \in \Gamma_{\bar{U}} \cap B_1(0)$ such that $|y - \bar{y}| \leqslant \bar{\varepsilon}/4$. Moreover, by making use of the result proved in the previous paragraph, we can take a sequence $\bar{y}_k \in \Gamma_{U_k} \cap B_1(0)$ such that $|\bar{y}_k - \bar{y}| \leqslant \bar{\varepsilon}/4$ for large k. But then

$$\operatorname{dist}(y_k, \Gamma_{U_k} \cap B_1(0)) \leqslant |y_k - \bar{y}_k| \leqslant |y_k - y| + |y - \bar{y}| + |\bar{y} - \bar{y}_k| \leqslant 3\bar{\varepsilon}/4 < \bar{\varepsilon}$$

for large k, a contradiction. □

With the $(N-1)$–dimensional (δ, R)–Reifenberg flat condition we are able to prove a local separation result. We quote Theorem 4.1 in [71] for a result in the same direction.

Proposition 3.38 (Local separation property)**.** *Assume that property (P) holds. Given $x_0 \in \Gamma^\star$ there exists a radius $R_0 > 0$ such that $B_{R_0}(x_0) \cap \Gamma^\star = B_{R_0}(x_0) \cap \Gamma_U$ and $B_{R_0}(x_0) \setminus \Gamma_U = B_{R_0}(x_0) \cap \{U > 0\}$ has exactly two connected components Ω_1, Ω_2. Moreover, for sufficiently small $\delta > 0$, we have that given $y \in \Gamma_U \cap B_{R_0}(x_0)$ and $0 < r < R_0 - |y - x_0|$*

there exist a hyper-plane $H_{y,r}$ (passing through y) and a unitary vector $\nu_{y,r}$ (orthogonal to $H_{y,r}$) such that

$$\{x+t\nu_{y,r} \in B_r(y) : \, x \in H_{y,r}, \, t \geqslant \delta r\} \subset \Omega_1, \qquad \{x-t\nu_{y,r} \in B_r(y) : \, x \in H_{y,r}, \, t \geqslant \delta r\} \subset \Omega_2.$$

Proof. Let s be such that $B_{2s}(x_0) \cap \Gamma^\star = B_{2s}(x_0) \cap \Gamma_U$ (which exists since $\Gamma^\star$ is a relatively open set in Γ_U) and fix $\delta < 1/6$. With the notations of Lemma 3.37, for $\tilde{\Omega} := B_s(x_0)$ there exists $R > 0$ such that $\Gamma_U \cap B_s(x_0)$ satisfies the $(N-1)$–dimensional (δ, R)–Reifenberg flat condition. We show that Proposition 3.38 holds with the choice $R_0 := \min\{R, s\}$.

Lemma 3.37 yields the existence of a hyper-plane H_{x_0,R_0} containing x_0 that

$$d_{\mathscr{H}}(\Gamma_U \cap B_{R_0}(x_0), H_{x_0,R_0} \cap B_{R_0}(x_0)) \leqslant \delta R_0. \tag{3.26}$$

Thus the set $B_{R_0}(x_0) \setminus N_{2\delta R_0}(H_{x_0,R_0})$ is made of two connected components, say A_1 and A_2, which do not intersect Γ_U (Fig. 1). Define the function

$$\sigma(x) = \begin{cases} 1 & \text{if} \quad x \in A_1, \\ -1 & \text{if} \quad x \in A_2. \end{cases}$$

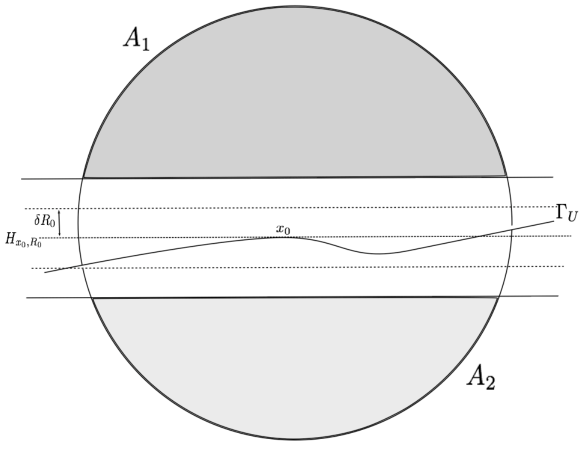

Fig. 1. $B_{R_0}(x_0) \setminus N_{2\delta R_0}(H_{x_0,R_0})$.

Now take any point $x_1 \in \Gamma_U \cap B_{R_0}(x_0) \subseteq N_{\delta R_0}(H_{x_0,R_0}) \cap B_{R_0}(x_0)$ and consider a ball of radius $R_0/2$ centered at x_1. Once again by Lemma 3.37 we have the existence of a hyper-plane $H_{x_1,R_0/2}$ such that

$$d_{\mathscr{H}}(\Gamma_U \cap B_{R_0/2}(x_1), H_{x_1,R_0/2} \cap B_{R_0/2}(x_1)) \leqslant \delta R_0/2.$$

This inequality together with (3.26) yields that

$$N_{\delta R_0/2}(H_{x_1,R_0/2} \cap B_{R_0/2}(x_1)) \cap B_{R_0}(x_0) \subseteq N_{2\delta R_0}(H_{x_0,R_0}) \cap B_{R_0}(x_0).$$

Hence $B_{R_0}(x_0)\cap B_{R_0/2}(x_1)\setminus N_{\delta R_0}(H_{x_1,R_0/2})$ has exactly two connected components where one intersects A_1 but not A_2, and the other intersects A_2 but not A_1 (Fig. 2). Thus the set

$$\Big(\cup_{x_1\in\Gamma_U\cap B_{R_0}(x_0)} B_{R_0}(x_0)\cap B_{R_0/2}(x_1)\setminus N_{\delta R_0}(H_{x_1,R_0/2})\Big)\cup A_1\cup A_2$$

has exactly 2 connected components which do not intersect Γ_U and hence we can continuously extend (by ± 1) the function σ to this set.

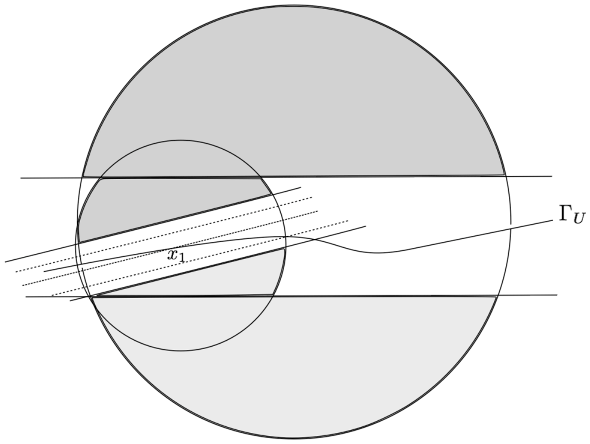

Fig. 2. $(B_{R_0}(x_0)\cap B_{R_0/2}(x_1)\setminus N_{\delta R_0}(H_{x_1,R_0/2}))\cup A_1\cup A_2$.

Now we iterate this process: in the k–th step, we apply the previous reasoning to a ball of radius $R_0/2^k$ centered at a point of Γ_U. In this way we find two connected and disjoint sets Ω_1,Ω_2 such that $B_{R_0}(x_0)\setminus\Gamma_U=\Omega_1\cup\Omega_2$, $A_1\subseteq\Omega_1$, $A_2\subseteq\Omega_2$. Moreover, the map $\sigma : B_1(0)\setminus\Gamma_U\to\{-1,1\}$ defined as $\sigma(x)=1$ if $x\in\Omega_1$, $\sigma(x)=-1$ if $x\in\Omega_2$ is continuous and thus $B_{R_0}(x_0)\setminus\Gamma_U$ has exactly two connected components. In order to check the continuity, take $x\in B_{R_0}(x_0)$ such that $\mathrm{dist}(x,\Gamma_U\cap B_{R_0}(x_0)) =: \gamma>0$, let $\bar{x}\in\Gamma_U\cap B_{R_0}(x_0)$ be a point of minimum distance and take k so large that $R_0/2^{k+1}\leqslant\gamma<R_0/2^k$; then $x\in B_{R_0/2^k}(\bar{x})\setminus N_{\delta R_0/2^{k-1}}(H_{\bar{x},R_0/2^k})$ (due to the choice of δ) and hence σ is constant (recall the construction of this function) in a small neighborhood of x. □

From now on we fix $x_0\in\Gamma^\star$ and take $R_0>0$ as in Proposition 3.38. Denote by Ω_1,Ω_2 the two connected components of $B_{R_0}(x_0)\cap\{U>0\}$ and by u and v the two functions amongst the components of the vector map U that satisfy $B_{R_0}(x_0)\cap\{u>0\}=\Omega_1$, $B_{R_0}(x_0)\cap\{v>0\}=\Omega_2$. Two situations may occur:

1. $u=u_i$ and $v=u_j$ in $B_{R_0}(x_0)$ for some $i\neq j$. In this case $u_k\equiv 0$ in $B_{R_0}(x_0)$ for $k\notin\{i,j\}$ and $(u,v)=(u_i,u_j)\in\mathcal{G}(B_{R_0}(x_0))$.
2. $u_k\equiv 0$ for all $k\neq i$ for some i. In this case we take

$$u(x)=\begin{cases} u_i(x) & \text{if } x\in\Omega_1\\ 0 & \text{if } x\in B_{R_0}(x_0)\setminus\Omega_1\end{cases}\qquad v(x)=\begin{cases} u_i(x) & \text{if } x\in\Omega_2\\ 0 & \text{if } x\in B_{R_0}(x_0)\setminus\Omega_2\end{cases}$$

The next statement shows that $(u,v)\in\mathcal{G}(B_{R_0}(x_0))$ also in this situation.

Lemma 3.39. *Under the situation of case 2 described before we obtain* $u, v \in H^1(B_{R_0}(x_0))$, $\nabla u = \nabla u_i \chi_{\Omega_1}$, $\nabla v = \nabla u_i \chi_{\Omega_2}$ *and the existence of nonnegative Radon measures* λ, μ *such that* $\mu_i = \lambda + \mu$ *and*

$$\begin{cases} -\Delta u = f_i(x,u) - \lambda \\ -\Delta v = f_i(x,v) - \mu \end{cases} \qquad \text{in } B_{R_0}(x_0).$$

Proof. We prove the result for u only. Take $\varphi \in \mathscr{D}(B_{R_0}(x_0))$ and consider a sequence $\varepsilon_n \to 0$ such that the sets $\{u > \varepsilon_n\}$ are regular (which exists by Sard's Theorem). We have

$$\begin{aligned} \int_{B_{R_0}(x_0)} u \nabla \varphi &= \int_{\Omega_1} u_i \nabla \varphi = \lim_n \int_{\Omega_1 \cap \{u_i > \varepsilon_n\}} u_i \nabla \varphi \\ &= \lim_n \int_{\Omega_1 \cap \{u_i > \varepsilon_n\}} -\nabla u_i \varphi + \lim_n \int_{\Omega_1 \cap \partial\{u_i > \varepsilon_n\}} u_i \varphi \nu \\ &= \int_{\Omega_1} -\nabla u_i \varphi + \lim_n \int_{\Omega_1 \cap \{u > \varepsilon_n\}} \varepsilon_n \nabla \varphi = \int_{\Omega_1} -\nabla u_i \varphi \end{aligned}$$

and hence $\nabla u = \nabla u_i \chi_{\Omega_1}$. On the other hand the existence of the measure λ comes from the fact that $\Delta u + f_i(x,u) \geqslant 0$ in $\mathscr{D}'(B_{R_0}(x_0))$: taking $\varphi \geqslant 0$,

$$\begin{aligned} \int_{B_{R_0}(x_0)} (u \Delta \varphi + f_i(x,u)\varphi) &= \lim_n \int_{\Omega_1 \cap \{u > \varepsilon_n\}} (u_i \Delta \varphi + f_i(x,u_i)\varphi) \\ &= \lim_n \int_{\Omega_1 \cap \partial\{u_i > \varepsilon_n\}} (u_i \partial_\nu \varphi - \partial_\nu u_i \varphi). \end{aligned}$$

Now the result follows because

$$\lim_n \int_{\Omega_1 \cap \partial\{u_i > \varepsilon_n\}} u_i \partial_\nu \varphi = \lim_n \int_{\Omega_1 \cap \{u_i > \varepsilon_n\}} \varepsilon_n \Delta \varphi = 0, \qquad \text{and} \qquad \int_{\Omega_1 \cap \partial\{u_i > \varepsilon_n\}} -\partial_\nu u_i \varphi \geqslant 0.$$

□

Hence in both cases the situation is the following: we have two nonnegative H^1-functions u, v such that $u \cdot v = 0$ in $B_{R_0}(x_0)$, $B_{R_0}(x_0) \cap \{u > 0\} = \Omega_1$, $B_{R_0}(x_0) \cap \{v > 0\} = \Omega_2$, $B_{R_0}(x_0) \setminus \Gamma_U = \Omega_1 \cup \Omega_2$, and there exist functions f, g satisfying (G1) and nonnegative Radon measures λ, μ satisfying (G2) such that

$$\begin{cases} -\Delta u = f(x,u) - \lambda \\ -\Delta v = g(x,v) - \mu \end{cases} \qquad \text{in } B_{R_0}(x_0).$$

Moreover condition (G3) holds. To end this section we will prove that in fact $\lambda = \mu$ in $B_{R_0}(x_0)$, which will moreover imply that $\Gamma^\star \cap B_{R_0}(x_0) = \Gamma_U \cap B_{R_0}(x_0)$ is a hyper-surface of class $C^{1,\alpha}$.

Lemma 3.40 (Reflection Principle)**.** *Let* $\bar{u}, \bar{v} \in H^1_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *be two non zero and nonnegative functions in* $\mathbb{R}^N$ *such that* $\bar{u} \cdot \bar{v} = 0$ *and*

$$\begin{cases} \Delta \bar{u} = \bar{\lambda} \\ \Delta \bar{v} = \bar{\mu} \end{cases} \qquad \text{in } \mathbb{R}^N$$

for some $\bar\lambda, \bar\mu \in \mathcal{M}_{loc}(\mathbb{R}^N)$*, locally nonnegative Radon measures satisfying (G2). Suppose moreover that* $\Gamma_{(\bar u,\bar v)} = \partial\{\bar u > 0\} = \partial\{\bar v > 0\}$ *is a hyper-plane and that* $(G3)$ *holds, that is*

$$\frac{d}{dr}E(x_0,(\bar u,\bar v),r) = \frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}((\partial_\nu \bar u)^2+(\partial_\nu \bar v)^2)\,d\sigma \qquad \text{for every } x_0\in\mathbb{R}^N, r>0 \quad (3.27)$$

(where we recall that $E(x_0,(\bar u,\bar v),r) = \frac{1}{r^{N-2}}\int_{B_r(x_0)}(|\nabla\bar u|^2+|\nabla\bar v|^2)$ *in this case). Then for every Borel set* $E\subseteq\mathbb{R}^N$ *it holds*

$$\bar\lambda(E) = \int_{E\cap\partial\{\bar u>0\}} -\partial_\nu \bar u\,d\sigma = \int_{E\cap\partial\{\bar v>0\}} -\partial_\nu\bar v\,d\sigma = \bar\mu(E)$$

and in particular $\Delta(\bar u-\bar v)=0$ *in* $\mathbb{R}^N$*.*

Proof. Suppose without loss of generality that $\Gamma := \Gamma_{(\bar u,\bar v)} = \mathbb{R}^{N-1}\times\{0\}$ and that $u\not\equiv 0$ in $\{x_N>0\}$, $v\not\equiv 0$ in $\{x_N<0\}$. In this case we observe that $\bar u\in C^\infty(\{x_N\geqslant 0\})$, $\bar v\in C^\infty(\{x_N\leqslant 0\})$ and that our goal is to check that

$$\bar\lambda(E) = \int_{E\cap\Gamma}\partial_{e_N}\bar u\,d\sigma = \int_{E\cap\Gamma}-\partial_{e_N}\bar v\,d\sigma = \bar\mu(E),$$

where e_N is the vector $(0,\dots,0,1)$. We divide the proof in two steps.
Step 1. For every E Borel set of $\mathbb{R}^N$ it holds

$$\bar\lambda(E) = \int_{E\cap\Gamma}\partial_{e_N}\bar u\,d\sigma \qquad \text{and} \qquad \bar\mu(E) = \int_{E\cap\Gamma}-\partial_{e_N}\bar v\,d\sigma. \quad (3.28)$$

We present the proof of this claim for $\bar\lambda$ only - for $\bar\mu$ the computations are analogous. It suffices to prove that (3.28) holds for every open ball $B_r(x_0)$. If $B_r(x_0)\cap\Gamma=\emptyset$ then $\bar\lambda(B_r(x_0))=0$ and equality holds. If on the other hand $B_r(x_0)\cap\Gamma\neq\emptyset$ then for any given $\delta>0$ take φ_δ to be a smooth cut-off function such that $\varphi_\delta=1$ in $B_{r-\delta}(x_0)$, $\varphi_\delta=0$ in $\mathbb{R}^N\setminus B_r(x_0)$. We have

$$\begin{aligned}
\int_{B_r(x_0)}\varphi_\delta\,d\bar\lambda &= -\int_{B_r(x_0)}\langle\nabla\bar u,\nabla\varphi_\delta\rangle = -\int_{B_r(x_0)\cap\{\bar u>0\}}\langle\nabla\bar u,\nabla\varphi_\delta\rangle\\
&= \int_{B_r(x_0)\cap\{\bar u>0\}}\Delta\bar u\varphi_\delta - \int_{B_r(x_0)\cap\Gamma}(\partial_{-e_N}\bar u)\varphi_\delta\,d\sigma\\
&= \int_{B_r(x_0)\cap\Gamma}(\partial_{e_N}\bar u)\varphi_\delta\,d\sigma.
\end{aligned}$$

Thus

$$\bar\lambda(B_r(x_0)) = \lim_{\delta\to0}\int_{B_r(x_0)}\varphi_\delta\,d\bar\lambda = \int_{B_r(x_0)\cap\Gamma}\partial_{e_N}\bar u\,d\sigma.$$

Step 2. $\partial_{e_N}\bar u = -\partial_{e_N}\bar v$ in Γ.

By using the regularity of $\bar u,\bar v$ together with the fact that Γ is a hyper-plane, we will compute the derivative of E directly, and compare afterwards the result with expression

(3.27). Since $\bar u, \bar v \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, then

$$\begin{aligned}\frac{d}{dr}E(x_0,(\bar u,\bar v),r) &= \frac{2-N}{r^{N-1}}\int_{B_r(x_0)}(|\nabla \bar u|^2+|\nabla \bar v|^2)+\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}(|\nabla \bar u|^2+|\nabla \bar v|^2)\,d\sigma\\ &= \frac{2-N}{r^{N-1}}\int_{B_r(x_0)\cap\{\bar u>0\}}|\nabla \bar u|^2+\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)\cap\{\bar u>0\}}|\nabla \bar u|^2\,d\sigma+\\ &\quad+\frac{2-N}{r^{N-1}}\int_{B_r(x_0)\cap\{\bar v>0\}}|\nabla \bar v|^2+\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)\cap\{\bar v>0\}}|\nabla \bar v|^2\,d\sigma.\end{aligned}$$

In order to rewrite the integrals on $\partial B_r(x_0)$, we use the following Rellich–type identity

$$\operatorname{div}\big((x-x_0)|\nabla \bar u|^2-2\langle x-x_0,\nabla \bar u\rangle\nabla \bar u\big)=(N-2)|\nabla \bar u|^2-2\langle x-x_0,\nabla\bar u\rangle\Delta \bar u \tag{3.29}$$

in $B_r(x_0)\cap\{\bar u>0\}$ (recall that $\bar u$ is smooth in this set). By the fact that $\Delta \bar u=0$ in the latter set and that $\nabla \bar u=(\partial_{e_N}\bar u)e_N$ on $\partial\{\bar u>0\}=\Gamma$, we have

$$\begin{aligned}\int_{\partial B_r(x_0)\cap\{\bar u>0\}}|\nabla \bar u|^2 = 2\int_{\partial B_r(x_0)\cap\{\bar u>0\}}(\partial_\nu \bar u)^2-\frac{1}{r}\int_{B_r(x_0)\cap\Gamma}(\partial_{e_N}\bar u)^2\langle e_N,x-x_0\rangle+\\ +\frac{N-2}{r}\int_{B_r(x_0)\cap\{\bar u>0\}}|\nabla \bar u|^2\end{aligned}$$

and analogously

$$\begin{aligned}\int_{\partial B_r(x_0)\cap\{\bar v>0\}}|\nabla \bar v|^2 = 2\int_{\partial B_r(x_0)\cap\{\bar v>0\}}(\partial_\nu \bar v)^2+\frac{1}{r}\int_{B_r(x_0)\cap\Gamma}(\partial_{e_N}\bar v)^2\langle e_N,x-x_0\rangle+\\ +\frac{N-2}{r}\int_{B_r(x_0)\cap\{\bar v>0\}}|\nabla \bar v|^2.\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dr}E(x_0,(\bar u,\bar v),r)=\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}((\partial_\nu\bar u)^2+(\partial_\nu \bar v)^2)\,d\sigma+\\ +\frac{1}{r^{N-1}}\int_{B_r(x_0)\cap\Gamma}[(\partial_{e_N}\bar v)^2-(\partial_{e_N}\bar u)^2]\langle e_N,x-x_0\rangle\end{aligned}$$

which, when compared with (3.27), yields that

$$\int_{B_r(x_0)\cap\Gamma}[(\partial_{e_N}\bar v)^2-(\partial_{e_N}\bar u)^2]\langle e_N,x-x_0\rangle=0 \qquad \text{for every } x_0\in\mathbb{R}^N,\ r>0,$$

and therefore $(\partial_{e_N}\bar v)^2=(\partial_{e_N}\bar u)^2$ on Γ. Finally we just have to observe that $|\partial_{e_N}\bar u|=\partial_{e_N}\bar u$ and $|\partial_{e_N}\bar v|=-\partial_{e_N}\bar v$. □

Theorem 3.41. *Assume that property (P) holds. With the previous notations we have $\lambda(E)=\mu(E)$ for every E Borel set of $B_{R_0}(x_0)$, and in particular*

$$-\Delta(u-v)=f(x,u)-g(x,v) \quad \textit{in } B_{R_0}(x_0). \tag{3.30}$$

Proof. We claim that

$$\lim_{r\to 0}\frac{\lambda(\bar B_r(y))}{\mu(\bar B_r(y))}=1 \qquad \text{for every } y\in\Gamma_U\cap B_{R_0}(x_0).$$

Fix $y\in\Gamma_U\cap B_{R_0}(x_0)$ and consider any arbitrary sequence $r_k\downarrow 0$. If we define $u_k(x)=u(y+r_kx)/\rho_k$, $v_k(x)=v(y+r_kx)/\rho_k$ as a usual blowup sequence at a point y, and consider λ_k,μ_k to be the associated rescaled measures, then Theorem 3.18 yields the existence of a pair of functions $(\bar u,\bar v)\in\mathcal{G}_{\rm loc}(\mathbb{R}^N)$ and measures $(\bar\lambda,\bar\mu)$ such that

$$\begin{aligned} &u_k\to\bar u, &&v_k\to\bar v &&\text{in } H^1_{\rm loc}\cap C^{0,\alpha}_{\rm loc}\\ &\lambda_k\rightharpoonup\bar\lambda, &&\mu_k\rightharpoonup\bar\mu &&\text{in the measure sense,}\end{aligned}$$

and $\Delta\bar u=\bar\lambda$, $\Delta\bar v=\bar\mu$ in $\mathbb{R}^N$. Property (P) implies that $\Gamma_{(\bar u,\bar v)}$ is a hyper-plane passing through the origin. From this fact, the uniform convergence of u_k,v_k to $\bar u,\bar v$, and the second statement of Proposition 3.38, we deduce also that $\bar u,\bar v\neq 0$. Thus we can apply Lemma 3.40 to the functions $\bar u,\bar v$, which provides

$$\bar\lambda(E)=\int_{E\cap\partial\{\bar u>0\}}-\partial_\nu\bar u\,d\sigma=\int_{E\cap\partial\{\bar v>0\}}-\partial_\nu\bar v\,d\sigma=\bar\mu(E)$$

for every Borel set E of $\mathbb{R}^N$. In particular $\bar\lambda(\bar B_1(0))=\bar\mu(\bar B_1(0))\neq 0$ and $\bar\lambda(\partial B_1(0))=\bar\mu(\partial B_1(0))=0$, thus

$$\lambda_k(\bar B_1(0))\to\bar\lambda(\bar B_1(0)),\qquad \mu_k(\bar B_1(0))\to\mu(\bar B_1(0))$$

(see Theorem A.18) and

$$1=\frac{\bar\lambda(\bar B_1(0))}{\bar\mu(\bar B_1(0))}=\lim_k\frac{\lambda_k(\bar B_1(0))}{\mu_k(\bar B_1(0))}=\lim_k\frac{\lambda(\bar B_{r_k}(y))}{\mu(\bar B_{r_k}(y))},$$

as claimed.

Therefore $D_\mu\lambda(y)=1$ for μ–a.e. $y\in B_{R_0}(x_0)$ and $D_\lambda\mu(y)=1$ for λ–a.e. $y\in B_{R_0}(x_0)$ (recall that both λ and μ are supported on Γ_U), and hence the Radon-Nikodym Decomposition Theorem (check Theorem A.15) yields that for every Borel set $E\subseteq B_{R_0}(x_0)$

$$\begin{aligned}\lambda(E)&=\lambda_s(E)+\mu(E)\geqslant\mu(E)\\ \mu(E)&=\mu_s(E)+\lambda(E)\geqslant\lambda(E)\end{aligned}$$

(where $\lambda_s\geqslant 0$ represents the singular part of λ with respect to μ and $\mu_s\geqslant 0$ represents the singular part of μ with respect to λ). Hence $\lambda(E)=\mu(E)$, which concludes the proof of the theorem. □

With the following result we end the proof of Theorem 3.35.

Corollary 3.42. *Under the previous notations, $u-v\in C^{1,\alpha}(B_{R_0}(x_0))$ for every $0<\alpha<1$, and*

$$\nabla(u-v)(x_0)\neq 0.$$

Proof. Since $w = u - v$ solves $-\Delta w = f(x, w^+) - g(x, w^-)$ and $f(x, w^+) - g(x, w^-) \in L^\infty(B)$, then standard elliptic regularity yields $w \in C^{1,\alpha}(B_{R_0}(x_0))$ for all $0 < \alpha < 1$. Now if we consider a blowup sequence centered at x_0, namely $w_k(x) := (u(x_0 + t_k x) - v(x_0 + t_k x))/\rho_k$ then

$$\begin{gathered} w_k \to \bar{w} := \bar{u} - \bar{v} \qquad \text{in } H^1_{\mathrm{loc}} \cap C^{0,\alpha}_{\mathrm{loc}}(B_2(0)) \\ -\Delta w_k = f_k(x, u_k) - g_k(x, v_k) \to 0 \qquad \text{in } L^\infty(B_2(0)) \\ \Delta \bar{w} = 0 \qquad \text{in } B_2(0) \end{gathered}$$

and hence

$$\|w_k - \bar{w}\|_{C^{1,\alpha}(B_1(0))} \leqslant C(\|w_k - \bar{w}\|_{L^\infty(B_2(0))} + \|f_k(x, u_k) - g_k(x, v_k))\|_{L^\infty(B_2(0))}) \to 0.$$

Since (by Corollary 3.27) $\bar{w}$ is a homogeneous function of degree one, then $\nabla \bar{w}(0) \neq 0$ and thus also $\nabla w_k(0) = t_k \nabla w(x_0)/\rho_k \neq 0$ for large k. □

Proof of Theorem 3.35. Corollary 3.42 combined with the Implicit Function Theorem implies that $\Gamma^\star$ is indeed a hyper-surface of class $C^{1,\alpha}$. Furthermore, equation (3.30) implies the reflection principle (3.23). □

3.6 Proof of the main result in any dimension $N \geqslant 2$: iteration argument

Given $N \geqslant 2$, by taking in consideration Theorems 3.32 and 3.35 as well as Remark 3.34, we deduce that in order to prove our main result (Theorem 3.1) it is enough to check the following.

Lemma 3.43. *Let $N \geqslant 2$. Given $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\Delta \bar{U} = 0$ in $\{\bar{U} > 0\}$, then either $\alpha = 1$ or $\alpha \geqslant 1 + \delta_N$ for some universal constant δ_N depending only on the dimension. Moreover if $\alpha = 1$ then $\Gamma_{\bar{U}}$ is a hyper-plane.*

In fact, assuming for the moment that Lemma 3.43 holds:

Proof of Theorem 3.1. 1. Fix $N \geqslant 2$, $\Omega \subseteq \mathbb{R}^N$ and let $U \in \mathcal{G}(\Omega)$. By Theorem 3.32-1 we have $\mathscr{H}_{\dim}(\Gamma_U) \leqslant N - 1$. Next, for each $x_0 \in \Omega$, take a blowup sequence $U_k(x) = U(x_0 + t_k x)/\rho_k$. Theorem 3.18 and Corollary 3.27 (case 1) together imply the existence of a blowup limit $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\Delta \bar{U} = 0$ in $\{\bar{U} > 0\}$, and $\alpha = N(x_0, U, 0^+)$. Thus we can apply Lemma 3.43 which allows us to deduce that either $N(x_0, U, 0^+) = 1$ or $N(x_0, U, 0^+) \geqslant 1 + \delta_N$, for some universal constant $\delta_N > 0$. In this way, being S_U, Σ_U the sets introduced in Definition 3.28, we obtain (by repeating word by word the proofs of Corollary 3.30 and Theorem 3.32-2) that S_U is closed, Σ_U is relatively open in Γ_U, and that $\mathscr{H}_{\dim}(S_U) \leqslant N - 2$. Finally, Corollary 3.27 (case 2) and Lemma 3.43 imply that $\Gamma^\star := \Sigma_U$ satisfies condition (P) in Theorem 3.35, which shows that Σ_U is a hyper-surface of class $C^{1,\alpha}$ and that (3.1) holds.
2. Let us now pass to the proof of (3.2). This proof follows the lines of the one of Proposition 2.25. In view of a contradiction, suppose the existence of $x_n \in \Omega$ with $d(x_n, S_U) \to 0$ and $r_n \to 0$ such that

$$\frac{1}{r_n^N} \int_{B_{r_n}(x_n)} |\nabla U|^2 \not\to 0 \qquad \text{as } n \to \infty. \tag{3.31}$$

Fix $\tilde{\Omega} \Subset \Omega$ where $x_n \in \mathrm{int}(\tilde{\Omega})$ for every n. Since $N(x,U,0^+) \geqslant 1+\delta$ for every $x \in S_U$, we deduce from Theorem 3.9 the existence of constants $\bar{r}, \tilde{C} > 0$ such that

$$H(x,U,r) \leqslant \tilde{C} r^{2(1+\delta)} \qquad \text{for every } x \in \tilde{\Omega} \cap S_U,\ 0<r<\bar{r}. \tag{3.32}$$

Moreover,

$$\frac{1}{r^N}\int_{B_r(x)} |\nabla U|^2 \leqslant \frac{2}{r^2}(E(x,U,r)+H(x,U,r)) = \frac{2}{r^2}(N(x,U,r)+1)H(x,U,r) \leqslant C r^{2\delta} \tag{3.33}$$

for every $x \in \tilde{\Omega} \cap S_U,\ 0<r<\bar{r}$. Now we have two cases.

Case a. $\mathrm{d}(x_n, S_U)/r_n \leqslant K$ for some constant $K>0$.

Let $y_n \in S_U$ be such that $|y_n - x_n| = \mathrm{d}(x_n, S_U)$ and define $s_n = r_n + \mathrm{d}(x_n, S_U)$ so that $B_{r_n}(x_n) \subseteq B_{s_n}(y_n)$. From (3.33) we see that

$$\frac{1}{r_n^N}\int_{B_{r_n}(x_n)} |\nabla U|^2 \leqslant \frac{(K+1)^N}{s_n^N}\int_{B_{s_n}(y_n)} |\nabla U|^2 \to 0,$$

which contradicts (3.31).

Case b. $\mathrm{d}(x_n, S_U)/r_n \to \infty$.

Denote $R_n = \mathrm{d}(x_n, S_U)/2 \to 0$. Two situations may occur. Either $B_{R_n}(x_n) \cap \{U>0\}$ has only one connected component, and hence only one density (say u_1) is non trivial and

$$-\Delta u_1 = f_1(x,u_1) \qquad \text{in } B_{R_n}(x_n) \text{ for every } n,$$

or else $B_{R_n}(x_n) \cap \{U>0\}$ has exactly two connected components and there exist u, v such that $u \cdot v \equiv 0$, $-\Delta u = f_i(x,u)$ in $B_{R_n}(x_n) \cap \{u>0\}$ and $-\Delta v = f_j(x,v)$ in $B_{R_n}(x_n) \cap \{v>0\}$ (for some $i,j \in \{1,\dots,m\}$) and $\sum_{i=1}^m u_i = u+v$ in $B_{R_n}(x_n)$. In this latter case, from (3.1) we see that

$$-\Delta(u-v) = f_i(x,u) - f_j(x,v) \qquad \text{in } B_{R_n}(x_n) \text{ for every } n.$$

In both cases we can use classical elliptic regularity theory and deduce that for each $q>N$ there exists a constant $C>0$ independent of n such that

$$\begin{aligned}
[U]_{C^{0,1}(\bar{B}_{R_n/2}(x_n))} &\leqslant C R_n^{-1}(R_n^{-N/q}\|U\|_{L^q(B_{R_n}(x_n))} + R_n^{2-N/q}\|F(x,U)\|_{L^q(B_{R_n}(x_n))}) \\
&\leqslant C' R_n^{-1}(\|U\|_{L^\infty(B_{R_n}(x_n))} + R_n^2).
\end{aligned}$$

Now we show that $\|U\|_{L^\infty(B_{R_n}(x_n))} \leqslant \mathrm{o}(1) R_n$ as $n \to \infty$, which gives a contradiction since $B_{r_n}(x_n) \subseteq B_{R_n/2}(x_n)$ for large n. From (3.32) we deduce that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} U^2 \leqslant C r^{2(1+\delta)} \qquad \text{for every } x \in \tilde{\Omega} \cap S_U \text{ and } 0<r<\bar{r}.$$

Take an arbitrary sequence $y_n \in B_{R_n}(x_n)$ and denote $s_n = \mathrm{d}(y_n, S_U)/2 \to 0$. Take moreover $w_n \in S_U$ such that $\mathrm{d}(y_n, S_U) = |y_n - w_n|$. From Lemma 3.24 we have that, for n large,

$$\begin{aligned}
U^2(y_n) &\leqslant \frac{1}{|B_{s_n}|}\int_{B_{s_n}(y_n)} U^2 + C s_n^2 \|U\|_{L^\infty(B_{s_n}(y_n))} \\
&\leqslant \frac{3^N}{|B_{3s_n}|}\int_{B_{3s_n}(w_n)} U^2 + C s_n^2 \|U\|_{L^\infty(B_{s_n}(y_n))} \\
&\leqslant C' s_n^{2(1+\delta)} + \mathrm{o}(1) s_n^2 = \mathrm{o}(1) s_n^2
\end{aligned}$$

as $n \to \infty$, which shows that $\|U\|^2_{L^\infty(B_{R_n}(x_n))} \leqslant o(1)s_n^2$, as claimed. □

Proof of Theorem 3.2. Furthermore, in dimension $N = 2$, we know from Theorem 3.32-2 that S_U is locally a finite set. For each $y_0 \in S_U$ take a small radius such that $S_U \cap B_r(y_0) = \{y_0\}$. Since (3.1) holds, we can apply the exact same reasoning of [47, Theorem 9.6] to the ball $B_r(y_0)$, proving in this way that $\Sigma_U \cap B_r(y_0)$ consists of a finite collection of curves meeting with equal angles at y_0, which is a singular point. □

The remainder of this section is devoted to the proof of Lemma 3.43. Its proof follows by induction in the dimension N. For $N = 2$ the statement holds by Proposition 3.29 and Remark 3.31. Suppose now that the claim holds in dimension $N - 1$ and take $\bar{U} = r^\alpha G(\theta) \in \mathcal{G}_{\rm loc}(\mathbb{R}^N)$ such that $\Delta \bar{U} = 0$ in $\{\bar{U} > 0\}$. We first treat the case in which the positive set has three or more connected components. In the three-dimensional case the exact value of δ_N has been proven to be 1/2 in [70].

Lemma 3.44. *If $\{G > 0\}$ has at least three connected components then there exists a universal constant $\bar{\delta}_N > 0$ such that $\alpha \geqslant 1 + \bar{\delta}_N$.*

Proof. We argue exactly as in the first part of the proof of Proposition 3.29 (from which we also recall the definition of $E(\theta)$). Note that for every connected component $A \subseteq \{g_i > 0\} \subset S^{N-1}$ it holds

$$-\Delta_{S^{N-1}} g_i = \lambda g_i \quad \text{in } A, \qquad \text{with } \lambda = \alpha(\alpha + N - 2) \text{ and } \lambda = \lambda_1(A).$$

At least one of the connected components, say C, must satisfy $\mathscr{H}^{N-1}(C) \leqslant \mathscr{H}^{N-1}(S^{N-1})/3$, and hence $\lambda = \lambda_1(C) \geqslant \lambda_1(E(\pi/3))$. Moreover it is well known that $\lambda_1(E(\pi/2)) = N - 1$. This implies the existence of $\gamma > 0$ such that $\lambda_1(E(\pi/3)) = N - 1 + \gamma$, and thus $\alpha = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} - \frac{N-2}{2} \geqslant 1 + \bar{\delta}_N$ for some $\bar{\delta}_N > 0$. □

From now on we suppose that $\{G > 0\}$ has at most two connected components. In order to prove Lemma 3.43 the next step is to study the local behavior of the function $\bar{U}$ at its non zero nodal points $y_0 \in \Gamma_{\bar{U}} \setminus \{0\}$. This study is accomplished by performing a new blowup analysis. Since $\bar{U}$ is homogeneous it suffices to take blowup sequences centered at $y_0 \in \Gamma_{\bar{U}} \cap S^{N-1} = \Gamma_G$.

Fix $y_0 \in \Gamma_{\bar{U}} \cap S^{N-1}$ and consider $V_k(x) := \bar{U}(y_0 + t_k x)/\rho_k$ for some $t_k \downarrow 0$ and $\rho_k = \|\bar{U}(y_0 + t_k \cdot)\|_{L^2(\partial B_1(0))}$. Theorem 3.18 and Corollary 3.27 provide the existence of a blowup limit $\bar{V} = r^\gamma H(\theta) \in \mathcal{G}_{\rm loc}(\mathbb{R}^N)$, with $\gamma = N(y_0, \bar{U}, 0^+)$. By the homogeneity of $\bar{U}$ we are able to prove that $\bar{V}$ actually depends only on $N - 1$ variables.

Lemma 3.45. *It holds $\bar{V}(x + \lambda y_0) = \bar{V}(x)$ for every $\lambda > 0$, $x \in \mathbb{R}^N$.*

Proof. Fix $x \in \mathbb{R}^N$ and $\lambda > 0$. Recall that $V_k \to \bar{V}$ in $C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$, which in particular implies pointwise convergence. Hence in particular $V_k(x) \to \bar{V}(x)$ and $V_k(x + \lambda y_0) \to \bar{V}(x + \lambda y_0)$. In order to prove the lemma it is enough to check that $\lim_k (V_k(x + t y_0) - V_k(x)) = 0$. From the homogeneity of $\bar{U}$ one obtains

$$\begin{aligned} V_k(x + \lambda y_0) &= \frac{1}{\rho_k} \bar{U}(y_0 + t_k(x + \lambda y_0)) = \frac{1}{\rho_k} \bar{U}((1 + \lambda t_k) y_0 + t_k x) \\ &= \frac{(1 + \lambda t_k)^\alpha}{\rho_k} \bar{U}\Big(y_0 + \frac{t_k}{1 + \lambda t_k} x\Big) = (1 + \lambda t_k)^\alpha V_k\Big(\frac{x}{1 + t_k \lambda}\Big). \end{aligned}$$

Take a compact set K containing x and $x/(1+\lambda t_k)$ for large k. There exists a constant $C = C(K)$ such that

$$\begin{aligned}
|V_k(x+\lambda y_0) - V_k(x)| &= \Big|(1+\lambda t_k)^\alpha V_k\Big(\frac{x}{1+t_k\lambda}\Big) - V_k(x)\Big| \\
&\leqslant \Big|(1+\lambda t_k)^\alpha V_k\Big(\frac{x}{1+\lambda t_k}\Big) - V_k\Big(\frac{x}{1+\lambda t_k}\Big)\Big| + \Big|V_k\Big(\frac{x}{1+\lambda t_k}\Big) - V_k(x)\Big| \\
&\leqslant C|(1+\lambda t_k)^\alpha - 1| + C\Big|\frac{1}{1+\lambda t_k} - 1\Big|^\alpha |x|^\alpha \to 0.
\end{aligned}$$

□

Next we use the induction hypothesis in order to prove a jump condition of the possible values of $\gamma = N(y_0, \bar{U}, 0^+)$.

Lemma 3.46. *With the previous notations, either $\gamma \geqslant 1$ or $\gamma \geqslant 1 + \delta_{N-1}$. Furthermore if $\gamma = 1$ then $\Gamma_{\bar{V}}$ is a hyper-plane.*

Proof. Up to a rotation we can suppose that $y_0 = (0, \dots, 0, 1)$. Hence by Lemma 3.45 $\bar{V}(x) = \bar{V}(x_1, \dots, x_{N-1}) = |(x_1, \dots, x_{N-1})|^\gamma H\left(\frac{(x_1,\dots,x_{N-1})}{|(x_1,\dots,x_{N-1})|}\right)$, $\Delta_{\mathbb{R}^{N-1}} \bar{V} = 0$ in $\{\bar{V} > 0\}$ and $\bar{V}_{|\mathbb{R}^{N-1}\times\{0\}} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^{N-1})$. Hence by the induction hypothesis either $\gamma = 1$ or $\gamma \geqslant 1 + \delta_{N-1}$. Moreover if $\gamma = 1$ then $\Gamma_{\bar{V}} \cap \big(\mathbb{R}^{N-1} \times \{0\}\big)$ is an $(N-2)$–dimensional subspace of $\mathbb{R}^{N-1}$ and hence $\Gamma_{\bar{V}}$ is a hyper-plane in $\mathbb{R}^N$. □

The previous result shows that given $y_0 \in \Gamma_{\bar{U}} \cap S^{N-1}$ then either $N(y_0, \bar{U}, 0^+) = 1$ or $N(y_0, \bar{U}, 0^+) \geqslant 1 + \delta_{N-1}$.

Lemma 3.47. *Suppose there exists $y_0 \in \Gamma_{\bar{U}} \cap S^{N-1}$ such that $N(y_0, \bar{U}, 0^+) \geqslant 1 + \delta_{N-1}$. Then $\alpha = N(0, \bar{U}, 0^+) \geqslant 1 + \delta_{N-1}$.*

Proof. Consider, for every $t > 0$, the rescaled function $\bar{U}_{0,t}(x) := \bar{U}(tx) = t^\alpha \bar{U}(x)$. By taking into account identity (3.17) we obtain that for every $r > 0$

$$N(y_0, \bar{U}, r) = N(y_0, t^\alpha \bar{U}, r) = N(y_0, \bar{U}_{0,t}, r) = N(ty_0, \bar{U}, tr).$$

Therefore $N(ty_0, \bar{U}, 0^+) = N(y_0, \bar{U}, 0^+) \geqslant 1 + \delta_{N-1}$ and the conclusion of the lemma follows from the upper semi-continuity of the map $y \mapsto N(y, \bar{U}, 0^+)$ (Corollary 3.15). □

From now on we suppose that the set $\{G > 0\}$ has at most two connected components and that $N(y_0, \bar{U}, 0^+) = 1$ for every $y_0 \in \Gamma_G$. Let us prove that $\alpha \in \mathbb{N}$ and that if $\alpha = 1$ then $\Gamma_{\bar{U}}$ is a hyper-plane (in the remaining cases we have shown that $\alpha \geqslant 1 + \min\{\bar{\delta}_N, \delta_{N-1}\}$).

Observe that the second conclusion in Lemma 3.46 shows that property (P) holds at every point $y_0 \in \Sigma_{\bar{U}} \cap S^{N-1} = \Gamma_{\bar{U}} \cap S^{N-1}$. Hence Theorem 3.35 yields that $\nabla \bar{U}(y_0) \neq 0$ whenever $y_0 \in \Gamma_{\bar{U}} \cap S^{N-1}$, and in particular $\nabla_\theta \bar{U}(y_0) \neq 0$ since $\bar{U}$ is a homogeneous function and $\bar{U}(y_0) = 0$. In this way we conclude that the set $\Gamma_{\bar{U}} \cap S^{N-1}$ is a compact $(N-2)$– dimensional sub-manifold of S^{N-1} without boundary, and by a generalization of the Jordan Curve Theorem (see [115, Theorem 7.1]) we conclude that in fact $S^{N-1} \setminus \Gamma_{\bar{U}}$ - as well as $\mathbb{R}^N \setminus \Gamma_{\bar{U}}$ - is made of two connected components.

Denote by Ω_1, Ω_2 the two connected components of $\mathbb{R}^N \setminus \Gamma_{\bar U}$ and respectively by u, v the non trivial components of $\bar U$ in these sets. Once again by Theorem 3.35 we obtain that $\nabla u = -\nabla v$ on $\Gamma_{\bar U} \setminus \{0\}$ and hence $\Delta(u - v) = 0$ in $\mathbb{R}^N$, and $(u, v) = r^\alpha G(\theta)$. Thus $\alpha \in \mathbb{N}$ and if $\alpha = 1$ then $\nabla(u - v)(0) \neq 0$ and $\Gamma_{\bar U}$ is a hyper-plane.

In conclusion we have proved the conclusion of Lemma 3.43 in any dimension N, more precisely we have shown that either $\alpha \geqslant 1 + \min\{2, \bar\delta_N, \delta_{N-1}\}$ or else $\alpha = 1$ and $\Gamma_{\bar U}$ is a hyper-plane.

3.7 Applications

In this section we provide two applications of the previously developed theory. In both cases we prove that the functions in consideration belong to the class $\mathcal{G}(\Omega)$, and hence Theorems 3.1 and 3.2 apply.

3.7.1 Asymptotic limits of a system of Gross-Pitaevskii equations

We recall from Chapter 2 the following system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_{i,\beta} u_i = \omega_i u_i^3 - \beta u_i \displaystyle\sum_{\substack{j=1\\ j\neq i}}^m u_j^2 & i = 1, \dots, m \\ u_i \in H_0^1(\Omega),\ u_i > 0 \text{ in } \Omega, \end{cases} \tag{3.34}$$

for $\beta > 0$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain ($N = 2, 3$), $\omega_i \in \mathbb{R}$ are fixed constants and $(\lambda_{i,\beta})_\beta$ are bounded sequences in $\mathbb{R}$ for $i = 1, \dots, m$. In Theorem 2.1 we have proved that given a uniformly bounded sequence $(U_\beta) = (u_{1,\beta}, \dots, u_{m,\beta})$ of solutions of (3.34) there exists $U = (u_1, \dots, u_m)$ such that $U_\beta \to U$ in $C^{0,\alpha}(\bar\Omega) \cap H_0^1(\Omega)$ for every $0 < \alpha < 1$. From now on we will call such $U's$ the *limiting profiles* of (3.34). We have the following result.

Theorem 3.48. *Let U be a limit limiting profile of* (3.34)*. Then $U \in \mathcal{G}(\Omega)$.*

Proof. This fact was already observed in Section 2.5. The proofs of the several conditions can be found along the second chapter, namely in the proofs of Theorem 2.2 and Propositions 2.25 and 2.27. Here we summarize the steps needed in the proof of such result in order to clarify some ideas and to keep this chapter as self contained as possible.

Let $U = (u_1, \dots, u_m)$ be a limiting profile for (3.34) and let $(U_\beta)_\beta$ be a sequence of solutions of (3.34) such that

$$U_\beta \to U \qquad \text{in } C^{0,\alpha}(\bar\Omega) \cap H_0^1(\Omega) \text{ for every } 0 < \alpha < 1.$$

From Theorem 2.2 we see that U is Lipschitz continuous, $u_i \cdot u_j \equiv 0$ whenever $i \neq j$ and each component u_i is a nonnegative function satisfying

$$-\Delta u_i = f_i(u_i) \qquad \text{in } \{u_1 > 0\},$$

where $f_i(s) = \omega_i s^3 - \lambda_i s$, $\lambda_i = \lim_{\beta\to+\infty} \lambda_{i,\beta}$. Observe that clearly we have $f_i(s) = \mathrm{O}(s)$ as $s \to 0$. Thus it only remains to prove that U satisfies condition (G3) in Definition 3.3.

The proof of this fact is contained in the proof of Proposition 2.25. Here we present a more direct approach which makes use of the identity (3.6). Fix $x_0 \in \Omega$ and let

$$\tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 \qquad \text{for every } 0 < r < \mathrm{d}(x_0, \partial\Omega).$$

We have

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2-N}{r^{N-1}} \int_{B_r(x_0)} |\nabla U|^2 + \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} |\nabla U|^2 \, d\sigma. \tag{3.35}$$

We use identity (3.6) for $u = u_{i,\beta}$ and combine it with (3.34), obtaining

$$\begin{aligned}\operatorname{div}\left((x-x_0)|\nabla u_{i,\beta}|^2 - 2\langle x - x_0, \nabla u_{i,\beta}\rangle \nabla u_{i,\beta}\right) &= (N-2)|\nabla u_{i,\beta}|^2 - 2\langle x - x_0, \nabla u_{i,\beta}\rangle \Delta u_{i,\beta} \\ &= (N-2)|\nabla u_{i,\beta}|^2 + 2\langle x - x_0, \nabla u_{i,\beta}\rangle \Big(\omega_i u_{i,\beta}^3 - \lambda_{i,\beta} u_{i,\beta} - \beta u_{i,\beta} \sum_{\substack{j=1 \\ j\neq i}}^m u_{j,\beta}^2\Big).\end{aligned}$$

By integrating by parts the previous identity in $B_r(x_0)$, it follows that

$$\begin{aligned} r\int_{\partial B_r(x_0)} |\nabla u_{i,\beta}|^2 \, d\sigma &= 2r\int_{\partial B_r(x_0)} (\partial_\nu u_{i,\beta})^2 \, d\sigma + (N-2)\int_{B_r(x_0)} |\nabla u_{i,\beta}|^2 + \\ &\quad + 2\int_{B_r(x_0)} \langle x - x_0, \nabla u_{i,\beta}\rangle \Big(\omega_i u_{i,\beta}^3 - \lambda_{i,\beta} u_{i,\beta} - \beta u_{i,\beta} \sum_{\substack{j=1 \\ j\neq i}}^m u_{j,\beta}^2\Big). \end{aligned} \tag{3.36}$$

Observe that due to the variational structure of (3.34) we have

$$\begin{aligned} &\sum_{i=1}^m 2\int_{B_r(x_0)} \langle x - x_0, \nabla u_{i,\beta}\rangle \beta u_{i,\beta} \sum_{\substack{j=1 \\ j\neq i}}^m u_{j,\beta}^2 = \beta \sum_{i=1}^m \int_{B_r(x_0)} \langle x - x_0, \nabla (u_{i,\beta}^2)\rangle \sum_{\substack{j=1 \\ j\neq i}}^m u_{j,\beta}^2 \\ &= \beta \sum_{\substack{i,j=1 \\ i<j}}^m \int_{B_r(x_0)} \langle x - x_0, \nabla(u_{i,\beta}^2 u_{j,\beta}^2)\rangle = -\sum_{\substack{i,j=1 \\ i<j}}^m \int_{B_r(x_0)} N\beta u_{i,\beta}^2 u_{j,\beta}^2 + \sum_{\substack{i,j=1 \\ i<j}}^m \int_{\partial B_r(x_0)} r\beta u_{i,\beta}^2 u_{j,\beta}^2 \, d\sigma \to 0 \end{aligned}$$

as $\beta \to +\infty$, as it can be deduced from Theorem 2.2-(ii). Thus by summing up in i the identities (3.36) and by making $\beta \to +\infty$ we obtain

$$\begin{aligned} r\int_{\partial B_r(x_0)} |\nabla U|^2 \, d\sigma &= 2r\int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + (N-2)\int_{B_r(x_0)} |\nabla U|^2 + \\ &\quad + 2\int_{B_r(x_0)} \sum_{i=1}^m \langle x - x_0, \nabla u_i\rangle (\omega_i u_i^3 - \lambda_i u_i) \end{aligned}$$

which, replaced in (3.35), gives

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_{i=1}^m \langle x - x_0, \nabla u_i\rangle (\omega_i u_i^3 - \lambda_i u_i).$$

□

Hence Theorems 3.1, 3.2 and 3.48 provide a new regularity result for the asymptotic limits of families of uniformly bounded excited state solutions of (3.34).

Corollary 3.49. *Let U be a limiting profile of* (3.34)*. Then $\mathscr{H}_{\dim}(\Gamma_U) \leqslant N-1$ and there exists a set $\Sigma_U \subseteq \Gamma_U$, relatively open in Γ_U, such that*

- *$\mathscr{H}_{\dim}(\Gamma_U \setminus \Sigma_U) \leqslant N-2$, and if $N = 2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- *Σ_U is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \Sigma_U$*
$$\lim_{x\to x_0^+} |\nabla U(x)| = \lim_{x\to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface. Moreover,
$$\lim_{x\to x_0} |\nabla U(x)| = 0 \qquad \textit{for every } x_0 \in \Gamma_U \setminus \Sigma_U.$$
If $N = 2$, then Σ_U consists actually of a locally finite collection of curves meeting with equal angles at singular points.

We observe once again that Caffarelli and Lin [31] and Conti, Terracini and Verzini [46, 47] have obtained similar results but only for the case when U is a solution of (3.34) having minimal energy.

3.7.2 The class $\mathcal{S}(\Omega)$

Recall from Section 1.3 the following class

$$\mathcal{S}(\Omega) = \left\{ U \in \left(H^1(\Omega) \cap L^\infty(\Omega)\right)^m : \begin{array}{c} u_i \geqslant 0,\ u_i \cdot u_j \equiv 0 \text{ for } i \neq j,\ -\Delta u_i \leqslant f_i(u_i), \\ -\Delta\left(u_i - \sum_{j\neq i} u_j\right) \geqslant f_i(u_i) - \sum_{j\neq i} f_j(u_j) \end{array} \right\}.$$

where $f(s) = \mathrm{O}(s)$ as $s \to 0^+$. As observed in Chapter 1, this class is associated with the minimal solution of (3.34). We show that the theory that we have developed in the present chapter contains the one that was already developed for the class $\mathcal{S}(\Omega)$. In fact, we have the following somehow surprising result.

Theorem 3.50. *$S(\Omega) \subseteq \mathcal{G}(\Omega)$.*

Proof. 1. Let $U \in \mathcal{S}(\Omega)$. We have $u_i \geqslant 0$ for every i, $u_i \cdot u_j \equiv 0$ whenever $i \neq j$ and, by Theorem 1.11 (which proof can be found in [47]), U is Lipschitz continuous. In each set $\{u_i > 0\}$ we have

$$-\Delta u_i \leqslant f_i(u_i) \qquad \text{and} \qquad -\Delta u_i = -\Delta\Big(u_i - \sum_{\substack{j=1\\ j\neq i}}^m u_j\Big) \geqslant f_i(u_i) - \sum_{\substack{j=1\\ j\neq i}}^m f_j(u_j) = f_i(u_i),$$

and hence
$$-\Delta u_i = f_i(u_i) \qquad \text{in } \{u_i > 0\}.$$

Thus, in order to conclude that $U \in \mathcal{G}(\Omega)$ the remaining thing to prove is that U satisfies property (G3) of Definition 3.3. To prove this fact we follow the ideas contained in [29, Theorem 15].
2. Consider $\delta > 0$ in such a way that each set $\{u_i > \delta\}$ is regular; moreover take $x_0 \in \Omega$ and $r > 0$. For simplicity we consider the case $F \equiv 0$. By using once again the Pohǒzaev–type identity (3.29), integrating it in each set $\{u_i > \delta\} \cap B_r(x_0)$ we have

$$r \int_{\partial B_r(x_0)\cap\{u_i>\delta\}} |\nabla u_i|^2 \, d\sigma = 2r \int_{\partial B_r(x_0)\cap\{u_i>\delta\}} (\partial_\nu u_i)^2 \, d\sigma -$$
$$- \int_{B_r(x_0)\cap\partial\{u_i>\delta\}} \Big(\langle \nu, x - x_0\rangle |\nabla u_i|^2 - 2\langle x - x_0, \nabla u_i\rangle (\partial_\nu u_i) \Big) \, d\sigma + (N-2) \int_{B_r(x_0)\cap\{u_i>\delta\}} |\nabla u_i|^2$$
$$= 2r \int_{\partial B_r(x_0)\cap\{u_i>\delta\}} (\partial_\nu u_i)^2 \, d\sigma + \int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i|^2 \langle \nu, x - x_0\rangle \, d\sigma + (N-2) \int_{B_r(x_0)\cap\{u_i>\delta\}} |\nabla u_i|^2 .$$

Thus

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2-N}{r^{N-1}} \int_{B_r(x_0)} |\nabla U|^2 + \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} |\nabla U|^2 \, d\sigma$$
$$= \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{1}{r^{N-1}} \sum_{i=1}^m \limsup_{\delta\to 0^+} \int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i|^2 \langle \nu, x - x_0\rangle \, d\sigma.$$

We claim that

$$\sum_{i=1}^m \limsup_{\delta\to 0^+} \int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i|^2 \langle \nu, x - x_0\rangle = 0, \tag{3.37}$$

which finishes the proof of the theorem. This will require two more steps. Fix $\varepsilon > 0$ and define the set $S_\varepsilon = \{x \in \Omega \setminus \Gamma_U : \sum_i |\nabla u_i| \leqslant \varepsilon\}$.
3. Since each component u_i is harmonic in $\{u_i > 0\}$, we have that

$$\int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i| \, d\sigma = - \int_{\partial B_r(x_0)\cap\{u_i>\delta\}} \partial_\nu u_i \, d\sigma \leqslant \int_{\partial B_r(x_0)} |\partial_\nu u_i| \, d\sigma.$$

Thus there exists a constant $C > 0$, independent of δ, such that

$$\int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i| \, d\sigma \leqslant C$$

and hence

$$\limsup_{\varepsilon\to 0^+} \limsup_{\delta\to 0^+} \Big| \int_{B_r(x_0)\cap\partial\{u_i>\delta\}\cap S_\varepsilon} |\nabla u_i|^2 \langle \nu, x - x_0\rangle \, d\sigma \Big|$$
$$\leqslant \limsup_{\varepsilon\to 0^+} \Big| \int_{B_r(x_0)\cap\partial\{u_i>\delta\}} |\nabla u_i| \varepsilon r \Big| \leqslant \limsup_{\varepsilon\to 0^+} C'\varepsilon = 0. \tag{3.38}$$

4. In [29, Theorem 14] it is shown that $|\nabla U|$ is a continuous function in Ω. Fix $x \in \Gamma_U \cap (\overline{\Omega \setminus S_\varepsilon})$. Then we have $|\nabla U|(x) \neq 0$ and thus in a small neighborhood of x there exist

exactly two components $u_{\kappa_1}, u_{\kappa_2}$, with $\kappa_1, \kappa_2 \in \{1, \dots, m\}$, $\kappa_1 \neq \kappa_2$. Hence $\Delta(u_{\kappa_1} - u_{\kappa_2}) = 0$ in $B_\gamma(x)$ and

$$\sum_{i=1}^m \limsup_{\delta \to 0^+} \int_{B_\gamma(x) \cap \partial\{u_i > \delta\} \cap (\Omega \setminus S_\varepsilon)} |\nabla u_i|^2 \langle \nu, x - x_0 \rangle \, d\sigma =$$

$$\int_{B_\gamma(x) \cap \partial\{u_{\kappa_1} > 0\} \cap (\Omega \setminus S_\varepsilon)} |\nabla u_{\kappa_1}|^2 \langle \nu, x - x_0 \rangle \, d\sigma + \int_{B_\gamma(x) \cap \partial\{u_{\kappa_2} > 0\} \cap (\Omega \setminus S_\varepsilon)} |\nabla u_{\kappa_2}|^2 \langle \nu, x - x_0 \rangle \, d\sigma = 0.$$

Therefore

$$\sum_{i=1}^m \limsup_{\delta \to 0^+} \int_{B_r(x_0) \cap \partial\{u_i > \delta\} \cap (\Omega \setminus S_\varepsilon)} |\nabla u_i|^2 \langle \nu, x - x_0 \rangle \, d\sigma = 0,$$

and by combining this with (3.38) we finally obtain (3.37), as claimed. □

Corollary 3.51. *Let $U \in \mathcal{S}(\Omega)$. Then $\mathscr{H}_{\dim}(\Gamma_U) \leqslant N-1$ and there exists a set $\Sigma_U \subseteq \Gamma_U$, relatively open in Γ_U, such that*

- *$\mathscr{H}_{\dim}(\Gamma_U \setminus \Sigma_U) \leqslant N-2$, and if $N = 2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- *Σ_U is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \Sigma_U$*
$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^\pm$ are taken from the opposite sides of the hyper-surface. Moreover,

$$\lim_{x \to x_0} |\nabla U(x)| = 0 \qquad \text{for every } x_0 \in \Gamma_U \setminus \Sigma_U.$$

If $N = 2$, then Σ_U consists actually of a locally finite collection of curves meeting with equal angles at singular points.

As observed in [29, 46, 47, 48, 49, 69, 70, 131], the class $\mathcal{S}(\Omega)$ is also related to the asymptotic limits of reaction-diffusion systems with a Lotka-Volterra–type competition terms, as well as to certain optimal partition problems. As can be seen in Corollaries 3.49 and 3.51, the regularity results of Theorems 3.1 and 3.2 apply to different contexts, and hence we believe that our approach has the advantage of showing some new connections between several problems. The connection point is the fact that the solutions to all these problems satisfy what we have called a Reflection Principle (property (G3)).

Next we recall with some detail two problems already studied in the literature which are related to the class $\mathcal{S}(\Omega)$.

Lotka-Volterra competitive interactions

Consider the following Lotka-Volterra model for the competition between m different species

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j \neq i}}^m u_j & \\ u_i = \varphi_i \text{ on } \partial\Omega, \; u_i > 0 \text{ in } \Omega, & \end{cases} \qquad i = 1, \dots, m \tag{3.39}$$

with $\Omega \subset \mathbb{R}^N$ a smooth bounded domain and φ_i positive $W^{1,\infty}(\partial\Omega)$–functions such that $\varphi_i \cdot \varphi_j \equiv 0$ for $i \neq j$. We remark that this system does not possess a variational structure, in opposition to (3.34). The asymptotic behavior of its solutions (as $\beta \to +\infty$) has been the object of recent study, see for instance [29, 49, 131] and references therein. In [49, Theorem 1], it is shown that all possible H^1–limits U of a given sequence of solutions $(U_\beta)_{\beta>0}$ of (3.39) (as $\beta \to +\infty$) belong to $\mathcal{S}(\Omega)$, under suitable hypotheses for the f_i's. Such proof takes advantage of the fact that the competition terms in (3.39) are simpler than the ones in (3.34). More precisely, the competition between two different components u_i and u_j is symmetric (the effect of u_j over u_i is modeled by the term $\beta u_i u_j$, as well as the effect of u_j over u_i). In fact, to better illustrate our previous statements we observe that due to the shape of the competition terms, we have that for each $\beta > 0$ fixed,

$$\begin{aligned} -\Delta\Big(u_{i,\beta} - \sum_{\substack{j=1\\ j\neq i}}^m u_{j,\beta}\Big) &= f_i(u_{i,\beta}) - \sum_{\substack{j=1\\ j\neq i}}^m f_j(u_{j,\beta}) - \beta \sum_{\substack{j=1\\ j\neq i}}^m u_{i,\beta}u_{j,\beta} + \beta \sum_{\substack{j=1\\ j\neq i}}^m \sum_{\substack{k=1\\ k\neq j}}^m u_{j,\beta}u_{k,\beta} \\ &\geqslant f_i(u_{i,\beta}) - \sum_{\substack{j=1\\ j\neq i}}^m f_j(u_{j,\beta}) \end{aligned}$$

which, as $\beta \to +\infty$, provides (1.14), the key property in $\mathcal{S}(\Omega)$.

Regularity of interfaces in optimal partition problems

Next we consider some optimal partition problems involving eigenvalues. For any integer $m \geqslant 0$, we define the set of m–partitions of Ω as

$$\mathfrak{B}_m = \{(\omega_1, \dots, \omega_m) : \ \omega_i \text{ measurable },\ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\}.$$

Consider the following optimization problems: for any positive real number $p \geqslant 1$,

$$\mathfrak{L}_{m,p} := \inf_{\mathfrak{B}_m} \left(\frac{1}{m} \sum_{i=1}^m (\lambda_1(w_i))^p \right)^{1/p}, \tag{3.40}$$

and

$$\mathfrak{L}_m := \inf_{\mathfrak{B}_m} \max_{i=1,\dots,m} (\lambda_1(\omega_i)), \tag{3.41}$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\omega)$ in a generalized sense (check [69, Definition 3.1]). We refer to the papers [30, 48, 69] for a more detailed description of these problems (in [48], for instance, it is shown that (3.41) is a limiting problem for (3.40), in the sense that $\lim_{p\to+\infty} \mathfrak{L}_{m,p} = \mathfrak{L}_m$). Our theory applies to opportune multiples of solutions of (3.40) and (3.41). More precisely, in [48, Lemma 2.1] it is shown that

- let $p \in [1, +\infty)$ and let $(\omega_1, \dots, \omega_m) \in \mathfrak{B}_m$ be any minimal partition associated with $\mathfrak{L}_{m,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\omega_i))_i$. Then there exist $a_i > 0$ such that the functions $u_i = a_i\phi_i$ verify in Ω, for every $i = 1, \dots, m$, the variational inequalities $-\Delta u_i \leqslant \lambda_1(\omega_i)u_i$ and $-\Delta(u_i - \sum_{j\neq i} u_j) \geqslant \lambda_1(\omega_i)u_i - \sum_{j\neq i} \lambda_1(\omega_i)u_j$ in the distributional sense;

and in [69, Theorem 3.4]:

- let $(\tilde{\omega}_1, \ldots, \tilde{\omega}_m) \in \mathfrak{B}_m$ be any minimal partition associated with $\mathfrak{L}_m$ and let $(\tilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\tilde{\omega}_i))_i$. Then there exist $a_i \geqslant 0$, not all vanishing, such that the functions $\tilde{u}_i = a_i \tilde{\phi}_i$ verify in Ω, for every $i = 1, \ldots, m$, the variational inequalities $-\Delta \tilde{u}_i \leqslant \mathfrak{L}_m \tilde{u}_i$ and $-\Delta(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j) \geqslant \mathfrak{L}_m(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j)$ in the distributional sense.

In particular the functions $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ and $U = (u_1, \ldots, u_m)$ belong to $\mathcal{S}(\Omega)$.

We refer to the book [24] for other interesting optimization problems. It is our belief that the solutions to some of these problems should belong to the class $\mathcal{G}(\Omega)$.

3.8 Additional comments. An open question

1. In the previous section we have shown that $\mathcal{S}(\Omega) \subseteq \mathcal{G}(\Omega)$. Let us show that an equality between these two classes of functions does not happen to be true. For that, we will take $m = 2$ from now on. Take $(u_1, u_2) \in \mathcal{S}(\Omega)$ such that $u_1, u_2 \neq 0$. Then $u_1 \cdot u_2 \equiv 0$, $-\Delta(u_1 - u_2) = f_1(u_1) - f_2(u_2)$ in Ω and in particular

$$|\nabla u_1| = |\nabla u_2| \qquad \text{on } \Gamma_{(u_1,u_2)} \neq \emptyset.$$

Define $u = u_1 + u_2$, $v \equiv 0$. Since $(u_1, u_2) \in \mathcal{G}(\Omega)$, we also have that $(u, v) \in \mathcal{G}(\Omega)$. However, $(u, v) \notin \mathcal{S}(\Omega)$, because we cannot have

$$-\Delta(u - v) = -\Delta u = f_1(u) + f_2(u) \qquad \text{in } \Omega$$

by the maximum principle. Hence we have shown that

$$S(\Omega) \subsetneqq \mathcal{G}(\Omega) \qquad \text{for every bounded domain } \Omega \subseteq \mathbb{R}^N.$$

2. The element (u, v) that we have built in the previous paragraph has the following property:

There exist $x_0 \in \Gamma_U$ such that $v \equiv 0$ in a small neighborhood of x_0.

What happens if we consider a vector function $U \in \mathcal{G}(\Omega)$ such that none of its components satisfy such a condition in the regular part Σ_U? Again we consider $m = 2$ and take $(u, v) \in \mathcal{G}(\Omega)$ satisfying the following:

(C) For every $x_0 \in \Sigma_U$ and every $\delta > 0$ we have $u, v \not\equiv 0$ in $B_\delta(x_0)$.

We claim that then

$$-\Delta(u - v) = f_1(u) - f_2(v), \qquad \text{and hence } (u, v) \in \mathcal{S}(\Omega).$$

Let $x_0 \in \Omega$.
Case a) If $x_0 \in \{u > 0\} \cap \{v > 0\}$, then either $-\Delta u = f_1(u)$ or $-\Delta v = f_2(v)$ in a small neighborhood of x_0, and hence there exists $\delta > 0$ such that

$$-\Delta(u - v) = f_1(u) - f_2(v) \qquad \text{in } B_\delta(x_0).$$

Case b) Suppose now that $x_0 \in \Sigma_U$ where Σ_U is the regular part of Γ_U given by Theorem 3.1. By condition (C) we have that $x_0 \in \partial\{u > 0\} \cap \partial\{v > 0\}$, and there exists $\delta > 0$ such that

$$|\nabla U| \neq 0 \qquad \text{on } \Gamma_U \cap B_\delta(x_0).$$

Thus

$$|\nabla u| = |\nabla v| \qquad \text{on } \Gamma_U \cap B_\delta(x_0), \quad \text{ and } \quad -\Delta(u - v) = f_1(u) - f_2(v) \text{ in } B_\delta(x_0).$$

By combining the cases a) and b) we conclude that

$$-\Delta(u - v) = f_1(u) - f_2(v) \qquad \text{in } \Omega \setminus S_U,$$

where $\mathscr{H}^{N-1}(S_U) \leqslant N - 2$, and thus actually

$$-\Delta(u - v) = f_1(u) - f_2(v) \qquad \text{in } \Omega,$$

as claimed.

3. The previous discussions lead to the formulation of an open question concerning the system (3.34) for the case of $m = 2$ equations. Take

$$\begin{cases} -\Delta u + \lambda_\beta u = \omega_1 u^3 - \beta u v^2 \\ -\Delta v + \mu_\beta v = \omega_2 v^3 - \beta u^2 v \\ u, v \in H_0^1(\Omega), \ u, v > 0 \ \text{ in } \Omega, \end{cases} \tag{3.42}$$

and let (u_β, v_β) be a sequence of solutions, uniformly bounded in $L^\infty(\Omega)$. Denote by (u, v) an associated limiting profile. If (u_β, v_β) are minimal energy solutions as in Sections 1.1 and 1.2, then $(u, v) \in \mathcal{S}(\Omega)$ and in particular

$$-\Delta(u - v) + \lambda u - \mu v = \omega_1 u^3 - \omega_2 v^3 \qquad \text{in } \Omega. \tag{3.43}$$

What if (u_β, v_β) are excited state solutions of (3.42)? Is it still true that as $\beta \to +\infty$ we always get (3.43)? As we have seen, in general we can only guarantee that $(u, v) \in \mathcal{S}(\Omega)$. Hence the question is whether property (C) is always satisfied for the limiting profiles of (3.42). As a first step it would be even interesting to find conditions on (u_β, v_β), more general than a minimality assumption, that would imply that the associated limiting profiles satisfy (3.43).

We are not able to solve this problem. However, it is possible in some cases to deduce more connections between the system (3.42) and the equation (3.43). This will be the topic of the next chapter.

Chapter 4

Convergence of minimax levels and continuation of critical points for singularly perturbed systems

4.1 Introduction

4.1.1 Motivations

We are interested in the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + u^3 + \beta u v^2 = \lambda u \\ -\Delta v + v^3 + \beta u^2 v = \mu v \\ u, v \in H_0^1(\Omega), \quad u, v > 0, \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N = 2, 3$, and λ, μ, β are positive parameters. We observe that (4.1) is a particular case of (1.1) of defocusing type. Exactly as in Section 1.2, we study solutions of (4.1) as nonnegative critical points of the coercive energy functional

$$J_\beta(u, v) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) + \frac{1}{4} \int_\Omega \left(u^4 + v^4\right) + \frac{\beta}{2} \int_\Omega u^2 v^2$$

constrained to the manifold

$$\mathcal{M} = \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : \int_\Omega u^2 = \int_\Omega v^2 = 1 \right\},$$

so that λ and μ in (4.1) are understood as Lagrange multipliers. This constraint represents a standard mass conservation law. In this chapter we will study the relation between suitable solutions (u_β, v_β) of (4.1), for β large, and the pairs (w^+, w^-) where w solves

$$-\Delta w + w^3 = \lambda w^+ - \mu w^-, \qquad w \in H_0^1(\Omega); \tag{4.2}$$

(for suitable λ, μ).

Besides the existence of minimal energy solutions, because of the invariance of J_β and $\mathcal{M}$ under the $\mathbb{Z}_2$ action $(u, v) \mapsto (v, u)$ we expect a multiplicity of critical points of minimax type for each β. We are mainly interested in the behavior of such solutions as $\beta \to +\infty$. In contrast with the approaches followed in the previous chapters and in the existing litterature (see [31, 34, 74, 83, 127, 132, 133], among others), here we analyze the asymptotic of the whole minimax structure, and we approach the problem from the point of view of Γ–convergence.

Consider the pointwise limit of J_β as β goes to infinity. Such limit is the extended valued functional defined (for $(u, v) \in \mathcal{M}$) as

$$J_\infty(u, v) = \sup_{\beta>0} J_\beta(u, v) = \begin{cases} J_0(u, v) & \text{when } \int_\Omega u^2v^2 = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

This functional turns out to be also the Γ–limit of J_β (for the definition of Γ–convergence we refer, for instance, to the book by Braides [22]). In this framework, while it is immediate to check the convergence of the minima of J_β on $\mathcal{M}$ to minima of J_∞, it is not even obvious what should be understood as critical point of J_∞ (because of its strong irregularity). Also in the case when a notion of critical point is established for the limiting functional, there needs not to be convergence: Jerrard and Sternberg in [73, Remark 4.5] exhibit an example of family of functions $(f^\varepsilon)_\varepsilon$ Γ–converging to f, in which a sequence of critical points of f^ε does not converge to a critical point of f .

Similarly to [73], we face the problem from an abstract point of view (see Section 4.2 ahead). In fact, we consider a general family of functionals, depending on a parameter β, and its Γ–limit. These functionals share the basic property of being lower semi-continuous (with respect to a suitable topology) and non decreasing with respect to β. After the introduction of a common minimax class, we provide a notion of critical point in connection with a choice of decreasing flows. The main problem is that, in our application, the limit of the gradient flows as $\beta \to +\infty$ needs not to be itself a continuous decreasing flow for the limiting functional (such limits were studied in [32] for the related heat equation). This prevents us to apply the recent theory of Γ–convergence of gradient flows developed in [113]. We stress that, for this reason, *we do not assume any relation between the flows for $\beta < \infty$ and the limit flow.*

To construct the flow for $\beta = +\infty$, we use the equation (4.2) which is related to the functional J_∞, in the case $u \cdot v \equiv 0$, by the identity

$$J_\infty(u, v) = J^*(u - v), \qquad \text{where } J^*(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega w^4 \tag{4.3}$$

(with $\int_\Omega (w^+)^2 = \int_\Omega (w^-)^2 = 1$).

Thus we will understand the critical points of J_∞ constrained to $\mathcal{M}$ as the pairs (u, v), such that $u \cdot v \equiv 0$ and $u - v$ satisfies equation (4.2). As a matter of fact, we will establish, also for minimax critical points, a relation between suitable solutions of (4.1), for β large, and the solutions of (4.2). Other results in the same direction have been obtained for radial functions in the recent papers [127, 133] for non minimal solutions, whereas, up to our knowledge, there are no results concerning non-radial excited state solutions.

4.1.2 A class of minimax problems

To proceed with the exposition of our main results, we need to introduce some suitable minimax framework which is admissible for the whole family of functionals. We are inspired by a recent work by Dancer, Wei and Weth [55], where infinitely many critical levels are found, in the focusing case, by exploiting the Krasnoselskii genus technique (see, for instance, the book by Struwe [121]) associated with the invariance of the problem when interchanging the roles of u and v.

In carrying on our asymptotic analysis, we will take advantage of a strong compactness property that goes beyond the usual Palais–Smale condition; to this aim we are led to set the genus theory in the L^2–topology. This is the main reason why we are addressing here the defocusing case: in the focusing one, indeed, the fact that the associated Nehari manifold is not closed for the L^2–norm seems to prevent us from performing an analogous analysis. Let us consider the involution

$$\sigma : H^1_0(\Omega) \times H^1_0(\Omega) \to H^1_0(\Omega) \times H^1_0(\Omega), \quad (u, v) \mapsto \sigma(u, v) = (v, u),$$

and the class of sets

$$\mathcal{F}_0 = \left\{ A \subset \mathcal{M} : \begin{array}{l} \bullet\ A \text{ is closed in the } L^2\text{–topology} \\ \bullet\ u \geqslant 0, v \geqslant 0\ \forall (u, v) \in A \\ \bullet\ \sigma(A) = A \end{array} \right\}$$

(observe that $\mathcal{M}$ is L^2–closed and that $\sigma(\mathcal{M}) = \mathcal{M}$). We can define the *Krasnoselskii L^2–genus* in $\mathcal{F}_0$ in the following way.

Definition 4.1. *Let $A \in \mathcal{F}_0$. The* L^2–genus *of A, denoted by $\gamma_2(A)$, is defined as*

$$\gamma_2(A) := \inf \left\{ m \in \mathbb{N} : \begin{array}{c} \text{there exists } f : A \to \mathbb{R}^m \setminus \{0\} \text{ such that} \\ \bullet\ f \text{ is continuous in the } L^2\text{–topology} \\ \bullet\ f(\sigma(u, v)) = -f(u, v) \text{ for every } (u, v) \in A \end{array} \right\}.$$

If no f as above exists, then $\gamma_2(A) := +\infty$, while $\gamma_2(\emptyset) := 0$. The set of subsets with L^2–genus at least k will be denoted by

$$\mathcal{F}_k = \{A \in \mathcal{F}_0 : \gamma_2(A) \geqslant k\}.$$

Under the previous notations we define, for $k \in \mathbb{N}$ and $0 < \beta \leqslant +\infty$, the minimax levels

$$c^k_\beta = \inf_{A \in \mathcal{F}_k} \sup_{(u,v) \in A} J_\beta(u, v). \tag{4.4}$$

In order to simplify notations, for $\beta < \infty$ we introduce a function $S_\beta : H^1_0(\Omega) \times H^1_0(\Omega) \times \mathbb{R}^2 \to \mathbb{R}^2$, in such a way that system (4.1) rewrites as follows

$$\begin{cases} S_\beta(u, v; \lambda, \mu) = \begin{pmatrix} -\Delta u + u^3 + \beta u v^2 - \lambda u \\ -\Delta v + v^3 + \beta u^2 v - \mu v \end{pmatrix} = 0 \\ u, v \in H^1_0(\Omega), \quad u, v > 0. \end{cases} \tag{4.5}$$

When $\beta < +\infty$, we define the critical set associated with c_β^k in the standard way

$$\mathcal{K}_\beta^k = \left\{ (u,v) \in \mathcal{M} : \begin{array}{l} u, v \geqslant 0, \\ J_\beta(u,v) = c_\beta^k, \text{ and} \\ \text{there exist } \lambda, \mu \text{ such that } S_\beta(u,v;\lambda,\mu) = (0,0) \end{array} \right\}.$$

Coming to the limiting problem, for the reasons we exposed before, we define the critical set as

$$\mathcal{K}_\infty^k = \left\{ (u,v) \in \mathcal{M} : \begin{array}{l} u, v \geqslant 0, \\ J_\infty(u,v) = c_\infty^k, \text{ and} \\ \text{there exist } \lambda, \mu \text{ such that } u - v \text{ solves (4.2)} \end{array} \right\}.$$

Our first main result states the existence of critical points and of *optimal sets*, in the following sense.

Theorem 4.2. *Let $k \in \mathbb{N}$ and $0 < \beta \leqslant +\infty$ be fixed. Then:*

1. *$\mathcal{K}_\beta^k$ is non empty and compact (with respect to the H_0^1–topology);*
2. *there exist $A_\beta^k \in \mathcal{F}_k$ and $(u_\beta^k, v_\beta^k) \in A_\beta^k \cap \mathcal{K}_\beta^k$ such that*
$$c_\beta^k = \max_{A_\beta^k} J_\beta = J_\beta(u_\beta^k, v_\beta^k).$$

As in the usual genus theory, one may also prove that, if c_β^k is the same for different k's, then the genus of $\mathcal{K}_\beta^k$ is large. This, together with suitable conditions which allow to avoid fixed points of σ (namely β large enough, see Lemma 4.20 ahead), provides the existence of many distinct critical points of J_β.

4.1.3 Limits as $\beta \to +\infty$

Since the same variational argument applies both to the β–finite and to the limiting case, the next step is to compare the limiting behavior of the variational structure as $\beta \to +\infty$ with the actual behavior at $\beta = +\infty$. This involves the study of the critical levels, of the optimal sets (in the sense of Theorem 4.2) and, finally, of the critical sets. Regarding the first two questions we have full convergence.

Theorem 4.3. *Let $k \in \mathbb{N}$ be fixed. As $\beta \to +\infty$ we have*

1. *$c_\beta^k \to c_\infty^k$;*
2. *if A_n^k is any optimal set for $c_{\beta_n}^k$, and $\beta_n \to +\infty$, then the set $\limsup_n A_n^k$ is optimal for c_∞^k (where the limit is intended in the L^2–sense).*

It is worthwhile to notice that, in general, the convergence of the critical levels is a delicate fact to prove (for instance, it remains an open problem in [92]). As previously mentioned, up to now there already existed results in this direction concerning the radial case only (is this case the nodal sets of the limiting equation are easier to handle). Coming to the convergence of the critical sets, we obtain the following relation.

Theorem 4.4. *Let*

$$\mathcal{K}_*^k = \left\{ (u,v) : \begin{array}{l} \text{there exist sequences } (u_n,v_n) \in \mathcal{M} \text{ with } u_n, v_n \geqslant 0, \\ \text{and } \beta_n \to +\infty \text{ such that} \\ \quad \bullet\ (u_n,v_n) \to (u,v) \text{ in } L^2, \\ \quad \bullet\ J_{\beta_n}(u_n,v_n) \to c_\infty^k, \text{ and} \\ \quad \bullet\ S_{\beta_n}(u_n,v_n) \to (0,0) \text{ in } L^2 \end{array} \right\}.$$

Then

$$\mathcal{K}_*^k \cap \mathcal{K}_\infty^k \quad \text{is not empty.}$$

This result can be better understood in the formulation below, which makes use of the uniform Hölder bounds obtained in Chapter 2. Such bounds imply that the L^2–convergence $(u_n,v_n) \to (u,v)$ in the definition of $\mathcal{K}_*^k$ is in fact strong in $H^1(\Omega) \cap C^{0,\alpha}(\bar\Omega)$.

Corollary 4.5. *For every integer k there exist pairs (u_∞, v_∞), $(\lambda_\infty, \mu_\infty)$ satisfying*

$$-\Delta(u_\infty - v_\infty) + (u_\infty - v_\infty)^3 + \lambda_\infty u_\infty - \mu_\infty v_\infty = 0,$$

at level c_∞^k, and (sub)sequences (u_β, v_β), (h_β, k_β), $(\lambda_\beta, \mu_\beta)$ satisfying

$$\begin{cases} -\Delta u_\beta + u_\beta^3 + \beta u_\beta v_\beta^2 - \lambda_\beta u_\beta = h_\beta \\ -\Delta v_\beta + v_\beta^3 + \beta u_\beta^2 v_\beta - \mu_\beta v_\beta = k_\beta \\ u_\beta, v_\beta \in H_0^1(\Omega), \quad u, v > 0, \end{cases}$$

such that $(\lambda_\beta, \mu_\beta) \to (\lambda_\infty, \mu_\infty)$,

$$(h_\beta, k_\beta) \to (0,0) \text{ in } L^2, \quad \text{and} \quad (u_\beta, v_\beta) \to (u_\infty, v_\infty) \text{ in } H^1(\Omega) \cap C^{0,\alpha}(\bar\Omega).$$

We address the open question of finding under which conditions a solution of (4.2) is the limit of a sequence of a solution of (4.1) (see also Section 3.8 for a related discussion).

This chapter is structured as follows. In Section 4.2 we present an abstract framework of variational type; we introduce a family of functionals enjoying suitable properties and perform an asymptotic analysis. Section 4.3 is devoted to fit (4.1) into the abstract setting; this immediately provides the convergence of the critical levels and of the optimal sets. Finally, in Section 4.4 we conclude the proof of the main results: we address existence and asymptotics of the critical points, leaving to Section 4.5 the technical details about the flows used in the deformation lemmas.

Notations. In this chapter we define $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx$ (sometimes it will also denote the vectorial norm). We will refer to the topology induced on $H_0^1(\Omega) \times H_0^1(\Omega)$ by the $L^2(\Omega) \times L^2(\Omega)$ norm as the "L^2–topology" (and we shall denote by $\langle \cdot, \cdot \rangle_2$, dist_2 the associated inner product and distance respectively). On the other hand, we will denote the usual topology on $H_0^1(\Omega) \times H_0^1(\Omega)$ the "H_0^1–topology". Finally, recall that, for a sequence of sets $(A_n)_n$,

$$x \in \limsup_n A_n \iff \text{for some } n_k \to +\infty \text{ there exist } x_{n_k} \in A_{n_k} \text{ such that } x_{n_k} \to x.^1$$

[1]This is the limit superior in the framework of the Kuratowski convergence.

4.2 Topological setting of a class of minimax principles

In this section we will introduce an abstract setting of minimax type in order to obtain critical values (in a suitable sense) of a given functional. Our aim is to consider a class of functionals, each of which fitting in such a setting, and to perform an asymptotic analysis of the variational structure. The asymptotic convergence requires some additional compactness, in the form of assumptions ($\mathcal{F}2$), ($\mathcal{F}2$') below. Later on, when applying these results, the compactness will be achieved by weakening the topology; the price to pay will be a loss of regularity of the functional involved. For this reason, with respect to the usual variational schemes, our main issue is to work with functionals that are merely lower semi-continuous.

Let $\mathcal{M}$ be a metric space and let us consider a set of subsets of $\mathcal{M}$, $\mathcal{F} \subseteq P(\mathcal{M})$. Given a lower semi-continuous functional $J : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$, we define the minimax level

$$c = \inf_{A \in \mathcal{F}} \sup_{x \in A} J(x),$$

and make the following assumptions:

($\mathcal{F}1$) A is closed in $\mathcal{M}$ for every $A \in \mathcal{F}$;

($\mathcal{F}2$) there exists $c' > c$ such that for any given $(A_n)_n \subseteq \mathcal{F}$ with $A_n \subseteq \mathcal{M}^{c'}$ for every n, it holds $\limsup_n A_n \in \mathcal{F}$,

where we denote

$$\mathcal{M}^{c'} = \left\{ x \in \mathcal{M} : \, J(x) \leqslant c' \right\}.$$

Moreover from now on we will suppose that $c \in \mathbb{R}$, which in particular implies that $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$. A first consequence of the compactness assumption ($\mathcal{F}2$) is the existence of an optimal set of the minimax procedure.

Proposition 4.6. *Let $J : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional, assume ($\mathcal{F}2$) and suppose moreover that $c \in \mathbb{R}$. Then there exists $\bar{A} \in \mathcal{F}$ such that* $\sup_{\bar{A}} J = c$. *In this situation, we will say that $\bar{A}$* is optimal for J at c.

Proof. For every $n \in \mathbb{N}$ let $A_n \in \mathcal{F}$ be such that

$$\sup_{A_n} J \leqslant c + \frac{1}{n}$$

and consider $\bar{A} := \limsup_n A_n$. On one hand we have $\bar{A} \in \mathcal{F}$ by assumption ($\mathcal{F}2$), which provides $\sup_{\bar{A}} J \geqslant c$. On the other hand, by the definition of lim sup, for any $x \in \bar{A}$ there exists a sequence $(x_n)_n$, $x_n \in A_n$, such that $x_n \to x$, up to a subsequence. The lower semi-continuity implies

$$J(x) \leqslant \liminf_n J(x_n) \leqslant \liminf_n \Big(\sup_{A_n} J \Big) \leqslant c,$$

and the proposition follows by taking the supremum in $x \in \bar{A}$. $\square$

Due to the lack of regularity of the functional it is not obvious what should be understood as critical set. We will give a very general definition of critical set at level c by means of a "deformation", defined in some sub-level of J, under which the functional decreases. To this aim we consider, for some $c' > c$, a map $\eta : \mathcal{M}^{c'} \to \mathcal{M}$ such that

(η1) $\eta(A) \in \mathcal{F}$ whenever $A \in \mathcal{F}$, $A \subseteq \mathcal{M}^{c'}$;

(η2) $J(\eta(x)) \leqslant J(x)$, for every $x \in \mathcal{M}^{c'}$.

We define the critical set of J (relative to η) at level c as

$$\mathcal{K}_c = \{x \in \mathcal{M} : \, J(x) = J(\eta(x)) = c\}$$

(notice that $x \in \mathcal{M}^c$ and hence $\eta(x)$ is well defined). We remark that the previous definition depends on the choice of η. In a quite standard way, some more compactness is needed in the form of a Palais–Smale type assumption.

Definition 4.7. *We say that the pair (J, η) satisfies $(PS)_c$ if for any given sequence $(x_n)_n \subset \mathcal{M}$ such that $J(x_n) \to c$, $J(\eta(x_n)) \to c$, there exists $\bar{x} \in \mathcal{K}_c$ such that, up to a subsequence, $x_n \to \bar{x}$.*

Remark 4.8. Incidentally we observe that, if in $(\mathrm{PS})_c$ one would require $\bar{x}$ to be also the limit of $\eta(x_n)$ (we do not assume it in this section, but it will turn out to be true in the subsequent application), as a consequence $\mathcal{K}_c$ would coincide with the set of the fixed points of η at level c, providing an alternative definition – perhaps more intuitive – of "critical set" (relative to η).

As usual, $(\mathrm{PS})_c$ immediately implies the compactness of $\mathcal{K}_c$. This assumption also implies the fact that every optimal set for J at level c intersects $\mathcal{K}_c$ (which in particular is non empty). More precisely

Theorem 4.9. *Let $J : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional, assume ($\mathcal{F}$1) and ($\mathcal{F}$2) and let $\eta : \mathcal{M}^{c'} \to \mathcal{M}$ be a map satisfying (η1) and (η2). Suppose moreover that (J, η) verifies $(PS)_c$ and that $c \in \mathbb{R}$. Then for every $A \in \mathcal{F}$ such that $\sup_A J = c$ there exists $\bar{x} \in A \cap \mathcal{K}_c$. In particular, K_c is non empty.*

Proof. Let $A \in \mathcal{F}$ be such that $\sup_A J = c$ (which exists by Proposition 4.6). By assumptions (η1) and (η2), $\eta(A) \in \mathcal{F}$ and $\sup_{\eta(A)} J \leqslant c$, hence $\sup_{\eta(A)} J = c$. Thus we can find a sequence $(x_n)_n \subset A$ such that $J(\eta(x_n)) \to c$. By using again assumption (η2), we infer

$$c \geqslant J(x_n) \geqslant J(\eta(x_n)) \to c,$$

and therefore (up to a subsequence) $x_n \to \bar{x} \in \mathcal{K}_c$ by $(\mathrm{PS})_c$. On the other hand, since $A \in \mathcal{F}$, assumption ($\mathcal{F}$1) implies that $\bar{x} \in A$, which concludes the proof of the theorem. □

Let us now turn to the asymptotic analysis. First of all we introduce a family of functionals parameterized on $\beta \in (0, +\infty)$, namely $J_\beta : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$, each of which is lower semi-continuous and moreover

(J) $J_{\beta_1}(x) \leqslant J_{\beta_2}(x)$ for every $x \in \mathcal{M}$, whenever $0 < \beta_1 \leqslant \beta_2 < +\infty$.

In such a framework we define the limit functional

$$J_\infty(x) := \sup_{\beta>0} J_\beta(x).$$

Lemma 4.10. *For every $x_n, x \in \mathcal{M}$ and $(\beta_n)_n \subseteq \mathbb{R}^+$ such that $x_n \to x$ and $\beta_n \to +\infty$, it holds*

$$J_\infty(x) \leqslant \liminf_n J_{\beta_n}(x_n).$$

In particular, J_∞ is lower semi-continuous, and J_β Γ–converges to J_∞.

Proof. For every fixed $0 < \beta < +\infty$ we see that

$$J_\beta(x) \leqslant \liminf_n J_\beta(x_n) \leqslant \liminf_n J_{\beta_n}(x_n) \leqslant \liminf_n J_\infty(x_n)$$

by using the lower semi-continuity of J_β and the fact that $J_\beta \leqslant J_{\beta_n}$ for n sufficiently large. Then by taking the supremum in β the lemma follows. □

Consequently, for $0 < \beta \leqslant +\infty$, we define the minimax levels

$$c_\beta = \inf_{A\in\mathcal{F}} \sup_{x\in A} J_\beta(x).$$

Remark 4.11. Assumption (J) clearly yields that

$$\beta_1 < \beta_2 < +\infty \qquad \Longrightarrow \qquad c_{\beta_1} \leqslant c_{\beta_2} \leqslant c_\infty.$$

The previous remark suggests that any constant greater than c_∞ is a suitable common bound for all the functionals. Hence we replace ($\mathcal{F}$2) with the following assumption.

($\mathcal{F}$2') for any given $(A_n)_n \subset \mathcal{F}$ such that, for some β, $A_n \subset \mathcal{M}_\beta^{c_\infty+1}$ for every n, it holds $\limsup_n A_n \in \mathcal{F}$,

where

$$\mathcal{M}_\beta^{c'} = \left\{x \in \mathcal{M} :\ J_\beta(x) \leqslant c'\right\}.$$

Observe that condition ($\mathcal{F}$2') implies that J_β satisfies ($\mathcal{F}$2) for each fixed $0 < \beta \leqslant +\infty$. Our first main result concerns the convergence of both the critical levels and the optimal sets (recall Proposition 4.6).

Theorem 4.12. *Let $J_\beta : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ ($0 < \beta < +\infty$) be a family of lower semi-continuous functionals satisfying (J) and let J_∞ be as before. Moreover suppose that condition ($\mathcal{F}$2') holds, and that $c_\beta \in \mathbb{R}$ for every $0 < \beta \leqslant +\infty$. Then*

1. *for every $0 < \beta < +\infty$ there exists an optimal set for J_β at c_β;*
2. *$c_\beta \to c_\infty$ as $\beta \to +\infty$;*
3. *if $A_n \in \mathcal{F}$ is optimal for J_{β_n} at c_{β_n} and $\beta_n \to +\infty$, then $A_\infty := \limsup_n A_n$ is optimal for J_∞ at c_∞.*

Proof. The first point is a direct consequence of Proposition 4.6. Now by Remark 4.11 we know that c_β is monotone in β and that $\lim_\beta c_\beta \leqslant c_\infty < +\infty$ by assumption. Let β_n, A_n, A_∞ be as in the statement of the theorem. We have that $\sup_{A_n} J_{\beta_1} \leqslant \sup_{A_n} J_{\beta_n} \leqslant c_\infty$, therefore $A_n \subset \{J_{\beta_1} \leqslant c_\infty + 1\}$ and assumption ($\mathcal{F}$2') provides $A_\infty \in \mathcal{F}$. For every $x \in A_\infty$ there exists a (sub)sequence $x_n \to x$, with $x_n \in A_n$. By taking into account Lemma 4.10 we have

$$J_\infty(x) \leqslant \liminf_n J_{\beta_n}(x_n) \leqslant \liminf_n \left(\sup_{A_n} J_{\beta_n} \right) = \lim_n c_{\beta_n} \leqslant c_\infty.$$

By taking the supremum in $x \in A_\infty$ (and by recalling that $A_\infty \in \mathcal{F}$), the theorem follows. □

Next we turn to the study of the corresponding critical sets, by introducing a family of maps $\eta_\beta : \mathcal{M}_\beta^{c_\infty+1} \to \mathcal{M}$ satisfying

$(\eta 1)_\beta$ $\eta_\beta(A) \in \mathcal{F}$ whenever $A \in \mathcal{F}$, $A \subseteq \mathcal{M}_\beta^{c_\infty+1}$;

$(\eta 2)_\beta$ $J_\beta(\eta_\beta(x)) \leqslant J_\beta(x)$, for every $x \in \mathcal{M}_\beta^{c_\infty+1}$.

Just as we did before, we define, for every $0 < \beta \leqslant +\infty$

$$\mathcal{K}_\beta = \mathcal{K}_{c_\beta} = \{x \in \mathcal{M} : J_\beta(x) = J_\beta(\eta_\beta(x)) = c_\beta\}. \tag{4.6}$$

As a straightforward consequence of Theorem 4.9, the following holds.

Theorem 4.13. *Let $J_\beta : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ $(0 < \beta < +\infty)$ be a family of lower semi-continuous functionals satisfying (J) and let J_∞ be as before. Suppose that ($\mathcal{F}$1), ($\mathcal{F}$2') hold and that, for every $0 < \beta \leqslant +\infty$, $c_\beta \in \mathbb{R}$ and the maps $\eta_\beta : \mathcal{M}_\beta^{c_\infty+1} \to \mathcal{M}$ verify $(\eta 1)_\beta$ and $(\eta 2)_\beta$. Suppose moreover that the pair (J_β, η_β) satisfies $(PS)_{c_\beta}$. Then every optimal set for J_β at c_β intersect $\mathcal{K}_\beta$, which in particular is non empty (for $\beta \leqslant +\infty$).*

It is now natural to wonder what is the relation between $\limsup_\beta \mathcal{K}_\beta$ and $\mathcal{K}_\infty$. The desired result would be the equality of the sets, which could be obtained under some suitable relations between the deformations η_β and η_∞. However, as we mentioned in the introduction of the chapter, in our application such relations do not seem to hold. Instead we will assume a uniform Palais–Smale type condition, which will lead us to consider a slightly larger set than $\limsup_\beta \mathcal{K}_\beta$. Let us assume that the following holds:

(UPS) if the sequences $(x_n)_n \subset \mathcal{M}$, $(\beta_n)_n \subset \mathbb{R}^+$ are such that $\beta_n \to +\infty$ and $J_{\beta_n}(x_n) \to c_\infty$, $J_{\beta_n}(\eta_{\beta_n}(x_n)) \to c_\infty$, then there exists $\bar{x} \in \mathcal{M}$ such that, up to a subsequence, $x_n \to \bar{x}$ and $\eta_{\beta_n}(x_n) \to \bar{x}$

(observe that $\eta_{\beta_n}(x_n)$ is well defined for large n since $J_{\beta_n}(x_n) \to c_\infty$). It is worthwhile to point out explicitly the two main differences between (PS) and (UPS), apart from the dependence on β. First, in the latter we do not obtain $\bar{x} \in \mathcal{K}_\infty$ – see Remark 4.15 below. Second, in (UPS) we require not only x_n but also $\eta_{\beta_n}(x_n)$ to converge, and the limit to be the same (to enlighten this choice, see also Remark 4.8). Condition (UPS) suggests the introduction of the set

$$\mathcal{C}_* = \left\{ x \in \mathcal{M} : \begin{array}{l} \text{there exist sequences } (x_n)_n \subseteq \mathcal{M} \text{ and } (\beta_n)_n \subseteq \mathbb{R}^+ \\ \text{with } \beta_n \to +\infty \text{ such that} \\ \quad \bullet\ x_n \to x, \\ \quad \bullet\ J_{\beta_n}(x_n) \to c_\infty, \\ \quad \bullet\ J_{\beta_n}(\eta_{\beta_n}(x_n)) \to c_\infty \end{array} \right\}.$$

Remark 4.14. If $(x_n)_n$ is a uniform Palais–Smale sequence in the sense of assumption (UPS), then (up to a subsequence) $x_n \to \bar{x} \in \mathcal{C}_*$.

Remark 4.15. By Theorem 4.12-2, it is immediate to verify that

$$\limsup_{\beta \to +\infty} \mathcal{K}_\beta \subset \mathcal{C}_*.$$

Furthermore, if $x \in \mathcal{C}_*$ then by Lemma 4.10 we obtain $J_\infty(x) \leqslant c_\infty$. Observe that the inequality may be strict (nonetheless, the following theorem will imply that for some point the equality holds).

Our final result of this section is the following.

Theorem 4.16. *Under the assumptions of Theorem 4.13, suppose moreover that (UPS) holds. Then we have that*

$$\mathcal{C}_* \cap K_\infty \neq \emptyset.$$

More precisely, for every $(A_n)_n \subset \mathcal{F}$, with A_n optimal for J_{β_n} at c_{β_n}, and $\beta_n \to +\infty$, there exists $\bar{x} \in \mathcal{C}_* \cap K_\infty \cap \limsup_n A_n$.

Proof. Let A_n be as in the statement, and take $B_n = \eta_{\beta_n}(A_n)$, which is also optimal for J_{β_n} at c_{β_n} by assumptions $(\eta 1)_{\beta_n}$, $(\eta 2)_{\beta_n}$. Theorem 4.12 then yields that $\limsup_n B_n =: B_\infty \in \mathcal{F}$ is optimal for J_∞ at c_∞, and there exists

$$\bar{y} \in B_\infty \cap K_\infty.$$

By definition, up to a subsequence there exists $x_n \in A_n$ such that $\eta_{\beta_n}(x_n) \to \bar{y}$. Then assumption $(\eta 2)_{\beta_n}$ together with Lemma 4.10 provides

$$c_\infty = J_\infty(\bar{y}) \leqslant \liminf_n J_{\beta_n}(\eta_{\beta_n}(x_n)) \leqslant \liminf_n J_{\beta_n}(x_n) \leqslant \lim_n \left(\sup_{A_n} J_{\beta_n} \right) = \lim_n c_n = c_\infty.$$

In particular this implies that $(x_n)_n$ is a uniform Palais–Smale sequence in the sense of assumption (UPS); by using Remark 4.14 we then infer that (again up to a subsequence)

$$x_n \to \bar{x} \in \limsup_n A_n \cap \mathcal{C}_*.$$

But (UPS) also implies that $\eta_{\beta_n}(x_n) \to \bar{x}$ and hence $\bar{x} = \bar{y}$, which concludes the proof of the theorem. $\square$

4.3 Convergence of the minimax levels

The rest of the chapter is devoted to apply (and refine) the results obtained in the previous section to the problem discussed in the introduction. In order to apply the abstract results of Section 4.2 we need to introduce $\mathcal{M}$, $\mathcal{F}$ and η_β for the present case. In this section we deal with the asymptotics of the minimax levels and prove Theorem 4.3. The proof of the remaining results, and in particular the construction of the deformations, will be the object of the subsequent sections. Since the proof is independent of k, from now on and throughout all the chapter we assume that

$$k \in \mathbb{N} \text{ is fixed (and will often be omitted).}$$

We define the metric space

$$\mathcal{M} = \left\{ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) : u, v \geqslant 0 \text{ in } \Omega, \int_\Omega u^2 = \int_\Omega v^2 = 1 \right\},$$

equipped with the distance

$$\begin{aligned} \mathrm{dist}_2\left((u_1,v_1),(u_2,v_2)\right) &= \left(\|u_1-u_2\|_2^2 + \|v_1-v_2\|_2^2\right)^{1/2} \\ &= \left(\int_\Omega (u_1-u_2)^2 + \int_\Omega (v_1-v_2)^2\right)^{1/2}, \end{aligned}$$

and take

$$J_\beta(u,v) = \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right) + \frac{1}{4}\int_\Omega \left(u^4+v^4\right) + \frac{\beta}{2}\int_\Omega u^2 v^2$$

for $0 < \beta < +\infty$. Notice that the limiting functional (as introduced in Section 4.2) coincides with the one defined in the introduction, *i.e.*,

$$J_\infty(u,v) = \sup_{\beta>0} J_\beta(u,v) = \begin{cases} J_0(u,v) & \text{when } \int_\Omega u^2v^2\,dx = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover we set

$$\mathcal{F} = \mathcal{F}_k = \{A \in \mathcal{F}_0 :\ \gamma_2(A) \geqslant k\} \quad \text{(as in Definition 4.1)},$$

which implies that the critical values c_β introduced in Section 4.2 coincide with the values c_β^k defined in the introduction.

Remark 4.17. It is worthwhile to stress that for any given $c' \in \mathbb{R}$ and $0 < \beta \leqslant \infty$ the set

$$\mathcal{M}_\beta^{c'} = \{(u,v) \in \mathcal{M} :\ J_\beta(u,v) \leqslant c'\}$$

is L^2–compact. This is a consequence of the coercivity of the functionals J_β together with the Sobolev embeddings. This motivates our decision of working with this topology.

We start by presenting some properties of the L^2–genus (recall Definition 4.1).

Proposition 4.18. *(i) Take $A \in \mathcal{F}_0$ and let S^{k-1} be the standard $(k-1)$–sphere in $\mathbb{R}^k$. If there exists an L^2–homeomorphism $\psi : S^{k-1} \to A$ satisfying $\psi(-x) = \sigma(\psi(x))$ then $\gamma_2(A) = k$.*

(ii) *Consider* $A \in \mathcal{F}_k$ *and let* $\eta : A \to \mathcal{M}$ *be an* L^2*–continuous,* σ*–equivariant and sign–preserving map. Then* $\overline{\eta(A)} \in \mathcal{F}_k$*.*

(iii) *If* $A \in \mathcal{F}_0$ *is an* L^2*–compact set, then there exists* $\delta > 0$ *such that* $\gamma_2\left(N_\delta(A) \cap \mathcal{M}\right) = \gamma_2(A)$*.*

(iv) *Let* $(A_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{F}_k$ *and let* X *be an* L^2*–compact subset of* $\mathcal{M}$ *such that* $A_n \subset X$ $\forall n$*. Then* $\limsup A_n \in \mathcal{F}_k$*.*

Proof. (i) Consider the inverse function $\psi^{-1} : A \to S^{k-1} \subseteq \mathbb{R}^k \setminus \{0\}$. Since $\sigma(u, v) = \sigma(\psi \circ \psi^{-1}(u, v)) = \psi(-\psi^{-1}(u, v))$, we get $\psi^{-1}(\sigma(u, v)) = -\psi^{-1}(u, v)$ for every $(u, v) \in A$ and hence $\gamma_2(A) \leqslant k$. On the other hand, if $\gamma_2(A) = l \leqslant k - 1$, then there exists a continuous function $f : A \to \mathbb{R}^l \setminus \{0\}$ satisfying $f(\sigma(u, v)) = -f(u, v)$. But then we get the existence of a continuous odd function $f \circ \psi : S^{k-1} \subseteq \mathbb{R}^k \setminus \{0\} \to \mathbb{R}^l \setminus \{0\}$, a contradiction by Borsuk-Ulam's Theorem.
(ii) First of all we observe that from the hypothesis made on η it is straightforward to show that $\overline{\eta(A)} \in \mathcal{F}$. Suppose now that $\gamma_2(\overline{\eta(A)}) = m < \infty$ and let $f : \overline{\eta(A)} \to \mathbb{R}^m \setminus \{0\}$ be a continuous function satisfying $f(\sigma(u, v)) = -f(u, v)$ for every $(u, v) \in \overline{\eta(A)}$. The map $f \circ \eta : A \to \mathbb{R}^m \setminus \{0\}$ is continuous and verifies $f \circ \eta(\sigma(u, v)) = f(\sigma(\eta(u, v))) = -f(\eta(u, v))$ for every $(u, v) \in A$, whence $\gamma_2(A) \leqslant m$. In particular $\overline{\eta(A)} \in \mathcal{F}_k$ whenever $A \in \mathcal{F}_k$.
(iii) From $A \subseteq N_\delta(A) \cap \mathcal{M}$ we know that $\gamma_2(A) \leqslant \gamma_2(N_\delta(A) \cap \mathcal{M})$ $\forall \delta$. Hence the conclusion follows in the case that $\gamma_2(A) = +\infty$. Now if $\gamma_2(A) = m$, there is a continuous function $f : A \to \mathbb{R}^m \setminus \{0\}$ such that $f \circ \sigma = -f$. Consider an extension F of f to $\mathcal{M}$ satisfying $F \circ \sigma = -F$. Since A is compact, there exists $\delta > 0$ such that $F \neq 0$ in $N_\delta(A) \cap \mathcal{M}$, and hence $\gamma_2\left(N_\delta(A) \cap \mathcal{M}\right) \leqslant m = \gamma_2(A)$.
(iv) Recalling that

$$\limsup A_n = \{(u, v) \in \mathcal{M} : \exists n_k \to +\infty,\ (u_{n_k}, v_{n_k}) \in A_{n_k} \text{ such that } (u_{n_k}, v_{n_k}) \to (u, v)\},$$

it is easy to check that $\limsup A_n \in \mathcal{F}_0$. We now claim that for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$A_n \subseteq (N_\delta(\limsup A_n) \cap \mathcal{M}) \qquad \text{for } n \geqslant n_0,$$

which, together with the previous point, implies the desired result. Suppose that our claim is false. Then there is a $\delta > 0$ and $n_k \to +\infty$, $(u_{n_k}, v_{n_k}) \in A_{n_k}$ such that $(u_{n_k}, v_{n_k}) \notin N_\delta(\limsup A_n)$. But since X is compact, there is a $(u, v) \in X \subset M$ such that, up to a subsequence, $(u_{n_k}, v_{n_k}) \to (u, v)$, hence $(u, v) \in \limsup A_n$, a contradiction. □

Lemma 4.19. *For every* β *finite it holds* $0 \leqslant c_\beta \leqslant c_\infty < +\infty$*.*

Proof. The proof of the lemma relies on the fact that , given any $k \in \mathbb{N}$, we can construct a set $G_k \in \mathcal{F}_0$ with $\gamma_2(G_k) = k$. Here we use some ideas presented in [55, Proposition 4.3]. Indeed, consider k functions $\phi_1, \ldots, \phi_k \in H^1_0(\Omega)$ such that $\phi_i \cdot \phi_j = 0$ a.e. for any $i \neq j$, with $\phi_i^+, \phi_i^- \neq 0$. Define

$$\psi : S^{k-1} \to \mathcal{M}, \qquad (t_1, \ldots, t_k) \mapsto \left(\bar{t}\left(\sum_{i=1}^k t_i\phi_i\right)^+, \bar{s}\left(\sum_{i=1}^k t_i\phi_i\right)^-\right),$$

where

$$\bar{t}^2 = \frac{1}{\left\|(\sum_i t_i\phi_i)^+\right\|_2^2} = \frac{1}{\left(\sum_i t_i^2 \left\|\phi_i^+\right\|_2^2\right)}, \qquad \bar{s}^2 = \frac{1}{\left\|(\sum_i t_i\phi_i)^-\right\|_2^2} = \frac{1}{\left(\sum_i t_i^2 \left\|\phi_i^-\right\|_2^2\right)},$$

and $G_k = \psi(S^{k-1})$. It is easy to verify that $G_k \in \mathcal{F}_0$. Since ψ is an L^2–homeomorphism between S^{k-1} and G_k, and $\sigma(\psi(t_1, \dots, t_k)) = \psi(-t_1, \dots, -t_k)$, then Proposition 4.18-(i) provides that $\gamma_2(G_k) = k$. Since $u \cdot v \equiv 0$ whenever $(u, v) \in G_k$, then

$$c_\infty \leqslant \sup_{G_k} J_\infty < +\infty.$$

Finally, Remark 4.11 allows to conclude the proof. □

We are already in a position to prove the convergence of the minimax levels.

Proof of Theorem 4.3. This is a direct consequence of Theorem 4.12. Let us check its hypotheses. Under the above definitions, assumption (J) easily holds. For every $0 < \beta \leqslant +\infty$, $c_\beta \in \mathbb{R}$ (by Lemma 4.19), and moreover ($\mathcal{F}2$') holds (by recalling Remark 4.17, Proposition 4.18-(iv) and by using the fact that $c_\infty \in \mathbb{R}$). Finally let us check that each J_β is a lower semi-continuous functional in $(\mathcal{M}, \mathrm{dist}_2)$, for $0 < \beta < +\infty$. Indeed, let (u_n, v_n), $(\bar{u}, \bar{v})$ be couples of H_0^1 functions such that $\mathrm{dist}_2((u_n, v_n), (\bar{u}, \bar{v})) \to 0$. If $\liminf_n J_\beta(u_n, v_n) = +\infty$ then there is nothing left to prove, otherwise, by passing to the subsequence that achieves the $\liminf$, we have that $\|(u_n, v_n)\|$ is bounded. Thus, again up to a subsequence, $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v})$ weakly in H_0^1. Then we can conclude by using the weak lower semicontinuity of $\|\cdot\|$ (and the weak continuity of the other terms in J_β). □

Let us conclude this section recalling that, if β is sufficiently large, then we can exclude the presence of fixed points of σ in the set $\mathcal{K}_\beta^k$. As in the usual genus theory, this insures that, if two (or more) critical values coincide, then $\mathcal{K}_\beta^k$ contains an infinite number of elements.

Lemma 4.20. *Let $k \in \mathbb{N}$ be fixed. There exists a (finite) number $\bar{\beta}(k) > 0$, depending only on k, such that, for every $\bar{\beta}(k) \leqslant \beta \leqslant +\infty$, we have*

$$\mathcal{K}_\beta^k \cap \{(u, u) \in \mathcal{M}\} = \emptyset.$$

Proof. When $\beta = +\infty$ the assertion holds true with no limitations on β, since $J_\infty(u, u) < +\infty$ implies $u \equiv 0$, and $(0, 0) \notin \mathcal{M}$. For $\beta < +\infty$ we have that

$$\inf_{(u,u)\in\mathcal{M}} J_\beta(u, v) = \inf_{\|u\|_2=1} \left(\|u\|^2 + \frac{1+\beta}{2}\int_\Omega u^4\right) \geqslant \inf_{\|u\|_2=1} \frac{1+\beta}{2|\Omega|}\left(\int_\Omega u^2\right)^2 = \frac{1+\beta}{2|\Omega|}.$$

Taking into account Lemma 4.19, the assertion of the lemma is proved once

$$\frac{1+\beta}{2|\Omega|} > c_\infty^k,$$

which holds true for $\beta \geqslant \bar{\beta}(k) = 2|\Omega| c_\infty^k$. □

4.4 Existence and asymptotics of the critical points

In this section we prove the remaining results stated in the introduction. To this aim we will define suitable deformations η_β, which will allow us to apply the abstract results of Section 4.2 that concern the critical sets – namely Theorems 4.13 and 4.16. Afterwards, we will establish the equivalence between the critical sets defined in the introduction and the ones of Section 4.2.

As we have already mentioned, in order to fulfill our purposes we need to choose different deformations for the case $\beta < +\infty$ and $\beta = +\infty$. Let us start with the definition of η_β for each $0 < \beta < +\infty$ fixed. The desired map will make use of the parabolic flow associated to J_β on $\mathcal{M}$. In order to do so, first we need to fix a relation between (λ, μ) and (u, v).

Remark 4.21. If $(u, v) \in \mathcal{M}$ satisfies (4.5) then, by testing the equations with u and v respectively, one immediately obtains

$$\lambda = \lambda(u,v) = \frac{\int_\Omega \left(|\nabla u|^2 + u^4 + \beta u^2 v^2\right)}{\int_\Omega u^2} = \int_\Omega \left(|\nabla u|^2 + u^4 + \beta u^2 v^2\right),$$

$$\mu = \mu(u,v) = \frac{\int_\Omega \left(|\nabla v|^2 + v^4 + \beta u^2 v^2\right)}{\int_\Omega v^2} = \int_\Omega \left(|\nabla v|^2 + v^4 + \beta u^2 v^2\right).$$

Motivated by the previous remark and by the definition of S_β (see (4.5)), we write, with some abuse of notations,

$$S_\beta(u,v) = S_\beta(u,v;\lambda(u,v),\mu(u,v)) = \begin{pmatrix} -\Delta u + u^3 + \beta u v^2 - \lambda(u,v) u \\ -\Delta v + v^3 + \beta u^2 v - \mu(u,v) v \end{pmatrix}, \tag{4.7}$$

with λ, μ as above. Then, for $(u, v) \in \mathcal{M}$, we consider the initial value problem with unknowns $U(x, t)$, $V(x, t)$,

$$\begin{cases} \partial_t(U,V) = -S_\beta(U,V) \\ U(\cdot,t), V(\cdot,t) \in H_0^1(\Omega) \\ U(x,0) = u(x), \quad V(x,0) = v(x), \end{cases} \tag{4.8}$$

We have the following existence result.

Lemma 4.22. *For every $(u, v) \in \mathcal{M}_\beta^{c_\infty+1}$ problem (4.8) has exactly one solution*

$$(U(t),V(t)) \in C^1\left((0,+\infty); L^2(\Omega)\times L^2(\Omega)\right) \cap C\left([0,+\infty); H_0^1(\Omega)\times H_0^1(\Omega)\right).$$

Moreover, for every $t > 0$, $\|(U(t), V(t))\|_2 = 1$ and

$$\frac{d}{dt} J_\beta(U(t),V(t)) = -\left\|S_\beta(U(t),V(t))\right\|_2^2 \leqslant 0.$$

We postpone to Section 4.5 the proofs of this result and of the subsequent properties.

Proposition 4.23. *With the notations of Lemma 4.22, the following properties hold* *(i)* $U(t) \geqslant 0, V(t) \geqslant 0$, *for every* $(u,v) \in \mathcal{M}_\beta^{c_\infty+1}$ *and* $t > 0$*;*

(ii) *for every fixed* $t > 0$ *the map* $(u,v) \mapsto (U(t), V(t))$ *is* L^2*–continuous from* $\mathcal{M}_\beta^{c_\infty+1}$ *into itself;*

(iii) *if* $(u,v) \in \mathcal{M}_\beta^{c_\infty+1}$ *and* $s, t \in [0, +\infty)$ *then*

$$\mathrm{dist}_2\left((U(s),V(s)),(U(t),V(t))\right) \leqslant |t-s|^{1/2} |J_\beta(U(s),V(s)) - J_\beta(U(t),V(t))|^{1/2}.$$

The previous results allow us to define an appropriate deformation, along with some key properties.

Proposition 4.24. *Let us define, under the above notations,*

$$\eta_\beta : \mathcal{M}_\beta^{c_\infty+1} \to \mathcal{M}_\beta^{c_\infty+1}, \qquad (u,v) \mapsto \eta_\beta(u,v) = (U(1),V(1)).$$

Then η_β *satisfies assumptions* $(\eta 1)_\beta$ *and* $(\eta 2)_\beta$*.*

Proof. Lemma 4.22 implies that

$$J_\beta(\eta_\beta(u,v)) = J_\beta(U(1),V(1)) \leqslant J_\beta(U(0),V(0)) = J_\beta(u,v)$$

for every (u,v), which corresponds exactly to condition $(\eta 2)_\beta$. Now consider $A \in \mathcal{F}_k$ such that $A \subseteq \mathcal{M}_\beta^{c_\infty+1}$. Observe that η_β is $\sigma-$ equivariant (by the uniqueness of the initial value problem (4.8) with respect to the initial datum) and that it is L^2–continuous and sign-preserving (by Proposition 4.23-(i),(ii)). Thus Proposition 4.18-(ii) applies, yielding $\overline{\eta_\beta(A)} \in \mathcal{F}_k$. Since A is L^2–compact in $\mathcal{M}$ (indeed it is a closed subset of the L^2–compact $\mathcal{M}_\beta^{c_\infty}$) then $\eta_\beta(A)$ is closed, and therefore assumption $(\eta 1)_\beta$ holds. □

Before moving to the infinite case, let us prove the validity of a Palais–Smale type condition. It will be the key ingredient in order to show that (J_β, η_β) satisfies $(\mathrm{PS})_{c_\beta}$ in the sense of Definition 4.7.

Lemma 4.25. *Let* $(u_n, v_n) \in \mathcal{M}$ *be such that, as* $n \to +\infty$*,*

$$J_\beta(u_n,v_n) \to c \qquad \text{and} \qquad \|S_\beta(u_n,v_n)\|_2 \to 0$$

for some $c \geqslant 0$*. Then there exists* $(\bar u, \bar v) \in \mathcal{M} \cap (H^2(\Omega) \times H^2(\Omega))$ *such that, up to a subsequence,*

$$(u_n,v_n) \to (\bar u,\bar v) \text{ strongly in } H_0^1(\Omega) \qquad \text{and} \qquad S_\beta(\bar u,\bar v) = 0.$$

Proof. Since $J_\beta(u_n,v_n) \to c$, then we immediately infer the existence of $(\bar u,\bar v) \in \mathcal{M}$ such that $(u_n,v_n) \rightharpoonup (\bar u,\bar v)$ weakly in $H_0^1(\Omega)$, up to a subsequence. Let us first prove the H_0^1–strong convergence. From the fact that $\|S_\beta(u_n,v_n)\|_2 \to 0$ and that $u_n - \bar u$ is L^2–bounded, we deduce

$$\langle S_\beta(u_n,v_n),(u_n-\bar u,0)\rangle_2 = \int_\Omega (\langle \nabla u_n, \nabla(u_n-\bar u)\rangle + (u_n^3 + \beta u_n v_n^2 - \lambda(u_n,v_n)u_n)(u_n-\bar u)) \to 0.$$

This, together with

$$\left|\int_\Omega (u_n^3+\beta u_n v_n^2-\lambda(u_n,v_n)u_n)(u_n-\bar u)dx\right| \leqslant \|u_n^3+\beta u_n v_n^2-\lambda(u_n,v_n)u_n\|_2\|u_n-\bar u\|_2$$
$$\leqslant C\|u_n-\bar u\|_2\to 0,$$

implies that $\int_\Omega\langle\nabla u_n,\nabla(u_n-\bar u)\rangle\to 0$, which yields the desired convergence. The fact that $v_n\to\bar v$ strongly in $H^1_0(\Omega)$ can be proved in a similar way.

Now we pass to the proof of the last part of the statement. A first observations is that

$$\int_\Omega(\Delta u_n)^2+\int_\Omega(\Delta v_n)^2\leqslant 2\|S_\beta(u_n,v_n)\|_2^2+2\|u_n^3+\beta u_n v_n^2-\lambda(u_n,v_n)u_n\|_2^2+$$
$$+2\|v_n^3+\beta u_n^2 v_n-\mu(u_n,v_n)v_n\|_2^2\leqslant C,$$

which yields the weak H^2–convergence $u_n\rightharpoonup\bar u, v_n\rightharpoonup\bar v$ (up to a subsequence). As a consequence, we have that $\langle S_\beta(u_n,v_n),(\phi,\psi)\rangle_2\to\langle S_\beta(\bar u,\bar v),(\phi,\psi)\rangle_2$ for any given $(\phi,\psi)\in L^2(\Omega)\times L^2(\Omega)$. On the other hand, $\|S_\beta(u_n,v_n)\|_2\to 0$ provides that

$$\langle S_\beta(u_n,v_n),(\phi,\psi)\rangle_2\to 0,$$

and thus $S_\beta(\bar u,\bar v)=0$. □

Let us turn to the definition of the deformation η_∞. The main difficulty in this direction is that J_∞ is finite if and only if $uv\equiv 0$, thus any flow that we wish to use must preserve the disjointness of the supports. As we said in the introduction, here the criticality condition will be given by equation (4.2). In order to overcome the lack of regularity due to the presence of the positive/negative parts in the equation, we will use a suitable gradient flow, instead of a parabolic flow. More precisely we define

$$S_\infty: H^1_0(\Omega)\to H^1_0(\Omega)$$

to be the gradient of the functional $J^*(w)$ (see equation (4.3)) constrained to the set $\int_\Omega(w^+)^2=\int_\Omega(w^-)^2=1$. If $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ with Dirichlet boundary conditions, then we will prove in Section 4.5 the following result.

Lemma 4.26. *Let $R_1,R_2>0$ be fixed. For every $w\in H^1_0(\Omega)$ such that*

$$\|w^+\|_2,\|w^-\|_2\geqslant R_1\qquad\text{and}\qquad\|w\|\leqslant R_2$$

there exist unique $\tilde\lambda=\tilde\lambda(w)$, $\tilde\mu=\tilde\mu(w)$ such that

$$S_\infty(w)=w+(-\Delta)^{-1}\left(w^3-\tilde\lambda w^++\tilde\mu w^-\right).$$

Moreover, $\tilde\lambda$ and $\tilde\mu$ are Lipschitz continuous in w with respect to the L^2–topology, with Lipschitz constants only depending on R_1,R_2.

For every $(u,v)\in\mathcal{M}_\infty^{c_\infty+1}$ we consider the initial value problem (with unknown $W=W(t,x)$)

$$\begin{cases}\partial_t W=-S_\infty(W)\\ W(\cdot,t)\in H^1_0(\Omega)\\ W(x,0)=u(x)-v(x),\end{cases}\tag{4.9}$$

and prove existence and regularity of the solution.

Lemma 4.27. *For every $(u,v) \in \mathcal{M}_\infty^{c_\infty+1}$ problem* (4.9) *has exactly one solution*

$$W(t) \in C^1\left((0,+\infty); H_0^1(\Omega)\right) \cap C\left([0,+\infty); H_0^1(\Omega)\right).$$

Moreover, for every t, $(W^+(t), W^-(t)) \in \mathcal{M}_\infty^{c_\infty+1}$ and

$$\frac{d}{dt} J_\infty(W^+(t), W^-(t)) = -\|S_\infty(W(t))\|^2 \leqslant 0.$$

Again, the proof of this result can be found in Section 4.5, together with the proof of the following properties.

Proposition 4.28. *Using the notations of Lemma 4.27, the following properties hold*

(i) for every fixed $t > 0$ the map $(u,v) \mapsto (W^+(t), W^-(t))$ is L^2–continuous from $\mathcal{M}_\infty^{c_\infty+1}$ into itself;

(ii) if $(u,v) \in \mathcal{M}_\infty^{c_\infty+1}$ and $s,t \in [0,+\infty)$ then[2]

$$\begin{aligned}\mathrm{dist}_2((W^+(s), W^-(s)), (W^+(t), W^-(t)))& \\ \leqslant C_S|t-s|^{1/2}|J_\infty(W^+(s), W^-(s)) &- J_\infty(W^+(t), W^-(t))|^{1/2}.\end{aligned}$$

Similarly to the case β finite, the previous properties allow us to define a suitable deformation

Proposition 4.29. *Let us define, under the above notations,*

$$\eta_\infty : \mathcal{M}_\infty^{c_\infty+1} \to \mathcal{M}_\infty^{c_\infty+1}, \qquad (u,v) \mapsto \eta_\infty(u,v) = (W^+(1), W^-(1)).$$

Then η_∞ satisfies assumptions $(\eta 1)_\infty$ and $(\eta 2)_\infty$.

We omit the proof of the previous result since it is similar to the case β finite. Turning to the Palais–Smale condition, now we present a preliminary result.

Lemma 4.30. *Let $(u_n, v_n) \in \mathcal{M}_\infty^{c_\infty+1}$ be such that, as $n \to +\infty$,*

$$J_\infty(u_n, v_n) \to c_\infty \qquad \text{and} \qquad \|S_\infty(u_n - v_n)\| \to 0.$$

Then there exists $\bar{w} \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n - v_n \to \bar{w} \text{ strongly in } H_0^1(\Omega) \qquad \text{and} \qquad S_\infty(\bar{w}) = 0.$$

Proof. Let (w_1, w_2) be such that, up to subsequences, $u_n \rightharpoonup w_1$ and $v_n \rightharpoonup w_2$ in $H_0^1(\Omega)$. Since $J_\infty(u_n, v_n) < \infty$, then $u_n \cdot v_n = 0$ and therefore also $w_1 \cdot w_2 = 0$. Denote $w_n = u_n - v_n$ and $\bar{w} = w_1 - w_2$ in such a way that

$$S_\infty(u_n - v_n) = w_n + (-\Delta)^{-1}(w_n^3 - \tilde{\lambda}(w_n)w_n^+ + \tilde{\mu}(w_n)w_n^-).$$

[2] Here C_S is the Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Observe that for large n we have $\|w_n^+\|, \|w_n^-\| \leqslant 2c_\infty + 1 =: R_2$ and $\|w_n^+\|_2 = \|w_n^-\|_2 = 1 =: R_1$, and hence $\tilde\lambda(w_n)$ and $\tilde\mu$ are bounded sequences. By combining the boundedness of $w_n - \bar w$ in $H_0^1(\Omega)$ with the convergence $\|S_\infty(u_n - v_n)\| \to 0$ we obtain

$$\int_\Omega \langle \nabla(w_n - \bar w), \nabla S_\infty(u_n - v_n)\rangle = \int_\Omega (\langle \nabla w_n, \nabla(w_n - \bar w)\rangle + w_n^3(w_n - \bar w) - \\ - \tilde\lambda(w_n) w_n^+(w_n - \bar w) + \tilde\mu(w_n) w_n^-(w_n - \bar w))\, dx \to 0.$$

This, together with the fact that

$$\left| \int_\Omega (w_n^3(w_n - \bar w) - \tilde\lambda(w_n) w_n^+(w_n - \bar w) + \tilde\mu(w_n) w_n^-(w_n - \bar w))\, dx \right| \leqslant \\ \leqslant \|w_n^3 - \tilde\lambda(w_n) w_n^+ + \tilde\mu(w_n) w_n^-\|_2 \|w_n - \bar w\|_2 \to 0$$

gives $\displaystyle\int_\Omega \langle \nabla w_n, \nabla(w_n - \bar w)\rangle \to 0$, which yields the H_0^1–convergence of w_n to $\bar w$.

In order to conclude the proof of the lemma it remains to show that $S_\infty(\bar w) = 0$. But the continuity of $(-\Delta)^{-1}$ considered as an operator from $L^2(\Omega)$ to $H_0^1(\Omega)$ implies that

$$(-\Delta)^{-1}(w_n^3 - \tilde\lambda(w_n) w_n^+ + \tilde\mu(w_n) w_n^-) \to (-\Delta)^{-1}(\bar w^3 - \tilde\lambda(\bar w)\bar w^+ + \tilde\mu(\bar w)\bar w^-) \text{ in } H_0^1(\Omega).$$

and hence also $S_\infty(u_n - v_n) \to S_\infty(\bar w)$ in $H_0^1(\Omega)$, which concludes the proof. □

We are ready to show that the deformations we have defined satisfy the remaining abstract properties required in Section 4.2.

Proposition 4.31. *For every $0 < \beta \leqslant +\infty$, the pair (J_β, η_β) satisfies $(PS)_{c_\beta}$ (in the sense of Definition 4.7).*

Proof. Let first $\beta < \infty$ be fixed. Let $(u_n, v_n) \in \mathcal{M}$ be a Palais–Smale sequence in the sense of Definition 4.7, that is, $J_\beta(u_n, v_n) \to c_\beta$ and $J_\beta(\eta_\beta(u_n, v_n)) \to c_\beta$. Let then $(\bar u, \bar v) \in \mathcal{M}$ be such that, up to a subsequence, $(u_n, v_n) \to (\bar u, \bar v)$ in $L^2(\Omega)$. Define $(U_n(t), V_n(t))$ as the solution of (4.8) with initial datum (u_n, v_n) (recall that therefore $\eta_\beta(u_n, v_n) = (U_n(1), V_n(1))$). By applying Proposition 4.23-(iii) with $(s, t) = (0, 1)$ we obtain

$$\mathrm{dist}_2((u_n, v_n), \eta_\beta(u_n, v_n)) \leqslant |J_\beta(u_n, v_n) - J_\beta(\eta_\beta(u_n, v_n))|^{1/2} \to 0,$$

which, together with the L^2–continuity of η_β, yields $(\bar u, \bar v) = \eta_\beta(\bar u, \bar v)$. It only remains to show that $J_\beta(\bar u, \bar v) = c_\beta$. Notice that

$$\int_0^1 \|S_\beta(U_n(t), V_n(t))\|_2^2 dt = J_\beta(u_n, v_n) - J_\beta(\eta_\beta(u_n, v_n)) \to 0,$$

(by Lemma 4.22) and hence $\|S_\beta(U_n(t), V_n(t))\|_2 \to 0$ for almost every $t \in (0, 1)$. Fix one of such t's. Being J_β a decreasing functional under the heat flow, it holds $J_\beta(U_n(t), V_n(t)) \to c_\beta$. Now Lemma 4.25 applies providing the existence of $(u, v) \in \mathcal{M}$ such that $(U_n(t), V_n(t)) \to (u, v)$ in $H_0^1(\Omega)$, and in particular $J_\beta(u, v) = c_\beta$. Finally the use of Proposition 4.23-(iii) with $(s, t) = (0, t)$ allows us to conclude that $(u, v) = (\bar u, \bar v)$, and the proof is completed.

The case $\beta = +\infty$ can be treated similarly, by replacing $(U_n(t), V_n(t))$ with $(W_n^+(t), W_n^-(t))$ and $\|S_\beta(U_n(t), V_n(t))\|_2$ with $\|S_\infty(W_n(t))\|$. □

A uniform Palais–Smale condition also holds, in the sense of assumption (UPS). The proof of this fact is very similar to the first part of the proof of Proposition 4.31, and hence we omit it.

Proposition 4.32. *Condition (UPS) holds.*

The properties collected in this section show that Theorems 4.13 and 4.16 apply to this framework. Thus we are in a position to conclude the proofs of the results stated in the introduction.

End of the proof of Theorem 4.2. As Theorem 4.13 holds, the last thing we have to check is that the critical set $\mathcal{K}_\beta$ (according to (4.6)) coincides with the one defined in the introduction. Again, we only present a proof in the case $\beta < +\infty$. We have to show that $J_\beta(u,v) = J_\beta(U(1), V(1))$ if and only if $S_\beta(u,v) = 0$. But this readily follows from the fact that, for $t \in [0,1]$,

$$\begin{aligned} \operatorname{dist}_2^2((u,v),(U(t),V(t))) &= \left\| \int_0^t \partial_\tau (U(\tau), V(\tau))\, d\tau \right\|_2^2 \\ &\leqslant \left(\int_0^t \|S_\beta(U(\tau),V(\tau))\|_2 \, d\tau \right)^2 \\ &\leqslant |t| \int_0^t \|S_\beta(U(\tau),V(\tau))\|_2^2 \, d\tau \\ &\leqslant \int_0^1 \|S_\beta(U(\tau),V(\tau))\|_2^2 \, d\tau = J_\beta(u,v) - J_\beta(U(1),V(1)), \end{aligned}$$

once one observes that $(U(t),V(t)) = (u,v)\ \forall t \in (0,1)$ if and only if $S_\beta(u,v) = 0$. Finally, the H^1–compactness of $\mathcal{K}_\beta$ comes directly from Lemmas 4.25 (for $\beta < +\infty$) and 4.30 (for $\beta = +\infty$). □

Proof of Theorem 4.4. As Theorem 4.16 holds, the result is proved once we show that $\mathcal{C}_* \subset \mathcal{K}_*$. To this aim, let us consider $(u,v) \in \mathcal{C}_*$ and let, by definition, $(u_n, v_n) \in \mathcal{M}$ be such that $(u_n,v_n) \to (u,v)$ in $L^2(\Omega)$, $J_{\beta_n}(u_n,v_n) \to c_\infty$ and $J_{\beta_n}(U_n(1), V_n(1)) \to c_\infty$. By arguing exactly as in the proof of Proposition 4.31, we infer the existence of $0 \leqslant t \leqslant 1$ such that it holds $(U_n(t),V_n(t)) \to (u,v)$, $J_{\beta_n}(U_n(t),V_n(t)) \to c_\infty$ and $\|S_{\beta_n}(U_n(t),V_n(t))\|_2 \to 0$. Therefore $(u,v) \in \mathcal{K}_*$. □

Proof of Corollary 4.5. The only thing left to prove is that, given any $(u_n,v_n) \in \mathcal{M}$ and $\beta_n \to +\infty$ such that $(u_n,v_n) \to (\bar{u},\bar{v})$ in $L^2(\Omega)$, $J_{\beta_n}(u_n,v_n) \to c_\infty$ and $\|S_{\beta_n}(u_n,v_n)\|_2 \to 0$, then in fact $(u_n,v_n) \to (\bar{u},\bar{v})$ in $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$. We will prove that the sequence (u_n,v_n) is uniformly bounded in $L^\infty(\Omega)$. This, together with the fact that, by assumption,

$$\begin{aligned} -\Delta u_n + u_n^3 + \beta_n u_n v_n^2 - \lambda(u_n,v_n) u_n &= h_n \to 0 \quad \text{in } L^2(\Omega) \\ -\Delta v_n + v_n^3 + \beta_n u_n^2 v_n - \mu(u_n,v_n) v_n &= k_n \to 0 \quad \text{in } L^2(\Omega), \end{aligned}$$

allows us to apply Theorem 2.4, which implies the desired result.

Since $J_{\beta_n}(u_n,v_n) \to c_\infty$, we infer the existence of $\lambda_{\max}, \mu_{\max} \in \mathbb{R}$ such that, up to a subsequence,

$$(u_n,v_n) \rightharpoonup (\bar{u},\bar{v}) \text{ weakly in } H_0^1(\Omega), \qquad \lambda(u_n,v_n) \leqslant \lambda_{\max},\ \mu(u_n,v_n) \leqslant \mu_{\max}, \quad \forall n.$$

In order to prove uniform bounds in the L^∞–norm, we apply a Brezis-Kato type argument to the sequence (u_n, v_n). Suppose that $u_n \in L^{2+2\delta}(\Omega)$ for some $\delta > 0$; we can test with $u_n^{1+\delta}$ the inequality

$$-\Delta u_n \leqslant \lambda(u_n, v_n)u_n + h_n,$$

obtaining

$$\frac{1+\delta}{\left(1+\frac{\delta}{2}\right)^2}\int_\Omega |\nabla(u_n^{1+\frac{\delta}{2}})|^2 \leqslant \lambda(u_n, v_n)\int_\Omega u_n^{2+\delta} + \int_\Omega h_n u_n^{1+\delta}.$$

Hence we have[3]

$$\|u_n\|_{6+3\delta} \leqslant \left(C_S^2\frac{\left(1+\frac{\delta}{2}\right)^2}{1+\delta}\right)^{\frac{1}{2+\delta}}\left(\lambda(u_n, v_n)\int_\Omega u_n^{2+\delta} + \int_\Omega h_n u_n^{1+\delta}\right)^{\frac{1}{2+\delta}}.$$

Suppose that $\int_\Omega u_n^{2+2\delta}\,dx \geqslant 1$. From the Hölder inequality and the fact that $\frac{1}{2} \leqslant \frac{2+\delta}{2+2\delta}$ we deduce that

$$\lambda(u_n, v_n)\int_\Omega u_n^{2+\delta} \leqslant \lambda_{\max}\Big(\int_\Omega u_n^2\Big)^{1/2}\Big(\int_\Omega u_n^{2+2\delta}\Big)^{1/2} \leqslant C_1\lambda_{\max}\|u_n\|_{2+2\delta}^{2+\delta}$$

and

$$\int_\Omega h_n u_n^{1+\delta} \leqslant \|h_n\|_2\|u_n\|_{2+2\delta}^{2+\delta}.$$

Since $\|h_n\|_2 \to 0$, we obtain the existence of a constant C, not depending on n and δ, such that

$$\|u_n\|_{6+3\delta} \leqslant \left(C\frac{\left(1+\frac{\delta}{2}\right)^2}{1+\delta}\right)^{\frac{1}{2+\delta}}\|u_n\|_{2+2\delta}. \tag{4.10}$$

Now we iterate, by letting

$$\delta(1) = 2,\ 2 + 2\delta(k+1) = 6 + 3\delta(k).$$

Observe that $\delta(k) \to \infty$ since $\delta(k) \geqslant (3/2)^{k-1}$.

If there exist infinite values of $\delta(k)$ such that $\int_\Omega u_n^{2+2\delta(k)}\,dx < 1$, then the desired L^∞–estimate is trivially proved. Otherwise without loss of generality we can suppose that the estimate (4.10) holds for every $\delta = \delta(k)$, implying that

$$\begin{aligned}\|u_n\|_{6+\delta(k)} &\leqslant \prod_{j=1}^k\left[C\frac{\left(1+\frac{\delta(j)}{2}\right)^2}{1+\delta(j)}\right]^{\frac{1}{2+\delta(j)}}\|u_n\|_6\\ &\leqslant \exp\left(\sum_{j=1}^\infty\frac{1}{2+\delta(j)}\log\left[C\frac{\left(1+\frac{\delta(j)}{2}\right)^2}{1+\delta(j)}\right]\right)\|u_n\|_6\end{aligned}$$

[3]Here C_S denotes the Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$.

Since $\delta(j) \geqslant (3/2)^{j-1}$, it is not hard to verify that

$$\sum_{j=1}^{\infty} \frac{1}{2+\delta(j)} \log \left[C \frac{\left(1+\frac{\delta(j)}{2}\right)^2}{1+\delta(j)} \right] < \infty,$$

which provides the uniform bound in $L^\infty(\Omega)$. The same calculations clearly hold for v_n. □

Remark 4.33. For $k = 1$ the theory developed so far provides an alternative proof of the following result for minimal energy solutions.

> Let $(u_\beta, v_\beta) \in \mathcal{M}$ be a minimizer of J_β constrained to $\mathcal{M}$ for $\beta \in (0, +\infty)$. Then, up to a subsequence, $(u_\beta, v_\beta) \to (u_\infty, v_\infty)$ strongly in $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ and (u_∞, v_∞) is a minimizer of J_∞ constrained to $\mathcal{M}$. Moreover $u_\infty - v_\infty$ solves (4.2).

Recall that such theorem was already proved in Section 1.2. In the framework of this chapter the key remark is that for every $0 < \beta \leqslant +\infty$ we can write

$$c_\beta^1 = \inf_{(u,v)\in\mathcal{M}} J_\beta(u,v).$$

More precisely,

$$(u_\beta, v_\beta) \text{ achieves } c_\beta^1 \quad \Longrightarrow \quad A_\beta = \{(u_\beta, v_\beta), (v_\beta, u_\beta)\} \text{ is an optimal set for } J_\beta \text{ at } c_\beta^1.$$

Now, the L^2–convergence of the minima follows by the convergence of the optimal sets (Theorem 4.3), while the converge in $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ can be obtained exactly as in the previous proof.

4.5 Construction of the flows

Proof of Lemma 4.22. In order to prove local existence, we want to apply Theorem 2, b) in [134], to which we refer for further details. Let us rewrite the problem as

$$w' = \Delta w + F(w),$$

where $w = (U, V)$, $w' = \partial_t(U, V)$, Δ is understood in the vectorial sense and F contains all the remaining terms. Using the notations of [134] we set $E = L^2(\Omega) \times L^2(\Omega)$ and $E_F = H_0^1(\Omega) \times H_0^1(\Omega)$. We obtain that $e^{t\Delta}$ is an analytic semigroup both on E and on E_F, satisfying[4]

$$\|e^{t\Delta} w_0\| \leqslant C t^{-1/2} \|w_0\|_2 \qquad \text{for every } w_0 \in E,$$

so that (2.1) in [134] holds with $a = 1/2$. Moreover, since all the terms in F are of polynomial type, it is easy to see that $F : E_F \to E$ is locally Lipschitz continuous, and

$$\|F(w_0) - F(z_0)\|_2 \leqslant \ell(r) \|w_0 - z_0\|, \text{ with } \ell(r) = O(r^p) \text{ as } r \to +\infty,$$

[4] Denoting by λ_k and φ_k the k–th eigenvalue and the L^2-normalized eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ respectively, one can see that $e^{t\Delta} u = \sum_{k=1}^{\infty} \langle u, \varphi_k \rangle_2 e^{-\lambda_k t} \varphi_k \forall u \in H_0^1(\Omega)$. After this observation it is not hard to check that the required inequality holds true with $C = (2e)^{-1/2}$.

whenever $\|w_0\| \leqslant r$, $\|z_0\| \leqslant r$ (for example, arguing as in Lemma 4.37 ahead, the previous estimate holds for $p = 4$). Now, choosing $b = 1/(2p) < a$, it is immediate to check that

$$\ell(r) = O\left(r^{(1-a)/b}\right),$$

thus (2.3) in [134] is also satisfied. In order to apply Theorem 2, b) the last assumption we need to verify is that, for every $w_0 \in H_0^1$ (which is our regularity assumption for the initial data in (4.8)), it holds

$$\limsup_{t\to 0^+} \|t^b e^{t\Delta} w_0\| = 0;$$

but this follows recalling that[5] $\|e^{t\Delta} w_0\| \leqslant \|w_0\|$. Therefore Theorem 2, b) and Corollary 2.1, b) and c) in [134] apply, providing the existence of a (unique) maximal solution of (4.8)

$$(U(t), V(t)) \in C^1\left((0, T_{\max}); L^2(\Omega) \times L^2(\Omega)\right) \cap C\left([0, T_{\max}); H_0^1(\Omega) \times H_0^1(\Omega)\right),$$

with the property that if $T_{\max} < +\infty$ then $\|(U, V)\| \to +\infty$ as $t \to T_{\max}^-$.

Now we want to prove that $(U(t), V(t)) \in \mathcal{M}$ in its interval of definition. To this aim let us consider the C^1–function

$$\rho(t) = \int_\Omega U^2(x, t)\, dx,$$

which is continuous at $t = 0$. The derivative of $\rho(t)$ verifies

$$\begin{aligned}
\rho'(t) &= 2\int_\Omega U(x,t)U_t(x,t)\, dx \\
&= -2\int_\Omega (|\nabla U(x,t)|^2 + U^4(x,t) + \beta U^2(x,t)V^2(x,t))\, dx + \\
&\quad +2\lambda(U(t), V(t))\int_\Omega U^2(x,t)\, dx \\
&= a(t)(\rho(t) - 1),
\end{aligned}$$

where $a(t) = 2\lambda(U(t), V(t))$ is a continuous function. Since $\rho(0) = 1$, then $\rho(t) \equiv 1$ in $[0, T_{\max})$ (and an analogous result holds for $V(t)$). Finally, by integrating by parts (observe that by standard regularity, $(U(t), V(t))$ belongs to H^2 for $t > 0$) and by using the fact that $\int_\Omega UU_t\, dx = \int_\Omega VV_t\, dx = 0$, one can easily deduce

$$\begin{aligned}
\frac{d}{dt} J_\beta(U(t), V(t)) &= \int_\Omega U_t(x,t)(-\Delta U(x,t) + U^3(x,t) + \beta U(x,t)V^2(x,t)) + \\
&\quad + \int_\Omega V_t(x,t)(-\Delta V(x,t) + V^3(x,t) + \beta U^2(x,t)V(x,t))\, dx \\
&= \int_\Omega \langle (U_t(x,t), V_t(x,t)), S_\beta(U(t), V(t))\rangle\, dx \\
&= -\|S_\beta(U(t), V(t))\|_2^2 \leqslant 0.
\end{aligned}$$

[5]Again, one can obtain this inequality by using an expansion in eigenfunctions.

This implies

$$\|(U(t),V(t))\|^2 \leqslant 2J_\beta(U(t),V(t)) \leqslant 2J_\beta(u,v) < +\infty \tag{4.11}$$

for every $t < T_{\max}$, which provides $T_{\max} = +\infty$. $\square$

Remark 4.34. Given $(u,v) \in \mathcal{M}$ let (U,V) be the corresponding solution of (4.8). By taking in consideration inequality (4.11) we see that the quantities $\|(U(t),V(t))\|$, $\|(U(t),V(t))\|_p$ (with $p \leqslant 6$), $\lambda(U(t),V(t))$ and $\mu(U(t),V(t))$ are bounded by constants which only depend on $J_\beta(u,v)$ (in particular, they are independent of t).

Lemma 4.35. *Let* $c \in C\left([0,T];L^{3/2}(\Omega)\right)$ *and let* $U \in C^1\left((0,T];L^2(\Omega)\right) \cap C\left([0,T];H_0^1(\Omega)\right)$ *be a solution of*

$$\partial_t U - \Delta U = c(x,t)U, \quad U(\cdot,t) \in H_0^1(\Omega), \quad U(x,0) \geqslant 0.$$

Then $U(x,t) \geqslant 0$ *for every* t.

Proof. Since $c : [0,T] \to L^{3/2}$, we can write $|c(x,t)| \leqslant k + c_1(x,t)$, where k is constant and $\|c_1(t)\|_{3/2} < 1/C_S^2$ (here C_S denotes the Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$). Let

$$\rho(t) = \frac{1}{2}\int_\Omega |U^-(x,t)|^2\,dx.$$

We know that $\rho \in C^1((0,T]) \cap C([0,T])$ and $\rho(0) = 0$; moreover,

$$\begin{aligned}
\rho'(t) &= -\int_\Omega U^-(x,t)U_t(x,t)\,dx \\
&= -\int_\Omega \left(U^-(x,t)\Delta U(x,t) - c(x,t)(U^-(x,t))^2\right)\,dx \\
&\leqslant -\|U^-(t)\|^2 + k\|U^-(t)\|_2^2 + \|c_1(t)\|_{3/2}\|U^-(t)\|_6^2 \\
&\leqslant \left(-1 + C_S^2\|c_1(t)\|_{3/2}\right)\|U^-(t)\|^2 + k\|U^-(t)\|_2^2 \\
&\leqslant 2k\rho(t).
\end{aligned}$$

Thus we deduce that $\rho(t) \leqslant e^{2kt}\rho(0)$ and the lemma follows. $\square$

Lemma 4.36. *Let* $w = (w_1,w_2) \in C^1\left((0,+\infty);L^2(\Omega)\times L^2(\Omega)\right) \cap C\left([0,+\infty);H_0^1(\Omega)\times H_0^1(\Omega)\right)$ *be a solution of*

$$\begin{cases} \partial_t w - \Delta w = F(w) \\ w(0) = w_0, \end{cases} \tag{4.12}$$

where there exists a positive constant C *such that*

$$\int_\Omega \langle F(w(x,t)), w(x,t)\rangle\,dx \leqslant \frac{1}{2}\|w(t)\|^2 + C\|w(t)\|_2^2 \quad \textit{for every } t \geqslant 0. \tag{4.13}$$

Then there exists a constant $C(t)$ *such that*

$$\|w(t)\|_2 \leqslant C(t)\|w_0\|_2.$$

Proof. Let
$$E(t)=\frac{1}{2}\|w(t)\|_2^2=\frac{1}{2}\int_\Omega (w_1^2(t,x)+w_2^2(t,x))\,dx.$$
A straightforward computation yields
$$\begin{aligned}E'(t) &= -\int_\Omega (|\nabla w_1(x,t)|^2+|\nabla w_2(x,t)|^2)\,dx+\int_\Omega \langle F(w(x,t)),w(x,t)\rangle\,dx\\ &\leqslant -\frac{1}{2}\|w(t)\|^2+C\|w(t)\|_2^2\\ &\leqslant 2CE(t),\end{aligned}$$
from which we obtain $E(t)\leqslant e^{2Ct}E(0)$, concluding the proof. □

Lemma 4.37. *For $i=1,2$ take $(u_i,v_i)\in\mathcal{M}$ and let $(U_i(t),V_i(t))$ be the corresponding solution of* (4.8)*. There exists a constant C, only depending on* $\max_i J_\beta(u_i,v_i)$*, such that, for every t*

1. $|\lambda(U_1(t),V_1(t))-\lambda(U_2(t),V_2(t))|\leqslant C\,(\|U_1(t)-U_2(t)\|+\|V_1(t)-V_2(t)\|_2)$;
2. $|\mu(U_1(t),V_1(t))-\mu(U_2(t),V_2(t))|\leqslant C\,(\|V_1(t)-V_2(t)\|+\|U_1(t)-U_2(t)\|_2)$.

Proof. We prove only the first inequality, since the second one is analogous. We have

$$\begin{aligned}&|\lambda(U_1(t),V_1(t))-\lambda(U_2(t),V_2(t))|\\ &\quad\leqslant \int_\Omega \left||\nabla U_1(x,t)|^2-|\nabla U_2(x,t)|^2\right|\,dx+\int_\Omega \left|U_1^4(x,t)-U_2^4(x,t)\right|\,dx+\\ &\qquad+\beta\int_\Omega \left|U_1^2(x,t)V_1^2(x,t)-U_2^2(x,t)V_2^2(x,t)\right|\,dx\\ &\quad\leqslant \int_\Omega |\nabla U_1(x,t)+\nabla U_2(x,t)|\,|\nabla U_1(x,t)-\nabla U_2(x,t)|\,dx+\\ &\qquad+\int_\Omega (U_1^2(x,t)+U_2^2(x,t))|U_1(x,t)+U_2(x,t)|\,|U_1(x,t)-U_2(x,t)|\,dx+\\ &\qquad+\beta\int_\Omega U_1^2(x,t)|V_1(x,t)+V_2(x,t)|\,|V_1(x,t)-V_2(x,t)|\,dx+\\ &\qquad+\beta\int_\Omega V_2^2(x,t)|U_1(x,t)+U_2(x,t)|\,|U_1(x,t)-U_2(x,t)|\,dx\\ &\quad\leqslant \|U_1(t)+U_2(t)\|\|U_1(t)-U_2(t)\|+\|(U_1^2(t)+U_2^2(t))(U_1(t)+U_2(t))\|_2\|U_1(t)-U_2(t)\|_2+\\ &\qquad+\|\beta U_1^2(t)(V_1(t)+V_2(t))\|_2\|V_1(t)-V_2(t)\|_2+\|\beta V_2^2(t)(U_1(t)+U_2(t))\|_2\|U_1(t)-U_2(t)\|_2,\end{aligned}$$

from which we can conclude the proof by recalling Remark 4.34. □

Corollary 4.38. *For $i=1,2$ consider $(u_i,v_i)\in\mathcal{M}$ and let $(U_i(t),V_i(t))$ be the corresponding solution of* (4.8)*. There exists a constant $C=C(t)$, depending on t (and also on* $\max_i J_\beta(u_i,v_i)$*) such that*
$$\|(U_1(t),V_1(t))-(U_2(t),V_2(t))\|_2\leqslant C(t)\|(u_1,v_1)-(u_2,v_2)\|_2.$$

Proof. We want to apply Lemma 4.36 to $w = (w_1, w_2) = (U_1 - U_2, V_1 - V_2)$. Subtracting the equations for (U_1, V_1) and (U_2, V_2) we end up with a system like (4.12), and thus we only need to check that

$$F = \begin{pmatrix} U_2^3 - U_1^3 + \beta(U_2V_2^2 - U_1V_1^2) + \lambda(U_1(t), V_1(t))U_1 - \lambda(U_2(t), V_2(t))U_2 \\ V_2^3 - V_1^3 + \beta(U_2^2V_2 - U_1^2V_1) + \mu(U_1(t), V_1(t))V_1 - \mu(U_2(t), V_2(t))V_2 \end{pmatrix},$$

satisfies (4.13). In order to make the calculations easier, we split F into four terms, after adding and subtracting some suitable quantities. The first term is

$$F_1 = -\begin{pmatrix} (U_1^2 + U_1U_2 + U_2^2)w_1 \\ (V_1^2 + V_1V_2 + V_2^2)w_2 \end{pmatrix},$$

from which we obtain, by recalling Remark 4.34,

$$\begin{aligned} \int_\Omega \langle F_1(w(x,t)), w(x,t)\rangle \, dx \quad &\leqslant \quad \|U_1(t)U_2(t)\|_3\|w_1(t)\|_6\|w_1(t)\|_2 + \\ &\quad\quad +\|V_1(t)V_2(t)\|_3\|w_2(t)\|_6\|w_2(t)\|_2 \\ &\leqslant \quad C(\|w_1(t)\|\|w_1(t)\|_2 + \|w_2(t)\|\|w_2(t)\|_2) \\ &\leqslant \quad \frac{1}{6}\left(\|w_1(t)\|^2 + \|w_2(t)\|^2\right) + C'\left(\|w_1(t)\|_2^2 + \|w_2(t)\|_2^2\right) \end{aligned}$$

(where in the last step we have used Young's inequality). The second term is

$$F_2 = -\beta\begin{pmatrix} U_1(V_1 + V_2)w_2 + V_2^2w_1 \\ V_1(U_1 + U_2)w_1 + U_2^2w_2 \end{pmatrix},$$

which immediately gives, reasoning in the same way as above

$$\begin{aligned} &\int_\Omega \langle F_2(w(x,t)), w(x,t)\rangle \, dx \\ &\quad \leqslant \beta\Big(\|U_1(t)(V_1(t) + V_2(t))\|_3\|w_2\|_6\|w_1\|_2 + \|V_2^2(t)\|_3\|w_1(t)\|_6\|w_1(t)\|_2 + \\ &\quad\quad\quad + \|V_1(t)(U_1(t) + U_2(t))\|_3\|w_1(t)\|_6\|w_2\|_2 + \|U_2^2(t)\|_3\|w_2(t)\|_6\|w_2(t)\|_2\Big) \\ &\quad \leqslant C\Big[\|w_1(t)\|(\|w_1(t)\|_2 + \|w_2(t)\|_2) + \|w_2\|(\|w_1(t)\|_2 + \|w_2(t)\|_2)\Big] \\ &\quad \leqslant \frac{1}{6}\left(\|w_1(t)\|^2 + \|w_2(t)\|^2\right) + C'\left(\|w_1(t)\|_2^2 + \|w_2(t)\|_2^2\right). \end{aligned}$$

The third term is

$$F_3 = \begin{pmatrix} \lambda(U_1(t), V_1(t))w_1 \\ \mu(U_1(t), V_1(t))w_2 \end{pmatrix}, \quad \text{from which} \quad \int_\Omega \langle F_3(w(x,t)), w(x,t)\rangle \, dx \leqslant C\|w(t)\|_2^2$$

(where we have used Remark 4.34 again). Finally, the last term is

$$F_4 = \begin{pmatrix} (\lambda(U_1(t), V_1(t)) - \lambda(U_2(t), V_2(t)))U_2 \\ (\mu(U_1(t), V_1(t)) - \mu(U_2(t), V_2(t)))V_2 \end{pmatrix},$$

which can be ruled out by using Lemma 4.37. We obtain

$$\int_\Omega \langle F_4(w(x,t)), w(x,t)\rangle\, dx \leqslant C\left(\|U_1(t)-U_2(t)\| + \|V_1(t)-V_2(t)\|_2\right)\int_\Omega |U_2(x,t)w_1(x,t)|\,dx +$$
$$+ C\left(\|V_1(t)-V_2(t)\| + \|U_1(t)-U_2(t)\|_2\right)\int_\Omega |V_2(x,t)w_2(x,t)|\,dx$$
$$\leqslant \frac{1}{6}\left(\|w_1(t)\|^2 + \|w_2(t)\|^2\right) + C'\left(\|w_1(t)\|_2^2 + \|w_2(t)\|_2^2\right).$$

Therefore $F = F_1 + F_2 + F_3 + F_4$ satisfies (4.13), and hence Lemma 4.36 yields the desired result. □

Proof of Proposition 4.23. Properties (i) and (ii) have been proved in Lemma 4.35 and Corollary 4.38 respectively. Let us now prove (iii), which is a direct consequence of the estimate on the derivative of $J_\beta(U(t),V(t))$ expressed in Lemma 4.22. In fact, for each $t > s$ the following holds

$$\begin{aligned}\mathrm{dist}_2\left((U(s),V(s)),(U(t),V(t))\right) &= \left\| \int_s^t \partial_\tau (U(\tau),V(\tau))d\tau \right\|_2 \\ &\leqslant |t-s|^{1/2}\left(\int_s^t \|S_\beta(U(\tau),V(\tau))\|_2^2 d\tau\right)^{1/2} \\ &= |t-s|^{1/2}|J_\beta(U(s),V(s)) - J_\beta(U(t),V(t))|^{1/2}. \end{aligned}$$ □

We now turn to the construction of the flow η_∞.

Proof of Lemma 4.26. By definition, $S_\infty(w)$ is the projection of the gradient of J^* at w on the tangent space of the manifold $\left\{w \in H_0^1(\Omega) : (w^+, w^-) \in \mathcal{M}\right\}$ at w, thus

$$S_\infty(w) = w + (-\Delta)^{-1}w^3 - \tilde\lambda(-\Delta)^{-1}w^+ + \tilde\mu(-\Delta)^{-1}w^-,$$

where the coefficients $\tilde\lambda, \tilde\mu$ satisfy $\int_\Omega w^+ S_\infty(w) = \int_\Omega w^- S_\infty(w) = 0$, that is

$$\begin{pmatrix} \int_\Omega w^+ (-\Delta)^{-1} w^+ & -\int_\Omega w^+ (-\Delta)^{-1} w^- \\ -\int_\Omega w^- (-\Delta)^{-1} w^+ & \int_\Omega w^- (-\Delta)^{-1} w^- \end{pmatrix}\begin{pmatrix}\tilde\lambda \\ \tilde\mu\end{pmatrix} = \begin{pmatrix} \int_\Omega \left(w + (-\Delta)^{-1} w^3\right) w^+ \\ -\int_\Omega \left(w + (-\Delta)^{-1} w^3\right) w^- \end{pmatrix}.$$

Denoting by A the coefficient matrix, we compute[6]

$$\det A = \left(\int_\Omega |\nabla (-\Delta)^{-1} w^+|^2\right)\left(\int_\Omega |\nabla (-\Delta)^{-1} w^-|^2\,dx\right) - \\ - \left(\int_\Omega \langle \nabla (-\Delta)^{-1} w^+, \nabla (-\Delta)^{-1} w^-\rangle\right)^2 \geqslant 0,$$

[6]By using the identity $\int_\Omega f\,(-\Delta)^{-1} g\,dx = \int_\Omega \langle \nabla (-\Delta)^{-1} f, \nabla (-\Delta)^{-1} g\rangle$.

by the Hölder inequality, and $\det A = 0$ if and only if $a\nabla(-\Delta)^{-1} w^+ + b\nabla(-\Delta)^{-1} w^- \equiv 0$, for some a, b not both zero. But such an equality would imply that the $H_0^1(\Omega)$–function $(-\Delta)^{-1}(aw^+ + bw^-)$ would have an identically zero gradient and therefore $aw^+ + bw^- \equiv 0$, in contradiction with the fact that, by assumption, $\|aw^+ + bw^-\|_2^2 \geqslant (a^2+b^2)R_1^2$. Thus the L^2–continuous function $\det A$ is strictly positive on the L^2–compact set $\{w : \|w^\pm\|_2 \geqslant R_1,\ \|w\| \leqslant R_2\}$, i.e. it is larger than a strictly positive constant (only depending on R_1, R_2). This provides (existence, uniqueness and) an explicit expression of $\tilde\lambda(w)$ and $\tilde\mu(w)$ for any w satisfying the previous assumptions. The regularity of these functions descends from such explicit expressions, once one notices that they are both products of Lipschitz continuous functions (and therefore bounded when $\|w\| \leqslant R_2$). Just as an example, we prove the Lipschitz continuity of the term $\int_\Omega w^+(-\Delta)^{-1} w^3\,dx$. We have[7]

$$\begin{aligned}\left|\int_\Omega w_1^+(-\Delta)^{-1} w_1^3 - \int_\Omega w_2^+(-\Delta)^{-1} w_2^3\right| \\ \leqslant \int_\Omega |w_1^+ - w_2^+|\,|(-\Delta)^{-1} w_1^3| + \int_\Omega w_2^+|(-\Delta)^{-1}(w_1^3 - w_2^3)| \\ \leqslant C\left\|w_1^+ - w_2^+\right\|_2 \left\|w_1^3\right\|_2 + \left\|w_2^+\right\|_{6/5}\left\|(-\Delta)^{-1}(w_1^3 - w_2^3)\right\|_6 \\ \leqslant CR_2^3\left\|w_1 - w_2\right\|_2 + CR_2\left\|w_1^3 - w_2^3\right\|_{6/5} \leqslant CR_2^3\left\|w_1 - w_2\right\|_2.\end{aligned}$$

All the other terms can be treated the same way. □

Remark 4.39. By reasoning as in the end of the previous proof, it can be proved that, whenever w_1, w_2 belong to the set

$$\left\{w \in H_0^1(\Omega) :\ \|w^\pm\|_2 \geqslant R_1,\ \|w\| \leqslant R_2\right\},$$

there exists a constant L, only depending on R_1, R_2, such that

$$\left\|S_\infty(w_1) - S_\infty(w_2)\right\|_2 \leqslant L\left\|w_1 - w_2\right\|_2,$$

$$\left\|S_\infty(w_1) - S_\infty(w_2)\right\| \leqslant L\left\|w_1 - w_2\right\|.$$

Proof of Lemma 4.27. Let us fix $0 < R_1 < 1$ and $R_2 > 2(c_\infty + 1)$. By Remark 4.39 we have that $-S_\infty$, as a map from $H_0^1(\Omega)$ into itself, is H_0^1–Lipschitz continuous on the mentioned set, with Lipschitz constant only depending on R_1, R_2; we infer existence (and uniqueness) of a maximal solution of the Cauchy problem, defined on $[0, T_{\max})$. Moreover, for any $t \in (0, T_{\max})$, we have

$$\frac{d}{dt}\|W^\pm(t)\|_2^2 = \pm 2\int_\Omega W^\pm(x,t) W_t(x,t)\,dx = \mp 2\int_\Omega W^\pm(x,t) S_\infty(W(t))\,dx = 0$$

[7] Remember that, by standard elliptic regularity results, both $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ and $(-\Delta)^{-1} : L^{6/5}(\Omega) \to L^6(\Omega)$ are continuous.

(by the definition of S_∞), and

$$\begin{aligned}
\frac{d}{dt}J_\infty(W^+(t),W^-(t)) &= \frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\nabla W(x,t)|^2 + \frac{1}{4}W^4(x,t)\right)\,dx \\
&= \int_\Omega \left(-\Delta W(x,t) + W^3(x,t)\right) W_t(x,t)\,dx \\
&= \int_\Omega -\Delta\left(W(x,t) + (-\Delta)^{-1}\,W^3(x,t)\right) W_t(x,t)\,dx \\
&= \int_\Omega \left\langle \nabla\left(S_\infty(W(t)) + (-\Delta)^{-1}\,(\tilde\lambda W^+(x,t) - \tilde\mu W^-(x,t))\right), \nabla\left(-S_\infty(W(t))\right)\right\rangle \\
&= -\|S_\infty(W(t))\|^2.
\end{aligned}$$

Thus, for any $t \in (0, T_{\max})$, we obtain $\|W^\pm(t)\|_2 = 1 > R_1$ and $\|W(t)\| \leqslant 2J_\infty(W^+(t),W^-(t)) \leqslant 2J_\infty(u,v) < R_2$. In particular this implies that $T_{\max} = +\infty$, concluding the proof of the lemma. □

Proof of Proposition 4.28. (i) Consider $(u_1,v_1),(u_2,v_2) \in \mathcal{M}_\infty^{c_\infty+1}$ and let $W_1(t), W_2(t)$ be the corresponding solutions of (4.9). We notice first of all that Remark 4.39 applies, providing the existence of $L = L(c_\infty)$ such that

$$\frac{d}{dt}\,\|W_1(t) - W_2(t)\|_2^2 \leqslant 2L\,\|W_1(t) - W_2(t)\|_2^2\,,$$

which implies

$$\|W_1(t) - W_2(t)\|_2^2 \leqslant e^{2Lt}\,\|W_1(0) - W_2(0)\|_2^2\,.$$

Therefore

$$\begin{aligned}
\mathrm{dist}_2^2((W_1^+(1),W_1^-(1)),(W_2^+(1),W_2^-(1))) &\leqslant \|W_1(1) - W_2(1)\|_2^2 \\
&\leqslant e^{2L}\,\|W_1(0) - W_2(0)\|_2^2 \\
&\leqslant 2e^{2L}(\|u_1 - v_1\|_2^2 + \|u_2 - v_2\|_2^2).
\end{aligned}$$

(ii) Notice first of all that

$$\mathrm{dist}_2^2((W^+(s),W^-(s)),(W^+(t),W^-(t))) \leqslant \|W(s) - W(t)\|_2^2\,.$$

Now, Lemma 4.27 allows us to deduce that, for $t > s$,

$$\begin{aligned}
\|W(s) - W(t)\|_2 \leqslant C\|W(s) - W(t)\| &= C\left\|\int_s^t \partial_\tau W(\tau)d\tau\right\| \\
&\leqslant C|t-s|^{1/2}\left(\int_s^t \|S_\infty(W(\tau))\|^2 d\tau\right)^{1/2} \\
&= C|t-s|^{1/2}|J_\infty(W^+(s),W^-(s)) - J_\infty(W^+(t),W^-(t))|^{1/2},
\end{aligned}$$

and the two inequalities together imply the statement of the proposition. □

Part II

Asymptotic study of a strongly coupled elliptic system

Chapter 5

Solutions with multiple spike patterns for a strongly coupled elliptic system

5.1 Introducing the problem

Let Ω be a domain of $\mathbb{R}^N$, $N \geqslant 3$, not necessarily bounded, with smooth or empty boundary. We consider an elliptic system of the form

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(v) & \text{in } \Omega \\ -\varepsilon^2 \Delta v + V(x)v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $u, v > 0$ in Ω and $\varepsilon > 0$ is a small parameter. Here $V(x)$ is locally Hölder continuous and $\inf_\Omega V > 0$, while $f, g \in C^1(\mathbb{R})$ satisfy the following assumptions.

$(fg1)$ $f(0) = g(0) = f'(0) = g'(0) = 0$;

$(fg2)$ $\displaystyle\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = \lim_{s\to+\infty} \frac{g(s)}{s^{q-1}} = 0$, for some $p, q > 2$ with $\dfrac{1}{p} + \dfrac{1}{q} > \dfrac{N-2}{N}$;

$(fg3)$ $0 < (1+\delta')f(s)s \leqslant f'(s)s^2$ and $0 < (1+\delta')g(s)s \leqslant g'(s)s^2$, for every $s > 0$, for some $\delta' > 0$.

We look for positive solutions of (5.1) and therefore we let $f(s) = g(s) = 0$ for $s \leqslant 0$.

Our motivation for the study of such a problem goes back to the work of Rabinowitz [101] and Wang [130] concerning the single equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \qquad \text{in } \mathbb{R}^N. \tag{5.2}$$

In [101] a mountain-pass type argument is used in order to find a ground state solution for $\varepsilon > 0$ sufficiently small, when V is such that $\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V(x) > 0$. In [130] it is proved that this mountain-pass solutions concentrate around a global minimum point of V as ε tends to 0.

It should be stressed that in these papers no nondegeneracy assumptions were made upon the minimum points of V; this is in contrast with previous works (see e.g. [5, 63, 95]) where solutions with a spike shape which concentrate around nondegenerate critical points of V were constructed. A related problem concerns the case where $V(x) \equiv 1$ in a bounded domain Ω under Neumann or Dirichlet boundary conditions, the main issue being then the location of the concentrating points of the least energy solutions, see e.g. [60, 78, 89, 90]. In [26] the function V is allowed to vanish at some points of $\mathbb{R}^N$. Problems with another type of nonlinearities were considered by many authors, see e.g. [36, 54, 56, 53] and their references.

A further step in the study of such problems was made by Del Pino and Felmer in [57], where the degenerate case in (5.2) was considered in a local setting. Namely, by assuming that $\inf_\Lambda V < \inf_{\partial\Lambda} V$ with respect to a bounded open set $\Lambda \subset \Omega$, a family u_ε exhibiting a single spike in Λ, at a point x_ε such that $V(x_\varepsilon) \to \inf_\Lambda V$, is constructed. In [59, 61] the author's approach was extended to the construction of a family of solutions with several spikes located around any prescribed finite set of local minima of V.

There are at least three difficulties in extending the quoted results to the system (5.1). Firstly, no uniqueness results seem to be known for the "limit problem"

$$-\Delta u + u = g(v), \qquad -\Delta v + v = f(u) \quad \text{in } \mathbb{R}^N$$

and this is in some cases a crucial assumption in the single equation case (compare e.g. with [53, Assumption (f5)], [36, p. 268], [59, Assumption (f4)], [78, p. 1448]).

On the other hand, let us introduce the associated energy functional $I_\varepsilon : H \times H \to \mathbb{R}$,

$$I_\varepsilon(u, v) := \int_\Omega \big(\varepsilon^2 \langle \nabla u, \nabla v\rangle + V(x)uv\big) - \int_\Omega F(u) - \int_\Omega G(v),$$

where $F(s) := \int_0^s f(\xi)\, d\xi$, $G(s) := \int_0^s g(\xi)\, d\xi$, and H is the Hilbert space $H := \{u \in H_0^1(\Omega) : \int_\Omega V(x)u^2 < +\infty\}$, with the inner product $\langle u, v\rangle := \int_\Omega (\langle \nabla u, \nabla v\rangle + V(x)uv)$ (at this point we assume that I_ε is well defined, *i.e.* that the constants p and q in assumption (fg2) are such that $2 < p, q < 2^* := 2N/(N-2)$; see below for a discussion on this). It is known that positive solutions of (5.1) correspond to critical points of the functional I_ε. However we see that, with respect to the single equation case, the quadratic part of the energy functional has no longer a positive sign. From a technical point of view, this causes some difficulties; for example, it is not clear whether the penalization method as used in [59, p. 138] can be applied to our problem.

From a more conceptual point of view, in the case of a system we also have to face the indefinite character of the energy functional, since ground-state critical points of I_ε are no longer expected to be generated from a direct (essentially) finite dimensional argument. This difficulty was bypassed in [4, 9] by means of a dual variational formulation of the problem while in [97, 98, 106, 110] a direct approach was proposed, based on a new variational characterization of the ground-state critical levels associated to (5.1). In these papers either the case $V(x) \equiv 1$ or the "coercive" case $\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V(x) > 0$ is considered.

Our goal here is to establish for the system (5.1) the analog of the results in [59, 61] concerning a single equation. Namely, we assume that V is locally Hölder continuous and

$(V1)$ $V(x) \geqslant \alpha > 0$, for all $x \in \Omega$;

$(V2)$ there exist bounded domains Λ_i, mutually disjoint, compactly contained in Ω, $i = 1, \dots, k$, such that

$$\inf_{\Lambda_i} V < \inf_{\partial \Lambda_i} V$$

(i.e. V admits at least k local strict minimum points, possibly degenerate).

We prove the following result.

Theorem 5.1. *Under assumptions* $(V1), (V2), (fg1) - (fg3)$, *there exists* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$, *problem* (5.1) *admits classical positive solutions* $u_\varepsilon, v_\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap H_0^1(\Omega)$, *and:*

(i) $u_\varepsilon + v_\varepsilon$ *posses exactly* k *local maximum points* $x_{i,\varepsilon} \in \Lambda_i$, $i = 1, \dots, k$;

(ii) $u_\varepsilon(x_{i,\varepsilon}) + v_\varepsilon(x_{i,\varepsilon}) \geqslant b > 0$, *and* $V(x_{i,\varepsilon}) \to \inf_{\Lambda_i} V$ *as* $\varepsilon \to 0$;

(iii) $u_\varepsilon(x) + v_\varepsilon(x) \leqslant \gamma e^{-\frac{\beta}{\varepsilon}|x - x_{i,\varepsilon}|}$, $\forall x \in \Omega \setminus \cup_{j \neq i} \Lambda_j$;

for some positive constants b, γ, β.

This chapter is devoted to the proof of Theorem 5.1. In Section 5.2 we set a general framework suitable for our proposals, mainly Proposition 5.8. The underlying ideas are already present in [110] but we provide here a more concise approach which in particular avoids an extra technical assumption that was needed in [110] and subsequently in [97, 98, 106] (namely, $f^2(s) \leqslant 2f'(s)F(s)$ and similarly for $g(s)$; we also avoid the assumption $f'(s)s \leqslant Cf(s)$, $0 < s < 1$, in [61, p. 3]). In Section 5.3, similarly to [61, p. 25] we seek for k-spike solutions by minimizing the energy functional over the product of k suitable "local" Nehari manifolds. Roughly speaking, each of these manifolds localizes I_ε near $H_0^1(\Lambda_i) \times H_0^1(\Lambda_i)$ $(i = 1, \dots, k)$, thanks to a technical condition in its definition which ensures that the manifold is weakly closed. As it will be clear later on, the main estimate in the proof of Theorem 5.1 is contained in Eq. (5.55) and it turns out that our setting is rather effective in providing it.

Once these preliminary settings are established, the proof of Theorem 5.1 follows by simple arguments, as shown in Section 5.4. Before the final Section 5.6, where some recent developments are discussed, Section 5.5 concerns the following question. Under the assumption $(fg2)$, the functional I_ε may not be well defined in the space $H \times H$, because it can happen that, say, $p < 2^* = \frac{2N}{N-2} < q$. However, as explained in Section 5.5, we only have to prove Theorem 5.1 in the case where $2 < p = q < 2^*$. In fact, given $n \in \mathbb{N}$ we can define the truncated functions

$$f_n(s) = \begin{cases} f(s) & \text{for } s \leqslant n \\ A_n s^{p-1} + B_n & \text{for } s > n \end{cases} \qquad g_n(s) = \begin{cases} g(s) & \text{for } s \leqslant n \\ \tilde{A}_n s^{p-1} + \tilde{B}_n & \text{for } s > n \end{cases} \tag{5.3}$$

with $A_n = f'(n)/((p-1)n^{p-2})$, $B_n = f(n) - (f'(n)n)/(p-1)$, $\tilde{A}_n = g'(n)/((p-1)n^{p-2})$, $\tilde{B}_n = g(n) - (g'(n)n)/(p-1)$; we show in Section 5.5 that the solutions $(u_{\varepsilon_n}, v_{\varepsilon_n})$ of the corresponding system obtained by means of Theorem 5.1 applied to the truncated problem are such that $||u_{\varepsilon_n}||_\infty, ||v_{\varepsilon_n}||_\infty \leqslant C$ for some $C > 0$ independent of n, and therefore they solve the original problem (5.1) if n is taken sufficiently large. Thanks to this remark, in Sections 5.2, 5.3 and 5.4 we assume that $2 < p = q < 2^*$. In particular, we may assume that the following holds:

$(fg4)$ $|f'(s)| + |g'(s)| \leqslant C(1 + |s|^{p-2})$ with $2 < p < 2N/(N-2)$.

$(fg5)$ For every $\mu > 0$ there exists $C_\mu > 0$ such that

$$|f(s)t| + |g(t)s| \leqslant \mu(s^2 + t^2) + C_\mu(f(s)s + g(t)t), \qquad s, t \in \mathbb{R}.$$

Moreover, we observe that condition $(fg3)$ implies the so called Ambrosetti-Rabinowitz condition, namely

(AR) $(2 + \delta')F(s) \leqslant f(s)s$ and $(2 + \delta')G(s) \leqslant g(s)s$, for every $s > 0$.

In particular, $F(s) \geqslant a(s^+)^{2+\delta'} - b$ and $G(s) \geqslant c(s^+)^{2+\delta'} - d$, for some positive constants a, b, c, d.

Remark 5.2. The work we present in this chapter is based in the paper [107], written by myself in collaboration with M. Ramos. It should be pointed out however that the statement of the main theorem in [107] (namely, Theorem 1.1) is slightly incorrect. In fact, there it is said that $u_\varepsilon, v_\varepsilon$ concentrate at k *common* local maximums, but the proof contains a mistake, more precisely at Proposition 4.3. The correct statement corresponds to Theorem 5.1 of this chapter, where conclusions for the sum $u_\varepsilon + v_\varepsilon$ are drawn. Although the uniqueness of the maximums of u_ε and v_ε does not seem to hold, nevertheless at the end of Section 5.4 we will show that they become very close as $\varepsilon \to 0$ (*cf.* Remark 5.33).

5.2 A variational framework for superlinear systems

In this section we establish some preliminary results which are needed for the proof of Theorem 5.1. Given $f, g \in C^1(\mathbb{R}, \mathbb{R})$ and V as in the previous section (we recall that without loss of generality we also assume that (fg4) and (fg5) hold), we consider the system

$$\begin{cases} -\Delta u + V(x)u = g(v) \\ -\Delta v + V(x)v = f(u) \\ u, v \in H_0^1(\Omega), \end{cases}$$

and the associated energy functional $I : H \times H \to \mathbb{R}$,

$$I(u, v) := \int_\Omega (\langle \nabla u, \nabla v\rangle + V(x)uv) - \int_\Omega F(u) - \int_\Omega G(v).$$

We are interested in the study of the quantity[1]

$$\sup_{\mathbb{R}^+(u,v)\oplus H^-} I = \sup\{I(t(u, v) + (\phi, -\phi)) : \ t \geqslant 0, \ \phi \in H\} \tag{5.4}$$

as a function of $(u, v) \in H \times H$. We start by making the following observation.

[1]Here, $H^- := \{(\phi, -\phi), \ \phi \in H\}$.

Remark 5.3. The supremum (5.4) might be equal to plus infinity. However,

$$\begin{aligned} I(t(u,v)+(\phi,-\phi)) &= \langle tu+\phi, tv-\phi\rangle - \int_\Omega (F(tu+\phi)+G(tv-\phi)) \\ &= t^2\langle u,v\rangle + t\langle \phi, v-u\rangle - \|\phi\|^2 - \int_\Omega (F(tu+\phi)+G(tv-\phi)), \end{aligned}$$

and if there exists a set $K\subseteq\Omega$ of positive measure such that $u+v>0$ in K, then (by supposing without loss of generality that $|K|<\infty$)

$$\begin{aligned} I(t(u,v)+(\phi,-\phi)) &\leqslant t^2\langle u,v\rangle + t\langle \phi, v-u\rangle - \|\phi\|^2 - \\ &\quad -a\int_K ((tu+\phi)^+)^{2+\delta'} - b\int_K ((tv-\phi)^+)^{2+\delta'} + C \\ &\leqslant t^2\langle u,v\rangle + t\langle \phi, v-u\rangle - \|\phi\|^2 - C' t^{2+\delta'} + C \to -\infty \end{aligned}$$

as $t\to+\infty$. Hence in such a situation the supremum is finite.

Lemma 5.4. *Given $(u,v)\in H\times H$ there exists a unique function $\Psi_{u,v}\in H$ such that*

$$\sup\{I((u,v)+(\phi,-\phi)):\ \phi\in H\} = I(u+\Psi_{u,v}, v-\Psi_{u,v}).$$

Moreover, $\Psi_{u,v}$ is uniquely characterized by

$$I'((u,v)+(\Psi_{u,v},-\Psi_{u,v}))(\phi,-\phi)=0, \qquad \forall\phi\in H. \tag{5.5}$$

Proof. Define $\Lambda(\phi) := I((u,v)+(\phi,-\phi)) = -\|\phi\|^2 + \langle\phi, v-u\rangle + \langle u,v\rangle - \int_\Omega F(u+\phi) - \int_\Omega G(v-\phi)$. We have that $\Lambda(\phi)\to-\infty$ as $\|\phi\|\to\infty$, and hence $s:=\sup\Lambda(\phi)<\infty$. Take a maximizing sequence ϕ_n, $\Lambda(\phi_n)\to s$; then $\|\phi_n\|$ is bounded and there exists $\bar\phi\in H$ such that, up to a subsequence, $\phi_n\rightharpoonup\bar\phi$ in H. Moreover,

$$\|\bar\phi\|^2 \leqslant \liminf\|\phi_n\|^2, \qquad \langle\phi, v-u\rangle = \lim\langle\phi_n, v-u\rangle$$

and, by Fatou's Lemma,

$$\int_\Omega F(u+\phi) \leqslant \liminf\int_\Omega F(u+\phi_n), \qquad \int_\Omega G(v-\phi)\leqslant\liminf\int_\Omega G(v-\phi_n).$$

So,

$$s=\limsup I((u,v)+(\phi_n,-\phi_n)) \leqslant I((u,v)+(\bar\phi,-\bar\phi)) \leqslant s$$

and thus s is attained at $\bar\phi$.

As for the uniqueness of $\bar\phi$, observe that

$$\begin{aligned} \Lambda''(\phi)(\varphi)(\varphi) &= I''((u,v)+(\phi,-\phi))(\varphi,-\varphi)(\varphi,-\varphi) \\ &= -2\|\varphi\|^2 - \int_\Omega f'(u+\phi)\varphi^2 - \int_\Omega g'(v-\phi)\varphi^2 < 0, \text{ for every } \varphi\not\equiv 0 \end{aligned}$$

and hence $\Lambda(\phi)$ has at most one single critical point. $\square$

Proposition 5.5. *The map $\Theta: H\times H\to H$, $(u,v)\mapsto\Theta(u,v)=\Psi_{u,v}$ is C^1.*

Proof. We apply the implicit function theorem (see for instance [138, Theorem 4.E]) to the map $\overline{\Theta} : (H \times H) \times H^- \to H^-$, $\overline{\Theta}((u,v),(\Psi,-\Psi)) = PI'((u,v)+(\Psi,-\Psi))$, where P is the orthogonal projection of $H \times H$ onto $H^- = \{(\phi,-\phi),\ \phi \in H\}$ and $I'((u,v)+(\Psi,-\Psi)) \in H \times H$ has a meaning according to Riesz's representation theorem. For any fixed pair $(u+\psi, v-\psi)$, the derivative of $\overline{\Theta}$ with respect to $(\phi,-\phi)$ evaluated at $(u+\psi, v-\psi)$ is given by the linear map

$$(\phi,-\phi) \mapsto T(\phi,-\phi) = PI''(u+\psi, v-\psi)(\phi,-\phi),$$

that is

$$\langle T(\phi,-\phi),(\varphi,-\varphi)\rangle = -2\langle \phi,\varphi\rangle - \int_\Omega f'(u+\psi)\phi\varphi - \int_\Omega g'(v-\psi)\phi\varphi, \quad \forall \phi,\varphi.$$

Since $f'(0)=g'(0)=0$ and $|f'(s)|,|g'(s)| \leqslant C|s|^{2^*}$ for $|s| \geqslant 1$, we have that $Id - T$ is a compact operator. The operator T is one-to-one, since if $T(\phi,-\phi)=0$, then

$$-2\|\phi\|^2 = \int_\Omega f'(u+\psi)\phi^2 + \int_\Omega g'(v-\psi)\phi^2 \geqslant 0$$

and so $\phi = 0$. Thus by the Fredholm's alternative theorem (see for instance [23, Theorem VI.6]) T is also onto and hence we can apply the implicit function theorem and obtain the desired result. □

Lemma 5.6. *If $(u,v) \neq (0,0)$ is such that $I'(u,v)(u,v) = 0$ and $I'(u,v)(\phi,-\phi) = 0$ for every $\phi \in H$, then*

$$\sup_{\phi \in H} I''(u,v)(u+\phi, v-\phi)(u+\phi, v-\phi) < 0.$$

Proof. We have

$$\begin{aligned} I''(u,v)(u+\phi,v-\phi)(u+\phi,v-\phi) &= 2\langle u+\phi, v-\phi\rangle - \int_\Omega f'(u)(u+\phi)^2 - \int_\Omega g'(v)(v-\phi)^2 \\ &= -2\|\phi\|^2 + 2\langle u,v\rangle + 2\langle \phi, v-u\rangle - \int_\Omega f'(u)(u+\phi)^2 - \int_\Omega g'(v)(v-\phi)^2. \end{aligned}$$

On the other hand, $I'(u,v)(u,v) = 0$ and $2I'(u,v)(\phi,-\phi) = 0$ are respectively equivalent to

$$2\langle u,v\rangle = \int_\Omega f(u)u + \int_\Omega g(v)v \qquad \text{and} \qquad 2\langle \phi, v-u\rangle = \int_\Omega 2f(u)\phi - \int_\Omega 2g(v)\phi,$$

and thus

$$\begin{aligned} I''(u,v)(u+\phi,v-\phi)(u+\phi,v-\phi) = -2||\phi||^2 - \int_\Omega \Big(\frac{f(u)}{u} + \frac{g(v)}{v}\Big)\phi^2 - \\ - \int_\Omega \Big(f'(u) - \frac{f(u)}{u}\Big)(u+\phi)^2 - \int_\Omega \Big(g'(v) - \frac{g(v)}{v}\Big)(v-\phi)^2. \end{aligned}$$

By condition $(fg3)$ we have

$$f'(u) - \frac{f(u)}{u} \geqslant \delta' \frac{f(u)}{u} \geqslant 0, \qquad g'(v) - \frac{g(v)}{v} \geqslant \delta' \frac{g(v)}{v} \geqslant 0$$

and hence the function $\phi \mapsto I''(u,v)(u+\phi, v-\phi)(u+\phi, v-\phi)$ is negative, tends to $-\infty$ as $\|\phi\| \to \infty$, and hence we can prove that its supremum is attained and it is negative, by reasoning exactly as in the proof of Lemma 5.4. $\square$

From now on, given $u, v \in H$ and $t \geqslant 0$, we denote $\Psi_t := \Psi_{tu,tv}$ according to the definition in (5.5), *i.e.*

$$I'(t(u,v) + (\Psi_t, -\Psi_t))(\phi, -\phi) = 0, \qquad \forall \phi \in H. \tag{5.6}$$

Lemma 5.7. *Given $u, v \in H$ such that $u \neq -v$, the map*

$$\alpha(t) := I(t(u,v) + (\Psi_t, -\Psi_t))$$

is C^2 and, for any $t > 0$,

$$\alpha'(t) = 0 \Rightarrow \alpha''(t) < 0.$$

In particular $\alpha(t)$ admits at most one positive critical point. Moreover, $\alpha'(0) = 0$ and $\alpha''(0) > 0$.

Proof. For any $t \in \mathbb{R}$ we denote by $\Psi_t' \in H$ the derivative of the map $t \mapsto \Psi_t$ evaluated at the point t (which has a sense according to Proposition 5.5). From (5.6) we see that

$$\alpha'(t) \quad = \quad I'(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t') = I'(t(u,v) + (\Psi_t, -\Psi_t))(u,v),$$

and hence

$$\alpha''(t) = I''(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t')(u,v).$$

On the other hand, it follows also from (5.6) that

$$I''(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t')(\phi, -\phi) = 0 \quad \text{for every } t > 0 \text{ and } \phi \in H \tag{5.7}$$

and so, by letting $\phi = t^2 \Psi_t'$ in the previous equality,

$$I''(t(u,v) + (\Psi_t, -\Psi_t))(tu + t\Psi_t', tv - t\Psi_t')(t\Psi_t', -t\Psi_t') = 0.$$

Thus it follows that

$$t^2 \alpha''(t) = I''(t(u,v) + (\Psi_t, -\Psi_t))(tu + t\Psi_t', tv - t\Psi_t')(tu + t\Psi_t', tv - t\Psi_t') \quad \text{for every } t. \tag{5.8}$$

Now, suppose that $\alpha'(t_1) = 0$ for some $t_1 > 0$ and denote $u_1 := t_1 u + \Psi_{t_1}$ and $v_1 := t_1 v - \Psi_{t_1}$. We have that $(u_1, v_1) \neq (0,0)$ (because $u \neq -v$),

$$I'(u_1, v_1)(\phi, -\phi) = I'(t_1(u,v) + (\Psi_{t_1}, -\Psi_{t_1}))(\phi, -\phi) = 0 \quad \text{for every } \phi,$$

and

$$\begin{aligned} I'(u_1, v_1)(u_1, v_1) \quad &= \quad I'(t_1(u,v) + (\Psi_{t_1}, -\Psi_{t_1}))(t_1 u + \Psi_{t_1}, t_1 v - \Psi_{t_1}) \\ &= \quad t_1 I'(t_1(u,v) + (\Psi_{t_1}, -\Psi_{t_1}))(u,v) = t_1 \alpha'(t_1) = 0. \end{aligned}$$

Hence by Lemma 5.6 it follows that

$$I''(u_1, v_1)(u_1 + \phi, v_1 - \phi)(u_1 + \phi, v_1 - \phi) < 0 \quad \text{for every } \phi.$$

By letting $\phi = t_1\Psi'_{t_1} - \Psi_{t_1}$ we conclude from (5.8) that $\alpha''(t_1) < 0$, as claimed.

As for the behavior of $\alpha(t)$ at $t = 0$, since $I'(0,0) = 0$ we have by definition that $\Psi_0 = 0$ and so $\alpha'(0) = I'(0,0)(u,v) = 0$. From (5.7) it follows that

$$0 = I''(0,0)(u + \Psi'_0, v - \Psi'_0)(\phi, -\phi) = \langle \phi, v - u - 2\Psi'_0 \rangle \qquad \text{for every } \phi$$

and hence $\Psi'_0 = (v - u)/2$, so that

$$\alpha''(0) = I''(0,0)\Big(\frac{u+v}{2}, \frac{u+v}{2}\Big)(u,v) = 2\left\|\frac{u+v}{2}\right\|^2 > 0,$$

since $u \neq -v$. □

Proposition 5.8. *Let $u, v \in H$ be such that $u \neq -v$, $I'(u,v)(u,v) = 0$ and $I'(u,v)(\phi, -\phi) = 0$ for every ϕ, and denote*

$$\theta(t) := I'(t(u,v) + (\Psi_t, -\Psi_t))(u,v).$$

Then there exists $\delta = \delta(u,v) > 0$ such that

$$\theta(t) = -\delta(t-1) + \mathrm{o}(t-1) \qquad \text{as } t \to 1. \tag{5.9}$$

Moreover, if the supremum in (5.4) *is finite, then*

$$I(u,v) = \sup\{I(t(u,v) + (\phi, -\phi)) : \ t \geqslant 0, \ \phi \in H\}. \tag{5.10}$$

Proof. The map θ is of class C^1 and

$$\theta(t) = \theta(1) + \theta'(1)(t-1) + \mathrm{o}(t-1).$$

Since $I'(u,v)(\phi, -\phi) = 0$, we deduce that $\Psi_1 = 0$ and hence $\theta(1) = \alpha'(1) = I'(u,v)(u,v) = 0$. Moreover, $\theta'(1) = \alpha''(1) < 0$ by Lemma 5.7, and so the conclusion in (5.9) follows.

As for (5.10), suppose that the supremum is attained at some $t_0 > 0, \phi_0 \in H$. Then we must have $\phi_0 = \Psi_{t_0}$ and $\alpha'(t_0) = 0$. By Lemma 5.7, the function α has at most one positive critical point, and so we must have that $t_0 = 1$ (and $\phi_0 = \Psi_1 = 0$). □

For future purposes, we state a variant of (5.10) which is essentially proved in [106, Lemma 2.1] under additional assumptions on f and g.

Proposition 5.9. *Let (u_n, v_n) be a Palais-Smale sequence for the functional I, namely*

$$0 < \liminf I(u_n, v_n) \leqslant \limsup I(u_n, v_n) < +\infty \qquad \text{and} \qquad I'(u_n, v_n) \to 0 \ \text{in } (H \times H)'.$$

Then

$$\sup_{\mathbb{R}^+(u_n,v_n)\oplus H^-} I = \sup\{I(t(u_n,v_n) + (\phi, -\phi)) : \ t \geqslant 0, \ \phi \in H\} = I(u_n, v_n) + \mathrm{O}(\mu_n^2), \tag{5.11}$$

where $\mu_n := \|I'(u_n, v_n)\|_{(H\times H)'} = \sup\{|I'(u_n,v_n)(\phi,\psi)|, \ \phi, \psi \in H, \ \|\phi\| + \|\psi\| \leqslant 1\} \to 0.$

Proof. We denote $\Psi_t^n := \Psi_{tu_n,tv_n}$ (*cf.* (5.5)) while $\Psi_t'^n$ stands for the derivative of the map $t \mapsto \Psi_t^n$ evaluated at the point t. Since the proof is quite long, we divide it into several steps.

Step 1. The sequence (u_n, v_n) is bounded in $H \times H$.

We start by observing that

$$\begin{aligned}\|u_n\|^2 + \|v_n\|^2 &= I'(u_n,v_n)(v_n,u_n) + \int_\Omega (f(u_n)v_n + g(v_n)(u_n)) \\ &= \mathrm{o}(\|(u_n,v_n)\|) + \int_\Omega (f(u_n)v_n + g(v_n)u_n) \\ &\leqslant \mathrm{o}(\|(u_n,v_n)\|) + \frac{\alpha}{2}\int_\Omega (u_n^2+v_n^2) + C\int_\Omega (f(u_n)u_n + g(v_n)v_n)\end{aligned} \tag{5.12}$$

for some $C > 0$ (where we have used $(fg5)$ with $\mu = \alpha/2$). Moreover, since $I(u_n, v_n) = \mathrm{O}(1)$ and $I'(u_n, v_n)(u_n, v_n) = \mathrm{o}(\|(u_n, v_n)\|)$, we obtain

$$\langle u_n, v_n\rangle - \int_\Omega F(u_n) - \int_\Omega G(v_n) = \mathrm{O}(1)$$

and

$$2\langle u_n, v_n\rangle - \int_\Omega f(u_n)u_n - \int_\Omega g(v_n)v_n = \mathrm{o}(\|(u_n,v_n)\|).$$

From this it follows that (by using the condition (AR))

$$\begin{aligned}\mathrm{O}(1) &= \mathrm{o}(\|(u_n,v_n)\|) + \int_\Omega (f(u_n)u_n - 2F(u_n)) + \int_\Omega (g(u_n)u_n - 2G(u_n)) \\ &\geqslant \mathrm{o}(\|(u_n,v_n)\|) + \frac{\delta'}{2+\delta'}\int_\Omega (f(u_n)u_n + g(v_n)v_n),\end{aligned}$$

and hence

$$\int_\Omega (f(u_n)u_n + g(v_n)v_n) \leqslant \mathrm{o}(\|(u_n,v_n)\|) + \mathrm{O}(1). \tag{5.13}$$

By combining (5.12) with (5.13) we obtain

$$\frac{1}{2}\|(u_n,v_n)\|^2 \leqslant \mathrm{o}\left(\|(u_n,v_n)\|\right) + \mathrm{O}(1),$$

and hence (u_n, v_n) is indeed a bounded sequence in $H \times H$.

Step 2. For $\alpha_n(t) := I(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n))$ there exists $t_n \geqslant 0$ such that

$$\alpha_n(t_n) = \sup_{t\geqslant 0}\alpha_n(t) = \sup\{I(t(u_n,v_n) + (\phi,-\phi)) :\ t \geqslant 0,\ \phi \in H\}.$$

There must be a region $K \subseteq \Omega$ having positive measure where both $u_n, v_n > 0$, otherwise by testing $I'(u_n, v_n) = \mathrm{o}(1)$ with (v_n, u_n) and by taking in consideration the Step 1, we would obtain

$$\mathrm{o}(1) = I'(u_n,v_n)(v_n,u_n) = \|(u_n,v_n)\|^2 - \int_\Omega f(u_n)v_n - \int_\Omega g(v_n)u_n \geqslant \|(u_n,v_n)\|^2$$

and hence $(u_n, v_n) \to (0,0)$ in $H \times H$ and $I(u_n, v_n) \to 0$, a contradiction. Thus the supremum in consideration is finite (*cf.* Remark 5.3) and it is easy to check that it is attained at a positive point t_n.

Step 3. $\|\Psi_1^n\| = \mathrm{O}(\mu_n)$.

By the definition of Ψ_1^n we have

$$I'((u_n, v_n) + (\Psi_1^n, -\Psi_1^n))(\phi, -\phi) = 0 \qquad \text{for every } \phi \in H.$$

By letting $\phi = \Psi_1^n$, we obtain

$$I'((u_n, v_n) + (\Psi_1^n, -\Psi_1^n))(\Psi_1^n, -\Psi_1^n) = 0,$$

which yields (by using the fact that f', g' are nonnegative)

$$\begin{aligned} 2\|\Psi_1^n\|^2 &= \langle \Psi_1^n, v_n - u_n\rangle - \int_\Omega f(u_n + \Psi_1^n)\Psi_1^n + \int_\Omega g(v_n - \Psi_1^n)\Psi_1^n \\ &= I'(u_n, v_n)(\Psi_1^n, -\Psi_1^n) + \int_\Omega (f(u_n)\Psi_1^n - f(u_n + \Psi_1^n)\Psi_1^n) + \int_\Omega (g(u_n - \Psi_1^n)\Psi_1^n - g(v_n)\Psi_1^n) \\ &\leqslant \mathrm{O}(\mu_n)\|\Psi_1^n\|, \end{aligned}$$

which proves the claim.

Step 4. $\alpha_n(1) = I(u_n, v_n) + \mathrm{O}(\mu_n^2)$ and $\alpha_n'(1) = \mathrm{O}(\mu_n)$.

We have

$$\begin{aligned} \alpha_n(1) &= I(u_n + \Psi_1^n, v_n - \Psi_1^n) = \langle u_n + \Psi_1^n, v_n - \Psi_1^n\rangle - \int_\Omega F(u_n + \Psi_1^n) - \int_\Omega G(v_n - \Psi_1^n) \\ &= \langle u_n, v_n\rangle + \langle \Psi_1^n, v_n - u_n\rangle - \|\Psi_1^n\|^2 - \int_\Omega F(u_n + \Psi_1^n) - \int_\Omega G(v_n - \Psi_1^n). \end{aligned}$$

From the identities

$$\langle u_n, v_n\rangle = I(u_n, v_n) + \int_\Omega F(u_n) + \int_\Omega G(v_n)$$

and

$$\langle \Psi_1^n, v_n - u_n\rangle = I'(u_n, v_n)(\Psi_1^n, -\Psi_1^n) + \int_\Omega f(u_n)\Psi_1^n - \int_\Omega g(v_n)\Psi_1^n,$$

we obtain that

$$\begin{aligned} \alpha_n(1) &= I(u_n, v_n) + I'(u_n, v_n)(\Psi_1^n, -\Psi_1^n) - \|\Psi_1^n\|^2 + \int_\Omega (F(u_n) - F(u_n + \Psi_1^n) + f(u_n)\Psi_1^n) + \\ &\quad + \int_\Omega (G(v_n) - G(v_n - \Psi_1^n) - g(v_n)\Psi_1^n) \\ &= I(u_n, v_n) + \mathrm{O}(\mu_n^2) \end{aligned}$$

from Step 3 and condition $(fg4)$.

Moreover, recall that

$$\alpha_n'(t) = I'(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n))(u_n + \Psi_t'^n, v_n - \Psi_t'^n) = I'(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n))(u_n, v_n)$$

and hence

$$\begin{aligned}
\alpha_n'(1) &= I'((u_n, v_n) + (\Psi_1^n, -\Psi_1^n))(u_n, v_n)\\
&= 2\langle u_n, v_n\rangle + \langle \Psi_1^n, v_n - u_n\rangle - \int_\Omega f(u_n + \Psi_1^n)u_n - \int_\Omega g(v_n - \Psi_1^n)v_n\\
&= I'(u_n, v_n)(u_n, v_n) + \langle \Psi_1^n, v_n - u_n\rangle + \int_\Omega (f(u_n)u_n - f(u_n + \Psi_1^n)u_n) +\\
&\quad + \int_\Omega (g(v_n)v_n - g(v_n - \Psi_1^n)v_n)\\
&= \mathrm{O}(\mu_n).
\end{aligned}$$

Step 5. $(\Psi_t^n)_n$ and $(\Psi_t'^n)_n$ are bounded as long as t remains bounded. Moreover, $\Psi_t^n \to 0$ in H as $t \to 1$ and $n \to \infty$.

Suppose $|t| \leqslant \bar{t}$. By letting $\phi = \Psi_t^n$ in (5.6) and by using $(fg1)$, $(fg2)$, Step 1, and the fact that f, g are non decreasing functions we obtain

$$\begin{aligned}
2\|\Psi_t^n\|^2 &= t\langle \Psi_t^n, v_n - u_n\rangle - \int_\Omega f(tu_n + \Psi_t^n)\Psi_t^n + \int_\Omega g(tv_n - \Psi_t^n)\Psi_t^n\\
&\leqslant t\langle \Psi_t^n, v_n - u_n\rangle - \int_\Omega f(tu_n)\Psi_t^n + \int_\Omega g(tv_n)\Psi_t^n\\
&\leqslant Ct\|\Psi_t^n\| + C\left(t\int_\Omega (|u_n| + |v_n|)|\Psi_t^n| + t^{p-1}\int_\Omega \left(|u_n|^{p-1} + |v_n|^{p-1}\right)|\Psi_t^n|\right)\\
&\leqslant C(\bar{t})\|\Psi_t^n\|
\end{aligned}$$

and hence $(\Psi_t^n)_n$ is bounded. As for $(\Psi_t'^n)_n$, by differentiating (5.6) and by letting $\phi = \Psi_t'^n$ we see that

$$I''(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n))(u_n + \Psi_t'^n, v_n - \Psi_t'^n)(\Psi_t'^n, -\Psi_t'^n) = 0,$$

and hence

$$\begin{aligned}
2||\Psi_t'^n||^2 &= \langle \Psi_t'^n, v_n - u_n\rangle - \int f'(tu_n + \Psi_t^n)(u_n + \Psi_t'^n)\Psi_t'^n + \int g'(tv_n - \Psi_t^n)(v_n - \Psi_t'^n)\Psi_t'^n\\
&\leqslant \langle \Psi_t'^n, v_n - u_n\rangle - \int f'(tu_n + \Psi_t^n)u_n\Psi_t'^n + \int g'(tv_n - \Psi_t^n)v_n\Psi_t'^n\\
&\leqslant C\|\Psi_t'^n\| + C\left(\int_\Omega (|u_n| + |v_n|)|\Psi_t'^n| + t^{p-2}\left(|u_n|^{p-1} + |v_n|^{p-1}\right)|\Psi_t'^n|\right) +\\
&\quad + C\int_\Omega (|u_n| + |v_n|)|\Psi_t^n|^{p-2}|\Psi_t'^n|\\
&\leqslant C(\bar{t})\|\Psi_t'^n\|,
\end{aligned}$$

where we have used $(fg4)$. Thus $(\Psi_t'^n)_n$ is also bounded. Finally, this fact combined with Step 3 provides that

$$\|\Psi_t^n\| \leqslant \|\Psi_t^n - \Psi_1^n\| + \|\Psi_1^n\| \leqslant C|t-1| + \mathrm{O}(\mu_n) \to 0$$

as $t \to 1$ and $n \to \infty$.

Step 6. There exist $\delta, L > 0$ such that

$$\sup_{t\in[1-\delta,1+\delta]} \alpha_n''(t) \leqslant -L \qquad \text{for large } n. \tag{5.14}$$

We have

$$\begin{aligned}
\alpha_n''(t) &= I''(t(u_n,v_n) + (\Psi_t^n, -\Psi_t^n))(u_n + \Psi_t'^n, v_n - \Psi_t'^n)(u_n + \Psi_t'^n, v_n - \Psi_t'^n) \\
&= -2\|\Psi_t'^n\|^2 + 2\langle u_n, v_n\rangle + 2\langle \Psi_t'^n, v_n - u_n\rangle - \int_\Omega f'(tu_n + \Psi_t^n)(u_n + \Psi_t'^n)^2 - \\
&\quad - \int_\Omega g'(tv_n - \Psi_t^n)(v_n - \Psi_t'^n)^2 \\
&= -2\|\Psi_t'^n\|^2 + \int_\Omega \big(f(u_n)u_n + 2f(u_n)\Psi_t'^n - f'(tu_n + \Psi_t^n)(u_n + \Psi_t'^n)^2\big) + \\
&\quad + \int_\Omega \big(g(v_n)v_n - 2g(v_n)\Psi_t'^n - g'(tv_n - \Psi_t^n)(v_n - \Psi_t'^n)^2\big) + \mathrm{o}(1)
\end{aligned}$$

as $t \to 1$, uniformly in n, where we have used the identities $J'(u_n, v_n)(u_n, v_n) = \mathrm{o}(1)$ and $J'(u_n, v_n)(\Psi_t'^n, -\Psi_t'^n) = \mathrm{o}(1)$ (which hold by Steps 1 and 5).

Let $\bar u_n := tu_n + \Psi_t^n$. Since $\Psi_t^n \to 0$ in H as $t \to 1$ and $n \to \infty$, and $u_n, \Psi_t'^n$ are bounded in H, it is straightforward to see that

$$\int_\Omega \left(f(u_n) - \frac{f(\bar u_n)}{\bar u_n} u_n \right) u_n \to 0 \qquad \int_\Omega \left(f(u_n) - \frac{f(\bar u_n)}{\bar u_n} u_n \right) \Psi_t'^n \to 0$$

as $t \to 1$ and $n \to \infty$. Analogously, for $\bar v_n := tv_n - \Psi_t^n$,

$$\int_\Omega \left(g(v_n) - \frac{g(\bar v_n)}{\bar v_n} v_n \right) v_n \to 0 \qquad \int_\Omega \left(g(v_n) - \frac{g(\bar v_n)}{\bar v_n} v_n \right) \Psi_t'^n \to 0$$

as $t \to 1$ and $n \to \infty$. Thus

$$\begin{aligned}
\alpha_n''(t) &= -2\|\Psi_t'^n\|^2 + \int_\Omega \left(\frac{f(\bar u_n)}{\bar u_n} u_n^2 + 2\frac{f(\bar u_n)}{\bar u_n} u_n \Psi_t'^n - f'(\bar u_n)(u_n + \Psi_t'^n)^2 \right) + \\
&\quad + \int_\Omega \left(\frac{g(\bar v_n)}{\bar v_n} v_n^2 - 2\frac{g(\bar v_n)}{\bar v_n} v_n \Psi_t'^n - g'(\bar v_n)(v_n + \Psi_t'^n)^2 \right) + \mathrm{o}(1) \\
&= -2\|\Psi_t'^n\|^2 + \int_\Omega \left(\frac{f(\bar u_n)}{\bar u_n} - f'(\bar u_n) \right)(u_n + \Psi_t'^n)^2 + \\
&\quad + \int_\Omega \left(\frac{g(\bar v_n)}{\bar v_n} - g'(\bar v_n) \right)(v_n - \Psi_t'^n)^2 - \int_\Omega \left(\frac{f(\bar u_n)}{\bar u_n} + \frac{g(\bar v_n)}{\bar v_n} \right)(\Psi_t'^n)^2 + \mathrm{o}(1).
\end{aligned}$$

as $t \to 1$ and $n \to \infty$. By using condition $(fg3)$, it follows that

$$\alpha_n''(t) \leqslant -\delta' \int_\Omega \left(\frac{f(\bar u_n)}{\bar u_n} u_n^2 + \frac{g(\bar v_n)}{\bar v_n} v_n^2 \right) + \mathrm{o}(1) = -\delta' \int_\Omega (f(u_n)u_n + g(v_n)v_n) + \mathrm{o}(1)$$

as $t \to 1$ and $n \to \infty$. Finally, since

$$\begin{aligned}
\int_\Omega (f(u_n)u_n + g(v_n)v_n) &= 2\langle u_n, v_n\rangle - I'(u_n,v_n)(u_n,v_n) \\
&\geqslant 2I(u_n,v_n) - I'(u_n,v_n)(u_n,v_n) = 2I(u_n,v_n) + \mathrm{o}(1)
\end{aligned}$$

and $\liminf I(u_n, v_n) > 0$, we have $\liminf \int_\Omega (f(u_n)u_n + g(v_n)v_n) > 0$ and the claim follows.
Step 7. $t_n - 1 = \mathrm{O}(\mu_n)$

From (5.14) we obtain that

$$\alpha_n'(1) - \alpha_n'(1-\delta) \leqslant -L\delta$$

and hence (by using Step 4)

$$\mathrm{O}(\mu_n) + L\delta = \alpha_n'(1) + L\delta \leqslant \alpha_n'(1-\delta).$$

Since $L > 0$ and $\mu_n \to 0$, we have $\alpha_n'(1-\delta) > 0$ for large n. Analogously, $\alpha_n'(1+\delta) < 0$ for large n. This implies that (always for large n) there exists a critical point of $\alpha_n(t)$ lying in the interval $(1-\delta, 1+\delta)$. Since $\alpha_n'(t_n) = 0$, by Lemma 5.7 we deduce that $t_n \in (1-\delta, 1+\delta)$. If $t_n < 1$, then we obtain

$$L|t_n - 1| \leqslant \alpha_n'(t_n) - \alpha_n'(1) = -\alpha_n'(1) = \mathrm{O}(\mu_n).$$

If $t_n > 1$ we obtain an analogous result.
Step 8. End of the proof.

In conclusion, we have

$$\begin{aligned} \alpha_n(t) &= \alpha_n(1) + \alpha_n'(1)(t_n - 1) + \mathrm{O}\left((t_n-1)^2\right) \\ &= I(u_n, v_n) + \mathrm{O}(\mu_n^2) \end{aligned}$$

by Steps 4 and 7. □

In the case $\Omega = \mathbb{R}^N$, for a given $\lambda > 0$ let us consider the problem

$$-\Delta u + \lambda u = g(v), \qquad -\Delta v + \lambda v = f(u), \qquad u, v \in H^1(\mathbb{R}^N), \tag{5.15}$$

and the associated energy functional

$$I_\lambda(u,v) = \int_{\mathbb{R}^N} (\langle \nabla u, \nabla v\rangle + \lambda uv) - \int_{\mathbb{R}^N} F(u) - \int_{\mathbb{R}^N} G(v), \qquad u, v \in H^1(\mathbb{R}^N).$$

We denote by $c_\lambda = c(\lambda)$ the corresponding ground-state critical level, that is

$$c(\lambda) = \inf\{I_\lambda(u,v) : (u,v) \neq (0,0) \text{ and } I_\lambda'(u,v) = 0\}.$$

It is not hard to prove that $c(\lambda) > 0$ is attained at a pair (u,v) with $u, v > 0$, solution of (5.15) (see for instance [117, Theorem 3] or [124, Section 1.2]). Moreover, from the Benci-Rabinowitz linking theorem [19], we see that

$$c(\lambda) \leqslant \sup\{I_\lambda(t(u,v) + (\phi, -\phi)) :\ t \geqslant 0,\ \phi \in H\} \tag{5.16}$$

for every $u, v \in H$ such that $u, v > 0$.

Corollary 5.10. *The map $\mathbb{R}^+ \to \mathbb{R}^+$, $\lambda \mapsto c(\lambda)$ is continuous and increasing.*

Proof. We apply the argument used in [106, Lemma 3.1]. Take $\lambda > 0$ and consider $\lambda_n \to \lambda$. Fix $u_0, v_0 > 0$ and let $t_n \in \mathbb{R}$, $\phi_n \in H$ be such that

$$\begin{aligned} c(\lambda_n) &\leqslant \sup\{I_{\lambda_n}(t(u_0,v_0)+(\phi,-\phi)) :\ t \geqslant 0,\ \phi \in H\} \\ &= I_{\lambda_n}(t_n(u_0,v_0)+(\phi_n,-\phi_n)). \end{aligned}$$

By using the arguments of Steps 1 and 3 in the proof of Proposition 5.9 we see that t_n and $\|\phi_n\|$ are bounded, whence $c(\lambda_n)$ is also bounded in $\mathbb{R}$. Let (u_n, v_n) be a ground-state for I_{λ_n} with $u_n, v_n > 0$. Since

$$I_{\lambda_n}(u_n,v_n) = c(\lambda_n) > 0 \quad \text{is bounded, and} \quad I'_{\lambda_n}(u_n,v_n) = 0,$$

by adapting the reasoning of Step 1 in the proof of Proposition 5.9 (here we have $\lambda_n \to \lambda$) it follows that (u_n, v_n) is a bounded sequence in $H \times H$. Hence

$$I_\lambda(u_n,v_n) = I_{\lambda_n}(u_n,v_n) + (\lambda-\lambda_n)\int_{\mathbb{R}^N} u_n v_n = c(\lambda_n) + \mathrm{o}(1)$$

and

$$\|I'_\lambda(u_n,v_n)\| \leqslant \max_{\|\phi\|+\|\psi\|\leqslant 1} |\lambda-\lambda_n| \Big|\int_{\mathbb{R}^N} (u_n\psi + v_n\phi)\Big| \leqslant C|\lambda_n-\lambda| \to 0$$

and therefore we conclude from (5.16) and Proposition 5.9 that

$$\begin{aligned} c(\lambda) &\leqslant \sup\{I_\lambda(t(u_n,v_n)+(\phi,-\phi)) :\ t \geqslant 0,\ \phi \in H^1(\mathbb{R}^N)\} \\ &= I_\lambda(u_n,v_n) + \mathrm{o}(1) = c(\lambda_n) + \mathrm{o}(1), \end{aligned}$$

which yields that

$$c(\lambda) \leqslant \liminf c(\lambda_n). \tag{5.17}$$

On the other hand, let (u, v) be a ground-state for I_λ, with $u, v > 0$. We have

$$I_{\lambda_n}(u,v) = I_\lambda(u,v) + (\lambda_n-\lambda)\int_{\mathbb{R}^N} uv = I_\lambda(u,v) + \mathrm{o}(1)$$

and

$$\|I'_{\lambda_n}(u,v)\| \leqslant \max_{\|\phi\|+\|\psi\|\leqslant 1} |\lambda_n-\lambda| \Big|\int_{\mathbb{R}^N} (u\psi + v\phi)\Big| \leqslant C|\lambda_n-\lambda| \to 0,$$

which yields

$$\begin{aligned} c(\lambda_n) &\leqslant \sup\{I_{\lambda_n}(t(u,v)+(\phi,-\phi)) :\ t \geqslant 0,\ \phi \in H^1(\mathbb{R}^N)\} \\ &= I_{\lambda_n}(u,v) + \mathrm{O}(\|I'_{\lambda_n}(u,v)\|^2) \\ &\leqslant I_{\lambda_n}(u,v) + C(\lambda_n-\lambda)^2 \\ &= c(\lambda) + (\lambda_n-\lambda)\int_{\mathbb{R}^N} uv + C(\lambda_n-\lambda)^2, \end{aligned}$$

whence $\lim c(\lambda_n) \leqslant c(\lambda)$ which, together with (5.17), yields

$$c(\lambda) \to c(\lambda).$$

Moreover, from the inequality

$$c(\lambda_n) \leqslant c(\lambda) + (\lambda_n - \lambda)\Big(C(\lambda_n - \lambda) + \int_{\mathbb{R}^N} uv\Big)$$

we see that $c(\lambda_n) < c(\lambda)$ whenever $\lambda_n < \lambda$ and λ_n is sufficiently close to λ. Thus the map $\lambda \mapsto c(\lambda)$ is locally increasing and since it is continuous it is actually increasing. $\square$

We present one further monotonicity result for ground-states $c(\lambda)$.

Remark 5.11. As a simple consequence of (5.10) and (5.16), we see that the ground-state critical levels are monotonic with respect to the potentials F and G.

5.3 Nehari manifold

In the following we fix bounded domains $\widetilde{\Lambda}_i$, mutually disjoint, such that $\Lambda_i \Subset \widetilde{\Lambda}_i \Subset \Omega$ for each $i = 1, \ldots, k$. We take cut-off functions ϕ_i such that $\phi_i = 1$ in Λ_i and $\phi_i = 0$ in $\mathbb{R}^N \setminus \widetilde{\Lambda}_i$, and suppose without loss of generality that $|\nabla \phi_i| \leqslant C\phi_i$ for some $C > 0$ (by replacing if necessary ϕ_i by ϕ_i^2). We also denote $\Lambda := \cup_i \Lambda_i$, $\widetilde{\Lambda} := \cup_i \widetilde{\Lambda}_i$. Following an idea introduced in [57, 59], we will truncate the functions f and g outside of Λ. To do so, we will need the following technical result.

Lemma 5.12. *Let $h : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a continuous function such that*

$$h(0) = 0 \qquad \text{and} \qquad \lim_{s \to +\infty} h(s) = +\infty.$$

Then for every $\delta > 0$ there exists $a > 0$ such that

$$h(a) \leqslant \delta \qquad \text{and} \qquad h(s) \geqslant h(a)\ \forall s \geqslant a.$$

Proof. Let a_0 be such that $h(s) \leqslant \delta$ for every $0 < s \leqslant a_0$. Then either $h(s) \geqslant h(a_0)$ for every $s \geqslant a_0$ (and we can take $a := a_0$) or there exists $s_0 > a_0$ such that $h(s_0) < h(a_0)$ and $h(s_0) = \min\{h(s) :\ s \geqslant a_0\}$. In the latter case, it is easy to prove that $a := \max\{s < a_0 : h(s) = h(s_0)\}$ satisfies the conclusions of the lemma. $\square$

The previous lemma implies that for every $\delta > 0$ there exist a_1, a_2 such that

$$f'(a_1) \leqslant \delta, \quad f'(s) \geqslant f'(a_1)\ \forall s \geqslant a_1, \qquad g'(a_2) \leqslant \delta, \quad g'(s) \geqslant g'(a_2)\ \forall s \geqslant a_2.$$

For such a_1, a_2, we define the following truncated functions

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{if } s \leqslant a_1 \\ f'(a_1)s + (f(a_1) - a_1 f'(a_1)) & \text{if } s \geqslant a_1, \end{cases}$$

$$\widetilde{g}(s) = \begin{cases} g(s) & \text{if } s \leqslant a_2 \\ g'(a_2)s + (g(a_2) - a_2 g'(a_2)) & \text{if } s \geqslant a_2. \end{cases}$$

Observe that $\tilde{f}(s) \leqslant f(s)$ and $\tilde{g}(s) \leqslant g(s)$ for every s. Then we introduce

$$f(x, s) := \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\widetilde{f}(s), \quad g(x, s) := \chi_\Lambda(x) g(s) + (1 - \chi_\Lambda(x))\widetilde{g}(s),$$

and the corresponding energy functional

$$J_\varepsilon(u,v) := \int_\Omega (\varepsilon^2\langle\nabla u,\nabla v\rangle + V(x)uv) - \int_\Omega F(x,u) - \int_\Omega G(x,v),$$

where $F(x,s) := \int_0^s f(x,\xi)\,d\xi$, $G(x,s) := \int_0^s g(x,\xi)\,d\xi$. Similarly to [57, 59], this truncation technique will be helpful in both bringing compactness to the problem and locating the maximum points of our solutions.

Notations. For every fixed $\varepsilon > 0$ we denote $\langle u,v\rangle_\varepsilon := \int_\Omega(\varepsilon^2\langle\nabla u,\nabla v\rangle + V(x)uv)$ and $\|(u,v)\|_\varepsilon^2 = \|u\|_\varepsilon^2 + \|v\|_\varepsilon^2 = \int_\Omega(\varepsilon^2|\nabla u|^2+\varepsilon^2|\nabla v|^2)+\int_\Omega V(x)(u^2+v^2)$. The partial derivatives $\frac{\partial f}{\partial s}(x,s)$ and $\frac{\partial g}{\partial s}(x,s)$ will be denoted respectively by $f'(x,s)$ and $g'(x,s)$). The relevant properties of $f(x,s)$ and $g(x,s)$ are displayed in the next lemma, whose proof is elementary.

Lemma 5.13. *The function f(x,s) (and also g(x,s)) satisfies:*

(i) $f(x,s) = \mathrm{o}(s)$ *as* $s \to 0$*, uniformly in* $x \in \Omega$*;*

(ii) $|f'(x,s)| \leqslant C(1+|s|^{p-2})$ *with* $2 < p < 2N/(N-2)$*,* $\forall x \in \Omega,\ s \in \mathbb{R}$*;*

(iii) $(1+\delta')f(x,s)s \leqslant s^2 f'(x,s)$*, with* $\delta' > 0$*,* $\forall x \in \Lambda,\ s \in \mathbb{R}$*;*

(iv) $0 < f(x,s)s \leqslant s^2 f'(x,s)$*,* $\forall x \in \Omega\setminus\Lambda,\ s \in \mathbb{R}, s \neq 0$*;*

(v) for every $\delta > 0$ *there exist* $a_1, a_2 > 0$ *such that*

$$|f(x,s)| + |g(x,s)| \leqslant \delta|s|,\quad \forall x \in \Omega\setminus\Lambda,\ s \in \mathbb{R}; \tag{5.18}$$

(vi) for every $\mu > 0$ *there exists* $C_\mu > 0$ *such that*

$$|f(x,s)t| + |g(x,t)s| \leqslant \mu\,(s^2+t^2) + C_\mu\,(f(x,s)s + g(x,t)t),\ \forall x \in \Omega,\ s,t \in \mathbb{R}. \tag{5.19}$$

We denote by N_ε the set

$$N_\varepsilon = \left\{ (u,v) \in H \times H : \begin{array}{c} J'_\varepsilon(u,v)(\phi,-\phi) = 0 \quad \forall \phi \in H, \\ J'_\varepsilon(u,v)(u\phi_i, v\phi_i) = 0, \quad \text{and} \quad \int_{\Lambda_i}(u^2+v^2) > \varepsilon^{N+1},\ \forall i = 1,...,k \end{array} \right\}$$

and by c_ε the infimum

$$c_\varepsilon := \inf_{N_\varepsilon} J_\varepsilon\,.$$

It can be shown that N_ε is non-empty. Indeed, let us fix points $x_i \in \Lambda_i$ such that $V(x_i) = \inf_{\Lambda_i} V$ and let us consider a fixed pair of positive solutions $u_i, v_i \in H^1(\mathbb{R}^N)$ of the system

$$\begin{cases} -\Delta u_i + V(x_i)u_i = g(v_i) \\ -\Delta v_i + V(x_i)v_i = f(u_i) \end{cases} \quad \text{in } \mathbb{R}^N,$$

corresponding to the ground-state critical level (*cf.* the paragraph preceding Corollary 5.10)

$$c_i = I_{V(x_i)}(u_i, v_i).$$

We let

$$u_{i,\varepsilon}(x) := \phi_i(x)u_i\Big(\frac{x-x_i}{\varepsilon}\Big), \qquad v_{i,\varepsilon}(x) := \phi_i(x)v_i\Big(\frac{x-x_i}{\varepsilon}\Big).$$

Our next proposition shows that N_ε is nonempty if ε is sufficiently small, and provides a crucial upper estimate for c_ε. We postpone its proof to the end of the current section.

Proposition 5.14. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \leqslant \varepsilon_0$ *and every* $i = 1, \dots, k$ *there is* $\Psi_\varepsilon \in H$ *and points* $t_{1,\varepsilon}, \dots, t_{k,\varepsilon} \in [0,1]$ *such that the functions*

$$\overline{u}_\varepsilon := \sum_{i=1}^{k} t_{i,\varepsilon} u_{i,\varepsilon} + \Psi_\varepsilon \quad \text{and} \quad \overline{v}_\varepsilon := \sum_{i=1}^{k} t_{i,\varepsilon} v_{i,\varepsilon} - \Psi_\varepsilon$$

satisfy

$$J_\varepsilon'(\overline{u}_\varepsilon, \overline{v}_\varepsilon)(\overline{u}_\varepsilon \phi_i, \overline{v}_\varepsilon \phi_i) = 0, \quad \forall i = 1, \dots, k, \tag{5.20}$$

$$J_\varepsilon'(\overline{u}_\varepsilon, \overline{v}_\varepsilon)(\phi, -\phi) = 0, \quad \forall \phi \in H, \tag{5.21}$$

and

$$J_\varepsilon(\overline{u}_\varepsilon, \overline{v}_\varepsilon) = \varepsilon^N \Big(\sum_{i=1}^{k} c_i + \mathrm{o}(1) \Big) \quad \text{as } \varepsilon \to 0. \tag{5.22}$$

Moreover,

$$\int_{\Lambda_i} (\overline{u}_\varepsilon^2 + \overline{v}_\varepsilon^2) \geqslant \eta \varepsilon^N \tag{5.23}$$

for some $\eta > 0$*, and so in particular* $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \in N_\varepsilon$*.*

We will show (via Ekeland's variational principle, for which we refer to Appendix B) that c_ε is indeed a critical point of J_ε over the space $H \times H$. In view of Theorem B.9, in this subsection we make some preliminary considerations. We know from Proposition 5.14 that

$$c_\varepsilon \leqslant \varepsilon^N \Big(\sum_{i=1}^{k} c_i + \mathrm{o}(1) \Big). \tag{5.24}$$

Hence it will follow from Proposition 5.15 bellow that $c_\varepsilon \geqslant 0$ and that the set $N_\varepsilon \cap \{J_\varepsilon(u,v) \leqslant 2\varepsilon^N \left(\sum_i c_i\right)\}$ is closed in $H \times H$.

Functions in N_ε are zero points of the functional $K_\varepsilon : H \times H \to \mathbb{R}^k \oplus H^-$,

$$K_\varepsilon(u,v) := (J_\varepsilon'(u,v)(u\phi_1, v\phi_1), \dots, J_\varepsilon'(u,v)(u\phi_k, v\phi_k), PJ_\varepsilon'(u,v)),$$

where $P : H \times H \to H^- = \{(\phi, -\phi),\ \phi \in H\}$ denotes the orthogonal projection. For any $(u,v) \in N_\varepsilon$, its derivative $K_\varepsilon'(u,v) : H \times H \to \mathbb{R}^k \oplus H^-$ is given by

$$K_\varepsilon'(u,v)(\zeta, \xi) = (\mu_1(\zeta,\xi), \dots, \mu_k(\zeta,\xi), PJ_\varepsilon''(u,v)(\zeta,\xi)), \tag{5.25}$$

where $PJ_\varepsilon''(u,v)(\zeta,\xi)$ has a meaning according to Riesz's theorem and

$$\mu_i(\zeta,\xi) := J_\varepsilon'(u,v)(\phi_i\zeta, \phi_i\xi) + J_\varepsilon''(u,v)(\phi_i u, \phi_i v)(\zeta,\xi), \qquad i = 1, \dots, k.$$

Let us concentrate on $K_\varepsilon'(u,v)$ restricted to the subspace $Z := \mathrm{span}\{(u\phi_i, v\phi_i),\ i = 1, \dots, k\} \oplus H^-$ of $H \times H$, which we can identify with $\mathbb{R}^k \oplus H^-$. Similarly to the proof of Proposition 5.5 we can check that $Id - K_\varepsilon'(u,v)$ is a compact operator. An element $(\bar{u}, \bar{v}) \in Z$ writes as $(\bar{u}, \bar{v}) = (u\Phi + \psi, v\Phi - \psi)$, with the notation $\Phi := \sum_i \lambda_i \phi_i$. If $K_\varepsilon'(u,v)(\bar{u},\bar{v}) = 0$, then in particular

$$0 = \langle K_\varepsilon'(u,v)(\overline{u},\overline{v}), (\overline{u},\overline{v}) \rangle_{\mathbb{R}^k \oplus H^-} = J_\varepsilon'(u,v)(u\Phi^2, v\Phi^2) + J_\varepsilon''(u,v)(\overline{u},\overline{v})(\overline{u},\overline{v}). \tag{5.26}$$

We shall prove below (*cf.* Proposition 5.20 and Remark 5.21) that for ε small enough such an identity holds if and only if $\lambda_1 = \ldots = \lambda_k = 0$ and $\psi = 0$. Hence $K'_\varepsilon(u,v)|_Z$ is one-to-one, and by the Fredholm's alternative theorem it is also onto. We have thus concluded that $K'_\varepsilon(u,v) : H \times H \to \mathbb{R}^k \oplus H^-$ is onto, and as a consequence the tangent space of the manifold N_ε at the point (u,v) is given by Ker $K'_\varepsilon(u,v)$ (see Theorem B.7). Then, according to the Lagrange multiplier rule, N_ε is a natural constraint for the functional J_ε; namely, if the infimum c_ε is achieved at $(u,v) \in N_\varepsilon$ then there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\psi \in H$ such that

$$J'_\varepsilon(u,v)(\zeta,\xi) = J'_\varepsilon(u,v)(\Phi\zeta,\Phi\xi) + J''_\varepsilon(u,v)(\overline{u},\overline{v})(\zeta,\xi), \qquad \forall \zeta,\xi \in H.$$

By letting $(\zeta,\xi) = (\overline{u},\overline{v})$, so that $J'_\varepsilon(u,v)(\overline{u},\overline{v}) = 0$, we conclude from the previous observation that we must have $\psi = 0$ and $\lambda_1 = \ldots = \lambda_k = 0$, hence (u,v) is indeed a critical point of J_ε. We now make these ideas precise.

Proposition 5.15. *For every $C_0 > 0$ there exist ε_0, D_0, $\eta_0 > 0$ such that, for every $0 < \varepsilon \leqslant \varepsilon_0$,*

$$(u,v) \in N_\varepsilon, \quad J_\varepsilon(u,v) \leqslant C_0\varepsilon^N \;\Rightarrow\; ||(u,v)||_\varepsilon^2 \leqslant D_0\varepsilon^N \;\; \textit{and} \;\; \int_{\Lambda_i} (u^2+v^2) \geqslant \eta_0\varepsilon^N,$$

for every $i = 1, \ldots, k$. Also, $J_\varepsilon(u,v) \geqslant \eta_0\varepsilon^N$ (whence $c_\varepsilon \geqslant \eta_0\varepsilon^N$).

Proof. 1. Let $\Phi := \sum_i \phi_i$ and $\xi := 1 - \Phi$, so that $\xi \geqslant 0$ in Ω and $\xi = 0$ in Λ. Since $J'_\varepsilon(u,v)(\Phi u, \Phi v) = 0$, we have that

$$\begin{aligned} \int_\Lambda (f(u)u + g(v)v) &\leqslant \int_\Omega (f(x,u)u + g(x,v)v)\Phi \\ &= -J'_\varepsilon(u,v)(\Phi u,\Phi v) + \langle u, \Phi v\rangle_\varepsilon + \langle v, \Phi u\rangle_\varepsilon = \langle u, \Phi v\rangle_\varepsilon + \langle v, \Phi u\rangle_\varepsilon \\ &= 2\langle u,v\rangle_\varepsilon + \langle v-u, \xi(v-u)\rangle_\varepsilon - \langle u,\xi u\rangle_\varepsilon - \langle v, \xi v\rangle_\varepsilon. \end{aligned} \tag{5.27}$$

Now, since $\xi \geqslant 0$,

$$\langle u, \xi u\rangle_\varepsilon = \int_\Omega \varepsilon^2|\nabla u|^2\xi + \int_\Omega V(x)u^2\xi + \varepsilon^2\int_\Omega u\langle\nabla u, \nabla\xi\rangle \geqslant \varepsilon^2 \int_\Omega u\langle \nabla u, \nabla \xi\rangle$$

and, as

$$\left|\varepsilon^2 \int_\Omega u\langle\nabla u,\nabla\xi\rangle\right| \leqslant C\varepsilon \int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2),$$

we obtain

$$\langle u,\xi u\rangle_\varepsilon \geqslant \mathrm{o}_\varepsilon(1)\|u\|_\varepsilon^2 \qquad \text{and (similarly)} \qquad \langle v,\xi v\rangle_\varepsilon \geqslant \mathrm{o}_\varepsilon(1)\|v\|_\varepsilon^2, \tag{5.28}$$

as $\varepsilon \to 0$. On the other hand, since $J'_\varepsilon(u,v)((v-u)\xi,(u-v)\xi) = 0$ and $\xi \geqslant 0$, we have that

$$\begin{aligned} \langle v-u,(v-u)\xi\rangle_\varepsilon &= \int_\Omega f(x,u)(v-u)\xi + \int_\Omega g(x,v)(u-v)\xi \\ &\leqslant \int_{\Omega\setminus\Lambda} (f(x,u)v\xi + g(x,v)u\xi) \leqslant \delta\int_\Omega (u^2+v^2) \end{aligned} \tag{5.29}$$

for a small $\delta > 0$ (by (5.18)). Finally, since $J_\varepsilon(u,v) \leqslant C_0\varepsilon^N$,

$$\begin{aligned}
2\langle u,v\rangle_\varepsilon &= 2\int_\Omega (F(x,u)+G(x,v)) + 2J_\varepsilon(u,v) \\
&\leqslant 2\int_\Lambda (F(u)+G(v)) + 2\int_{\Omega\setminus\Lambda}(F(x,u)+G(x,v)) + 2C_0\varepsilon^N \\
&\leqslant 2\int_\Lambda (F(u)+G(v)) + \delta\int_\Omega (u^2+v^2) + 2C_0\varepsilon^N \\
&\leqslant \frac{2}{2+\delta'}\int_\Lambda (f(u)u+g(v)v) + \delta\int_\Omega (u^2+v^2) + 2C_0\varepsilon^N. \qquad (5.30)
\end{aligned}$$

We see from (5.27) – (5.30) that

$$\int_\Lambda (f(u)u+g(v)v) \leqslant C(\delta + \mathrm{o}_\varepsilon(1))\,||(u,v)||_\varepsilon^2 + C_0'\varepsilon^N, \qquad (5.31)$$

for some $\delta > 0$ and some $C, C_0' > 0$ (independent of δ and ε). Thus, going back to (5.30),

$$2\langle u,v\rangle_\varepsilon - \langle v,\Phi u\rangle_\varepsilon - \langle u,\Phi v\rangle_\varepsilon \leqslant 2\langle u,v\rangle_\varepsilon \leqslant C'(\delta+\mathrm{o}_\varepsilon(1))\,||(u,v)||_\varepsilon^2 + C_0''\varepsilon^N. \qquad (5.32)$$

2. Next, from $J_\varepsilon'(u,v)((v-u)\Phi,(u-v)\Phi) = 0$ and $J_\varepsilon'(u,v)(u\Phi, v\Phi) = 0$ it follows that $J_\varepsilon'(u,v)(\Phi v, \Phi u) = 0$, which implies that

$$\begin{aligned}
\langle u,\Phi u\rangle_\varepsilon + \langle v,\Phi v\rangle_\varepsilon &= \int_\Omega (f(x,u)v + g(x,v)u)\Phi \\
&\leqslant \mu\int_\Omega (u^2+v^2) + C_\mu\int_\Lambda (f(u)u+g(v)v) + C_\mu\int_{\Omega\setminus\Lambda}(f(x,u)u+g(x,v)v) \\
&\leqslant \frac{\mu}{\alpha}||(u,v)||_\varepsilon^2 + C_\mu\left(C(\delta+\mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + C_0'\varepsilon^N\right) + \frac{C_\mu\delta}{\alpha}\|(u,v)\|_\varepsilon^2 \\
&\leqslant \frac{\mu}{\alpha}||(u,v)||_\varepsilon^2 + C_\mu'(\delta+\mathrm{o}_\varepsilon(1))\|(u,v)\|_\varepsilon^2 + C_\mu C_0'\varepsilon^N \qquad (5.33)
\end{aligned}$$

where we have used (5.18), (5.19) and (5.31).
3. Finally, by recalling the estimates (5.29), (5.32) and (5.33) and from the fact that $\langle v,\Phi u\rangle + \langle u,\Phi v\rangle = \int_\Omega (f(x,u)u + g(x,v)v)\Phi \geqslant 0$, we deduce that

$$\begin{aligned}
||(u,v)||_\varepsilon^2 &= \langle u,\Phi u\rangle_\varepsilon + \langle v,\Phi v\rangle_\varepsilon + \langle v-u,\xi(v-u)\rangle_\varepsilon + 2\langle u,v\rangle_\varepsilon - \\
&\quad -\langle v,\Phi u\rangle_\varepsilon - \langle u,\Phi v\rangle_\varepsilon \\
&\leqslant C(\mu+\delta+\mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + C_\mu(\delta+\mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + \\
&\quad +(C_\mu C_0' + C_0'')\varepsilon^N. \qquad (5.34)
\end{aligned}$$

By choosing a small $\mu > 0$ and, subsequentely, a small $\delta > 0$ we conclude that

$$||(u,v)||_\varepsilon^2 \leqslant D_0\varepsilon^N \qquad (5.35)$$

for every small ε. This proves the first part of Proposition 5.15.

4. Let us prove the existence of $\eta > 0$ such that

$$\varepsilon^2 \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\phi_i \geqslant \eta\varepsilon^N \qquad \text{for small } \varepsilon. \tag{5.36}$$

Going back to identity $J'_\varepsilon(u,v)(v\phi_i, u\phi_i) = 0$, it is straightforward to obtain

$$\int_\Omega \left(\varepsilon^2|\nabla u|^2 + V(x)u^2\right)\phi_i + \int_\Omega (\varepsilon^2|\nabla v|^2 + V(x)v^2)\phi_i = \int_\Omega (f(x,u)v\phi_i + g(x,v)u\phi_i) - \\ - \varepsilon^2 \int_\Omega u\langle \nabla u, \nabla \phi_i\rangle - \varepsilon^2 \int_\Omega v\langle \nabla v, \nabla\phi_i\rangle.$$

Since $|\nabla\phi_i|^2 \leqslant C\phi_i$, we have

$$\left|\varepsilon^2 \int_\Omega u\langle\nabla u, \nabla\phi_i\rangle\right| \leqslant C'\varepsilon^2 \int_\Omega u^2 + \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2\phi_i;$$

moreover, $\int_{\Lambda_i}(u^2+v^2) > \varepsilon^{N+1}$ and (5.35) imply that (recall that $V(x) \geqslant \alpha > 0$)

$$C'\varepsilon^2\int_\Omega u^2 \leqslant C''\varepsilon^{N+2} \leqslant D'_0\varepsilon \int_\Omega V(x)(u^2+v^2)\phi_i \leqslant \frac{1}{4}\int_\Omega V(x)(u^2+v^2)\phi_i$$

for sufficiently small $\varepsilon > 0$. By proceeding in a similar way with the function v we conclude that

$$-\varepsilon^2\int_\Omega u\langle\nabla u,\nabla\phi_i\rangle - \varepsilon^2\int_\Omega v\langle\nabla v, \nabla\phi_i\rangle \leqslant \frac{1}{2}\int_\Omega \varepsilon^2\left(|\nabla u|^2 + |\nabla v|^2\right)\phi_i + \frac{1}{2}\int_\Omega V(x)(u^2+v^2)\phi_i.$$

Then,

$$\frac{1}{2}\left(\int_\Omega \varepsilon^2\left(|\nabla u|^2+|\nabla v|^2\right)\phi_i + \int_\Omega V(x)(u^2+v^2)\phi_i\right) \leqslant \int_\Omega (f(x,u)v + g(x,v)u)\phi_i. \tag{5.37}$$

By (5.19) and from the fact that $|f(s)|, |g(s)| \leqslant \delta|s| + C_\delta|s|^{2^*-1}$,

$$\begin{aligned}\int_\Omega (f(x,u)v + g(x,v)u)\phi_i &\leqslant \delta\int_\Omega V(x)(u^2+v^2)\phi_i + \int_{\Lambda_i}(f(u)v + g(v)u)\\ &\leqslant 2\delta\int_\Omega V(x)(u^2+v^2)\phi_i + C_\delta\int_{\Lambda_i}\left(|u|^{2^*-1}|v| + |u||v|^{2^*-1}\right)\\ &\leqslant 2\delta\int_\Omega V(x)(u^2+v^2)\phi_i + C'_\delta\int_{\Lambda_i}\left(|u|^{2^*} + |v|^{2^*}\right)\end{aligned}$$

and hence, if we choose δ to be sufficiently small we get

$$\int_\Omega \varepsilon^2(|\nabla u|^2 + |\nabla v|^2)\phi_i + \int_\Omega V(x)(u^2+v^2)\phi_i \leqslant C\int_{\Lambda_i}\left(|u|^{2^*}+|v|^{2^*}\right).$$

Now, by recalling that $\phi_i = 1$ in Λ_i and by using the embedding $H_0^1(\tilde{\Lambda}_i) \hookrightarrow L^{2^*}(\tilde{\Lambda}_i)$,

$$\begin{aligned}\int_{\Lambda_i} \left(|u|^{2^*} + |v|^{2^*}\right) &= \int_{\Lambda_i} \left(|u\phi_i|^{2^*} + |v\phi_i|^{2^*}\right) \\ &\leqslant C\left(\int_\Omega |\nabla(u\phi_i)|^2\right)^{2^*/2} + C\left(\int_\Omega |\nabla(v\phi_i)|^2\right)^{2^*/2} \\ &\leqslant C'\left(\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\phi_i^2\right)^{2^*/2} + C'\left(\int_\Omega |\nabla\phi_i|^2(u^2+v^2)\right)^{2^*/2} \\ &\leqslant C'\left(\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\phi_i\right)^{2^*/2} + C''\left(\int_\Omega V(x)(u^2+v^2)\phi_i\right)^{2^*/2}.\end{aligned}$$

Since $2^*N/2 \geqslant N+2$, from (5.35) and $\int_{\Lambda_i}(u^2+v^2) \geqslant \varepsilon^{N+1}$ we deduce that

$$\left(\int_\Omega V(x)(u^2+v^2)\phi_i\right)^{2^*/2} \leqslant D_0^{2^*/2}\varepsilon^{2^*N/2} \leqslant D_0^{2^*/2}\varepsilon^{N+2} \leqslant \frac{1}{2}\int_\Omega V(x)(u^2+v^2)\phi_i$$

for sufficiently small ε. Therefore we see that

$$\int_\Omega \varepsilon^2(|\nabla u|^2 + |\nabla v|^2)\phi_i + \int_\Omega V(x)(u^2+v^2)\phi_i \leqslant 2C'\left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\phi_i\right)^{2^*/2}$$

and hence, since $u^2+v^2 \not\equiv 0$ in Λ_i,

$$\frac{\varepsilon^2}{2C'} \leqslant \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\phi_i\right)^{2^*/2-1} = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\phi_i\right)^{2/(N-2)},$$

which corresponds to (5.36)

5. Suppose now by contradiction that there exists $(u_\varepsilon, v_\varepsilon) \in N_\varepsilon$ such that

$$\varepsilon^{-N}\int_{\Lambda_i}(u_\varepsilon^2 + v_\varepsilon^2) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Denote $\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, $\bar{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x)$, $\Omega^\varepsilon = \Omega/\varepsilon$ and $\Lambda_i^\varepsilon = \Lambda_i/\varepsilon$. Then $\int_{\Lambda_i^\varepsilon}(\bar{u}_\varepsilon^2 + \bar{v}_\varepsilon^2) \to 0$ and, from (5.35),

$$\left(\int_{\Lambda_i^\varepsilon} |\bar{u}_\varepsilon|^{2^*}\right)^{2/2^*} \leqslant C\int_{\Omega^\varepsilon}\left(|\nabla\bar{u}_\varepsilon|^2 + \bar{u}_\varepsilon^2\right) \leqslant CD_0$$

where we have used the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Thus

$$\varepsilon^{-N}\int_{\Lambda_i}|u_\varepsilon|^p = \int_{\Lambda_i^\varepsilon}|\bar{u}_\varepsilon|^p \to 0$$

since $2 < p < 2^*$, and analogously for v_ε. Coming back to (5.37), since by (i) and (ii) of Lemma 5.13 it holds $|f(x,s)|, |g(x,s)| \leqslant \delta|s| + C_\delta|s|^{p-1}$, it follows that

$$\int_\Omega \varepsilon^2\left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right)\phi_i + \int_\Omega V(x)(u_\varepsilon^2 + v_\varepsilon^2)\phi_i \leqslant C\int_{\Lambda_i}\left(|u_\varepsilon|^p + |v_\varepsilon|^p\right).$$

But then

$$\varepsilon^{-N}\int_{\Lambda_i}(|u_\varepsilon|^p+|v_\varepsilon|^p)\to 0 \quad \text{implies} \quad \varepsilon^{2-N}\int_\Omega\left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\phi_i\to 0,$$

which is in contradiction with (5.36). □

Remark 5.16. For future reference, observe that under the conditions of Proposition 5.15 we have

$$\int_{\Lambda_i}(f(u)u+g(v)v)\geqslant \eta_2\varepsilon^N \qquad \text{for every } i. \tag{5.38}$$

Indeed, from (5.18), (5.19) and (5.37) we see that

$$\frac{1}{2}\left(\int_\Omega \varepsilon^2(|\nabla u|^2+|\nabla v|^2)\phi_i+\int_\Omega(u^2+v^2)\phi_i\right)\leqslant\int_\Omega(f(x,u)v+g(x,v)u)\phi_i\leqslant$$
$$(\mu+\delta)\int_\Omega(u^2+v^2)\phi_i+C_\mu\int_{\Lambda_i}(f(u)u+g(v)v),$$

which, together with (5.36), implies our claim.

Lemma 5.17. *Under the conditions of Proposition 5.15, let $i\in\{1,\dots,k\}$ and $\psi\in H$. Then*

$$\alpha_{i,\varepsilon}:=\langle(v-u)\phi_i,\psi\rangle_\varepsilon-\langle v-u,\psi\phi_i\rangle_\varepsilon=\mathrm{o}_\varepsilon(1)\varepsilon^{N/2}||\psi||_\varepsilon,$$

where $\mathrm{o}_\varepsilon(1)\to 0$ as $\varepsilon\to 0$, uniformly in u,v,ψ.

Proof. Clearly,

$$\begin{aligned}
|\alpha_{i,\varepsilon}| &= \left|\varepsilon^2\int_\Omega(v-u)\langle\nabla\phi_i,\nabla\psi\rangle-\varepsilon^2\int_\Omega\psi\langle\nabla\phi_i,\nabla(v-u)\rangle\right|\\
&\leqslant C\varepsilon\left(\int_\Omega|v-u||\nabla\psi|\varepsilon+\int_\Omega|\psi|\,|\nabla(v-u)|\varepsilon\right)\\
&\leqslant C\varepsilon\left(\int_\Omega\varepsilon^2|\nabla\psi|^2\right)^{1/2}\left(\int_\Omega|v-u|^2\right)^{1/2}+C\varepsilon\left(\int_\Omega\varepsilon^2|\nabla(v-u)|^2\right)^{1/2}\left(\int_\Omega|\nabla\psi|^2\right)^{1/2}\\
&\leqslant C'\varepsilon\|\psi\|_\varepsilon\|(u,v)\|_\varepsilon
\end{aligned}$$

for some $C>0$ not depending on u,v,ψ,ε, and the conclusion follows from Proposition 5.15. □

Lemma 5.18. *Under the conditions of Proposition 5.15, let $i\in\{1,\dots,k\}$ and denote*

$$\beta_{i,\varepsilon}:=\langle u,v\phi_i^2\rangle_\varepsilon+\langle v,u\phi_i^2\rangle_\varepsilon+2\langle u\phi_i,v\phi_i\rangle_\varepsilon-2\int_\Omega(f(x,u)u+g(x,v)v)\phi_i^2.$$

Then

$$\beta_{i,\varepsilon}\leqslant\mathrm{o}_\varepsilon(1)\varepsilon^N,$$

where $\mathrm{o}_\varepsilon(1)\to 0$ as $\varepsilon\to 0$, uniformly in u,v.

Proof. We may subtract the quantity $2J_\varepsilon'(u,v)(u\phi_i, v\phi_i) = 0$ in the expression of $\beta_{i,\varepsilon}$, obtaining

$$\begin{aligned}
\beta_{i,\varepsilon} &= \beta_{i,\varepsilon} - 2J_\varepsilon'(u,v)(u\phi_i, v\phi_i) \\
&= \langle u, v\phi_i^2\rangle_\varepsilon - 2\langle u, v\phi_i\rangle_\varepsilon + \langle v, u\phi_i^2\rangle_\varepsilon - 2\langle v, u\phi_i\rangle_\varepsilon + \\
&\quad +2\langle u\phi_i, v\phi_i\rangle_\varepsilon + 2\int_\Omega (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) \\
&= 2\langle u, v(\phi_i^2 - \phi_i)\rangle_\varepsilon + 2\langle v, u(\phi_i^2 - \phi_i)\rangle_\varepsilon + 2\int_\Omega (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) + \\
&\quad +2\langle u\phi_i, v\phi_i\rangle_\varepsilon - \langle u, v\phi_i^2\rangle_\varepsilon - \langle v, u\phi_i^2\rangle_\varepsilon \\
&= 2\langle u, u(\phi_i^2 - \phi_i)\rangle_\varepsilon + 2\langle v, v(\phi_i^2 - \phi_i)\rangle_\varepsilon + 2\langle v-u, (v-u)(\phi_i - \phi_i^2)\rangle_\varepsilon + \\
&\quad +2\int_\Omega (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) + 2\langle u\phi_i, v\phi_i\rangle_\varepsilon - \langle u, v\phi_i^2\rangle_\varepsilon - \langle v, u\phi_i^2\rangle_\varepsilon.
\end{aligned}$$

First of all, observe that

$$2\langle u\phi_i, v\phi_i\rangle_\varepsilon - \langle u, v\phi_i^2\rangle_\varepsilon - \langle v, u\phi_i^2\rangle = 2\varepsilon^2 \int_\Omega uv|\nabla\phi_i|^2 \leqslant \varepsilon \int_\Omega V(x)(u^2+v^2) \leqslant C\varepsilon^{N+1} = \mathrm{o}_\varepsilon(1)\varepsilon^N$$

for sufficiently small ε. Moreover, since $\phi_i - \phi_i^2 \geqslant 0$ in Ω and $\phi_i - \phi_i^2 = 0$ in Λ, we obtain from (5.18) that

$$\begin{aligned}
2\int_\Omega (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) &= 2\int_{\Omega\setminus\Lambda} (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) \\
&\leqslant \frac{1}{2}\int_\Omega V(x)(u^2+v^2)(\phi_i - \phi_i^2)
\end{aligned}$$

and

$$\begin{aligned}
&2\langle v-u, (v-u)(\phi_i - \phi_i^2)\rangle_\varepsilon = 2J_\varepsilon'(u,v)((v-u)(\phi_i - \phi_i^2), (u-v)(\phi_i - \phi_i^2)) + \\
&+ 2\int_\Omega (f(x,u) + g(x,v))(v-u)(\phi_i - \phi_i^2) = 2\int_{\Omega\setminus\Lambda} (f(x,u) + g(x,v))(v-u)(\phi_i - \phi_i^2) \\
&\qquad \leqslant \frac{1}{2}\int_\Omega V(x)(u^2+v^2)(\phi_i - \phi_i^2).
\end{aligned}$$

Finally,

$$\begin{aligned}
2\langle u, u(\phi_i^2 - \phi_i)\rangle_\varepsilon &= 2\int_\Omega \big(\varepsilon^2|\nabla u|^2(\phi_i^2 - \phi_i) + V(x)u^2(\phi_i^2 - \phi_i)\big) + 2\varepsilon^2\int_\Omega \langle\nabla u, \nabla(\phi_i^2 - \phi_i)\rangle u \\
&\leqslant -2\int_\Omega V(x)u^2(\phi_i - \phi_i^2) + C\varepsilon\int_\Omega \big(\varepsilon^2|\nabla u|^2 + u^2\big) \\
&\leqslant -2\int_\Omega V(x)u^2(\phi_i - \phi_i^2) + \mathrm{o}_\varepsilon(1)\varepsilon^N
\end{aligned}$$

and analogously $2\langle v, v(\phi_i^2 - \phi_i)\rangle_\varepsilon \leqslant -2\int_\Omega V(x)v^2(\phi_i - \phi_i^2) + \mathrm{o}_\varepsilon(1)\varepsilon^N$. Hence we deduce that

$$\beta_{i,\varepsilon} \leqslant -\int_\Omega V(x)(u^2+v^2)(\phi_i - \phi_i^2) + \mathrm{o}_\varepsilon(1)\varepsilon^N \leqslant \mathrm{o}_\varepsilon(1)\varepsilon^N.$$

□

Lemma 5.19. *Under the conditions of Proposition 5.15, let $\psi \in H$ and, for $i \in \{1, \dots, k\}$, denote*

$$\gamma_{i,\varepsilon} := 2\|\psi\|_\varepsilon^2 + \int_\Omega \Big(\frac{f(x,u)}{u} + \frac{g(x,v)}{v}\Big)\psi^2 + \int_\Omega \Big(f'(x,u) - \frac{f(x,u)}{u}\Big)(u\phi_i + \psi)^2 + \\ + \int_\Omega \Big(g'(x,v) - \frac{g(x,v)}{v}\Big)(v\phi_i - \psi)^2.$$

Then there exists $\eta > 0$, independent of u, v, ψ and ε such that

$$\gamma_{i,\varepsilon} \geqslant \eta\varepsilon^N + \|\psi\|_\varepsilon^2.$$

Proof. By recalling (iii) in Lemma 5.13, we see that

$$\gamma_{i,\varepsilon} - \|\psi\|_\varepsilon^2 \geqslant \\ \geqslant \|\psi\|_\varepsilon^2 + \int_{\Lambda_i} \left(f'(x,u) - \frac{f(x,u)}{u}\right)(u\phi_i + \psi)^2 + \int_{\Lambda_i} \left(g'(x,v) - \frac{g(x,v)}{v}\right)(v\phi_i - \psi)^2 \\ \geqslant \|\psi\|_\varepsilon^2 + \delta' \int_{\Lambda_i} \left(\frac{f(u)}{u}(u+\psi)^2 + \frac{g(v)}{v}(v-\psi)^2\right).$$

Suppose by contradiction that there exists $(u_\varepsilon, v_\varepsilon) \in N_\varepsilon$, $\psi_\varepsilon \in H$ such that

$$\varepsilon^{-N}\|\psi_\varepsilon\|_\varepsilon^2 \to 0 \quad \text{and} \quad \varepsilon^{-N}\int_{\Lambda_i} \left(\frac{f(u_\varepsilon)}{u_\varepsilon}(u_\varepsilon + \psi_\varepsilon)^2 + \frac{g(v_\varepsilon)}{v_\varepsilon}(v_\varepsilon - \psi_\varepsilon)^2\right) \to 0$$

as $\varepsilon \to 0$. We use the change of variables $\bar u_\varepsilon(x) := u_\varepsilon(\varepsilon x)$, $\bar v_\varepsilon(x) := v_\varepsilon(\varepsilon x)$, $\bar\psi_\varepsilon(x) := \psi_\varepsilon(\varepsilon x)$, and $\Lambda_i^\varepsilon = \Lambda_i/\varepsilon$. The above convergences are then equivalent to

$$\|\bar\psi_\varepsilon\|_1^2 \to 0 \quad \text{and} \quad \int_{\Lambda_i^\varepsilon} (f(\bar u_\varepsilon)\bar u_\varepsilon + 2f(\bar u_\varepsilon)\bar\psi_\varepsilon + g(\bar v_\varepsilon)\bar v_\varepsilon - 2g(\bar v_\varepsilon)\bar\psi_\varepsilon) \to 0,$$

and hence

$$\int_{\Lambda_i^\varepsilon} (f(\bar u_\varepsilon)\bar u_\varepsilon + g(\bar v_\varepsilon)\bar v_\varepsilon) \to 0,$$

which contradicts (5.38). □

Proposition 5.20. *Under the conditions of Proposition 5.15, there exist $\varepsilon_0, \eta > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, any $\psi \in H$, and any $i \in \{1, \dots, k\}$,*

$$J_\varepsilon'(u,v)(u\phi_i^2, v\phi_i^2) + J_\varepsilon''(u,v)(u\phi_i + \psi, v\phi_i - \psi)(u\phi_i + \psi, v\phi_i - \psi) \leqslant -\eta\varepsilon^N - \frac{\|\psi\|_\varepsilon^2}{2} < 0.$$

Proof. We can add to the above expression the quantity

$$2\langle u - v, \psi\rangle_\varepsilon + 2\int_\Omega f(u)\phi_i\psi - 2\int_\Omega g(v)\phi_i\psi = -J_\varepsilon'(u,v)(\phi_i\psi, -\phi_i\psi) = 0,$$

obtaining this way $2\alpha_{i,\varepsilon} + \beta_{i,\varepsilon} - \gamma_{i,\varepsilon}$, where these quantities were defined respectively in Lemmas 5.17, 5.18 and 5.19 . According to these lemmas, for ε small enough this quantity is bounded from above by

$$\mathrm{o}_\varepsilon(1)\varepsilon^{N/2}\|\psi\|_\varepsilon + \mathrm{o}_\varepsilon(1)\varepsilon^N - \eta\varepsilon^N - \|\psi\|_\varepsilon^2 \leqslant \mathrm{o}_\varepsilon(1)\varepsilon^N + \mathrm{o}_\varepsilon(1)\|\psi\|_\varepsilon^2 - \eta\varepsilon^N - \|\psi\|_\varepsilon^2 \leqslant -\frac{1}{2}\|\psi\|_\varepsilon^2 - \frac{\eta}{2}\varepsilon^N,$$

and the conclusion follows. □

Remark 5.21. The previous proposition implies that the map

$$K_\varepsilon'(u,v)|_Z : Z \simeq \mathbb{R}^k \oplus H \to \mathbb{R}^k \oplus H$$

given by (5.25) is one-to-one. In fact, suppose that (5.26) holds and let us prove that $(\bar u, \bar v) = (\sum_i \lambda_i u\phi_i + \psi, \sum_i \lambda_i v\phi_i - \psi) = (0,0)$. Let $K = \{i \in \{1,\ldots,k\} :\ \lambda_i \neq 0\}$. If $K \neq \emptyset$, then

$$\begin{aligned} 0 &= \langle K_\varepsilon'(u,v)(\bar u,\bar v),(\bar u,\bar v)\rangle_{\mathbb{R}^k\oplus H^-} = J_\varepsilon'(u,v)(u\Phi^2, v\Phi^2) + J_\varepsilon''(u,v)(\bar u,\bar v)(\bar u,\bar v) \\ &= \sum_{i\in K} \lambda_i^2 \Big(J_\varepsilon'(u,v)(u\phi_i^2, v\phi_i^2) + J_\varepsilon''(u,v)(u\phi_i + \psi/\lambda_i, v\phi_i - \psi/\lambda_i)(u\phi_i + \psi/\lambda_i, v\phi_i - \psi/\lambda_i)\Big) \leqslant \\ &\leqslant \sum_{i\in K} \lambda_i^2 \Big(-\eta\varepsilon^N - \frac{1}{2}\Big\|\frac{\psi}{\lambda_i}\Big\|_\varepsilon^2 \Big) < 0, \end{aligned}$$

a contradiction. Thus $\lambda_i = 0$ for every i and hence

$$\begin{aligned} 0 = \langle K_\varepsilon'(u,v)(\bar u,\bar v),(\bar u,\bar v)\rangle_{\mathbb{R}^k\oplus H^-} &= J_\varepsilon''(u,v)(\psi,-\psi)(\psi,-\psi) \\ &= -2\|\psi\|^2 - \int_\Omega f'(x,u)\psi^2 - \int_\Omega g'(x,v)\psi^2, \end{aligned}$$

which yields that also $\psi = 0$. Thus $K_\varepsilon'(u,v)|_Z$ is one-to-one.

Proof of Proposition 5.14. Although technical, the idea of this proof is actually quite simple. The starting point is the observation that it is not very hard to prove that

$$J_\varepsilon\Big(\sum_{i=1}^k u_{i,\varepsilon}, \sum_{i=1}^k v_{i,\varepsilon}\Big) = \varepsilon^N\Big(\sum_{i=1}^k c_i + \mathrm{o}(1)\Big) \quad \text{and} \quad J_\varepsilon'(u_{i,\varepsilon}, v_{i,\varepsilon})(u_{i,\varepsilon}\phi_i, v_{i,\varepsilon}\phi_i) = \mathrm{o}(1)$$

as $\varepsilon \to 0$. Introducing parameters $t_{1,\varepsilon},\ldots,t_{k,\varepsilon}$ and a function Ψ_ε, we will perturb the pair $(\sum_i u_{i,\varepsilon}, \sum_i v_{i,\varepsilon})$ in order to find an element $(\bar u_\varepsilon, \bar v_\varepsilon) = (\sum_i t_{i,\varepsilon}u_{i,\varepsilon} + \Psi_\varepsilon, \sum_i t_{i,\varepsilon}v_{i,\varepsilon} - \Psi_\varepsilon)$ which belongs to N_ε. The final step will be to show that $t_{i,\varepsilon} \to 1$, $\Psi_\varepsilon \to 0$, and that (5.22), (5.23) hold.

1. For every $\bar t = (t_1,\ldots,t_k) \in [0,2]\times\cdots\times[0,2]$, let

$$u_{\varepsilon,\bar t} := \sum_{i=1}^k t_i u_{i,\varepsilon}, \qquad v_{\varepsilon,\bar t} := \sum_{i=1}^k t_i v_{i,\varepsilon}, \tag{5.39}$$

and let $\Psi_{\varepsilon,\bar t}$ be such that

$$J_\varepsilon'((u_{\varepsilon,\bar t}, v_{\varepsilon,\bar t}) + (\Psi_{\varepsilon,\bar t}, -\Psi_{\varepsilon,\bar t}))(\phi,-\phi) = 0, \qquad \forall \phi \in H, \tag{5.40}$$

that is, $\Psi_{\varepsilon,\bar t} \in H$ is such that

$$\begin{aligned} -2\varepsilon^2\Delta\Psi_{\varepsilon,\bar t} + 2V(x)\Psi_{\varepsilon,\bar t} &= -\varepsilon^2\Delta v_{\varepsilon,\bar t} + V(x)v_{\varepsilon,\bar t} + \varepsilon^2\Delta u_{\varepsilon,\bar t} - V(x)u_{\varepsilon,\bar t} \\ &\quad -f(x, u_{\varepsilon,\bar t} + \Psi_{\varepsilon,\bar t}) + g(x, v_{\varepsilon,\bar t} - \Psi_{\varepsilon,\bar t}) \quad \text{in } H'. \end{aligned} \tag{5.41}$$

We start by proving some estimates for $\Psi_{\varepsilon,\bar t}$. Let us show that

$$\int_\Omega \Big(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|^2 + V(x)\Psi_{\varepsilon,\bar t}^2\Big) \leqslant C\varepsilon^N \tag{5.42}$$

and

$$\varepsilon^{-N}\int_{\Omega\setminus\Lambda}\left(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|^2+V(x)\Psi^2_{\varepsilon,\bar t}\right)\to 0\quad\text{as }\varepsilon\to 0, \tag{5.43}$$

uniformly in $\bar t$. We multiply (5.41) by $\Psi_{\varepsilon,\bar t}$, obtaining

$$2\|\Psi_{\varepsilon,\bar t}\|^2_\varepsilon=\langle v_{\varepsilon,\bar t}-u_{\varepsilon,\bar t},\Psi_{\varepsilon,\bar t}\rangle_\varepsilon-\int_\Omega f(x,u_{\varepsilon,\bar t}+\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}+\int_\Omega g(x,v_{\varepsilon,\bar t}-\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}. \tag{5.44}$$

In view of the change of variables $y=\varepsilon x+x_i$, we define $\phi_i^\varepsilon(x)=\phi_i(\varepsilon x+x_i)$. Throughout this proof we will denote $A^\varepsilon:=(A-x_i)/\varepsilon$ for every $A\subseteq\Omega$ (we omit the dependence on x_i, which will be clear at each step). Since

$$\begin{aligned}\|u_{\varepsilon,\bar t}\|^2_\varepsilon=&\int_\Omega(\varepsilon^2|\nabla u_{\varepsilon,\bar t}|^2+V(x)u^2_{\varepsilon,\bar t})\\&=\sum_{i=1}^k\varepsilon^N\int_{\tilde\Lambda_i^\varepsilon}(|\nabla(\phi_i^\varepsilon u_i)|^2+V(\varepsilon x+x_i)(\phi_i^\varepsilon)^2u_i^2)\leqslant C\varepsilon^N\end{aligned}$$

and, analogously, $\|v_{\varepsilon,\bar t}\|^2_\varepsilon\leqslant C\varepsilon^N$, we have

$$\langle v_{\varepsilon,\bar t}-u_{\varepsilon,\bar t},\Psi_{\varepsilon,\bar t}\rangle^2_\varepsilon\leqslant\|\Psi_{\varepsilon,\bar t}\|^2_\varepsilon\|v_{\varepsilon,\bar t}-u_{\varepsilon,\bar t}\|^2_\varepsilon\leqslant 2\|\Psi_{\varepsilon,\bar t}\|^2_\varepsilon(\|u_{\varepsilon,\bar t}\|^2_\varepsilon+\|v_{\varepsilon,\bar t}\|^2_\varepsilon)\leqslant 4C\|\Psi_{\varepsilon,\bar t}\|^2_\varepsilon\varepsilon^N.$$

Moreover,

$$\begin{aligned}-\int_\Omega f(x,u_{\varepsilon,\bar t}+\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}&\leqslant -\int_\Omega f(x,u_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}\leqslant\int_\Omega\delta u_{\varepsilon,\bar t}\Psi_{\varepsilon,\bar t}+C\int_\Omega u^{2^*-1}_{\varepsilon,\bar t}\Psi_{\varepsilon,\bar t}\\&\leqslant\|u_{\varepsilon,\bar t}\|_\varepsilon\|\Psi_{\varepsilon,\bar t}\|_\varepsilon+C\left(\int_\Omega u^{2^*}_{\varepsilon,\bar t}\right)^{\frac{2^*-1}{2^*}}\left(\int_\Omega|\Psi_{\varepsilon,\bar t}|^{2^*}\right)^{\frac{1}{2^*}}\\&\leqslant\|u_{\varepsilon,\bar t}\|_\varepsilon\|\Psi_{\varepsilon,\bar t}\|_\varepsilon+C'\left(\int_\Omega|\nabla u_{\varepsilon,\bar t}|^2\right)^{\frac{2^*-1}{2}}\left(\int_\Omega|\nabla\Psi_{\varepsilon,\bar t}|^2\right)^{\frac12}\\&\leqslant\|u_{\varepsilon,\bar t}\|_\varepsilon\|\Psi_{\varepsilon,\bar t}\|_\varepsilon+C'\varepsilon^{1-2^*}\|u_{\varepsilon,\bar t}\|^{2^*-1}_\varepsilon\varepsilon^{-1}\|\Psi_{\varepsilon,\bar t}\|_\varepsilon\leqslant C''\|\Psi_{\varepsilon,\bar t}\|\varepsilon^{N/2}.\end{aligned}$$

and, analogously,

$$\int_\Omega g(x,v_{\varepsilon,\bar t}-\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}\leqslant C''\|\Psi_{\varepsilon,\bar t}\|\varepsilon^{N/2}.$$

Going back to (5.44), it is now easy to obtain (5.42). As for property (5.43), we observe that

$$-2\varepsilon^2\Delta\Psi_{\varepsilon,\bar t}+2V(x)\Psi_{\varepsilon,\bar t}=-f(x,\Psi_{\varepsilon,\bar t})+g(x,-\Psi_{\varepsilon,\bar t})\quad\text{ in }\Omega\setminus\widetilde\Lambda$$

(because $u_{\varepsilon,\bar t}=v_{\varepsilon,\bar t}=0$ in $\Omega\setminus\widetilde\Lambda$). For every $\xi\in H$ such that $\xi=0$ in $\Omega\setminus\widetilde\Lambda$, it follows

$$\begin{aligned}&2\int_{\Omega\setminus\tilde\Lambda}\left(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|^2+V(x)\Psi^2_{\varepsilon,\bar t}\right)\xi=\\&\quad=-2\int_{\Omega\setminus\tilde\Lambda}\varepsilon^2\langle\nabla\Psi_{\varepsilon,\bar t},\nabla\xi\rangle\Psi_{\varepsilon,\bar t}-\int_{\Omega\setminus\tilde\Lambda}f(x,\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}\xi+\int_{\Omega\setminus\tilde\Lambda}g(x,-\Psi_{\varepsilon,\bar t})\Psi_{\varepsilon,\bar t}\xi\\&\qquad\leqslant 2\int_{\Omega\setminus\tilde\Lambda}\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|\,|\nabla\xi|\,|\Psi_{\varepsilon,\bar t}|\leqslant C\varepsilon\left(\int_\Omega\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|^2+V(x)\Psi^2_{\varepsilon,\bar t}\right)=\mathrm{o}(\varepsilon^N)\end{aligned}$$

as $\varepsilon \to 0$, and thus the estimate (5.43) holds over each proper subset of $\Omega \setminus \tilde{\Lambda}$. Let us now cover the remaining case of a small neighborhood of each $\tilde{\Lambda}_i \setminus \Lambda_i$. We denote $\Psi^\varepsilon_{\bar{t},i}(x) = \Psi_{\varepsilon,\bar{t}}(\varepsilon x + x_i)$. Over the set $\Omega^\varepsilon \setminus \cup_{j\neq i}\tilde{\Lambda}^\varepsilon_j$ we have

$$
\begin{aligned}
-2\Delta\Psi^\varepsilon_{\bar{t},i}+2V(\varepsilon x+x_i)\Psi^\varepsilon_{\bar{t},i} &= -\Delta(t_i v_i\phi^\varepsilon_i)+V(\varepsilon x+x_i)t_i v_i\phi^\varepsilon_i+\Delta(t_i u_i\phi^\varepsilon_i)-V(\varepsilon x+x_i)t_i u_i\phi^\varepsilon_i-\\
&\quad - f(\varepsilon x + x_i, t_i u_i\phi^\varepsilon_i + \Psi^\varepsilon_{\bar{t},i}) + g(\varepsilon x + x_i, t_i v_i\phi^\varepsilon_i - \Psi^\varepsilon_{\bar{t},i})\\
&= t_i f(u_i) - f(\varepsilon x + x_i, t_i u_i\phi^\varepsilon_i + \Psi^\varepsilon_{\bar{t},i}) + g(\varepsilon x + x_i, t_i v_i\phi^\varepsilon_i - \Psi^\varepsilon_{\bar{t},i}) - t_i g(v_i) + \mathrm{o}(1) \quad (5.45)
\end{aligned}
$$

as $\varepsilon \to 0$, by recalling the equation for (u_i, v_i) and the fact that $V(\varepsilon x + x_i) \to V(x_i)$, and $\|(1-\phi^\varepsilon_i)u_i\|_{H^1(\mathbb{R}^N)}$, $\|(1-\phi^\varepsilon_i)v_i\|_{H^1(\mathbb{R}^N)} \to 0$. Let us fix two sets $\omega_i \Subset \Lambda_i \Subset \tilde{\Lambda}_i \Subset \tilde{\omega}_i$ and let ξ_i be a cut-off function such that $\xi_i = 1$ in $\tilde{\omega}_i \setminus \Lambda_i$, $\xi_i = 0$ in ω_i and outside a small neighborhood of $\tilde{\omega}_i$. If we multiply the equation (5.45) by $\Psi^\varepsilon_{\bar{t},i}\xi^\varepsilon_i$, where $\xi^\varepsilon_i(x) = \xi_i(\varepsilon x + x_i)$, then we see that

$$
\begin{aligned}
&2\int_{\mathbb{R}^N}(|\nabla\Psi^\varepsilon_{\bar{t},i}|^2 + V(\varepsilon x + x_i)(\Psi^\varepsilon_{\bar{t},i})^2)\xi^\varepsilon_i = -2\int_{\mathbb{R}^N}\langle\nabla\Psi^\varepsilon_{\bar{t},i}, \nabla\xi^\varepsilon_i\rangle\Psi^\varepsilon_{\bar{t},i}+\\
&+\int_{\mathbb{R}^N}(t_i f(u_i)-f(\varepsilon x+x_i, t_i u_i\phi^\varepsilon_i+\Psi^\varepsilon_{\bar{t},i}))\Psi^\varepsilon_{\bar{t},i}\xi^\varepsilon_i+\int_{\mathbb{R}^N}(g(\varepsilon x+x_i, t_i v_i\phi^\varepsilon_i-\Psi^\varepsilon_{\bar{t},i})-t_i g(v_i))\Psi^\varepsilon_{\bar{t},i}\xi^\varepsilon_i\\
&\leqslant \int_{\mathbb{R}^N}(t_i f(u_i) - f(\varepsilon x + x_i, t_i u_i\phi^\varepsilon_i))\Psi^\varepsilon_{\bar{t},i}\xi^\varepsilon_i + \int_{\mathbb{R}^N}(g(\varepsilon x + x_i, t_i v_i\phi^\varepsilon_i) - t_i g(v_i))\Psi^\varepsilon_{\bar{t},i}\xi^\varepsilon_i + \mathrm{o}(1)\\
&\leqslant C\int_{\mathbb{R}^N\setminus\omega^\varepsilon_i}(|u_i|+|v_i|)|\Psi^\varepsilon_{\bar{t},i}|+\mathrm{o}(1) \leqslant C\left(\int_{\mathbb{R}^N\setminus\omega^\varepsilon_i}(u_i^2+v_i^2)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|\Psi^\varepsilon_{\bar{t},i}|^2\right)^{\frac{1}{2}}+\mathrm{o}(1) = \mathrm{o}(1)
\end{aligned}
$$

as $\varepsilon \to 0$, where we have used the fact that $u_i, v_i \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f(s), g(s) = \mathrm{o}(s)$ as $s \to 0$, and $\int_{\mathbb{R}^N}|\Psi^\varepsilon_{\bar{t},i}|^2$ is bounded (*cf.* (5.42)). Thus

$$
\int_{\tilde{\omega}^\varepsilon_i\setminus\Lambda^\varepsilon_i}(|\nabla\Psi^\varepsilon_{\bar{t},i}|^2 + V(\varepsilon x + x_i){\Psi^\varepsilon_{\bar{t},i}}^2) \to 0 \qquad \text{as } \varepsilon \to 0, \quad \text{uniformly in } \bar{t},
$$

and (5.43) holds.

2. Having understood the asymptotic behavior of $\Psi^\varepsilon_{\bar{t},i}$ outside Λ^ε_i, let us control it over Λ^ε_i. With this in mind, we introduce Ψ_{t_i} such that

$$
I'_{V(x_i)}(t_i(u_i, v_i) + (\Psi_{t_i}, -\Psi_{t_i}))(\phi, -\phi) = 0 \qquad \text{for every } \phi \in H^1(\mathbb{R}^N),
$$

that is

$$
\begin{aligned}
-2\Delta\Psi_{t_i}+2V(x_i)\Psi_{t_i} &= -\Delta t_i v_i+V(x_i)t_i v_i+\Delta t_i u_i-V(x_i)t_i u_i-f(t_i u_i+\Psi_{t_i})+g(t_i v_i-\Psi_{t_i}) =\\
&= t_i f(u_i) - f(t_i u_i + \Psi_{t_i}) + g(t_i v_i - \Psi_{t_i}) - t_i g(v_i) \qquad \text{in } \mathbb{R}^N. \quad (5.46)
\end{aligned}
$$

By combining (5.45) and (5.46) we conclude that, over $\Omega^\varepsilon \setminus \cup_{j\neq i}\tilde{\Lambda}^\varepsilon_j$,

$$
\begin{aligned}
-2\Delta(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i}) + 2V(\varepsilon x + x_i)(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i}) &= 2(V(x_i) - V(\varepsilon x + x_i))\Psi_{t_i}+\\
&\quad + f(t_i u_i + \Psi_{t_i}) - f(\varepsilon x + x_i, t_i u_i\phi^\varepsilon_i + \Psi^\varepsilon_{\bar{t},i})+\\
&\quad + g(\varepsilon x + x_i, t_i v_i\phi^\varepsilon_i - \Psi^\varepsilon_{\bar{t},i}) - g(t_i v_i - \Psi_{t_i}) + \mathrm{o}(1) \quad (5.47)
\end{aligned}
$$

as $\varepsilon \to 0$. Since Ψ_{t_i} is a continuous map as a function of t_i, by recalling estimate (5.42) we see that

$$\Big|\int_{\tilde{\omega}_i^\varepsilon} (V(x_i) - V(\varepsilon x + x_i))\Psi_{t_i}(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i})\Big| \leqslant$$

$$\left(\int_{\tilde{\omega}_i^\varepsilon} ((V(x_i) - V(\varepsilon x + x_i))\Psi_{t_i})^2\right)^{\frac{1}{2}} \left(\int_{\tilde{\omega}_i^\varepsilon} |\Psi^\varepsilon_{\bar{t},i}|^2\right)^{\frac{1}{2}} + \int_{\tilde{\omega}_i^\varepsilon} |(V(x_i) - V(\varepsilon x + x_i))\Psi^2_{t_i}| \to 0$$

as $\varepsilon \to 0$, uniformly in $\bar{t}$, by the Lebesgue dominated convergence theorem. By recalling moreover (5.43), and by observing that $f(\varepsilon x + x_i, s) = f(s)$ whenever $x \in \Lambda_i^\varepsilon$, and that $f'(s) \geqslant 0$, we have

$$\int_{\tilde{\omega}_i^\varepsilon} (f(t_i u_i + \Psi_{t_i}) - f(\varepsilon x + x_i, t_i u_i + \Psi^\varepsilon_{\bar{t},i}))(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i})$$
$$= \int_{\Lambda_i^\varepsilon} (f(t_i u_i + \Psi_{t_i}) - f(t_i u_i + \Psi^\varepsilon_{\bar{t},i}))(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i}) + \mathrm{o}(1) \leqslant \mathrm{o}(1)$$

and analogously for g. At this point it is not so hard to check that

$$\int_{\widetilde{\Lambda}_i^\varepsilon} |\nabla(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i})|^2 + V(\varepsilon x + x_i)(\Psi^\varepsilon_{\bar{t},i} - \Psi_{t_i})^2 \to 0 \quad \text{as } \varepsilon \to 0, \text{ uniformly in } \bar{t}. \tag{5.48}$$

3. Now, let us introduce the continuous function

$$\begin{aligned}
\theta_{i,\varepsilon}(\bar{t}) &:= J'_\varepsilon(\bar{u}_\varepsilon, \bar{v}_\varepsilon)(\bar{u}_\varepsilon\phi_i, \bar{v}_\varepsilon\phi_i) \\
&= J'_\varepsilon((u_{\varepsilon,\bar{t}}, v_{\varepsilon,\bar{t}}) + (\Psi_{\varepsilon,\bar{t}}, -\Psi_{\varepsilon,\bar{t}}))((u_{\varepsilon,\bar{t}} + \Psi_{\varepsilon,\bar{t}})\phi_i, (v_{\varepsilon,\bar{t}} - \Psi_{\varepsilon,\bar{t}})\phi_i), \\
&= J'_\varepsilon((u_{\varepsilon,\bar{t}}, v_{\varepsilon,\bar{t}}) + (\Psi_{\varepsilon,\bar{t}}, -\Psi_{\varepsilon,\bar{t}}))(u_{\varepsilon,\bar{t}}\phi_i, v_{\varepsilon,\bar{t}}\phi_i)
\end{aligned} \tag{5.49}$$

where $u_{\varepsilon,\bar{t}}$, $v_{\varepsilon,\bar{t}}$ and $\Psi_{\varepsilon,\bar{t}}$ were defined in (5.39) and (5.40) respectively. We claim that there exist $\varepsilon_0, \mu > 0$ such that, for any $0 < \varepsilon \leqslant \varepsilon_0$ and every points $t_j \in [0,2]$, $j \neq i$,

$$\theta_{i,\varepsilon}(t_1, \dots, t_{i-1}, 1-\mu, t_{i+1}, \dots, t_k) > 0 > \theta_{i,\varepsilon}(t_1, \dots, t_{i-1}, 1+\mu, t_{i+1}, \dots, t_k). \tag{5.50}$$

We observe that, by Miranda's theorem, this yields the desired conclusion (5.20) (and also (5.21), thanks to (5.40)). We have

$$\begin{aligned}
\theta_{i,\varepsilon}(\bar{t}) &= J'_\varepsilon(u_{\varepsilon,\bar{t}} + \Psi_{\varepsilon,\bar{t}}, v_{\varepsilon,\bar{t}} - \Psi_{\varepsilon,\bar{t}})(u_{\varepsilon,\bar{t}}\phi_i, v_{\varepsilon,\bar{t}}\phi_i) \\
&= J'_\varepsilon(t_i u_{i,\varepsilon} + \Psi_{\varepsilon,\bar{t}}, t_i v_{i,\varepsilon} - \Psi_{\varepsilon,\bar{t}})(t_i u_{i,\varepsilon}\phi_i, t_i v_{i,\varepsilon}\phi_i) \\
&= t_i^2\langle u_{i,\varepsilon}, v_{i,\varepsilon}\phi_i\rangle_\varepsilon + t_i^2\langle v_{i,\varepsilon}, u_{i,\varepsilon}\phi_i\rangle_\varepsilon + t_i\langle \Psi_{\varepsilon,\bar{t}}, (v_{i,\varepsilon} - u_{i,\varepsilon})\phi_i\rangle_\varepsilon - \\
&\quad - \int_\Omega f(x, t_i u_{i,\varepsilon} + \Psi_{\varepsilon,\bar{t}})t_i u_{i,\varepsilon}\phi_i - \int_\Omega g(x, t_i v_{i,\varepsilon} - \Psi_{\varepsilon,\bar{t}})t_i v_{i,\varepsilon}\phi_i
\end{aligned}$$

and hence

$$\begin{aligned}\varepsilon^{-N}\theta_{i,\varepsilon}(\bar t) &= t_i^2\int_{\tilde\Lambda_i^\varepsilon}(\langle\nabla(u_i\phi_i^\varepsilon),\nabla(v_i(\phi_i^\varepsilon)^2)\rangle+V(\varepsilon x+x_i)u_iv_i(\phi_i^\varepsilon)^2)+\\ &\quad +t_i^2\int_{\tilde\Lambda_i^\varepsilon}(\langle\nabla(v_i\phi_i^\varepsilon),\nabla(u_i(\phi_i^\varepsilon)^2)\rangle+V(\varepsilon x+x_i)u_iv_i(\phi_i^\varepsilon)^2)+\\ &\quad +t_i\int_{\tilde\Lambda_i^\varepsilon}(\langle\nabla\Psi_{\bar t,i}^\varepsilon,\nabla((v_i-u_i)(\phi_i^\varepsilon)^2)\rangle+V(\varepsilon x+x_i)(v_i-u_i)\Psi_{\bar t,i}^\varepsilon(\phi_i^\varepsilon)^2)-\\ &\quad -t_i\int_{\Omega^\varepsilon}f(\varepsilon x+x_i,t_iu_i\phi_i^\varepsilon+\Psi_{\bar t,i}^\varepsilon)u_i(\phi_i^\varepsilon)^2-t_i\int_{\Omega^\varepsilon}g(\varepsilon x+x_i,t_iv_i\phi_i^\varepsilon-\Psi_{\bar t,i}^\varepsilon)v_i(\phi_i^\varepsilon)^2.\end{aligned}$$

Defining

$$\begin{aligned}\theta_i(t_i) &= I'_{V(x_i)}(t_i(u_i,v_i)+(\Psi_{t_i},-\Psi_{t_i}))(u_i,v_i)\\ &= 2t_i\int_{\mathbb{R}^N}(\langle\nabla u_i,\nabla v_i\rangle+2V(x_i)u_iv_i)+\int_{\mathbb{R}^N}(\langle\nabla\Psi_{t_i},\nabla(v_i-u_i)\rangle+V(x_i)\Psi_{t_i}(v_i-u_i))-\\ &\quad -\int_{\mathbb{R}^N}f(t_iu_i+\Psi_{t_i})u_i-\int_{\mathbb{R}^N}g(t_iv_i-\Psi_{t_i})v_i,\end{aligned}$$

we observe that, as $\varepsilon\to 0$, we have

$$\begin{aligned}\int_{\tilde\Lambda_i^\varepsilon}&(\langle\nabla(u_i\phi_i^\varepsilon),\nabla(v_i(\phi_i^\varepsilon)^2)\rangle+2V(\varepsilon x+x_i)u_iv_i(\phi_i^\varepsilon)^2)+\langle\nabla(v_i\phi_i^\varepsilon),\nabla(u_i(\phi_i^\varepsilon)^2)\rangle)\\ &=2\int_{\mathbb{R}^N}(\langle\nabla u_i,\nabla v_i\rangle+V(x_i)u_iv_i)+\mathrm{o}(1),\end{aligned}$$

$$\begin{aligned}\int_{\tilde\Lambda_i^\varepsilon}&(\langle\nabla\Psi_{\bar t,i}^\varepsilon,\nabla((v_i-u_i)(\phi_i^\varepsilon)^2)\rangle+V(\varepsilon x+x_i)(v_i-u_i)\Psi_{\bar t,i}^\varepsilon(\phi_i^\varepsilon)^2)\\ &=\int_{\mathbb{R}^N}\langle\nabla\Psi_{t_i},\nabla(v_i-u_i)\rangle+V(x_i)\Psi_{t_i}(v_i-u_i)+\mathrm{o}(1)\end{aligned}$$

(where we have used (5.48)), and

$$\int_{\Omega^\varepsilon}f(\varepsilon x+x_i,t_iu_i\phi_i^\varepsilon+\Psi_{\bar t,i}^\varepsilon)u_i(\phi_i^\varepsilon)^2=\int_{\Lambda_i^\varepsilon}f(t_iu_i+\Psi_{\bar t,i}^\varepsilon)u_i+\mathrm{o}(1)=\int_{\mathbb{R}^N}f(t_iu_i+\Psi_{t_i})u_i+\mathrm{o}(1),$$

and analogously for g. Thus, by combining these identities with Proposition 5.8 we see that

$$\varepsilon^{-N}\theta_{i,\varepsilon}(\bar t)=t_i\theta_i(t_i)+\mathrm{o}_\varepsilon(1)=-t_i\delta_i(t_i-1)+\mathrm{o}_{t_i}(t_i-1)+\mathrm{o}_\varepsilon(1)$$

for some $\delta_i>0$, where

$$\frac{\mathrm{o}_{t_i}(t_i-1)}{t_i-1}\to 0\quad\text{as } t_i\to 1,\ \text{ and }\quad \mathrm{o}_\varepsilon(1)\to 0\quad \text{as }\varepsilon\to 0,\text{uniformly in }\bar t.$$

By choosing μ in a way that $|\mathrm{o}_{t_i}(t_i-1)|\leqslant\delta_it_i|t_i-1|/2$ for $t_i\in[1-\mu,1+\mu]$, there exists $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ there exists $\bar t_\varepsilon=(t_{1,\varepsilon},\dots,t_{k,\varepsilon})\in[1-\mu,1+\mu]^k$ such that (by Miranda's Theorem) $\theta_{i,\varepsilon}(t_{1,\varepsilon},\ldots,t_{k,\varepsilon})=0$ for every $i=1,\ldots,k$. Moreover,by changing μ, we can take $t_{i,\varepsilon}\to 1$, and so $\Psi_{t_{i,\varepsilon}}\to\Psi_1=0$. Moreover,

$$\begin{aligned}\int_{\Lambda_i}(\overline{u}_\varepsilon^2+\overline{v}_\varepsilon^2) &= \int_{\Lambda_i}\Big((t_{i,\varepsilon}u_{i,\varepsilon}+\Psi_{\varepsilon,\bar{t}_\varepsilon})^2+(t_{i,\varepsilon}v_{i,\varepsilon}-\Psi_{\varepsilon,\bar{t}_\varepsilon})^2\Big)\\ &= \varepsilon^N\int_{\Lambda_i^\varepsilon}\Big((t_{i,\varepsilon}u_i+\Psi^\varepsilon_{\bar{t}_\varepsilon,i})^2+(t_{i,\varepsilon}v_i-\Psi^\varepsilon_{\bar{t}_\varepsilon,i})^2\Big)\\ &= \varepsilon^N\int_{\Lambda_i^\varepsilon}\Big((t_{i,\varepsilon}u_i+\Psi_{t_{i,\varepsilon}})^2+(t_{i,\varepsilon}v_i-\Psi_{t_{i,\varepsilon}})^2\Big)+\mathrm{o}(\varepsilon^N)\\ &\geqslant \eta\varepsilon^N,\end{aligned}$$

for some $\eta>0$.

4. Finally, the only thing left to prove is estimate (5.22). It suffices to check that

$$\varepsilon^{-N}J_\varepsilon(\overline{u}_\varepsilon,\overline{v}_\varepsilon)=\sum_{i=1}^k I_{V(x_i)}(t_{i,\varepsilon}(u_i,v_i)+(\Psi_{t_{i,\varepsilon}},-\Psi_{t_{i,\varepsilon}}))+\mathrm{o}(1). \tag{5.51}$$

Now,

$$\begin{aligned}J_\varepsilon(\bar{u}_\varepsilon,\bar{v}_\varepsilon)=\sum_{i=1}^k t_{i,\varepsilon}^2\langle u_{i,\varepsilon},v_{i,\varepsilon}\rangle_\varepsilon-\|\Psi_{\varepsilon,\bar{t}_\varepsilon}\|^2_\varepsilon+\sum_{i=1}^k t_{i,\varepsilon}\langle\Psi_{\varepsilon,\bar{t}_\varepsilon},v_{i,\varepsilon}-u_{i,\varepsilon}\rangle_\varepsilon-\\ -\int_\Omega F\Big(x,\sum_{i=1}^k t_{i,\varepsilon}u_{i,\varepsilon}+\Psi_{\varepsilon,\bar{t}_\varepsilon}\Big)-\int_\Omega G\Big(x,\sum_{i=1}^k t_{i,\varepsilon}v_{i,\varepsilon}-\Psi_{\varepsilon,\bar{t}_\varepsilon}\Big),\end{aligned}$$

and it is straightforward to see that

$$\begin{aligned}\varepsilon^{-N}\sum_{i=1}^k t_{i,\varepsilon}^2\langle u_{i,\varepsilon},v_{i,\varepsilon}\rangle_\varepsilon &= \sum_{i=1}^k t_{i,\varepsilon}^2\int_{\tilde{\Lambda}_i^\varepsilon}(\langle\nabla(u_i\phi_i^\varepsilon),\nabla(v_i\phi_i^\varepsilon)\rangle+V(\varepsilon x+x_i)u_iv_i(\phi_i^\varepsilon)^2)\\ &= \sum_{i=1}^k t_{i,\varepsilon}^2\int_{\mathbb{R}^N}(\langle\nabla u_i,\nabla v_i\rangle+V(x_i)u_iv_i)+\mathrm{o}(1),\end{aligned}$$

$$\begin{aligned}&\varepsilon^{-N}\|\Psi_{\varepsilon,\bar{t}}\|^2_\varepsilon=\varepsilon^{-N}\int_{\Lambda_i}(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar{t}_\varepsilon}|^2+V(x)\Psi^2_{\varepsilon,\bar{t}_\varepsilon})+\mathrm{o}(1)\\ &=\sum_{i=1}^k\int_{\Lambda_i^\varepsilon}(|\nabla\Psi^\varepsilon_{\bar{t}_\varepsilon,i}|^2+V(\varepsilon x+x_i)|\Psi^\varepsilon_{\bar{t}_\varepsilon,i}|^2)+\mathrm{o}(1)=\sum_{i=1}^k\int_{\Lambda_i^\varepsilon}(|\nabla\Psi_{t_{i,\varepsilon}}|^2+V(x_i)\Psi^2_{t_{i,\varepsilon}})+\mathrm{o}(1)\\ &=\sum_{i=1}^k\int_{\mathbb{R}^N}(|\nabla\Psi_{t_{i,\varepsilon}}|^2+V(x_i)\Psi^2_{t_{i,\varepsilon}})+\mathrm{o}(1),\end{aligned}$$

and

$$\varepsilon^{-N}\sum_{i=1}^k\langle\Psi_{\varepsilon,\bar{t}_\varepsilon},v_{i,\varepsilon}-u_{i,\varepsilon}\rangle_\varepsilon=\sum_{i=1}^k\int_{\mathbb{R}^N}(\langle\nabla\Psi_{t_{i,\varepsilon}},v_i-u_i\rangle+V(x_i)\Psi_{t_{i,\varepsilon}}(v_i-u_i))+\mathrm{o}(1).$$

Finally,

$$\varepsilon^{-N}\int_\Omega F(x,\sum_i t_{i,\varepsilon}u_{i,\varepsilon}+\Psi_{\varepsilon,\bar t_\varepsilon})=\varepsilon^{-N}\sum_{i=1}^k\int_{\Lambda_i}F(x,t_{i,\varepsilon}u_{i,\varepsilon}+\Psi_{\varepsilon,\bar t_\varepsilon})+\varepsilon^{-N}\int_{\Omega\setminus\Lambda}F(x,\Psi_{\varepsilon,\bar t_\varepsilon})=$$
$$\sum_{i=1}^k\int_{\tilde\Lambda_i^\varepsilon}F(\varepsilon x+x_i,t_{i,\varepsilon}\phi_i^\varepsilon u_i+\Psi^\varepsilon_{\bar t_\varepsilon,i})+\mathrm{o}(1)=\sum_{i=1}^k\int_{\tilde\Lambda_i^\varepsilon}F(t_{i,\varepsilon}u_i+\Psi_{t_{i,\varepsilon}})+\mathrm{o}(1)$$
$$=\sum_{i=1}^k\int_{\mathbb{R}^N}F(t_{i,\varepsilon}+\Psi_{t_{i,\varepsilon}})+\mathrm{o}(1),$$

and analogously for G, and hence (5.51) follows. $\square$

5.4 Proof of Theorem 5.1

We use the same notations as in the previous section.We recall that $c_\varepsilon:=\inf_{N_\varepsilon}J_\varepsilon$. We are now in a position to prove the existence of minimizers.

Theorem 5.22. *For every small $\varepsilon>0$ there exists $(u_\varepsilon,v_\varepsilon)\in N_\varepsilon$ such that*

$$J_\varepsilon(u_\varepsilon,v_\varepsilon)=c_\varepsilon\qquad\textit{and}\qquad J_\varepsilon'(u_\varepsilon,v_\varepsilon)=0.$$

Proof. 1. From Proposition 5.15 we know that $c_\varepsilon\geqslant 0$. Hence by the Ekeland's variational principle (*cf.* Theorem B.9) there exists a constrained Palais-Smale sequence for the functional J_ε at the level c_ε, that is, there exist sequences $(u_n,v_n)\in N_\varepsilon$, $\lambda_i^n\in\mathbb{R}$ $(i=1,\dots,k)$ and $\psi_n\in H$ such that, for any $\zeta,\xi\in H$,

$$J_\varepsilon(u_n,v_n)\to c_\varepsilon,$$

and

$$\begin{aligned}J_\varepsilon'(u_n,v_n)(\zeta,\xi)\ &=\ J_\varepsilon'(u_n,v_n)(\Phi_n\zeta,\Phi_n\xi)+\\&\quad+J_\varepsilon''(u_n,v_n)(\Phi_nu_n+\psi_n,\Phi_nv_n-\psi_n)(\zeta,\xi)+\mathrm{o}_n(1)\end{aligned}$$

where $\mathrm{o}_n(1)\to 0$ uniformly for bounded ζ,ξ as $n\to\infty$, and $\Phi_n:=\sum_i\lambda_i^n\phi_i$. Let $\lambda_n:=(\sum_i(\lambda_i^n)^2)^{1/2}$. We claim that

$$\lambda_n\to 0\qquad\text{and}\qquad\psi_n\to 0.\tag{5.52}$$

Indeed, we let $\zeta=(\Phi_nu_n+\psi_n)/\lambda_n,\xi=(\Phi_nv_n-\psi_n)/\lambda_n$ so that $J_\varepsilon'(u_n,v_n)(\zeta,\xi)=0$. We see that

$$\begin{aligned}0=J_\varepsilon''(u_n,v_n)(\zeta,\xi)&=\sum_{i=1}^k\frac{(\lambda_i^n)^2}{\lambda_n}\Big(J_\varepsilon'(u_n,v_n)(u_n\phi_i^2,v_n\phi_i^2)+\\&\quad+J_\varepsilon''(u_n,v_n)(u_n\phi_i+\frac{\Psi_n}{\lambda_i^n},v_n\phi_i-\frac{\Psi_n}{\lambda_i^n})(u_n\phi_i+\frac{\Psi_n}{\lambda_i^n},v_n\phi_i-\frac{\Psi_n}{\lambda_i^n})\Big)+\mathrm{o}_n(1)\\&\leqslant\sum_{i=1}^k\frac{(\lambda_i^n)^2}{\lambda_n}\Big(-\frac12\Big\|\frac{\Psi_n}{\lambda_i^n}\Big\|_\varepsilon^2-\eta\varepsilon^N\Big)+\mathrm{o}_n(1)=-\frac{k}{2\lambda_n}\|\Psi_n\|_\varepsilon^2-\lambda_n\varepsilon^N+\mathrm{o}_n(1).\end{aligned}$$

Thus $k\|\Psi_n\|_\varepsilon^2 + 2\lambda_n^2\varepsilon^N \leqslant \mathrm{o}(\lambda_n)$ and hence (5.52) follows (in case $\lambda_n = 0$ by letting $(\zeta, \xi) = (\psi_n, -\psi_n)$ we immediately get that $\psi_n \to 0$).
2. It follows from (5.52) that (u_n, v_n) is a (bounded) Palais-Smale sequence for J_ε, namely $J_\varepsilon(u_n, v_n) \to c_\varepsilon$ and $J_\varepsilon'(u_n, v_n) \to 0$ as $n \to \infty$. Up to a subsequence, let $(u_\varepsilon, v_\varepsilon)$ be a weak limit of the sequence $(u_n, v_n)_n$. Of course, $J_\varepsilon'(u_\varepsilon, v_\varepsilon) = 0$ and $(u_\varepsilon, v_\varepsilon) \in N_\varepsilon$. Moreover, since

$$\begin{aligned} 2J_\varepsilon(u_n, v_n) &= 2J_\varepsilon(u_n, v_n) - J_\varepsilon'(u_n, v_n)(u_n, v_n) + \mathrm{o}_n(1) \\ &= \int_\Omega (f(x, u_n)u_n - 2F(x, u_n)) + \int_\Omega (g(x, v_n)v_n - 2G(x, v_n)) + \mathrm{o}_n(1), \end{aligned}$$

we obtain by Fatou's Lemma

$$\begin{aligned} 2J_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_\Omega (f(x, u_\varepsilon)u_\varepsilon - 2F(x, u_\varepsilon)) + \int_\Omega (g(x, v_\varepsilon) - 2G(x, v_\varepsilon)) \\ &= \int_\Omega \lim_n (f(x, u_n)u_n - 2F(x, u_n)) + \int_\Omega \lim_n (g(x, v_n) - 2G(x, v_n)) \\ &\leqslant \liminf_n (2J_\varepsilon(u_n, v_n) + \mathrm{o}(1)) = 2c_\varepsilon, \end{aligned}$$

and hence $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$. □

From now on we consider the positive functions $u_\varepsilon, v_\varepsilon > 0$ given by Theorem 5.22, which satisfy $(u_\varepsilon, v_\varepsilon) \in H^1(\mathbb{R}^N)$ and

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon + V(x)u_\varepsilon = g(x, v_\varepsilon) \\ -\varepsilon^2 \Delta v_\varepsilon + V(x)v_\varepsilon = f(x, u_\varepsilon) \end{cases} \quad \text{in } \Omega. \tag{5.53}$$

In the remaining part of this section we will prove that $(u_\varepsilon, v_\varepsilon)$ solves the original problem (5.1) and that the properties $(i) - (iii)$ of Theorem 5.1 hold true. After establishing local and global energy estimates for $(u_\varepsilon, v_\varepsilon))$ (see the next theorem), the rest of the proof will be carried out in a series of lemmas.

Introduce the functional

$$J_\varepsilon^i(u, v) := \int_{\widetilde{\Lambda}_i} \left(\varepsilon^2 \langle \nabla u, \nabla v\rangle + V(x)uv - F(x, u) - G(x, v)\right).$$

The following estimates hold.

Theorem 5.23. *If $(u_\varepsilon, v_\varepsilon)$ is as in Theorem (5.22) , then*

$$J_\varepsilon^i(u_\varepsilon, v_\varepsilon) = \varepsilon^N(c_i + \mathrm{o}(1)) \quad \forall i = 1, \dots, k \qquad \textit{and} \qquad c_\varepsilon = \varepsilon^N \left(\sum_{i=1}^k c_i + \mathrm{o}(1)\right) \tag{5.54}$$

as $\varepsilon \to 0$.

Proof. 1. Let ξ be a cut-off function such that $\xi = 0$ in Λ and $\xi = 1$ in $\Omega \setminus \widetilde{\Lambda}$. By testing $J'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ with $(v_\varepsilon\xi, u_\varepsilon\xi)$, we see that

$$\begin{aligned}
&\int_\Omega (\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2)\xi + \int_\Omega (\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2)\xi \\
&\quad = \int_{\Omega\setminus\Lambda} (f(x,u_\varepsilon)v_\varepsilon + g(x,v_\varepsilon)u_\varepsilon)\xi - \int_\Omega \varepsilon^2\langle\nabla u_\varepsilon, \nabla\xi\rangle u_\varepsilon - \int_\Omega \varepsilon^2\langle\nabla v_\varepsilon, \nabla\xi\rangle v_\varepsilon \\
&\qquad\qquad \leqslant \delta\int_\Omega (u_\varepsilon^2 + v_\varepsilon^2)\xi + C\varepsilon\|(u_\varepsilon, v_\varepsilon)\|_\varepsilon^2
\end{aligned}$$

and hence, by choosing δ and ε sufficiently small, we obtain from Proposition 5.15 that

$$\int_{\Omega\setminus\widetilde{\Lambda}} (\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2) = \mathrm{o}_\varepsilon(\varepsilon^N) \quad \text{as } \varepsilon \to 0.$$

This, in turn, readily implies that

$$J_\varepsilon(u_\varepsilon, v_\varepsilon) = \sum_{i=1}^k J_\varepsilon^i(u_\varepsilon, v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N) \quad \text{as } \varepsilon \to 0.$$

Now, if we able to prove that, for every $i \in \{1, \dots, k\}$,

$$J_\varepsilon^i(u_\varepsilon, v_\varepsilon) \geqslant \varepsilon^N(c_i + \mathrm{o}_\varepsilon(1)) \quad \text{as } \varepsilon \to 0, \tag{5.55}$$

then

$$\begin{aligned}
\varepsilon^N\Big(\sum_{i=1}^k c_i + \mathrm{o}_\varepsilon(1)\Big) &\leqslant \sum_{i=1}^k J_\varepsilon^i(u_\varepsilon, v_\varepsilon) = J_\varepsilon(u_\varepsilon, v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N) \\
&= c_\varepsilon + \mathrm{o}_\varepsilon(\varepsilon^N) \leqslant \varepsilon^N\Big(\sum_{i=1}^k c_i + \mathrm{o}_\varepsilon(1)\Big),
\end{aligned}$$

and so (5.54) follows. The rest of the proof is dedicated to check the validity of (5.55).
2. We denote $u^\varepsilon(x) = u_\varepsilon(\varepsilon x)$ and $v^\varepsilon(x) := v_\varepsilon(\varepsilon x)$. In view of a contradiction, suppose that for every $S > 0$,

$$\sup_{y\in\Lambda_i/\varepsilon} \int_{B_S(y)} (u^\varepsilon)^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Take Ψ_R^ε to be a cut-off function such that $0 \leqslant \Psi_R^\varepsilon \leqslant 1$, $|\nabla\Psi_R^\varepsilon| \leqslant C/R$ and $\Psi_R^\varepsilon = 1$ in $N_R(\Lambda_i/\varepsilon) := \{x : \ \mathrm{dist}(x, \Lambda_i/\varepsilon) \leqslant R\}$, $\Psi_R^\varepsilon = 0$ in $\mathbb{R}^N \setminus N_{2R}(\Lambda_i/\varepsilon)$. By Proposition 5.15 we see that, for every $R > 0$, the sequence $(u^\varepsilon\Psi_R^\varepsilon)_\varepsilon$ is bounded in $H^1(\mathbb{R}^N)$. Moreover, for every $S > 0$,

$$\sup_{y\in\mathbb{R}^N} \int_{B_S(y)} (u^\varepsilon\Psi_R^\varepsilon)^2 \to 0 \qquad \text{as } \varepsilon \to 0.$$

Hence by the P. L. Lions' concentration lemma (see for instance [136, Theorem 1.34]) it follows that, for each fixed $R > 0$, $u^\varepsilon\Psi_R^\varepsilon \to 0$ in $L^q(\mathbb{R}^N)$ for every $2 < q < 2^*$, as $\varepsilon \to 0$. In particular, for each fixed $R > 0$,

$$\int_{N_R(\Lambda_i/\varepsilon)} (u^\varepsilon)^p \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.56}$$

By using $(v^\varepsilon\Psi_R^\varepsilon, u^\varepsilon\Psi_R^\varepsilon)$ as test function in $J'_\varepsilon(u_\varepsilon, v_\varepsilon)=0$, we obtain

$$\begin{aligned}
&\int_{\Omega/\varepsilon}(|\nabla u^\varepsilon|^2+V(\varepsilon x)(u^\varepsilon)^2)\Psi_R^\varepsilon+\int_{\Omega/\varepsilon}(|\nabla v^\varepsilon|^2+V(\varepsilon x)(v^\varepsilon)^2)\Psi_R^\varepsilon\\
&\quad=\int_{\Omega/\varepsilon}(f(\varepsilon x,u^\varepsilon)v^\varepsilon+g(\varepsilon x,v^\varepsilon)u^\varepsilon)\Psi_R^\varepsilon-\int_{\Omega/\varepsilon}\langle\nabla u^\varepsilon,\nabla\Psi_R^\varepsilon\rangle u^\varepsilon-\int_{\Omega/\varepsilon}\langle\nabla v^\varepsilon,\nabla\Psi_R^\varepsilon\rangle v^\varepsilon\\
&\qquad=\delta\int_{\Omega/\varepsilon}((u^\varepsilon)^2+(v^\varepsilon)^2)\Psi_R^\varepsilon+C(\delta)\int_{N_{2R}(\Lambda_i/\varepsilon)}(|u^\varepsilon|^{p-1}|v^\varepsilon|+|v^\varepsilon|^{p-1}|u^\varepsilon|)+\frac{C}{R}
\end{aligned}$$

and thus, by choosing $\delta\leqslant\alpha/2$,

$$\begin{aligned}
&\int_{N_R(\Lambda_i/\varepsilon)}(|\nabla u^\varepsilon|^2+V(\varepsilon x)(u^\varepsilon)^2+|\nabla v^\varepsilon|^2+V(\varepsilon x)(v^\varepsilon)^2)\leqslant\\
&C\Big(\int_{N_{2R}(\Lambda_i/\varepsilon)}|u^\varepsilon|^p\Big)^{\frac{p-1}{p}}\Big(\int_{N_{2R}(\Lambda_i/\varepsilon)}|v^\varepsilon|^p\Big)^{\frac1p}+C\Big(\int_{N_{2R}(\Lambda_i/\varepsilon)}|v^\varepsilon|^p\Big)^{\frac{p-1}{p}}\Big(\int_{N_{2R}(\Lambda_i/\varepsilon)}|u^\varepsilon|^p\Big)^{\frac1p}+\frac{C}{R}.
\end{aligned}$$

By taking R large and afterwards ε small, recalling (5.56) we obtain a contradiction with the second conclusion of Proposition 5.15. Thus there exist $x_\varepsilon\in\Lambda_i$ and $S_1,\rho_1>0$ such that

$$\int_{B_{\varepsilon S_1}(x_\varepsilon)}u_\varepsilon^2\geqslant\rho_1\varepsilon^N. \tag{5.57}$$

We suppose that $x_\varepsilon\to\bar x\in\bar\Lambda_i$, up to a subsequence.

3. Take a blowup sequence centered at x_ε, namely $\bar u_\varepsilon(x):=u_\varepsilon(\varepsilon x+x_\varepsilon)$ and $\bar v_\varepsilon(x):=v_\varepsilon(\varepsilon x+x_\varepsilon)$, which solves

$$\begin{cases}-\Delta\bar u_\varepsilon+V(\varepsilon x+x_\varepsilon)\bar u_\varepsilon=g(\varepsilon x+x_\varepsilon,\bar v_\varepsilon)\\ -\Delta\bar v_\varepsilon+V(\varepsilon x+x_\varepsilon)\bar v_\varepsilon=f(\varepsilon x+x_\varepsilon,\bar u_\varepsilon)\end{cases}\quad\text{in }\Omega^\varepsilon:=\frac{\Omega-x_\varepsilon}{\varepsilon}. \tag{5.58}$$

From Proposition 5.15, we know that $\|(\bar u_\varepsilon,\bar v_\varepsilon)\|_{H^1}\leqslant C$, and thus there exists $\bar u,\bar v\in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $\bar u_\varepsilon\rightharpoonup\bar u$, $\bar v_\varepsilon\rightharpoonup\bar v$ in $H^1(\mathbb{R}^N)$ (by elliptic regularity, actually the convergence is strong in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$). Moreover, from (5.57) we see that $(\bar u,\bar v)\neq(0,0)$. Since $\chi_\Lambda(\varepsilon x+x_\varepsilon)$ is bounded, it converges weakly in L^p_{loc} to a function χ such that $0\leqslant\chi\leqslant1$. Therefore we conclude that

$$\begin{cases}-\Delta\bar u+V(\bar x)\bar u=\bar g(x,\bar v)\\ -\Delta\bar v+V(\bar x)\bar v=\bar f(x,\bar u)\end{cases}\quad\text{in }\mathbb{R}^N,$$

where $\bar f(x,s)=\chi(x)f(s)+(1-\chi(x))\tilde f(s)$, $\bar g(x,s)=\chi(x)g(s)+(1-\chi(x))\tilde g(s)$. Let us prove that

$$\bar J_{V(\bar x)}(\bar u,\bar v)\leqslant\liminf\varepsilon^{-N}J^i_\varepsilon(u_\varepsilon,v_\varepsilon), \tag{5.59}$$

where

$$\bar J_{V(\bar x)}(\bar u,\bar v):=\int_{\mathbb{R}^N}(\langle\nabla\bar u,\nabla\bar v\rangle+V(\bar x)\bar u\bar v)-\int_{\mathbb{R}^N}\bar F(x,\bar u)-\int_{\mathbb{R}^N}\bar G(x,\bar v),$$

with $\bar F(x,s):=\int_0^s\bar f(x,\xi)\,d\xi$, $\bar G(x,s):=\int_0^s\bar g(x,\xi)\,d\xi$.

We have $\varepsilon^{-N}J^i_\varepsilon(u_\varepsilon,v_\varepsilon)=\int_{\tilde\Lambda_i^\varepsilon}h_\varepsilon$, with $h_\varepsilon:=\langle\nabla\bar u_\varepsilon,\nabla\bar v_\varepsilon\rangle+V(\varepsilon x+x_\varepsilon)\bar u_\varepsilon\bar v_\varepsilon-F(\varepsilon x+x_\varepsilon,\bar u_\varepsilon)-G(\varepsilon x+x_\varepsilon,\bar v_\varepsilon)$ and $\tilde\Lambda_i^\varepsilon=(\tilde\Lambda_i-x_\varepsilon)/\varepsilon$. For each $R>0$ fixed, we have that

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{B_R(0)}h_\varepsilon &= \int_{B_R(0)}(\langle\nabla\bar u,\nabla\bar v\rangle+V(\bar x)\bar u\bar v-\bar F(x,\bar u)-\bar G(x,\bar v))\\ &= \bar J_{V(\bar x)}(\bar u,\bar v)-\left(\int_{\mathbb{R}^N\setminus B_R(0)}(\langle\nabla\bar u,\nabla\bar v\rangle+V(\bar x)\bar u\bar v-\bar F(x,\bar u)-\bar G(x,\bar v))\right).\end{aligned}$$

Hence for any given $\delta>0$ we conclude that

$$\lim_{\varepsilon\to0}\int_{B_R(0)}h_\varepsilon\geqslant\bar J_{V(\bar x)}(\bar u,\bar v)-\delta\qquad\text{for large }R>0.\tag{5.60}$$

On the other hand, take $0\leqslant\phi_R\leqslant1$ a cut-off function such that $|\nabla\phi_R|\leqslant C/R$ and $\phi_R=1$ in $\mathbb{R}^N\setminus B_R(0)$, $\phi_R=0$ in $B_{R/2}(0)$. Let K_i be a set such that $\tilde\Lambda_i\Subset K_i$ and $K_i\cap\tilde\Lambda_j=\emptyset$ for $j\neq i$. Take ξ_i such that $0\leqslant\xi_i\leqslant1$, $|\nabla\xi_i|\leqslant C$, $\xi_i=1$ in $\tilde\Lambda_i$ and $\xi_i=0$ in $\mathbb{R}^N\setminus K_i$. We define $\xi_i^\varepsilon(x)=\xi_i(\varepsilon x+x_\varepsilon)$ and use $(\bar v_\varepsilon\xi_i^\varepsilon\phi_R,\bar u_\varepsilon\xi_i^\varepsilon\phi_R)$ as a test function in (5.58). We see that

$$\begin{aligned}&2\int_{\tilde\Lambda_i^\varepsilon\setminus B_R(0)}h_\varepsilon\\ &\geqslant-\int_{K_i^\varepsilon\setminus\tilde\Lambda_i^\varepsilon\cup B_R(0)\setminus B_{R/2}(0)}(\langle\nabla\bar u_\varepsilon,\nabla(\bar v_\varepsilon\xi_i^\varepsilon\phi_R)\rangle+\langle\nabla\bar v_\varepsilon,\nabla(\bar u_\varepsilon\xi_i^\varepsilon\phi_R)\rangle+2V(\varepsilon x+x_\varepsilon)\bar u_\varepsilon\bar v_\varepsilon\xi_i^\varepsilon\phi_R)\\ &\qquad+\int_{K_i^\varepsilon\setminus\tilde\Lambda_i^\varepsilon\cup B_R(0)\setminus B_{R/2}(0)}(f(\varepsilon x+x_\varepsilon,\bar u_\varepsilon)\bar u_\varepsilon\xi_i^\varepsilon\phi_R+g(\varepsilon x+x_\varepsilon,\bar v_\varepsilon)\bar v_\varepsilon\xi_i^\varepsilon\phi_R)\end{aligned}$$

and thus

$$\liminf_{\varepsilon\to0}\int_{\tilde\Lambda_i^\varepsilon\setminus B_R(0)}h_\varepsilon\geqslant-\delta\quad\text{ for sufficiently large }R>0.$$

By combining the previous inequality with (5.60), we obtain (5.59).
4. Finally, since $(\bar u,\bar v)\neq(0,0)$, $\bar F(x,s)\leqslant F(x,s)$, $\bar G(x,s)\leqslant G(x,s)$, and $\inf_{\Lambda_i}V\leqslant V(\bar x)$, then by Corollary 5.10 and Remark 5.11 we conclude that

$$c_i\leqslant c_{V(\bar x)}\leqslant\bar J_{V(\bar x)}(\bar u,\bar v)\leqslant\liminf\varepsilon^{-N}J^i_\varepsilon(u_\varepsilon,v_\varepsilon),$$

which is (5.55). □

Remark 5.24. Recalling (5.57), we observe that we could have performed the same reasoning with the function v_ε. Hence there exist points $x_\varepsilon,y_\varepsilon\in\Lambda_i$ and constants $\rho_1,\rho_2,S_1,S_2>0$ such that

$$\int_{B_{\varepsilon S_1}(x_\varepsilon)}u_\varepsilon^2\geqslant\rho_1\varepsilon^N\qquad\text{and}\qquad\int_{B_{\varepsilon S_2}(y_\varepsilon)}v_\varepsilon^2\geqslant\rho_2\varepsilon^N.$$

The next result concludes the proof of Theorem 5.1.

Proposition 5.25. *The functions $u_\varepsilon,v_\varepsilon$ satisfy the properties (i)-(iii) stated in Theorem 5.1.*

The proof of this result will be divided in several lemmas. The arguments used are quite standard and are similar to the ones of [57, 59].

Lemma 5.26. *If* $z_\varepsilon \in \bar{\Lambda}_i$ *is such that* $\liminf_{\varepsilon\to 0} \max\{u_\varepsilon(z_\varepsilon), v_\varepsilon(z_\varepsilon)\} > 0$*, then* $\lim_{\varepsilon\to 0} V(z_\varepsilon) = \inf_{\Lambda_i} V$*.*

Proof. Up to subsequences we have that $z_\varepsilon \to \bar{z} \in \bar{\Lambda}_i$, and the functions $\bar{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + z_\varepsilon)$, $\bar{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x + z_\varepsilon)$ (which are bounded in $H^1(\mathbb{R}^N)$ by Proposition 5.15) converge weakly in $H^1(\mathbb{R}^N)$ and strongly in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to $(\bar{u}, \bar{v})$. From the hypothesis of the lemma we conclude that $\max\{\bar{u}(0), \bar{v}(0)\} > 0$, and hence $(\bar{u}, \bar{v}) \neq (0,0)$ and, by reasoning exactly as in the proof of the previous theorem, we obtain

$$c_{V(\bar{z})} \leqslant \bar{J}_{V(\bar{z})}(\bar{u}, \bar{v}) \leqslant \liminf \varepsilon^{-N} J^i_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Hence from Theorem 5.23 we deduce that $c_{V(\bar{z})} \leqslant c_i$ and thus $V(\bar{z}) \leqslant \inf_{\Lambda_i} V$, whence $\inf_{\Lambda_i} V = V(\bar{z})$. □

Lemma 5.27. *We have* $\lim_{\varepsilon\to 0} \sup_{\Omega\setminus\Lambda}\{u_\varepsilon, v_\varepsilon\} = 0$*. In particular,* $(u_\varepsilon, v_\varepsilon)$ *solves* (5.1)*.*

Proof. It follows directly from the previous lemma and from our basic assumption $\inf_{\Lambda_i} V < \inf_{\partial\Lambda_i} V$ that $\sup_{\partial\Lambda_i} u_\varepsilon \to 0$ and $\sup_{\partial\Lambda_i} v_\varepsilon \to 0$ for every $i = 1, \dots, k$. Since, according to (5.53), $-\Delta(u_\varepsilon + v_\varepsilon) \leqslant 0$ over $\Omega \setminus \Lambda$ (by choosing $\delta \leqslant \alpha/2$ in Lemma 5.13), the first conclusion follows from the maximum principle. By recalling that $f(x,u) = f(u)$ if either $x \in \Lambda$, or $x \in \Omega \setminus \Lambda$ and u small, and similarly for $g(x,v)$, we have that $(u_\varepsilon, v_\varepsilon)$ solves (5.1). □

Lemma 5.28. *We have* $\liminf_{\varepsilon\to 0} \min\{\sup_{\Lambda_i} u_\varepsilon, \sup_{\Lambda_i} v_\varepsilon\} > 0$.

Proof. Suppose that, say, $\sup_{\Lambda_i} u_\varepsilon \to 0$. Then by Lemma 5.27 we obtain $u_\varepsilon \to 0$ in Ω, which contradicts Remark 5.24. Hence $\sup_{\Lambda_i} u_\varepsilon, \sup_{\Lambda_i} v_\varepsilon \geqslant b > 0$ for small ε. □

Lemma 5.29. *If* $z_\varepsilon \in \Omega$ *is a local maximum of* $u_\varepsilon + v_\varepsilon$*, then*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(z_\varepsilon) + v_\varepsilon(z_\varepsilon) > 0.$$

Proof. At z_ε we have

$$\begin{aligned} 0 < \alpha \quad &\leqslant \quad V(z_\varepsilon) \leqslant \frac{-\Delta(u_\varepsilon + v_\varepsilon)(z_\varepsilon)}{u_\varepsilon(z_\varepsilon) + v_\varepsilon(z_\varepsilon)} + V(z_\varepsilon) \\ &= \quad \frac{f(u_\varepsilon(z_\varepsilon))}{u_\varepsilon(z_\varepsilon) + v_\varepsilon(z_\varepsilon)} + \frac{g(v_\varepsilon(z_\varepsilon))}{u_\varepsilon(z_\varepsilon) + v_\varepsilon(z_\varepsilon)} \leqslant \frac{f(u_\varepsilon(z_\varepsilon))}{u_\varepsilon(z_\varepsilon)} + \frac{g(v_\varepsilon(z_\varepsilon))}{v_\varepsilon(z_\varepsilon)}, \end{aligned}$$

and hence $u_\varepsilon(z_\varepsilon) + v_\varepsilon(z_\varepsilon) \geqslant b > 0$ for some $b > 0$, otherwise we would obtain a contradiction from the fact that $f(s) = g(s) = \mathrm{o}(s)$ as $s \to 0$. □

Remark 5.30. By combining Lemmas 5.27 and 5.29 we get that $u_\varepsilon + v_\varepsilon$ does not admit local maximums in $\Omega \setminus \Lambda$ for sufficiently small ε.

Lemma 5.31. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *and every* $i = 1, \dots, k$*, the function* $u_\varepsilon + v_\varepsilon$ *has exactly one local maximum* $x_{i,\varepsilon}$ *in* Λ_i*.*

Proof. The existence of local maximums is guaranteed by Lemma 5.28. Let us check its uniqueness. Let $x_{i,\varepsilon}, y_{i,\varepsilon} \in \Lambda_i$ be local maximums of $u_\varepsilon + v_\varepsilon$. By Lemmas 5.26 and 5.29 we have

$$u_\varepsilon(x_{i,\varepsilon}) + v_\varepsilon(x_{i,\varepsilon}) \geqslant b > 0, \qquad u_\varepsilon(y_{i,\varepsilon}) + v_\varepsilon(y_{i,\varepsilon}) \geqslant b > 0$$

for some constant $b > 0$, and $V(x_{i,\varepsilon}), V(y_{i,\varepsilon}) \to \inf_{\Lambda_i} V =: V_i$. Moreover, we can suppose that $x_{i,\varepsilon} \to \bar{x} \in \Lambda_i$, $y_{i,\varepsilon} \to \bar{y} \in \Lambda_i$. Define the rescaled functions

$$\bar{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_{i,\varepsilon}), \qquad \bar{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x + x_{i,\varepsilon}),$$

and

$$\tilde{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + y_{i,\varepsilon}), \qquad \tilde{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x + y_{i,\varepsilon}),$$

and take $(\bar{u}, \bar{v}), (\tilde{u}, \tilde{v}) \neq (0, 0)$, solutions of

$$\begin{cases} -\Delta u + V_i u = g(u) \\ -\Delta v + V_i v = f(u) \end{cases} \quad \text{in } \mathbb{R}^N,$$

such that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \to (\bar{u}, \bar{v})$, $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \to (\tilde{u}, \tilde{v})$. Observe that $\bar{u}, \bar{v}, \tilde{u}, \tilde{v}$ are radially symmetric with respect to the origin, and are radially decreasing. Moreover, either $\bar{u}''(0) \neq 0$ or $\bar{v}''(0) \neq 0$ (see [110, page 3276]), whence $\tilde{u}''(0) + \tilde{v}''(0) \neq 0$.

Take $z_\varepsilon := \frac{y_{i,\varepsilon} - x_{i,\varepsilon}}{\varepsilon}$. We claim that $|z_\varepsilon|$ is a bounded sequence. Suppose in view of a contradiction that $|z_\varepsilon| \to +\infty$. Then for every $R > 0$ the sets $B_{\varepsilon R}(x_{i,\varepsilon})$ and $B_{\varepsilon R}(y_{i,\varepsilon})$ are disjoint for sufficiently small ε, and hence

$$\varepsilon^{-N} J^i_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_{B_R(0)} h_\varepsilon + \int_{B_R(0)} \tilde{h}_\varepsilon + \int_{\tilde{\Lambda}^\varepsilon_i \setminus (B_R(0) \cup B_R(z_\varepsilon))} h_\varepsilon,$$

with

$$h_\varepsilon = \langle \nabla \bar{u}_\varepsilon, \nabla \bar{v}_\varepsilon \rangle + V(\varepsilon x + x_{i,\varepsilon}) \bar{u}_\varepsilon \bar{v}_\varepsilon - F(\bar{u}_\varepsilon) - G(\bar{v}_\varepsilon),$$

and

$$\tilde{h}_\varepsilon = \langle \nabla \tilde{u}_\varepsilon, \nabla \tilde{v}_\varepsilon \rangle + V(\varepsilon x + y_{i,\varepsilon}) \tilde{u}_\varepsilon \tilde{v}_\varepsilon - F(\tilde{u}_\varepsilon) - G(\tilde{v}_\varepsilon).$$

Hence, by reasoning as in the proof of Theorem 5.23, Step 3, we can prove that for every $\delta > 0$ there exists $R > 0$ such that

$$\lim_{\varepsilon \to 0} \int_{B_R(0)} h_\varepsilon \geqslant I_{V_i}(\bar{u}, \bar{v}) - \delta \geqslant c_i - \delta, \qquad \lim_{\varepsilon \to 0} \int_{B_R(0)} \tilde{h}_\varepsilon \geqslant I_{V_i}(\tilde{u}, \tilde{v}) - \delta \geqslant c_i - \delta$$

and

$$\liminf_{\varepsilon \to 0} \int_{\tilde{\Lambda}^\varepsilon_i \setminus (B_R(0) \cup B_R(z_\varepsilon))} h_\varepsilon \geqslant -\delta$$

and therefore

$$-3\delta + 2c_i \leqslant \liminf \varepsilon^{-N} J^i_\varepsilon(u_\varepsilon, v_\varepsilon) = c_i$$

for every $\delta > 0$, a contradiction.

Let z_0 be such that $z_\varepsilon \to z_0$. From the fact that

$$\begin{aligned} \bar{u}_\varepsilon(x) + \bar{v}_\varepsilon(x) &= u_\varepsilon(\varepsilon x + x_{i,\varepsilon}) + v_\varepsilon(\varepsilon x + y_{i,\varepsilon}) \\ &= u_\varepsilon\Big(y_{i,\varepsilon} + \varepsilon\Big(x - \frac{y_{i,\varepsilon} - x_{i,\varepsilon}}{\varepsilon}\Big)\Big) + v_\varepsilon\Big(y_{i,\varepsilon} - \varepsilon\Big(x - \frac{y_{i,\varepsilon} - x_{i,\varepsilon}}{\varepsilon}\Big)\Big) \\ &= \tilde{u}_\varepsilon\Big(x - \frac{y_{i,\varepsilon} - x_{i,\varepsilon}}{\varepsilon}\Big) + \tilde{v}_\varepsilon\Big(x - \frac{y_{i,\varepsilon} - x_{i,\varepsilon}}{\varepsilon}\Big) \end{aligned}$$

we see that

$$\bar{u}(x) + \bar{v}(x) = \tilde{u}(x - z_0) + \tilde{v}(x - z_0)$$

and hence $z_0 = 0$. Since $\bar{u}''(0) + \bar{v}''(0) \neq 0$, then $x_{i,\varepsilon} = y_{i,\varepsilon}$ for sufficiently small $\varepsilon > 0$. $\square$

Lemma 5.32. *There exists $\gamma, \beta > 0$ such that*

$$u_\varepsilon(x) + v_\varepsilon(x) \leqslant \gamma e^{\frac{\beta}{\varepsilon}|x - x_{i,\varepsilon}|} \qquad \textit{for every } x \in \Omega \setminus \cup_{j\neq i}\Lambda_j.$$

Proof. 1. First of all, take $\Lambda_i' \Subset \Lambda_i$ such that $\inf_{\Lambda_i} V = \inf_{\Lambda_i'} V < \inf_{\partial\Lambda_i'} V$. Reasoning as before we have that $\sup_{\Omega\setminus\Lambda'}(u_\varepsilon + v_\varepsilon) \to 0$ and $u_\varepsilon + v_\varepsilon$ does not admit any local maximum over $\Omega \setminus \Lambda'$ for sufficiently small ε ($\Lambda' := \cup_{i=1}^k \Lambda_i'$).
2. For every fixed i, consider the rescaled functions

$$\bar{u}_\varepsilon(x) + \bar{v}_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_{i,\varepsilon}) + v_\varepsilon(\varepsilon x + x_{i,\varepsilon})$$

and let $(\bar{u}, \bar{v}) \neq (0,0)$ be such that $\bar{u}_\varepsilon \to \bar{u}$, $\bar{v}_\varepsilon \to \bar{v}$ in C^2_{loc}, and

$$-\Delta\bar{u} + V(x_i)\bar{u} = g(\bar{v}), \qquad -\Delta\bar{v} + V(x_i)\bar{v} = f(\bar{u}),$$

with $V(x_i) = \inf_{\Lambda_i} V$. Take $b > 0$ such that

$$c := \inf_\Omega V - \frac{f(b)}{b} - \frac{g(b)}{b} > 0.$$

We claim the existence of $R, \varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,

$$\bar{u}_\varepsilon(x) + \bar{v}_\varepsilon(x) \leqslant b \qquad \text{for every } x \in \omega := \mathbb{R}^N \setminus \Big(\cup_{j\neq i}\frac{\Lambda_j' - x_{i,\varepsilon}}{\varepsilon} \cup B_R(0)\Big).$$

In view of a contradiction, let $\varepsilon_n \to 0$ and $|x_n| \to \infty$ be such that $\varepsilon_n x_n + x_{i,\varepsilon} \notin \Lambda_j$ for every $j \neq i$, and

$$\bar{u}_{\varepsilon_n}(x_n) + \bar{v}_{\varepsilon_n}(x_n) > b. \tag{5.61}$$

Since $\bar{u}$ and $\bar{v}$ decay exponentially to 0 as $|x| \to \infty$, there exists $R_0 > 0$ such that, for ε small,

$$\bar{u}_{\varepsilon_n}(x) + \bar{v}_{\varepsilon_n}(x) < b \qquad \text{for } |x| = R_0.$$

Thus for ε_n small we have

$$\bar{u}_{\varepsilon_n}(x) + \bar{v}_{\varepsilon_n}(x) < b \qquad \text{for } x \in \partial\omega = \cup_{j\neq i}\frac{\partial\Lambda_j' - x_{i,\varepsilon}}{\varepsilon} \cup \partial B_{R_0}(0)$$

which, together with (5.61), yields the existence of a local maximum point of $u_{\varepsilon_n} + v_{\varepsilon_n}$ on the set $\Omega \setminus (\cup_{j\neq i}\Lambda_j' \cup B_{\varepsilon R_0}(x_{i,\varepsilon}))$, contradicting the first paragraph of this proof. In conclusion, we have

$$-\Delta(\bar{u}_\varepsilon + \bar{v}_\varepsilon) + c(\bar{u}_\varepsilon + \bar{v}_\varepsilon) \leqslant 0 \quad \text{in } \omega, \qquad \text{and} \qquad 0 \leqslant \bar{u}_\varepsilon + \bar{v}_\varepsilon \leqslant b \quad \text{in } \partial\omega.$$

3. Let $0 < \delta' < \delta < \sqrt{c}$ and $y_1, \dots, y_{l_0}$ be such that

$$B_\delta(y_l) \subseteq \cup_{j\neq i}\Lambda_j, \qquad \partial\Big(\cup_{j\neq i}\Lambda_j'\Big) \subseteq \cup_{l=1}^{l_0}(B_\delta(y_l) \setminus B_{\delta'}(y_l)),$$

and

$$|x - y_l| \geqslant 2\delta + \mu' \qquad \forall l,\ \forall x \in \Omega \setminus \cup_{j\neq i}\Lambda_j \qquad \text{for some } \mu' > 0.$$

We claim that there exists $\mu > 0$ such that

$$\delta|x - y_l| - 2\delta^2 \geqslant \mu|x - x_{i,\varepsilon}| \qquad \forall x \in \Omega \setminus \cup_{j\neq i}\Lambda_j. \tag{5.62}$$

In fact, on the one hand

$$|x - x_{i,\varepsilon}| + 4\delta \leqslant |x - y_l| + |y_l - x_{i,\varepsilon}| + 4\delta \leqslant |x - y_l| + C \leqslant 2|x - y_l|$$

for all $|x| \geqslant R$ for sufficiently large R (whence $\delta|x - y_l| - 2\delta^2 \geqslant \delta|x - x_{i,\varepsilon}|/2$ for $|x| \geqslant R$). On the other hand, if $|x| \leqslant R$, then for sufficiently small $\mu > 0$ we have

$$\mu|x - x_{i,\varepsilon}| \leqslant \mu R' \leqslant \mu' \leqslant |x - y_l| - 2\delta.$$

4. Let $w \in H^1(\mathbb{R}^N \setminus B_1(0))$ be such that

$$-\Delta w + \delta^2 w = 0 \qquad \text{and} \qquad a_1 e^{-2\delta|x|} \leqslant w(x) \leqslant a_2 e^{-\delta|x|}\ \forall |x| \geqslant 1,$$

for some $a_1, a_2 > 0$. Given constants $\lambda_0, \lambda_1, \dots, \lambda_{l_0} > 0$, we consider the function

$$z(x) = \lambda_0 w(x) + \sum_{l=1}^{l_0} \lambda_l w\Big(x - \frac{y_l - x_{i,\varepsilon}}{\varepsilon}\Big) \qquad \text{in } \omega.$$

Such function is well defined since $\varepsilon x - x_{i,\varepsilon} \notin \Lambda_j'$ for $j \neq i$ whenever $x \in \omega$, and hence $|\varepsilon x - x_{i,\varepsilon} - y_l| \geqslant \delta' \geqslant \varepsilon$ for small ε. We have

$$-\Delta(\bar u_\varepsilon + \bar v_\varepsilon) + \delta^2(\bar u_\varepsilon + \bar v_\varepsilon) \leqslant -\Delta z + \delta^2 z \qquad \text{in } \omega$$

and we claim that there exists $\lambda_0, \lambda_1, \dots, \lambda_{l_0}$ such that

$$\bar u_\varepsilon(x) + \bar v_\varepsilon(x) \leqslant b \leqslant z(x) \qquad \text{on } \partial\omega.$$

In fact, we can choose $\lambda_0 = \lambda_0(R)$ such that $b \leqslant \lambda_0 w$ on $\partial B_R(0)$; moreover, for every $x \in (\partial\Lambda_j' - x_{i,\varepsilon})/\varepsilon$, let l' be such that $\varepsilon x + x_{i,\varepsilon} \in B_\delta(y_{l'})$. By taking

$$\lambda_{l'} := \frac{b}{a_1} e^{\frac{2\delta^2}{\varepsilon}}, \qquad \text{we obtain} \qquad b \leqslant \lambda_{l'} w\Big(x - \frac{y_{l'} - x_{i,\varepsilon}}{\varepsilon}\Big)$$

and the claim follows. Therefore by the maximum principle we have that for every $x \in \omega$,

$$\begin{aligned} u_\varepsilon(x) + v_\varepsilon(x) &\leqslant \lambda_0 w\Big(\frac{x - x_{i,\varepsilon}}{\varepsilon}\Big) + \sum_{l=1}^{l_0} \lambda_l w\Big(\frac{x - y_l}{\varepsilon}\Big) \\ &\leqslant \lambda_0 a_2 e^{-\frac{\delta}{\varepsilon}|x - x_{i,\varepsilon}|} + \frac{b a_2}{a_1} e^{-\frac{\mu}{\varepsilon}|x - x_{i,\varepsilon}|}, \end{aligned}$$

where we have used (5.62). Finally, since $\bar u \to \bar u$,$\bar v_\varepsilon \to \bar v$ in $C^2(B_R(0))$, we also have that

$$u_\varepsilon(\varepsilon x + x_{i,\varepsilon}) + v_\varepsilon(\varepsilon x + x_{i,\varepsilon}) \leqslant C \leqslant \gamma' e^{-\beta'|x|}.$$

These inequalities together imply the desired result. □

Remark 5.33. By following the reasoning of the proof of Lemma 5.31, one can conclude that, given $y_{i,\varepsilon}$ a local maximum of u_ε in Λ_i and $z_{i,\varepsilon}$ a local maximum of v_ε in Λ_i (which exist, by Lemma 5.28), then

$$\frac{|y_{i,\varepsilon} - z_{i,\varepsilon}|}{\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0^+.$$

5.5 The case $p \neq q$

In Section 5.4 we have proved Theorem 5.1 except that we have worked with a truncated problem, as explained at the end of Section 5.1. The full statement of Theorem 5.1 will be established once we prove uniform bounds in $L^\infty(\Omega)$ of the solutions constructed so far. So, let us suppose that $p, q > 2$ are such that $1/p + 1/q > (N-2)/N$ with, say, $2 < p < 2^* = 2N/(N-2)$ and $p < q$.

Given $n \in \mathbb{N}$, recall the functions f_n and g_n already defined in (5.3),

$$f_n(s) = \begin{cases} f(s) & \text{for } s \leqslant n \\ A_n s^{p-1} + B_n & \text{for } s > n \end{cases} \qquad g_n(s) = \begin{cases} g(s) & \text{for } s \leqslant n \\ \tilde{A}_n s^{p-1} + \tilde{B}_n & \text{for } s > n. \end{cases}$$

Then, for a fixed n, thanks to Theorem 5.1 there exists $\varepsilon_{0,n} > 0$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ there are positive solutions $u_\varepsilon, v_\varepsilon$ of the problem

$$-\varepsilon^2 \Delta u + V(x)u = g_n(v), \quad -\varepsilon^2 \Delta v + V(x)v = f_n(u) \ \text{ in } \Omega, \qquad u, v \in H_0^1(\Omega), \tag{5.63}$$

satisfying the conclusions of that theorem; at this point, all the quantities appearing in the theorem depend of n. Moreover, by Theorem 5.23 we have

$$I_\varepsilon^n(u_\varepsilon, v_\varepsilon) = \varepsilon^N \left(\sum_{i=1}^k c_{i,n} + o_n(1) \right) \qquad \text{as } \varepsilon \to 0,$$

where $I_\varepsilon^n(u_\varepsilon, v_\varepsilon) = \int_{\mathbb{R}^N} (\varepsilon^2 \langle \nabla u, \nabla v \rangle + V(x)uv - F_n(u) - G_n(v))$ with obvious notations, and $c_{i,n}$ is the ground-state critical level of

$$-\Delta u + V(x_i)u = g_n(v), \quad -\Delta v + V(x_i)v = f_n(u) \qquad \text{in } \mathbb{R}^N. \tag{5.64}$$

Therefore, given n we can consider $\varepsilon_{0,n} > 0$ small enough such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have $I_\varepsilon^n(u_\varepsilon, v_\varepsilon) \leqslant 2\varepsilon^N \sum_{i=1}^k c_{i,n}$.

Lemma 5.34. *For every $n \in \mathbb{N}$ there exists $\varepsilon_{0,n} > 0$ such that, for every $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$I_\varepsilon^n(u_\varepsilon, v_\varepsilon) \leqslant C\varepsilon^N,$$

for some $C > 0$ independent of n and ε. In particular, also

$$\int_{\mathbb{R}^N} (f_n(u_\varepsilon)u_\varepsilon + g_n(v_\varepsilon)v_\varepsilon) \leqslant C\varepsilon^N. \tag{5.65}$$

Proof. We only have to prove that $c_{i,n} \leqslant C$, with $C > 0$ independent of n, for any fixed $i = 1, \ldots, k$. We recall that from our assumptions on f we have that $f'(s) \geqslant (1+\delta')f(s)/s \geqslant (1+\delta')f(1)s^{\delta'}$ for some $0 < \delta' < p-2$. We set

$$h_f(s) := \begin{cases} f(s) & , \ s \leqslant 1 \\ \frac{f'(1)}{1+\delta'} s^{1+\delta'} + f(1) - \frac{f'(1)}{1+\delta'} & , \ s > 1. \end{cases}$$

Then, for a small $\lambda > 0$ (namely, $\lambda < (1+\delta')f(1)/f'(1)$) it follows easily that $\lambda h_f' \leqslant f_n'$, thus also $\lambda h_f \leqslant f_n$. We proceed in a similar way with the function g, yielding some function h_g such that $\lambda h_g \leqslant g_n$. Then, according to Remark 5.11, we conclude that $c_{i,n} \leqslant c_{\lambda h_f, \lambda h_g}$, where the latter quantity refers to the ground-state critical level associated to the problem

$$-\Delta u + V(x_i)u = \lambda h_g(v), \qquad -\Delta v + V(x_i)v = \lambda h_f(u), \qquad u, v \in H^1(\mathbb{R}^N).$$

The final conclusion follows from the fact that the left-hand side of (5.65) is bounded by $\frac{2(2+\delta')}{\delta'} I_\varepsilon^n(u_\varepsilon, v_\varepsilon)$, according to our assumption (fg3). □

We denote by $x_{i,\varepsilon}$ the maximum points of $u_\varepsilon + v_\varepsilon$ over Λ_i, as mentioned in Theorem5.1.

Lemma 5.35. *Given $\rho > 0$, $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have $u_\varepsilon(x), v_\varepsilon(x) \leqslant 1$, for all $x \in \Omega \backslash \cup_{j \neq i} \Lambda_j$ such that $|x - x_{i,\varepsilon}| \geqslant \rho$.*

Proof. According to Theorem 5.1 we have $u_\varepsilon(x), v_\varepsilon(x) \leqslant \gamma_n e^{-\frac{\beta_n}{\varepsilon}|x - x_{i,\varepsilon}|}$, for all $x \in \Omega \backslash \cup_{j \neq i} \Lambda_j$. Then we just have to choose $\varepsilon_{0,n} \leqslant \rho\beta_n / \log \gamma_n$. □

Taking the previous lemma into account, we are left to the analysis of the behavior of $u_\varepsilon(x), v_\varepsilon(x)$ over small neighborhoods of the points $x_{i,\varepsilon}$. To that purpose, we take $\rho > 0$ such that $B_{2\rho}(x_{i,\varepsilon}) \subseteq \Lambda_i$ and we introduce cut-off functions ϕ_i such that $\phi_i = 1$ in $B_\rho(x_{i,\varepsilon})$, $\phi_i = 0$ in $\mathbb{R}^N \setminus B_{2\rho}(x_{i,\varepsilon})$, and denote $\phi_{i,\varepsilon}(x) = \phi_i(\varepsilon x + x_{i,\varepsilon})$. We also consider the functions

$$\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_{i,\varepsilon}), \qquad \bar{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x + x_{i,\varepsilon}),$$

which satisfy, in the whole space $\mathbb{R}^N$,

$$\left\{ \begin{array}{lcl} -\Delta(\bar{u}_\varepsilon \phi_{i,\varepsilon}) + V(\varepsilon x + x_{i,\varepsilon})\bar{u}_\varepsilon \phi_{i,\varepsilon} & = & g_n(\bar{v}_\varepsilon)\phi_{i,\varepsilon} - \bar{u}_\varepsilon \Delta\phi_{i,\varepsilon} - 2\langle \nabla \bar{u}_\varepsilon, \nabla \phi_{i,\varepsilon} \rangle, \\ -\Delta(\bar{v}_\varepsilon \phi_{i,\varepsilon}) + V(\varepsilon x + x_{i,\varepsilon})\bar{v}_\varepsilon \phi_{i,\varepsilon} & = & f_n(\bar{u}_\varepsilon)\phi_{i,\varepsilon} - \bar{v}_\varepsilon \Delta\phi_{i,\varepsilon} - 2\langle \nabla \bar{v}_\varepsilon, \nabla \phi_{i,\varepsilon} \rangle. \end{array} \right. \tag{5.66}$$

We now use the same variational setting as in [117]. Given $r \geqslant 0$, recall the definition of the Sobolev space

$$H^r(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \ (1 + |\xi|^2)^{\frac{r}{2}} \widehat{u}(\xi) \in L^2(\mathbb{R}^N)\},$$

where $\widehat{}$ is the Fourier transform and $^\vee$ is its inverse. Define the linear map $A^r : H^r \to L^2$ as the isomorphism $A^r(u) := ((\alpha + |\xi|^2)^{\frac{r}{2}} \widehat{u}(\xi))^\vee$, with $\alpha := \inf_\Omega V$. We observe that $A^2 = -\Delta + \alpha$,

$$\int_{\mathbb{R}^N} A^r(u)v = \int_{\mathbb{R}^N} uA^r(v) \qquad \text{for every } u, v \in H^r(\mathbb{R}^N), \ r \geqslant 0,$$

and that $H^r(\mathbb{R}^N)$ is a Hilbert space when endowed with the inner product

$$\langle u, v \rangle_{H^r} = \int_{\mathbb{R}^N} A^r(u)A^r(v).$$

Define s, t such that $s + t = 2$, $s, t < \frac{N}{2}$ and $p < \frac{2N}{N-2s}$, $q < \frac{2N}{N-2t}$ (see [117, p. 1453]). This implies the following continuous injections $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $H^t(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$.

Lemma 5.36. *Given $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$\|\bar{u}_\varepsilon \phi_{i,\varepsilon}\|_{H^s(\mathbb{R}^N)} + \|\bar{v}_\varepsilon \phi_{i,\varepsilon}\|_{H^t(\mathbb{R}^N)} \leqslant C, \qquad \textit{with } C > 0 \textit{ independent of } n \textit{ and } \varepsilon.$$

Proof. We first add on both sides of the first equation in (5.66) the term $V(x_{i,\varepsilon})\bar{u}_\varepsilon\phi_{i,\varepsilon}$ and use the test function $A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon})$. It follows

$$\begin{aligned}\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 \leqslant & \int_{\mathbb{R}^N}(g_n(\bar{v}_\varepsilon)\phi_{i,\varepsilon} - \bar{u}_\varepsilon\Delta\phi_{i,\varepsilon} - 2\langle\nabla\bar{u}_\varepsilon,\nabla\phi_{i,\varepsilon}\rangle)A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}) + \\ & + \int_{\mathbb{R}^N}(V(x_{i,\varepsilon}) - V(\varepsilon x + x_{i,\varepsilon}))\bar{u}_\varepsilon\phi_{i,\varepsilon}A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}).\end{aligned}$$

We know from (5.65) that $\int_{\mathbb{R}^N}(f_n(\bar{u}_\varepsilon)\bar{u}_\varepsilon + g_n(\bar{v}_\varepsilon)\bar{v}_\varepsilon) \leqslant C$. Also, for every $s \geqslant 1$, $|f_n(s)| \leqslant C|s|^{p-1}$ and $|g_n(s)| \leqslant C|s|^{q-1}$. Thus

$$\begin{aligned}&\int_{\mathbb{R}^N} g_n(\bar{v}_\varepsilon)\phi_{i,\varepsilon}A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}) \\ &\leqslant \frac{1}{4}(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}^2) + C\left(\int_{\{\bar{v}_\varepsilon\geqslant\delta'\}}|g_n(\bar{v}_\varepsilon)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\|A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon})\|_{L^q} \\ &\leqslant \frac{1}{4}(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}^2) + C'\left(\int_{\mathbb{R}^N}|g_n(\bar{v}_\varepsilon)|\bar{v}_\varepsilon\right)^{\frac{q-1}{q}}\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s} \\ &\leqslant \frac{1}{4}(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}^2) + C''\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}.\end{aligned}$$

Moreover,

$$-\int_{\mathbb{R}^N}\Delta\phi_{i,\varepsilon}\bar{u}_\varepsilon A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}) \leqslant C\varepsilon^2\int_{\mathbb{R}^N}|\bar{u}_\varepsilon|\,|A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon})| \leqslant C_n\varepsilon^2\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}$$

and

$$-2\int_{\mathbb{R}^N}\langle\nabla\bar{u}_\varepsilon,\nabla\phi_{i,\varepsilon}\rangle A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}) \leqslant C\varepsilon\int_{\mathbb{R}^N}|\nabla\bar{u}_\varepsilon|\,|A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon})| \leqslant C_n\varepsilon\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s},$$

for some positive constant C_n depending on n but not on ε. Finally, since V is locally Hölder continuous (for some $\alpha > 0$) we have

$$\int_{\mathbb{R}^N}(V(x_{i,\varepsilon}) - V(\varepsilon x + x_{i,\varepsilon}))\bar{u}_\varepsilon\phi_{i,\varepsilon}A^{-t}A^s(\bar{u}_\varepsilon\phi_{i,\varepsilon}) \leqslant \rho^\alpha C\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2.$$

Therefore, proceeding similarly with the second equation in (5.66) and by choosing $\varepsilon_{0,n}$ small enough, we deduce that

$$\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}^2 \leqslant \rho^\alpha C(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s}^2 + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}^2) + C(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}).$$

So, provided $\rho > 0$ is chosen sufficiently small, the conclusion follows. □

Lemma 5.37. *Given $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \leqslant C, \qquad \text{with } C > 0 \text{ independent of } n \text{ and } \varepsilon.$$

Proof. Thanks to Lemma 5.36, we can bootstrap similarly to [117, p. 1450 & 1451]. After a finite number of steps and by taking if necessary a smaller $\varepsilon_{0,n}$, we conclude that $\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^{N/2}} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^{N/2}} \leqslant C$. The conclusion follows from the imbedding $H^{\frac{N}{2}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. □

Our final result completes the proof of Theorem 5.1 in its full generality.

Proposition 5.38. *There exist* $n_0 \in \mathbb{N}$ *and* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *our solutions* $u_\varepsilon, v_\epsilon$ *of problem* (5.63) *with* $n = n_0$ *satisfy all the assertions of Theorem 5.1.*

Proof. According to Lemma 5.37 we may choose $n_0 \in \mathbb{N}$ large enough so that $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \leqslant n_0$ for every $0 < \varepsilon < \varepsilon_{0,n_0}$ and in this way we solve the original problem (5.1). The conclusion follows then from Proposition 5.25. $\square$

5.6 Additional Comments

1. In order to prove Theorem 5.1, one could have used the following alternative (but equivalent) approach, which takes in consideration the decomposition $H \times H = H^+ \oplus H^-$, with $H^+ = \{(\phi, \phi) : \phi \in H\}$ and $H^- = \{(\phi, -\phi) : \phi \in H\}$. After the truncation of the functions f, g at the beginning of Section 5.3, one could have considered the *reduced* functional

$$\tilde{J}_\varepsilon : H \to \mathbb{R}, \qquad \tilde{J}_\varepsilon(u) = J_\varepsilon((u, u) + (\Psi_{u,u}, -\Psi_{u,u}))$$

(recall that $\Psi_{u,u}$ was defined in (5.5)). Observe that $\tilde{J}_\varepsilon$ is a C^1 functional, and that

$$\tilde{J}'_\varepsilon(u)\varphi = J'_\varepsilon(u + \Psi_{u,u}, u - \Psi_{u,u})(\varphi, \varphi), \quad \forall u, \varphi \in H.$$

Hence the map

$$\eta : H \to H \times H, \qquad u \mapsto (u + \Psi_{u,u}, u - \Psi_{u,u})$$

is a homeomorphism between the critical points of $\tilde{J}_\varepsilon$ and of J_ε, respectively. Under this framework, the "localized" Nehari manifold of Section 5.3 becomes

$$N_\varepsilon = \left\{ u \in H : \ \tilde{J}'_\varepsilon(u)(u\phi_i) = 0, \quad \text{and} \quad \int_{\Lambda_i} u^2 > \varepsilon^{N+1}, \ \forall i = 1, \dots, k \right\}$$

(which is a manifold with finite co-dimension). All the remaining arguments would apply, *mutatis mutandis*. One would therefore deduce the existence of w_ε, critical point for $\tilde{J}_\varepsilon$, such that w_ε concentrates at exactly k local maximum points $x_{i,\varepsilon} \in \Lambda_i$, $i = 1, \dots, k$. Then $(u_\varepsilon, v_\varepsilon) = \eta(w_\varepsilon)$ would be a critical point for J_ε, and $u_\varepsilon + v_\varepsilon = w_\varepsilon$ would satisfy the conclusions of Theorem 5.1.

2. The previous approach has been used in more recent literature. We will give an example of this fact at the end of the next chapter. Here, we refer to the work by Ramos [104], where the author (under the dimensional restriction $3 \leqslant N \leqslant 6$) exhibits solutions of (5.1) that concentrate around a prescribed critical point of V which is not necessarily a minimum. It remains an open question whether there exist multi-peak solutions of (5.1) concentrating around non-minimal critical points of V, in the sense of [61, Theorem 1.2].

3. A related reduction method was also used by Szulkin and Weth [122] in order to find ground-state solutions of the equation

$$-\Delta u + V(x)u = f(x, u), \qquad u \in H^1(\mathbb{R}^N),$$

with f a superlinear, subcritical nonlinearity, f and V periodic in x, and 0 not belonging to the spectum of $-\Delta + V$. The authors use a reduction based on the decomposition $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ related to the positive and negative parts of the spectrum of $-\Delta + V$.

4. Finally, we would like to point out that the occurrence of concentration phenomena was also studied for the systems considered in the first part of this text. More precisely, for the system

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_1 u = \mu_1 u^3 - \beta u v^2 \\ -\varepsilon^2 \Delta v + \lambda_2 v = \mu_2 v^3 - \beta u^2 v \\ u, v \in H_0^1(\Omega), \ u, v > 0, \end{cases}$$

with Ω a bounded regular domain of $\mathbb{R}^N$, $N \leqslant 3$, and $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$, Lin and Wei [82] study the shape of ground-state solutions as $\varepsilon \to 0^+$, drawing conclusions that depend on the sign of β.

Part III

Multiplicity of sign-changing solutions for second and fourth order equations

Chapter 6

A Bahri-Lions theorem revisited

> *"(...) several examples have been given of the existence of multiple critical points for functionals invariant under a group of symmetries. A natural question is: What happens when such a functional is subjected to a perturbation which destroys the symmetry?"* [1]

6.1 Introduction

In the celebrated paper [12], A. Bahri and P. L. Lions studied the following semilinear elliptic problem

$$-\Delta u = |u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega), \tag{6.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N (N \geqslant 2)$; $2 < p < 2N/(N-2)$ ($p < \infty$ if $N = 2$); $f(x,u)$ is a Carathéodory function in $\Omega \times \mathbb{R}$, not necessarily odd symmetric in u and satisfies

(B1) $|f(x,t)| \leqslant h(x) + C|t|^q$, for some $C \geqslant 0$, $h \in L_+^r(\Omega)$, where $q = (N+2)/(N-2)$ ($q < \infty$ if $N \leqslant 2$), $r = 2N/(N+2)$ ($r > 1$ if $N \leqslant 2$);

(B2) $|F(x,t)| \leqslant a(x) + b(x)|t|^\nu$ for some $0 \leqslant \nu < 2$, where $a \in L_+^1(\Omega)$, $b \in L_+^\beta(\Omega)$ with $\beta > 1$, $\beta < 2N/(N-2)(1/\nu)$ ($\beta > 1$ if $N \leqslant 2$), $F(x,t) = \int_0^t f(x,s)ds$.

Under these assumptions, they obtained the following existence result.

Bahri-Lions Theorem. *Let*

$$2 < p < \frac{2N - 2\nu}{N-2}\cdot \tag{6.2}$$

Then equation (6.1) *admits an unbounded sequence of solutions* $u_n \in H_0^1(\Omega)$.

A natural, still open question is to know whether this sequence of solutions has an increasing number of nodal domains (this is the case for the dimension $N = 1$). In this

[1] in P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. 65, Amer. Math. Soc., Providence, R.I., 1986.

chapter, we give a first positive answer to this question. Precisely, we will obtain infinitely many sign-changing solutions for the (slightly more general) problem

$$-\Delta u = g(x,u) + f(x,u), \quad u \in H_0^1(\Omega), \tag{6.3}$$

where $g, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that

(H1) $g(x,s)$ is odd in s and $g(x,s)/s \to 0$ as $s \to 0$ uniformly in x;

(H2) $0 \leqslant g(x,s)s \leqslant C(|s|^p + 1)$, $C > 0$, $2 < p < 2^* := 2N/(N-2)$, $N \geqslant 3$;

(H3) $g(x,s)s \geqslant \mu G(x,s) - C$, $C > 0$, $\mu > 2$, where $G(x,s) := \int_0^s g(x,\xi)d\xi$;

(H4) $f(x,s)/s \to 0$ as $s \to 0$ uniformly in x, and $0 \leqslant f(x,s)s \leqslant C(|s|^\nu + 1)$ $\forall s$, for some $C > 0$, $0 < \nu < \mu$.

We will prove

Theorem 6.1. *Assume* (H1)–(H4)*. If moreover*

$$2 < p < \frac{2N\mu}{N\mu - 2\mu + 2\nu}\,, \tag{6.4}$$

then the problem (6.3) *admits a sequence of sign-changing solutions* $(u_{k_n})_{n\in\mathbb{N}}$ *whose energy levels* $J(u_{k_n})$ *satisfy*

$$c_1 k_n^{\frac{2p}{N(p-2)}} \leqslant J(u_{k_n}) \leqslant c_2 k_n^{\frac{2\mu}{N(\mu-2)}}$$

for some $c_1, c_2 > 0$ *independent of* n*, where* J *is the energy functional*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u), \quad u \in H_0^1(\Omega).$$

Furthermore, we have the following information on the number of nodal domains of the sign-changing solutions obtained in Theorem 6.1.

Corollary 6.2. *Under the assumptions of Theorem 6.1, suppose moreover that for a.e.* x *the following partial derivatives exist, are continuous and*

$$g(x,s)s + f(x,s)s < \frac{\partial g}{\partial s}(x,s)s^2 + \frac{\partial f}{\partial s}(x,s)s^2 \leqslant C(|s|^p + 1), \quad \forall s.$$

Then the sign-changing solutions u_{k_n} *can be chosen with the further property that* u_{k_n} *has at most* $k_n + 1$ *nodal domains.*

Clearly, if we set $g(x,u) = |u|^{p-2}u$ then $\mu = p$ and (6.4) reduces to (6.2). Note that we may also replace the constant C in (H2)–(H4) by some coefficient functions as in (B1)–(B2). However, our main concern here consists in showing that this type of problems does admit a sequence of sign-changing solutions.

Since $f(x,u)$ is not assumed to be odd symmetric in u, this kind of semilinear elliptic problems is often referred to as “perturbation from symmetry” problems; here the symmetry of the corresponding functional is completely broken. A long standing open question is whether the symmetry of the functional is crucial for the existence of infinitely many critical points (*cf.* M. Struwe [121, page 118]). Several partial answers have been obtained in the past 30 years. Let us sketch some history. Previously to the work of A. Bahri and P.L. Lions [12] mentioned above, the special case

$$-\Delta u = |u|^{p-2}u + h(x), \quad u \in H_0^1(\Omega), \tag{6.5}$$

was first studied by A. Bahri and H. Berestycki [11] and M. Struwe [120] independently. In [10], A. Bahri considered (6.5) and proved that there is an open dense set of h in $W^{-1,2}(\Omega)$ such that (6.5) has infinitely many solutions if $2 < p < 2N/(N-2)$. In [100, 102], P. Rabinowitz considered (6.3) under the assumption

$$2 < p < \frac{4N\mu - 2N\nu}{2N\mu - 2\mu - N\nu + 2\nu}. \tag{6.6}$$

It can be checked that (6.6) implies (6.4). In [123], K. Tanaka studied (6.3) by Morse index methods under (6.4) with $f(x,u) = f(x)$, $\nu = 1$. In [126], H. Tehrani considered the case of a sign-changing potential. P. Bolle, N. Ghoussoub and H. Tehrani [20] proved existence results for non-homogeneous boundary conditions

$$-\Delta u = |u|^{p-2}u + h(x) \quad \text{in } \Omega, \qquad u = u_0 \quad \text{on } \partial\Omega,$$

where $u_0 \in C^2(\bar{\Omega}, \mathbb{R})$ with $\Delta u_0 = 0$ and $2 < p < \frac{2N}{N-1}$. Y. Long [84] considered periodic solutions of perturbed superquadratic second order Hamiltonian systems. We emphasize that the papers mentioned above are mainly concerned with the existence of infinitely many solutions only. In the past years, this question has raised the attention of other authors; see for example the survey paper [16] as well as the recent work in [35, 37, 38, 109, 114] and their references.

Going back to equation (6.2) under assumptions (B1)–(B2), we observe that the results by Gidas and Spruck [66] yield *a priori* bounds for positive solutions. This fact combined with Bahri-Lions' result immediately provides the existence of infinitely many sign-changing solutions for (6.2). However, this (indirect) argument is quite restrictive and fails to work in a variety of situations. For instance, it does not yield any conclusion for the equation

$$-\Delta u = |u|^{p-2}u(2 + \sin u) + f(x,u)$$

(which can be treated by Theorem 6.1), or for strongly coupled elliptic systems (as the ones considered in Part 2 and Section 6.4), where in general no *a priori* bounds for positive solutions are known (we refer to Section 6.4 for more details).

We hope that our (direct) method of proof provides a better understanding of the structure of the solutions' set of perturbed symmetric elliptic problems. One of the advantages of it is that it is flexible enough to deal with other boundary value problems with variational structure enjoying a maximum principle. We illustrate this by considering the fourth order problem (see also Remark 6.24)

$$\Delta^2 u = g(x,u) + f(x,u) \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{6.7}$$

where $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying (H1)–(H4). For definiteness, we let $N \geqslant 5$, so that now (cf. (H2)) $2 < p < 2^* := 2N/(N-4)$. We need a further restriction on f and g.

(H5) $g(x, s)$ and $f(x, s)$ are nondecreasing in s, for a.e. $x \in \Omega$.

Theorem 6.3. *Under assumptions* (H1)–(H5)*, if moreover*

$$2 < p < \frac{2N\mu}{N\mu - 4\mu + 4\nu} \,,$$

then (6.7) *admits an unbounded sequence of sign-changing solutions* $u_n \in H^2(\Omega) \cap H_0^1(\Omega)$.

The case where the problem is symmetric (namely $f \equiv 0$) was recently studied by Weth [135]. We point out that sign-changing solutions for fourth order equations are harder to exhibit; loosely speaking, this is due to the fact that the usual decomposition $u = u^+ - u^-$, where $u^\pm := \max\{\pm u, 0\}$, is no longer available in the space $H^2(\Omega)$. In the quoted paper [135], Weth was able to bypass this difficulty by using a decomposition method in dual cones.

To prove Theorem 6.1, we first study the corresponding even functional for the unperturbed (symmetric) equation and provide a precise estimate on the lower and upper growth of the Morse index of a sequence of sign-changing solutions (see Theorem 6.4). For that, we will introduce a suitable notion of linking. Based on this information together with a perturbation argument (on level subsets of the energy functional) and a very recent result on odd continuous extensions due to Castro and Clapp [37], we will construct a sequence of critical values with sign-changing critical points. The proof is presented in Section 6.2, while Section 6.3 is devoted to the biharmonic operator. Finally, in Section 6.4 we give an idea of how our ideas can be applied to strongly coupled elliptic systems.

We conclude this introduction by referring the readers to T. Bartsch *et al.* [13, 15, 17, 79] for the study of sign-changing solutions in the symmetric case (even symmetric functionals); see also [135] for the fourth order case (6.7). Particularly, by computing critical groups of the energy functional, in the paper of T. Bartsch, K. C. Chang and Z.-Q. Wang [14] two different estimates for the Morse indices of sign-changing solutions were obtained: one concerns a sign-changing solution of mountain pass type with Morse index less than or equal to one; another one concerns a possibly degenerate critical point having Morse index two. In the present chapter, we will provide Morse index estimates for higher dimension situations.

The content of this chapter is based on the work [108], written in collaboration with M. Ramos and W. Zou.

6.2 Proof of Theorem 6.1

Consider the Hilbert space $H_0^1(\Omega)$ equipped with the norm $\|u\| := (\int_\Omega |\nabla u|^2)^{1/2}$ and inner product $\langle u, v\rangle := \int_\Omega \langle \nabla u, \nabla v\rangle$; also, $\|u\|_p := (\int_\Omega |u|^p)^{1/p}$. In the following we denote $P := \{u \in H_0^1(\Omega) : u \geqslant 0\}$, $\mathcal{P} := P \cup (-P)$ and, for any $\delta > 0$, $\mathcal{P}_\delta := \{u \in H_0^1(\Omega) : \mathrm{dist}(u, \mathcal{P}) < \delta\}$.

6.2.1 The symmetric case

If we let $f \equiv 0$, the functional J reduces to an even functional, denoted

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x, u), \quad u \in H_0^1(\Omega).$$

The critical points of I correspond to the weak solutions of the problem

$$-\Delta u = g(x, u), \qquad u \in H_0^1(\Omega). \tag{6.8}$$

It is known that under the above assumptions (H1)–(H3) this problem admits infinitely many (pairs of) sign-changing solutions (cf. [13, 15]). However, the precise estimate on their Morse indices seems not to have been settled so far. We will need this information in order to prove Theorem 6.1, and this is the content of the following theorem.

Theorem 6.4. *Assume* (H1)–(H3) *and moreover that* $g(x,\cdot)$ *is* C^1 *and*

$$\Big|\frac{\partial g}{\partial s}(x,s)\Big| \leqslant C(1+|s|^{p-2}) \qquad \text{for every } x, s. \tag{6.9}$$

Then for every $k \in \mathbb{N}, k \geqslant 2$, *the problem* (6.8) *has a sign-changing solution* u_k *such that* $m(u_k) \leqslant k \leqslant m^*(u_k)$.

In the above statement, $m(u)$ stands for the Morse index of the critical point u of I, namely

$$m(u) = \sup\{\dim Y:\ Y \subseteq H_0^1(\Omega) \text{ is a subspace such that } I''(u)(\varphi,\varphi) < 0,\ \forall \varphi \in Y \setminus \{0\}\}$$

(observe that under the assumptions of Theorem 6.4, I is a C^2 function). The augmented Morse index is defined as $m^*(u) = m(u) + \dim \operatorname{Ker}(I''(u))$, that is

$$m^*(u) = \sup\{\dim Y:\ Y \subseteq H_0^1(\Omega) \text{ is a subspace such that } I''(u)(\varphi,\varphi) \leqslant 0,\ \forall \varphi \in Y\}.$$

Remark 6.5. Since $I''(u)(\varphi,\psi) = \int_\Omega \langle \nabla\varphi, \nabla\psi\rangle - \frac{\partial g}{\partial s}(x,u)\varphi\psi$, we see that $I''(u) = Id - T_u$, where $T_u : H_0^1(\Omega) \to H_0^1(\Omega)$ is the compact symmetric operator $T_u\varphi = (-\Delta)^{-1}(\frac{\partial g}{\partial s}(x,u)\varphi)$. Then $I''(u)(\varphi,\varphi) = \langle (Id - T_u)\varphi, \varphi\rangle$ and it is not hard to see that $m(u)$ is equal to the number of negative eigenvalues of $I''(u)$, while $m^*(u)$ corresponds to the number of non-positive eigenvalues of $I''(u)$.

We stress that in Theorem 6.4 we assume that g is C^1, but not in Theorem 6.1. In order to prove Theorem 6.4, we need a few lemmas. The following one is already present in the works [15, 43, 44]; however, we recall its proof since we will use a variant of it later on. It shows that a small neighborhood of $\mathcal{P}$ is *positively invariant* for every gradient flow associated with I.

Lemma 6.6. *There exists* $\bar\delta_1 > 0$ *such that, for* $0 < \delta < \bar\delta_1$, *any solution* $\sigma(t,u)$ *of*

$$\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{\nabla I(\sigma(t,u))}{\|\nabla I(\sigma(t,u))\|}, \quad \sigma(0,u) = u,$$

satisfies $\sigma(t,u) \in \mathcal{P}_\delta$ *for all* $u \in \mathcal{P}_\delta$ *and all* $t \geqslant 0$. *Here* $\chi : H_0^1(\Omega) \to [0,1]$ *is any smooth function such that* σ *is well defined in* $\mathbb{R} \times H_0^1(\Omega)$, *and* $\bar\delta_1$ *does not depend on the choice of* χ.

Proof. We show this result for P, the proof for $-P$ follows in an analogous way. We can write $\nabla I = Id - K$ where K is the compact operator in $H_0^1(\Omega)$ given by $v = Ku$ if and only if $-\Delta v = g(x,u)$, $v \in H_0^1(\Omega)$. We claim that P is K–invariant, namely that there exists a small $\bar{\delta} > 0$ such that $K(P_\delta) \subseteq P_{\delta/2}$ for every $0 < \delta < \bar{\delta}$. First of all observe that

$$\mathrm{dist}(u,P) = \min_{w\in P} \|u-w\| \leqslant \|u-u^+\| = \|u^-\|,$$

$$\|u^-\|_s = \min_{w\in P} \|u-w\|_s \leqslant \min_{w\in P} C_s\|u-w\| = C_s \mathrm{dist}(u,P) \qquad \text{for every } s \in (2,2^*),$$

and that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|g(x,s)| \leqslant \varepsilon|s| + C_\varepsilon|s|^{p-1}$ (where the last observation follows from (H1)–(H2)). Thus, if $\mathrm{dist}(u,P) < \delta$ and $v := Ku$ then, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{dist}(v,P)\|v^-\| &\leqslant \|v^-\|^2 = -\int_\Omega g(x,u)v^- \leqslant -\int_\Omega g(x,-u^-)v^- \\
&\leqslant \varepsilon\|u^-\|_2\|v^-\|_2 + C_\varepsilon\|u^-\|_p^{p-1}\|v^-\|_p \\
&\leqslant \varepsilon C \mathrm{dist}(u,P)\|v^-\| + C'_\varepsilon \mathrm{dist}(u,P)^{p-1}\|v^-\|,
\end{aligned}$$

whence $\mathrm{dist}(v,P) < \varepsilon C\delta + C'_\varepsilon \delta^{p-1} \leqslant \delta/2$ by choosing first $\varepsilon \leqslant 1/(4C)$ and afterwards δ such that $C'_\varepsilon \delta^{p-1} < \delta/4$, and the claim follows. As a consequence of that, since

$$\begin{aligned}
\sigma(t,u) &= \sigma(0,u) + t\dot{\sigma}(0,u) + \mathrm{o}(t) \\
&= u - \lambda t \nabla I(u) + \mathrm{o}(t) \qquad \text{as } t \to 0,
\end{aligned}$$

with $\lambda = \chi(u)/\|\nabla I(u)\|$ we see that, for any $u \in P_\delta$,

$$\begin{aligned}
\mathrm{dist}(\sigma(t,u),P) &= \mathrm{dist}(u - \lambda t(u-Ku) + \mathrm{o}(t), P) \\
&= \mathrm{dist}((1-\lambda t)u + \lambda t Ku + \mathrm{o}(t), P) \\
&\leqslant (1-\lambda t)\mathrm{dist}(u,P) + \lambda t \mathrm{dist}(Ku,P) + \mathrm{o}(t) \\
&< (1-\lambda t)\delta + \lambda t\delta/2 + \mathrm{o}(t) \\
&= \delta - \lambda\delta t/2 + \mathrm{o}(t) < \delta
\end{aligned}$$

for sufficiently small $t > 0$, and the conclusion of Lemma 6.6 follows. □

Remark 6.7. In the previous lemma we have shown that $\nabla I = Id - K$, with $K(\mathcal{P}_\delta) \subseteq \mathcal{P}_{\delta/2}$ for $0 < \delta < \bar{\delta}$. Thus in particular $\|\nabla I(u)\| \geqslant \delta/2$ for every $u \in \partial\mathcal{P}_\delta$, so that I has no critical points lying in $\mathcal{P}_{\bar{\delta}} \setminus \mathcal{P}$.

For any $k \in \mathbb{N}$, $k \geqslant 2$, we denote by E_k the space $E_k = \mathrm{span}\{\varphi_1, \ldots, \varphi_k\}$ of dimension k, where φ_i is an eigenfunction corresponding to the i-th eigenvalue of $(-\Delta, H_0^1(\Omega))$. In order to define min-max levels for the functional I which will ultimately provide sign-changing solutions to our problem, we need to find a closed set $S \subset H_0^1(\Omega)$ which intersects $\gamma(B_R(0) \cap E_k)$ far away from the cones P and $-P$, for any large $R > 0$ and any continuous and odd map γ which leaves invariant the boundary of $B_R(0) \cap E_k$. The natural choice would be to take for S the unit sphere in the orthogonal space $E_{k-1}^\perp$; however, $\mathrm{dist}(S,\mathcal{P}) = 0$ for such an S [2]. Besides, finite dimensional reductions do not seem compatible with flow

[2] In fact, take $v_n \geqslant 0$ such that $\|v_n\| = 1$ and $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$. Then $u_n := v_n - P_{E_{k-1}}v_n$ is such that $\mathrm{dist}(u_n,P) \to 0$ and $\|u_n\| \to 1$, whence $\mathrm{dist}(S,P) = 0$.

invariance in the restricted cones $\mathcal{P} \cap E_m$, $m \in \mathbb{N}$. On the other hand, since there is a compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, it is natural to work with the closed set

$$S_k := \{u \in E_{k-1}^{\perp} : \|u\|_p = 1\}.$$

Recall the Gagliardo–Nirenberg inequality

$$\|u\|_p \leqslant C\|u\|_2^{\alpha}\|u\|^{1-\alpha}, \qquad \text{for every } u \in H_0^1(\Omega),$$

where $\alpha \in (0,1)$ is defined by

$$\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*}, \qquad \text{that is } \alpha = \frac{2}{p}\frac{2^*-p}{2^*-2}.$$

Denoting by $(\lambda_k)_{k\in\mathbb{N}}$ the non-increasing sequence of eigenvalues of the operator $(-\Delta, H_0^1(\Omega))$, we see that, for every $u \in S_k$,

$$\lambda_k^{\alpha/2} \leqslant C\|u\| \qquad \text{and} \qquad I(u) \geqslant \frac{1}{2}\|u\|^2 - C\Big(1 + \int_\Omega u^p\Big) = \frac{1}{2}\|u\|^2 - 2C \geqslant C_1\lambda_k^{\alpha} - C_2.$$

Therefore

$$\inf_{S_k} I \geqslant C_1\lambda_k^{\frac{2}{p}\frac{2^*-p}{2^*-2}} - C_2 \to +\infty \qquad \text{as } k \to \infty,$$

and in particular we can fix a constant $c_0 > 0$ (independent of k) such that

$$\inf_{S_k} I > -c_0, \qquad \forall k.$$

For a given positive constant R_k, we denote

$$Q_k := B_{R_k}(0) \cap E_k, \qquad \partial Q_k := \partial B_{R_k}(0) \cap E_k.$$

Since $\sup_{E_k \setminus B_R(0)} I \to -\infty$ as $R \to +\infty$, we can fix R_k so large that

$$\sup_{\partial Q_k} I < -c_0 \qquad \text{and} \qquad \inf\{\|u\|_p : u \in \partial Q_k\} > 1.$$

We also fix any number

$$M_k > \sup_{E_k} I.$$

Lemma 6.8. *There exists $\mu_k > 0$ such that* $\operatorname{dist}(u, \mathcal{P}) \geqslant 2\mu_k$ *whenever $u \in S_k$ and $I(u) \leqslant M_k$.*

Proof. Assuming the contrary we find a sequence $(u_n) \subset S_k$ such that $I(u_n) \leqslant M_k$ and $\operatorname{dist}(u_n, \mathcal{P}) \to 0$. The sequence $(\|u_n\|_p)_n$ is bounded and $(I(u_n))_n$ is bounded from above, whence $(||u_n||)_n$ is also bounded. Since $E_{k-1}^{\perp} \cap \mathcal{P} = \{0\}$, this implies that, up to a subsequence, $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Using the compact imbedding $H_0^1(\Omega) \subset L^p(\Omega)$, this contradicts the fact that $\|u_n\|_p = 1$ for every n. $\square$

Let μ_k be given by Lemma 6.8 and let us set

$$\mathcal{U}_k := \{u \in H_0^1(\Omega) : \mathrm{dist}(u, \mathcal{P}) \geqslant \mu_k\}.$$

By possibly taking a smaller μ_k, we can assume that the conclusion of Lemma 6.6 applies to μ_k ($\mu_k \leqslant \bar{\delta}_1$). We denote

$$\Gamma_k := \{\gamma \in C(Q_k, H_0^1(\Omega)) : \ \gamma \text{ is odd }, \gamma|_{\partial Q_k} = Id, \sup_{\gamma(Q_k)} I < M_k\},$$

and

$$c_k := \inf_{\gamma \in \Gamma_k} \sup_{\gamma(Q_k) \cap \mathcal{U}_k} I.$$

Observe that $\Gamma_k \neq \emptyset$, as $Id \in \Gamma_k$. We prove that c_k is a critical value for I which corresponds to a sign-changing solution of our original problem.

Proposition 6.9. *Under the assumptions of Theorem 6.4, for any $k \in \mathbb{N}, k \geqslant 2$, there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = c_k$, $I'(u_k) = 0$ and $m(u_k) \leqslant k$. Moreover,* $\inf_{S_k} I \leqslant c_k < M_k$.

Proof. Since $Id \in \Gamma_k$, it is clear that

$$c_k \leqslant \sup_{Q_k \cap \mathcal{U}_k} I \leqslant \sup_{E_k} I < M_k.$$

On the other hand, given $\gamma \in \Gamma_k$ we consider the map

$$D := \{u \in Q_k : \|\gamma(u)\|_p < 1\} \longrightarrow E_{k-1}, \quad u \mapsto P_{E_{k-1}}(\gamma(u)),$$

where $P_{E_{k-1}}$ denotes de orthogonal projection of $H_0^1(\Omega)$ onto E_{k-1}. The set D is a bounded, symmetric neighborhood of the origin in E_k and so, according to the Borsuk-Ulam theorem, its boundary contains a point u such that $P_{E_{k-1}}(\gamma(u)) = 0$, *i.e.* $\gamma(u) \in E_{k-1}^{\perp}$. Our choice of the constant R_k implies that $u \notin \partial Q_k$ and so $\|\gamma(u)\|_p = 1$, and $\gamma(u) \in S_k$. Since moreover $I(\gamma(u)) \leqslant \sup_{\gamma(Q_k)} I < M_k$, we deduce from Lemma 6.8 that $\gamma(u) \in S_k \cap \mathcal{U}_k$. This shows that

$$c_k \geqslant \inf_{S_k} I.$$

Let us now show that c_k is indeed a critical value corresponding to a critical point in $\mathcal{U}_k$. In view of a contradiction, suppose there exists $0 < \varepsilon \leqslant \frac{1}{2}(c_k + c_0)$ such that

$$\|\nabla I(u)\| \geqslant 2\varepsilon, \quad \forall u : |I(u) - c_k| \leqslant 2\varepsilon, \mathrm{dist}(u, \mathcal{P}) \geqslant \mu_k. \tag{6.10}$$

Then we consider two closed, symmetric and disjoint sets

$$\begin{aligned} A &:= \{u : \nabla I(u) = 0\} \cup \{u : |I(u) - c_k| \geqslant 2\varepsilon\} \cup \{u : \mathrm{dist}(u, \mathcal{P}) \leqslant \mu_k/2\}, \\ B &:= \{u : |I(u) - c_k| \leqslant \varepsilon, \ \mathrm{dist}(u, \mathcal{P}) \geqslant \mu_k\}, \end{aligned}$$

together with a smooth, even cut-off function $\chi : H_0^1(\Omega) \to [0, 1]$ such that $\chi = 0$ in A and $\chi = 1$ in B. According to Lemma 6.6, this gives rise to a flow $\sigma : \mathbb{R} \times H_0^1(\Omega) \to H_0^1(\Omega)$, solution of

$$\frac{d}{dt}\sigma(t, u) = -\chi(\sigma(t, u)) \frac{\nabla I(\sigma(t, u))}{\|\nabla I(\sigma(t, u))\|}, \quad \sigma(0, u) = u,$$

for which $\mathcal{P}_{\mu_k}$ is positively invariant.

Let us take any $\gamma \in \Gamma_k$ such that $\sup_{\gamma(Q_k)\cap\mathcal{U}_k} I \leqslant c_k + \varepsilon$. Denoting $\sigma_1(u) := \sigma(1, u)$, we claim that $\sigma_1 \circ \gamma \in \Gamma_k$. Indeed, σ_1 is odd since χ is even and ∇I is odd, and hence $\sigma_1 \circ \gamma \in C(Q_k, H_0^1(\Omega))$ is an odd function. Moreover, for every $u \in \partial Q_k$ we have

$$I(u) \leqslant \sup_{\partial Q_k} I < -c_0 \leqslant c_k - 2\varepsilon$$

and hence $\chi(u) = 0$ and $\sigma_1(u) = u$, whence $\sigma_1 \circ \gamma|_{\partial Q_k} = Id$. Finally, it is easy to see that $t \mapsto I(\sigma_1(t, u))$ is a non-increasing map, and therefore

$$\sup_{(\sigma_1\circ\gamma)(Q_k)} I \leqslant \sup_{\gamma(Q_k)} I < M_k.$$

Thus our claim follows and

$$c_k \leqslant \sup_{(\sigma_1\circ\gamma)(Q_k)\cap\mathcal{U}_k} I.$$

Next, we show that

$$\sup_{(\sigma_1\circ\gamma)(Q_k)\cap\mathcal{U}_k} I \leqslant c_k - \varepsilon,$$

which provides a contradiction. Let $u \in Q_k$ with $(\sigma_1 \circ \gamma)(u) \in \mathcal{U}_k$ be such that

$$\sup_{(\sigma_1\circ\gamma)(Q_k)\cap\mathcal{U}_k} I = I(\sigma_1(\gamma(u))).$$

Since $(\sigma_1 \circ \gamma)(u) \in \mathcal{U}_k$, by the invariance property of the flow (Lemma 6.6) we must have that $\gamma(u) \in \mathcal{U}_k$, and in fact $\sigma(t, \gamma(u)) \in \mathcal{U}_k$ for every $t \in [0, 1]$. Moreover,

$$c_k \leqslant I((\sigma_1 \circ \gamma)(u)) \leqslant I(\sigma(t, \gamma(u))) \leqslant I(\gamma(u)) \leqslant \sup_{\gamma(Q_k)\cap\mathcal{U}_k} I \leqslant c_k + \varepsilon \qquad \text{for every } t \in [0, 1].$$

Thus $\chi(\sigma(t, \gamma(u)) = 1$ for every $t \in [0, 1]$ and so

$$\begin{aligned} \frac{d}{dt} I(\sigma(t, \gamma(u)) &= \left\langle \nabla I(\sigma(t, \gamma(u))), -\chi(\sigma(t, \gamma(u))) \frac{\nabla I(\sigma(t, \gamma(u)))}{\|\nabla I(\sigma(t, \gamma(u)))\|} \right\rangle \\ &= -\|\nabla I(\sigma(t, \gamma(u)))\| \leqslant -2\varepsilon, \end{aligned}$$

and

$$I((\sigma_1 \circ \gamma)(u)) \leqslant I(\gamma(u)) - \int_0^1 2\varepsilon \, dt \leqslant c_k + \varepsilon - 2\varepsilon = c_k - \varepsilon.$$

Thus (6.10) implies a contradiction and therefore, for some sequence $\varepsilon_n \to 0$, we can find a Palais-Smale sequence $(u_n) \subseteq H_0^1(\Omega)$ such that

$$I(u_n) \to c_k, \qquad I'(u_n) \to 0 \qquad \text{and} \qquad \operatorname{dist}(u_n, \mathcal{P}) \geqslant \mu_k.$$

It is standard to check that I satisfies the Palais-Smale condition, namely that, up to a subsequence, $u_n \to u$ in $H_0^1(\Omega)$ for some $u \in H_0^1(\Omega)$. In particular, $I(u) = c_k$, $I'(u) = 0$ and $\operatorname{dist}(u, \mathcal{P}) \geqslant \mu_k > 0$, as claimed.

As for the information on the Morse index, let C be the (non empty) compact symmetric set $C := \{u \in \mathcal{U}_k : I(u) = c_k, I'(u) = 0\}$. Using the symmetric version of Marino-Prodi's perturbation method (cf. [129] or [103, Theorem 12.7 & Corollary 12.8]), we may assume that C consists of a finite number of non-degenerate critical points of I. Since Q_k is contained in a vector space of dimension k, classical arguments such as the ones in e.g. [12, 39, 77, 105, 119, 123] immediately yield the conclusion that some point $u \in C$ must have Morse index less than or equal to k. $\square$

Next we will be concerned with the problem of finding sign-changing solutions having (augmented) Morse index greater than or equal to k. The sets $S_k, \partial Q_k$ and the constant M_k were defined above and we introduce now the corresponding notion of linking.

Definition 6.10. *Given $A \subset H_0^1(\Omega)$, we say that A and S_k* link *if A is compact, symmetric, $\partial Q_k \subset A$, $\sup_A I < M_k$, and moreover $\eta(A) \cap S_k \neq \emptyset$ for every odd and continuous map $\eta : A \to H_0^1(\Omega)$ such that $\eta|_{\partial Q_k} = Id$.*

Let
$$c_k^* := \inf_{A \in \mathcal{L}_k} \sup_{A \cap \mathcal{U}_k} I,$$
where $\mathcal{L}_k := \{A \subset H_0^1(\Omega) : A \text{ and } S_k \text{ link }\}$. We have
$$\inf_{S_k} I \leqslant c_k^* \leqslant c_k < M_k.$$
In fact, for any $A \in \mathcal{L}_k$ we see that $A \cap S_k \neq \emptyset$ (by taking $\eta = Id$ in Definition 6.10) and, thanks to Lemma 6.8, $A \cap S_k \subseteq \mathcal{U}_k$; hence $\inf_{S_k} I \leqslant c_k^*$. Moreover, it is easy to see that $\gamma(Q_k)$ and S_k link for every $\gamma \in \Gamma_k$, and thus $c_k^* \leqslant c_k < M_k$.

This allows us to derive a lower bound on the Morse index of a sequence of solutions of our problem, much in the spirit of the abstract results in [77, 105, 119]. We recall here a particular case of [105, Definition 2.4].

Definition 6.11. *We denote by $m(S_k, H_0^1(\Omega))$ the supremum of the integers $m \in \mathbb{N}_0$ such that the following holds:*

> *Given compact sets $C \subseteq \tilde{C}$ of $\mathbb{R}^m$, every continuous map $h : C \to H_0^1(\Omega) \setminus S_k$ admits a continuous extension $H : \tilde{C} \to H_0^1(\Omega) \setminus S_k$.*

Proposition 6.12. *Under the assumptions of Theorem 6.4, for any $k \in \mathbb{N}$, $k \geqslant 2$, there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = c_k^*$, $I'(u_k) = 0$ and $m^*(u_k) \geqslant k$.*

Proof. By taking into account our considerations in the proof of Proposition 6.9, we can apply [105, Theorem 2.5], which in our context states that $m^*(u_k) > m(S_k, H_0^1(\Omega))$. Thus it remains to show that $m(S_k, H_0^1(\Omega)) \geqslant k-1$ which, in turn, can be proved by a slight modification in [105, Proposition 3.1]. Let $C \subseteq \tilde{C}$ be two compact sets of $\mathbb{R}^{k-1}$ and let $h : C \to H_0^1(\Omega) \setminus S_k$ be a continuous map. Take a continuous extension of h, namely
$$\tilde{h} : \tilde{C} \to H_0^1(\Omega),$$
which we write as
$$\tilde{h}(x) = \alpha(x) + \beta(x) \in E_{k-1} \oplus E_{k-1}^{\perp},$$
where $\alpha(x), \beta(x)$ denote the projections of $\tilde{h}(x)$ onto E_{k-1} and $E_{k-1}^{\perp}$, respectivelly. Observing that $\tilde{h}(x) \in S_k$ if and only if $\alpha(x) = 0$ and $\|\beta(x)\|_p = 1$, we let
$$F := \{x \in \tilde{C} : \alpha(x) = 0 \text{ and } \|\beta(x)\|_p = 1\}.$$

By assumption, F is a compact set disjoint from C. We can thus choose an ε-neighborhood F_ε of F in such a way that $C \cap F_\varepsilon = \emptyset$ and $\beta(x) \neq 0\ \forall x \in F_\varepsilon$. Since $F_\varepsilon \subset \mathbb{R}^{k-1}$ and $\dim(E_{k-1} \times \mathbb{R}) = k$, the map

$$\partial F_\varepsilon \to (E_{k-1} \times \mathbb{R})\backslash\{(0,1)\}, \quad x \mapsto (\alpha(x), \|\beta(x)\|_p)$$

admits a continuous extension $F_\varepsilon \to (E_{k-1} \times \mathbb{R})\backslash\{(0,1)\}$, say $x \mapsto (\tilde{\alpha}(x), \rho(x))$. The desired extension map

$$H : \tilde{C} \to H_0^1 \setminus S_k$$

is then given by

$$H(x) = \begin{cases} \tilde{\alpha}(x) + \frac{\rho(x)}{\|\beta(x)\|_p}\beta(x), & \text{if } x \in F_\varepsilon \\ \tilde{h}(x), & \text{if } x \notin F_\varepsilon, \end{cases}$$

and this completes the proof of Proposition 6.12. □

Proof of Theorem 6.4 completed. Following an idea introduced in [77, 119], we restrict further the class $\mathcal{L}_k$ by setting[3]

$$\tilde{\mathcal{L}}_k := \{A \in \mathcal{L}_k : \mathscr{H}_{\dim}(A) \leqslant k\} \quad \text{and} \quad \tilde{c}_k := \inf_{A \in \tilde{\mathcal{L}}_k} \sup_{A \cap \mathcal{U}_k} I.$$

We have

$$c_k^* \leqslant \tilde{c}_k \leqslant c_k.$$

In fact, the inclusion $\tilde{\mathcal{L}}_k \subseteq \mathcal{L}_k$ readily implies that $c_k^* \leqslant \tilde{c}_k$. Since $\mathscr{H}_{\dim}(\eta(A)) \leqslant \mathscr{H}_{\dim}(A)$ whenever η is a Lipschitz map, the inequality $\tilde{c}_k \leqslant c_k$ follows from the facts that any odd continuous map in Q_k can be approximated by odd and Lipschitz continuous ones, and $Q_k \in \tilde{\mathcal{L}}_k$. Now, it follows from [77, Theorem 2.6] (see also [105, Theorem 2.9] or [119, Theorem 3]) combined with our previous arguments that there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = \tilde{c}_k$, $I'(u_k) = 0$ and $m(u_k) \leqslant k \leqslant m^*(u_k)$. □

Our proof of Theorem 6.1 will need a variant of Propositions 6.9 and 6.12 in which we consider a slight perturbation of the previous min-max levels (see Proposition 6.14 ahead). We anticipate that this is due to the fact that the extension theorem in [37] (which we state in Lemma 6.17) cannot be applied directly to the functionals J or I, but rather to the functional I_0 defined below.

To be precise, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H1)-(H3), and let us fix any number $2 < q < \mu$ and a corresponding functional

$$I_0(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q}\int_\Omega |u|^q, \quad u \in H_0^1(\Omega).$$

In the sequel, we also fix a small number λ_0 in such a way that $0 < \lambda_0 < (\mu - 2)/2$. The space $E_k = \operatorname{span}\{\varphi_1, ..., \varphi_k\}$ and its subset $Q_k = B_{R_k}(0) \cap E_k$ have been defined previously, as well as the closed set $\mathcal{U}_k = \{u \in H_0^1(\Omega) : d(u, \mathcal{P}) \geqslant \mu_k\}$ for some $\mu_k > 0$. For a given $M_k > 0$, we introduce the number

$$d_k := \inf_{\gamma \in \Gamma_k^1} \sup_{\gamma(Q_k) \cap \mathcal{U}_k} I$$

[3] Here $\mathscr{H}_{\dim}(A)$ denotes the Hausdorff dimension of the set A.

with

$$\Gamma_k^1 = \{\gamma \in C(Q_k, H_0^1(\Omega)) : \ \gamma \text{ is odd }, \gamma|_{\partial Q_k} = Id, \sup_{\gamma(Q_k)} I_1 < M_k\},$$

where

$$I_1(u) := I^+(u) + \lambda_0 I_0(u), \quad u \in H_0^1(\Omega), \qquad \text{and} \qquad I^+(u) := \max\{I(u), 0\}.$$

Now, we need a more general version of Lemma 6.6 involving both I and I_1.

Lemma 6.13. *There exists $\bar{\delta}_2 > 0$ such that, for $0 < \delta < \bar{\delta}_2$, any solution of*

$$\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{V(\sigma(t,u))}{\|V(\sigma(t,u))\|}, \qquad \sigma(0,u) = u, \tag{6.11}$$

with

$$V(u) = \frac{1}{2}\frac{\nabla I(u)}{\|\nabla I(u)\|} + \frac{1}{2}\frac{\theta(u)}{\|\nabla I(u)\|}\frac{\nabla I_1(u)}{\|\nabla I_1(u)\|}, \quad \nabla I(u), \nabla I_1(u) \neq 0,$$

satisfies $\sigma(t,u) \in \mathcal{P}_\delta$ for all $u \in \mathcal{P}_\delta$ and all $t \geqslant 0$. Here $\chi, \theta : H_0^1(\Omega) \to [0,1]$ are any smooth functions such that σ is well defined in $\mathbb{R} \times H_0^1(\Omega)$ and moreover $\theta(u) = 0$ whenever $I(u) \leqslant 0$.

Proof. We recall that $\nabla I(u) = u - Ku$, where $v := Ku$ is such that $-\Delta v = g(x,u)$. Moreover, since $\theta(u) = 0$ over the set $\{u : I(u) \leqslant 0\}$, we have $\theta(u)\nabla I_1(u) = \theta(u)(\nabla I(u) + \lambda_0 \nabla I_0(u)) = \theta(u)((1+\lambda_0)u + \bar{K}u)$, where $v := \bar{K}u$ is defined by $-\Delta v = g(x,u) + \lambda_0 |u|^{q-1}u$, $v \in H_0^1(\Omega)$. We can find a $\bar{\delta} > 0$ such that $K(\mathcal{P}_\delta) \subseteq \mathcal{P}_{\delta/2}$ and $\bar{K}(\mathcal{P}_\delta) \subseteq \mathcal{P}_{\delta/2}$ for every $\delta < \bar{\delta}$, and the rest of the proof follows exactly as the one of Lemma 6.6. □

Proposition 6.14. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (H1)–(H3). *With the above notations, suppose $M_k > (\sup_{E_k} I_0)^2$. Then, provided k is sufficiently large, for some choice of μ_k (arbitrary small) there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = d_k$ and $I'(u_k) = 0$. Moreover,*

$$d_k \geqslant ck^{2p/N(p-2)}, \qquad \textit{for some } c > 0 \textit{ independent of } k \textit{ and } \mu_k.$$

Proof. 1. Since $q < \mu$ and Ω is a bounded set, we have that

$$I(u) \leqslant I_0(u) + C_0, \ u \in H_0^1(\Omega), \qquad \text{for some fixed } C_0 > 0. \tag{6.12}$$

Therefore

$$\sup_{Q_k} I_1 \leqslant \sup_{Q_k} I + \lambda_0 \sup_{Q_k} I_0 \leqslant (1+\lambda_0)\sup_{E_k} I_0 + C_0 < M_k$$

provided k is sufficiently large. Thus $Id \in \Gamma_k^1$ and d_k is well defined for large k. By reasoning as in Lemma 6.8, we may choose $0 < \mu_k \leqslant \bar{\delta}_2$ (where $\bar{\delta}_2$ is the constant defined in Lemma 6.13) such that

$\operatorname{dist}(u, \mathcal{P}) \geqslant 2\mu_k$ *whenever $u \in S_k$ and $I_1(u) = I^+(u) + \lambda_0 I(u) \leqslant M_k$.*

2. It follows also as in Proposition 6.9 that $d_k \geqslant \inf_{S_k} I \to +\infty$ as $k \to \infty$. In order to prove that d_k is indeed a critical value for I we must be able to build a flow for which both

the sublevel set $\{I_1 = I^+ + \lambda_0 I_0 < M_k\}$ and the cone $\mathcal{P}$ are invariant. In order to do this, we will use an argument suggested by the one in [86], in a rather different context. Let

$$D_k := \{u \in H_0^1(\Omega) : |I(u) - d_k| \leqslant 1,\ |I_1(u) - M_k| \leqslant 1\}.$$

We claim that

$$\alpha \nabla I(u) + \beta \nabla I_1(u) \neq 0, \qquad \forall u \in D_k,\ \alpha, \beta \geqslant 0,\ \alpha^2 + \beta^2 = 1. \tag{6.13}$$

Observe that for large k one has $I(u) \geqslant d_k - 1 > 0$ for every $u \in D_k$, and hence $I_1(u) = I(u) + \lambda_0 I_0(u)$ and $\nabla I_1(u) = \nabla I(u) + \lambda_0 \nabla I_0(u)$. With (6.13) in mind, let us first show that

$$\nabla I(u) + \lambda \nabla I_0(u) \neq 0, \qquad \forall u \in D_k,\ 0 \leqslant \lambda \leqslant \lambda_0. \tag{6.14}$$

In case $\nabla I(u) + \lambda \nabla I_0(u) = 0$, by multiplying the equation by u we see that

$$(1 + \lambda_0)\|u\|^2 \geqslant (1 + \lambda)\|u\|^2 = \int_\Omega g(x, u)u + \lambda \int_\Omega |u|^q \geqslant \mu \int_\Omega G(x, u) - C_1$$

and hence

$$\int_\Omega G(x, u) \leqslant \frac{1 + \lambda_0}{\mu}\|u\|^2 + \frac{C_1}{\mu}.$$

Since

$$\frac{1}{2}\|u\|^2 - \int_\Omega G(x, u) = I(u) \leqslant d_k + 1,$$

this leads to

$$0 < \Big(\frac{1}{2} - \frac{1 + \lambda_0}{\mu}\Big)\|u\|^2 \leqslant C(d_k + 1) \qquad \text{for some } C > 0,$$

by the choice of λ_0. Since, by assumption,

$$d_k \leqslant \sup_{E_k} I \leqslant \sup_{E_k} I_0 + C_0 \leqslant 2\sqrt{M_k} \qquad \text{for large } k,$$

we get $\|u\|^2 \leqslant C'\sqrt{M_k}$, which is incompatible with

$$M_k - 1 \leqslant I_1(u) = I(u) + \lambda_0 I_0(u) \leqslant \frac{1 + \lambda_0}{2}\|u\|^2,$$

for large k. This proves (6.14), which in turn implies that for every $\alpha, \beta \geqslant 0$,

$$\alpha \nabla I(u) + \beta \nabla I_1(u) = (\alpha + \beta)\Big(\nabla I(u) + \frac{\lambda_0 \beta}{\alpha + \beta}\nabla I_0(u)\Big) \neq 0,$$

and thus our claim (6.13) follows.
3. From the previous considerations we get the existence of $\theta_k \in [0, 1)$ such that

$$\inf_{u \in D_k} \frac{\langle \nabla I(u), \nabla I_1(u)\rangle}{\|\nabla I(u)\|\|\nabla I_1(u)\|} > -\theta_k.$$

Indeed, if this condition is violated then we can find a sequence $(u_n) \subset D_k$ such that

$$\frac{\langle \nabla I(u_n), \nabla I_1(u_n)\rangle}{\|\nabla I(u_n)\|\|\nabla I_1(u_n)\|} \to -1$$

and hence we see that

$$\|v_n\|^2 \to 0, \qquad \text{for } v_n := \frac{\nabla I(u_n)}{\|\nabla I(u_n)\|} + \frac{\nabla I_1(u_n)}{\|\nabla I_1(u_n)\|}.$$

Since $d_k - 1 \leqslant I(u_n)$, $\lambda_0 I_0(u_n) \leqslant I_1(u_n) \leqslant M_k + 1$ and $q < \mu$, it follows easily that $(\|u_n\|)_n$ is bounded and therefore, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. In particular, also

$$\left\| \frac{\|\nabla I(u_n)\|\|\nabla I_1(u_n)\|}{\sqrt{\|\nabla I(u_n)\|^2 + \|\nabla I_1(u_n)\|^2}} v_n \right\|^2 \leqslant C\|v_n\|^2 \to 0,$$

that is

$$\alpha_n \nabla I(u_n) + \beta_n \nabla I_1(u_n) \to 0, \tag{6.15}$$

where $u_n \in D_k$, and

$\alpha_n = \|\nabla I_1(u_n)\| / \sqrt{\|\nabla I(u_n)\|^2 + \|\nabla I_1(u_n)\|^2}$, $\beta_n = \|\nabla I(u_n)\| / \sqrt{\|\nabla I(u_n)\|^2 + \|\nabla I_1(u_n)\|^2}$

satisfy $\alpha_n, \beta_n \geqslant 0$, $\alpha_n^2 + \beta_n^2 = 1$. Multiplying (6.15) by $u_n - u$ yields that actually $u_n \to u$ strongly in $H_0^1(\Omega)$. Therefore, in this way we find $u \in D_k$ such that $\alpha \nabla I(u) + \beta \nabla I_1(u) = 0$ for some $\alpha, \beta \geqslant 0$, $\alpha^2 + \beta^2 = 1$, which we already proved to be impossible (see (6.13)). This establishes the existence of the number $\theta_k \in [0, 1)$ mentioned above.

4. Now, assume first that g is smooth enough, so that ∇I is locally Lipschitz continuous. In this case we consider the vector field

$$V(u) := \frac{1}{2}\frac{\nabla I(u)}{\|\nabla I(u)\|^2} + \frac{1}{2}\frac{\theta(u)}{\|\nabla I(u)\|}\frac{\nabla I_1(u)}{\|\nabla I_1(u)\|}, \quad u \in H_0^1(\Omega), \nabla I(u) \neq 0,$$

where $\theta : H_0^1(\Omega) \to [0, \theta_k]$ is a cut-off function such that $\theta(u) = \theta_k$ if $u \in D_k$ and $\theta(u) = 0$ if u lies in a closed small neighborhood of the set $\{u : \nabla I_1(u) = 0\} \cup \{u : I(u) \leqslant 0\}$. This is easily shown to be a pseudo-gradient vector for I, namely

$$0 < \frac{1 - \theta_k}{2} \leqslant \langle V(u), \nabla I(u)\rangle \leqslant \|V(u)\|\|\nabla I(u)\| < 1, \quad \forall u \in H_0^1(\Omega), \nabla I(u) \neq 0,$$

and moreover

$$\langle V(u), \nabla I_1(u)\rangle > 0, \quad \forall u \in D_k.$$

5. Let us now show the existence of $u_k \in \mathcal{U}_k$ such that $I(u_k) = d_k$ and $I'(u_k) = 0$. In view of a contradiction, suppose there exists $0 < \varepsilon < \min\{\frac{1}{2}(d_k + c_0), \frac{1}{2}\}$ such that

$$\|\nabla I(u)\| \geqslant 2\varepsilon, \qquad \forall u : \ |I(u) - d_k| \leqslant 2\varepsilon, \ \mathrm{dist}(u, \mathcal{P}) \geqslant \mu_k.$$

Consider the sets

$$\begin{aligned} A &:= \{u : \nabla I(u) = 0\} \cup \{u : |I(u) - d_k| \geqslant 2\varepsilon\} \cup \{u : \mathrm{dist}(u, \mathcal{P}) \leqslant \mu_k/2\} \\ &\qquad \cup \{u : I_1(u) \geqslant M_k + 2\}, \\ B &:= \{u : |I(u) - d_k| \leqslant \varepsilon, \ \mathrm{dist}(u, \mathcal{P}) \geqslant \mu_k, \ I_1(u) \leqslant M_k + 1\}, \end{aligned}$$

and let $\chi : H_0^1(\Omega) \to [0,1]$ be an even smooth cut-off function such that $\chi = 0$ in A and $\chi = 1$ in B, and take $\sigma(t,u)$ to be the associated flow defined by (6.11). Define $\sigma_{\bar t}(u) = \sigma(\bar t, u)$ with $\bar t = (1-\theta_k)/2$.

Let us take $\gamma \in \Gamma_k^1$ such that

$$\sup_{\gamma(Q_k)\cap\mathcal{U}_k} I \leqslant d_k + \varepsilon.$$

We claim that $\sigma_{\bar t}\circ\gamma \in \Gamma_k^1$. Indeed, $\sigma_{\bar t}\circ\gamma$ is an odd and continuous function, and for every $u \in \partial Q_k$ we have

$$\sup_{\partial Q_k} I < -c_0 < d_k - 2\varepsilon,$$

which implies that $\chi(u) = 0$ and $\sigma_{\bar t}\circ\gamma(u) = \sigma_{\bar t}(u) = u$ for every $u \in \partial Q_k$. Finally, suppose by contradiction the existence of $t > 0$ and $u \in Q_k$ such that $I_1(\sigma(t,\gamma(u))) = M_k$. For such u, let t^* be the smallest t satisfying that property. Then either $\sigma(t^*,\gamma(u)) \in \mathrm{int}(D_k)$ and

$$\frac{d}{dt} I_1(\sigma(t,\gamma(u)) = -\chi(\sigma(t,\gamma(u)))\Big\langle \nabla I_1(\sigma(t,\gamma(u))), V(\sigma(t,\gamma(u)))\Big\rangle \leqslant 0$$

for t in a neighborhood of t^* (contradicting the definition of t^*), or $\sigma(t^*,\gamma(u)) \notin \mathrm{int}(D_k)$ and $|I(\sigma(t^*,\gamma(u))) - d_k| \geqslant 1 > 2\varepsilon$. In the latter case we have $\chi(\sigma(t,\gamma(u))) = 0$ for t in a small neighborhood of t^*, and hence $M_k = I_1(\sigma(t^*,\gamma(u))) = I_1(\sigma(t,\gamma(u)))$ for t close to t^*, again a contradiction. Thus, in fact

$$\sup_{(\sigma_{\bar t}\circ\gamma)(Q_k)} I_1 < M_k, \qquad \text{and in particular} \qquad \sigma_{\bar t}\circ\gamma \in \Gamma_k^1.$$

Therefore,

$$d_k \leqslant \sup_{(\sigma_{\bar t}\circ\gamma)(Q_k)\cap\mathcal{U}_k} I.$$

Let $u \in Q_k$ with $(\sigma_{\bar t}\circ\gamma)(u) \in \mathcal{U}_k$ be such that

$$\sup_{(\sigma_{\bar t}\circ\gamma)(Q_k)\cap\mathcal{U}_k} I = I((\sigma_{\bar t}\circ\gamma)(u)).$$

From the positively invariance of $\mathcal{P}_{\mu_k}$ with respect to σ we see that $\gamma(u) \in \mathcal{U}_k$ and $\sigma(t,\gamma(u)) \in \mathcal{U}_k$ for every $t \in [0,\bar t]$. Moreover,

$$d_k \leqslant I((\sigma_{\bar t}\circ\gamma)(u)) \leqslant I(\sigma(t,\gamma(u))) \leqslant I(\gamma(u)) \leqslant d_k + 1$$

and (as we have seen before)

$$\sup_{\sigma(t,\gamma(Q_k))} I_1 < M_k < M_k + 1.$$

Therefore $\chi(\sigma(t,\gamma(u))) = 1$ for every $t \in [0,\bar t]$, and in particular

$$\begin{aligned}\frac{d}{dt} I(\sigma(t,\gamma(u))) &= -\frac{1}{\|V(\sigma(t,\gamma(u)))\|}\Big\langle V(\sigma(t,\gamma(u))), \nabla I(\sigma(t,\gamma(u)))\Big\rangle \\ &\leqslant -\frac{2}{1-\theta_k}\|\nabla I(\sigma(t,\gamma(u)))\| \leqslant -\frac{2}{1-\theta_k}2\varepsilon\end{aligned}$$

Hence

$$I(\sigma_{\bar{t}} \circ \gamma(u)) \leqslant I(\gamma(u)) - 2\varepsilon \leqslant d_k - \varepsilon,$$

a contradiction, and thus $\mathcal{U}_k$ contains a critical point of I at level d_k.
6. In the general case where g is merely assumed to be a Carathéodory function, the map $K : H_0^1(\Omega) \to H_0^1(\Omega)$ mentioned in the proof of Lemma 6.6 needs not to be locally Lipschitz continuous; however, given $\eta > 0$ we can find such a map $K_\eta : H_0^1(\Omega) \to H_0^1(\Omega)$ such that $\|K_\eta(u) - K(u)\| \leqslant \eta \ \forall u \in H_0^1(\Omega)$. Then, since $\|\nabla I\|$ is bounded both from below and from above in the set $H_0^1(\Omega) \setminus A$, provided η is small enough we can define $V(u)$ with a similar expression as above, with ∇I (resp. ∇I_1) replaced by $W_\eta = \nabla I + K - K_\eta$ (resp. $W_\eta^1 = \nabla I_1 + K - K_\eta$).
7. As for the final conclusion in Proposition 6.14, let

$$\bar{I}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\kappa}{p}\int_\Omega |u|^p, \quad u \in H_0^1(\Omega),$$

so where we choose κ in such a way that there exists $C_0 > 0$ such that,

$$\bar{I} \leqslant I + C_0 \quad \text{and} \quad \{I^+ + \lambda_0 I_0 < M_k\} \subset \{\bar{I} < M_k\},$$

with $\bar{M}_k := 2C_0 + M_k/\lambda_0$. This implies that $d_k \geqslant \bar{b}_k - C_0$, where

$$\bar{b}_k := \inf\Big\{ \sup_{\gamma(Q_k)\cap\mathcal{U}_k} \bar{I} : \ \gamma \in C(Q_k, H_0^1(\Omega)) \text{ is an odd map }, \gamma|_{\partial Q_k} = Id, \sup_{\gamma(Q_k)} \bar{I} < \bar{M}_k \Big\}$$

By eventually taking a smaller μ_k (more precisely in such a way that Lemma 6.8 holds for the sublevel $\{\bar{I} < \bar{M}_k\}$ and that Lemma 6.6 holds with $\bar{I}$ in the place of I), we can apply Proposition 6.12 and show the existence of a level $\bar{b}_k^* \leqslant \bar{b}_k$ and $w_k \in \mathcal{U}_k$ satisfying $\bar{I}(w_k) = \bar{b}_k^*$, $\bar{I}'(w_k) = 0$ and $m^*(w_k) \geqslant k$. Observe that critical points of $\bar{I}$ solve $-\Delta u = h(x,u)$ with $h(x,s) = \kappa|s|^{p-2}s$, which is a C^1 function satisfying (6.9). The conclusion now follows from Lemma C.3 (recall also Remark 6.5), which yields that

$$k \leqslant m^*(u) \leqslant C_N[\kappa(p-1)]^{\frac{N}{2}} \int_\Omega |u|^{\frac{N(p-2)}{2}} \leqslant C'\Big(\int_\Omega |u|^p\Big)^{\frac{N(p-2)}{2p}} = C''(\bar{b}_k^*)^{\frac{N(p-2)}{2p}}$$

□

Remark 6.15. Observe that the lower estimate obtained via the augmented Morse index is better than the one obtained by simply using the lower bound

$$d_k \geqslant \inf_{S_k} I \geqslant C_1 \lambda_k^{\frac{2}{p}\frac{2^*-p}{2^*-2}} - C_2 \simeq k^{\frac{4}{pN}\frac{2^*-p}{2^*-2}}$$

($\lambda_k \simeq k^{2/N}$, see [51]). In fact, it is easy to check that

$$\frac{2p}{N(p-1)} > \frac{4}{pN}\frac{2^*-p}{2^*-2}.$$

6.2.2 Perturbations from symmetry

We recall the definition of $J : H_0^1(\Omega) \to \mathbb{R}$,

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u), \qquad u \in H_0^1(\Omega),$$

and define $J_1(u) = J^+(u) + \lambda_0 I_0(u)$. Similarly to Lemmas 6.6 and 6.13, we have

Lemma 6.16. *There exists $\bar{\mu} > 0$ such that, for $0 < \mu < \bar{\mu}$, any solution of*

$$\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{V(\sigma(t,u))}{\|V(\sigma(t,u))\|}, \qquad \sigma(0,u) = u, \tag{6.16}$$

with

$$V(u) = \frac{1}{2}\frac{\nabla J(u)}{\|\nabla J(u)\|} + \frac{1}{2}\frac{\theta(u)}{\|\nabla J(u)\|}\frac{\nabla J_1(u)}{\|\nabla J_1(u)\|}, \quad \nabla J(u), \nabla J_1(u) \neq 0, \tag{6.17}$$

satisfies $\sigma(t,u) \in \mathcal{P}_\mu$ for all $u \in \mathcal{P}_\mu$ and all $t \geqslant 0$. Here $\chi, \theta : H_0^1(\Omega) \to [0,1]$ are any smooth functions such that σ is well defined in $\mathbb{R} \times H_0^1(\Omega)$ and moreover $\theta(u) = 0$ whenever $J(u) \leqslant 0$.

Next, we recall an extension result due to Castro and Clapp [37, Corollary 2.2], which can be applied to the functional I_0.

Lemma 6.17. *There are constants $\alpha, \beta > 0$, depending only on Ω and q, with the following property:*

> *For every pair of finite-dimensional subspaces $V \subseteq W$ of $H_0^1(\Omega)$ with* $\dim W = \dim V + 1$, *every odd map $\varphi : V \to H_0^1(\Omega)$ and every $R > 0$ such that $\varphi(v) = v$ if $\|v\| \geqslant R$, there is $\tilde{R} \geqslant R$ and an odd map $\tilde{\varphi} : W \to H_0^1(\Omega)$ which satisfies:*
>
> *(i)* $\tilde{\varphi}(v) = \varphi(v)$ *for every* $v \in V$,
>
> *(ii)* $\tilde{\varphi}(w) = w$ *for every* $w \in W$ *with* $\|w\| \geqslant \tilde{R}$,
>
> *(iii)* $\max_{w \in W} I_0(\tilde{\varphi}(w)) \leqslant \alpha \max_{v \in V} I_0(\varphi(v)) + \beta$.

In Lemma 6.23 below we state a similar extension result for the biharmonic operator. Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. For the sake of clarity we divide the proof in several steps.
Step 1. We may assume without loss of generality that

$$|J(u) - J(-u)| \leqslant \beta_1(|J(u)|^{\nu/\mu} + 1), \quad \forall u \in H_0^1(\Omega).$$

Indeed, otherwise, as in [100] or [102, Chapter 10], we can replace J by a penalized function $\tilde{J}$ given by $\tilde{J}(u) = J(u) + (1 - \theta(u))\int_\Omega F(x,u)$, where

$$\theta(u) := \chi\Big(\frac{\delta\int_\Omega G(x,u)}{\sqrt{I^2(u) + 1}}\Big)$$

and $I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u)$ is the even symmetric part of J; here $\chi \in \mathcal{D}((-2,2))$ is a smooth cut-off function, $0 \leqslant \chi \leqslant 1$, with $\chi = 1$ in $[-1,1]$, and δ is a small positive constant. The functional $\tilde{J}$ does satisfy $|\tilde{J}(u)-\tilde{J}(-u)| \leqslant \beta_1(|\tilde{J}(u)|^{\mu/\nu}+1)$ and moreover $\tilde{J}'(u)\varphi = (1+o(1))\langle u,\varphi\rangle - (1+o(1))\int_\Omega g(x,u)\varphi - \theta(u)\int_\Omega f(x,u)\varphi,\ \forall u,\varphi \in H_0^1(\Omega)$, where $o(1) \to 0$ as $\tilde{J}(u) \to +\infty$. In particular, critical points of J and $\tilde{J}$ coincide at high levels of the energy. We mention that, in order to preserve the property described in Lemma 6.6, our penalized term differs from Rabinowitz's one (in [100, 102], $\theta(u) = \chi\Big(\delta\int_\Omega G(x,u)/\sqrt{J^2(u)+1}\Big)$).

Step 2. We use notations similar to the ones in Proposition 6.14. For any large integer $k \in \mathbb{N}$ and any given $M_k > (\sup_{E_k} I_0)^2$, let

$$b_k := \inf\Big\{\sup_{\gamma(Q_k)\cap\mathcal{U}_k} J : \gamma \in C(Q_k; H_0^1(\Omega)) \text{ is odd },\gamma|_{\partial Q_k} = Id,\ \sup_{\gamma(Q_k)} J_1 < M_k\Big\},$$

Since J might not be even, b_k will not be in general a critical point of J. However, there exists $C_0 > 0$ such that

$$J(u) \leqslant I_0(u) + C_0 \qquad \text{and} \qquad \bar{I}(u) \leqslant J(u) + C_0,$$

for

$$\bar{I}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\kappa}{p}\int_\Omega |u|^p.$$

As we have seen at the end of the proof of Proposition 6.14, there exists $c_1 > 0$ such that, for every μ_k sufficiently small,

$$b_k \geqslant c_1 k^{2p/N(p-2)}$$

(we stress the fact that c_1 is independent of μ_k and k). Moreover, it is known that $b_k \leqslant c_2 k^{2\mu/N(\mu-2)}$ (cf. [12, p.1035]). We will fix μ_k in the next step.

Step 3. In order to fix μ_k, we need to make some algebraic considerations. Consider the inequality

$$b_{k+1} \leqslant b_k + 2 + \beta_1\Big((b_k+2)^{\nu/\mu}+1\Big). \tag{6.18}$$

Since $b_k \geqslant c_1 k^{2p/N(p-2)}$, for k sufficiently large (6.18) yields

$$b_{k+1} \leqslant b_k + c b_k^{\nu/\mu}, \qquad \text{where } c \text{ is independent of } k \text{ and } \mu_k. \tag{6.19}$$

Now, for any given k_0 sufficiently large, there exists $m_{k_0} \in \mathbb{N}$ such that it cannot happen (6.19) for $k_0, k_0+1, \dots, k_0+m_{k_0}$, otherwise by Lemma C.6 $b_k \leqslant C_{k_0} k^{\mu/(\mu-\nu)}$ for every such k, contradicting the facts that[4] $2p/N(p-2) > \mu/(\mu-\nu)$ and $b_k \geqslant c k^{2p/N(p-2)}$ for every large k.

From now on, for every k sufficiently large we fix μ_k small so that $2^{m_k+1}\mu_k \leqslant \bar{\mu}$ (where $\bar{\mu}$ was defined in Lemma 6.16) and so that the conclusion of Lemma 6.8 holds true for $2^{m_k}\mu_k$ and $\{\bar{I} < \bar{M}_k^{3^{m_k}}\}$, and the one of Lemma 6.6 holds with $\bar{I}$ instead of I.

Step 4. Let us fix $\gamma : Q_k \to H_0^1(\Omega)$ continuous and odd, $\gamma|_{\partial Q_k} = Id$ and $\sup_{\gamma(Q_k)} J_1 < M_k$, such that

$$\sup_{\gamma(Q_k)\cap\mathcal{U}_k} J \leqslant b_k + 1.$$

[4] Observe that $\frac{2p}{N(p-2)} > \frac{\mu}{\mu-\nu}$ follows from our basic assumption (6.4).

According to Lemma 6.17, γ has an odd continuous extension (which we still denote by γ), $\gamma : E_{k+1} \to H_0^1(\Omega)$ such that $\gamma(u) = u$ for every $u \in E_{k+1}$ with large norm (say, $\|u\| \geqslant R_{k+1} > R_k$) and every $u \in E_k \setminus B_{R_k}(0)$. We have $\sup_{\gamma(E_k)} I_0 < M_k$ by eventually taking a larger M_k, and thus

$$\sup_{\gamma(E_{k+1})} I_0 \leqslant \alpha M_k + \beta,$$

where α, β are positive constants depending only on Ω and q. In particular, $\sup_{\gamma(E_{k+1})} J \leqslant \alpha' M_k + \beta'$ and $\sup_{\gamma(E_{k+1})} J_1 \leqslant \alpha'' M_k + \beta'' < M_k^2$ (by eventually choosing a larger M_k, depending only on α'' and β''). The following number is therefore well-defined:

$$\bar{c}_k := \inf_{\gamma \in \Lambda_k} \sup_{\gamma(Q_k^+) \cap \overline{\mathcal{U}}_k} J,$$

where we have denoted

$$\overline{\mathcal{U}}_k := \{u : \operatorname{dist}(u, \mathcal{P}) \geqslant 2\mu_k\} \subset \mathcal{U}_k = \{u : \operatorname{dist}(u, \mathcal{P}) \geqslant \mu_k\}, \quad Q_k^+ := (E_k \oplus \mathbb{R}^+ \varphi_{k+1}) \cap B_{R_{k+1}}(0),$$

and

$$\partial Q_k^+ := \big((B_{R_{k+1}}(0) \backslash B_{R_k}(0)) \cap E_k\big) \cup \big((E_k \oplus \mathbb{R}^+ \varphi_{k+1}) \cap \partial B_{R_{k+1}}(0)\big).$$

By definition,

$$\Lambda_k := \left\{ \gamma \in C(Q_k^+, H_0^1(\Omega)) : \begin{array}{c} \text{(i) } \gamma|_{Q_k} \text{ is odd;} \quad \text{(ii) } \gamma|_{\partial Q_k^+} = Id; \quad \text{(iii) } \sup_{\gamma(Q_k^+)} J_1 < M_k^2 \\ \text{(iv) } \sup_{\gamma(Q_k) \cap \mathcal{U}_k} J \leqslant b_k + 1; \end{array} \right\}.$$

Step 5. Suppose first that

$$b_k + 1 < \bar{c}_k$$

and let us prove that $\bar{c}_k$ is a critical value for J. Arguing as in the fourth step of the proof of Proposition 6.14, for

$$D_k = \{u \in H_0^1(\Omega) : \ |J(u) - \bar{c}_k| \leqslant 1, \ |J_1(u) - M_k^2| \leqslant 1\},$$

we deduce the existence of $\theta_k \in [0, 1)$ such that

$$\inf_{u \in D_k} \frac{\langle \nabla J(u), \nabla J_1(u) \rangle}{\|\nabla J(u)\| \|\nabla J_1(u)\|} > -\theta_k.$$

Next, we take $V(u)$ as in (6.17) with θ a smooth cut-off function such that $\theta = \theta_k$ in D_k, $\theta = 0$ in a closed small neighborhood of $\{u : \nabla J_1(u) = 0\} \cup \{u : J(u) \leqslant 0\}$. We have

$$0 < \frac{1 - \theta_k}{2} \leqslant \langle V(u), \nabla J(u) \rangle \leqslant \|V(u)\| \|\nabla J(u)\| < 1, \quad \forall u \in H_0^1(\Omega), \nabla J(u) \neq 0,$$

and

$$\langle V(u), \nabla J_1(u) \rangle > 0, \quad \forall u \in D_k.$$

Suppose in view of a contradiction the existence of $0 < \varepsilon < \min\{\frac{1}{2}(\bar{c}_k - b_k - 1), \frac{1}{2}\}$ such that

$$\|\nabla J(u)\| \geqslant 2\varepsilon \qquad \forall u : |J(u) - \bar{c}_k| \leqslant 2\varepsilon, \ \operatorname{dist}(u, \mathcal{P}) \geqslant 2\mu_k$$

and take the following two closed disjoint sets

$$\begin{aligned} A &= \{u : \nabla J(u) = 0\} \cup \{u : |J(u) - \bar{c}_k| \geqslant 2\varepsilon\} \cup \{u : \operatorname{dist}(u, \mathcal{P}) \leqslant 3\mu_k/2\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \cup \{u : J_1(u) \geqslant M_k^2 + 2\}, \\ B &= \{u : |J(u) - \bar{c}_k| \leqslant \varepsilon, \operatorname{dist}(u, \mathcal{P}) \geqslant 2\mu_k, J_1(u) \leqslant M_k^2 + 1\}. \end{aligned}$$

Let $\chi : H_0^1(\Omega) \to [0,1]$ be a smooth cut-off function such that $\chi = 0$ in A, $\chi = 1$ in B, and consider $\sigma(t,u)$ to be the solution of (6.16) associated with the previously defined θ and χ. We observe that the choice of μ_k (made in Step 3) implies that $2\mu_k \leqslant \bar{\mu}$.

Let $\gamma \in \Lambda_k$ be such that

$$\sup_{\gamma(Q_k^+) \cap \overline{\mathcal{U}}_k} J \leqslant \bar{c}_k + \varepsilon.$$

For $\bar{t} = (1 - \theta_k)/2$ and $\sigma_{\bar{t}}(u) = \sigma(\bar{t}, u)$, we claim that $\sigma_{\bar{t}} \circ \gamma \in \Lambda_k$. In fact, by arguing as in the proof of Proposition 6.14 we have that $\sigma_{\bar{t}} \circ \gamma$ satisfies the properties (ii), (iii) and (iv) of Λ_k. Since $\sigma(t, \cdot)$ is not in general an odd map, the proof of (i) is more delicate. Let $u \in Q_k$ be such that $\gamma(u) \in \mathcal{U}_k$. Then

$$J(\gamma(u)) \leqslant \sup_{\gamma(Q_k) \cap \mathcal{U}_k} J \leqslant b_k + 1 \leqslant \bar{c}_k - 2\varepsilon$$

and hence $\gamma(u) \in A$, which implies $\chi(\gamma(u)) = 0$ and $\sigma(t, \gamma(u)) = \gamma(u)$ for all $t \in [0, \bar{t}]$. On the other hand, if $u \in Q_k$ verifies $\operatorname{dist}(\gamma(u), \mathcal{P}) < \mu_k \leqslant 3\mu_k/2$ then also $\gamma(u) \in A$ and $\sigma(t, \gamma(u)) = \gamma(u)$ for all $t \in [0, \bar{t}]$. In particular we have that $\sigma_{\bar{t}} \circ \gamma|_{Q_k} = \gamma|_{Q_k}$, which is an odd function. Thus

$$\bar{c}_k \leqslant \sup_{(\sigma_{\bar{t}} \circ \gamma)(Q_k^+) \cap \overline{\mathcal{U}}_k} J.$$

Take now $u \in Q_k^+$ such that $\sigma(\bar{t}, \gamma(u)) \in \overline{\mathcal{U}}_k$ and

$$J(\sigma(\bar{t}, \gamma(u))) = \sup_{(\sigma_{\bar{t}} \circ \gamma)(Q_k^+) \cap \overline{\mathcal{U}}_k} J.$$

Since $2\mu \leqslant \bar{\mu}$, we obtain that $\gamma(u) \in \overline{\mathcal{U}}_k$ and $\sigma(t, \gamma(u)) \in \overline{\mathcal{U}}_k$ for every $t \in [0, \bar{t}]$. Moreover,

$$\bar{c}_k \leqslant J(\sigma(\bar{t}, \gamma(u))) \leqslant J(\sigma(t, \gamma(u))) \leqslant J(\gamma(u)) \leqslant \bar{c}_k + \varepsilon$$

and

$$J_1(\gamma(u)) < M_k^2 < M_k^2 + 1.$$

Thus $\chi(\sigma(t, \gamma(u))) = 1$ for every $t \in [0, \bar{t}]$,

$$\frac{d}{dt} J(\sigma(t, \gamma(u))) \leqslant \frac{2}{\theta_k - 1} 2\varepsilon,$$

and

$$J((\sigma_{\bar{t}} \circ \gamma)(u)) \leqslant J(\gamma(u)) - 2\varepsilon \leqslant \bar{c}_k - \varepsilon,$$

a contradiction. Hence there exists $u_k \in \overline{\mathcal{U}}_k$ such that $J(u_k) = c_k$ and $J'(u_k) = 0$. As for the growth estimate $\bar{c}_k \leqslant ck^{2\mu/N(\mu-2)}$, it is enough to observe that rather than taking squares we can increase the value of constraint of J_1 linearly, and so $\bar{c}_k \leqslant C(b_k + 1)$ for every k.

Step 6. Suppose now that
$$\bar{c}_k \leqslant b_k + 1.$$

Then we can find $\gamma \in \Lambda_k$ such that $\sup_{\gamma(Q_k^+)\cap\overline{\mathcal{U}}_k} J \leqslant b_k + 2$. Since $Q_{k+1} = Q_k^+ \cup (-Q_k^+)$, we can extend γ by an odd symmetry to Q_{k+1}; we still denote by γ this extension. By taking into account the property mentioned in the first step of the proof, we see that

$$\sup_{\gamma(Q_{k+1})\cap\overline{\mathcal{U}}_k} J \leqslant b_k + 2 + \beta_1\Big((b_k+2)^{\nu/\mu} + 1\Big).$$

In fact, let $u \in Q_{k+1}$ with $\gamma(u) \in \overline{\mathcal{U}}_k$ be such that

$$\sup_{\gamma(Q_{k+1})\cap\overline{\mathcal{U}}_k} J = J(\gamma(u)).$$

If $u \in Q_k^+$, then $J(\gamma(u)) \leqslant b_k + 2$; otherwise, $-u \in Q_k^+$ and

$$J(\gamma(u)) \leqslant J(\gamma(-u)) + \beta_1\Big(|J(\gamma(-u)|^{\nu/\mu} + 1\Big) \leqslant b_k + 2 + \beta_1\Big((b_k+2)^{\nu/\mu} + 1\Big).$$

Thus our claim follows and since moreover $\sup_{\gamma(Q_{k+1})} J_1 < M_k^3$, we may define

$$b_{k+1} := \inf\left\{ \sup_{\gamma(Q_{k+1})\cap\overline{\mathcal{U}}_k} J : \begin{array}{l} \gamma \in C(Q_{k+1}; H_0^1(\Omega)) \text{ is odd }, \gamma|_{\partial Q_{k+1}} = Id, \\ \sup_{\gamma(Q_{k+1})} J_1 < M_k^3, \end{array} \right\}$$

obtaining that
$$b_{k+1} \leqslant b_k + 2 + \beta_1\Big((b_k+2)^{\nu/\mu} + 1\Big).$$

Starting from b_{k+1}, we iterate this process as in Step 4 above. Now, as seen in Step 3, given any k_0 large this process will stop at least after m_{k_0} steps, and we must have

$$b_k + 1 < \bar{c}_k \qquad \text{for some } k \in [k_0, k_0 + m_{k_0}],$$

where b_k is redefined with $\mu_k := 2^{k-k_0}\mu_{k_0} \leqslant \bar{\mu}$ and $M_k := M_{k_0}^{3^{k-k_0}}$. Thus, similarly to Step 5, we can conclude that $\bar{c}_k$ is a critical value for J and the proof of Theorem 6.1 is complete. $\square$

Proof of Corollary 6.2. It follows by our assumptions that J is a C^2 functional. Then, similarly to Lemma 6.9, the critical point u_{k_n} of J at level $\bar{c}_{k_n}$ can be chosen in such a way that its Morse index is less than or equal to $k_n + 1$ (observe that Q_{k_n+1} is contained in a vector space of dimension $k_n + 1$). The conclusion now follows from Lemma C.5, which states that the number of nodal domains of u_{k_n} is bounded from above by $m(u_{k_n})$. $\square$

6.3 Fourth order equations

This section is devoted to the proof of Theorem 6.3. The proof parallels the one of Theorem 6.1 and therefore we only point out some extra tools that are needed in order to deal with the biharmonic operator.

Solutions of (6.7) correspond now to critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u), \qquad u \in H := H^2(\Omega)\cap H_0^1(\Omega). \tag{6.20}$$

The Hilbert space H is endowed with the product $\langle u,v\rangle = \int_\Omega \Delta u \Delta v$ and corresponding norm $||\cdot||$.

With this new framework in mind and going back to Section 6.2, one realizes that all the arguments are still true except for three issues:

1. In $H^2(\Omega)$ the decomposition $u = u^+ - u^-$ is not available (in the sense that in general $u \in H^2(\Omega)$ does not imply $u^+, u^- \in H^2(\Omega)$) and hence we need a new tool to prove a version of Lemma 6.6 for (6.20). We will use Weth's argument [135] with dual cones, replacing u^+, u^- by appropriate projections of u (see bellow).

2. The lower estimate for the energy levels d_k defined in Lemma 6.14 uses an estimate due to Cwikel, Lieb and Rosenbljum which concerns the eigenvalues of an operator of the form $-\Delta + V$ in $L^2(\mathbb{R}^N)$. Instead, we must use a version of such inequality that is suitable for the biharmonic operator (Lemma C.4).

3. As we saw in the proof of Theorem 6.1 (Step 4), in order to control the invariant neighborhoods of the cone $\mathcal{P}$, we need an extension theorem in the spirit of the one in Lemma 6.17, which concerns the space $H_0^1(\Omega)$. This turns out to be a delicate question for the biharmonic operator. We will be able to prove a weaker version of this result (Lemma 6.23), which will be enough for our purposes.

Let us start by dealing with point 1. First, we need to recall some definitions and results. Take $P = \{u \in H : \ u \geqslant 0 \ \text{ a. e.}\}$ and consider its dual cone

$$P^* := \{u \in H : \ \langle u, v\rangle \leqslant 0 \ \forall v \in P\}.$$

Both P and P^* are closed convex subsets of H, and hence we may consider the corresponding projection operators

$$A : H \to P, \qquad A^* : H \to P^*.$$

Due to a classical result by Moreau [87], we can write any $u \in H$ as

$$u = Au + A^*u, \qquad \text{and } \langle Au, A^*u\rangle = 0.$$

In particular, $\|A^*u\| = \text{dist}(u, P)$.

Lemma 6.18. *With the previous notations, we have*

(a) $P^* \subseteq -P$

(b) $u \geqslant A^*u$ *for every* $u \in H$.

Proof. (a) This proof is taken from [65, Lemma 2.2]. Consider $u \in H$ such that $\Delta^2 u \leqslant 0$ and let us prove that $u \leqslant 0$ a.e. in Ω. Take an arbitrary nonnegative funtion $h \in L^\infty(\Omega)$ and let $v \in C^3(\bar{\Omega})$ be a solution of the problem

$$\Delta^2 v = h \text{ in } \Omega, \qquad v = \Delta v = 0 \text{ on } \partial\Omega.$$

By applying twice the maximum principle for the harmonic operator, from the boundary conditions we see that $v \geqslant 0$ a.e. in Ω. Therefore

$$\int_\Omega uh = \int_\Omega u\Delta^2 v = \int_\Omega (\Delta^2 u)v \leqslant 0$$

and the conclusion follows.

(b) This fact is a direct conclusion of the decomposition $u = Au + A^*u$ together with point (a). □

The same considerations can be made for the cone $-P$. We are now in a position to prove a variant of Lemma 6.6 for (6.20). As explained in the proof of Proposition 6.9, it is sufficient to consider the case where f and g are smooth.

Lemma 6.19. *Under assumptions* (H1)-(H5) *with C^1 functions f and g, there exists $\bar{\delta} > 0$ such that for any $0 < \delta < \bar{\delta}$, any solution $\sigma(t,u)$ of*

$$\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{\nabla J(\sigma(t,u))}{||\nabla J(\sigma(t,u))||}, \qquad \sigma(0,u) = u,$$

satisfies $\sigma(t,u) \in \mathcal{P}_\delta$ for all $u \in \mathcal{P}_\delta$ and all $t \geqslant 0$. Here $\chi : H \to [0,1]$ is any smooth function such that σ is well defined in $\mathbb{R} \times H$.

Proof. For simplicity, we assume $f \equiv 0$ and prove the invariance only for the cone P. Let K be the compact operator defined in H by $v = Ku$ if and only if $\Delta^2 v = g(x,u)$, $v \in H$. From our previous considerations, we see that

$$\begin{aligned}
\operatorname{dist}(v,P)\|A^*v\| &= \|A^*v\|^2 = \langle A^*v, v - Au\rangle = \langle A^*v, v\rangle \\
&= \int_\Omega \Delta(A^*v)\Delta v = \int_\Omega (A^*v)g(x,u) \\
&= -\int_\Omega |A^*v|g(x,u) \leqslant -\int_\Omega |A^*v|g(x,,A^*u) \\
&\leqslant \varepsilon\|A^*u\|_2\|A^*v\|_2 + C_\varepsilon\|A^*u\|^{p-1}\|A^*v\|
\end{aligned}$$

which yields that

$$\operatorname{dist}(v,P) \leqslant \varepsilon\operatorname{dist}(u,P) + C_\varepsilon(\operatorname{dist}(u,P))^{p-1}.$$

Thus there exists $\bar{\delta} > 0$ such that $K(P_\delta) \subseteq P_{\delta/2}$ for every $\delta < \bar{\delta}$, and the proof can be finished as in Lemma 6.6. □

Concerning the basic estimate on the Morse index, we have the following result.

Lemma 6.20. *Given $2 < p < 2^* = 2N/(N-4)$, let u be a solution of the problem*

$$\Delta^2 u = |u|^{p-2}u, \qquad u \in H,$$

with augmented Morse index greater than or equal to $k \in \mathbb{N}$. Then there exists a universal constant $c > 0$ such that

$$\bar{I}(u) \geqslant \kappa c^{\frac{4p}{N(p-2)}},$$

where $\bar{I}(u) = \dfrac{1}{2}\displaystyle\int_\Omega (\Delta u)^2 - \dfrac{1}{p}\int_\Omega |u|^p$ is the corresponding energy of u.

Proof. From the estimate of Lemma C.4, we see that

$$k \leqslant m^*(u) \leqslant D_1 \int_\Omega |u|^{N(p-2)/4} \leqslant D_2\Big(\int_\Omega |u|^p\Big)^{N(p-2)/4p} = D_3(\bar{I}(u))^{N(p-2)/4p}$$

and the conclusion follows. □

Finally we deal with point 3 of the list presented at the beginning of this section. Let

$$I_r(u) := \frac{1}{2}||u||^2 - \frac{1}{r}\int_\Omega |u|^r, \qquad u \in H,\ 2 < r < \mu.$$

In Section 6.2 we have worked with a fixed "bareer" functional $I_0(u) = \frac{1}{2}||u||^2 - \frac{1}{q}\int_\Omega |u|^q$ for some $2 < q < \mu$ but now it will be more convenient to look at q as a parameter varying in the interval $(2, \mu)$ (see Remark 6.22 bellow). Besides, we will be forced to work with the $L^1(\Omega)$-norm rather than the $L^r(\Omega)$-norm, and therefore we need to consider the auxiliary functionals

$$\hat{I}_r(u) := \frac{1}{2}||u||^2 - \frac{1}{r}\left(\int_\Omega |u|\right)^r, \qquad u \in H,\ 2 < r < \mu.$$

We can assume without loss of generality that Ω has Lebesgue measure 1, so that $I_r \leqslant \hat{I}_r$. As in Section 6.2, λ_0 will denote a fixed constant $0 < \lambda_0 < (\mu - 2)/2$.

Lemma 6.21. *Given $r \in (2, \mu)$ there exists $\alpha, \beta > 0$ with the following property*

> *For every pair of finite-dimensional subspaces $V \subset W$ of H with* $\dim W = \dim V + 1$*, every odd map $\varphi : V \to H$ and every $R > 0$ such that $\varphi(v) = v$ if $||v|| \geqslant R$, there exist $\widetilde{R} \geqslant R$ and an odd map $\widetilde{\varphi} : W \to H$ which satisfy:*
>
> *(i)* $\widetilde{\varphi}(v) = \varphi(v)$, $\forall v \in V$;
>
> *(ii)* $\widetilde{\varphi}(w) = w$, $\forall w \in W : ||w|| \geqslant \widetilde{R}$;
>
> *(iii)* $\sup\limits_{\widetilde{\varphi}(W)} \hat{I}_r \leqslant \alpha \sup\limits_{\varphi(V)} \hat{I}_r + \beta$.

Proof. A similar result was proved in [37, Corollary 2.2] (recall also Lemma 6.17), but working with functions in $H_0^1(\Omega)$. All their arguments adapt perfectly to our case, except in one point. In fact, one of the main ideas is to open a hole in the support of any function $u \in H_0^1(\Omega)$, using the following homotopy

$$u_s(x) = \begin{cases} u(x', x_N) & \text{if } x_N < 0 \\ u(x', 2x_N) & \text{if } x_N > 0 \end{cases} \qquad 1 \leqslant s \leqslant 2,$$

extended by 0 to the whole Ω. Such homotopy satisfies the key estimate

$$\bar{I}(u_s) := \frac{1}{2}\int_\Omega |\nabla u_s|^2 - \frac{1}{p}\int_\Omega |u_s|^p \leqslant a\int_\Omega |\nabla u|^2 - b\int_\Omega |u|^p \tag{6.21}$$

for some constants $a, b > 0$ independent of u (*cf.* page 173 of [37]). However, in our case in general $u \in H$ does not imply $u(x', 2x_N) \in H$. So our goal is to build another kind of function u_2 such that $\mathrm{supp} u_2 \subseteq \Omega_2 \Subset \Omega$ for all u, and a homotopy u_s between u and the new u_2 satisfying a version of (6.21) for the functionals $\bar{I}_r$. If we are able to do that, then as in [37] our result is true.

In order to prove the claim, let us fix a small $\delta > 0$. For every $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) \leqslant 2\delta$ we denote by $z(x) \in \partial\Omega$ the unique point such that $|z(x)-x| = \mathrm{dist}(x, \partial\Omega)$ and by $n(z(x))$ the unit outward normal at the point $z(x)$, that is $n(z(x)) = (z(x) - x)/|z(x) - x|$; in the sequel we assume $\partial\Omega$ is smooth enough (C^5 regularity will be sufficient). Let $\phi \in C^\infty(\mathbb{R};\mathbb{R})$, $0 \leqslant \phi \leqslant 1$, be a smooth function such that $\phi(t) = 1$ if $t \leqslant \delta$ and $\phi(t) = 0$ if $t \geqslant 2\delta$, and

$$\lambda_s(x) := x + s\mathrm{d}(x)\phi(\mathrm{d}(x))n(z(x)) = x + s\phi(\mathrm{d}(x))(z(x) - x), \qquad x \in \Omega,\ 0 \leqslant s \leqslant 1,$$

where $\mathrm{d}(x) = \mathrm{dist}(x, \partial\Omega)$. We observe that $\lambda_s(x) \in \Omega$ and that $\mathrm{dist}(\lambda_s(x), \partial\Omega) = \mathrm{d}(x)(1 - s\phi(\mathrm{d}(x)))$. Let

$$\alpha_s(t) = t(1 - s\phi(t)).$$

It is trivial to check that for $s < 1$ the map α is a diffeomorphism of the interval $(0, 2\delta)$ onto itself, while if $s = 1$ then $\alpha : (\delta, 2\delta) \to (0, 2\delta)$ is a diffeomorphism. It follows that if $s < 1$ then λ_s is a diffeomorphism of the open set $\{x \in \Omega : \mathrm{d}(x) < 2\delta\}$ onto itself, while if $s = 1$ then we have a diffeomorphism $\lambda_1 : \{x \in \Omega : \delta < \mathrm{d}(x) < 2\delta\} \to \{x \in \Omega : \mathrm{d}(x) < 2\delta\}$; the inverse of λ_s is explicitly given by

$$\lambda_s^{-1}(x) = x - (\alpha^{-1}(\mathrm{d}(x)) - \mathrm{d}(x))n(z(x)),$$

so that $\mathrm{dist}(\lambda_s^{-1}(x), \partial\Omega) = \alpha_s^{-1}(\mathrm{d}(x))$. Finally, we define

$$u_s(x) = \det(D\lambda_s(x))\, u(\lambda_s(x)), \qquad 0 \leqslant s \leqslant 1,\ x \in \Omega.$$

It is clear that $u_0 = u$, $\mathrm{supp}\ u_1 \subseteq \Omega_2 := \{x \in \Omega : \mathrm{d}(x) \geqslant \delta\}$ and that we have

$$\int_\Omega |u_s| = \int_\Omega |u| \qquad \text{for every } s. \tag{6.22}$$

We claim that $u_s \in H$ and that

$$\int_\Omega (\Delta u_s)^2 \leqslant a\int_\Omega (\Delta u)^2 \qquad \text{for some } a > 0. \tag{6.23}$$

To that purpose, using the change of variables $y = \lambda_s(x)$, it is enough to show that

$$|\nabla(\det(D\lambda_s))|^2 + |D^2(\det(D\lambda_s))|^2 \leqslant C|\det(D\lambda_s)|.$$

This property is invariant by local diffeomorphisms and therefore we can assume that $\partial\Omega$ is flat near a given point $x_0 \in \partial\Omega$, say $x_0 = 0$ and $\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$,

$\partial\Omega = \{(x', 0), x' \in \mathbb{R}^{N-1}\}$ near the origin. Then $\det D\lambda_s(x', x_N) = \alpha'(x_N)$ near the origin and the conclusion follows by a direct computation.

By combining (6.22) with (6.23) we get

$$\hat{I}_r(u_s) \leqslant \frac{a}{2}\int_\Omega (\Delta u)^2 - \frac{1}{r}\Big(\int_\Omega |u|\Big)^r$$

(compare with (6.21)) and thus the proof of the lemma is complete. □

Remark 6.22. In the previous result we considered $u_s(x) = \det(D\lambda_s(x))u(\lambda_s(x))$, which yields an isometry for the $L^1(\Omega)$– norm. If we had aimed for an isometry in $L^p(\Omega)$, the natural choice would have been $u_s(x) = \det(D\lambda_s(x))^{1/p}u(\lambda_s(x))$, leading to inconclusive and hard computations in view of (6.23). This justifies our choice of working with $\hat{I}_r$ rather than I_r. The price to pay is that in order to obtain an extension result for $J^+ + \lambda_0 I_r$ we still need some extra work.

Lemma 6.23. *Given $r_1 \in (2, \mu)$ sufficiently close to 2, there exist $\alpha, \beta > 0$ and $r \in (r_1, \mu)$ with the following property:*

> *For every pair of finite-dimensional subspaces $V \subset W$ of H with $\dim W = \dim V + 1$, every odd map $\varphi : V \to H$ and every $R > 0$ such that $\varphi(v) = v$ if $||v|| \geqslant R$, there exist $\widetilde{R} \geqslant R$ and an odd map $\widetilde{\varphi} : W \to H$ which satisfy:*
>
> *(i)* $\widetilde{\varphi}(v) = \varphi(v)$, $\forall v \in V$;
>
> *(ii)* $\widetilde{\varphi}(w) = w$, $\forall w \in W : ||w|| \geqslant \widetilde{R}$;
>
> *(iii)* $\sup\limits_{\widetilde{\varphi}(W)} (J^+ + \lambda_0 I_r) \leqslant \alpha \sup\limits_{\varphi(V)} (J^+ + \lambda_0 I_{r_1}) + \beta$.

Moreover, $r \to 2$ as $r_1 \to 2$.

Proof. Given $r_1 \in (2, \mu)$, let

$$r_2 := \frac{2(2^* - r_1)}{(3 - r_1)2^* - 2}.$$

We have indeed that $r_2 < \mu$ if r_1 is close to 2 (since $r_2 = 2$ if $r_1 = 2$), while the requirement that $r_2 > r_1$ is equivalent to $(r_1 - 1)(r_1 - 2) > 0$. Let us fix any $r \in (r_2, \mu)$ and let $\tilde{\varphi}, \alpha, \beta$ be as in the preceding lemma. Since $r < \mu$ and $I_r \leqslant \hat{I}_r$, there exists a constant $c(r) > 0$ such that

$$\sup_{\tilde{\varphi}(W)} (J^+ + \lambda_0 I_r) \leqslant (1 + \lambda_0) \sup_{\tilde{\varphi}(W)} \hat{I}_r + c(r) \leqslant (1 + \lambda_0)\alpha \sup_{\varphi(V)} \hat{I}_r + (1 + \lambda_0)\beta + c(r).$$

Thus, for $M := \sup_{\varphi(V)}(J^+ + \lambda_0 I_{r_1})$, it is sufficient to prove that

$$\hat{I}_r(u) \leqslant 2M + C'(r, r_1, r_2), \qquad \forall u : I_{r_1}(u) \leqslant M.$$

Now, if $I_{r_1}(u) \leqslant M$ then either $\frac{1}{2}||u||^2 \leqslant 2M$ or else $||u||^2 \leqslant \frac{4}{r_1}\int_\Omega |u|^{r_1}$. Assuming the latter (otherwise the conclusion is obvious) we see from the Gagliardo-Nirenberg inequality that

$$\|u\|^2 \leqslant \kappa\Big(\int_\Omega |u|\Big)^{\lambda r_1}\Big(\int_\Omega |u|^{2^*}\Big)^{(1-\lambda)r_1/2^*} \leqslant \kappa\left(\int_\Omega |u|\right)^{\lambda r_1} \|u\|^{(1-\lambda)r_1},$$

where λ is defined by the condition $\frac{1}{r_1} = \lambda + \frac{1-\lambda}{2^*}$. Thus

$$\|u\|^2 \leqslant \kappa''\Big(\int_\Omega |u|\Big)^{2\lambda r_1/(2-r_1(1-\lambda))} = \kappa''\Big(\int_\Omega |u|\Big)^{r_2}$$

and

$$\hat{I}_r(u) \leqslant \frac{1}{2}\|u\|^2 - \frac{1}{r}\Big(\int_\Omega |u|\Big)^r \leqslant \frac{\kappa''}{2}\Big(\int_\Omega |u|\Big)^{r_2} - \frac{1}{r}\left(\int_\Omega |u|\right)^r \leqslant C'(r, r_1, r_2)$$

by recalling that $r > r_2$.

□

Proof of Theorem 6.3 completed. Thanks to Lemmas 6.19, 6.20 and 6.23, the proof of Theorem 6.3 follows word by word the argument in the proof of Theorem 6.1, the only difference being that the "bareer" functional $I_0(u) = \frac{1}{2}||u||^2 - \frac{1}{q}\int_\Omega |u|^q$ is now replaced by a finite sequence of functionals

$$I_{r_i}(u) = \frac{1}{2}||u||^2 - \frac{1}{r_i}\int_\Omega |u|^{r_i} \qquad \text{with } 2 < r_{k_0} < \ldots < r_i < \ldots < r_{k_0+m_0} < \mu.$$

We stress that since a prescribed finite number of functionals I_{r_i} is to be considered, this construction is well-defined.

In conclusion, we obtain a sequence u_{k_n} of sign-changing critical points of the functional J satisfying

$$c_1 k_n^{\frac{4p}{N(p-2)}} \leqslant J(u_{k_n}) \leqslant c_2 k_n^{\frac{4\mu}{N(\mu-2)}}$$

for some $c_1, c_2 > 0$ independent of n. □

Remark 6.24. In case of equation (6.7) with Dirichlet boundary conditions $u = \frac{\partial u}{\partial \nu} = 0$, we can obtain a similar result by working in the space $H_0^2(\Omega)$, *provided* the corresponding Green function on Ω is positive. We refer the reader to [135] and its references for a discussion on this subject.

6.4 Further developments

Consider the following superlinear elliptic system

$$-\Delta u = |v|^{q-2}v, \qquad -\Delta v = |u|^{p-2}u, \qquad (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{6.24}$$

with both $p, q > 2$. In [108, Section 4] the following result is proved.

Theorem 6.25. *Assume $p, q > 2$ and $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$. Then* (6.24) *admits an unbounded sequence of sign-changing solutions (u_k, v_k), in the sense that both $(u_k + v_k)^+ \neq 0$ and $(u_k + v_k)^- \neq 0$ for every k.*

We observe that it has been proved in [8] by means of homology theory that under the assumptions of Theorem 6.25 the problem (6.24) admits an unbounded sequence of solutions. However, even for this simple model case, no *a priori* bounds for the positive solutions are known, except in special cases (namely if $N = 3$ or if $p, q < 2N/(N-2)$, see *e.g.* [99] for recent developments).

Let us say some words about the proof of the previous result. Exactly as in Chapter 5, it is enough to consider the case $p, q < 2^*$. Then the solutions of problem (6.24) are given by the critical points of the C^2 functional

$$I(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle - \int_\Omega F(u) - \int_\Omega G(v), \qquad (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

where we have denoted $F(u) = |u|^p/p$ and $G(v) = |v|^q/q$. Consider the (even symmetric) reduced functional

$$J(u) := I(u + \Psi_{u,u}, u - \Psi_{u,u}), \qquad u \in H_0^1(\Omega),$$

where $\Psi_{u,u}$ is defined by $I(u + \Psi_{u,u}, u - \Psi_{u,u}) = \max_{\psi \in H_0^1(\Omega)} I(u + \psi, u - \psi)$. It turns out that $J \in C^2(H_0^1(\Omega); \mathbb{R})$ and

$$J'(u)\varphi = I'(u + \Psi_{u,u}, u - \Psi_{u,u})(\varphi, \varphi), \qquad \forall u, \varphi \in H_0^1(\Omega)$$

(see also Section 5.6) In particular, u is a critical point of J if and only if $(u + \Psi_{u,u}, u - \Psi_{u,u})$ is a critical point of I, hence a solution of (6.24). In this way Theorem 6.25 is proved in [108] by applying the arguments of Section 6.2 to the funcional J, which enables to find an unbounded sequence of critical points of J.

We end this chapter by mentioning the work by Bonheure and Ramos [21], where the authors establish the existence of infinitely many solutions (not necessarily sign-changing) for the perturbed problem

$$-\Delta u = |v|^{q-2}v + k(x), \qquad -\Delta v = |u|^{p-2}u + h(x)$$

in a regular bounded domain $\Omega \subseteq \mathbb{R}^N$, under the assumptions $h, k \in L^2(\Omega)$, $N \geqslant 3$ and $\frac{N}{2}(1 - \frac{1}{q} - \frac{1}{q}) < \frac{p-1}{p}$.

Appendices

Appendix A

Some notes on measure theory

A.1 General measure theory

In this appendix we review some definitions and results of measure theory, and state and prove in detail the less standard *Federer's Reduction Principle*. With the exception of Theorem A.26, all the statements are taken from [62].

Following [62], we will only deal with measures defined in $\mathbb{R}^N$. However, all results hold true for a bounded domain as well.

Definition A.1. *A map* $\mu : P(\mathbb{R}^N) \to [0, +\infty]$ *is called a* measure *on* $\mathbb{R}^N$ *if*

(i) $\mu(\emptyset) = 0$*;*

(ii) $\mu(A) \leqslant \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subseteq \cup_{k=1}^{\infty} A_k$*.*

Remark A.2. We will always deal with nonnegative measures, and therefore we will omit the term nonnegative. Moreover, it is worthwhile noticing that, as also pointed out in [62], here the terminology varies with respect to standard usage; in fact, what we define as measure is usually called an *outer measure*.

Definition A.3. *Let* μ *be a measure on* $\mathbb{R}^N$ *and* $A \subseteq \mathbb{R}^N$*. Then* μ restricted to A*, written*

$$\mu\lfloor A,$$

is the measure defined by

$$(\mu\lfloor A)(B) = \mu(A \cap B) \qquad \text{for all } B \subseteq \mathbb{R}^N.$$

Moreover a measure μ *is said to* concentrate *on* A *if* $\mu = \mu\lfloor A$*.*

Definition A.4. *A set* $A \subseteq \mathbb{R}^N$ *is* μ*–measurable if for each set* $B \subseteq \mathbb{R}^N$*,*

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A).$$

Definition A.5. *A collection of subsets* $\mathcal{A} \subset P(\mathbb{R}^N)$ *is a* σ*–algebra provided that*

(i) $\emptyset,\ \mathbb{R}^N \in \mathcal{A}$*;*

(ii) $A \in \mathcal{A}$ *implies* $\mathbb{R}^N \setminus A \in \mathcal{A}$*;*

(iii) $A_k \in \mathcal{A}$ $\forall k \in \mathbb{N}$ *implies* $\cup_{k=1}^{\infty} A_k \in \mathcal{A}$.

We observe that the collection of all μ–measurable subsets forms a σ–algebra.[1]

Definition A.6. *The* Borel σ–algebra *of* $\mathbb{R}^N$ *is the smallest* σ*–algebra of* $\mathbb{R}^N$ *containing the open subsets of* $\mathbb{R}^N$.

Definition A.7. *(i) A measure* μ *on* $\mathbb{R}^N$ *is* Borel *if every Borel set is* μ*–measurable.*

(ii) A measure μ *on* $\mathbb{R}^N$ *is* Borel regular *if* μ *is Borel and for each* $A \subseteq \mathbb{R}^N$ *there exists a Borel set* B *such that* $A \subseteq B$ *and* $\mu(A) = \mu(B)$.

(iii) A measure μ *on* $\mathbb{R}^N$ *is a* Radon *measure if* μ *is Borel regular and* $\mu(K) < \infty$ *for each compact set* $K \subseteq \mathbb{R}^N$.

Theorem A.8. *Let* μ *be a Borel regular measure on* $\mathbb{R}^N$*. Suppose that* $A \subseteq \mathbb{R}^N$ *is* μ*–measurable and* $\mu(A) < \infty$*. Then* $\mu \lfloor A$ *is a Radon measure.*

Theorem A.9. *Let* μ *be a Radon measure on* $\mathbb{R}^N$*. Then*

(i) for each set $A \subseteq \mathbb{R}^N$,

$$\mu(A) = \inf\{\mu(U) : \ A \subseteq U, \ U \text{ open}\},$$

(ii) for each μ*–measurable set* $A \subseteq \mathbb{R}^N$,

$$\mu(A) = \sup\{\mu(K) : \ K \subseteq A, \ K \text{ compact}\}.$$

Theorem A.10. *Assume* $L : C_c^\infty(\mathbb{R}^N) \to \mathbb{R}$ *is linear and nonnegative, so that*

$$L(f) \geqslant 0 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^N), f \geqslant 0.$$

Then there exists a Radon measure μ *on* $\mathbb{R}^N$ *such that*

$$L(f) = \int_{\mathbb{R}^N} f \, d\mu \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^N).$$

Next we present some concepts concerning the differentiation of Radon measures. Let μ and ν be measures on $\mathbb{R}^N$.

Definition A.11. *For each point* $x \in \mathbb{R}^N$*, define*

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r \to 0} \frac{\nu(\bar{B}_r(x))}{\mu(\bar{B}_r(x))} & \text{if } \mu(\bar{B}_r(x)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(\bar{B}_r(x)) = 0 \text{ for some } r > 0, \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r \to 0} \frac{\nu(\bar{B}_r(x))}{\mu(\bar{B}_r(x))} & \text{if } \mu(\bar{B}_r(x)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(\bar{B}_r(x)) = 0 \text{ for some } r > 0. \end{cases}$$

[1] Actually, if we restrict the domain of μ to be the set of all μ–measurable sets, μ turns out to be a *measure* in the usual sense. Indeed, it can be shown that if $(U_k)_k$ is a family of disjoint μ–measurable sets, then $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$.

If $\overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x) < +\infty$, *we say that* ν *that is* differentiable *with respect to* μ *at* x *and write*

$$D_\mu\nu(x) = \overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x).$$

$D_\mu\nu$ *is called the* derivative *(or* density*) of* ν *with respect to* μ.

Theorem A.12. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^N$. *Then* $D_\mu\nu$ *exists and is finite* μ*–a.e.. Furthermore,* $D_\mu\nu$ *is* μ*–measurable.*

Definition A.13. *The measure* ν *is* absolutely continuous *with respect to* μ, *written*

$$\nu \ll \mu,$$

provided that $\mu(A) = 0$ *implies* $\nu(A) = 0$ *for all* $A \subseteq \mathbb{R}^N$.

Definition A.14. *The measures* ν *and* μ *are* mutually singular, *written*

$$\nu \perp \mu,$$

if there exists a Borel subset $B \subseteq \mathbb{R}^N$ *such that*

$$\mu(\mathbb{R}^N \setminus B) = \nu(B) = 0.$$

Theorem A.15. *Let* ν *and* μ *be Radon measures on* $\mathbb{R}^N$.

(i) Then $\nu = \nu_{\rm ac} + \nu_{\rm s}$, *where* $\nu_{\rm ac}$ *and* $\nu_{\rm s}$ *are Radon measures on* $\mathbb{R}^N$ *with*

$$\nu_{\rm ac} \ll \mu \qquad \text{and } \nu_s \perp \mu.$$

(ii) Furthermore,

$$D_\mu\nu = D_\mu\nu_{\rm ac}, \qquad D_\mu\nu_{\rm s} = 0 \qquad \mu\text{–a.e.},$$

and

$$\nu(A) = \int_A D_\mu\nu\, d\mu + \nu_{\rm s}(A)$$

for each Borel set $A \subseteq \mathbb{R}^N$.

Next we introduce a weak notion of convergence of measures

Definition A.16. *Let* μ, μ_k $(k \in \mathbb{N})$ *be Radon measures on* $\mathbb{R}^N$. *We say that the measures* μ_k *converge weakly to the measure* μ, *written*

$$\mu_k \rightharpoonup \mu,$$

if

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} f\, d\mu_k = \int_{\mathbb{R}^N} f\, d\mu_k \qquad \text{for all } f \in C_{\rm c}(\mathbb{R}^N).$$

Theorem A.17. *Let* $(\mu_k)_k$ *be a sequence of Radon measures on* $\mathbb{R}^N$ *satisfying*

$$\sup_k \mu_k(K) < \infty \text{ for each compact set } K \subseteq \mathbb{R}^N.$$

Then there exists a Radon measure μ *such that, up to a subsequence,*

$$\mu_k \rightharpoonup \mu.$$

Theorem A.18. *Let μ, μ_k $(k \in \mathbb{N})$ be such that $\mu_k \rightharpoonup \mu$. Then*

$$\lim_{k\to\infty} \mu_k(B) = \mu(B) \text{ for each Borel set } B \subseteq \mathbb{R}^N \text{ with } \mu(\partial B) = 0.$$

Proof. This is perhaps a less known fact, and hence we present its proof. From Theorem A.9 it is not hard to prove that

$$\limsup_{k\to\infty} \mu_k(K) \leqslant \mu(K) \qquad \text{for each compact set } K \subseteq \mathbb{R}^N,$$

and

$$\mu(U) \leqslant \liminf_{k\to\infty} \mu_k(U) \qquad \text{for each open set } U \subseteq \mathbb{R}^N.$$

Then, for B in the hypotheses of the statement, we see that

$$\begin{aligned} \mu(B) = \mu(\mathrm{int}(B)) &\leqslant \liminf_{k\to\infty} \mu_k(\mathrm{int}(B)) \\ &\leqslant \limsup_{k\to\infty} \mu_k(\bar{B}) \\ &\leqslant \mu(\bar{B}) = \mu(B). \end{aligned}$$

□

A.2 Hausdorff measure. Federer's Reduction Principle

The purpose of this section is to introduce the definition of Hausdorff measure as well as to present some of its basic properties, in view of proving the Federer's Reduction Principle.

Definition A.19. *Let $A \subseteq \mathbb{R}^N$, $0 \leqslant s < \infty$, $0 < \delta \leqslant \infty$. Define*

$$\mathscr{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\mathrm{diam} C_j}{2} \right)^s : \; A \subseteq \cup_{j=1}^{\infty} C_j, \; \mathrm{diam}\; C_j \leqslant \delta \right\},$$

where

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}.$$

Here $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$ $(0 < s < \infty)$ is the usual gamma function.

Definition A.20. *For $A \subseteq \mathbb{R}^N$ and $0 \leqslant s < \infty$, define*

$$\mathscr{H}^s(A) = \lim_{\delta\to 0^+} \mathscr{H}^s_\delta(A) = \sup_{\delta>0} \mathscr{H}^s_\delta(A).$$

*We call $\mathscr{H}^s$ the s–*dimensional Hausdorff measure *on $\mathbb{R}^N$.*

Theorem A.21. *$\mathscr{H}^s$ is a Borel regular measure $(0 \leqslant s < \infty)$.*

Lemma A.22. *Suppose that $A \subseteq \mathbb{R}^N$ and $\mathscr{H}^s_\delta(A) = 0$ for some $0 < \delta \leqslant \infty$. Then $\mathscr{H}^s(A) = 0$.*

Lemma A.23. *Let $A \subseteq \mathbb{R}^N$ and $0 \leqslant s < t < \infty$.*

(i) *If* $\mathscr{H}^s(A) < \infty$*, then* $\mathscr{H}^t(A) = 0$.

(ii) *If* $\mathscr{H}^t(A) > 0$*, then* $\mathscr{H}^s(A) = +\infty$.

Definition A.24. *The* Hausdorff dimension *of a set* $A \subseteq \mathbb{R}^n$ *is defined to be*

$$\mathscr{H}_{\dim}(A) = \inf\{0 \leqslant s < \infty : \ \mathscr{H}^s(A) = 0\}.$$

The following result will be used ahead in this section.

Lemma A.25. *Assume* $E \subseteq \mathbb{R}^N$*,* E *is* $\mathscr{H}^s$*–measurable, and* $\mathscr{H}^s(E) < \infty$*. Then*

$$\frac{1}{2^s} \leqslant \limsup_{r \to 0} \frac{\mathscr{H}^s(B_r(x) \cap E)}{\alpha(s) r^s} \leqslant 1$$

for $\mathscr{H}^s$*–a.e.* $x \in E$.

Now we are in a condition to state and prove the Federer's Reduction Principle. The statement we present here is based in the version contained in [41, 42, 116]. Our proof follows very closely the one of [116, Appendix A].

Theorem A.26. *Let* $\mathcal{F} \subseteq \{\Phi : \mathbb{R}^N \to \mathbb{R}^m\}$ *and define, for any given* $\Phi \in \mathcal{F}$*,* $x_0 \in \mathbb{R}^N$ *and* $\lambda > 0$*, the rescaled and translated function*

$$\Phi_{x_0,\lambda}(x) := \Phi(x_0 + \lambda x) \qquad \textit{for every } x \in \mathbb{R}^N.$$

(A0) (Technical assumption) We assume that $\mathcal{F}$ *is contained in a topological space in which the convergence satisfies the following: given* $\Phi_k, \Psi \in \mathcal{F}$ *such that* $\Phi_k \to \Psi$*, then* $(\Phi_k)_{y,1} \to \Psi_{y,1}$ *and* $a_k(\Phi_k)_{0,b_k} \to \Psi$*, for all* $y \in \mathbb{R}^N$*,* $a_k, b_k \to 1$.

Assume that $\mathcal{F}$ *satisfies the following assumptions:*

(A1) (Closure under rescaling, translation and normalization) Given $|x_0| \leqslant 1 - \lambda$*,* $0 < \lambda < 1$*,* $\rho > 0$ *and* $\Phi \in \mathcal{F}$*, we have that also* $\rho \cdot \Phi_{x_0,\lambda} \in \mathcal{F}$.

(A2) (Existence of a homogeneous "blowup") Given $|x_0| < 1, \lambda_k \downarrow 0$ *and* $\Phi \in \mathcal{F}$*, there exist a sequence* $\rho_k \in (0, +\infty)$*, a real number* $\alpha \geqslant 0$ *and a function* $\Psi \in \mathcal{F}$ *satisfying the identity* $\Psi_{0,\lambda} = \lambda^\alpha \Psi$ *for any* $\lambda > 0$*, such that up to a subsequence there holds*

$$\rho_k \cdot \Phi_{x_0,\lambda_k} \to \Psi.$$

(A3) (Singular set hypotheses) There exists a map $\mathscr{S} : \mathcal{F} \to \mathcal{C}$ *(where* $\mathcal{C} := \{A \subset \mathbb{R}^N : A \cap B_1(0)$ *is closed in* $B_1(0)\}$*) such that*

(i) Given $|x_0| \leqslant 1 - \lambda$*,* $0 < \lambda < 1$*,* $\rho > 0$ *and* $\Phi \in \mathcal{F}$*, it holds*

$$\mathscr{S}(\rho \cdot \Phi_{x_0,\lambda}) = (\mathscr{S}(\Phi))_{x_0,\lambda} := \frac{\mathscr{S}(\Phi) - x_0}{\lambda}.$$

(ii) Given $|x_0| < 1$*,* $\lambda_k \downarrow 0$ *and* $\Phi, \Psi \in \mathcal{F}$ *such that there exists* $\rho_k \geqslant 0$ *satisfying* $\Phi_k := \rho_k \cdot \Phi_{x_0,\lambda_k} \to \Psi$ *in* $\mathcal{F}$*, it holds the following "continuity" property:*

$$\forall \varepsilon > 0 \, \exists k(\varepsilon) > 0 : \ k \geqslant k(\varepsilon) \Rightarrow \mathscr{S}(\Phi_k) \cap B_1(0) \subseteq \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \mathscr{S}(\Phi)) < \varepsilon\}.$$

Then, if we define

$$d = \max\big\{\dim L : \ L \text{ is a subspace of } \mathbb{R}^N \text{ and there exist } \Phi \in \mathcal{F} \text{ and } \alpha \geqslant 0 \\ \text{such that } \mathscr{S}(\Phi) \neq \emptyset \text{ and } \Phi_{y,\lambda} = \lambda^\alpha \Phi \ \forall y \in L, \ \lambda > 0\big\},$$

either $\mathscr{S}(\Phi) \cap B_1(0) = \emptyset$ *for every* $\Phi \in \mathcal{F}$*, or else* $\mathscr{H}_{\dim}(\mathscr{S}(\Phi) \cap B_1(0)) \leqslant d$ *for every* $\Phi \in \mathcal{F}$*. Moreover there exist a function* $\Psi \in \mathcal{F}$*, a d-dimensional subspace* $L \leqslant \mathbb{R}^N$ *and a real number* $\alpha \geqslant 0$ *such that*

$$\Psi_{y,\lambda} = \lambda^\alpha \Psi, \qquad \forall y \in L, \ \lambda > 0, \tag{A.1}$$
$$\mathscr{S}(\Psi) \cap B_1(0) = L \cap B_1(0).$$

If in addition $d = 0$ *then* $\mathscr{S}(\Phi) \cap B_\rho(0)$ *is a finite set for each* $\Phi \in \mathcal{F}$ *and* $0 < \rho < 1$*.*

Remark A.27. If condition (A.1) holds for a function Φ, this means that it is a homogeneous function of some degree $\alpha \geqslant 0$ for every center $x_0 \in L$. The idea of the theorem, in a not very precise way, is that in order to control the dimensions of $\mathscr{S}(\Phi)$ for every $\Phi \in \mathcal{F}$, we just need to control the Φ's in $\mathcal{F}$ that are homogeneous.

Definition A.28. *1. We define, for every* $\Phi \in \mathcal{F}$ *and* $x_0 \in B_1(0)$*, the set of all possible homogeneous blowup's at* x_0 *by*

$$T(\Phi, x_0) = \Big\{\Psi \in \mathcal{F} : \ \text{ there exists } \alpha \geqslant 0 \text{ such that } \Psi_{0,\lambda} = \lambda^\alpha \Psi \ \forall \lambda > 0 \text{ and} \\ \Psi = \lim_k \rho_k \cdot \Phi_{x_0,\lambda_k} \text{ for some sequences } \lambda_k \downarrow 0, \ \rho_k \in (0, +\infty)\Big\}. \tag{A.2}$$

Observe that $T(\Phi, x_0) \neq \emptyset$ *by property (A2).*

2. Given any $l \geqslant 0$*, we define the set* $\mathcal{F}^l := \big\{\Phi \in \mathcal{F} : \ \mathscr{H}^l(\mathscr{S}(\Phi) \cap B_1(0)) > 0\big\}$*.*

Before passing to the proof of the theorem, we will first present two auxiliary lemmas, whose proofs we postpone to the end of this section.

Lemma A.29. *For every* $\Phi \in \mathcal{F}^l$ *there exists* $\Psi \in T(\Phi, x) \cap \mathcal{F}^l$ *for* $\mathscr{H}^l$*–a.e.* $x \in \mathscr{S}(\Phi) \cap B_1(0)$ *(that is,* $\mathscr{H}^l(\{x \in \mathscr{S}(\Phi) \cap B_1(0) : \ T(\Phi, x) \cap \mathcal{F}^l = \emptyset\}) = 0$*).*

Lemma A.30. *Let* $\Phi \in \mathcal{F}$ *and suppose that there exist* L *(a subspace of* $\mathbb{R}^N$*) and* $\alpha \geqslant 0$ *such that*

$$\Phi_{y,\lambda} = \lambda^\alpha \Phi \qquad \text{for all } \ y \in L, \ \lambda > 0.$$

Then for any $\Psi \in T(\Phi, x)$ *there exists* $h \geqslant 0$ *such that*

$$\Psi_{y,\lambda} = \lambda^h \Psi \qquad \text{for all } \ y \in \operatorname{span}(L \cup \{x\}), \ \lambda > 0.$$

Proof of Theorem A.26. In order to prove the first part of the theorem it is equivalent to check that

$$\mathcal{F}^l = \emptyset \qquad \text{for every } l > d. \tag{A.3}$$

Suppose then, by contradiction, that there exists $l > d$ such that $\mathcal{F}^l \neq \emptyset$. Then by Lemma A.29 there exist $\Phi \in \mathcal{F}^l$ and $\alpha \geqslant 0$ such that $\Phi_{0,\lambda} = \lambda^\alpha \Phi$ for every $\lambda > 0$. Let us now take any L (subspace of $\mathbb{R}^N$) of dimension k such that

$$\Phi_{y,\lambda} = \lambda^\alpha \Phi \qquad \text{for all } y \in L, \lambda > 0$$

(which exists, at least for $k = 0$). Since $l > d$, then by the definition of d we must have $k < l$, which implies that $\mathscr{H}^l(L) = 0$. This, together with the fact that $\Phi \in \mathcal{F}^l$, insures the existence (by using Lemma A.29) of a point $x \in \mathscr{S}(\Phi) \cap B_1(0) \setminus L$ and of a function Ψ such that

$$\Psi \in T(\Phi, x) \cap \mathcal{F}^l.$$

Observe now that by Lemma A.30 Ψ satisfies the identity (for some $h \geqslant 0$)

$$\Psi_{y,\lambda} = \lambda^h \Psi \qquad \text{for every} \quad y \in T, \lambda > 0,$$

where $T := \operatorname{span}(L \cup \{x\})$ (which is a subspace of dimension $k+1$). Again by the definition of d, and since $l > d$, we must have $k + 1 < l$. But then by iterating this process we will get a contradiction in a finite number of steps.

As for the second part of the theorem, we know by the definition of d that there exist $\Psi \in \mathcal{F}$ with $\mathscr{S}(\Psi) \neq \emptyset$, a d–dimensional subspace L and $\alpha \geqslant 0$ verifying (A.1). Then (A3)-(i) together with the fact that $\mathscr{S}(\mathcal{F}) \in \mathcal{C}$ yields

$$L \cap B_1(0) \subseteq \mathscr{S}(\Psi) \cap B_1(0).$$

But now observe that these sets must coincide, since if there was an $x \in \mathscr{S}(\Psi) \cap B_1(0) \setminus L \cap B_1(0)$ we could argue as above (observe that $\mathscr{H}^d(L \cap B_1(0)) < +\infty$ and $\mathscr{H}^d(\mathscr{S}(\Psi) \cap B_1(0)) = +\infty$, as $\operatorname{span}(L \cup \{x\}) \cap B_1(0) \subseteq \mathscr{S}(\Psi) \cap B_1(0)$), constructing a function $\bar{\Psi} \in T(\Phi, x)$ with $\mathscr{S}(\bar{\Psi}) \neq \emptyset$ which, by the conclusions of Lemma A.30, would contradict the definition of d.

Finally suppose that $d = 0$; then in particular $\mathscr{H}^1(\mathscr{S}(\Phi) \cap B_1(0)) = 0$ for every $\Phi \in \mathcal{F}$. If $\mathscr{S}(\Phi) \cap B_\rho(0)$ is not finite for some $\Phi \in \mathcal{F}$ and $\rho > 0$, choose $x \in \bar{B}_\rho(0) \subseteq B_1(0)$ and take a sequence $x_k \in \mathscr{S}(\Phi) \cap B_\rho(0)$ such that $x_k \to x$ and $x_k \neq x$ for every k. By taking $\lambda_k = 2|x_k - x| \to 0$ there exists exist a sequence $\rho_k > 0$ and a function $\Psi \in T(\Phi, x)$ such that, up to a subsequence,

$$\rho_k \Phi_{x,\lambda_k} \to \Psi \quad \text{in } \mathcal{F} \qquad \text{and} \qquad \frac{x_k - x}{\lambda_k} \to \xi \in \partial B_{1/2}(0).$$

Observe that $(x_k - x)/\lambda_k \in \mathscr{S}(\rho_k \Phi_{x,\lambda_k}) \cap B_1(0)$ and hence $\xi \in \mathscr{S}(\Psi)$. Furthermore, by taking into account assumption (A3)-(i) and the fact that $\Psi_{0,\lambda} = \lambda^\alpha \Psi$ for some $\alpha \geqslant 0$, we conclude that the segment that connects the origin to ξ is contained in $\mathscr{S}(\Psi) \cap B_1(0)$. Thus $\mathscr{H}^1(\mathscr{S}(\Psi) \cap B_1(0)) > 0$, which is a contradiction. $\square$

Let us now pass to the proof of the two lemmas stated above.

Proof of Lemma A.29. We start with the observation that by taking in consideration the definition of Hausdorff measure as well as Lemma A.22 we can rewrite $\mathcal{F}^l$ as follows:

$$\mathcal{F}^l = \{\Phi \in \mathcal{F} : \ \mathscr{H}^l_\infty(\mathscr{S}(\Phi) \cap B_1(0)) > 0\}.$$

By Lemma A.25 we have that

$$\limsup_{\rho \to 0} \frac{\mathscr{H}^l_\infty(\mathscr{S}(\Phi) \cap B_1(0) \cap B_\rho(x))}{\rho^l} > 0$$

for $\mathscr{H}^l$–a.e $x \in \mathscr{S}(\Phi) \cap B_1(0)$. In particular for any such $x \in \mathscr{S}(\Phi) \cap B_1(0)$ we can fix a sequence $\lambda_k \downarrow 0$ such that

$$\lim_k \frac{\mathscr{H}^l_\infty(\mathscr{S}(\Phi) \cap B_{\lambda_k}(x))}{\lambda_k^l} > 0. \tag{A.4}$$

Up to a subsequence, we can also suppose, by (A2), that the sequence $(\lambda_k)_k$ is such that there are $\Psi \in T(\Phi, x)$ and $\rho_k \in (0, +\infty)$ with $\lim_k \rho_k \Phi_{x,\lambda_k} = \Psi$. So, in order to conclude the proof, we need only to check that $\Psi \in \mathcal{F}^l$. Suppose that this does not hold; then for every $\varepsilon > 0$ there exist balls $\{B_{\rho_j}(x_j)\}_j$ such that

$$\mathscr{S}(\Psi) \cap B_1(0) \subseteq \cup_j B_{\rho_j}(x_j) \qquad \text{and} \qquad \sum_j \mathscr{H}^l_\infty(B_{2\rho_j}(x_j)) < \varepsilon.$$

By (A3)-(ii), we conclude that there exists $k(\varepsilon) > 0$ such that whenever $k > k(\varepsilon)$ we have

$$\mathscr{S}(\rho_k \Phi_{x,\lambda_k}) \cap B_1(0) \subset \cup_j B_{2\rho_j}(x_j).$$

Therefore

$$\mathscr{H}^l_\infty(\mathscr{S}(\rho_k \Phi_{x,\lambda_k}) \cap B_1(0)) \leqslant \mathscr{H}^l_\infty(\cup_j B_{2\rho_j}(x_j)) \leqslant \sum_j \mathscr{H}^l_\infty(B_{2\rho_j}(x_j)) < \varepsilon.$$

In conclusion, we have that for every $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that $k > k(\varepsilon)$ implies

$$\mathscr{H}^l_\infty(\mathscr{S}(\rho_k \Phi_{x,\lambda_k}) \cap B_1(0))) < \varepsilon.$$

But now since $\mathscr{H}^l_\infty(\mathscr{S}(\rho_k \cdot \Phi_{x,\lambda_k}) \cap B_1(0)) = \mathscr{H}^l_\infty(\mathscr{S}(\Phi) \cap B_{\lambda_k}(x))/\lambda_k^l$, we get a contradiction with (A.4). □

Proof or Lemma A.30. Since Ψ is a homogeneous function centered at 0 of degree $h > 0$, it is enough to prove that

$$\Psi_{y,1} = \Psi \quad \forall y \in L, \qquad \text{and} \qquad \Psi_{\beta x,1} = \Psi \quad \forall \beta \in \mathbb{R}.$$

In fact if such a claim holds then for every $y \in L$, $\lambda, \beta > 0$ and $\xi \in \mathbb{R}^N$,

$$\Psi_{y+\beta x,\lambda}(\xi) = \Psi(y + \beta x + \lambda\xi) = \Psi(\beta x + \lambda\xi) = \Psi(\lambda\xi) = \lambda^h \Psi(\xi).$$

It is at this point that we will need the technical assumptions made for the convergence in $\mathcal{F}$. Taking $y \in L$, we have

$$\begin{aligned} \Psi_{y,1}(\xi) &= \Psi(y + \xi) = \lim_k \Phi_{x,\lambda_k}(y + \xi) = \lim_k \Phi(x + \lambda_k y + \lambda_k \xi) = \lim_k \Phi(x + \lambda_k \xi) \\ &= \lim_k \Phi_{x,\lambda_k}(\xi) = \Psi(\xi). \end{aligned}$$

As for the second equality we have, given $\beta \in \mathbb{R}$,

$$\begin{aligned} \Psi_{\beta x,1}(\xi) &= \Psi(\beta x + \xi) = \lim_k \Phi_{x,\lambda_k}(\beta x + \xi) = \lim_k \Phi(x + \lambda_k \beta x + \lambda_k \xi) \\ &= \lim_k (1 + \lambda_k \beta)^\alpha \Phi\Big(x + \frac{\lambda_k}{1 + \lambda_k \beta}\xi\Big) = \lim_k (1 + \lambda_k \beta)^\alpha \Phi_{x, \frac{\lambda_k}{1+\lambda_k\beta}}(\xi) = \Psi(\xi). \end{aligned}$$

□

Appendix B

Ekeland's variational principle on C^1 Hilbert manifolds

The purpose of this appendix is to state a general version of the Ekeland's variational principle (Theorem B.5) and to prove two consequences of this result in the context of C^1 manifolds which are very useful when solving minimizing problems (check Corollary B.6 and Theorem B.9 ahead).

For our purposes, it will be enough to work with manifolds lying in a fixed real Hilbert space. This allows a simpler and faster presentation of the preliminary definitions needed to understand and prove the desired results. For the general theory of Hilbert manifolds (or even Banach manifolds) we advice the reading of [137, Section 4.17] and to check the references mentioned therein.

Throughout this section X and Y will always denote real Hilbert spaces. Let $N \subseteq X$ be an arbitrary set. A map $f : N \to Y$ is said to be C^1 if for each $u \in N$ there is an open set $U \subseteq X$ containing u and a C^1 map $F : U \to Y$ such that $F = f$ over $N \cap U$. The map F is called an extension of f at u.

Definition B.1. *Let $N \subseteq X$ and $M \subseteq Y$ be arbitrary sets. A map $f : N \to M$ is called a C^1 diffeomorphism if f is bijective and both f and f^{-1} are C^1 maps.*

Definition B.2. *A set $N \subseteq X$ is called a (Hilbert) C^1 manifold if is locally diffeomorphic to a Hilbert space. This means that there is a Hilbert space Y such that every $u \in N$ admits a neighborhood V in N which is C^1 diffeomorphic to an open set U in Y. Any particular C^1 diffeomorphism*

$$\varphi : U \subseteq Y \to V \subseteq N$$

is called a parametrization of N.

Now, given $N \subseteq X$ a C^1 manifold we associate to each $u \in N$ a vector subspace of X, $T_u(N)$, called the *tangent space of N at u*. Choosing a parametrization $\varphi : U \subseteq Y \to N$ such that $\varphi(y) = u$ for some $y \in U$, we define $T_u(M)$ as being $d\varphi_y(Y)$, the image of the continuous linear map $d\varphi_y : Y \to X$.

It is easy to prove that this definition does not depend on the choice of the parametrization. Moreover, $d\varphi_y : Y \to T_u(M)$ is a continuous isomorphism (*i.e.*, it is a bijective linear map between vector spaces).

Definition B.3. *Let $N \subseteq X$, $M \subseteq Y$ be two C^1 manifolds and take a C^1 map*

$$f : N \to M.$$

The derivative of f at $u \in N$ is a map

$$df_u : T_u(N) \to Y$$

defined as follows. Take an open set W containing $u \in N$ and a C^1 extension $F : W \to Y$ such that $F_{|W\cap N} = f_{|W\cap N}$. Then we define $df_u(v) = dF_u(v)$ for all $v \in T_u(N)$.

It can be proved (by parametrizing N and M) that df_u is an isomorphism between $T_u(N)$ and $T_{f(u)}(M)$, and that its definition does not depend on the choice of the extension F.

Remarks 1. If $U \subseteq X$ is an open set, then U is a C^1 manifold and $T_u(U) = X$ for every $u \in U$.

2. It holds the so called *chain rule*: if $f : N \to M$ and $g : M \to Y$ are two C^1 map and N, M are C^1 manifolds, then $g \circ f$ is C^1 and

$$d(g \circ f)_u(v) = dg_{f(u)} \circ df_u(v) \qquad \text{for every } v \in T_u(N).$$

3. If $f : X \to Y$ is a C^1 map and $N \subseteq X$ is a C^1 manifold, then the restricted function $f|_N : N \to Y$ is also C^1 and $d(f|_N)_u(v) = df_u(v)$ for every $v \in T_u(N)$.

Let us give an equivalent characterization of tangent space.

Proposition B.4. *Let $N \subseteq X$ be a C^1 manifold. Then, for each $u \in N$,*

$$T_u(N) = \left\{ v \in X : \begin{array}{c} \text{there exists } \varepsilon > 0 \text{ and a } C^1 \text{ curve } \alpha : (-\varepsilon, \varepsilon) \to N \\ \text{such that } \alpha(0) = u,\ \alpha'(0) = v \end{array} \right\}.$$

Proof. Let $\varphi : U \subseteq Y \to N \subseteq X$ be a parametrization of N such that $\varphi(y) = u$ for some $y \in U$. If $\alpha : (-\varepsilon, \varepsilon) \to N$ is a curve such that $\alpha(0) = u$ and $\alpha'(0) = v$, then consider the curve $t \mapsto \varphi^{-1}(\alpha(t))$ in U (for $|t|$ sufficiently small we have $\alpha(t) \in \varphi(U)$). Since $\varphi \circ \varphi^{-1}(\alpha(t)) = \alpha(t)$, we have that

$$d\varphi_y \left(\frac{d}{dt} \varphi^{-1}(\alpha(t))_{|t=0} \right) = \alpha'(0) = v$$

and hence $v \in T_u(N)$. Reciprocally, for each $v \in T_u(N)$ let $w \in Y$ be such that $d\varphi_y(w) = v$ and consider the curve $\alpha(t) := \varphi(y + tw) \in N$ for $|t|$ sufficiently small. We have $\alpha(0) = \varphi(y) = u$ and $\alpha'(0) = d\varphi_y(w) = v$. $\square$

The following result is a particular case of a more general principle proved by Ekeland in 1974. Its proof can be found for instance in [138, Theorem 2.H].

Theorem B.5 (Ekeland's Variational Principle)**.** *Assume that N is a nonempty closed subset of X and take a lower semi-continuous functional $f : N \to \mathbb{R}$ such that* $\inf_{w\in N} f(w) > -\infty$. *For every $\varepsilon > 0$ we choose a point $v \in N$ such that*

$$f(v) \leqslant \inf_{w\in N} f(w) + \varepsilon.$$

Then there exists $u \in N$ such that

$$f(u) \leqslant f(w) + \varepsilon\|u - w\| \qquad \text{for all } w \in N$$

and $\|u - v\| \leqslant 1$, $f(u) \leqslant f(v)$.

Corollary B.6. *Let $N \subseteq X$ be a closed C^1 manifold and let $f : N \to \mathbb{R}$ be a C^1 function such that $\inf_{w\in N} f(w) > -\infty$. Then for each $\varepsilon > 0$ there exists a point $u \in N$ such that*

$$f(u) \leqslant \inf_{w\in N} f(w) + \varepsilon \qquad \textit{and} \qquad \|df_u\|_{(T_u(N))'} \leqslant \varepsilon.$$

Proof. By applying the previous theorem we easily get the existence of u such that

$$f(u) \leqslant \inf_{w\in N} f(w) + \varepsilon \qquad \text{and} \qquad f(u) \leqslant f(w) + \varepsilon\|u - w\| \ \text{ for all } w \in N.$$

For every $v \in T_u(N)$ let $\alpha : (-\varepsilon, \varepsilon) \to N$ be a curve such that $\alpha(0) = u$ and $\alpha'(0) = v$. We have

$$f(\alpha(t)) - f(u) \geqslant -\varepsilon\|u - \alpha(t)\| = -\varepsilon\|tv + \mathrm{o}(t)\| \qquad \text{as } t \to 0,$$

and hence $df_u(v) \geqslant -\varepsilon\|v\|$. Thus $|df_u(v)| \leqslant \varepsilon\|v\|$ for every $v \in T_u(N)$. □

B.1 A particular type of manifolds: the set of the solutions of an equation

Let $g : U \subseteq X \to Y$ be a C^1 function, where U is an open set of X. Take the set

$$N = \{u \in U : \ g(u) = 0\}$$

and assume that the linear map $dg_u : X \to Y$ is surjective for each $u \in N$.

Theorem B.7. *With the previous notations and assumptions, N is a C^1 manifold with*

$$T_u(N) = Ker(dg_u) := \{v \in X : \ dg_u(v) = 0\}$$

for each $u \in N$.

Proof. Since X is a Hilbert space and $Ker(dg_u)$ is a closed vector space, then we can split X in $X = Ker(dg_u) \oplus Ker(dg_u)^\perp$. Fix a point $u \in N$ and write it as $u = u_0 + u_1$, with $u_0 \in Ker(dg_u)$, $u_1 \in Ker(dg_u)^\perp$. Consider the map

$$F : Ker(dg_u) \times Ker(dg_u)^\perp \to Y, \quad (v, w) \mapsto g(v + w),$$

defined in a neighborhood of (u_0, u_1). Since $dg_u(X) = Y$, the map

$$F_w(u_0, u_1) = dg_u|_{Ker(dg_u)^\perp} : Ker(dg_u)^\perp \to Y$$

is bijective. Hence by the implicit function theorem (see for instance [138, Theorem 4.E]) there exist V an open set of X containing u, W an open set of $Ker(dg_u)$ containing u_0, and a C^1 map $\Psi : W \to Ker(dg_u)^\perp$ such that $\Psi(u_0) = u_1$ and

$$F(v, w) = g(v + w) = 0, \ v + w \in V \quad \Longleftrightarrow \quad w = \Psi(v), \ v \in W.$$

Thus

$$\varphi : W \subseteq Ker(dg_u) \to X, \quad v \mapsto v + \Psi(v)$$

is a parametrization of N around u and since $d\Psi_{u_0}(v) = 0$ for every $v \in Ker(dg_u)$, we have $T_{u_0}(N) = d\varphi_{u_0}(Ker(dg_u)) = Ker(dg_u)$. □

Lemma B.8. *With the previous notations and assumptions, let $f : X \to \mathbb{R}$ be a C^1 map. Then $f|_N$ is a C^1 function and for each $u \in N$ there exists $\lambda \in Y'$ such that*

$$df_u(v) = d(f|_N)_u(Pv) + \lambda \circ dg_u(v)$$

for every $v \in X$, where $P : X \to Ker(dg_u)$ denotes the orthogonal projection of X onto $Ker(dg_u)$.

Proof. Recall from the proof of the previous result that the linear map $dg_u|_{Ker(dg_u)^\perp} : Ker(dg_u)^\perp \to Y$ is bijective. Let $G := \left(dg_u|_{Ker(dg_u)^\perp}\right)^{-1} : Y \to Ker(dg_u)^\perp$. From the closed graph theorem we deduce that G is a linear continuous function (see for instance [23, Corollary II.6]) and hence $\lambda := df_u \circ G : Y \to \mathbb{R}$ is an element of Y'. We have $df_u(w) = \lambda \circ dg_u(w)$ for every $w \in Ker(dg_u)^\perp$, and $\lambda \circ dg_u(w) = 0$ for $w \in Ker(dg_u)$. Moreover, $df_u(v) = d(f|_N)_u(v)$ for every $v \in T_u(N) = Ker(dg_u)$, and thus the conclusion of the lemma follows. $\square$

By combining Corollary B.6 with the two previous results, we obtain the following statement.

Theorem B.9. *Let $g : U \subseteq X \to Y$ be a C^1 function, U being an open set of X. Suppose that*

$$N = \{u \in U : \ g(u) = 0\}$$

is a closed set, and that for each $u \in N$, $dg_u : X \to Y$ is a surjective map. Consider furthermore a C^1 map $f : X \to \mathbb{R}$ such that $c := \inf_{w \in N} f(w) > -\infty$.

Then there exist sequences $(u_n)_n \subseteq N$ and $(\lambda_n)_n \subseteq Y'$ such that

$$f(u_n) \to c \qquad \text{and} \qquad df_{u_n} - \lambda_n \circ dg_{u_n} \to 0 \ \text{in } X'. \tag{B.1}$$

In the literature, a sequence $(u_n)_n$ satisfying (B.1) is usually called a Palais-Smale sequence for f in N (or, more generally, satisfying $f(u_n) \to c$ and $\|df_{u_n}\|_{(T_{u_n}(N))'} \to 0$). Hence we have shown that, under suitable assumptions, any *minimal energy level* admits an approximating Palais-Smale sequence.

Appendix C

Some mathematical results

In this appendix we collect several results which are used during this dissertation.

Lemma C.1. *Suppose that u is an $H^2(\Omega)$ function such that $u \geqslant 0$ in Ω and $u = 0$ on $\partial\Omega$. Then $\partial_\nu u \leqslant 0$ on $\partial\Omega$.*

Proof. First of all we observe that by the trace embeddings we have that $\nabla u \in L^2(\partial\Omega)$ and $\partial_\nu u = \langle \nabla u, \nu \rangle$ a.e. on $\partial\Omega$. Moreover, we remark that if $u \in C^1(\Omega)$ then the conclusion of Lemma C.1 is easy to obtain, since in such a case for every $x_0 \in \partial\Omega$ it is immediate to see that

$$\partial_\nu u(x_0) = \lim_{t \to 0^+} \frac{u(x_0) - u(x_0 - t\nu)}{t} = \lim_{t \to 0^+} \frac{-u(x_0 - t\nu)}{t} \leqslant 0.$$

Step 1. Let us analyze the case where $u \in H^2(\Omega)$ with $\Omega = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$. Consider the "odd" extension of u to the whole $\mathbb{R}^N$,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ -u(x_1, \dots, x_{N-1}, -x_N) & \text{if } x \notin \Omega. \end{cases}$$

Let ρ_n denote a sequence of radially symmetric *mollifiers* (see for instance [23]), decreasing in $|x|$. We recall that

$$\operatorname{supp} \rho_n \subseteq (0, 1/n), \qquad \int_{\mathbb{R}^N} \rho_n = 1, \qquad \text{and} \qquad \rho_n \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

Take the sequence $\rho_n * \tilde{u} \in C_c^\infty(\mathbb{R}^N)$, where

$$(\rho_n * \tilde{u})(x) = \int_{\mathbb{R}^N} \rho_n(x - y)\tilde{u}(y)\, dy$$

denotes the convolution of ρ_n with $\tilde{u}$. Let σ denote the reflection in $\mathbb{R}^N$ with respect to $\partial\Omega$. For each $x = (x_1, \dots, x_{N-1}, 0) \in \partial\Omega$, by using the symmetry of $\tilde{u}$ and ρ_n and the fact that $\sigma(\Omega^C) = \bar{\Omega}$, it follows that

$$(\rho_n * \tilde{u})(x) = \int_\Omega \rho_n(x' - y', -y_N)\tilde{u}(y)\, dy + \int_\Omega \rho_n(x' - y', y_N)\tilde{u}(y', -y_N)\, dy = 0.$$

Now take $x \in \Omega$. If $B_{1/n}(x) \subseteq \Omega$, then

$$\rho_n * \tilde{u}(x) = \int_{B_{1/n}(x)} \rho_n(x-y)u(y)\,dy \geqslant 0.$$

On the other hand if $B_{1/n}(x) \cap \Omega^C \neq \emptyset$ then we see that

$$\rho_n * \tilde{u}(x) \geqslant \int_{B_{1/n}(x)\setminus((\Omega^C \cap B_{1/n}(x)) \cup \sigma(\Omega^C \cap B_{1/n}(x)))} \rho_n(x-y)u(y)\,dy \geqslant 0,$$

where the first inequality is a consequence of the fact that $\rho_n(x-\cdot)$ is radially decreasing with respect to x.

We have concluded that $\rho_n * \tilde{u}$ is a regular function such that $\rho_n * \tilde{u} \geqslant 0$ in Ω and $\rho_n * \tilde{u} = 0$ on $\partial\Omega$, and hence

$$\partial_\nu(\rho_n * \tilde{u}) = \langle \nabla(\rho_n * \tilde{u}), \nu \rangle \leqslant 0 \qquad \text{on } \partial\Omega.$$

Since $\rho_n * \tilde{u} \to \tilde{u}$ in $H^2(\Omega)$, then also $\partial_\nu(\rho_n * \tilde{u}) \to \partial_\nu \tilde{u}$ in $L^2(\partial\Omega)$, which yields that $\partial_\nu \tilde{u} \leqslant 0$ a.e. on $\partial\Omega$.
Step 2. Finally, take $u \in H^2(\Omega)$ with Ω a general regular domain and fix $x_0 \in \partial\Omega$. Let A be a neighborhood of the origin in $\mathbb{R}^N$ and let $\Psi : B_\delta(x_0) \to A$ be a diffeomorphism such that $\Psi(x_0) = 0$, $\Psi(B_\delta(x_0) \cap \Omega) = A \cap \{x_N > 0\}$, $\Psi(B_\delta(x_0) \cap \partial\Omega) = A \cap \{x_N = 0\}$ and $\Psi(B_\delta(x_0) \cap \Omega^C) = A \cap \{x_N < 0\}$. Define the function $v(\Psi(x)) = u(x)$. We have, for $x \in B_\delta(x_0) \cap \partial\Omega$,

$$\partial_\nu u(x) = \langle \nabla u(x), \nu(x) \rangle = \langle \nabla v(\Psi(x)), d\Psi_x(\nu(x)) \rangle = \partial_\nu v(\Psi(x)),$$

which is non positive by Step 1. This ends the proof of the lemma. $\square$

Lemma C.2. *Let $u \in C^2(\Omega)$ satisfy $-\Delta u \leqslant au$ for some $a > 0$. Then for any ball $B_R(x_0) \Subset \Omega$ we have*

$$u(x_0) \leqslant \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u + \frac{a}{2(N+2)} R^2 \|u\|_{L^\infty(B_R(x_0))}.$$

Proof. This proof is an easy adaptation of the standard proof of the mean value theorem for subharmonic function [67, Theorem 2.1]. Let $\rho \in (0,R)$. From the divergence theorem we see that

$$\int_{\partial B_\rho(x_0)} \partial_\nu u\,d\sigma = \int_{B_\rho(x_0)} \Delta u \geqslant -\int_{B_\rho(x_0)} au. \tag{C.1}$$

On the other hand, denoting the polar coordinates centered at x_0 by (r,θ),

$$\int_{\partial B_\rho(x_0)} \partial_\nu u\,d\sigma = \rho^{N-1} \int_{\partial B_1(0)} \partial_r u(\rho,\theta)\,d\sigma = \rho^{N-1} \frac{d}{d\rho}\left(\frac{1}{\rho^{N-1}} \int_{\partial B_\rho(x_0)} u\,d\sigma \right). \tag{C.2}$$

We combine (C.2) with (C.1) obtaining

$$\frac{d}{d\rho}\left(\frac{1}{\rho^{N-1}} \int_{\partial B_\rho(x_0)} u\,d\sigma \right) \geqslant -\frac{1}{\rho^{N-1}} \int_{B_\rho(x_0)} au \qquad \text{for every } \rho \in (0,R).$$

By fixing $r < R$ and integrating the previous inequality between r and R, we obtain that

$$\begin{aligned}\frac{1}{r^{N-1}}\int_{\partial B_r(x_0)} u\,d\sigma &\leqslant \frac{1}{R^{N-1}}\int_{\partial B_R(x_0)} u\,d\sigma + \int_r^R \left(\frac{1}{\rho^{N-1}}\int_{B_\rho(x_0)} au\right) d\rho \\ &\leqslant \frac{1}{R^{N-1}}\int_{\partial B_R(x_0)} u\,d\sigma + \frac{a|B_1|}{2}(R^2 - r^2)\|u\|_{L^\infty(B_R(x_0))}\end{aligned}$$

and thus, as $r \to 0$,

$$|\partial B_1|u(x_0) \leqslant \frac{1}{R^{N-1}}\int_{\partial B_R(x_0)} u\,d\sigma + \frac{a|B_1|}{2}R^2\|u\|_{L^\infty(B_r(x_0))}.$$

Hence

$$R^{N-1}|\partial B_1|u(x_0) \leqslant \int_{\partial B_R(x_0)} u\,d\sigma + \frac{a|B_1|}{2}R^{N+1}\|u\|_{L^\infty(B_r(x_0))}$$

and now the result follows after a simple integration. □

The next two results give two estimates which, combined with lower estimates on the augmented Morse index, yield a lower bound for the energy of some critical levels. The first one is a classical estimate due to Cwickel [52] Lieb [80] and Rosenbljum [111].

Lemma C.3. *Suppose that $N \geqslant 3$ and $V \in L^{N/2}(\mathbb{R}^N)$. Then there exists a universal constant $C_N > 0$ such that the number of non-positive eigenvalues of the operator $-\Delta - V$ in $L^2(\mathbb{R}^N)$ is bounded from above by*

$$C_N\int_\Omega |V^+(x)|^{N/2}.$$

Taking now the biharmonic case

$$\Delta^2 u - V(x)u = \lambda u \quad \text{in } \Omega, \qquad u \in H^2(\Omega)\cap H^1_0(\Omega). \tag{C.3}$$

we have the following (*cf.* [52, 111] and also [75, Eq. (34)]).

Lemma C.4. *Suppose that $N \geqslant 5$ and $V \in L^{N/4}(\Omega)$. Then there exists a universal constant $D_N > 0$ such that the number of non-positive eigenvalues of the operator $\Delta^2 - V$ in $L^2(\mathbb{R}^N)$ is bounded from above by*

$$D_N\int_\Omega |V^+(x)|^{N/4}.$$

Now, we turn to a result which provides a relation between the number of nodal components of u and its Morse index. The proof we present is taken from [18].

Lemma C.5. *Consider a C^1 function $h : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfying*

$$h(x,s)s < \frac{\partial h}{\partial s}(x,s)s^2 \leqslant C(1+|s|^{2^*}) \qquad \textit{for every } x\in\Omega, s\neq 0.$$

Let $u \in H_0^1(\Omega)$ be a solution of

$$-\Delta u = h(x,u) \qquad x \in \Omega. \tag{C.4}$$

Then the number of connected components of u is bounded from above by $m(u)$, the Morse index of u.

Proof. Consider the C^2 energy functional of (C.4), namely

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega H(x,u),$$

with $H(x,s) = \int_0^s h(x,\xi)\,d\xi$. Let $\Omega_1, \Omega_2, \ldots, \Omega_k$ ($k \in \mathbb{N}$) be k connected components of $\Omega \setminus u^{-1}(\{0\})$ and set

$$v_j(x) = \begin{cases} u(x) & \text{if } x \in \Omega_j \\ 0 & \text{if } x \notin \Omega j \end{cases} \qquad j = 1, \ldots, k.$$

The functions $v_j \in H_0^1(\Omega)$ are clearly linearly independent and the conclusion of the lemma holds once we prove that

$$E''(u)(v,v) < 0 \qquad \text{for every } v \in \operatorname{span}\{v_1, \ldots, v_k\} \setminus \{0\}.$$

This in turn follows after a direct computation. For $v = \sum_{j=1}^k \alpha_j v_j \neq 0$,

$$\begin{aligned} E''(u)(v,v) &= \int_\Omega \Big(|\nabla v|^2 - \frac{\partial h}{\partial s}(x,u)v^2\Big) \\ &= \sum_{j=1}^k \alpha_j^2 \int_{\Omega_j} \Big(|\nabla v_j|^2 - \frac{\partial h}{\partial s}(x,u)v_j^2\Big) \\ &= \sum_{j=1}^k \alpha_j^2 \int_{\Omega_j} \Big(|\nabla u|^2 - \frac{\partial h}{\partial s}(x,u)u^2\Big) \\ &< \sum_{j=1}^k \alpha_j^2 \int_{\Omega_j} \Big(|\nabla u|^2 - h(x,u)u\Big) \\ &= \sum_{j=1}^k \alpha_j^2 \int_{\Omega_j} \Big(\langle \nabla u, \nabla v_j\rangle - g(x,u)v_j\Big) = \sum_{j=1}^k \alpha_j^2 E'(u)v_j = 0. \end{aligned}$$

□

Finally, we present an algebric lemma (*cf.* [11, Lemma 5.3] or [102, Proposition 10.46]).

Lemma C.6. *Le $(b_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that*

$$b_{k+1} \leqslant b_k + \delta d_k^\theta \qquad \forall k = k_0, \ldots, k_0 + m_0,$$

where $k_0, m_0 \in \mathbb{N}$, $\delta > 0$ and $0 < \theta < 1$. Then there exists $c = c(b_{k_0}, \delta, \theta)$ such that

$$b_k \leqslant c k^{1/(1-\theta)} \qquad \forall k = k_0, \ldots, k_0 + m_0.$$

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