Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Positivity and Variational Methods in Second and Fourth Order Boundary Value Problems

Ricardo Mariano Roque Capela Enguiça

Doutoramento em Matemática

Especialidade: Análise Matemática

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Abstract

We study the existence of solutions for a nonlocal singular second order ordinary differential equation. We obtain results through Krasnoselskii's fixed point Theorem and using some properties of the eigenvalues of the underlying singular linear problem and, on a different approach, through the monotone method associated with well-ordered lower and upper solutions.

We deal with second and fourth order problems in infinite intervals, where we prove the existence of an homoclinic or an heteroclinic solution. For the second order we consider both superlinear and bounded nonlinearities, and prove existence results through variational methods. A non-variational approach was made for a second order problem with a dissipative term and a p-laplacian problem was also adressed. Simpler fourth order bvp's were also tackled from a variational point of view.

We also analyse fourth order boundary value problems related to beam deflection theory, generalizing some well known results for the second order. We analysed two types of problems: the case where the correspondent fourth order operator can be decomposed in two positive second order operators and the case where that cannot be done. The results are obtained through topological arguments in association with lower and upper solutions.

Keywords

Second order boundary value problem, singularities, nonlocal problem, upper and lower solutions, fourth order boundary value problem, variational methods, unbounded domains, positivity.

Resumo

Nesta tese propomos demonstrar resultados de existência (e por vezes também de localização) para dois tipos de problemas:

(i) Problemas de valores na fronteira (pvf) menos analisados na literatura existente, como problemas singulares não-locais ou problemas em intervalos ilimitados;

(ii) Problemas de valores na fronteira de quarta ordem mais simples.

A teoria de pvf de segunda ordem num intervalo limitado tem sido desenvolvida há mais de um século de um modo bastante substancial. Provou-se a existência de soluções com propriedades específicas usando diversos métodos, como por exemplo a teoria do grau topológico, teoremas de ponto fixo, métodos variacionais, o método monótono, sub e sobre-soluções associadas a ferramentas topológicas, apenas para mencionar alguns. Nesta tese damos especial relevo à abordagem do ponto de vista das sub e sobre-soluções, uma vez que este método apoia-se em vários outros métodos mencionados e providencia informação sobre a localização das soluções.

Foi em 1893 que se provou pela primeira vez a existência de solução para um problema para o qual temos uma sub e (ou) uma sobre-solução. Nessa altura, Picard demonstrou a existência de uma solução para o problema de Dirichlet

$$-u'' = f(t, u), \quad u(a) = u(b) = 0,$$

onde f é crescente em u e $f(t, 0) = 0$. Provou-se a existência de solução no caso sublinear, caso existisse uma função $\alpha_0 > 0$ de classe C^2 tal que

$$-\alpha_0'' < f(t, \alpha_0), \ \forall t \in (a, b), \quad \alpha_0(a) = \alpha_0(b) = 0,$$

usando uma sucessão aproximante $0 < \alpha_0 \leq \alpha_1 \leq ...$, construída do seguinte modo:

$$-\alpha_n'' = f(t, \alpha_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0 \quad \forall n \in \mathbb{N}.$$

Nos anos 30 do século passado, Scorza Dragoni considerou o problema de Dirichlet

$$-u'' = f(t, u, u'), \quad u(a) = A, \, u(b) = B,$$

supondo que existiam α, $\beta \in C^2[a, b]$ com $\alpha \leq \beta$ tais que

$$\begin{aligned} -\alpha'' \leq & f(t, \alpha, y), \quad \forall t \in [a, b], \quad y \leq \alpha'(t) \\ \alpha(a) \leq & A, \quad \alpha(b) \leq B; \end{aligned}$$

$$
\begin{aligned}
-\beta'' \geq & f(t,\beta,y), \quad \forall t \in [a,b], \quad y \geq \beta'(t) \\
& \beta(a) \geq A, \quad \beta(b) \geq B.
\end{aligned}
$$

Demonstrou a existência de uma solução $u(t)$, com $\alpha \leq u \leq \beta$, considerando uma função f contínua em

$$E = \{(t,u,v) \colon \alpha(t) \leq u \leq \beta(t)\}\,.$$

Podemos também relacionar a existência de sub e sobre-soluções com o grau topológico. Usando uma definição adequada de sub e sobre-soluções estritas, é possível calcular o grau de um operator integral associado ao problema, no conjunto das funções contínuas entre a sub e a sobre-solução.

O caso das sub e sobre-soluções na ordem invertida só foi abordado no princípio dos anos 90, por Gossez and Omari, que demonstraram que em caso de não-ressonância com os valores próprios associados ao problema, existe uma solução. Nos anos 70, Amman já tinha apresentado alguns exemplos de problemas com sub e sobre-soluções não ordenadas, para os quais não existia qualquer solução.

Podemos também usar métodos variacionais para demonstrar a existência de soluções dadas sub e sobre-soluções ordenadas. Em muitos casos, o funcional de Euler-Lagrange associado ao pvf tem um mínimo local num conjunto de funções admissíveis entre a a sub e a sobre-solução.

Estes e outros métodos foram usados para garantir existência de soluções sem a presença de sub e sobre-soluções. Teoria do grau topológico, Teorema de Schauder, Teorema de Krasnoselskii, alternativa não-linear, teorema de Leggett-Williams, teoria de Sturm-Liouville são apenas alguns exemplos.

Relativamente a problemas de segunda ordem, nesta tese estudamos uma equação com uma singularidade na variável independente e uma dependência não-local da solução. As soluções consideradas podem ser vistas como soluções radiais de um problema de dimensão superior. Usamos o Teorema de Krasnoselskii para provar a existência de solução, e numa outra abordagem ao problema, usamos o método das sub e sobre-soluções com a ajuda de princípios de máximo não-locais. A equação estudada é a seguinte

$$-\Delta u = f\left(u, \int_U g(u)\right).$$

Procuramos soluções radiais desta equação numa bola de $\mathbb{R}^N$, reduzindo o problema a uma só variável, num intervalo limitado. A equação associada é

$$-v''(r) - \frac{n-1}{r}v'(r) = f\left(v(r), \omega_n \int_0^1 s^{n-1} g(v(s))\,ds\right),$$

singular em $r = 0$. Este problema foi abordado num artigo de Fijalkowski e Przeradski, onde a principal condição é $f(u,v) \leq Au + B$, com A relacionado com a função de Green associada. Abordando o problema com a mesma metodologia (Teorema de Krasnoselskii) e considerando os valores próprios do problema linear singular subjacente, demonstramos que é possível obter uma estimativa mais geral para a constante A. Abordamos também este problema do ponto de vista das sub e sobre-soluções. Provamos princípios de máximo não-locais e construimos uma sucessão monótona, convergente para uma solução radial do problema.

Os pvf de quarta ordem têm sido um domínio prolífico nas duas últimas décadas. Estes estão relacionados com aplicações importantes na teoria da deformação de vigas, mas a generalização de resultados bem conhecidos de segunda ordem é provavelmente a principal razão para o aumento do interesse nesta área.

Os pvf de quarta ordem têm uma estrutura mais complexa quando comparados com os de segunda ordem, e muitos resultados da segunda ordem não são facilmente generalizáveis. O facto de existirem mais derivadas intermédias não permite o uso das ferramentas topológicas disponíveis em ordens inferiores. Um exemplo disso é o teorema da existência de solução dadas sub e sobre-soluções na ordem invertida. De um modo geral, é verdade para a segunda ordem, mas são necessárias condições de monotonia para obter resultados na quarta ordem. Para quarta ordem, Cabada, Cid e Sanchez demonstraram resultados neste sentido apenas em 2007. No último capítulo analisamos problemas procurando obter resultados nesse sentido. Demonstramos resultados usando duas abordagens: decompondo operadores de quarta ordem em dois de segunda ordem quando isso for possível, ou uma abordagem mais directa analisando algumas propriedades das soluções. Os princípios de máximo têm um papel preponderante na possível aplicação do método das sub e sobre-soluções. Estudamos equações do tipo

$$u^{(4)} = f(x, u, u'')$$

no caso das condições de fronteira do tipo "viga apoiada" ($u(0) = u(1) = u''(0) = u''(1) = 0$) e periódicas, e com a não-linearidade sem dependência na segunda derivada no problema com condições de "viga encastrada" ($u(0) = u(1) = u'(0) = u'(1) = 0$).

O problema periódico sem dependência em u'' foi estudado por vários autores através de princípios de máximo e método monótono. Com dependência linear em u'', também foram obtidos resultados usando teoremas de ponto fixo. No caso das condições de "viga apoiada", resultados de existência e multiplicidade foram obtidos para equações não dependentes de u'', com dependência linear em u'' e no caso sobrelinear.

Neste trabalho abordamos os problemas periódico e de "viga apoiada" considerando condições de Lipschitz unilaterais para a não-linearidade em u e u'', supondo que existem sub e sobre-soluções (bem ordenadas ou na ordem invertida no caso periódico e bem ordenadas no caso da "viga apoiada"). A nossa abordagem é semelhante à usada num artigo de Gao, Jiang e Wan para equações de segunda ordem.

Para as condições do tipo "viga encastrada", existem menos resultados na literatura uma vez que a decomposição em dois operadores de segunda ordem não é apropriada. A condição imposta à não-linearidade é

$$f\,(x, \alpha(x)) + k\alpha(x) \leq f(x, u) + k\,u \leq f\,(x, \beta(x)) + k\beta(x), \quad \alpha(x) \leq u \leq \beta(x),$$

dadas sub e sobre-soluções α e β, e um domínio de variação para a constante k. A teoria de valores próprios e o comportamento oscilatório das soluções da equação

$$u^{(4)} + m^4 u = 0$$

desempenham um papel fundamental na nossa demonstração, que faz uso de um resultado bastante interessante de Schröder.

O estudo de problemas de segunda e quarta ordem em intervalos ilimitados mereceu também a nossa atenção nesta tese, onde os métodos variacionais são cruciais na demonstração de resultados de existência.

Considerando problemas autónomos de segunda ordem, a análise do plano de fases relativamente a existência de soluções homoclínicas e heteroclínicas é importante no sentido em que estas são separatrizes de diferentes tipos de comportamento de outras soluções do problema. Para problemas de quarta ordem não temos o plano de fases para fazer essa análise, mas os métodos analíticos da segunda ordem funcionam de um modo semelhante. O nosso objectivo é extrapolar resultados conhecidos em problemas autónomos para problemas não-autónomos, tanto na segunda como na quarta ordem.

O problema não-autónomo de segunda ordem

$$u'' = a(x)u - g\,(u) \quad u'(0) = u(+\infty) = 0$$

tem sido estudado nas duas últimas décadas, especialmente no caso em que $g(u)$ é uma potência sobrelinear. Nesta tese consideramos o caso $g(u)$ sobrelinear e o caso $g(u)$ limitado. Num artigo de Korman, Lazer e Yi são encontrados resultados de existência para $g(u) = u^p$, com $p > 1$ e $a(x)$ crescente em $[0, +\infty)$. Generalizamos alguns destes resultados, considerando a função $a(x) \to a \in \mathbb{R}$ não necessariamente monótona. Resolvemos uma sucessão de problemas num intervalo $[0, T]$ e considerando uma sucessão de Ts tendendo para ∞ adequada, encontramos uma solução do problema em $[0, +\infty)$ como limite das soluções correspondentes u_T.

Para o caso onde $g(u)$ é uma função limitada, a mesma abordagem não pode ser adaptada, tendo sido necessário considerar uma condição mais restritiva para a função $a(x)$. Estudamos também a equação

$$u' + c\,u' = a(x)u - g\,(u)\,,$$

e um problema autónomo com o operador p-Laplaciano. O teorema da passagem da montanha e o método diagonal são os principais mecanismos usados para provar a existência de solução por métodos variacionais.

Relativamente a problemas de quarta ordem em intervalos infinitos, estudamos o problema

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases}$$

considerando $a(x)$ em três situações: o caso $\lim_{x \to +\infty} a(x) = +\infty$, o caso crescente e o caso autónomo. Os métodos usados para obter resultados de existência são os mesmos dos da segunda ordem.

Palavras-chave

Problemas de valores na fronteira de segunda ordem, singularidades, problema não-local, sub e sobre soluções, problemas de valores na fronteira de quarta ordem, métodos variacionais, domínios ilimitados, positividade.

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Introduction

In this thesis we propose to prove existence results (in some cases some kind of localization results too) in two types of problems:

(i) Some second order boundary value problems (bvp) which are less covered by the existing literature, such as singular nonlocal problems, or problems in infinite intervals with certain types of nonlinearities;

(ii) Fourth order bvp's in simpler cases.

The theory of second order bvp's in a bounded interval has been developed for more than a century in a very substantial way. The existence of a solution with prescribed properties has been proved using many different tools such us the classical degree theory, fixed point theorems, variational methods, the monotone method, lower and upper solutions together with topological tools, just to name a few. In this thesis, the lower and upper solutions method was the preferred approach to prove existence results. It also relies on the other methods and provides some information on the localization of solutions.

The problem of proving the existence of a solution for a problem where we have an upper and (or) a lower solution has been tackled since 1893. By then, Picard searched for solutions of the Dirichlet problem

$$-u'' = f(t, u), \quad u(a) = u(b) = 0,$$

where f is increasing in the variable u and $f(t, 0) = 0$. He proved that in the sublinear case, if there exists a C^2 function $\alpha_0 > 0$ such that

$$-\alpha_0'' < f(t, \alpha_0),\ \forall t \in (a, b), \quad \alpha_0(a) = \alpha_0(b) = 0,$$

then there exists a monotone sequence of approximations $0 < \alpha_0 \leq \alpha_1 \leq ...$ converging to a nontrivial solution of the problem. The sequence was built by the following rule:

$$-\alpha_n'' = f(t, \alpha_{n-1}), \quad \alpha_n(a) = \alpha_n(b) = 0 \quad \forall n \in \mathbb{N}.$$

In the 30's of last century, Scorza Dragoni considered the general Dirichlet bvp

$$-u'' = f(t, u, u'), \quad u(a) = A,\ u(b) = B,$$

and assumed that there were functions α, $\beta \in C^2[a, b]$ with $\alpha \leq \beta$ such that

$$\begin{aligned} -\alpha'' \leq & f(t, \alpha, y), \quad \forall t \in [a, b], \quad y \leq \alpha'(t) \\ & \alpha(a) \leq A, \quad \alpha(b) \leq B; \end{aligned}$$

$$\begin{aligned} -\beta'' \geq & f(t, \beta, y), \quad \forall t \in [a, b], \quad y \geq \beta'(t) \\ & \beta(a) \geq A, \quad \beta(b) \geq B. \end{aligned}$$

He proved the existence of a solution $u(t)$ with $\alpha \leq u \leq \beta$ assuming that f was a continuous function on the set

$$E = \{(t, u, v) \colon \alpha(t) \leq u \leq \beta(t)\} .$$

Note that the differential inequalities satisfied by α and β are not only valid for $y = \alpha'(t)$, but for a much larger set of values, which is a much more restrictive condition than the standard definition of lower and upper solutions. To find such functions is not at all a trivial task and there are no methods to get them in a general case.

By 1937, Nagumo proved the existence of a solution for the problem above, but with the simpler notion of lower and upper solutions. He assumed that there exist $\alpha(t) < \beta(t)$ C^1 functions, α' and β' with left and right derivatives D_l, D_r,

$$-D_{l,r}\alpha' \leq f(t, \alpha, y), \quad \forall t \in [a, b], \qquad \alpha(a) \leq A, \quad \alpha(b) \leq B;$$

and the reversed inequalities are satisfied by β. Concerning the nonlinearity, he assumed that $f \colon E \to \mathbb{R}$, $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are continuous and f satisfies what became known as a Nagumo condition:

$$|f(t, u, v)| \leq \varphi\left(|v|\right),$$

where $\varphi \colon [0, +\infty) \to (0, +\infty)$ is continuous and is such that

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty.$$

Several other Nagumo type conditions have been considered to prove *a priori* bounds for the derivatives of possible solutions, which is fundamental to apply some usual topological tools to prove the existence of a solution.

We can also relate topological degree theory to the existence of lower and upper solutions. Basically, by defining strict lower and upper solutions in a convenient way, we are able to evaluate the degree of an operator associated with the boundary problem in the open bounded set of continuous functions between the strict lower and upper solutions. Existence and multiplicity results can be obtained by this approach, that was originally taken by Amann and more recently developed by De Coster and Habets.

Another relevant problem is to prove existence of a solution if there exist a lower and an upper solution, but not well-ordered (reversely ordered or not ordered at all). In the early 90's, Gossez and Omari dealt with this kind of problem and proved that under nonressonance with the associated eigenvalues, the problem with non well-ordered upper and lower solution has a solution. Earlier in the 70's, Amann had given some counter-examples for some problems where there were non well-ordered lower and upper solutions, but there were no solutions.

To prove the existence of a solution provided that an upper and a lower solution exist, we can use variational tools as well. In many cases, the Euler-Lagrange functional associated to the bvp has a local minimum in the set of admissible functions that lie between the lower and upper solutions.

These and other tools have been used to prove existence of solutions in general, without the existence of upper or lower solutions. The degree theory, Schauder's fixed point Theorem, Krasnoselskii's fixed point Theorem in its several versions, the nonlinear alternative, Leggett-Williams theorem, Sturm-Liouville comparison theorems, are just some examples used in a vast literature, to prove existence, and, in some cases, multiplicity or nonexistence.

Concerning second order problems, in this thesis we will study a differential equation with a singularity in the independent variable and a nonlocal dependence on the solution. The solutions to be considered can be seen as radial solutions of a higher dimension case. We will use the Krasnoselkii's fixed point Theorem to prove existence, and in another approach we will use new nonlocal maximum principles to prove existence via upper and lower solutions.

Singular boundary value problems arise naturally from physical models, both in the independent and the dependent variables. General existence results were difficult to prove, and until the 1990's, only very specific examples were examined. Usually, the techniques used in those cases were only applicable to that particular case. More general conclusions started to appear when new results in inequality and fixed point theory were available, specially by the end of last century. Concerning singularities in the independent variable, a very well known case is the L^p-Carathéodory nonlinearities (which is not the case of the second order singular problem studied here). We say that $f\colon I \times \mathbb{R}^N \to \mathbb{R}$ is called L^p-Carathéodory if

(i) the map $y \to f(t, y)$ is continuous for a.e. $t \in I$;

(ii) the map $t \to f(t, y)$ is measurable for all $y \in \mathbb{R}^N$;

(iii) for every $c > 0$, there exists $h_c \in L^p(I)$ such that

$$|y| \le c \Rightarrow |f(t, y)| \le h_c(t) \quad \text{for } a.e.\ t \in I$$

Problems of the type

$$u''(t) = f(t, u, u')$$

where f is L^p-Carathéodory have deserved the attention of researchers in the past two decades, and many classical results can also be established for this type of problems. Picard-Lindelöf theorem, Peano's theorem, local existence theorem in the Carathéodory setting provide us very general existence criteria. Bernstein-Nagumo theory was fundamental to prove existence results for general Sturm-Liouville problems. For some problems, the existence of ordered lower and upper solutions was enough to provide the existence of a solution between them, and the study of more adequate weighted Banach spaces to search solutions of singular problems was also fruitful.

Nonlocal problems have been given a great deal of attention lately. In these problems, the differential equation (or the boundary conditions) depends directly on the global behaviour of the dependent variable. The equation that we study in Chapter 2 is a good example of that:

$$-\Delta u = f\left(u, \int_U g(u)\right).$$

The Laplacian depends on an integral term of the solution u, which is not a pointwise dependence. This is a nonlinear Poisson-Boltzman equation, with many physically important examples associated. We will search for a radial solution of this problem in a sphere in $\mathbb{R}^N$, reducing the problem to one independent variable in a bounded interval. The differential equation for the one dimensional problem is

$$-v''(r) - \frac{n-1}{r}v'(r) = f\left(v(r), \omega_n \int_0^1 s^{n-1} g(v(s))\, ds\right),$$

with the obvious singularity for $r = 0$. This problem came to our attention through the paper by Fijalkowski and Przeradski [25], where the main assumption is that f grows at most like $Au + B$, A being computed by means of a Green's function. By using a similar theoretical background (Krasnoselskii's fixed point Theorem), together with the consideration of the eigenvalues of the underlying singular linear problem, we show that an improvement of that bound is possible. We also followed another approach for this problem: the upper and lower solution method. We establish nonlocal maximum principles and we use them to build a monotone approximation sequence converging to a radial solution. We follow an idea used by Jiang, et al. [31] in studying a fourth order periodic problem.

Fourth order bvp's has been a prolific domain in the theory of differential equations in the past two decades. Fourth order problems are related to important applications in the theory of beam-columns deflection, but the intention to generalize well-known results of second order problems was probably the main reason for the increase of interest in this area. Let us describe superficially the relation of fourth order boundary problems with beam deflection theory, based on a very simplistic model. Consider a beam with length $L = 1$ and a symmetric cross section, with end points $x = 0$ and $x = 1$. The unknown function is $u(x)$ and represents the vertical deflection of the point $x \in [0, 1]$, positive downwards. Assume that $u(0) = u(1) = 0$. When the beam is subjected to both axial and lateral loads, the bending moments, shear forces, stresses and deflections will not be proportional to the axial load. Considering an axial compressive force P and a constant lateral load Q (positive in the u-axis), an element dx between two cross sections has a shearing force V and a bending moment M, that satisfy the equations

$$Q = \frac{dV}{dx}, \qquad V = \frac{dM}{dx} - P\frac{du}{dx}.$$

By neglecting the effects of the shearing deformations and shortening of the beam axis, the expression for the curvature of the axis of the beam is

$$EI\frac{d^2u}{dx^2} = -M,$$

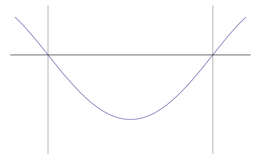
where EI represents the flexibility of the beam. Combining these equations, one gets the fourth order differential equation

$$EI\, u^{(4)} + P\, u'' = Q\,.$$

Let us now analyse some of the most common boundary conditions:

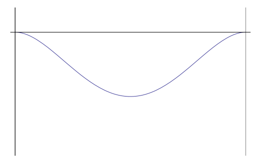
(i) Simply supported boundary conditions; $u(0) = u(1) = u''(0) = u''(1) = 0$.

These obviously represent a beam with both ends at the same level, with null bending moments, that is, assuming that the beam continued for $x < 0$ and $x > 1$, $x = 0$ and $x = 1$ would be inflection points of u.



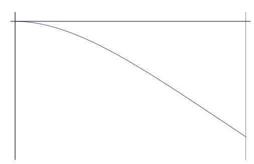
(ii) Clamped beam boundary conditions; $u(0) = u(1) = u'(0) = u'(1) = 0$.

These conditions represent a beam with both ends at the same level, clamped in a wall, and "leaving" the wall with horizontal tangent.

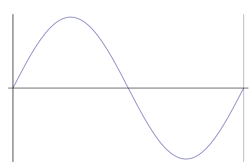


(iii) One side clamped beam boundary conditions; $u(0) = u'(0) = u''(1) = u'''(1) = 0$.

Null bending moment and shearing force at the right end point of the beam, and clamped on the left end.



(iv) Periodic boundary conditions; $u^{(i)}(0) = u^{(i)}(1)$ for $i = 0, 1, 2, 3$.



Fourth order boundary value problems have a more complex structure when compared to second order problems, and many results that are valid for second order are not easy to generalize. The fact that there are more intermediate derivatives does not allow us to use the usual topological tools available in the second order. An example of this is the theorem that states that if a given bvp has well ordered lower and upper solutions, then it has a solution lying between those two functions. In general, this is true in the second-order case, but one needs to add monotonicity assumptions to obtain some true statement in the fourth order case. For problems where there are no well-ordered upper and lower solutions, at least the paper of Cabada, Cid and Sanchez [11] gave a positive answer to the existence of a solution in this case.

In the last chapter we propose a step in the direction of establishing this type of conclusions. We prove some new results using two types of approaches: a decomposition of the fourth order operator into two second order operators, when that is possible (namely in the "simply supported" and periodic boundary conditions cases), and a more direct approach to the fourth order operator using some analytic properties of possible solutions of the bvp (this was done for the clamped beam problem). Maximum principles play a crucial role on the applicability of the monotone method in presence of lower and upper solutions. The equation studied is of the type

$$u^{(4)} = f(x, u, u'')$$

for the "simply supported" and periodic boundary conditions, and without the dependence on the second derivative in the harder to tackle "clamped beam" case.

The periodic problem with f not depending on u'' has been studied before via maximum principles and the monotone method. Using fixed point theory, the existence of a solution for the problem with a linear dependence of f on u'' was also obtained. For the "simply supported" problem, existence and multiplicity results for the nonlinearity without dependence on u'', with linear dependence on u'' or the superlinear case have been studied by several authors.

We consider the periodic as well as the "simply supported" boundary conditions, and prove existence results (considering f one-sided Lipschitz in both variables u and u'') if there exist lower and upper solutions (well-ordered or in reversed order for the periodic case, and ordered in the "simply supported" case). We deal with these problems in the same way as Gao, et. al. [26] did for the second order. Habets and Sanchez [30] have obtained similar results using Lipschitz conditions. The main difference is that in our case only localization is obtainable, no iterative technique is possible.

For the "clamped beam problem", there are less results in the literature since the decomposition into two second order operators is inappropriate. We impose that f is continuous and satisfies the inequality

$$f(x, \alpha(x)) + k\alpha(x) \leq f(x, u) + k\, u \leq f(x, \beta(x)) + k\beta(x), \quad \alpha(x) \leq u \leq \beta(x),$$

for given ordered lower and upper solutions α and β, and a given range of values of k. Eigenvalue theory and the oscillatory behaviour of solutions of the fourth order differential equation

$$u^{(4)} + m^4 u = 0$$

play a crucial role in our proof, where a very interesting result of Schröder in [49] was used.

The study of second and fourth order bvp's in infinite intervals has also deserved our attention in this thesis, where the variational methods play a central role in proving existence of solutions.

If we have constant solutions, it is important to know whether there exist solutions with phase plane trajectories (considering second order autonomous problems) that are a loop curve connecting a single equilibria (homoclinic) or a curve connecting two different equilibria. These types of trajectories provide us with important data, since they are separatrices of different types of behaviour of other solutions. For fourth order problems we do not have the phase plane for such analysis, but functional analytic methods work on the same basic ideas. Our objective is to extrapolate results of the autonomous problems for the non-autonomous case, in both second and fourth order problems.

A detailed compilation of the previous achievements in this area for autonomous problems can be found in the second chapter of [8].

The second order non-autonomous problem

$$u'' = a(x)u - g(u) \quad u'(0) = u(+\infty) = 0$$

has been studied in the last two decades, especially in the case where $g(u)$ is a superlinear power. Here we are interested in superlinear functions $g(u)$ and also the case where $g(u)$ is bounded. Korman, Lazer and Yi in [32],[33] gave a variational approach for $g(u) = u^p$, where $p > 1$ and $a(x)$ is increasing in $[0, +\infty)$. Here we partially generalize some of those results by allowing a to have a different behaviour: while having a limit at $+\infty$, $a(x)$ does not approach its limit in an increasing, or even monotonic way. We shall solve a sequence of boundary value problems in $[0, T]$ and if we consider an appropriate sequence of T's tending to $+\infty$, a nontrivial solution of the infinite interval problem will be found as the limit of the corresponding solutions u_T.

The autonomous problem has been completely solved by Berestycki and Lions [5] as they gave a necessary and sufficient condition for the problem

$$\begin{cases} -u'' = f(u) \\ u(\pm\infty) = 0 \end{cases}$$

to have a unique positive homoclinic, and gave some important results concerning the shape of that solution.

The case where the function $g(u)$ is bounded is also analysed, where the results do not follow in the same way as in the superlinear case. We also deal with the more general differential equation

$$u' + c\,u' = a(x)u - g(u)\,,$$

where the same ideas from the simpler case still work if we deal with weighted Banach spaces. Autonomous p-Laplacian problems are also analysed. Mountain-Pass Theorem and the diagonal method were the main results used to prove existence through our variational approach.

Concerning fourth order infinite interval problems, we deal with the problems

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases}$$

where we consider $a(x)$ in three different situations: the autonomous case, the case where $a(x)$ is nondecreasing and the case when $\lim_{x\to+\infty} a(x) = +\infty$. The methods used to prove the existence of a solution are the same as in the above mentioned second order problems.

Chapter 1

Some useful results from Functional Analysis

In this chapter we present some classical definitions and results from Functional Analysis that sometimes have several different versions in the literature. Since we will apply those results later in this thesis, we opted to include them here. We emphasize the Krasnoselskii's fixed point Theorem by presenting the proofs of two different versions and also some auxiliary results.

Definition 1.1. Let X, Y be Banach spaces and $T\colon D \subseteq X \to Y$ an operator. We say that T is completely continuous if

(i) T is continuous;

(ii) T maps bounded sets into relatively compact sets.

Theorem 1.2 (Schauder's fixed point Theorem)**.** *Let $M \neq \varnothing$ be a bounded closed convex set of a Banach space X and let $T\colon M \to M$ be a completely continuous operator. Then T has at least one fixed point in M.*

Mapping degree

Let us state some of the basic properties of the Degree theory.

If X is a Banach space and $G \subseteq X$ is a bounded open set, then a mapping $F\colon \overline{G} \to X$ is called admissible if it is completely continuous and $F(x) \neq x$ for all $x \in \partial G$. Two admissible mappings F_1, F_2 are called homotopic if there exists a completely continuous map $H\colon \overline{G} \times [0,1] \to X$ such that $H(x,t) \neq x$ for all $(x,t) \in \partial G \times [0,1]$ and $H(x,0) = F_1$, $H(x,1) = F_2$. We shall write $F_1 \cong F_2$. With the completely continuous perturbations of the identity of the form $I - F$, with F admissible, we can associate an integer $deg(I-F,G)$, which is uniquely defined if it satisfies the following properties:

(1) Taking $F \equiv 0$, we have

$$deg(I,G) = \begin{cases} 1, & 0 \in G \\ 0, & 0 \notin G. \end{cases}$$

(2) If $deg(I - F, G) \neq 0$, then there exists $x \in G$ such that $F(x) = x$.

(3) If $G = \cup_{i=1}^{n} G_i$ for some $n \in \mathbb{N}$ and F is admissible for G, G_i $(i = 1, ...n)$, then

$$deg(I - F, G) = \sum_{i=1}^{n} deg(I - F, G_i).$$

(4) If F_1 and F_2 are homotopic, then $deg(I - F_1, G) = deg(I - F_2, G)$.

(5) If $F(x) = x$ has no solutions in $\overline{G}$, then $deg(I - F, G) = 0$.

(6) If $F(x) \neq x$ for all $x \in C$, where C is a closed set, then $deg(I - F, G) = deg(I - F, G \setminus C)$ (excision property).

(7) The mapping degree $deg(I - F, G)$ depends only of the values of F in ∂G.

Recall that the topological index and the mapping degree are related by the formula

$$i(F, G) = deg(I - F, G).$$

Definition 1.3. A closed convex set K is called a cone in a Banach space X if

(i) $\lambda K \subset K$ for all $\lambda \geq 0$;

(ii) $K \cap (-K) = \{0\}$.

We say that $x \leq y$ for some $x, y \in X$ if $y - x \in K$.

Example 1.0.1. The set of all nonnegative continuous functions in the interval $[0, 1]$ is a cone in $C[0, 1]$.

Theorem 1.4. *[37] Let A be a positive completely continuous operator defined in a cone K, with $A \colon K \to K$ and $R > 0$ such that, for all $\epsilon > 0$*

$$Ax \not\geq (1 + \epsilon)x, \quad \forall x \in K \text{ such that } \|x\| = R. \tag{1.1}$$

Then the operator A has at least one fixed point $x_0 \in K$.

Proof. Let

$$\widetilde{A}\,x = \begin{cases} Ax, & if \;\; x \in K,\ \|x\| \leq R \\ A\left(\frac{R}{\|x\|}x\right), & if \;\; x \in K,\ \|x\| \geq R. \end{cases}$$

The operator $\widetilde{A}$ is also completely continuous and maps the cone K into a relatively compact subset of K. By Schauder's fixed point Theorem (K is convex), there exists $x_0 \in K$ such that $\widetilde{A}x_0 = x_0$.

Suppose towards a contradiction that $\|x_0\| > R$ and define

$$y_0 = \frac{R}{\|x_0\|}x_0.$$

We have $\|y_0\| = R$ e consequently

$$\widetilde{A}y_0 = Ay_0 = A\left(\frac{R}{\|x_0\|}x_0\right) = \widetilde{A}x_0 = x_0 = \frac{\|x_0\|}{R}y_0,$$

contradicting (1.1).

Then $\|x_0\| \leq R$, and consequently $Ax_0 = \widetilde{A}x_0 = x_0$. $\square$

Definition 1.5. We say that $A\colon K \to K$ is a cone compression if there exist $r, R > 0$ with $r < R$, such that

$$Ax \nleq x \quad \forall\, x \in K,\ \|x\| = r, \tag{1.2}$$

and for all $\epsilon > 0$

$$Ax \ngeq (1+\epsilon)x \quad \forall\, x \in K,\ \|x\| = R. \tag{1.3}$$

Theorem 1.6 (Krasnoselskii's fixed point Theorem 1)**.** *[37] Let A be a cone compression completely continuous operator. Then A has a nontrivial fixed point in K.*

Proof. With no loss of generality, we may assume that $r < 1 < R$.

Let $h_0 \in K$ be such that

$$\|h_0\| > \frac{1}{1-r}\left(r^2 + r \sup_{y\in K, \|y\|=r} \|Ay\|\right). \tag{1.4}$$

Setting

$$\widetilde{A}x = \begin{cases} \frac{\|x\|}{r} A\left(\frac{r}{\|x\|}x\right) + (1-r)\,h_0, & if\ \ \|x\| \le r^2 \\ \frac{\|x\|}{r} A\left(\frac{r}{\|x\|}x\right) + \frac{\|x\|}{r}\cdot\frac{r-\|x\|}{\|x\|}h_0, & if\ r^2 \le \|x\| \le r \\ Ax, & if\ r \le \|x\| \le R \\ A\left(\frac{R}{\|x\|}x\right), & if\ \ x \in K,\ \|x\| \ge R, \end{cases}$$

we have $\widetilde{A}$ completely continuous and mapping K into a relatively compact subset of K. By Schauder's fixed point Theorem, there exists a fixed point x_0 of $\widetilde{A}$.

Assume that $\|x_0\| \le r^2$. Then

$$x_0 = \frac{\|x_0\|}{r} A\left(\frac{r}{\|x_0\|}x_0\right) + (1-r)\,h_0,$$

and therefore

$$\|h_0\| \le \frac{1}{1-r}\left(r^2 + r\left\|A\left(\frac{r}{\|x_0\|}x_0\right)\right\|\right),$$

which contradicts (1.4).

Let us now assume that $r^2 \le \|x_0\| \le r$. We have

$$\frac{r}{\|x_0\|}x_0 = A\left(\frac{r}{\|x_0\|}x_0\right) + \frac{r-\|x_0\|}{\|x_0\|}h_0,$$

so $\frac{r}{\|x_0\|}x_0 \ge A\left(\frac{r}{\|x_0\|}x_0\right)$, which contradicts (1.2).

Finally, assuming that $\|x_0\| > R$, we have

$$x_0 = A\left(\frac{R}{\|x_0\|}x_0\right) = (1+\epsilon)\,\frac{R}{\|x_0\|}x_0, \quad (\text{with }\ \epsilon = \frac{\|x_0\|}{R} - 1 > 0)$$

and since $\left\|\frac{R}{\|x_0\|}x_0\right\| = R$, we get a contradiction with (1.3).

Excluded the cases above, we must have $r \le \|x_0\| \le R$, and consequently x_0 is a fixed point of A. □

We can consider a slightly simpler definition of compression, which is easier to work with:

Definition 1.7. A completely continuous operator $T\colon K \to K$ is a cone compression if there exist positive constants r, R with $r < R$ such that

$$Tx \nleq x \quad \forall x \in K \text{ such that } \|x\| = r, \tag{1.5}$$
$$Tx \ngeq x \quad \forall x \in K \text{ such that } \|x\| = R. \tag{1.6}$$

The cone expansion definition is somehow the opposite:

Definition 1.8. A completely continuous operator $T\colon K \to K$ is an cone expansion if there exist positive constants r, R with $r < R$ such that

$$Tx \ngeq x \quad \forall x \in K \text{ such that } \|x\| = r, \tag{1.7}$$
$$Tx \nleq x \quad \forall x \in K \text{ such that } \|x\| = R. \tag{1.8}$$

Definition 1.9. Let X be a Banach space and $r\colon X \to M \subseteq X$ a continuous function such that $r(x) = x$ for all $x \in M$. Then r is called a retraction and M a retract of X.

Proposition 1.10. *[52] Every closed convex set of a Banach space X is a retract of X.*

Theorem 1.11 (Krasnoselskii's fixed point Theorem 2)**.** *[52] Let X be a Banach space and K a cone in X. Let $T\colon K \to K$ be a cone compression or expansion. Then T has a fixed point x in K and $r < \|x\| < R$.*

Proof. We will only prove the result for the compression case, since the expansion case is similar.

From the previous proposition we know that there exists a retraction $r\colon X \to K$, and if we consider the operator $T \circ r$, it will coincide with T in K. In the following we will denote abusively the operator $T \circ r$ by T. Since r is continuous and can be taken mapping bounded sets into bounded sets, the new operator T is still completely continuous. Note that the fixed points of the new operator are obviously fixed points of the original operator.

Setting

$$U = \{x \in X\colon\ \|x\| < r\}, \quad \text{and}$$
$$V = \{x \in X\colon\ \|x\| < R\},$$

by the excision property of the topological degree, we know that

$$deg\left(I - T, V \setminus \overline{U}\right) = deg\,(I - T, V) - deg\,(I - T, U)\,.$$

Suppose that $deg\,(I - T, U) \neq 0$. We can pick a value $a > 0$ such that $\|Tx\| \leq a$ in $\overline{U}$, $x_0 \in K$ with $\|x_0\| > r + a$ and set $H(x,t) = Tx + tx_0$.

If $H(x,t) = x$ for some $(x,t) \in \partial U \times [0,1]$, we obviously have $x \in K$ and $(x - Tx) \in K$ (or equivalently $Tx \leq x$). But by the definition of cone compression that cannot happen, so H is an homotopy and

$$deg\,(I - H(\cdot,1), U) = deg\,(I - T, U) \neq 0.$$

Hence, there exists $x \in U$ such that $Tx + x_0 = x$, which implies $\|x_0\| \leq a + r$. Since this is a contradiction, we conclude that $deg\,(I - T, U) = 0$.

Let us prove that $deg\,(I - T, V) = 1$. Setting

$$H(x,t) = tTx,$$

if $H(x,t) = x$ for some $(x,t) \in \partial V \times [0,1]$, then $t \neq 0$ and $x \in K$, and therefore $Tx = x/t \geq x$. Again the definition of cone compression rules out this possibility, so we conclude that

$$deg\,(I - T, V) = deg\,(I - H(\cdot, 0), V) = deg\,(I, V) = 1.$$

Consequently

$$deg\,\left(I - T, V \setminus \overline{U}\right) = 1,$$

which implies the existence of a fixed point satisfying the required properties. $\square$

Definition 1.12. A cone P of a Banach space is called normal if for all $u, v \in P$, with $u \leq v$, we have $\|u\| \leq C\,\|v\|$, for some constant $C > 0$. If $C = 1$ the cone is called monotonic.

We now state a corollary of the Krasnoselskii's fixed point Theorem, which is the most used version of that Theorem in the literature:

Corollary 1.13. *Let P be a monotonic cone in a Banach space and $T\colon P \to P$ a completely continuous operator. If there exist positive constants $r < R$ such that*

$$\|Tx\| \geq \|x\|\,, \quad \textit{for all } x \in P \textit{ such that } \|x\| = r,$$

$$\|Tx\| \leq \|x\|\,, \quad \textit{for all } x \in P \textit{ such that } \|x\| = R,$$

then T has a fixed point x in P such that $r < \|x\| < R$.

For completeness, we will also present the usual formulation of the Mountain-Pass Theorem. Given a Banach space X, we say that $f \in C^1(X, \mathbb{R})$ satisfies the *Palais-Smale* condition at the level $c \in \mathbb{R}$ if, for all the sequences $(u_n)_{n\in\mathbb{N}}$ such that $f(u_n) \to c$ and $f'(u_n) \to 0$, c is a critical value of f.

Theorem 1.14 (Mountain-Pass Theorem)**.** *Let $f \in C^1(X, \mathbb{R})$, $u, v \in X$, $r > 0$ such that*

$$\|u - v\| > r, \qquad \inf_{\|x-u\|=r} f(x) > \max\left(f(u), f(v)\right)$$

and consider the value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where

$$\Gamma = \left\{\gamma \in C([0,1], X)\colon \gamma(0) = u,\ \gamma(1) = v\right\}.$$

Then, if f satisfies the Palais-Smale condition at the level c, c is a critical value of f.

Chapter 2

Radial solutions for a second order nonlocal boundary value problem

2.1 Introduction

This chapter is dedicated to the nonlocal boundary value problem considering the domain $U = B(0, R)$ of $\mathbb{R}^n$:

$$-\Delta u = f\left(u, \int_U g(u)\right) \tag{2.1}$$

$$u|_{\partial U} = 0, \tag{2.2}$$

where f and g are continuous functions. For simplicity we shall take $R = 1$. We want to study the existence of positive radial solutions

$$u(x) = v(\|x\|). \tag{2.3}$$

of (2.1)-(2.2). This may be seen as the stationary problem corresponding to a class of nonlocal evolution (parabolic) boundary value problems related to relevant phenomena in Engineering and Physics. Some hints on the motivation for the study of this mathematical model can be found in the paper by Bebernes and Lacey [4] and more recent developments can be seen in [15] and the references therein.

When dealing with a nonlinear term with rather general dependence on the nonlocal functional as in (2.1), new difficulties arise with respect to the treatment of standard boundary value problems. Differences of behaviour which are met in general elliptic and parabolic problems are already present in simple models as those we shall analyse in this chapter. For instance, the use of the powerful lower and upper solutions method (good accounts of which can be consulted in the monographs of Pao [43] and De Coster and Habets [16]) is seriously limited by the absence of general maximum principles. Even for linear problems with nonlocal terms the issue of positivity is far from trivial and may require a detailed study via the analysis of the Green's operator.

The purpose of this chapter is twofold. First, we want to improve a quite recent result of P. Fijalkowski and B. Przeradski [25]: these authors have proved the existence of positive radial solutions of (2.1)-(2.2) by using Krasnoselski's fixed point Theorem in cones; the main assumption is that f may grow at most like $Au + B$, the bound on A

being computed by means of a Green's function. By using a similar theoretical background, together with the consideration of the eigenvalues of the underlying linear problem, we show that an improvement of that bound is possible. Second, while remaining in the same simple general setting, we shall handle (2.1)-(2.2) from the point of view of the lower and upper solution method. We establish a nonlocal maximum principle and we use it as a device to obtain a monotone approximation scheme for the radial solutions of (2.1)-(2.2) in presence of lower and upper solutions. We follow an idea used by D. Jiang, W. Gao, A. Wan [31] in studying a fourth order periodic problem.

Note that we could use similar methods to consider the case where $U = B(0,1)\backslash\overline{B}(0,\rho)$, with $0 < \rho < 1$. Similar results could then be reached. We remark also that for special classes of functions f and g, different approaches are needed. For instance, in [28] variational methods have been used to study existence and multiplicity when $f(u,v) = g(u)/v^p$ $(p > 0)$ and g behaves as an exponential function.

2.2 Nonlinearities with linear growth in u: a positive solution

It is well known that the existence of a solution for some boundary value problems is equivalent to the existence of a fixed point of a certain operator. For our purpose we need to consider a second order ordinary differential equation of the form

$$-\left(p\left(t\right)u'\left(t\right)\right)' = p(t)f\left(t,u(t)\right), \tag{2.4}$$

with boundary conditions

$$u'(0) = u(1) = 0, \tag{2.5}$$

where f is a continuous function in $[0,1]\times\mathbb{R}$ and $p \in C[0,1]$ is positive and increasing in $(0,1]$.

If $p > 0$ in $[0,1]$, it is well known that the problem is fully regular, having a standard reduction to a fixed point problem:

$$u = Tf(\cdot,u(\cdot)) \quad \text{in } C[0,1],$$

where T is the linear operator that takes $v \in C[0,1]$ into the unique solution u of

$$-\left(p\left(t\right)u'\left(t\right)\right)' = p(t)v(t), \quad u'(0) = u(1) = 0. \tag{2.6}$$

In addition, we can write explicitly

$$Tv(t) = \int_0^1 G(t,s)\,v(s)\,ds,$$

where $G(t,s)$ is the Green's function associated to the problem. The Green's function is continuous in $[0,1]\times[0,1]$, so T is a completely continuous linear operator in $C[0,1]$.

We are interested in the case where $p(t) > 0$ in $(0,1]$ only, that is, $p(0) = 0$. Under certain assumptions we still have a continuous Green's function for the linear problem (2.6). The reader can find a more general approach in [27], but for completeness we include here a simple version which is sufficient for our purpose:

Lemma 2.2.1. *Let p be continuous, increasing in $[0,1]$, $p(0)=0$ and $p>0$ in $(0,1]$. If the function*

$$p(s)\int_s^1 \frac{1}{p(\tau)}\,d\tau$$

is continuously extendible to $[0,1]$, then the operator $T\colon C[0,1]\to C[0,1]$ previously considered is well defined, linear and completely continuous.

Proof. Consider the equation

$$-\left(p\left(t\right)u'\left(t\right)\right)' = p(t)v(t), \tag{2.7}$$

with boundary conditions (2.5). Integrating both sides we get

$$-p(t)u'(t) = \int_0^t p(s)v(s)\,ds.$$

Integrating again, we obtain

$$\begin{aligned} u(t) &= \int_t^1 \frac{d\tau}{p(\tau)} \int_0^\tau p(s)v(s)\,ds \\ &= \int_0^t p(s)v(s)\,ds \int_t^1 \frac{1}{p(\tau)}\,d\tau + \int_t^1 p(s)v(s)\,ds \int_s^1 \frac{1}{p(\tau)}\,d\tau \\ &= \int_0^1 G(t,s)\,v(s)\,ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} p(s)\int_t^1 \frac{1}{p(\tau)}\,d\tau, & t\ge s \\ p(s)\int_s^1 \frac{1}{p(\tau)}\,d\tau, & t\le s \end{cases}$$

is clearly continuous in $[0,1]\times[0,1]$, so that the operator

$$Tv(t) = \int_0^1 G(t,s)\,v(s)\,ds = \int_0^t p(s)\int_t^1 \frac{1}{p(\tau)}\,d\tau\,v(s)\,ds + \int_t^1 p(s)\int_s^1 \frac{1}{p(\tau)}\,d\tau\,v(s)\,ds$$

is completely continuous in $C[0,1]$.

It is trivial to see that $Tv(1)=0$ and, if we differentiate the expression for $Tv(t)$, we obtain

$$\begin{aligned} (Tv)'(t) &= p(t)\int_t^1 \frac{1}{p(\tau)}\,d\tau\,v(t) + \int_0^t -\frac{p(s)v(s)}{p(t)}\,ds - p(t)\int_t^1 \frac{1}{p(\tau)}\,d\tau\,v(t) \\ &= -\int_0^t \frac{p(s)v(s)}{p(t)}\,ds, \end{aligned}$$

and thus

$$(Tv)'(0) = \lim_{t\to 0} -\int_0^t \frac{p(s)v(s)}{p(t)}\,ds = -\lim_{t\to 0} v(0)\frac{\int_0^t p(s)}{p(t)} = 0.$$

□

Remark 2.2.2. The continuous functions $p(t) = t^n$, with $n > 0$, satisfy the assumptions of the lemma.

Let $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous functions. The radial solutions v of the problem (2.1)–(2.2) solve the ordinary differential equation

$$-v''(r) - \frac{n-1}{r}v'(r) = f\left(v(r), \omega_n \int_0^1 s^{n-1} g(v(s))\,ds\right) \tag{2.8}$$

which is equivalent to

$$-\left(r^{n-1}v'(r)\right)' = r^{n-1} f\left(v(r), \omega_n \int_0^1 s^{n-1} g(v(s))\,ds\right),$$

with boundary conditions

$$\lim_{r\to 0^+} v'(r) = v(1) = 0, \tag{2.9}$$

where ω_n is the superficial measure of the unit sphere in $\mathbb{R}^n$.

The homogeneous equation $-v'' - (n-1)v'/r = 0$, with the boundary conditions (2.9), has only the trivial solution, and therefore there exists a Green's function associated to the linear problem. In fact, the Green's function may be written, according to lemma 2.2.1 (see also [25]), in the following way:

(i) for $n > 2$,

$$G(r,t) = \frac{t^{n-1}}{n-2}\left(\frac{1}{\max(r,t)^{n-2}} - 1\right);$$

(ii) and for $n = 2$,

$$G(r,t) = -t\ln\left(\max(r,t)\right).$$

Hence the boundary value problem (2.8)–(2.5) is equivalent to the integral equation

$$v(r) = \int_0^1 G(r,t) f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s))\,ds\right) dt. \tag{2.10}$$

In $C\,[0,1]$, the Banach space of continuous functions in $[0,1]$ with the usual norm, let P be the cone of the nonnegative functions. The radial solutions of (2.1)–(2.2) are exactly the fixed points of the completely continuous operator $S\colon P \to P$, defined by

$$S(v)(r) = \int_0^1 G(r,t) f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s))\,ds\right) dt. \tag{2.11}$$

In [25], the following theorem is proved:

Theorem 2.2.3. *Let $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous functions, and*

$$\gamma = \sup_{r\in[0,1]} \int_0^1 G(r,s)\,ds.$$

Suppose there exist constants $A, B \in \mathbb{R}$ such that $0 \le A < \gamma^{-1}$ and

$$f(v,y) \le Av + B$$

for all $v \ge 0$ and $y \in \mathbb{R}$.

Then the problem (2.1)–(2.2) has a positive radial solution.

We will show that the estimate on the constant A in the previous result can be improved.

Consider the problem (2.8)–(2.9) and the associated eigenvalue problem:

$$-v''(r) - \frac{n-1}{r}v'(r) = \lambda\, v(r), \text{ with } \lim_{r\to 0^+} v'(r) = 0 \text{ and } v(1) = 0. \tag{2.12}$$

We have

$$\begin{aligned} -v''(r) - \frac{n-1}{r}v'(r) = \lambda v(r) \Leftrightarrow \\ \Leftrightarrow \left(r^{n-1}v'(r)\right)' + \lambda r^{n-1}v(r) = 0. \end{aligned}$$

To find the eigenvalues, it is useful to consider the auxiliar initial value problem:

$$\left(r^{n-1}v'(r)\right)' + r^{n-1}v(r) = 0, \quad v(0) = 1 \text{ and } v'(0) = 0. \tag{2.13}$$

The solution $v(r)$ to this problem is well defined in $[0, +\infty)$, oscillates, and has zeros $\{\xi_n \mid n \in \mathbb{N}\}$ such that $0 < \xi_1 < \xi_2 < \ldots \to +\infty$, with $\xi_{n+1} - \xi_n \to \pi$ (see [51]).

Define $u(r) = v(\beta r)$. Then

$$u'(r) = \beta v'(\beta r) \qquad \text{and} \qquad u''(r) = \beta^2 v''(\beta r).$$

Using (2.13), we have

$$\begin{aligned} (n-1)(\beta r)^{n-2}v'(\beta r) + (\beta r)^{n-1}v''(\beta r) + (\beta r)^{n-1}v(\beta r) = 0 \Leftrightarrow \\ \Leftrightarrow \left(r^{n-1}u'(r)\right)' + \beta^2 r^{n-1}u(r) = 0. \end{aligned}$$

It is obvious that $u'(0) = 0$, so it remains to find β such that $u(1) = 0$. As $u(1) = v(\beta)$, we get $\beta = \xi_n$ for some $n \in \mathbb{N}$, hence $\beta = \xi_n$ and, therefore, the eigenvalues of (2.12) are

$$\lambda_n = \beta^2 = {\xi_n}^2.$$

Let us identify the zeros of the unique solution of (2.13). We have

$$\begin{aligned} \left(r^{n-1}v'(r)\right)' + r^{n-1}v(r) = 0 \Leftrightarrow \\ \Leftrightarrow r^{n-3}\left(r^2v'' + (n-1)rv' + r^2v\right) = 0, \end{aligned}$$

and the last equation has the form

$$t^2u'' + atu' + (b + ct^m)\,u = 0,$$

which is easily reduced to a Bessel equation (cf.[50]). Using the new independent variable

$$y = r^{\frac{n-2}{2}}v,$$

we obtain the transformed equation

$$r^2y'' + ry' + \left(r^2 - \left(\frac{n-2}{2}\right)^2\right)y = 0,$$

whose solutions are well known, and thus we get:

$$(i) \quad v(r) = c_1 r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(r) + c_2 r^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(r) \quad \text{if } n \text{ is even, or}$$

$$(ii) \quad v(r) = c_1 r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(r) + c_2 r^{-\frac{n-2}{2}} J_{\frac{2-n}{2}}(r) \quad \text{if } n \text{ is odd,}$$

where c_1, c_2 are constants and J_i, K_i are Bessel functions of order i, of the first and the second kind respectively.

Taking into consideration the boundary conditions, the constant c_2 must be zero in both cases (otherwise we would have $\lim_{r\to 0^+} v(r) = \infty$), so that

$$v(r) = c_1 r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(r).$$

For our boundary value problem we know that $\gamma^{-1} = 2n$ (see [25]). If we compare $\sqrt{2n}$ with ξ_1 – the zeros of these Bessel functions are well known – we can see that

$$\sqrt{2n} < \xi_1$$

and hence,

$$\gamma^{-1} < \lambda_1 \quad \text{(first eigenvalue of (2.12)).}$$

For instance, for $n = 2$ or $n = 4$ we have

$$\begin{aligned} \sqrt{4} = 2,000 &\quad < \quad \xi_1(J_0) \approx 2,404, \\ \sqrt{8} \approx 2,828 &\quad < \quad \xi_1(J_1) \approx 3,832. \end{aligned}$$

By adapting the approach of [25], we shall prove the following improved version of theorem 2.2.3:

Theorem 2.2.4. *Let $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous functions, and λ_1 defined as above.*

Suppose there exist constants $A, B \in \mathbb{R}$ such that $0 \le A < \lambda_1$, and

$$f(v,y) \le Av + B, \qquad \text{for all } v \ge 0 \text{ and } y \in \mathbb{R}.$$

Then the problem (2.1)–(2.2) has a positive radial solution.

Let ϕ be an eigenfunction associated with the first eigenvalue λ_1. We have

$$-\phi'' - \frac{n-1}{r}\phi' = \lambda_1 \phi \quad \text{and} \quad \phi'(0) = 0 = \phi(1). \tag{2.14}$$

Since our computation above shows that we may assume that $\phi(r) = v(\xi_1 r)$, where $v(r) = r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(r)$, it is clear that $\phi > 0$ in $[0,1)$, (and, by the way, $\phi'(1) < 0$). We may therefore consider the norm

$$\|v(r)\|_X = \sup_{[0,1)} \frac{|v(r)|}{\phi(r)}$$

in the Banach space

$$X = \left\{ v \in C[0,1] : \frac{|v(r)|}{\phi(r)} \text{ bounded} \right\}.$$

Then, as stated before, we can write problem (2.8)–(2.9) as $v = Sv$, where

$$S(v)(r) = \int_0^1 G(r,t) f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s))\, ds\right) dt, \text{ for } v \in X. \tag{2.15}$$

Let T denote the operator introduced above, with $p(s) = s^{n-1}$. This operator acts in $C[0,1]$. Let K be the restriction of T to X and $v \in X$. Since

$$|v(t)| \leq \|v\|_X\, \phi(t),$$

and

$$\int_0^1 G(r,t)\phi(t)\, dt = \frac{\phi(r)}{\lambda_1},$$

we have

$$|K\,(v)\,(r)| \leq \int_0^1 G(r,t)\, |v(t)|\ dt \leq \|v\|_X \int_0^1 G(r,t)\phi(t)\, dt$$

so that

$$\frac{|K(v)(r)|}{\phi(r)} \leq \frac{\|v\|_X}{\lambda_1}.$$

Taking the least upper bound in the left hand side of last inequality, we obtain

$$\|K(v)\|_X \leq \frac{\|v\|_X}{\lambda_1}. \tag{2.16}$$

Lemma 2.2.5. *The operator $S\colon X \to X$ is completely continuous.*

Proof. Since the embedding $i_1\colon X \to C[0,1]$ is obviously continuous, the Nemytskii operator $N\colon X \to C[0,1]$ given, for each $v \in X$, by

$$N(v) = f\left(v, \omega_n \int_0^1 s^{n-1} g(v(s))\, ds\right),$$

is also continuous. Moreover it takes bounded sets into bounded sets.

Now let us consider the following decomposition of T:

$$C[0,1] \xrightarrow{T_*} C^2_*[0,1] \xrightarrow{i_2} C^1_*[0,1] \xrightarrow{i_3} X, \tag{2.17}$$

where

$$C^2_*[0,1] = \left\{u \in C^2[0,1]\colon u'(0) = u(1) = 0\right\},$$
$$C^1_*[0,1] = \left\{u \in C^1[0,1]\colon u(1) = 0\right\},$$

i_2, i_3 are embeddings, and T_* is the operator T acting between those two spaces.

The operator $(T_*)^{-1}$ takes u into $-u'' - \frac{(n-1)}{r}u'$; it is obviously linear continuous and bijective and, therefore, using the Open Map Theorem, we get that T_* is continuous. The embedding i_2 is a well known completely continuous operator and using L'Hospital's rule we can prove that i_3 is also continuous. Since $S = i_3 i_2 T_* i_1$, the conclusion of the lemma is now straightforward. ∎

Proof of theorem 2.2.4. The proof is similar to that of theorem 2.2.3 and so we only outline it. If $f(0, \omega_n \frac{g(0)}{n}) = 0$, then $v \equiv 0$ is obviously a fixed point of the operator S, so let us suppose that $f(0, \omega_n \frac{g(0)}{n}) > 0$. Then there exist positive constants M and δ such that

$$f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s))\, ds\right) \geq M, \quad \text{for all } \|v\|_X \leq \delta.$$

A simple computation yields

$$\|Sv\|_X \geq M \sup_{r\in(0,1)} \int_0^1 \frac{G(r,t)}{\phi(r)}\, dt = M\epsilon,$$

if $\|v\|_X \leq \delta$, where we have set $\epsilon := \sup_{r\in(0,1)} \int_0^1 \frac{G(r,t)}{\phi(r)}\, dt$.

If we define $\Omega_1 = \{v \in X \mid \|v\|_X < \min(M\epsilon/2, \delta)\}$, in $\partial\Omega_1$ we have

$$\|Sv\|_X \geq M\epsilon > \|v\|_X \, .$$

Defining $\Omega_2 = \{v \in X \mid \|v\|_X < \|TB\|_X / (1 - A'/\lambda_1)\}$ with $A < A' < \lambda_1$, then for $v \in P \cap \partial\Omega_2$ we have (using the positivity of T and the estimate (2.16))

$$\begin{aligned} \|Sv\|_X &\leq \|T(Av + B)\|_X \leq \|AKv\|_X + \|TB\|_X \\ &< \frac{A'/\lambda_1 \|TB\|_X}{1 - A'/\lambda_1} + \frac{\|TB\|_X - A'/\lambda_1 \|TB\|_X}{1 - A'/\lambda_1} = \|v\|_X \, . \end{aligned}$$

Applying Krasnoselskii's fixed point Theorem 1.13 (compression version) we find a fixed point of S, and therefore a positive radial solution of (2.1)–(2.2).

□

In both theorems above, as mentioned in [25], the condition on f does not depend on the second variable, and, therefore, nothing is restraining the behaviour of g. The arguments used there are also valid for the same problem with $f(v(r), \alpha(v))$, for any continuous functional α in X.

A similar procedure allows us to us prove a result in the spirit of the one considered in [25] where g is restrained, but the condition on f is weakened:

Theorem 2.2.6. *Let $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous functions.*

Suppose there exist positive constants $A < \lambda_1$, B, C, D, p and q with $pq \leq 1$ such that

$$f(v, y) \leq Av + B + C\,|y|^p \quad \textit{for all} \quad v \geq 0 \textit{ and } y \in \mathbb{R}$$

and

$$|g(v)| \leq D\,|v|^q \quad \textit{for all} \quad v \in \mathbb{R}$$

where ϕ is the eigenfunction associated with λ_1.

Then problem (2.1)–(2.2) has a positive radial solution.

Remark 2.2.7. We could have considered in equation (2.8) a right-hand side of the form $f\left(r, v(r), \omega_n \int_0^1 s^{n-1} g(v(s))\, ds\right)$, continuous in $[0,1] \times \mathbb{R} \times \mathbb{R}$. Indeed we might even work with a nonlinear nonnegative function $f(r, v, w)$ continuous in (v, w) for a.e. $r \in [0,1]$, and measurable in r for all $(v, w) \in \mathbb{R} \times \mathbb{R}$. However in this case, defining

$$L_k^p(0,1) = \left\{ u\colon u \text{ is measurable in } (0,1),\ \int_0^1 s^k\, |u(s)|\, ds < +\infty \right\}$$

we should confine ourselves to $L_{n-1}^p(0,1)$ Carathéodory functions f, i.e.

$$\forall M > 0 \quad \sup_{|v|+|w| \le M} |f(\cdot, v, w)| \in L_{n-1}^p(0,1),$$

where $p > n$ is fixed. Under this restriction, it can still be shown that the analogue of Lemma 2.2.5 holds, because we can obtain an analogue of T acting compactly from $L_{n-1}^p(0,1)$ to $C_*^1[0,1]$.

2.3 Lower and upper solutions and monotone approximation

We will now apply the lower and upper solution method to find solutions of problem (2.8)–(2.9). We will use two different types of conditions concerning the given functions f and g, and construct monotone convergent sequences to solutions of the problem.

Let us define the linear operator

$$Lu(r) = -u''(r) - \frac{n-1}{r} u'(r) + \lambda\, u(r).$$

Lemma 2.3.1 (Maximum Principle 1)**.** *Let $\lambda \ge 0$, and $u \in C^1[0,1] \cap C^2(0,1)$ be such that $Lu(r) \ge 0$ in $(0,1]$, $u'(0) \le 0$ and $u(1) \ge 0$. Then $u(r) \ge 0$ for all $r \in [0,1]$.*

Proof. Towards a contradiction, assume that $u(r_0) < 0$ for some $r_0 \in (0,1)$. There are two cases to consider:

(i) $u(r) < 0$ in some interval $(c, d) \subset [0,1]$, with $u(c) = u(d) = 0$.

Let us consider first the case where $\lambda > 0$. Then there must exist $p \in (c, d)$ such that $u'(p) = 0$, and $u''(p) \ge 0$, and since $u(p) < 0$, we get $Lu(p) < 0$, which is a contradiction.

If $\lambda = 0$, integrating in $[c, d]$, we get the contradiction

$$0 < d^{n-1} u'(d) - c^{n-1} u'(c) \le 0.$$

(ii) $u(r) < 0$ in some interval $[0, c[\subset [0,1]$, with $u(c) = 0$.

If $u'(0) < 0$, the same argument applies. If $u'(0) = 0$, integrating in $[0, c]$, we get

$$0 > -c^{n-1} u'(c) + \lambda \int_0^c r^{n-1} u(r)\, dr \ge 0.$$

□

From now on we assume that $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}\to\mathbb{R}$ are continuous functions. Consider the boundary value problem

$$-u''(r)-\frac{n-1}{r}u'(r)=f\left(u(r),\omega_n\int_0^1 s^{n-1}g\left(u(s)\right)\,ds\right)\ \text{for } 0<r\le 1, \tag{2.18}$$

and

$$u'(0)=0=u(1). \tag{2.19}$$

We say that $\alpha(r)$ is a *lower solution* of (2.18)–(2.19) if

$$-\alpha''(r)-\frac{n-1}{r}\alpha'(r)\le f\left(\alpha(r),\omega_n\int_0^1 s^{n-1}g\left(\alpha\left(s\right)\right)\,ds\right),\ \text{for } 0<r\le 1,$$

$$\alpha'(0)\ge 0 \text{ and } \alpha(1)\le 0.$$

A function β satisfying the reversed inequalities is called an *upper solution*.

Let α_0 be a lower solution and β_0 an upper solution of (2.18)–(2.19). Consider the restriction L_0 of the operator L to the subspace $\{u\in C^1[0,1]\cap C^2(0,1)\colon u'(0)=u(1)=0\}$. With the notations above, to get a solution of problem (2.18)–(2.19) is equivalent to find a fixed point of the completely continuous operator in $C[0,1]$

$$\Phi u\equiv L_0^{-1}\left(f\left(u,\omega_n\int_0^1 s^{n-1}g\left(u\left(s\right)\right)\,ds\right)+\lambda\,u\right).$$

Let us define

$$R_f(u,v_1,v_2)=\frac{f(u,v_2)-f(u,v_1)}{v_2-v_1}\quad\text{and}\quad R_g(u_1,u_2)=\frac{g(u_2)-g(u_1)}{u_2-u_1}.$$

Lemma 2.3.2. *Let α_0 be a lower solution and β_0 an upper solution of (2.18)–(2.19) such that $\alpha_0\le\beta_0$ in $[0,1]$. Suppose $f(u,v)$ is such that*

$$f(u_2,v)-f(u_1,v)\ge-\lambda(u_2-u_1),$$

for some $\lambda\ge 0$, $v\in\mathbb{R}$, u_1,u_2 such that for some $r\in[0,1]$, $\alpha_0(r)\le u_1\le u_2\le\beta_0(r)$, and R_f,R_g have the same sign for all u_1,u_2 such that $\alpha_0(r)\le u_1,u_2\le\beta_0(r)$ for some $r\in[0,1]$.

Then, for any two functions $u_1(r)$, $u_2(r)\in C[0,1]$ such that

$$\alpha_0(r)\le u_1(r)\le u_2(r)\le\beta_0(r),$$

we have

$$\Phi u_1(r)\le\Phi u_2(r).$$

Proof. The Green's function G_λ associated with the operator L_0 is nonnegative according to Lemma 2.3.1. We have

$$\begin{aligned}
&\Phi u_2(r)-\Phi u_1(r)=\\
&=\int_0^1 G_\lambda(r,t)\left[f\left(u_2,\omega_n\int_0^1 s^{n-1}g\left(u_2\left(s\right)\right)\,ds\right)-f\left(u_1,\omega_n\int_0^1 s^{n-1}g\left(u_2\left(s\right)\right)\,ds\right)\right]dt+\\
&+\int_0^1 G_\lambda(r,t)\left[f\left(u_1,\omega_n\int_0^1 s^{n-1}g\left(u_2\left(s\right)\right)\,ds\right)-f\left(u_1,\omega_n\int_0^1 s^{n-1}g\left(u_1\left(s\right)\right)\,ds\right)\right]dt+\\
&+\int_0^1 G_\lambda(r,t)\lambda\left(u_2-u_1\right)\,dt\ge\int_0^1 G_\lambda(r,t)\left[-\lambda\left(u_2-u_1\right)+\lambda\left(u_2-u_1\right)\right]\,dt\ge 0.
\end{aligned}$$

$\square$

Remark 2.3.3. Clearly if f and g are C^1 functions, the hypotheses of the last theorem are satisfied provided that

$$\frac{\partial f}{\partial u} \geq -\lambda, \text{ and } \quad \frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \text{ have the same sign.}$$

Theorem 2.3.4. *Suppose that f and g satisfy the assumptions of Lemma 2.3.2. Let α_0, β_0 be lower and upper solutions, respectively, of* (2.18)–(2.19)*. If we put*

$$\alpha_{n+1} = \Phi\alpha_n \quad \textit{and} \quad \beta_{n+1} = \Phi\beta_n, \quad \textit{for all } n \in \mathbb{N}_0,$$

we obtain

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0.$$

The monotone bounded sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ defined above are convergent in $C[0,1]$ respectively to the minimal and maximal solutions of (2.18)–(2.19) *in the interval* $[\alpha_0, \beta_0]$.

Proof. Since

$$L\alpha_1 = f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g\left(\alpha_0\left(s\right)\right) \, ds\right) + \lambda\alpha_0, \text{ and } L\alpha_0 \leq f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g\left(\alpha_0\left(s\right)\right) \, ds\right) + \lambda\alpha_0,$$

we have

$$L\left(\alpha_1 - \alpha_0\right) \geq 0, \quad \left(\alpha_1 - \alpha_0\right)'(0) \leq 0, \quad \left(\alpha_1 - \alpha_0\right)(1) \geq 0,$$

and therefore, by Lemma 2.3.1, we have $\alpha_0 \leq \alpha_1$.

Using similar arguments, we can prove that $\alpha_1 \leq \beta_0$.

We are now able to apply Lemma 2.3.2 to α_0 and α_1 which gives $\alpha_1 \leq \alpha_2$. By iteration of this argument, we prove that $(\alpha_n)_{n\in\mathbb{N}_0}$ is an increasing sequence and stays below β_0. Analogously, we prove that $(\beta_n)_{n\in\mathbb{N}_0}$ is a decreasing sequence so that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0.$$

Concerning the convergence of the sequences, as the cone of positive functions in $C[0,1]$ is normal (since $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$), we can use the standard argument ([52], p.283), which gives the convergence of this iteration method to fixed points of Φ, and these are exactly the smallest and largest fixed points in $[\alpha_0, \beta_0] \subset C[0,1]$. □

Example 2.3.5. Let us consider the nonlocal differential equation

$$-u''(r) - \frac{2}{r}u'(r) = \frac{4}{3}\pi e^u \int_0^1 s^2\left(u(s) + 1\right) \, ds \tag{2.20}$$

with boundary conditions $u'(0) = u(1) = 0$.

In this case we have $n = 3$, $f(u,v) = \frac{e^u v}{3}$, and $g(u) = u + 1$.

Consider $\alpha_0 \equiv 0$ and $\beta_0 = 1 - r$. Then

$$-\alpha_0''(r) - \frac{2}{r}\alpha_0'(r) = 0 \leq \frac{4}{9}\pi = \frac{4}{3}\pi e^0 \int_0^1 s^2 \, ds$$

and $\alpha_0'(0) = \alpha_0(1) = 0$, so α_0 is a lower solutions of (2.20)–(2.19).

For $r \in [0,1]$ we have

$$-\left(r^2\beta_0'\right)' = 2r \geq \frac{5}{9}\pi r^2 e^{1-r} = \frac{4}{3}\pi r^2 e^{1-r}\int_0^1 s^2\,(1-s+1)\;ds,$$

$\beta_0'(0) = -1$ and $\beta_0(1) = 0$. Therefore β_0 is an upper solution of (2.20)–(2.19).

The conditions in the Theorem 2.3.2 are satisfied for α_0 and β_0, so there exists a solution u of (2.20)–(2.19), such that

$$0 \leq u(r) \leq 1-r, \text{ for all } r \in [0,1].$$

This solution is the limit of a monotone sequence constructed as in the statement of the theorem.

Let us now try another approach using the lower and upper solutions method, where we drop a part of the monotonicity assumptions.

Lemma 2.3.6 (Maximum Principle 2)**.** *Suppose that $u \in C^1[0,1]\cap C^2(0,1)$ satisfies*

$$-u''(r) - \frac{n-1}{r}u'(r) + \lambda\, u(r) + M\int_0^1 s^{n-1}\,|u(s)|\;ds \geq 0 \tag{2.21}$$

for some λ, $M>0$ such that $\lambda + M < 1$ and $u'(0)\leq 0$, $u(1)\geq 0$. Then we have $u(r)\geq 0$ for all $r\in[0,1]$.

Proof. Suppose by contradiction that there exists a function u_0 that satisfies the assumptions above and is negative at some point.

Normalizing u_0, we can assume that $\int_0^1 s^{n-1}\,|u_0(s)|\;ds = 1$ without loss of generality, which implies that $\left\|r^{n-1}u_0(r)\right\|_\infty \geq 1$.

Let us consider the auxiliary problem

$$-w''(r) - \frac{n-1}{r}w'(r) + M = 0, \quad w'(0) = w(1) = 0. \tag{2.22}$$

which is equivalent to

$$\left(r^{n-1}w'(r)\right)' = r^{n-1}M, \quad w'(0) = w(1) = 0. \tag{2.23}$$

Integrating (2.23), we get

$$w(r) = \frac{M}{2n}(r^2-1) \leq 0.$$

As u_0 satisfies

$$-u_0''(r) - \frac{n-1}{r}u_0'(r) + \lambda\, u_0(r) + M \geq 0,$$

with $u_0'(0)\leq 0$, $u_0(1)\geq 0$, we have

$$-(u_0-w)'' - \frac{n-1}{r}(u_0-w)' + \lambda(u_0-w) \geq -\lambda w, \quad (u_0-w)'(0)\leq 0,\ (u_0-w)(1)\geq 0,$$

and, therefore, applying Lemma 2.3.1, we get $u_0 \geq w$. We can easily see that

$$r^{n-1}u_0(r) \geq r^{n-1}w(r) \geq -\frac{M}{2n} > -1,$$

so the fact that $\left\|r^{n-1}u_0(r)\right\|_\infty \geq 1$ insures that there exists $a > 0$ such that $u_0(a) \geq \frac{1}{a^{n-1}}$.

If u_0 is negative at $b > a$, there exists $c \in (a,b)$ such that $u_0(c) = 0$ (we can assume that $u_0'(b) = 0$). Using Lagrange's Theorem, there exists $d \in [a,c]$ such that $u_0'(d) \leq -\frac{1}{a^{n-1}}$. As $d \geq a$, we have $d^{n-1}u_0'(d) \leq -1$ and therefore there exists $e \in [d,b]$ such that $(r^{n-1}u_0'(r))'|_{r=e} \geq 1$, (we can take e such that $e^{n-1}u_0(e) < 1$).

If u is negative at $b < a$, there exists $c < a$ such that $u_0(c) = 0$. As $u_0(a) > 1$, there exists $d \in (c,a)$ such that $u_0'(d) \geq 1$. Considering the boundary condition $u_0'(0) \leq 0$, there exists $e \in [0,d)$ such that $u_0'(e) = 0$ and $u_0'(r) > 0$ for all $r \in (e,d]$. Therefore there exists $f \in [e,d]$ such that $u_0''(f) \geq 1$ and $u_0'(f) > 0$ (we can take f such that $f^{n-1}u_0(f) < 1$).
In both cases, we know that for some r_0 we have $(r^{n-1}u_0'(r))'|_{r=r_0} \geq 1$, and $r_0^{n-1}u_0(r_0) < 1$. Therefore we would get

$$-(r^{n-1}u_0'(r))'|_{r=r_0} + \lambda r_0^{n-1}u_0(r_0) + M \leq -1 + \lambda + M < 0,$$

which is a contradiction. □

For a given function $u(r) \in C[0,1]$, consider the boundary value problem

$$-v''(r) - \frac{n-1}{r}v'(r) + \lambda\, v(r) = f\left(u(r), \omega_n \int_0^1 s^{n-1} g\left(v(s)\right)\, ds\right) + \lambda\, u(r),$$

with $v'(0) = 0 = v(1)$. Using the operator L defined in the beginning of this section, this equation is equivalent to the fixed point equation in C[0,1]

$$v = L_0^{-1}\left(f\left(u, \omega_n \int_0^1 s^{n-1} g\left(v\left(s\right)\right)\, ds\right) + \lambda\, u\right) \equiv \Phi_u v. \tag{2.24}$$

Remark 2.3.7. Using a comparison method as the one in the proof of Lemma 2.3.6, we get $\left\|L_0^{-1}\right\| \leq \frac{1}{2n}$ in $C[0,1]$.

Lemma 2.3.8. *If $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz, and $k_1k_2\omega_n < 2n^2$, then Φ_u has a unique fixed point.*

Proof. We have

$$\begin{aligned}|\Phi_u v_2(r) - \Phi_u v_1(r)| &\leq \frac{1}{2n}k_1 \left|\omega_n \int_0^1 s^{n-1} g(v_2(s))\, ds - \omega_n \int_0^1 s^{n-1} g(v_1(s))\, ds\right| \leq \\ &\leq \frac{1}{2n}k_1k_2\omega_n \int_0^1 s^{n-1}\left|v_2(s) - v_1(s)\right|\, ds \leq \frac{k_1k_2\omega_n}{2n^2}\left\|v_2 - v_1\right\|_\infty,\end{aligned}$$

so that

$$\left\|\Phi_u v_2(r) - \Phi_u v_1(r)\right\|_\infty \leq \frac{k_1k_2\omega_n}{2n^2}\left\|v_2 - v_1\right\|_\infty$$

and therefore Φ_u is a contraction mapping. □

Lemma 2.3.9. *Let f and g be functions defined as in the lemma above, $\lambda > 0$ such that $k_1 k_2 \omega_n + \lambda < 1$, and suppose that*

$$f(u_2, v) - f(u_1, v) \geq -\lambda(u_2 - u_1),$$

for all $r \in [0,1]$, $v \in \mathbb{R}$, and $u_1 \leq u_2$.

Let $u_1(r) \leq u_2(r)$ be two given functions defined in $[0,1]$ and $v_1(r)$, $v_2(r)$ the two respective solutions of (2.24). Then $v_1(r) \leq v_2(r)$.

Proof. We have

$$\begin{aligned}
&-(v_2 - v_1)'' - \frac{n-1}{r}(v_2 - v_1)' + \lambda(v_2 - v_1) = \\
&= \lambda(u_2 - u_1) + f\left(u_2, \omega_n \int_0^1 s^{n-1} g\left(v_2\right) ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g\left(v_2\right) ds\right) + \\
&\qquad + f\left(u_1, \omega_n \int_0^1 s^{n-1} g\left(v_2\right) ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g\left(v_1\right) ds\right) \geq \\
&\geq -k_1 k_2 \omega_n \int_0^1 s^{n-1} \left|v_2 - v_1\right| ds.
\end{aligned}$$

The conclusion follows from Lemma 2.3.6. □

Theorem 2.3.10. *Suppose that $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz. Assume that for some $\lambda > 0$ such that $k_1 k_2 \omega_n + \lambda < 1$, we have*

$$f(u_2, v) - f(u_1, v) \geq -\lambda(u_2 - u_1),$$

for all $v \in \mathbb{R}$, and $u_1 \leq u_2$. Let α_0 and β_0 be a lower and an upper solution of (2.18)–(2.19) respectively, with $\alpha_0 \leq \beta_0$ in $[0,1]$. If we take $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ such that, according to Lemma 2.3.8,

$$\alpha_{n+1} = \Phi_{\alpha_n} \alpha_{n+1} \quad \text{and} \quad \beta_{n+1} = \Phi_{\beta_n} \beta_{n+1}, \text{ for all } n \in \mathbb{N}_0,$$

we obtain

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0.$$

The monotone bounded sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ defined above are convergent in $C[0,1]$ to solutions of (2.18)–(2.19).

Proof. The computation used here is similar to another one used in [31]. We have

$$\begin{aligned}
L\left(\alpha_1 - \alpha_0\right) &\geq f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g\left(\alpha_1\left(s\right)\right) ds\right) - f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g\left(\alpha_0\left(s\right)\right) ds\right) \geq \\
&\geq -k_1 k_2 \omega_n \int_0^1 \left|\alpha_1(s) - \alpha_0(s)\right| ds
\end{aligned}$$

with

$$\left(\alpha_1 - \alpha_0\right)'(0) \leq 0, \quad \left(\alpha_1 - \alpha_0\right)(1) \geq 0,$$

and, therefore, using Lemma 2.3.6, we get $\alpha_0 \leq \alpha_1$. Let us prove that $\alpha_1 \leq \beta_0$. This comes from

$$\begin{aligned}
L(\beta_0 - \alpha_1) \geq\ & f\left(\beta_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s))\, ds\right) + \lambda\beta_0 - f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s))\, ds\right) + \\
& + f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s))\, ds\right) - f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_1(s))\, ds\right) - \lambda\alpha_0 \geq \\
& \geq -\lambda(\beta_0 - \alpha_0) + \lambda(\beta_0 - \alpha_0) - k_1 k_2 \omega_n \int_0^1 |\beta_0(s) - \alpha_1(s)|\, ds = \\
& = -k_1 k_2 \omega_n \int_0^1 |\beta_0(s) - \alpha_1(s)|\, ds.
\end{aligned}$$

Applying this Lemma in the following iterations, we prove that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0$$

as in the proof of Theorem 2.3.4.

Concerning the convergence of the sequences, there is a slight difference from the usual method, because in each iteration we use a different operator. But, as

$$\alpha_{n+1}(r) = L_0^{-1}\left(f\left(\alpha_n, \omega_n \int_0^1 s^{n-1} g(\alpha_{n+1}(s))\, ds\right) + \lambda\,\alpha_n\right)$$

and $\|\alpha_n\|_\infty \leq \max(\|\alpha_0\|_\infty, \|\beta_0\|_\infty)$, we have that $\|\alpha_{n+1}\|_{C^1}$ is bounded, and, therefore, using Àrzela-Ascoli Theorem, there exists a convergent subsequence of α_n. Considering the monotonicity of α_n, we get the conclusion by the standard argument. $\square$

Remark 2.3.11. It is not difficult to prove that the monotone sequences defined in theorem 4.10 converge in fact to *extremal solutions* of the boundary value problem (18)-(19).

Example 2.3.12. Suppose that

$$\liminf_{(a,b)\to(0^+,0^+)} \frac{f(a, \frac{\omega_n}{n} g(b))}{a} > \lambda_1.$$

and there exists $k > 0$ such that $f(k, \omega_n g(k)/n) < 0$. Suppose in addition that f and g satisfy the assumptions of Theorem 2.3.10.

Then there exists a *positive* solution of (2.18)–(2.19). This solution may be approximated by monotone sequences. In fact, a simple calculation shows that for $\epsilon > 0$ small enough, $\epsilon\phi$ is a positive lower solution of (2.18)–(2.19). The constant k is clearly an upper solution. The statement follows.

The fact that we needed the assumption $\lambda^2 + M < 1$ in 2.3.6 is a limitation in the strength of this maximum principle.

The purpose of what follows is to extend the nonlocal maximum principle so as to allow its applicability to a large range of values of $\lambda > 0$ and $M > 0$.

We will now investigate the admissible range of values in two cases:

(i) first we consider a simpler model and look for which values of $\lambda > 0$ and $M > 0$ do the inequalities

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(s)| \, ds \geq 0, \quad u(0) \geq 0, \, u(1) \geq 0$$

yield a maximum principle;

(ii) then we proceed to the inequality (2.21), related to the important class of radial problems in a ball.

It turns out that the two situations may be dealt in a similar way, although some computations are easier in the first case.

In the course of our approach we find it convenient to consider the linear singular differential equation

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \tag{2.25}$$

and find an expression for one of its solutions as

$$u(t) = \int_0^1 H_\lambda(t,s) h(s) ds,$$

where H_λ is a Green's function. The solution we have in mind exists for a certain class of right-hand sides h, and may satisfy boundary conditions $u'(0) = a$, $u(1) = 0$, where a needs not be zero.

Some remarks about the solutions of a linear problem

Let us consider the differential equation

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \quad t \in (0,1), \tag{2.26}$$

where $k > 1$, $\lambda > 0$ and $h \in L^2_{k+2}(0,1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 \tau^{k+2} h(\tau)^2 \, d\tau < \infty \right\}$.

We shall use the Hilbert Spaces

$$H_k(0,1) = \left\{ u \in AC\,(0,1] \colon \int_0^1 \tau^k u'(\tau)^2 \, d\tau < \infty, \quad u(1) = 0 \right\},$$

with the norm $\|u\| = \left(\int_0^1 \tau^k u'(\tau)^2 \, d\tau \right)^{1/2}$.

Following [7], for $u \in H_k(0,1)$, with $k > 1$, we have $\left(\int_0^1 \tau^{k-2} u(\tau)^2 \, d\tau \right)^{1/2} \leq \frac{2}{k-1} \|u\|$, so the functional

$$J(u) := \int_0^1 \left[\frac{1}{2} \left(t^k u'(t)^2 + \lambda^2 t^k u(t)^2 \right) + t^k h(t) u(t) \right] dt$$

is well defined in $H_k(0,1)$, since

$$\int_0^1 t^k h(t) u(t) \, dt \leq \left(\int_0^1 t^{k-2} u(t)^2 \, dt \right)^{1/2} \left(\int_0^1 t^{k+2} h(t)^2 \, dt \right)^{1/2}.$$

It is obvious that $J(u)$ is a coercive strictly convex functional, so that equation (2.26) has a unique solution in $H_k(0,1)$.

Proposition 2.3.13. *If* $h \in L^2_1(0,1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 \tau h(\tau)^2\, d\tau < \infty \right\}$, *then the unique solution* u *of (2.26) in* $H_k(0,1)$ *is in fact in* $C^1[0,1]$ *and it satisfies* $u'(0) = 0$ *(note that* $L^2_1(0,1) \subset L^2_{k+2}(0,1)$*).*

Proof. Equation (2.26) is obviously equivalent to

$$-\left(t^k u'(t)\right)' + \lambda^2 t^k u(t) = t^k h(t).$$

If $h \in L^2_1(0,1)$, it is easy to verify that $\left|t^k u'(t)\right|$ satisfies Cauchy's condition at $t = 0$, therefore there exists $L \in \mathbb{R}$ such that $\lim_{t\to 0} \left|t^k u'(t)\right| = L$. Necessarily $L = 0$, because otherwise we would not have $u \in H_k(0,1)$. Applying Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\left|t^k u'(t)\right| \le & \left|\int_0^t \lambda^2 \tau^k u(\tau)\, d\tau\right| + \left|\int_0^t \tau^k h(\tau)\, d\tau\right| \\ \le & c_1 \left(\int_0^t \tau^{k-2} u(\tau)^2\, d\tau\right)^{1/2} t^{\frac{k+3}{2}} + c_2 \left(\int_0^t \tau h(\tau)^2\, d\tau\right)^{1/2} t^k,\end{aligned}$$

for some constants $c_1, c_2 > 0$.

If $\frac{k+3}{2} \ge k$ $(k \le 3)$, it is obvious that $\lim_{t\to 0} u'(t) = 0$. Otherwise, if $k > 3$, we have

$$\left|t^k u'(t)\right| \le c\, t^{\frac{k+3}{2}}, \tag{2.27}$$

for some constant $c > 0$.

In general, if we have $\left|t^k u'(t)\right| \le c t^\alpha$, then $|u(t)| \le C + C t^{\alpha - k + 1}$, for some $C > 0$, hence, we can conclude that near $t = 0$, there exists a constant $c_3 > 0$ such that

$$\left(\int_0^t \tau^{k-2} u(\tau)^2\, d\tau\right)^{1/2} \le c_3\, t^{\min\left(\frac{k-1}{2}, \frac{2\alpha - k + 1}{2}\right)}.$$

Consequently, for some $c_4 > 0$, we have

$$\left|t^k u'(t)\right| \le c_4\, t^{\min(k+1, \alpha+2)} + c_2 \left(\int_0^t \tau h(\tau)^2\, d\tau\right)^{1/2} t^k,$$

and setting $\alpha = \frac{k+3}{2}$, it is easy to see that with a finite number of iterations of this process, we will get

$$\left|t^k u'(t)\right| \le c^*\, t^{k^*} + c_2 \left(\int_0^t \tau h(\tau)^2\, d\tau\right)^{1/2} t^k,$$

where $k^* > k$, and then the conclusion follows easily. □

It is a standard procedure in the literature to associate solutions of a boundary value problem to fixed points of some functional operator. In our case, the solutions of the second order homogeneous differential equation

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = 0, \tag{2.28}$$

which is equivalent to $\left(t^k u'(t)\right)' = \lambda^2 t^k u(t)$, with initial conditions $u(0) = 1$, $u'(0) = 0$, may be viewed as fixed points of the operator

$$Tu(t) = 1 + \int_0^t \frac{\lambda^2}{\tau^k} \int_0^\tau s^k u(s)\, ds\, d\tau,$$

defined in some functional space. Considering the space $Z = \{u \in C[0, t_0] \colon u(0) = 1\}$, for some t_0 small enough, T has a unique fixed point since it is a contraction. The singularity of equation (2.28) is at the point $t = 0$, so it is obvious that this solution can be extended to the interval $[0, 1]$. Let u_1 be this solution, and consider the function $v_1(t) = u_1(t) \int_t^1 \frac{ds}{s^k u_1(s)^2}$, which is the solution of (2.28) obtained by the standard method of reducing the order of an ordinary differential equation. The solutions u_1 and v_1 are linearly independent and their associated Wronskian is $W(t) = u_1(t)v_1'(t) - u_1'(t)v_1(t) = -t^{-k}$. Furthermore, they satisfy the following properties, which we shall use in the next proposition: $u_1'(t) \geq 0$, $v_1(1) = 0$, $v_1(t) \sim t^{-(k-1)}$, and $v_1'(t) \sim t^{-k}$ as $t \to 0$ (we write $f(t) \sim g(t)$ as $t \to 0$ if and only if $\lim_{t \to 0} \frac{f(t)}{g(t)} = L \in \mathbb{R} \setminus \{0\}$).

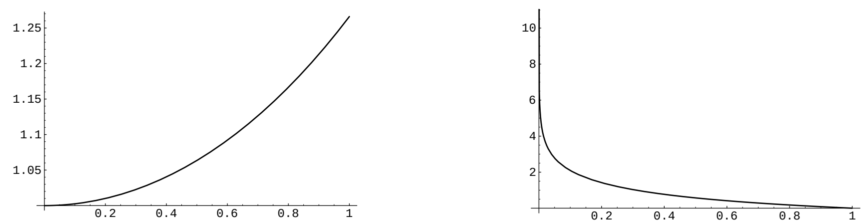


Figure 2.1: Graphics of u_1 and v_1 for a particular case

Let $X \equiv \left\{h(t) \text{ measurable} \colon \exists\, c \in \mathbb{R},\, h_0 \in L_1^2(0,1), h(t) = \frac{c}{t} + h_0(t)\right\}$.

Proposition 2.3.14. *The boundary value problem*

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t),\ u(1) = 0, \quad \textit{for } h \in X, \quad t \in (0, 1] \tag{2.29}$$

has a unique solution in $H_k(0, 1) \cap C^1[0, 1]$, *given by the integral expression*

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds. \tag{2.30}$$

Proof. Let us first note that $L_1^2(0, 1) \subset X \subset L_{k+2}^2(0, 1)$, so that equation (2.26) has a unique solution in $H_k(0, 1)$, that satisfies $u(1) = 0$.

Suppose that $h \in L_1^2(0, 1)$, that is, $c = 0$. Applying the method of undetermined coefficients, we see that the unique solution of

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in (0, 1], \qquad u'(0) = u(1) = 0, \tag{2.31}$$

is given by the well defined integral expression

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds. \tag{2.32}$$

If we differentiate this expression, we get

$$u'(t) = -u_1'(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1'(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds$$

from which, after some computation, we can confirm that $u'(0) = 0$.

Suppose now that $h(t) \notin L^2_1(0,1)$, that is, $h(t) = \frac{c}{t} + h_0(t)$ for some $c \neq 0$, $h_0 \in L^2_1(0,1)$. In this case, the integral expression (2.30) is still well defined, satisfies equation (2.26), and

$$u'(0) = -\lim_{t\to 0} v_1'(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds = \lim_{t\to 0} c\, v_1'(t) \int_0^t u_1(s) s^{k-1}\, ds = -\frac{c}{k}.$$

□

Remark 2.3.15. Expression (2.30) can obviously be written in the form

$$\int_0^1 H_\lambda(t,s)h(s)\, ds, \tag{2.33}$$

which allows us to get the explicit form of the Green's function associated to (2.31). From the expression of H_λ, it is a simple matter to verify that it is continuous in $[0,1]\times[0,1]$ and positive in $(0,1)\times(0,1)$.

From the proof of the previous proposition, we infer that formula (2.33), where the Green's function H_λ appears, provides us the unique solution of (2.26) for all the boundary conditions $u'(0) = a \in \mathbb{R}$, $u(1) = 0$, whenever $h(t) + \frac{ka}{t} \in L^2_1(0,1)$.

The boundary value problem

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \qquad u(1) = b,$$

with $b \neq 0$, has also a unique solution in $C^2(0,1] \cap C^1[0,1]$ (if we had two different solutions w_1, w_2, then $w_1 - w_2$ would be the unique solution of the homogeneous problem, which is identically zero), given by $u_0(t) + \frac{b}{u_1(1)} u_1(t)$, where $u_0(t)$ is the unique solution of

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t),$$

in $H_k(0,1)$. Note that for some functions $h(t) \notin X$ we can still obtain a solution of equation (2.26) via the Green's function, which possibly has infinite derivative at $t = 0$, or simply does not have derivative at $t = 0$, but we will not consider these cases.

Consider now the equation for $k = 1$

$$-u''(t) - \frac{1}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in (0,1], \tag{2.34}$$

where $\lambda > 0$ and $h \in L^q_1(0,1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 t h(t)^q\, dt < \infty \right\}$, for some $1 < q < 2$. Consider also the functional

$$J(u) := \int_0^1 \frac{1}{2}\left(t\, u'(t)^2 + \lambda^2 t\, u^2(t)\right) + t\, h(t)\, u(t)\, dt,$$

defined in $H_1(0,1)$. We have

$$\int_0^1 t\,h(t)\,u(t)\,dt \le \left(\int_0^1 tu(t)^p\,dt\right)^{1/p}\left(\int_0^1 th(t)^q\,dt\right)^{1/q}.$$

Following [23], since for any $p>2$ we have $u^p \le Ce^{|u|^{2-\eta}}$, for some $C,\eta>0$, we know that $\int_0^1 tu(t)^p\,dt<\infty$, and therefore the functional $J(u)$ is well defined in $H_1(0,1)$.

We can state exactly the same results obtained above for $k>1$ in the case $k=1$, just noticing that in this case $v_1(t)\sim \ln t$, and $v_1'(t)\sim t^{-1}$. The fact that $1<q<2$ allows us to conclude that with a function $h(t)\sim\frac{1}{t}$, $J(u)$ is well defined, and the associated solution is the one obtained via Green function, with non-zero derivative at $t=0$.

Nonlocal Linear Problems

Let us consider the linear boundary value problem in the interval $[0,1]$

$$-u''(t)+\lambda^2u(t)=h(t),\quad u(0)=u(1)=0, \tag{2.35}$$

where $\lambda>0$ and $h\in C[0,1]$.

This problem has a well known Green's function

$$G_\lambda(t,s)=\begin{cases}\frac{\sinh(\lambda)\cosh(\lambda t)\sinh(\lambda s)-\cosh(\lambda)\sinh(\lambda s)\sinh(\lambda t)}{\lambda\sinh(\lambda)}, & t\ge s\\ \frac{\sinh(\lambda)\cosh(\lambda s)\sinh(\lambda t)-\cosh(\lambda)\sinh(\lambda t)\sinh(\lambda s)}{\lambda\sinh(\lambda)}, & t\le s,\end{cases}$$

and therefore we have

$$u(t)=\int_0^1 G_\lambda(t,s)h(s)\,ds.$$

Proposition 2.3.16. *Let $w\in C[0,1]\cap C^2(0,1)$ be such that*

$$-w''(t)+\lambda^2w(t)+M\int_0^1 w(\tau)\,d\tau=0,\quad w(0)=w(1)=0, \tag{2.36}$$

for some $\lambda>0$, $M>0$. Then we have $w(t)=0$ for all $t\in[0,1]$.

Proof. Assume towards a contradiction that there exists $w(t)\ne 0$ satisfying (2.36).

If $w(t)\gneq 0$ (by $\gneq$ we mean $\ge$ and $\not\equiv$), then w reaches a positive maximum for some $t_0\in(0,1)$, where we would have the contradiction

$$0<-w''(t_0)+\lambda^2w(t_0)+M\int_0^1 w(\tau)\,d\tau=0.$$

If $w(t)\lneq 0$, we get a contradiction with a similar argument. So $w(t)$ must have a positive maximum for some $t_1\in(0,1)$ and a negative minimum for some $t_2\in(0,1)$. With $t=t_1$ in (2.36) we get $\int_0^1 w(\tau)\,d\tau<0$, and with $t=t_2$ in (2.36) we get $\int_0^1 w(\tau)\,d\tau>0$. The conclusion now follows. □

Lemma 2.3.17. *Let $u \in W^{2,1}(0,1)$ be such that*

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 u(\tau)\, d\tau = f(t) \geq 0, \quad u(0) = a \geq 0,\ u(1) = b \geq 0, \tag{2.37}$$

for some $\lambda > 0$, $M > 0$, and consider the $C^2[0,1]$ functions U, V, where $U(t)$ is the unique solution of (2.35) with $h(t) = 1$ and $V(t)$ is the unique solution of $-V''(t) + \lambda^2 V(t) = 0$, with boundary conditions $V(0) = a$, $V(1) = b$ (note that U and V depend on λ).

Suppose that

$$\frac{M}{1 + M\int_0^1 U(\tau)\, d\tau} \leq \inf_{0<t,s<1} \frac{G_\lambda(t,s)}{U(t)\,U(s)}, \quad \text{and} \quad \frac{M\,U(t)}{1 + M\int_0^1 U(\tau)\, d\tau} \leq \frac{V(t)}{\int_0^1 V(\tau)\, d\tau}. \tag{2.38}$$

Then we have $u(t) \geq 0$ for all $t \in [0,1]$.

Proof. Let v and w be such that

$$\begin{aligned} -\, v''(t) + \lambda^2 v(t) &= f(t), \qquad v(0) = a,\ v(1) = b, \\ -\, w''(t) + \lambda^2 w(t) &= \frac{M\int_0^1 v(\tau)\, d\tau}{1 + M\int_0^1 U(\tau)\, d\tau}, \qquad w(0) = w(1) = 0. \end{aligned}$$

As $w(t) = \frac{M\int_0^1 v(\tau)\, d\tau}{1+M\int_0^1 U(\tau)\, d\tau} U(t)$, it can be easily verified that $v - w$ satisfies (2.37). Proposition 2.3.16 allows us to conclude that $u = v - w$, so we only need to prove that $v \geq w$.

Using the Green's function G_λ defined above and the fact that $G_\lambda(t,s) = G_\lambda(s,t)$, we have

$$\begin{aligned} v(t) &= \int_0^1 G_\lambda(t,s) f(s)\, ds + V(t), \quad \text{and} \\ w(t) &= \frac{M}{1 + M\int_0^1 U(\tau)\, d\tau} \int_0^1 G_\lambda(t,\sigma)\, d\sigma \int_0^1 \left(\int_0^1 G_\lambda(\tau,s) f(s)\, ds + V(\tau) \right) d\tau \\ &= \frac{M}{1 + M\int_0^1 U(\tau)\, d\tau} \left(\int_0^1 U(t)U(s)f(s)\, ds + U(t) \int_0^1 V(\tau)\, d\tau \right), \end{aligned}$$

and therefore, if the conditions in (2.38) are verified, we have $v \geq w$. □

Remark 2.3.18. The explicit form of U and V is:

$$\begin{aligned} U(t) &= -\frac{e^{-\lambda t}\left(-1 + e^{\lambda t}\right)\left(-e^{\lambda} + e^{\lambda t}\right)}{\left(1 + e^{\lambda}\right)\lambda^2} \\ V(t) &= \frac{e^{-\lambda t}\left(-be^{\lambda} + ae^{2\lambda} - ae^{2\lambda t} + be^{\lambda + 2\lambda t}\right)}{-1 + e^{2\lambda}}. \end{aligned}$$

Let us now consider the linear boundary value problem

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \quad u(1) = b,\ u \in C^2(0,1] \cap C^1[0,1], \tag{2.39}$$

where $k \geq 1$, $\lambda > 0$ and $h \in X$.

As stated before, this problem has a unique solution given by

$$u(t)=\int_0^1 H_\lambda(t,s)h(s)\,ds+\frac{b}{u_1(1)}\,u_1(t),$$

where, as before, $u_1(t)$ is the solution of the homogeneous equation with $u_1(0)=1$, $u_1'(0)=0$.

Lemma 2.3.19. *The Green's function $H_\lambda(t,s)$ satisfies the following symmetry property:*

$$t^k H_\lambda(t,s)=s^k H_\lambda(s,t).$$

Proof. Let u_1,u_2 be such that

$$-u_i''(t)-\frac{k}{t}u_i'(t)+\lambda^2u_i(t)=f_i(t),\quad u_i'(0)=u_i(1)=0,\quad i=1,2$$

for some continuous functions f_1,f_2. The equations above are obviously equivalent to

$$-\left(t^k u_i'(t)\right)'+\lambda^2t^ku_i(t)=t^kf_i(t)$$

Using this form of the equations, integrating by parts we obtain

$$\int_0^1 t^kf_1(t)u_2(t)\,dt=\int_0^1 t^kf_2(t)u_1(t)\,dt,$$

and therefore

$$\int_0^1\int_0^1 t^kf_1(t)H_\lambda(t,s)f_2(s)\,ds\,dt=\int_0^1\int_0^1 t^kf_2(t)H_\lambda(t,s)f_1(s)\,ds\,dt.$$

Given the arbitrariness of f_1 and f_2, the conclusion follows now easily. □

Proposition 2.3.20. *Let $w\in C^2[0,1]$ be such that*

$$-w''(t)-\frac{k}{t}w'(t)+\lambda^2w(t)+M\int_0^1\tau^kw(\tau)\,ds=0,\quad w'(0)=w(1)=0,\tag{2.40}$$

for some $\lambda>0$, $M>0$. Then we have $w(t)=0$ for all $t\in[0,1]$.

Proof. We obtain $w(t)=0$ using similar arguments to those used in the proof of Proposition 2.3.16. □

Lemma 2.3.21. *Let $u\in C^2[0,1]$ be such that*

$$-u''(t)-\frac{k}{t}u'(t)+\lambda^2u(t)+M\int_0^1\tau^ku(\tau)\,ds=f(t)\ge0,\quad u'(0)=a\le0,\ u(1)=b\ge0,\tag{2.41}$$

for some $\lambda>0$, $M>0$. Suppose that

$$\frac{M}{1+M\int_0^1\tau^kU(\tau)\,d\tau}\le\inf_{0<t,s<1}\frac{H_\lambda(t,s)}{U(t)\,U(s)s^k},\quad \textit{and}\quad \frac{M\,U(t)}{1+M\int_0^1\tau^kU(\tau)\,d\tau}\le\frac{u_1(t)}{\int_0^1\tau^ku_1(\tau)\,d\tau}\tag{2.42}$$

where $U(t)$ is the unique solution of (2.39) with $h(t)=1$, a, $b=0$. Then we have $u(t)\ge0$ for all $t\in[0,1]$.

Proof. Note that $f \in X$. Let v and w be such that

$$
\begin{aligned}
&-v''(t) - \frac{k}{t}v'(t) + \lambda^2 v(t) = f(t), \qquad v \in C^2(0,1] \cap C^1[0,1],\ v(1) = b,\\
&-w''(t) - \frac{k}{t}w'(t) + \lambda^2 w(t) = \frac{M\int_0^1 \tau^k v(\tau)\,d\tau}{1 + M\int_0^1 \tau^k U(\tau)\,d\tau}, \qquad w'(0) = w(1) = 0.
\end{aligned}
$$

As $w(t) = \frac{M\int_0^1 \tau^k v(\tau)\,d\tau}{1+M\int_0^1 \tau^k U(\tau)\,d\tau} U(t)$, it can be easily verified that $v - w$ satisfies (2.41). Proposition 2.3.20 allows us to conclude that $u = v - w$, so we only need to prove that $v \geq w$.

Using the Green's function H_λ defined above and the previous lemma, we have

$$
\begin{aligned}
v(t) &= \int_0^1 H_\lambda(t,s) f(s)\,ds + \frac{b}{u_1(1)}\,u_1(t), \quad \text{and}\\
w(t) &= \frac{M}{1 + M\int_0^1 \tau^k U(\tau)\,d\tau} \int_0^1 H_\lambda(t,\sigma)\,d\sigma \int_0^1 \tau^k \left(\int_0^1 H_\lambda(\tau,s) f(s)\,ds + \frac{b}{u_1(1)}\,u_1(\tau)\right) d\tau\\
&= \frac{M}{1 + M\int_0^1 \tau^k U(\tau)\,d\tau} \left(\int_0^1 U(t)U(s)s^k f(s)\,ds + \frac{b\,U(t)}{u_1(1)} \int_0^1 \tau^k u_1(\tau)\,d\tau \right),
\end{aligned}
$$

and therefore, if the conditions in (2.42) are verified, we have $v \geq w$. □

Remark 2.3.22. In the two previous results we do not need to consider $C^2[0,1]$ functions, the same conclusions are valid in $C^1[0,1) \cap C^2(0,1)$.

Nonlocal Semi-Linear Problems

Consider the boundary value problem

$$
-u''(t) + \lambda^2 u(t) + M\int_0^1 |u(\tau)|\,d\tau = f(t), \quad u(0) = a \geq 0,\ u(1) = b \geq 0. \tag{2.43}
$$

Proposition 2.3.23. *If $f \in L^1(0,1)$ and*

$$
M < \min_{\substack{u \in H_0^1(0,1)\\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\,d\tau}{\left(\int_0^1 |u(\tau)|\,d\tau\right)^2},
$$

then problem (2.43) has a unique solution.

Proof. We shall consider two cases:

(i) If $f(t) = 0$, and $a = b = 0$, multiplying the equation in (2.43) by u and integrating by parts, we have

$$
\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\,d\tau = -M\int_0^1 |u(\tau)|\,d\tau \int_0^1 u(\tau)\,d\tau \leq M\left(\int_0^1 |u(\tau)|\,d\tau\right)^2,
$$

and the conclusion follows.

(ii) If $f(t) \neq 0$, let u_1, u_2 be such that

$$-u_i''(t) + \lambda^2 u_i(t) + M \int_0^1 |u_i(\tau)|\, d\tau = f(t), \quad u_i(0) = a, \;\; u_i(1) = b, \quad i = 1, 2.$$

Setting $w = u_1 - u_2$, we have

$$-w''(t) + \lambda^2 w(t) + M \int_0^1 \theta(\tau) w(\tau)\, d\tau = 0, \quad w(0) = w(1) = 0,$$

where $\theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)}$. Since $|\theta(\tau)| \leq 1$, using an argument similar to the one in (i), we get $w(t) = 0$, and therefore there is a unique solution to (2.43).

$\square$

Proposition 2.3.24. *We have*

$$\min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 |u(\tau)|\, d\tau\right)^2} = \min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 u(\tau)\, d\tau\right)^2}.$$

Proof. If a function u_0 minimizes the left-hand side, then, since $|u_0| \in H_0^1(0,1)$, the right-hand side has the same value. $\square$

Let

$$l_1 = \min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 u(\tau)\, d\tau\right)^2} = \min_{\substack{u \in H_0^1(0,1) \\ \int_0^1 u(\tau)\, d\tau = 1}} \int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau.$$

To find l_1, we need to solve a constrained extrema problem, which we can do using Lagrange Multipliers (the proposition above allows us to use a differentiable restriction). Our minimizer u_0 satisfies

$$-u_0''(t) + \lambda^2 u_0(t) = m, \quad u_0(0) = u_0(1) = 0,$$

where m is the Lagrange Multiplier, so $u_0(t) = mU(t)$. Since $\int_0^1 u_0(\tau)\, d\tau = 1$, we get $m = \left(\int_0^1 U(\tau)\, d\tau\right)^{-1}$, and consequently

$$l_1 = \int_0^1 {u_0'}^2(\tau) + \lambda^2 u_0^2(\tau)\, d\tau = \frac{1}{\int_0^1 U(\tau)\, d\tau}.$$

Theorem 2.3.25 (Maximum Principle 3)**.** *Let λ, M be positive constants, G_λ the Green's function associated to (2.35), $U(t) = \int_0^1 G_\lambda(t,s)\, ds$, and $V(t)$ the unique solution of $-V''(t) + \lambda^2 V(t) = 0$, with boundary conditions $V(0) = a \geq 0$, $V(1) = b \geq 0$. Suppose that*

$$\frac{M}{1 + M \int_0^1 U(\tau)\, d\tau} \leq \inf_{0<t,s<1} \frac{G_\lambda(t,s)}{U(t)\, U(s)}, \qquad \frac{V(t)}{\int_0^1 V(\tau)\, d\tau} \geq \frac{M\, U(t)}{1 + M \int_0^1 U(\tau)\, d\tau},$$

and

$$M < \frac{1}{\int_0^1 U(\tau)\, d\tau}.$$

Then, if $u \in C^2[0,1]$ *satisfies*

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(\tau)|\, d\tau \geq 0, \quad u(0) = a \geq 0,\ u(1) = b \geq 0, \tag{2.44}$$

we have $u(t) \geq 0$.

Proof. Let $f(t) = -u''(t) + \lambda^2 u(t) + M \int_0^1 |u(\tau)|\, d\tau$. By Lemma 2.3.17, we know that the linear problem (2.37) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (2.43). □

Using *Mathematica*, we have the following estimates relative to the first pair of conditions:

$\lambda = 0.2$	$M_{max} \approx 5.98$
$\lambda = 0.5$	$M_{max} \approx 5.92$
$\lambda = 1$	$M_{max} \approx 5.71$
$\lambda = 2$	$M_{max} \approx 4.89$
$\lambda = 4$	$M_{max} \approx 2.74$
$\lambda = 7$	$M_{max} \approx 0.62$
$\lambda = 10$	$M_{max} \approx 0.09$

The last condition is less restrictive, as it is shown by the following graph:

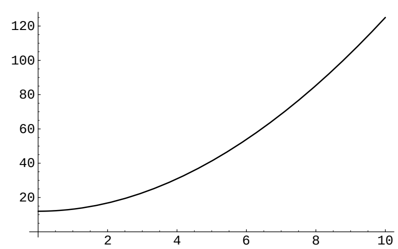


Figure 2.2: $l_1(\lambda) = \frac{1}{\int_0^1 U(\tau)\, d\tau}$

Using the same technique, we can reach similar results for the boundary value problem

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(\tau)|\, d\tau = f(t), \quad u'(0) = a \leq 0,\ u(1) = b \geq 0. \tag{2.45}$$

Let us consider the Hilbert Space

$$H_k(0,1) = \left\{ u \in AC\,(0,1]\colon \int_0^1 \tau^k u'^2(\tau)\, d\tau < \infty,\ \ u(1) = 0 \right\},$$

with the norm $\|u\| = \left(\int_0^1 \tau^k u'^2(\tau)\, d\tau \right)^{1/2}$. Following [7], for any $u \in H_k(0,1)$ with $k > 1$, we have $\int_0^1 \tau^k u^2 \leq C\, \|u\|^2$, for some $C > 0$.

Remark 2.3.26. Note that if $u \in H_k(0,1)$, then $|u| \in H_k(0,1)$.

Proposition 2.3.27. *If $f \in X$ and*

$$M < \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k |u(\tau)| \, d\tau\right)^2},$$

then problem (2.45) has a unique solution.

Proof. As stated before we can write equation (2.45) in the form

$$-\left(t^k u'(t)\right)' + \lambda^2 t^k u(t) + M t^k \int_0^1 \tau^k |u(\tau)| \, d\tau = t^k f(t)$$

We shall consider two cases:

(i) If $f(t) = 0$, a, $b = 0$, multiplying the equation in (2.45) by u and integrating by parts, we have

$$\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau = -M \int_0^1 \tau^k |u(\tau)| \, d\tau \int_0^1 \tau^k u(\tau) \, d\tau \leq M \left(\int_0^1 \tau^k |u(\tau)| \, d\tau\right)^2,$$

and the conclusion follows.

(ii) If $f(t) \neq 0$, let u_1, u_2 be such that

$$-u_i''(t) - \frac{k}{t} u_i'(t) + \lambda^2 u_i(t) + M \int_0^1 \tau^k |u_i(\tau)| \, d\tau = f(t), \quad u_i'(0) = a, \; u_i(1) = b,$$

Setting $w = u_1 - u_2$, we have

$$-\left(t^k w'(t)\right)' + \lambda^2 t^k w(t) + M t^k \int_0^1 |\theta(\tau)| \, \tau^k |w(\tau)| \, d\tau = 0, \quad w'(0) = w(1) = 0,$$

where $\theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)}$. Since $|\theta(\tau)| \leq 1$, using an argument similar to the one in (i), we get $w(t) = 0$, and therefore there is a unique solution to (2.43).

□

Proposition 2.3.28. *We have*

$$\min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k |u(\tau)| \, d\tau\right)^2} = \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k u(\tau) \, d\tau\right)^2}.$$

Proof. If a function u_0 minimizes the left-hand side, then, since $|u_0| \in H_k(0,1)$, the right-hand side has the same value. □

Let

$$l_2 = \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k u(\tau)\, d\tau\right)^2} = \min_{\substack{u \in H_k(0,1) \\ \int_0^1 \tau^k u(\tau)\, d\tau = 1}} \int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau.$$

So, to find l_2, we need to solve another constrained extrema problem. Our minimizer u_0 satisfies

$$-u_0''(t) - \frac{k}{t} u_0'(t) + \lambda^2 u_0(t) = m, \quad u_0'(0) = u_0(1) = 0,$$

where m is the Lagrange Multiplier, so $u_0(t) = mU(t)$. Since $\int_0^1 \tau^k u_0(\tau)\, d\tau = 1$, we get $m = \left(\int_0^1 \tau^k U(\tau)\, d\tau\right)^{-1}$, and consequently

$$l_2 = \int_0^1 \tau^k \left(u_0'^2(\tau) + \lambda^2 u_0^2(\tau)\right) d\tau = \frac{1}{\int_0^1 \tau^k U(\tau)\, d\tau}.$$

We can now state the following improved version of Maximum principle 2 (Lemma 2.3.6):

Theorem 2.3.29 (Maximum Principle 4). *Let λ, M be positive constants, H_λ the Green's function associated to (2.39), and $U(t) = \int_0^1 H_\lambda(t,s)\, ds$. Suppose that*

$$\frac{M}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \leq \inf_{0<t,s<1} \frac{H_\lambda(t,s)}{U(t)\, U(s) s^k}, \qquad \frac{M\, U(t)}{1 + M\int_0^1 U(\tau)\, d\tau} \leq \frac{u_1(t)}{\int_0^1 \tau^k u_1(\tau)\, d\tau},$$

and

$$M < \frac{1}{\int_0^1 \tau^k U(\tau)\, d\tau}.$$

Then, if for $0 < t \leq 1$, $u \in C^2[0,1]$ satisfies

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(\tau)|\, ds \geq 0, \quad u'(0) = a \leq 0,\ u(1) = b \geq 0, \quad (2.46)$$

we have $u(t) \geq 0$.

Proof. Let $f(t) = -u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(\tau)|\, ds$. By Lemma (2.3.21), we know that the linear problem (2.41) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (2.45). □

We have the following estimates relative to the cases $k = 1, 2, 3$:

(i) k=1:

$\lambda = 0.25$	$M_{max} \approx 15.95$
$\lambda = 1$	$M_{max} \approx 15.30$
$\lambda = 5$	$M_{max} \approx 5.71$

(ii) k=2:

$\lambda = 0.25$	$M_{max} \approx 29.9$
$\lambda = 1$	$M_{max} \approx 28.9$
$\lambda = 5$	$M_{max} \approx 12.2$

(iii) k=3:

$\lambda = 0.25$	$M_{max} \approx 47.9$
$\lambda = 0.5$	$M_{max} \approx 47.5$
$\lambda = 1$	$M_{max} \approx 46.5$
$\lambda = 3$	$M_{max} \approx 36.0$
$\lambda = 5$	$M_{max} \approx 21.5$
$\lambda = 10$	$M_{max} \approx 2.2$

The last condition is also less restrictive. We present here the graph of $l_2(\lambda)$ in the case $k = 3$:

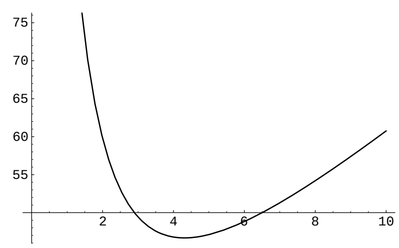


Figure 2.3: $l_2(\lambda) = \frac{1}{\int_0^1 \tau^3 U(\tau)\, d\tau}$

We can now use this Maximum Principle 4 to obtain more general results for the lower and upper solutions method in our problem.

For a given function $u(t) \in C[0,1]$, consider the boundary value problem

$$-v''(t) - \frac{n-1}{t} v'(t) + \lambda^2 v(t) = f\left(u(t), \omega_n \int_0^1 s^{n-1} g\left(v(s)\right)\, ds\right) + \lambda^2 u(t),$$

with $v'(0) = 0 = v(1)$. Using the operator $Lu = -u'' - \frac{n-1}{t} u' + \lambda^2 u$, in the space $C^* = \left\{u \in C^2[0,1] \colon u'(0) = u(1) = 0\right\}$, this problem is equivalent to the fixed point equation

$$v = L^{-1}\left(f\left(u, \omega_n \int_0^1 s^{n-1} g\left(v\left(s\right)\right)\, ds\right) + \lambda^2 u\right) \equiv \Phi_u v. \tag{2.47}$$

It turns out that it is advantageous to look at Φ_u as an operator from $L^2_{n-1}(0,1)$ into itself. Noticing that L^{-1} is a compact self-adjoint operator in this space with norm

$\|L^{-1}\| = (\xi_n^2+\lambda^2)^{-1}$ where ξ_n is the first positive zero of the Bessel function $J_{\frac{n-2}{2}}$, it is easy to see that if $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz, then Φ_u is Lipschitz with constant $\frac{\omega_n k_1 k_2}{(\xi_n^2+\lambda^2)n}$. In particular, when the condition

$$\frac{\omega_n k_1 k_2}{(\xi_n^2+\lambda^2)n} < 1 \tag{2.48}$$

is satisfied, Φ_u is a contraction mapping, and therefore has a unique fixed point.

Using maximum principle 2.3.29, we get the following improved version of Theorem 4.10 in [19]:

Theorem 2.3.30. *Suppose that $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz. Assume that $M \equiv k_1k_2\omega_n$ and λ satisfy the hypothesis of the Maximum Principle 2.3.29, condition (2.48) holds and*

$$f(u_2,v) - f(u_1,v) \geq -\lambda^2(u_2-u_1),$$

for all $v \in \mathbb{R}$, and $u_1 \leq u_2$. Let α_0 and β_0 be a lower and an upper solution of (2.18)–(2.19) *respectively, with $\alpha_0 \leq \beta_0$ in* $[0,1]$. *If we take $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ such that,*

$$\alpha_{n+1} = \Phi_{\alpha_n}\alpha_{n+1} \quad \textit{and} \quad \beta_{n+1} = \Phi_{\beta_n}\beta_{n+1}, \; \textit{for all } n \in \mathbb{N}_0,$$

we obtain

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0.$$

The monotone bounded sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ defined above are convergent in $C[0,1]$ to solutions of (2.18)–(2.19).

Example 2.3.31. Let us consider the nonlocal differential equation

$$-u''(t) - \frac{2}{t}u'(t) = f\left(u, 4\pi\int_0^1 s^2\left(\frac{u(s)^2+1}{3}\right)ds\right) \tag{2.49}$$

where

$$f(u,v) = \begin{cases} (\sqrt{u}+1)(\sin v+1)+4.1, & u \leq 1 \\ \left(\frac{1}{u}+1\right)(\sin v+1)+4.1, & u \geq 1, \end{cases}$$

with boundary conditions $u'(0) = u(1) = 0$ and $\alpha_0 = 1-t^2$ and $\beta_0 = \frac{4}{3}\left(1-t^2\right)$.

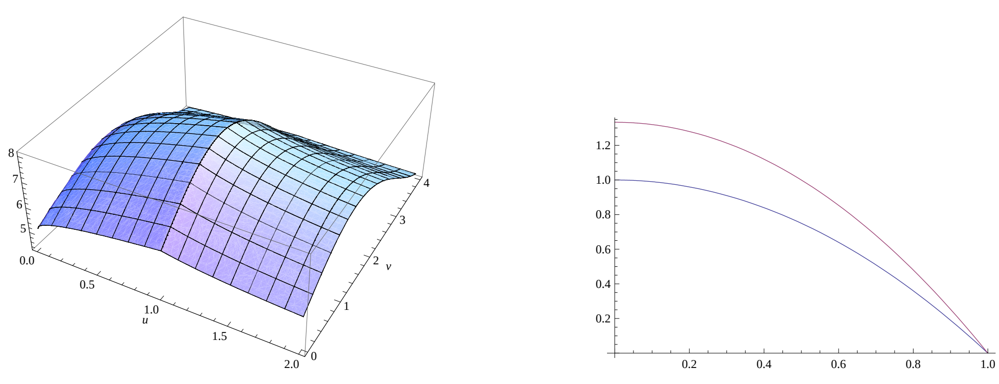


Figure 2.4: $f(u,v)$, α_0 and β_0

After some computation, we can verify that α_0, β_0 are respectively a lower and an upper solution of (2.49), both satisfying the considered boundary conditions (see pictures below). Since $0 \leq \alpha_0(t) \leq \beta_0(t) \leq \frac{4}{3}$, for all $t \in [0,1]$, we can consider $k_1 = 2$, $k_2 = \frac{8}{9}$, and $\lambda = 1$. Moreover $\xi_3 = \pi$. Setting $M = \frac{64\pi}{9}$, the conditions of theorem 2.3.30 are satisfied, and therefore, using the described iterative method, we can approximate a solution $u(t)$ of (2.49) satisfying $u'(0) = u(1) = 0$ and $1 - t^2 \leq u(t) \leq \frac{4}{3}\left(1 - t^2\right)$.

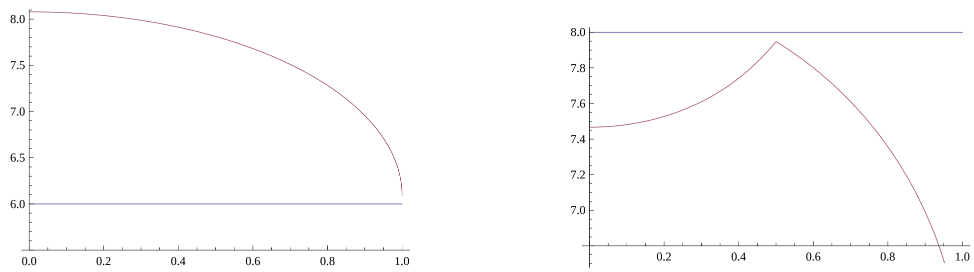


Figure 2.5: Both sides of (2.49) for α_0 and β_0

Final remarks

We opted to make a detailed analysis of a particular problem, but it would be interesting to study also different types of nonlocal problems. Nonlocal problems even without a singularity deserve more study, that would allow us to better understand the main differences with the more classical problems. Analysis of problems with nonlocal terms in the boundary conditions is also an area that seems to be of great interest.

Chapter 3

Boundary value problems in infinite intervals

3.1 Second order problems

3.1.1 Introduction

The study of existence of positive homoclinics of the ordinary differential equation

$$u''(x) = a(x)u - g(u) \tag{3.1}$$

where $g(0) = 0$ is partially motivated by a problem in higher dimensions: the search for special stationary states of the Klein-Gordon type equation

$$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi),$$

where $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex function, $a \in \mathbb{R}$ and $f(\rho\, e^{i\theta}) = f(\rho)e^{i\theta}$. Looking for a "standing wave" solution $\Phi(t, x) = e^{i\omega t}u(x)$, one is led to the equation

$$-\Delta u + \left(a^2 - \omega^2\right) u = f(u). \tag{3.2}$$

The corresponding Euler-Lagrange functional is

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \left(a^2 - \omega^2\right) u^2 - 2F(u)\right) dx,$$

where $F(u) = \int_0^u f(s)\, ds$, and for this integral to be well-defined, $|u|$ needs to vanish at $+\infty$. Our problem is somehow the corresponding in dimension one (the case of radial solutions of (3.2) when $N \geq 2$ yields a different kind of ODE), but working with a non-autonomous term $a(x)u$ instead. In [14], G. Cerami surveys the $\mathbb{R}^N$ non-autonomous case for $\mathbb{N} \geq 3$, under several conditions concerning the non-autonomous term. There, an overall picture is given of what is known in the cases where some symmetry properties in the domain and in the solution are required, in particular the case of radial solutions. Non symmetric problems are addressed as well.

Equations of type (3.1) have been studied in the last two decades, especially in the case where $g(u)$ is a superlinear power. Here we are interested not only in superlinear functions $g(u)$, but also in the case where $g(u)$ is bounded. P. Korman and A. Lazer gave a variational approach for the cases $g(u) = u^3$ in [32] and $g(u) = u^p$, where $p > 1$, in [33]. In these papers, the coefficient $a(x)$ is increasing in $[0, +\infty)$. Here we partially generalize some of those results by allowing a to have a different behaviour, although we confine ourselves to the case where a is even, thus reducing our problem to the half line $[0, +\infty)$. We shall solve a sequence of boundary value problems in $[0, T]$ and if we consider an appropriate sequence of T's tending to $+\infty$, a nontrivial solution of the infinite interval problem will be found as the limit of the corresponding solutions u_T. M. Grossinho, F. Minhós and S. Tersian also gave a similar variational approach for this problem in [29], but working with two simultaneous powers in the nonlinear term. Related with these problems we also mention the papers [3], [24], [41], [45] and [46].

The autonomous problem has been completely solved by H. Berestycki and P. Lions [5] as they gave a necessary and sufficient condition for the problem

$$-u'' = f(u), \qquad u(\pm\infty) = 0$$

to have a unique positive homoclinic (up to translation), and gave some important results concerning the shape of that solution, which will be used ahead. Some of the hypotheses used are reminiscent of those used by Berestycki and Lions. In Subsection 3.1.2 we recall some of those results for the autonomous equation $u'' = au - g(u)$ and considering the following hypotheses:

(H_1) There exists $q > 2$ such that

$$0 < q\, G(u) \leq u\, g(u), \quad \forall u \in \ (0, +\infty),$$

where $G(u) = \int_0^u g(s)\, ds$.

(H_2) $g(u) = o\,(u)$ at $u = 0$.

In Subsection 3.1.3, we treat the non-autonomous equation (3.1) in the case where $a(x)$ is positive and has a behaviour, which, as far as we know, has deserved less attention in the literature: we mean the case where $a(x)$, while having a limit at $+\infty$, does not approach its limit in an increasing, or even monotonic way. The arguments that we will use to deal with equation (3.1) are also valid for the more general equation

$$u'' = (a\,(x) + \epsilon\, b\,(x))\, u - c\,(x)\, g\,(u)\,,$$

where $a(x)$ is as above, ϵ is small enough, $b(x)$ and $c(x)$ are bounded functions, with $0 < \delta \leq c(x)$ for some constant δ.

Concerning the function g, we assume (H_1), (H_2). The assumptions for $a(x)$ will be

(A_1) there exist $0 < a < A$ such that $0 < a(x) \leq A\ \forall x \geq 0$ and $\lim_{x\to+\infty} a(x) = a$,

(A_2) ${J_A}^* < 2{J_a}^*$.

Here ${J_a}^* = 2\int_0^{u_a(0)} u\sqrt{a - \frac{2}{p+1}u^{p-1}}du$ is the value of the Euler-Lagrange functional associated with the autonomous problem

$$\begin{cases} u'' = au - u^p \\ u'(0) = u(+\infty) = 0, \end{cases}$$

computed at its nontrivial solution u_a.

We also analyse this problem using a different approach - a shooting technique - but only presenting the computations for the simpler case $g(u) = u^3$. The non-autonomous term $a(x)$ considered is positive and nondecreasing (does not necessarily have finite limit when x tends to $+\infty$). We use a connectedness argument used in the paper of H. Berestycki, P. Lions and L. Peletier [6] to prove the existence of a solution satisfying the requested conditions.

In Subsection 3.1.5 we deal with the problem with a bounded nonlinearity. Without assuming the strong assumption (H_1) (not even partially in an interval $[0, r]$), the estimates become less obvious. We restrict ourselves to the case where

(A_1') $0 < a(x) \leq A$ $\forall x \geq 0$ and there exists $x_0 > 0$ such that $a(x) \equiv a$ $\forall x \geq x_0$.

Subsections 3.1.6 and 3.1.7 treat the autonomous and non-autonomous problems for the differential equation with an extra dissipative term. Weighted Banach spaces play a crucial role in this generalization.

The last subsection concerns the search of an heteroclinic solution for the differential equation involving the p-Laplacian operator

$$\left(\left|u'\right|^{p-2} u'\right)' + cu' + g(u) = 0,$$

where $p > 1$ and $g(u)$ is a type A function in $[0, 1]$, that is, continuous, $g(0) = g(1) = 0$ and g is positive in $(0, 1)$. We will take a similar approach to the one used in [39].

3.1.2 Autonomous problem

In this subsection we make some considerations for the autonomous differential equation

$$u'' = a\,u - g\,(u)\,, \tag{3.3}$$

with $a \in \mathbb{R}^+$ and $g > 0$. We will divide the subsection into two parts: first we deal with an easier case where $g(u)$ is a power, and then we deal with the more general case. These are classic results (some of them based in the mentioned paper of H.Berestycki and P.Lions [5]), which we will use in the next subsection to deal with the non-autonomous case. Assume that conditions (H_1) and (H_2) are satisfied. The graph of $a\frac{u^2}{2} - G(u)$ and the phase plane for the admissible function $g(u) = u^3$ are:

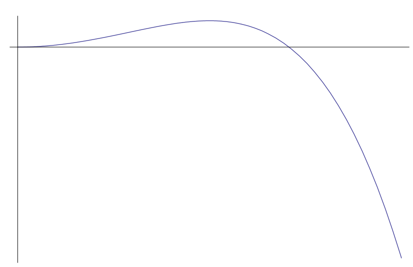


Figure 3.1: $a\frac{u^2}{2} - G(u)$

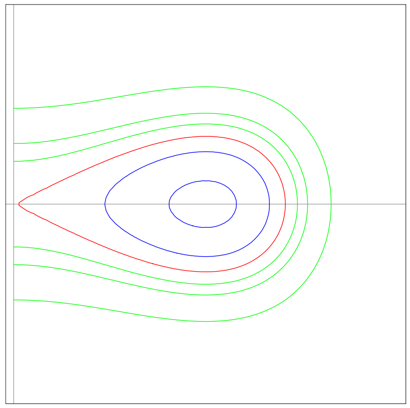


Figure 3.2: phase plane

Hypotheses (H_1) and (H_2) are well known sufficient conditions for the existence of a positive solution via the Mountain-pass Theorem for the equation (3.3) with boundary conditions $u'(0) = 0$ and $u(T) = 0$, for $1 \leq T < +\infty$. In fact, for the underlying Euler-Lagrange functional

$$J_{a,T}(u) = \int_0^T \left(u'(x)^2 + a\,u(x)^2 - 2G(u_+)\right)\,dx,$$

defined in the functional space $H_T^* \equiv \left\{H^1[0,T]\colon u(T) = 0\right\}$, we have $J_{a,T}(0) = 0$, and, for $\epsilon > 0$ small enough, if $\|u\| = \epsilon$, then $J_{a,T}(u) > \delta(\epsilon) > 0$. The Palais-Smale condition is satisfied and, setting $u_\lambda = \lambda(1 - x^2)$, it is easy to see that $J_{a,T}(u_\lambda^+) < 0$ for $\lambda > 0$ large enough (independent of $T > 1$). Since the autonomous problem has a unique solution, the positive solution obtained via mountain-pass is the well-known phase plane solution.

Consider first the case where $g(u) = u^p$, for $p > 1$, where hypotheses (H_1) and (H_2) are obviously verified. In this case the computations are easier to follow.

We know that there exists a positive homoclinic $u_a(x)$ at $u = 0$ passing through $(\zeta_0, 0)$, where $\zeta_0 = (\frac{a(p+1)}{2})^{\frac{1}{p-1}}$ (in the general case $g(u)$, ζ_0 will be the smallest positive value u such that $a\,u^2 - 2G(u) = 0$).

Multiplying the differential equation by u' and integrating, we get

$$u'^2 - a\,u^2 + \frac{2}{p+1}u^{p+1} = C, \quad C \in \mathbb{R}. \tag{3.4}$$

The homoclinic u_a corresponds to the constant $C = 0$, and therefore we have

$$u_a' = -u_a\sqrt{a - \frac{2}{p+1}\,u_a^{p-1}}, \quad u_a(0) = \zeta_0.$$

Considering the formally associated Euler-Lagrange functional

$$J_a(u) = \int_0^{+\infty} \left(u'(x)^2 + a\,u(x)^2 - \frac{2}{p+1}u_+(x)^{p+1}\right)\,dx,$$

in the functional space $H^1[0, +\infty)$, the "critical level" of u_a satisfies

$$J_a(u_a) = \int_0^{+\infty} \left(u_a'^2 + a\,u_a^2 - \frac{2}{p+1}u_a^{p+1}\right)\,dx = 2\int_0^{u_a(0)} u\sqrt{a - \frac{2}{p+1}u^{p-1}}du \equiv {J_a}^*.$$

Let us now consider the boundary value problem

$$\begin{cases} u''(x) = a\,u(x) - u(x)^p \\ u'(0) = 0, \quad u(T) = 0, \end{cases} \tag{3.5}$$

and the associated functional

$$J_{a,T}(u) = \int_0^T \left(u'^2 + a\,u^2 - \frac{2}{p+1} {u_+}^{p+1} \right) dx,$$

for $u \in H_T^*$. As we have seen, via the Mountain-pass Theorem, the boundary value problem (3.5) has a nontrivial positive solution $u_{a,T}$, which we identify also in the phase plane.

Proposition 3.1.1. *The solution $u_{a,T}$ of (3.5) satisfies $\lim_{T\to+\infty} u_{a,T}(0) = u_a(0)$.*

Proof. Using (3.4) we have ${u'_{a,T}}^2 = a\,{u_{a,T}}^2 - \frac{2}{p+1}{u_{a,T}}^{p+1} + C_T$ for some constant C_T, and consequently ${u'_{a,T}(T)}^2 = C_T > 0$. If there exists a sequence of $T's$ tending to $+\infty$ such that $u'_{a,T}(T) \to c < 0$, then, by a phase plane analysis, knowing that the trajectories in the phase plane cannot cross each other, we could easily see that trajectories that cross the u' axis close to c, could not be positive for an arbitrarily large interval $[0,T]$. So we must have $u'_{a,T}(T) \to 0$ and therefore $C_T \to 0$. Since $C_T = -a\,u_{a,T}(0)^2 + \frac{2}{p+1}u_{a,T}(0)^{p+1}$, we must have $u_{a,T}(0) \to \zeta_0 = u_a(0)$ and $u_{a,T}(0) > u_a(0)$. □

Proposition 3.1.2. *The critical value $J_{a,T}(u_{a,T})$ tends to ${J_a}^*$ as T tends to $+\infty$.*

Proof. We have

$$J_{a,T}(u_{a,T}) = \int_0^T \left(u'_{a,T}(x)^2 + a\,u_{a,T}(x)^2 - \frac{2}{p+1}{u_{a,T}}^{p+1} \right) dx =$$

$$= \int_0^{u_{a,T}(0)} \left(2\sqrt{u^2\left(a - \frac{2}{p+1}u^{p-1}\right) + C_T} - \frac{C_T}{\sqrt{u^2\left(a - \frac{2}{p+1}u^{p-1}\right) + C_T}} \right) du.$$

For simplicity let $f(u) = u^2\left(a - \frac{2}{p+1}u^{p-1}\right)$. Consider the following decomposition:

$$\int_0^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du = \int_0^{u_a(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du + \int_{u_a(0)}^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du.$$

Since $f(u) \geq 0$ for $u \in [0, u_a(0)]$, the first integral is smaller than $u_a(0)\sqrt{C_T}$. The second integral has a singularity at $u = u_{a,T}(0)$, and considering the Taylor expansion of $f(u)$ at $u = u_{a,T}(0)$, we easily check that there exists a constant $k > 0$ such that

$$\int_{u_a(0)}^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du \leq \int_{u_a(0)}^{u_{a,T}(0)} \frac{k\,c_T}{\sqrt{u_{a,T}(0)-u}}\,du.$$

It is now easy to conclude that $\int_0^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du \to 0$ and since $u_{a,T}(0) \to u_a(0)$ and $c_T \to 0$, the result follows. □

Consider now the differential equation (3.3), with $g(u)$ satisfying (H_1) and (H_2). In this case, multiplying the equation by u' and integrating gives us $u'^2 = a\,u^2 - 2G(u) + C$, $C \in \mathbb{R}$, and taking $C = 0$, we get the phase plane equation of an homoclinic solution u_a. Note that we can assume that $u_a'(0) = 0$, and, consequently, $u_a(0)$ is a zero of the function $a\,u^2 - 2G(u)$. The classic work of Berestycki and Lions [5] allows us to conclude it must be the first positive zero ζ_0 of that function.

The underlying Euler-Lagrange functional is

$$J_a(u) = \int_0^{+\infty} \left(u'(x)^2 + a\,u(x)^2 - 2G(u)\right)\,dx,$$

and it is easily seen that

$$J_a(u_a) = 2\int_0^{u_a(0)} \sqrt{a\,u^2 - 2G(u)}\,du \equiv {J_a}^*.$$

The boundary value problem

$$\begin{cases} u''(x) = a\,u(x) - g\left(u\left(x\right)\right) \\ u'(0) = 0, \quad u(T) = 0 \end{cases} \tag{3.6}$$

has a positive solution $u_{a,T}$ and

$$\begin{aligned} J_{a,T}(u_{a,T}) = & \int_0^{u_{a,T}(0)} 2\sqrt{(a\,u^2 - 2G(u)) - \left(a\,u_{a,T}(0)^2 - 2G(u_{a,T}(0))\right)}\,du + \\ & + \int_0^{u_{a,T}(0)} \frac{a\,u_{a,T}(0)^2 - 2G(u_{a,T}(0))}{\sqrt{(a\,u^2 - 2G(u)) - \left(a\,u_{a,T}(0)^2 - 2G(u_{a,T}(0))\right)}}\,du. \end{aligned}$$

A careful analysis of the phase plane implies that $\lim_{T\to+\infty} u_{a,T}(0) = u_a(0)$, and after some computation, knowing that $a\,\zeta_0 - g\left(\zeta_0\right) < 0$, we can conclude again that $J_{a,T}(u_{a,T}) \to {J_a}^*$.

3.1.3 Superlinear nonlinearity

In this subsection we will prove the existence of a solution for the non-autonomous problem in $\mathbb{R}^+$, with $u'(0) = 0$ and $u(+\infty) = 0$.

Let $a(x)$ be a continuous function defined in $\mathbb{R}^+$, satisfying (A_1) and (A_2). Note that we could as well have taken $a(x)$ to be a piecewise continuous function.

Remark 3.1.3. For $g(u) = u^3$ condition (A_2) is the inequality $A < 2^{2/3}a$.

The arguments given in the previous subsection, concerning the existence of positive mountain-pass solutions for the autonomous case, are also valid for the non-autonomous case.

Consider the boundary value problem

$$\begin{cases} u'' = a(x)\,u - u^p \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{3.7}$$

Setting

$$J_T(u) = \int_0^T \left(u'^2 + a(x)\, u^2 - \frac{2}{p+1} u_+^{p+1} \right) dx,$$

as in the constant case, this functional has a mountain-pass geometry relative to the local minimum $u = 0$ in $H_T^* \equiv \{H^1[0,T]\colon u(T) = 0\}$, and consequently (3.7) has a positive nontrivial solution u_T. We want to find a solution for the infinite domain problem as the limit of a sequence of solutions u_T, with $T \to \infty$. Let c_T be the mountain-pass critical value of J_T, that is, $c_T = J_T(u_T)$. Defining $\Gamma_T = \{\gamma(\tau)\colon [0,1] \to H_T^*\colon \gamma(0) = 0, \gamma(1) = u_\lambda^+\}$, we know that

$$c_T = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right).$$

Since $\Gamma_{T_1} \subseteq \Gamma_{T_2}$ for $T_1 < T_2$, we have $c_T \le c_1$ for $T \ge 1$. Also by comparison we prove the following result:

Lemma 3.1.4. *The critical values c_T are such that $c_T \le J_{A,T}(u_{A,T})$.*

Multiplying the differential equation by u and integrating, we get

$$-\int_0^T u_T'^2\, dx = \int_0^T \left(a(x) u_T^2 - u_T^{p+1} \right) dx,$$

and consequently, we have

$$J_T(u_T) = \frac{p-1}{p+1} \int_0^T u_T^{p+1}\, dx = \frac{p-1}{p+1} \int_0^T \left(u'^2 + a(x)\, u^2 \right) dx. \tag{3.8}$$

Extending u_T to $[0, +\infty)$ by $u_T(x) = 0$ for $x \ge T$, it follows that:

Proposition 3.1.5. *We have uniform estimates for the $L^{p+1}(0,+\infty)$ and $H^1(0,+\infty)$ norms of the solutions u_T (for $T \ge 1$).*

Proof. Since $J_T(u_T) \le c_1$ for all $T > 1$, (3.8) allows us to conclude the result. ☐

Corollary 3.1.6. *There exists $k > 0$ such that, for all $T > 1$,*

$$|u_T(x)|\,,\; \left|{u_T}'(x)\right|,\; \left|{u_T}''(x)\right| \le k \quad \forall x \in [0,T].$$

As a consequence, using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^1-uniformly in compact intervals and ${u_T}' \rightharpoonup u'$ weakly in $L^2(0,+\infty)$.

Proposition 3.1.7. ${u_T}'(T) \to 0$ *as* $T \to +\infty$.

Proof. If ${u_T}'(T) \not\to 0$, then, since ${u_T}'$ is bounded, there exists a sequence of T's tending to $+\infty$ such that ${u_T}'(T) \to d$ for some constant $d < 0$. Consider the initial value problem

$$\begin{cases} u'' = a\, u - u^p \\ u'(0) = d, \quad u(0) = 0. \end{cases} \tag{3.9}$$

Let $-2c$ be the largest negative zero of the solution $\bar{u}$ of (3.9). We have $\bar{u}'(-c) = 0$, $\bar{u}(-c) > 0$, $\bar{u}'(-3c) = 0$ and $\bar{u}(-3c) = -\bar{u}(-c)$. Defining $v_T(x) = u_T(x+T)$, we must have $v_T(x) \to \bar{u}(x)$ and consequently, for T large enough, we would have $u_T(T-3c) < 0$, which is a contradiction since u_T does nor vanish before T. Consequently, we must have ${u_T}'(T) \to 0$.

□

Corollary 3.1.8. *Setting l_T as the largest maximizer of u_T, we have $T - l_T \to +\infty$.*

In the following, let $J_T(u)\big|_{[m,n]} = \int_m^n \left(u'^2 + a(x)\,u^2 - \frac{2}{p+1} u_+^{p+1}\right)\,dx$.

Lemma 3.1.9. *Given an arbitrary positive constant ϵ, there exists x_ϵ such that for all $T > 0$ and all $x > x_\epsilon$ we have $u_T(x) \le \epsilon$.*

Proof. Let $x_{\epsilon,T} = \inf\{x : \forall t \ge x,\ u_T(t) \le \epsilon\}$. Suppose that $x_{\epsilon,T} \to \infty$ as $T \to \infty$. If $u_T \ge \epsilon$ in $[0, x_{\epsilon,T}]$ along a sequence of T's, then, because of (3.8), $J_T(u_T) \to \infty$, which is a contradiction.
Claim 1. Given $\epsilon > 0$, $J_T(u_T)\big|_{[x_{\epsilon,T},T]} \le \eta(\epsilon)$, where $\lim_{\epsilon\to 0}\eta(\epsilon) = 0$.
Proof of Claim 1. Multiplying the differential equation by u_T and integrating in $[x_{\epsilon,T}, T]$, we have

$$J_T(u_T)\big|_{[x_{\epsilon,T},T]} = -u_T(x_{\epsilon,T}){u_T}'(x_{\epsilon,T}) + \int_{x_{\epsilon,T}}^T \frac{p-1}{p+1} u_T^{p+1}\,dx \le$$
$$\le \epsilon\, {u_T}'(x_{\epsilon,T}) + \epsilon^{p-1}\frac{p-1}{p+1}\int_{x_{\epsilon,T}}^T u_T^2\,dx$$

Since $p - 1 > 0$, the conclusion follows easily using Proposition 3.1.5 and Corollary 3.1.6.
Claim 2. Let l_T be the largest maximizer of u_T. If $l_T \to +\infty$ as $T \to +\infty$, then

$$J_T(u_T)\big|_{[l_T,T]} \to {J_a}^* \text{ as } T \to +\infty.$$

Proof of Claim 2. Defining $v_T(x) = u_T(x + l_T)$, then, along a subsequence, we have $v_T \to v$ C^1-uniformly in compact intervals, where $v''(x) = a\,v(x) - v(x)^p$, $v'(0) = 0$ and $v > 0$ in $[0, +\infty)$ by Corollary 3.1.8, that is, $v = u_a$.

Let $\delta > 0$ be such that ${J_A}^* < 2{J_a}^* - 3\delta$. Given an arbitrary $\epsilon > 0$, there exists a constant $c = c(\epsilon)$ such that

$$|u_a(c)| < \epsilon,\ \left|u_a'(c)\right| < \epsilon, \tag{3.10}$$

and

$$J_a(u_a)\big|_{[0,c]} > {J_a}^* - \delta.$$

For T large enough, $u_T(x + l_T)$ converges uniformly in $C^1[0, c]$ to $u_a(x)$, so we also have

$$|v_T(x) - u_a(x)| < \epsilon,\ \left|v_T'(x) - u_a'(x)\right| < \epsilon \quad \forall x \in [0, c].$$

Since $l_T \to \infty$, we may assume that $|a(x) - a| < \epsilon$ for $x > l_T$, so we have

$$\left|J_T(u_T)\big|_{[l_T,l_T+c]} - J_a(u_a)\big|_{[0,c]}\right| \le \tag{3.11}$$
$$\le \int_0^c \left|v_T'(x)^2 - u_a'(x)^2\right| + |a(x + l_T) - a|\, v_T(x)^2 +$$
$$+\, a\left|v_T(x)^2 - u_a(x)^2\right| + \frac{2}{p+1}\left|v_T(x)^{p+1} - u_a(x)^{p+1}\right|\,dx \le K\epsilon$$

Using *Claim 1,* we have

$$\begin{aligned}&\left|J_T(u_T)\big|_{[l_T,T]} - {J_a}^*\right| \leq \\ &\quad\leq \left|J_T(u_T)\big|_{[l_T,l_T+c]} - {J_a}^*\big|_{[0,c]}\right| + \left|J_T(u_T)\big|_{[l_T+c,T]} - {J_a}^*\big|_{[c,T]}\right| \leq K\epsilon + \delta + \eta\left(\epsilon\right)\end{aligned}$$

and the conclusion follows.

Now, if there is a maximizer $l_T \to \infty$, $v_T(x)$ is well-defined in $[-c,c]$ and converges uniformly in $C^1[-c,c]$ to $u_a(x)$ (considering the even extension of u_a). This means that the solutions u_T will have an almost symmetric bell shape (the shape of u_a) around l_T for large T. The same arguments used in (3.11) provide us that

$$\left|J_T(u_T)\big|_{[l_T-c,l_T+c]} - 2J_a(u_a)\big|_{[0,c]}\right| \leq K\epsilon.$$

On the other hand, we have

$$\begin{aligned}J_T(u_T)\big|_{[0,l_T-c]} &= \int_0^{l_T-c} u_T'^2 - a(x)u_T^2 - u_T^{p+1}\,dx + \int_0^{l_T-c} \frac{p-1}{p+1}u_T^{p+1}\,dx = \\ &= u_T'(l_T-c)u_T(l_T-c) + \int_0^{l_T-c} \frac{p-1}{p+1}u_T^{p+1}\,dx > 0\end{aligned}$$

and consequently, we would conclude that

$${J_A}^* \geq J_T(u_T) \geq 2{J_a}^* - 2\delta - K\epsilon - \eta\left(\epsilon\right)$$

This fact contradicts assumption (A_2) by Proposition 3.1.2 and Lemma 3.1.4.

□

In order to show that the limit is not the trivial solution, we need the following

Proposition 3.1.10. *There exists a constant $c > 0$ such that $u_T(0) > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to $+\infty$ such that $u_T(0) \to 0$. Then it is obvious that l_T tends to $+\infty$ with T. Then we obtain the same contradiction as in the proof of Lemma 3.1.9. □

We are now able to to prove the main result:

Theorem 3.1.11. *Under the assumptions (A_1), (A_2) the boundary value problem*

$$\begin{cases} u'' = a(x)\,u - u^p \\ u'(0) = 0, \quad u(+\infty) = 0 \end{cases} \tag{3.12}$$

has a positive solution.

Proof. Using Proposition 3.1.10 and Lemma 3.1.9 we have $u_T(x) \to u(x)$ C^1-uniformly in compact intervals, with $u(x)$ a positive solution of (3.12). □

Remark 3.1.12. In the case where $g(u)$ is not a power, but satisfies the hypotheses (H_1) and (H_2), we have

$$J_T(u_T) = \int_0^T (g(u)u - 2G(u))\, dx.$$

Multiplying the differential equation by u and integrating, we get

$$-\int_0^T u_T'^2\, dx = \int_0^T \left(a(x)u_T^2 - u_T\, g(u_T)\right)\, dx,$$

and consequently, we have

$$J_T(u_T) \geq \left(1 - \frac{2}{q}\right) \int_0^T \left(u_T'^2 + a(x)\, u_T^2\right)\, dx = \left(1 - \frac{2}{q}\right) \int_0^T g(u_T)u_T\, dx,$$

so that the same arguments used in the case $g(u) = u^p$ will provide us similar conclusions.

3.1.4 Cubic nonlinearity and $a(x)$ nondecreasing

Let us focus on the problem

$$\begin{cases} u'' = a(x)u - u^3 = u\, \left(a(x) - u^2\right) \\ u'(0) = 0, \quad u(+\infty) = 0. \end{cases} \tag{3.13}$$

The arguments used below are still valid for a more general power u^p (with $p > 1$) instead of u^3, or even more general increasing functions $g(u)$, but for simplicity, we will only present the calculations for this particular case.

Lemma 3.1.13. *Let $a(x)$ be a positive and nondecreasing function defined in $[0, +\infty)$. If $u(x)$ is a solution of (3.13), the energy function $E(x) \equiv \frac{u'^2}{2} + \frac{u^4}{4} - \frac{a(x)u^2}{2}$ is decreasing in $\mathbb{R}^+$.*

Proof. Let $x_1 < x_2 \in \mathbb{R}^+$. Using the Stieltjes integral, we have

$$\begin{aligned} E(x_2) - E(x_1) &= \int_{x_1}^{x_2} dE = \int_{x_1}^{x_2} \left(\frac{u'^2}{2} + \frac{u^4}{4}\right)' dx - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} = \\ &= \int_{x_1}^{x_2} a(x)u\, u'\, dx - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} \frac{u^2}{2}\, da(x) \leq 0. \end{aligned}$$

□

Positive solutions of $u''(x) = a(x)u(x) - u(x)^3 = u(x)\left(a(x) - u(x)^2\right)$ are concave if $u(x) > \sqrt{a(x)}$ and convex if $u(x) < \sqrt{a(x)}$, therefore the graph of the solution of

$$\begin{cases} u''(x) = a(x)u(x) - u(x)^3 \\ u(0) = L, \quad u'(0) = 0 \end{cases} \tag{3.14}$$

where $L > \sqrt{a(0)}$, crosses the graph of $\sqrt{a(x)}$ at $x = c_L$ for some $c_L > 0$, and we may assume that c_L is the minimum value with this property.

Proposition 3.1.14. *As L tends to $+\infty$, c_L tends to* 0.

Proof. Let us first prove the result for $a(x)$ bounded. Let d_L be the minimum value such that $u_L(d_L) = \frac{L}{2}$. Suppose towards a contradiction that $d_L \not\to 0$. This means that there exists a sequence $L_n \to +\infty$ such that $d_{L_n} > k$ for some constant $k > 0$. Let $p = \frac{\pi}{2k} > \frac{\pi}{2d_{L_n}}$. Since $a(x)$ is bounded, for n large enough we have $a(x) - u_{L_n}^2(x) \le -p^2$ for $x \in [0, d_{L_n}]$, so the unique solution v of the initial value problem

$$\begin{cases} v''(x) = -p^2 v \\ v(0) = L_n, \quad v'(0) = 0 \end{cases} \tag{3.15}$$

is such that $v(x) \ge u_{L_n}(x)$ in the interval $[0, d_{L_n}]$. But $v(x) = L_n \cos(p\,x)$ vanishes at $x = \frac{\pi}{2\,p} = k < d_{L_n}$, which contradicts $v(x) \ge u_{L_n}(x)$. Consequently we have $d_L \to 0$ and the concavity of u_L on $[0, c_L]$ implies that $c_L \le 2d_L$, so we conclude that $c_L \to 0$.

In case $a(x)$ is unbounded, consider the bounded auxiliar function

$$\bar{a}(x) = \begin{cases} a(x), & x \le 1 \\ a(1), & x > 1. \end{cases}$$

Applying the result obtained for bounded functions, we get $c_L < 1$ for $L > L_0$ large enough and since the result only depends on the values of x smaller than c_L, the result holds for the unbounded function $a(x)$. □

Corollary 3.1.15. *As L tends to $+\infty$, ${u_L}'(c_L) \to -\infty$.*

Proposition 3.1.16. *For $L > \sqrt{a(0)}$ large enough, the solution of* (3.14) *has at least one zero.*

Proof. For simplicity, let us denote c_L by c. Given $L^* > \sqrt{a(0)}$ large, let c^* be the first value such that the graph of the solution of (3.14) with $L = L^*$ crosses the graph of $\sqrt{a(x)}$. Taking a sufficiently large $L > L^*$, the corresponding solution u_L of (3.14) satisfies $u_L(c) = \sqrt{a(c)}$, for some $c < c^*$. Suppose towards a contradiction that u_L does not vanish in $[0, c^*]$. Then, there exists $\hat{c} \in [c, c^*]$ such that ${u_L}'(\hat{c}) = -\frac{\sqrt{a(c)}}{c^*-c}$, which is the slope of the line connecting $(c, \sqrt{a(c)})$ and $(c^*, 0)$, and we have

$$-\frac{\sqrt{a(c)}}{c^* - c} - {u_L}'(c) = \int_c^{\hat{c}} {u_L}''(x)\,dx \le \int_c^{\hat{c}} a(x) u_L(x)\,dx \le \int_c^{\hat{c}} a(x)^{3/2}\,dx \le c^* \sqrt{a(c^*)}^3.$$

Taking in consideration last corollary, we have a contradiction. □

Proposition 3.1.17. *Consider the initial value problem* (3.14) *with $L > \sqrt{a(0)}$. If its solution u_L is positive and does not have a local minimum, then $u_L(+\infty) = 0$.*

Proof. It is obvious that the graph of u_L crosses the graph of $\sqrt{a(x)}$ with negative derivative and since the derivative does not vanish again and u_L is positive, we must have $u_L'(+\infty) = 0$ and therefore $u_L(+\infty) = k \ge 0$. If $k > 0$ then

$$u_L''(+\infty) = u_L(+\infty)\left(a(+\infty) - u_L^2(+\infty)\right) > 0$$

and therefore $u'(+\infty) = +\infty$. Then there would exist $c \in \mathbb{R}$ such that $u_L'(c) = 0$, which is a contradiction. □

Proposition 3.1.18. *If $0 < L < \sqrt{2a(0)}$ then the solution u_L of* (3.14) *is positive in $\mathbb{R}^+$ and attains a positive minimum m for some $x_m \geq 0$.*

Proof. Since $E(0) = \frac{L^2}{2}\left(\frac{L^2}{2} - a(0)\right) < 0$, we have $E(x) < 0$ for every $x > 0$. If there exists $x_0 > 0$ such that $u_L(x_0) = 0$, then $E(x_0) = \frac{u_L{}'(x_0)^2}{2} \geq 0$, which is a contradiction.

If u_L does not attain a positive minimum, then $u_L(+\infty) = 0$ and $u_L'(+\infty) = 0$, and therefore $E(+\infty) = 0$, which is again a contradiction. □

Proposition 3.1.19. *If the solution u_L of* (3.14) *attains a positive minimum m for some $x_m \geq 0$, then u_L is positive for $x > x_m$.*

Proof. We can conclude as above, since $E(x_m) = \frac{m^2}{2}\left(\frac{m^2}{2} - a(x_m)\right) < 0$. □

Theorem 3.1.20. *Let $a(x)$ be a positive nondecreasing function. Then problem* (3.13) *has at least one positive solution.*

Proof. We use a connectedness argument appearing in the paper of H. Berestycki, P. Lions and L. Peletier [6]. Consider the following subsets of $\mathbb{R}^+$:

$$A = \left\{L > \sqrt{a(0)} \colon u_L > 0 \text{ and } u_L \text{ has a positive minimum}\right\},$$

$$B = \left\{L > \sqrt{a(0)} \colon u_L(x_0) = 0 \text{ for some } x_0 > 0\right\}.$$

Both sets are nonempty, obviously disjoint, and, by the continuous dependence on the initial data, open in $\mathbb{R}$. Let $u_0 = \inf B$. Since u_0 does not belong neither to A or B, we must conclude that the solution of problem (3.14) with $L = u_0$ is positive and tends to 0 at ∞. □

3.1.5 Bounded nonlinearity and $a(x)$ constant in a neighborhood of ∞

In this subsection we prove the existence of a positive solution of the boundary value problem

$$u'' = a(x)u - g(u), \quad u'(0) = u(+\infty) = 0. \tag{3.16}$$

We will consider $a(x)$ satisfying (A_2) and a stronger hypothesis than (A_1): assume that

(A_1') $0 < a(x) \leq A$ for all $x \geq 0$ and there exists $x_0 > 0$ such that $a(x) \equiv a$ for all $x \geq x_0$.

The function $g \in C([0,\infty), [0,\infty))$ will be a bounded function that satisfies (H_2), and in addition:

(H_3) The function $f(u) := a\,u^2 - 2G(u)$ has only one negative minimum attained at $u = \eta$, and hence only one zero, say ξ in $(0, \eta)$.

(H_4) $A\,u^2 - 2G(u) = 0$ has also a negative minimum.

(H_5) There exists $\alpha > 0$ such that $|f(u) - f(v)| \geq \alpha|u - v|$ $\forall u$, v in a neighborhood of ξ,

(H_6) $\int_0^\eta \frac{du}{\sqrt{f(u)-f(\eta)}} = +\infty.$

Condition (H_3) is not absolutely necessary since we could reach the same conclusions in a more general context, but we included it for simplicity of notations and calculations. Note that (H_1) does not hold. Since we look for positive solutions, in what follows we set $g(u) = 0$ for $u < 0$.

Here we present the graphs of $a\frac{u^2}{2} - G(u)$ and the autonomous phase plane structure for an admissible function $g(u)$:

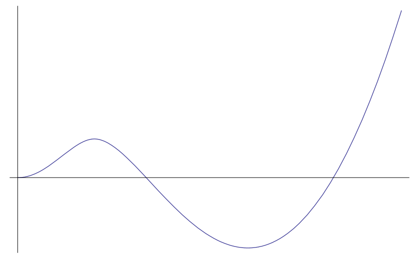


Figure 3.3: $a\frac{u^2}{2} - G(u)$

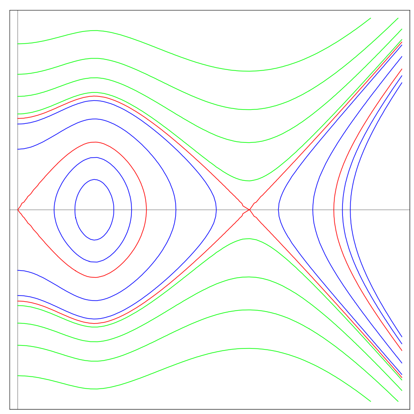


Figure 3.4: phase plane

For completeness, we present also the same graphs in a case where (H_3) is not satisfied, but the process to find a solution could be used:

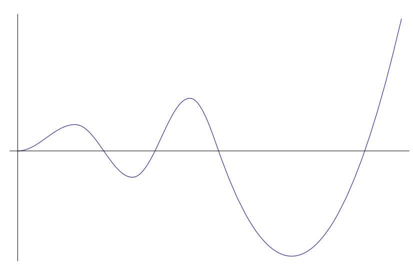


Figure 3.5: $a\frac{u^2}{2} - G(u)$

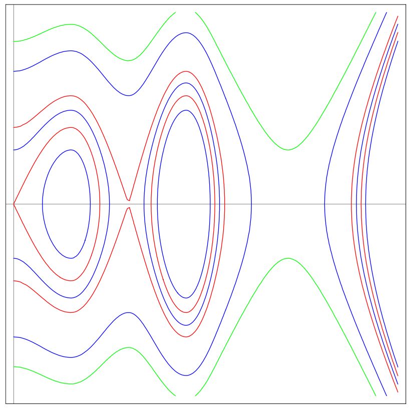


Figure 3.6: phase plane

Remark 3.1.21. Before we deal with the problem above, let us consider a slight variation. Suppose that instead of (A_2), $a(x)$ satisfies

$$\sqrt{A}\tanh(\sqrt{A}x_0) < \sqrt{a}. \tag{3.17}$$

If we consider the initial value problem

$$u'' = a(x)u - g(u), \quad u(0) = \zeta, \quad u'(0) = 0, \tag{3.18}$$

it is obvious that for ζ large the solutions must be convex and therefore larger than ζ for every $x > 0$. If $\zeta > 0$ is small enough, then, by (H_2), we have $(u(x,\zeta),\, u'(x,\zeta)) = \zeta(v(x),\, v'(x)) + \mathrm{o}(\zeta)$ uniformly in $[0, x_0]$, where v is the solution of the linear problem

$$v'' = a(x)v, \quad v(0) = 1, \quad v'(0) = 0.$$

Since $z(x) = \frac{v'(x)}{v(x)}$ satisfies $z' + z^2 = a(x)$, an elementary comparison theorem shows that $z(x_0) \leq \sqrt{A}\tanh(\sqrt{A}x_0)$.

Now the positive homoclinic at the origin for the autonomous equation $u'' = a\,u - g(u)$ has an image curve in the (u, u')-plane whose slope at the origin in the half-plane $u' > 0$ is precisely $\sqrt{a}$. Hence by (3.17), for ζ sufficiently small, $(u(x_0,\zeta), u'(x_0,\zeta))$ lies "inside" the homoclinic. Since for ζ large $(u(x_0,\zeta), u'(x_0,\zeta))$ is obviously "outside" the homoclinic, a connectedness argument based on the Peano phenomenon (see e. g. [44]) allows us to conclude that there exists a value ζ_0 such that $(u(x_0,\zeta_0), u'(x_0,\zeta_0))$ is a point of the homoclinic solution of the autonomous problem. Since for $x \geq x_0$ we have $a(x) = a$, there exists a positive solution of (3.16).

Note that estimate (3.17) works well only if x_0 is small.

Consider now the problem assuming conditions $(H_2) - (H_3) - (H_4) - (H_5) - (H_6)$ and $(A_1') - (A_2)$. Proceeding as above, we easily see that the boundary value problems

$$\begin{cases} u'' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(T) = 0 \end{cases} \tag{3.19}$$

have a positive solution u_T, because the associated modified Euler-Lagrange functionals

$$J_T(u) = \int_0^T \left(u'^2 + a(x)\,u^2 - 2G(u_+)\right)\,dx$$

have a mountain-pass geometry relative to the local minimum $u = 0$ in the space $H_T^* \equiv \left\{H^1[0,T] \colon u(T) = 0\right\}$. The mountain-pass critical values $c_T = J_T(u_T)$ are positive, decreasing in T and therefore, for $T > 1$, we have $c_T \le c_1$. The solution u_T must attain a maximum at a point where $u_T{}'' \le 0$ so $\|u_T\|_\infty$ is uniformly bounded in T. The differential equation allows us to conclude that $\|u_T{}''\|_\infty$ is bounded too and consequently the same is true for $\|u_T{}'\|_\infty$.

Proposition 3.1.22. *$u_T{}'(T) \to 0$ as $T \to +\infty$.*

Proof. If $u_T{}'(T) \not\to 0$, then there exists a sequence of T's tending to $+\infty$ such that $u_T{}'(T) \to d$ for some constant $d < 0$.

If we multiply the differential equation with $u = u_T$ by $u_T{}'$ and integrate, we get

$$u_T{}'^2 = a\,u_T{}^2 - 2G(u_T) + K_T, \quad \forall\, x \ge x_0, \tag{3.20}$$

where K_T is a constant.

Consider the autonomous initial value problem

$$\begin{cases} u'' = a\,u - g(u) \\ u'(0) = d, \quad u(0) = 0. \end{cases} \tag{3.21}$$

Recall that ξ is the smallest positive value such that $2G(u) - a\,u^2 = 0$ and η is the maximizer of $2G(u) - a\,u^2$. Let $d_\eta < 0$ be the value of the derivative when $u = 0$ for the trajectory that goes to $(\eta, 0)$ as $x \to -\infty$. This trajectory exists by virtue of (H_6) and is given by $u' < 0$ and

$$u'^2 = au^2 - 2G(u) + d_\eta^2,$$

where

$$d_\eta^2 = -a\eta^2 + 2G(\eta).$$

We will divide the proof into three cases, $d_\eta < d < 0$, $d = d_\eta$ and $d < d_\eta$:

(1) If $d_\eta < d < 0$, the correspondent solution u of the autonomous problem (3.21) has a largest negative zero $-c$ and $u'(-c) > 0$. For T large enough we have $T - c > x_0$, so the solutions u_T coincide with the autonomous solutions and consequently, since we have uniform convergence in compact intervals, we would have a contradiction with the positivity of the solutions u_T.

(2) If $d = d_\eta$, we will distinguish two cases: $u_T{}'(T) \to d_\eta$ from above and $u_T{}'(T) \to d_\eta$ from below. In the first situation, if there exists a local maximum point $x_T \ge x_0$ (let $u_T(x_T) \equiv \eta_T$) then $\eta_T < \eta$ and $f(\eta_T) + K_T = 0$, which implies that $\eta_T \to \eta$ as $T \to \infty$. We have

$$\begin{aligned} J_T(u_T) &= \int_0^T \left[u_T{}'^2 + a(x)\,u_T{}^2 - 2G(u_T)\right]\,dx = \\ &= \int_0^{x_0} \left[u_T{}'^2 + a(x)\,u_T{}^2 - 2G(u_T)\right]\,dx + \int_{x_0}^{x_T} \left[2\left(a\,u_T{}^2 - 2G(u_T)\right) + K_T\right]\,dx + \\ &\qquad + \int_{x_T}^{T} \left[2\left(a\,u_T{}^2 - 2G(u_T)\right) + K_T\right]\,dx \end{aligned} \tag{3.22}$$

The first integral is obviously uniformly bounded and, making a change of variable, we get for the third integral

$$\begin{aligned}\int_{x_T}^{T} \left[2\left(a\, {u_T}^2 - 2G(u_T)\right) + K_T\right]\, dx &= \\ = \int_0^{\eta_T} \left[\sqrt{f(u) - f(\eta_T)} + \frac{f(u)}{\sqrt{f(u) - f(\eta_T)}}\right] du.\end{aligned} \tag{3.23}$$

The first part of the integral is obviously bounded and using Fatou's Lemma and (H_6), we have

$$+\infty = \int_{\eta-\delta}^{\eta} \frac{du}{\sqrt{f(u) - f(\eta)}} \leq \liminf \int_{\eta-\delta}^{\eta_T} \frac{du}{\sqrt{f(u) - f(\eta_T)}},$$

where $\delta > 0$ is such that $f(u) < 0$ for $u \in [\eta - \delta, \eta + \delta]$. It is easy to see that this implies that the second part of integral tends to $-\infty$ and consequently (3.23) also tends to $-\infty$. For the second integral in (3.22), we have analogously

$$\begin{aligned}\int_{x_0}^{x_T} \left[2\left(a\, {u_T}^2 - 2G(u_T)\right) + K_T\right]\, dx &= \\ = \int_{u_T(x_0)}^{\eta_T} \left[\sqrt{a\, u^2 - 2G(u) + K_T} + \frac{a\, u^2 - 2G(u)}{\sqrt{a\, u^2 - 2G(u) + K_T}}\right] du,\end{aligned}$$

and if $u_T(x_0)$ does not tend to η, we also have this integral tending to $-\infty$ (otherwise it is bounded). This implies that $J_T(u_T)$ tends to $-\infty$, which contradicts the fact that the mountain pass critical level is positive. Consider now the case where the solution u_T is decreasing for every $x \geq x_0$. In this situation we have

$$J_T(u_T) = \int_0^{x_0} \left[{u_T'}^2 + a(x)\, {u_T}^2 - 2G(u_T)\right] dx + \int_{x_0}^{T} \left[2\left(a\, {u_T}^2 - 2G(u_T)\right) + K_T\right] dx,$$

where $K_T \to 2G(\eta) - a\,\eta^2$. Setting $u_T(x_0) = \eta_T$, we have

$$\begin{aligned}\int_{x_0}^{T} \left[2\left(a\, {u_T}^2 - 2G(u_T)\right) + K_T\right]\, dx &= \\ = \int_0^{\eta_T} \left[\sqrt{a\, u^2 - 2G(u) + K_T} + \frac{a\, u^2 - 2G(u)}{\sqrt{a(x)\, u^2 - 2G(u) + K_T}}\right] du,\end{aligned}$$

and since we must have $\eta_T \to \eta$ ($T - x_0 \to \infty$ implies it), we have a contradiction of the same type as above.

The case where ${u_T}'(T) \to d_\eta$ from below can also be treated in a similar way, since we also must have $K_T \to 2G(\eta) - a\,\eta^2$. Setting $u_T(x_0) = \eta_T$, it follows that $\eta_T \to \eta$ and therefore we would again reach the contradiction $J_T(u_T) \to -\infty$.

(3) If $d < d_\eta$, the correspondent solution w of the autonomous problem satisfies

$$w'^2 = aw^2 - 2G(w) + d^2.$$

This shows that $w'(x) < -\sqrt{d^2 - d_\eta^2}$ for all $x < 0$, and hence w is unbounded above. Again by uniform convergence in compact intervals, u_T would take arbitrarily large values for T sufficiently large. This is a contradiction with the uniform boundedness of u_T.

We can therefore conclude that ${u_T}'(T) \to 0$. □

Corollary 3.1.23. *Setting l_T as the largest maximizer of u_T, we have $T - l_T \to +\infty$.*

In the following, let $J_T(u)\big|_{[m,n]} = \int_m^n \left[u'^2 + a(x)\,u^2 - 2G(u_+)\right] dx$. In order to show that the limit of the solutions u_T cannot be the trivial solution, we need the following

Proposition 3.1.24. *There exists a constant $k > 0$ such that $u_T(0) > k$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to $+\infty$ such that $u_T(0) \to 0$. Then $|u_T(x)| + |u'_T(x)| \to 0$ uniformly in $[0, x_0]$ as $T \to \infty$. Since for u small we have $G(u) = o(u^2)$, then $J_T(u_T)\big|_{[0,x_0]} \to 0$.

Since $J_T(u_T)$ is bounded away from zero, there exists a maximizer $x_T > x_0$ (otherwise we easily would show that $J_T(u_T)$ becomes arbitrarily small). It is obvious that x_T tends to $+\infty$ with T. Now, setting $\xi_T = u_T(x_T)$, we compute

$$J_T(u_T)\big|_{[x_0,x_T]} = \int_{u(x_0)}^{\xi_T} \left(2\sqrt{a\,u^2 - 2G(u) + K_T} - \frac{K_T}{\sqrt{a\,u^2 - 2G(u) + K_T}}\right) du,$$

with $K_T = 2G(\xi_T) - a\,\xi_T^2$, and because of Proposition 3.1.22, it follows that $\xi_T \to \xi$ and $K_T \to 0$. For simplicity we can write the second integral in the simpler form

$$\int_{u(x_0)}^{\xi_T} \frac{K_T}{\sqrt{f(u) + K_T}}\, du = \int_{u(x_0)}^{\xi} \frac{K_T}{\sqrt{f(u) + K_T}}\, du + \int_{\xi}^{\xi_T} \frac{K_T}{\sqrt{f(u) - f(\xi_T)}}\, du.$$

Since $f(u) \geq 0$ for $u \in [0, \xi]$, the first integral is smaller than $\xi\sqrt{K_T}$. The second integral has a singularity at $u = \xi_T$, but using (H_5) we easily check that there exists a constant $k > 0$ such that

$$\int_{\xi}^{\xi_T} \frac{K_T}{\sqrt{f(u) - f(\xi_T)}}\, du \leq \int_{\xi}^{\xi_T} \frac{k\,K_T}{\sqrt{\xi_T - u}}\, du$$

and it follows that this integral tends to zero as well. This implies that

$$\lim_{T\to\infty} J_T(u_T)\big|_{[x_0,x_T]} = J_a^*.$$

The same computations are valid for the integral

$$J_T(u_T)\big|_{[x_T,T]} = \int_0^{\xi_T} \left(2\sqrt{a\,u^2 - 2G(u) + K_T} - \frac{K_T}{\sqrt{a\,u^2 - 2G(u) + K_T}}\right) du.$$

Since

$$J_T(u_T) = J_T(u_T)\big|_{[0,x_0]} + J_T(u_T)\big|_{[x_0,x_T]} + J_T(u_T)\big|_{[x_T,T]}$$

and as we have seen

$$\lim_{T\to\infty} J_T(u_T)\big|_{[0,x_0]} = 0,$$

$$\lim_{T\to\infty} J_T(u_T)\big|_{[x_0,x_T]} = \lim_{T\to\infty} J_T(u_T)\big|_{[x_T,T]} = J_a^*,$$

we conclude that:

$$\lim_{T\to\infty} J_T(u_T) = 2J_a^*. \tag{3.24}$$

Let $J_{A,T}(u) = \int_0^T u'^2 + A\,u^2 - 2G(u_+)\,dx$.
<u>*Claim*</u> We have

$$J_T(u_T) \le J_{A,T}(z_T),$$

where z_T is a solution to

$$\begin{cases} z'' = A\,z - g(z) \\ z'(0) = 0, \quad z(T) = 0, \quad z > 0 \quad \text{in } [0,T). \end{cases} \tag{3.25}$$

Proof of Claim: Let $\alpha > 0$ be such that $A\,\alpha^2 - 2G(\alpha) < 0$. Consider the function

$$\overline{u}(x) = \begin{cases} \alpha, & 0 \le x \le L \\ \alpha(L+1-x), & L \le x \le L+1 \\ 0, & x \ge L+1. \end{cases} \tag{3.26}$$

It is easy to see that for L large enough (and consequently, we take T large also) we have $J_{A,T}(\overline{u}) < 0$. It is obvious that for all $u \in H_T^*$ we have

$$J_T(u) \le J_{A,T}(u),$$

so $J_T(\overline{u})$ is also negative. Defining $\Gamma_T = \{\gamma(\tau)\colon [0,1] \to H_T^*\colon \gamma(0) = 0, \gamma(1) = \overline{u}\}$, we may assume that

$$J_T(u_T) = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right) \quad \text{and} \quad J_{A,T}(z_T) = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_{A,T}\left(\gamma(\tau)\right).$$

For a given $\gamma \in \Gamma_T$, we obviously have

$$\max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right) \le \max_{\tau\in[0,1]} J_{A,T}\left(\gamma(\tau)\right)$$

and taking the infimum of both sides of the inequality, the claim follows.
By arguments already used in the proof, we easily see that this solution z_T is given by

$$z_T'^2 = A\,z_T^2 - 2G(z_T) + d_T^2$$

where $d_T = z_T'(T) \to 0$ as $T \to \infty$. Therefore $z_T(0) \to \bar{\xi}$ as $T \to \infty$, where $\bar{\xi}$ is the smallest positive root of $Au^2 - 2G(u)$. We conclude that

$$\lim_{T\to\infty} J_T(u_T) \le J_A^*,$$

contradicting (3.24) and (A_2). $\square$

Theorem 3.1.25. *Let a and g satisfy $(A_1') - (A_2) - (H_3) - (H_4) - (H_5) - (H_6)$. Then the problem* (3.16) *has at least one positive solution.*

Proof. Applying the classical diagonal method, we know that there exist a sequence of T's and $u \in C^2[0, +\infty)$ such that $u_T \to u$ C^1-uniformly in compact intervals. Applying the arguments of the previous proposition, if there exists a maximizer $x_T > x_0$ of u_T, then these maximizers must be bounded from above and we must have $u_T(x_T) \to \xi$. It follows that $u \neq 0$ and consequently u must be a branch of the well known homoclinic solution u_a of the autonomous problem for $x \geq x_0$. Since $[0, x_0]$ is a compact interval, we conclude that u must be a solution of (3.16). □

3.1.6 Autonomous problem with a dissipative term

In this subsection we prove the existence of a positive nonincreasing solution of the autonomous problem

$$\begin{cases} u'' + c\,u' = h(u) \\ u'(0) = 0, \quad u(+\infty) = 0, \end{cases} \tag{3.27}$$

where c is a positive constant, $h(u)$ is a continuous function such that $h(0) = h(b) = 0$ for some $b > 0$ and $h(u) > 0$ for $u \in (0, b)$. We consider in addition that $\liminf_{u \to +\infty} \frac{h(u)}{u} = -\infty$. We follow a similar approach of the one in [8] (p.133) to reduce the order of this problem.

Remark 3.1.26. The function $h(u) = u - u^p$, where $p > 1$, satisfies the conditions above.

Lemma 3.1.27. *The derivative of a nonincreasing positive solution u of* (3.27) *does not vanish on* $(0, +\infty)$.

Proof. Suppose towards a contradiction that there exists $x_1 > 0$ such that $u'(x_1) = 0$. Since $u' \leq 0$, we must have $u''(x_1) = 0$, which implies that $u(x_1) = b$. By the uniqueness of the initial value problem we would have $u(x) \equiv b$, which contradicts the condition $u(+\infty) = 0$. □

Let $U(x)$ be a nonincreasing solution of the differential equation in (3.27) defined in the maximal interval $[0, x_+)$ where $U > 0$. Since $U'(x) < 0$ for $x \in (0, x_+)$ we can consider the inverse function $x(u)$ of $U(x)$ and define $\varphi(u) = U'(x(u))$. We have $\varphi'\varphi + c\varphi = f(u)$, and setting $\psi(u) = \varphi(u)^2$ (noting that $\varphi(u) = -\sqrt{\psi(u)}$), we have

$$\psi' = 2c\sqrt{\psi} + 2h(u), \quad \psi(0) = 0. \tag{3.28}$$

Let M be the maximum of $h(u)$ for $u \in (0, b)$ and consider the initial value problem

$$\hat{\psi}' = 2c\sqrt{\hat{\psi}} + 2M, \quad \hat{\psi}(0) = 0. \tag{3.29}$$

The solution of this problem is given implicitly by the expression

$$\frac{\sqrt{\hat{\psi}}}{c} - \frac{M}{c^2} \ln \left| c\sqrt{\hat{\psi}} + M \right| = u - \frac{M}{c^2} \ln(M).$$

By a well known comparison theorem, we have $\psi < \hat{\psi}$ and consequently, $\psi(u) \leq ku^2$ for some positive constant k. Hence $\psi'(u) \leq \tilde{k}u + 2h(u)$ for some constant $\tilde{k}$, from which we infer that $\lim_{u\to+\infty} \psi'(u) = -\infty$. We can now conclude that ψ vanishes at some positive value u^*.

Since there exists a solution ψ of (3.28) that vanishes at some positive value u^*, following the argument used in [8], we can conclude that $u(x)$ defined by

$$\begin{cases} u' = -\sqrt{\psi} \\ u(0) = u^* \end{cases} \tag{3.30}$$

is a solution of the differential equation in (3.27) in the interval $[0, x_+)$, where $x_+ = \int_0^{u^*} \frac{du}{\sqrt{\psi}}$. An easy computation gives $x_+ = +\infty$ and consequently we have proved the following

Theorem 3.1.28. *The autonomous boundary value problem* (3.27) *has a positive decreasing solution.*

3.1.7 Non-autonomous problem with a dissipative term

In this subsection we focus on finding a positive solution of the problem

$$\begin{cases} u'' + c\,u' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(+\infty) = 0, \end{cases} \tag{3.31}$$

where $a(x) > \delta > 0$ for all $x \geq 0$ and $g(u)$ satisfies the assumptions (H_1) and (H_2) mentioned above.

A simple example of functions satisfying these assumptions are the powers $g(u) = u^p$ where $p > 1$.

As in subsection 3.1.3, we will find a solution of (3.31) as the limit of positive solutions of the boundary value problems

$$\begin{cases} u'' + c\,u' = a(x)\,u - g(u) \\ u'(0) = 0, \quad u(T) = 0, \end{cases} \tag{3.32}$$

by taking a convenient sequence of T's tending to $+\infty$. Let us consider the underlying Euler-Lagrange functional

$$J_T(u) = \int_0^T e^{c\,x} \left(u'^2 + a(x)u^2 - 2G(u_+)\right)\, dx,$$

defined in the functional space $H_T^c \equiv \left\{u \in H^1(0,T)\colon\ \int_0^T e^{c\,x}u'^2\, dx < +\infty,\ u(T) = 0\right\}$, with the norm $\|u\| = \left(\int_0^T e^{c\,x}u'^2\, dx\right)^{1/2}$. We have $J_T(0) = 0$, and, for $\epsilon > 0$ small enough, if $\|u\| = \epsilon$, then $J_T(u) > \delta(\epsilon) > 0$. The Palais-Smale condition is satisfied and, setting $u_\lambda = \lambda(1 - x^2)$, it is easy to see that $J_{a,T}(u_\lambda^+) < 0$ for $\lambda > 0$ large enough (independent on $T > 1$). The Mountain-Pass Theorem allows us to conclude that the boundary value problems (3.32) have a positive solution. Let c_T be the mountain-pass critical value of

J_T, that is, $c_T = J_T(u_T)$. Defining $\Gamma_T = \{\gamma\colon [0,1] \to H_T^*\colon \gamma(0) = 0, \gamma(1) = u_\lambda^+\}$, we know that

$$c_T = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right).$$

Since $\Gamma_{T_1} \subseteq \Gamma_{T_2}$ for $T_1 < T_2$, we have $c_T \le c_1$ for $T \ge 1$.

Multiplying the differential equation by $e^{c\,x}$ and then by u and integrating, we get

$$-\int_0^T e^{c\,x} u_T'^2\,dx = \int_0^T e^{c\,x}\left(a(x)u_T^2 - u_T\,g(u_T)\right)\,dx,$$

and consequently, using (H_1), we have

$$J_T(u_T) \ge \left(1 - \frac{2}{q}\right)\int_0^T e^{c\,x}\left(u_T'^2 + a(x)\,u_T^2\right)\,dx = \left(1 - \frac{2}{q}\right)\int_0^T e^{c\,x} g(u_T)u_T\,dx. \qquad (3.33)$$

Extending u_T to $[0,+\infty)$ by $u_T(x) = 0$ for $x \ge T$, and considering the functional space

$$H_c(0,+\infty) \equiv \left\{u \in H^1_{loc}[0,+\infty)\colon\ \int_0^{+\infty} e^{c\,x}u'^2\,dx < +\infty,\ u(+\infty) = 0\right\}$$

with the norm $\|u\| = \left(\int_0^{+\infty} e^{c\,x}u'^2\,dx\right)^{1/2}$, the following result holds:

Proposition 3.1.29. *We have uniform estimates for the $H_c(0,+\infty)$ norms of the solutions u_T (for $T \ge 1$).*

Proof. Since $J_T(u_T) \le c_1$ for all $T > 1$, (3.33) allows us to conclude the result. □

The next lemma plays an important role in what follows.

Lemma 3.1.30. *[1] For $u \in H_c(0,+\infty)$ we have*

$$\|u\|_{L^\infty(s,+\infty)} \le \frac{e^{-\frac{c\,s}{2}}}{\sqrt{c}}\,\|u\|.$$

Proof. By Schwarz inequality we have

$$|u(T) - u(s)| = \int_s^T e^{-ct/2}e^{ct/2}u'(t)\,dt \le \left(\frac{e^{-cs} - e^{-cT}}{c}\int_s^T e^{ct}u'(t)^2\,dt\right)^{\frac{1}{2}}.$$

Taking $T \to +\infty$ in both sides of the inequality, the conclusion follows. □

Corollary 3.1.31. *There exists $k > 0$ such that, for all $T > 1$,*

$$|u_T|,\ |u_T'|,\ |u_T''| \le k \quad \forall x \in [0,T].$$

Proof. The previous result implies the uniform estimate for u_T. Setting $v = u_T'$, from the differential equation it follows that $|v' + cv|$ is uniformly bounded by some constant K: since $v(0) = 0$, this implies $|v| \le K/c$. Again using the differential equation, we conclude the uniform boundedness of u_T''. □

As a consequence, using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^1-uniformly in compact intervals and $u'_T \rightharpoonup u'$ weakly in $L^2(0,+\infty)$. With this it is easy to prove the following

Lemma 3.1.32. *Given an arbitrary positive constant ϵ, there exists x_ϵ such that for all $T \geq 1$ and all $x > x_\epsilon$ we have $u_T(x) \leq \epsilon$.*

Proof. By the previous lemma, for x_ϵ large enough we have

$$|u_T(x)| \leq \frac{e^{-\frac{c\,x_\epsilon}{2}} c_1}{\sqrt{c}} \leq \epsilon \quad \forall x > x_\epsilon,\ T \geq 1.$$

□

In order to show that the limit function u is not the trivial solution, we need the following

Proposition 3.1.33. *There exists a constant $c_0 > 0$ such that $u_T(0) > c_0$ for all $T \geq 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of T's tending to ∞ such that $u_T(0) \to 0$. We have $a(x) \geq \delta > 0$ and for $u_T(0)$ small enough we have $u''_T(0) > 0$, so the solutions u_T must have a local maximizer $l_T \to +\infty$. Using the differential equation it is easy to see that $\frac{g(u_T(l_T))}{u_T(l_T)} \geq a(l_T) > \delta$, and since for u close enough to 0 we have $\frac{g(u)}{u} < \delta$, we can conclude that $u_T(l_T)$ is bounded from below by a positive constant k_δ. Taking $\epsilon < k_\delta$, we have a contradiction with the fact that $u_T(l_T) < \epsilon$ when $l_T > x_\epsilon$. □

We are now able to prove the main result:

Theorem 3.1.34. *The boundary value problem* (3.31) *has a positive solution.*

Proof. Of course the limit function u is a nonnegative solution of the given equation. Now we only need to apply Lemma 3.1.32 and Proposition 3.1.33, so as to argue, respectively, that $u(+\infty) = 0$ and $u \not\equiv 0$. □

Remark 3.1.35. If instead of a positive constant c we take a continuous function $c(x)$ with $0 < c_1 \leq c(x) \leq c_2$, the arguments used above are still valid for the differential equation $u'' + c(x)u' = a(x)u - g(u)$.

3.1.8 Heteroclinics for some equations involving the p-Laplacian

Consider the p-Laplacian partial differential equation

$$u_t = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \tag{3.34}$$

where $p > 1$ and $g(u)$ is a type A function in $[0,1]$, that is, continuous, $g(0) = g(1) = 0$ and g is positive in $(0,1)$. A positive travelling wave solution of (3.34) is a positive solution of the form $u(x,t) = u(\xi)$ where $\xi = x - ct$ for some $c > 0$ (this value c is the propagation speed of the wave). We will focus on travelling wave solutions such that u is defined in $\mathbb{R}$,

$u(-\infty) = 1$ and $u(+\infty) = 0$. These travelling wave solutions will be heteroclinic solutions of the ordinary differential equation

$$\left(|u'|^{p-2} u'\right)' + cu' + g(u) = 0, \tag{3.35}$$

connecting the equilibria $u = 1$ and $u = 0$.

Our objective is to prove the existence of an heteroclinic of (3.35). We will take an approach somehow close to the one used in [39] and for simplicity, we will call the variable x instead of ξ.

We shall obtain existence results for $1 < p < 2$. Recently Pang et al. [42] have studied related problems for $p > 2$.

Lemma 3.1.36. *The derivative of a nonincreasing solution u of* (3.35) *with* $0 < u(x) < 1$ *does not vanish. We also have* $u'(\pm\infty) = 0$.

Proof. If there exists x_0 such that $u'(x_0) = 0$ and $0 < u(x_0) < 1$, using the differential equation we would have $\left(|u'|^{p-2} u'\right)' |_{(x=x_0)} < 0$ and $|u'(x_0)|^{p-2} u'(x_0) = 0$, which contradicts the fact that $|u'|^{p-2} u' \leq 0$ for all $x \in \mathbb{R}$.

Concerning the limit of the derivative we will only prove for $+\infty$, being the $-\infty$ case similar. Suppose towards a contradiction that $\liminf_{x\to+\infty} u'(x) = -\delta < 0$. We can take two sequences t_n and s_n tending to $+\infty$ such that $u'(t_n) \to 0$ and $u'(s_n) \to -\delta$. Integrating the differential equation in $[0, t_n]$, we easily conclude that the sequence $\int_0^{t_n} g(u(x))\, dx$ is bounded and therefore $\int_0^{+\infty} g(u(x))\, dx$ is convergent. Consequently we have

$$\begin{aligned}
&\int_{t_n}^{s_n} \left(|u'|^{p-2} u'\right)' + cu' + g(u)\, dx = \\
&= \left|u'(s_n)\right|^{p-2} u'(s_n) - \left|u'(t_n)\right|^{p-2} u'(t_n) + c\left(u(s_n) - u(t_n)\right) + \int_{t_n}^{s_n} g(u)\, dx = 0
\end{aligned}$$

and making $n \to \infty$ we would get the contradiction $-\delta^{p-1} = 0$. $\square$

Let $U(x)$ be a nonincreasing solution of the differential equation in (3.35) defined in the maximal interval $]x_-, x_+[$ where $0 < U(x) < 1$. Since $U'(x) < 0$ for $x \in (x_-, x_+)$ we can consider the inverse function $x(u)$ of $U(x)$ and define $\varphi(u) = U'(x(u))$. We have

$$\frac{d}{du}\left(|\varphi(u)|^{p-2} \varphi(u)\right) \frac{du}{dx} + c\varphi(u) + g(u) = (p-1)\, |\varphi|^{p-2} \varphi\varphi' + c\varphi + g(u) = 0,$$

and setting $\psi(u) = |\varphi(u)|^p$ (noting that $\varphi = -\psi^{1/p}$), we have

$$\begin{cases} \psi' = \frac{p}{p-1} c\, \psi^{\frac{1}{p}} - \frac{p}{p-1} g(u) \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{3.36}$$

that is, ψ is a positive type A solution of (3.36).

Conversely, if we have type A solution ψ of (3.36), defining $u(x)$ as the solution of the Cauchy problem

$$\begin{cases} u' = -\psi\,(u)^{\frac{1}{p}} \\ u(0) = \frac{1}{2} \end{cases} \tag{3.37}$$

then u is a solution of the differential equation in (3.35) in the interval (x_-, x_+), where $x_- = -\int_{1/2}^{1} \frac{du}{\psi^{1/p}}$ and $x_+ = \int_0^{1/2} \frac{du}{\psi^{1/p}}$. A solution of (3.36) satisfies $\psi' \leq \frac{p}{p-1} c\, \psi^{\frac{1}{p}}$ and therefore $\psi \leq c u^{\frac{p}{p-1}}$. Since $x'(u) = -\frac{1}{\psi^{1/p}}$, we have $x(0) \geq \frac{1}{c}\int_0^{1/2} \frac{du}{u^{1/(p-1)}}$, which is a divergent integral if $p \leq 2$. If we have an extra condition $g(u) \leq M(1-u)^{p-1}$, we are able to prove in a similar way that $x_- = -\infty$.

Proposition 3.1.37. *Suppose that $s(u)$ is a C^1-function in $[0,1]$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0,1)$ and for all $u \in [0,1]$,*

$$s'(u) \leq \frac{p}{p-1} c\, s(u)^{\frac{1}{p}} - \frac{p}{p-1} g(u). \tag{3.38}$$

Then (3.36) has a unique type A solution.

Proof. We prove this result by a simple adaptation of the arguments used in [8] (p.135). Since a linear function $k\,u$, with k large enough, is an obvious upper solution, it is well known that there exists a positive solution $\psi(u)$ of the differential equation in (3.36) with $\psi(0) = 0$ such that $s(u) \leq \psi(u) \leq k\,u$. If $\psi(1) = 0$ we already have a type A solution. If $\psi(1) > 0$, we consider the solution $\bar{\psi}$ of the initial value problem

$$\bar{\psi}' = \frac{p}{p-1} c\, |\bar{\psi}|^{\frac{1}{p}} - \frac{p}{p-1} g(u), \quad \bar{\psi}(1) = 0.$$

We may assume that $\bar{\psi} \geq 0$ in $[0,1]$ since $u = 0$ is an obvious lower solution. Let us prove that $0 < \bar{\psi}(u) < \psi(u)$ for all $u \in (0,1)$. If u_0 is the largest zero of $\bar{\psi}$ in $(0,1)$, then the differential equation implies $\bar{\psi}'(u_0) < 0$, which is a contradiction with the positivity of $\bar{\psi}$. If there exists $u_1 \in (0,1)$ such that $\bar{\psi}(u_1) = \psi(u_1)$, then by the uniqueness of solution we would have $\bar{\psi} = \psi$, which contradicts the fact that $\bar{\psi}(1) = 0$. By continuity we have $\bar{\psi}(0) = 0$, so it is a type A solution. Concerning the uniqueness, if we assume that there exist two distinct solutions ψ_1, ψ_2 of type A. We know that these solutions must be ordered for $u \in (0,1)$, so let us assume that $\psi_1 < \psi_2$. But the differential equation shows that $\psi_2 - \psi_1$ is increasing, so we cannot have $\psi_1(1) = \psi_2(1) = 0$, and therefore we have a contradiction. ☐

Proposition 3.1.38. *Assume that $g(u) \leq M u^{1/(p-1)}$ for $1 < p < 2$ and $M > 0$. Then there exists a constant $c^* > 0$ (depending on M) such that (3.36) admits a unique positive solution if and only if $c \geq c^*$.*

Proof. Given $M > 0$ it is obvious that for c large enough, the inequality $\beta - c\,\beta^{1/p} + M < 0$ has positive solutions β. For one such solution, let $s(u) = \beta u^{p/(p-1)}$. Then, for all $u \in [0,1]$, we have

$$s'(u) = \frac{p}{p-1} \beta\, u^{1/(p-1)} \leq \left(c\,\beta^{1/p} - M\right) \frac{p}{p-1} u^{1/(p-1)} \leq c \frac{p}{p-1} s(u)^{\frac{1}{p}} - \frac{p}{p-1} g(u).$$

The previous proposition allows us to conclude that for such value c, the boundary value problem (3.36) has a unique positive solution.

Now let c^* be the infimum of the values $c > 0$ such that problem (3.36) has a unique positive solution. Let us prove that for all $c > c^*$, problem (3.36) has a solution. Given

$\hat{c} > c^*$, let us consider a value $\tilde{c}$ such that (3.36) has a positive solution $\psi_{\tilde{c}}$ and $c^* < \tilde{c} < \hat{c}$. For all $u \in [0,1]$ we have

$$\psi'_{\tilde{c}} = \tilde{c}\frac{p}{p-1}\psi_{\tilde{c}}^{\frac{1}{p}} - \frac{p}{p-1}g(u) \leq \hat{c}\frac{p}{p-1}\psi_{\tilde{c}}^{\frac{1}{p}} - \frac{p}{p-1}g(u),$$

so $\psi_{\tilde{c}}$ is a lower solution for the problem with $c = \hat{c}$ and by the previous proposition, we conclude the solvability for (3.36) with $c = \hat{c}$.

To prove the solvability for $c = c^*$, consider a decreasing sequence c_n tending to c^* and the correspondent positive solutions ψ_n. For all $u \in [0,1]$ we have

$$\psi'_{n+1} = c_{n+1}\frac{p}{p-1}\psi_{n+1}{}^{\frac{1}{p}} - \frac{p}{p-1}g(u) \leq c_n\frac{p}{p-1}\psi_{n+1}{}^{\frac{1}{p}} - \frac{p}{p-1}g(u),$$

so again by the previous proposition we conclude that $\psi_{n+1} \leq \psi_n$, for all $n \in \mathbb{N}$, that is, ψ_n is a nonincreasing sequence. Let us define $\psi^*(u) := \inf_{n\in\mathbb{N}} \psi_n(u)$. By Lebesgue's dominated convergence Theorem, we can conclude that ψ^* is a solution of the differential equation in (3.36) for $c = c^*$ and satisfies $\psi^*(0) = 0$. Applying the same argument used in the proof of the previous proposition, we can conclude that there exists a solution $\hat{\psi}^*$ also satisfying the boundary condition $\hat{\psi}^*(1) = 0$, and, consequently, a positive solution of (3.36).

Let us now prove that $c^* \neq 0$. Consider the initial value problem

$$\psi' = \frac{p}{p-1}c\,\psi^{\frac{1}{p}} - \frac{p}{p-1}g(u), \quad \psi(1) = 0. \tag{3.39}$$

For $c = 0$ we have $\psi(u) = \int_u^1 \frac{p}{p-1}g(u)\,du$ and by the continuous dependence of the initial data, in a compact interval $[\delta, 1]$, for c close enough to 0 we have $\|\psi_c - \psi_0\|_\infty \leq \epsilon(\delta)$, where $\epsilon(\delta)$ tends to zero with δ. If there was a type A solution for such small values of c, then the derivative of the corresponding ψ_c, by the mean value theorem, would have to take values close to $\frac{\psi_0(0)}{\delta}$, which is as large as we want. A simple analysis of the differential equation rules out that possibility, and therefore there are no type A solutions of (3.39) for c small enough. □

We are now able to state the main result of this subsection.

Theorem 3.1.39. *If $c \geq c^*$, $1 < p < 2$ and $g(u) \leq Mu^{1/(p-1)}$, $g(u) \leq M(1-u)^{p-1}$ for some constant $M > 0$, then (3.35) has an heteroclinic decreasing solution such that $u(-\infty) = 1$, $u(+\infty) = 0$ and $0 < u(x) < 1$ for all $x \in \mathbb{R}$.*

Remark 3.1.40. The case $p = 2$ is the regular Laplacian which is already well studied in several papers by Malaguti and Marcelli [39],[40], and the for case $p > 2$ we cannot be sure anymore that $x(0) = +\infty$ and therefore it would be a “degenerated” heteroclinic.

3.2 Fourth order problems

3.2.1 Introduction

The phase plane plays for the second order autonomous problems a very important role to search possible homoclinic solutions, but when we deal with fourth order problems we do not have that possibility. The larger number of intermediate derivatives between u and $u^{(4)}$ is an obstacle for the generalization of such well-known results of the second order. In the following subsection we prove some results concerning the existence of a homoclinic solution for the autonomous problem, and in the final subsection we prove existence for some non-autonomous equations.

We deal with the problems

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases}$$

where we consider $a(x)$ in three different situations: the autonomous case, the case where $a(x)$ is nondecreasing and the case when $\lim_{x\to+\infty} a(x) = +\infty$.

3.2.2 Autonomous problem

In this subsection we prove the existence of a nontrivial solution of the problem

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases} \tag{3.40}$$

where a and c are positive constants and $p > 1$. The solution of (3.40) will be found as a limit of solutions to the boundary value problems

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(T) = u'(T) = 0, \end{cases} \tag{3.41}$$

by taking $T \to +\infty$.

Proposition 3.2.1. *The boundary value problem* (3.41) *has a nontrivial solution.*

Proof. Consider the functional

$$J_T(u) = \int_0^T \left(u''^2 + c\,u'^2 + a\,u^2\right)\,dx$$

defined in $H^2(0,T)$ and let us minimize it in the manifold

$$M_T = \left\{u \in H^2(0,T)\colon u'(0) = u(T) = u'(T) = 0,\ \int_0^T |u|^{p+1}\,dx = 1\right\}.$$

Since the interval is bounded, M_T is weakly closed in $H^2[0,T]$, so that there exists such a minimum u_T, and by the Lagrange multipliers theory, there exist $\lambda_T \in \mathbb{R}$ and $u_T \in M_T$ such that ${u_T}'''(0) = 0$ and

$$\int_0^T \left({u_T}''v'' + c\,{u_T}'v' + a\,u_T\,v\right)\,dx = \lambda_T \int_0^T |u_T|^{p-1}\,u_T\,v, \quad \forall v \in M_T,$$

that is, as it is well known from standard arguments, u_T is a classical solution of

$$u^{(4)} - c\,u'' + a\,u = \lambda_T\,|u|^{p-1}\,u, \quad u'(0) = u'''(0) = 0, \quad u(T) = u'(T) = 0.$$

By taking $v = u_T$, we get

$$\int_0^T \left({u_T}''^2 + c\,{u_T}'^2 + a\,{u_T}^2\right)\,dx = \lambda_T \int_0^T |u_T|^{p+1}\,dx = \lambda_T,$$

and since $\int_0^T |u_T|^{p+1}\,dx = 1$, u_T cannot be the trivial solution. □

Remark 3.2.2. If we take a sequence of values T tending to $+\infty$, the corresponding sequence λ_T is decreasing (if $T_1 < T_2$, then $M_{T_1} \subseteq M_{T_2}$). Considering the obvious extensions of the functions $u \in H^2(0,T)$, it is obvious that $J_T(u)$ is an equivalent norm of $H^2(0,+\infty)$, therefore u_T is a bounded sequence in $H^2(0,+\infty)$ and, consequently, there exists a constant $k > 0$ such that $\|u_T\|_\infty \le k$. We have

$$1 = \int_0^T |u_T|^{p+1}\,dx \le k^{p-1} \int_0^T {u_T}^2\,dx \le K J_T(u_T) \text{ with } K = k^{p-1}\frac{\lambda_T}{a} > 0,$$

so the sequence λ_T tends to a strictly positive value.

Proposition 3.2.3. *There exists a constant $c > 0$ such that $u_T(0) > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to $+\infty$ such that $u_T(0) \to 0$.

Consider the differential equation with $u = u_T$ and multiply it by ${u_T}'$. By simply integrating, we get

$$u_T{}'''u_T{}' - \frac{{u_T}''^2}{2} - c\,\frac{{u_T}'^2}{2} + a\frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1} = c_T, \tag{3.42}$$

for some constant c_T. If we integrate this expression in $[0,T]$, we get

$$\int_0^T -\frac{3}{2}{u_T''}^2 - c\,\frac{{u_T}'^2}{2} + a\frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1}\,dx = c_T\,T.$$

Since

$$\int_0^T \left| -\frac{3}{2}{u_T''}^2 - c\,\frac{{u_T}'^2}{2} + a\frac{{u_T}^2}{2} - \frac{|u_T|^{p+1}}{p+1} \right| dx$$

is bounded by $\frac{3}{2}\lambda_1 + 1$ and λ_T is decreasing in T, we conclude that c_T must tend to 0 as T tends to $+\infty$. Considering the initial sequence of T's, (3.42) implies that $u_T''(0) \to 0$ (we already know that $u_T'(0) = u_T'''(0) = 0$ and assumed that $u_T(0) \to 0$). As a consequence, by the continuous dependence on initial data, the solutions u_T have their last maximizer m_T tending to $+\infty$ (we may assume it is a maximizer since $-u_T$ is also a solution with the same properties). We have

$$1 = \int_0^T |u_T|^{p+1}\,dx \le \|u_T\|_\infty^{p-1} \int_0^T {u_T}^2\,dx \le \frac{\lambda_T}{a}\,\|u_T\|_\infty^{p-1},$$

so we can conclude that $\|u_T\|_\infty \ge \left(\frac{a}{\lambda_T}\right)^{\frac{1}{p-1}}$. We already know that $\|u_T\|_\infty$ is bounded independently of T.

<u>*Claim*</u> All the derivatives u_T', u_T'', u_T''' and $u_T^{(4)}$ are bounded, independently of T.
Proof of Claim: Setting $w = u_T''$, we have, taking (3.42) into account

$$w'' - cw \quad \text{is bounded}, \quad w'(0) = 0 \quad \text{and} \quad w(T) \quad \text{is bounded}.$$

Therefore u_T'' is also bounded independently of T, and considering the differential equation, we have that ${u_T}^{(4)}$ is also bounded. As a consequence, all the intermediate derivatives are bounded too, independently of T.

Let us now consider two auxiliar functions v_T and w_T defined in the following way:

$$v_T(x) = \begin{cases} u_T(x + m_T) & x \in [0, T - m_T] \\ 0, & x \in [T - m_T, T], \end{cases} \quad , \qquad w_T(x) = \begin{cases} u_T(m_T - x), & x \in [0, m_T] \\ \rho(x), & x \in [m_T, m_T + \eta] \\ 0 & x \in [m_T + \eta, T], \end{cases}$$

where $\eta > 0$ and $\rho(x) = \frac{u_T(0)}{2}(\cos(\frac{\pi}{\eta}(x - m_T)) + 1)$. Since $u_T(m_T) \geq \left(\frac{a}{2}\right)^{\frac{1}{p-1}}$ and ${u_T}'$ is bounded, we can take a constant η small enough such that $m_T + \eta < T$ for T large. Let $\alpha_T = \int_0^T |v_T|^{p+1}\, dx$ and $\beta_T = \int_0^T |w_T|^{p+1}\, dx$. The uniform boundedness of ${u_T}'$ implies that each of these integrals cannot be arbitrarily small. We have $\alpha_T + \beta_T = 1 + \delta(T)$, where $\delta(T) = \frac{u_T(0)}{2}^{p+1} \int_0^\eta \left(cos(\frac{\pi\, x}{\eta}) + 1\right)^{p+1} dx$. If $T \to +\infty$ then $\delta \to 0$. For all $z \in H^2(0,T)$ such that $z'(0) = z(T) = z'(T) = 0$ we have

$$J_T(z) \geq \lambda_T \, \|z\|^2_{L^{p+1}(0,T)} \,,$$

since $\frac{z}{\|z\|_{p+1}} \in M_T$ for $z \not\equiv 0$. Furthermore

$$J_T(u_T) = J_T(v_T) + J_T(w_T) - \delta_1(T),$$

where $\delta_1(T) \to 0$: indeed, for a suitable constant $C > 0$, we get $\delta_1(T) \leq C{R_T}^2$, where R_T stands for the norm of ρ in $L^2[0,\eta]$. It is now possible to conclude that

$$J_T(u_T) \geq \lambda_T \left(\|v_T\|^2_{L^{p+1}(0,T)} + \|w_T\|^2_{L^{p+1}(0,T)}\right) - \delta_1(T) = \lambda_T \left({\alpha_T}^{\frac{2}{p+1}} + {\beta_T}^{\frac{2}{p+1}}\right) - \delta_1(T).$$

The fact that α_T and β_T do not tend to 0 and

$$\alpha_T + \beta_T = 1 + \delta(T)$$

with $\delta(T)$ small enough, implies that

$$\left({\alpha_T}^{\frac{2}{p+1}} + {\beta_T}^{\frac{2}{p+1}}\right) > K > 1,$$

where K is independent of T. Indeed, let $\gamma \in (0,1)$ be a lower bound for α_T and β_T and recall that $\epsilon := \frac{2}{p+1} \in (0,1)$: then the ratio between $\alpha^\epsilon + \beta^\epsilon$ and $\alpha + \beta$ attains a minimum value $K > 1$ on the pairs $(\alpha, \beta) \in [\gamma, 1]^2$ such that $\alpha + \beta \leq \frac{3}{2}$, and the estimate above holds, provided that $\delta_T \leq \frac{1}{2}$. It follows that for T large

$$J_T(u_T) > \lambda_T,$$

which is a contradiction. □

Using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^3-uniformly in compact intervals and $u(x)$ is a solution of

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = \lambda\,|u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0, \end{cases} \tag{3.43}$$

where $\lambda = \lim_{T\to+\infty} \lambda_T$. Since $u_T(0) > c > 0$ for all $T > 1$, we have $u(x) \not\equiv 0$, so that $u^* := \lambda^{1/(p-1)}u$ is a nontrivial solution of the given equation, and we can conclude the main result of this subsection:

Theorem 3.2.4. *There exists a nontrivial solution of* (3.40).

Corollary 3.2.5. *The equation in* (3.40) *has a nontrivial homoclinic at* $u = 0$.

Proof. The function $u^*(x) = \lambda^{\frac{1}{p-1}} u(x)$ solves the half-line problem (3.40). Since $u^*(-x)$ is also a solution of the differential equation and $u'(0) = u'''(0) = 0$, the conclusion follows. □

If we consider the manifold

$$M_T^+ = \left\{ u \in H^2(0,T)\colon u'(0) = u(T) = u'(T) = 0, \int_0^T {u_+}^{p+1}\,dx = 1 \right\},$$

where $u_+ = \max(0,u)$, the arguments used above will still provide a solution of

$$\begin{cases} u^{(4)} - c\,u'' + a\,u = {u_+}^p \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0. \end{cases} \tag{3.44}$$

The following lemma allows us to prove that for c large enough, this solution is positive.

Lemma 3.2.6. *Let $y \in C^2(0,+\infty)$ be a bounded function such that $y'(0) = 0$ and $\mu > 0$ a constant. Then, if $y'' - \mu\, y = h(x) \geq 0$, we have $y \leq 0$.*

Proof. Assume towards a contradiction that $y \not\leq 0$. If $y(0) > 0$, then $y''(0) > 0$ and since $y'(0) = 0$, we must have $y(x) > y(0)$ for $x > 0$ close to 0. It is then obvious that y is a convex function and stays above a line of positive slope. This is a contradiction because y is bounded. If $y(0) = 0$ we obviously get $y \equiv h \equiv 0$. If $y(0) < 0$ and there exists a value $x_0 > 0$ such that $y(x_0) = 0$, then we could apply the same argument as above and reach a contradiction. □

Theorem 3.2.7. *If $c^2 \geq 4\,a$, then the boundary value problem* (3.40) *has a positive solution.*

Proof. Consider the solution u of (3.44). Let μ_1 and μ_2 be the solutions of $x^2 - c\,x + a = 0$. Since $c^2 > 4\,a$, $c > 0$ and $a > 0$, these values are positive and we can write the differential equation in the form

$$\big(D^2 - \mu_1\big)\,\big(D^2 - \mu_2\big)\,u = u_+(x)^p = h(x) \geq 0.$$

Setting $y(x) = \big(D^2 - \mu_2\big)\,u$, we have $y'(0) = 0$ and $y'' = \mu_1 y + h(x)$. Since u is a solution of (3.44), we know (by the arguments of the *Claim* in the proof of Proposition 3.2.3) that u and u'' are bounded and, therefore, y is bounded. Applying the previous lemma we have $y \leq 0$. Applying the same lemma to $-u$ we conclude that $u \geq 0$ and, therefore, is also a solution of (3.40). □

3.2.3 Non-autonomous problems

Consider the boundary value problem

$$\begin{cases} u^{(4)} - c\,u'' + a(x)\,u = |u|^{p-1}\,u \\ u'(0) = u'''(0) = 0, \quad u(+\infty) = u'(+\infty) = 0 \end{cases} \tag{3.45}$$

where $a(x)$ is a nondecreasing function with $\lim_{x\to+\infty} a(x) = a \in \mathbb{R}^+$, c is a positive constant and $p > 1$. We will follow the approach of the previous subsection, so let u_T be defined as above (with $a(x)$ substituted for a).

Proposition 3.2.8. *There exists a constant $c > 0$ such that $|u_T(0)| + |u_T''(0)| > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to $+\infty$ such that $|u_T(0)| + |u_T''(0)| \to 0$. Let $v_T(x)$ and $w_T(x)$ be defined in the proof of Proposition 3.2.3. We have

$$\begin{aligned} J_T(v_T) &= \int_{m_T}^{T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(x - m_T)u_T(x)^2\right]\,dx \le \\ &\le \int_{m_T}^{T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(x)u_T(x)^2\right]\,dx, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} J_T(w_T) &= \int_{0}^{m_T} \left[{u_T}''(x)^2 + c\,{u_T}'(x)^2 + a(m_T - x)u_T(x)^2\right]\,dx + \\ &+ \int_{m_T}^{m_T+\eta} \left[\rho''(x)^2 + c\rho'(x)^2 + a(x)\rho(x)^2\right]\,dx. \end{aligned} \tag{3.47}$$

Given $\epsilon > 0$, there exists $x_0(\epsilon)$ such that $a - a(x) < \epsilon$ if $x \ge x_0$.

By continuous dependence on parameters, for T large enough, we have $|u_T(x)| < \delta$ for all $x \in [0, x_0]$, therefore

$$\int_0^{x_0} a(m_T - x)u_T(x)^2\,dx \le a\,x_0\,\delta^2. \tag{3.48}$$

We can assume that $m_T \ge 2x_0$, so that both inequalities $x \ge x_0$ and $m_T - x \ge x_0$ hold for $x_0 \le x \le m_T/2$. By the uniform boundedness in T of the $L^2[0,T]$ norms (let K be such bound), we can conclude that

$$\int_{x_0}^{\frac{m_T}{2}} \left(a(m_T - x) - a(x)\right) u_T(x)^2\,dx \le \epsilon \int_{x_0}^{\frac{m_T}{2}} u_T(x)^2\,dx \le K\epsilon. \tag{3.49}$$

Since $a(m_T - x) \le a(x)$ for $x \in [\frac{m_T}{2}, m_T]$ we have

$$\int_{\frac{m_T}{2}}^{m_T} a(m_T - x)u_T(x)^2\,dx \le \int_{\frac{m_T}{2}}^{m_T} a(x)u_T(x)^2\,dx$$

and therefore

$$\int_0^{m_T} a(m_T - x)u_T(x)^2\,dx \le a\,x_0\,\delta^2 + \int_{x_0}^{m_T} a(x)u_T(x)^2\,dx + K\epsilon. \tag{3.50}$$

We can now make the following estimates:

$$\begin{aligned}
J_T(u_T) &\ge \int_0^{m_T} \left[{u_T''}^2 + c\,{u_T'}^2\right] dx + \int_0^{x_0} a(x){u_T}^2\,dx + \int_{x_0}^{m_T} a(x){u_T}^2\,dx + J_T(v_T) \ge \\
&\ge \int_0^{m_T} \left[{u_T''}^2 + c\,{u_T'}^2\right] dx + \int_0^{x_0} a(x){u_T}^2\,dx + \\
&\quad + \int_0^{m_T} a(m_T - x){u_T}^2\,dx - a\,x_0\,\delta^2 - K\epsilon + J_T(v_T) \ge \\
&\ge J_T(v_T) + J_T(w_T) - \int_{m_T}^{m_T+\eta} \left[\rho''^2 + c\rho'^2 + a(x)\rho^2\right] dx - a\,x_0\,\delta^2 - K\epsilon
\end{aligned}$$

(we have used (3.46) in the first inequality, (3.50) in the second one and (3.47) in the third one). The terms of negative sign can be taken arbitrarily small, so we can repeat the arguments from the previous subsection and reach a contradiction. □

Now the following result can be proved exactly as in the foregoing subsection.

Theorem 3.2.9. *Let $a(x)$ be a nondecreasing function with $\lim_{x\to+\infty} a(x) = a \in \mathbb{R}$, $c > 0$ and $p > 1$. Then problem (3.45) has a solution.*

Let us now consider the boundary value problem (3.45), but now assuming that $a(x)$ is a positive function in $\mathbb{R}^+$ such that $\lim_{x\to\infty} a(x) = +\infty$. We will prove the existence of a nontrivial solution by proving that the functional

$$J(u) = \int_0^{+\infty} \left[u''^2 + c\,u'^2 + a\,u^2\right] dx$$

defined in $H^2(0,+\infty)$ has a minimum in the manifold

$$M = \left\{ u \in H^2(0,+\infty) \colon u'(0) = 0,\ \int_0^{+\infty} \frac{u^{p+1}}{p+1}\,dx = 1 \right\}.$$

Let m be the infimum of $J(u)$ in M ($m \ge 0$) and consider a sequence u_n, with $n \in \mathbb{N}$ such that $J(u_n) \to m$. Obviously, $J(u_n)$ is bounded and u_n is bounded in $L^\infty(0,+\infty)$ (since it is bounded in $H^2(0,+\infty)$), so we have

$$p+1 = \int_0^{+\infty} {u_n}^{p+1}\,dx \le {\|u_n\|_\infty}^{p-1} \int_0^{+\infty} {u_n}^2\,dx,$$

and hence $\int_0^{+\infty} {u_n}^2\,dx \ge c_1 := (p+1)C^{1-p}$.

On the other hand, for all positive L, there exists $x_0(L) > 0$ such that $a(x) > L$ for $x > x_0(L)$, so

$$L \int_{x_0(L)}^{+\infty} {u_n}^2\,dx \le \int_{x_0(L)}^{+\infty} a(x){u_n}^2\,dx \le c_2,$$

where c_2 is the upper bound of $J(u_n)$, and consequently $\int_{x_0(L)}^{+\infty} {u_n}^2\,dx \le \frac{c_2}{L}$. Considering L large enough, we have $\int_{x_0(L)}^{+\infty} {u_n}^2\,dx \le \frac{c_1}{2}$, and therefore $\int_0^{x_0(L)} {u_n}^2\,dx \ge \frac{c_1}{2}$, which implies that the limit u of the convergent subsequence of u_n cannot be the trivial solution.

Now it is easy to get the following result:

Theorem 3.2.10. *Let $a(x)$ be a positive function in $\mathbb{R}^+$ such that $\lim_{x\to\infty} a(x) = +\infty$ and c a positive constant. Then problem (3.45) has a solution.*

Final remarks

In the problem studied in subsection 3.1.5 we would like to consider condition (A_1) instead of (A_1'), but we were not able to prove the existence result using similar arguments to the ones of subsection 3.1.3, so we had to go for a less general scenario. Nonetheless we feel that the more general result should hold without asking more from the nonlinearity.

Maybe similar ideas can be used with Φ-Laplacian cases. If we consider the autonomous boundary value problem

$$\big(\Phi(u')\big)' = a\psi(u) - g(u) = 0, \quad u'(0) = u(+\infty) = 0, \tag{3.51}$$

where Φ and ψ are homeomorphisms with $\psi(0) = \Phi(0) = 0$, the differential equation can be written in the system form

$$\begin{cases} u' = \Phi^{-1}(z) \\ z' = a\psi(u) - g(u). \end{cases}$$

It is easy to see that $\frac{dz}{du} = \frac{a\psi(u)-g(u)}{\Phi^{-1}(z)}$ and therefore $F(z) - a\Psi(u) + G(u) = c$, where $F(z) = \int_0^z \Phi^{-1}(s)\,ds$, $\Psi(u) = \int_0^u \psi(s)\,ds$ and $G(u) = \int_0^u g(s)\,ds$. The fact that $z(0) = 0$ implies that $c = 0$, so that the solutions of (3.51) have trajectories given by $F(z) = a\Psi(u) - G(u)$. The similarities to the second order problem conservation law (3.4) are evident. If we consider the Euler-Lagrange functionals

$$J_{a,T}(u) = \int_0^T \left(\tilde{F}(u'(x)) + a\Psi(u(x)) - G(u(x))\right)\,dx, \text{ where } \tilde{F}(z) = \int_0^z \Phi(s)\,ds$$

defined in the Orlicz spaces

$$O_T = \left\{u \in H^1[0,T]\colon \int_0^T \left(\tilde{F}(u'(x)) + a\Psi(u(x))\right)\,dx < +\infty,\ u(T) = 0\right\},$$

it seems that a mountain-pass geometry can be found with the adequate restrictions on Φ, ψ and g and the Palais-Smale condition can be proved.

The results obtained in subsection 3.1.8 suggest also that we investigate whether the heteroclinic connections for $p > 2$ may exist and be found by some different technique.

Finally, the study that we made for infinite domain non-autonomous fourth order problems is only a beginning and it may also be of interest to make a deeper analysis.

Chapter 4

Fourth order boundary value problems in a bounded interval

4.1 Introduction

It is well known that fourth order boundary value problems are related to the theory of beam deflection. Recently, several authors have studied existence and multiplicity of solutions of the equation

$$u^{(4)}(x) = f\left(x, u(x), u''(x)\right), \quad x \in [0, 2\pi] \tag{4.1}$$

with different boundary conditions. We will address the periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi),$$

the "simply supported" boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0$$

and the "clamped beam" boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

We considered different lengths of the interval for convenience and for no other special reason.

The periodic problem with f not depending on u'' has been studied by Cabada [10], via maximum principles and the monotone method. Jiang, Gao and Wan [31] obtained results for the full nonlinear problem using the monotone method. Allowing a linear dependence on f on u'', Li [35] and Liu and Li [36] have obtained existence results using fixed point theory.

In the "simply supported" case, Bai and Wang [2] have obtained existence and multiplicity results without dependence on u''. With linear dependence on u'', we can find results of existence in Li [35] and existence and multiplicity in Yao [48]. Cabada, Cid and Sanchez [11] obtained results for the problem without dependence on u'', using upper and lower solutions in reversed order. The superlinear case has been studied by B.

R. Rynne [47] using a bifurcation technique. The “clamped beam” problem has deserved less attention in the literature as far as we know.

In the first two sections, we deal with fourth order boundary value problems in a way that Gao, Weng, Jiang and Hou [26] did for second order. We consider equation (4.1) with periodic as well as “simply supported” boundary conditions, and prove existence results (considering f one-sided Lipschitz in both variables u and u'') if there exist lower and upper solutions (ordered or in reversed order for the periodic case, and ordered in the “simply supported” case).

Habets and Sanchez [30] have obtained similar results to the ones we obtain here, using Lipschitz conditions. The main difference is that in our case only localization is obtainable, no iterative technique is possible.

In these cases, the decomposition of the fourth order operator in two operators of second order was the key to prove monotonicity of the associated fourth order operator. In the case of clamped beam conditions, the fourth order problem

$$\begin{cases} u^{(4)}(x) + (m+M)u''(x) + m\,M\,u(x) = f(x), \\ u(0) = u'(0) = u(1) = u'(1) = 0, \end{cases} \tag{4.2}$$

for a given positive continuous function $f(x)$, can still be divided in two second order problems, which are

$$\begin{cases} u''(x) + m\,u(x) = v(x), \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} v''(x) + M\,v(x) = f(x), \\ \int_0^1 \sinh\left(\sqrt{|m|}(1-s)\right) v(s)\,ds = 0, \\ \int_0^1 \sinh\left(\sqrt{|m|}s\right) v(s)\,ds = 0, \end{cases}$$

if $m < 0$;

$$\begin{cases} v''(x) + M\,v(x) = f(x), \\ \int_0^1 s\,v(s)\,ds = 0, \\ \int_0^1 (1-s)\,v(s)\,ds = 0, \end{cases}$$

if $m = 0$; or

$$\begin{cases} v''(x) + M\,v(x) = f(x), \\ \int_0^1 \sin\left(2\,m\,\pi\,(1-s)\right) v(s)\,ds = 0, \\ \int_0^1 \sin\left(2\,m\,\pi\,s\right) v(s)\,ds = 0, \end{cases}$$

if $m > 0$.

Now if we look at the boundary conditions of the second operator one immediately realizes that there exists no possibility for v to be positive for all $m \leq 1/4$, so the arguments used for example in [20] can not be applied.

Concerning this problem, we will focus on the differential equation $u^{(4)} = f(x,u)$. Adding $k\,u$ to both sides of the equation, we will use topological arguments for negative values of k to prove the existence of a solution, and for $k > 0$ we will prove the positivity of the associated Green’s function using a very interesting result of Schröder’s paper [49]. The procedure after knowing the values of k for which we have positivity will be similar to the one taken in [11].

4.2 Existence and localization for periodic boundary conditions

We start by proving two maximum principles, that are the key to prove the existence results we look for.

Lemma 4.2.1 (Maximum Principle 1)**.** *Let $L < 0$, $p, q, r \in \mathbb{R}$ with $p < r < q$ and $y \in C[p,q] \cap W^{2,1}(p,r) \cap W^{2,1}(r,q)$ such that*

$$y''(x) + Ly(x) = f(x) \geq 0 \text{ a.e. } x \in (p,q), \quad y(p) = y(q), \;\; y'(p) \geq y'(q), \;\; y'(r^+) \geq y'(r^-).$$

Then $y(x) \leq 0$ for all $x \in [p,q]$. Moreover, if $y''(x) + Ly(x) \not\equiv 0$, then $y(x) < 0$ for all $x \in (p,q)$.

Proof. Suppose that $y(x) > 0$ for all $x \in (p,q)$. Then we would have the contradiction

$$0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) = \int_p^r f(x) - Ly(x)\,dx + \int_r^q f(x) - Ly(x)\,dx > 0.$$

If $y(p) > 0$, then $y(q) > 0$ and therefore there exist two intervals $[p, p_1]$ and $[q_1, q]$ where $y > 0$, $y(p_1) = y(q_1) = 0$, $y'(p_1) \leq 0$ and $y'(q_1) \geq 0$. If r belongs to one of the intervals, then we would have the contradiction

$$0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) + y'(p_1) - y'(q_1) = \int_p^{p_1} f(x) - Ly(x)\,dx + \int_{q_1}^q f(x) - Ly(x)\,dx > 0,$$

otherwise, the contradiction is the same, without the terms involving r.

If $y(p) < 0$ then the exists an interval (p_1, q_1) where $y > 0$ and $y(p_1) = y(q_1) = 0$, and we can apply the arguments used in the first case. □

Lemma 4.2.2 (Maximum Principle 2)**.** *Let $0 < L < \frac{1}{4}$, $p < r < q$ with $q - p \leq 2\pi$ and $y \in C[p,q] \cap W^{2,1}(p,r) \cap W^{2,1}(r,q)$ such that*

$$y''(x) + Ly(x) = f(x) \geq 0, \quad y(p) = y(q), \;\; y'(p) \geq y'(q), \;\; y'(r^+) \geq y'(r^-).$$

Then $y(x) \geq 0$ for all $x \in [p,q]$. Moreover, if $y''(x) + Ly(x) \not\equiv 0$, then $y(x) > 0$ for all $x \in (p,q)$.

Proof. It follows easily combining the arguments used in the proof of the previous lemma and the proof of Proposition 2.3 in [30]. □

Consider the fourth order equation

$$u^{(4)}(x) = f\left(x, u(x), u''(x)\right), \quad x \in [0, 2\pi] \tag{4.3}$$

where f is a L^1-Carathéodory function, with periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi). \tag{4.4}$$

We say that $\alpha \in W^{4,1}(0, 2\pi)$ is a lower solution of the boundary value problem (4.3)–(4.4) if

$$\alpha^{(4)}(x) \leq f\left(x, \alpha(x), \alpha''(x)\right), \quad x \in [0, 2\pi]$$

$$\alpha(0)=\alpha(2\pi),\quad \alpha'(0)=\alpha'(2\pi),\quad \alpha''(0)=\alpha''(2\pi),\quad \alpha'''(0)\le\alpha'''(2\pi).$$

A function $\beta\in W^{4,1}(0,2\pi)$ satisfying the reversed inequalities is called an upper solution.

Let α, β be respectively a lower and an upper solution of (4.3)–(4.4), such that $\alpha(x)\le\beta(x)$ for all $x\in[0,2\pi]$.

In the following, let us assume the hypothesis

(H1) there exist constants $C,D>0$ with $D^2>4C$, such that

$$f(x,u_2,v_2)-f(x,u_1,v_1)\ge -C(u_2-u_1)+D(v_2-v_1) \tag{4.5}$$

for *a.e.* $x\in[0,2\pi]$, $\alpha(x)\le u_1\le u_2\le\beta(x)$, $v_1\le v_2$.

Remark 4.2.3. If $f(x,u,v)$ is a C^1 function in (u,v), the inequality in (H1) is equivalent to $\frac{\partial f}{\partial u}\ge -C$ and $\frac{\partial f}{\partial v}\ge D$.

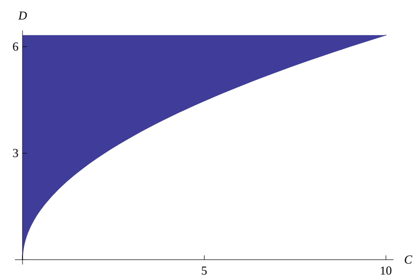


Figure 4.1: Admissible values C,D

Let $m,M<0$ be the two roots of the equation $x^2+Dx+C=0$ (note that $C=mM$, $D=-(m+M)$).

Setting $a(x)=\alpha''(x)+m\alpha(x)$ and $b(x)=\beta''(x)+m\beta(x)$, we have the following result

Proposition 4.2.4. *If f is a L^1-Carathéodory function satisfying (H1) for $\alpha(x),\beta(x)$ lower and upper solutions such that $\alpha(x)\le\beta(x)$ for all $x\in[0,2\pi]$, then $b(x)\le a(x)$.*

Proof. Setting $y(x)=b(x)-a(x)$, then $y(0)=y(2\pi)$ and $y'(0)\ge y'(2\pi)$. Suppose towards a contradiction that there exists $x_0\in[0,2\pi]$ such that $y(x_0)>0$.

If $y(x)>0$ for all $x\in[0,2\pi]$, we have (noting that $b(x)-m\beta(x)\ge a(x)-m\alpha(x)$ and $m^2+mD+C=0$)

$$\begin{aligned} y''(x)+My(x)=&b''(x)-a''(x)+Mb(x)-Ma(x)\\ =&\beta^{(4)}(x)+(m+M)\beta''(x)+(m+M)m\beta(x)-m^2\beta(x)-\\ &-(\alpha^{(4)}(x)+(m+M)\alpha''(x)+(m+M)m\alpha(x)-m^2\alpha(x))\\ \ge&f(x,\beta(x),b(x)-m\beta(x))-f(x,\alpha(x),a(x)-m\alpha(x))+\\ &+(m+M)y(x)-m^2(\beta(x)-\alpha(x))\ge 0, \end{aligned}$$

and this is a contradiction, since by the Maximum Principle 4.2.1 we would have $y(x)\le 0$.

Otherwise, considering if necessary the periodic extension of $y(x)$, there exists an interval $[p,q]$, with $q-p<2\pi$, such that $y(p)=y(q)=0$, $y'(p)>0>y'(q)$, and $y(x)>0$ for $x\in(p,q)$. Applying the same argument in $[p,q]$ as above, we reach again a contradiction.

$\square$

Let

$$p(x,z) = \begin{cases} b(x), & z < b(x) \\ z, & b(x) \le z \le a(x) \\ a(x), & z > a(x). \end{cases}$$

Consider the boundary value problem

$$u''(x) + mu(x) \equiv L_m\, u(x) = q(x), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

with $q \in L^1[0,1]$. Since $m < 0$, the operator L_m is invertible, so that we can write its unique solution u as $u = L_m^{-1} q$, and by the Maximum principle 4.2.1 we know that if $q(x) \ge 0$ then $u(x) \le 0$. Since $\alpha(x) = L_m^{-1} a(x)$, $\beta(x) = L_m^{-1} b(x)$ and $b(x) \le p(x, z(x)) \le a(x)$, for any function $z(x)$ we have

$$\alpha(x) \le L_m^{-1} p(x, z(x)) \le \beta(x).$$

Let us consider the modified problem

$$\begin{aligned} z''(x) + Mz(x) = (Fz)(x) \equiv f\left(x, L_m^{-1} p(x,z(x)), p(x,z(x)) - mL_m^{-1} p(x,z(x))\right) + \\ + (m+M)p(x,z(x)) - m^2 L_m^{-1} p(x,z(x)), \quad z(0) = z(2\pi), \quad z'(0) = z'(2\pi). \end{aligned} \tag{4.6}$$

Considering the operator $\Phi : C[0,2\pi] \to C[0,2\pi]$ with $\Phi z = L_M^{-1}(Fz)$, since $p(x,z(x))$ and $L_m^{-1} p(x,z(x))$ are bounded and f is a Carathéodory function, there exists a $L^1[0,2\pi]$ function $g(x)$ such that $|(Fz)(x)| \le g(x)$ for $a.e.\, x \in [0,2\pi]$. Therefore, applying Schauder's fixed point Theorem, we can conclude that Φ has a fixed point $z(x)$ which is a solution of the modified problem (4.6).

Proposition 4.2.5. *Let $z(x)$ be a solution of the modified problem* (4.6)*. Assuming (H1), for given lower and upper solutions $\alpha(x)$ and $\beta(x)$, with $\alpha \le \beta$ for all $x \in [0,2\pi]$, we have*

$$b(x) \le z(x) \le a(x).$$

Proof. We will only prove that $z(x) \le a(x)$, since the other inequality can be obtained with similar arguments.

We have

$$a''(x) + Ma(x) \le f\left(x, L_m^{-1} a(x), a(x) - mL_m^{-1} a(x)\right) + (m+M)a(x) - m^2 L_m^{-1} a(x),$$

$$a(0) = a(2\pi), \quad a'(0) \le a'(2\pi).$$

Setting $y(x) = z(x) - a(x)$, then $y(0) = y(2\pi)$, $y'(0) \ge y'(2\pi)$. Suppose towards a contradiction that there exists $x_0 \in [0,2\pi]$ such that $y(x_0) > 0$.

If $y(x) > 0$ for all x, then $z(x) > a(x)$ and, therefore, $p(x,z(x)) = a(x)$, so

$$\begin{aligned} z''(x) + Mz(x) =& f\left(x, L_m^{-1} a(x), a(x) - mL_m^{-1} a(x)\right) + (m+M)a(x) - m^2 L_m^{-1} a(x) \ge \\ \ge& a''(x) + Ma(x), \end{aligned}$$

which is a contradiction, since by the Maximum Principle 4.2.1 we would have $y(x) \le 0$.

Otherwise, considering if necessary the periodic extension of $y(x)$, there exists an interval $[p, q]$ with $q - p < 2\pi$ such that $y(p) = y(q) = 0$, $y'(p) > 0 > y'(q)$, $y(x) > 0$ for $x \in (p, q)$ and (recalling that $L_m^{-1}p(x, z(x)) \geq L_m^{-1}a(x)$)

$$\begin{aligned} y''(x) + My(x) =& z''(x) + Mz(x) - a''(x) - Ma(x) \geq \\ \geq& f\left(x, L_m^{-1}p(x, z(x)), p(x, z(x)) - mL_m^{-1}p(x, z(x))\right) - \\ &- f(x, L_m^{-1}a(x), a(x) - mL_m^{-1}a(x)) + \\ &+ (m + M)y(x) - m^2(L_m^{-1}p(x, z(x)) - \alpha(x)) \geq 0, \end{aligned}$$

so, we reach again a contradiction by the Maximum Principle 4.2.1. □

Theorem 4.2.6. *Assuming (H1), for given lower and upper solutions α and β, with $\alpha \leq \beta$, the boundary value problem (4.3)–(4.4) has a solution $u(x) \in W^{4,1}(0, 2\pi)$.*

Proof. Let $u(x) = L_m^{-1}z(x)$, where $z(x)$ is a solution of the modified problem (4.6). Since $z(x) = u''(x) + mu(x)$, we have $z''(x) + Mz(x) = u^{(4)}(x) - Du''(x) + Cu(x)$. On the other hand,

$$\begin{aligned} z''(x) + Mz(x) =& f\left(x, L_m^{-1}p(x, z(x)), p(x, z(x)) - mL_m^{-1}p(x, z(x))\right) + \\ &+ (m + M)p(x, z(x)) - m^2L_m^{-1}p(x, z(x)) = \\ =& f\left(x, z(x), u''(x)\right) - Du''(x) + Cu(x), \end{aligned}$$

so $u(x)$ satisfies (4.3)–(4.4). □

We can reach a similar conclusion, assuming the following hypothesis

(H1') there exist constants $C, D > 0$ with $D < 4C + 1/4$ and $D^2 > 4C$, such that

$$f(x, u_2, v_1) - f(x, u_1, v_2) \geq -C(u_2 - u_1) - D(v_1 - v_2) \tag{4.7}$$

for *a.e.* $x \in [0, 2\pi]$, $\alpha(x) \leq u_1 \leq u_2 \leq \beta(x)$, $v_1 \leq v_2$.

Remark 4.2.7. If $f(x, u, v)$ is a C^1 function in (u, v), the inequality in (H1') is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \leq -D$.

Figure 4.2: Admissible values C, D

Let $0 < m, M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM$, $D = m + M$)).

Defining $a(x)$ and $b(x)$ as above, we have the following result:

Proposition 4.2.8. *If f is a L^1-Carathéodory function satisfying (H1') for $\alpha(x), \beta(x)$ lower and upper solutions such that $\alpha(x) \leq \beta(x)$, then $b(x) \geq a(x)$.*

Proof. Setting $y(x) = b(x) - a(x)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $x_0 \in [0, 2\pi]$ such that $y(x_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 4.2.4, using instead the Maximum principle 4.2.2. □

Using the same arguments as in the previous case, we prove the following result:

Theorem 4.2.9. *Assuming (H1'), for given lower and upper solutions α and β, with $\alpha \leq \beta$, the boundary value problem (4.3)–(4.4) has a solution $u(x) \in W^{4,1}(0, 2\pi)$.*

Now we prove similar results from the ones above, but with lower and upper solutions in reversed order, that is $\beta(x) \leq \alpha(x)$, for all $x \in [0, 2\pi]$.

(H2) there exist constants C, D with $C < 0$ and $D > -4C - 1/4$, such that

$$f(x, u_1, v_2) - f(x, u_2, v_1) \geq -C(u_1 - u_2) + D(v_2 - v_1) \tag{4.8}$$

for *a.e.* $x \in [0, 2\pi]$, $\beta(x) \leq u_1 \leq u_2 \leq \alpha(x)$, $v_1 \leq v_2$.

Remark 4.2.10. If $f(x, u, v)$ is a C^1 function in (u, v), the inequality in (H2) is equivalent to $\frac{\partial f}{\partial u} \leq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Figure 4.3: Admissible values C, D

Let $M < 0$ and $0 < m < \frac{1}{4}$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m + M)$).

Defining $a(x) = \alpha''(x) + m\alpha(x)$ and $b(x) = \beta''(x) + m\beta(x)$, we have the following result:

Proposition 4.2.11. *If f is a L^1-Carathéodory function satisfying (H2) for $\alpha(x), \beta(x)$ lower and upper solutions such that $\beta(x) \leq \alpha(x)$, then $b(x) \leq a(x)$.*

Proof. Setting $y(x) = b(x) - a(x)$, then $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $x_0 \in [0, 2\pi]$ such that $y(x_0) > 0$.

If $y(x) > 0$ for all x, then (noting that $b(x) - m\beta(x) \geq a(x) - m\alpha(x)$)

$$\begin{aligned} y''(x) + My(x) \geq & f(x, \beta(x), b(x) - m\beta(x)) - f(x, \alpha(x), a(x) - m\alpha(x)) + \\ & + (m + M)y(x) - m^2(\beta(x) - \alpha(x)) \geq 0, \end{aligned}$$

and this is a contradiction, since by the Maximum Principle 4.2.1 we would have $y(x) \leq 0$.

Otherwise, considering if necessary the periodic extension of $y(x)$, there exists an interval $[p, q]$ with $q - p < 2\pi$ such that $y(p) = y(q) = 0$, $y'(p) > 0 > y'(q)$, and $y(x) > 0$ for $x \in (p, q)$. Applying the same argument in $[p, q]$ as above, we reach again a contradiction. □

Let

$$p(x, z) = \begin{cases} b(x), & z < b(x) \\ z, & b(x) \leq z \leq a(x) \\ a(x), & z > a(x). \end{cases}$$

Consider the boundary value problem

$$u''(x) + mu(x) \equiv L_m u(x) = q(x), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

with $q \in L^1[0, 1]$. Since $m < 1$, the operator L_m is invertible, so that we can write its unique solution u as $u = L_m^{-1} q$, and by the Maximum principle 4.2.2 we know that if $q(x) \geq 0$, then $u(x) \geq 0$. Since $\alpha(x) = L_m^{-1} a(x)$, $\beta(x) = L_m^{-1} b(x)$ and $b(x) \leq p(x, z(x)) \leq a(x)$ for any functions $z(x)$, we have

$$\beta(x) \leq L_m^{-1} p(x, z(x)) \leq \alpha(x).$$

Let us consider the modified problem

$$\begin{aligned} z''(x) + Mz(x) = & (Fz)(x) \equiv f\left(x, L_m^{-1} p(x, z(x)), p(x, z(x)) - mL_m^{-1} p(x, z(x))\right) + \\ & + (m + M)p(x, z(x)) - m^2 L_m^{-1} p(x, z(x)), \; z(0) = z(2\pi), \; z'(0) = z'(2\pi). \end{aligned} \tag{4.9}$$

Considering the operator $\Phi : C[0, 2\pi] \to C[0, 2\pi]$ with $\Phi z = L_M^{-1}(Fz)$ since $p(x, z(x))$ and $L_m^{-1} p(x, z(x))$ are bounded and f is a Carathéodory function, there exists a $L^1[0, 2\pi]$ function $g(x)$ such that $|(Fz)(x)| \leq g(x)$ for $a.e.\, x \in [0, 2\pi]$. Therefore, applying Schauder's fixed point Theorem, we can conclude that Φ has a fixed point $z(x)$ which is a solution of the modified problem (4.9).

Proposition 4.2.12. *Let $z(x)$ be a solution of the modified problem* (4.9)*. Assuming (H2), for given lower and upper solutions α and β, with $\alpha \leq \beta$, we have*

$$b(x) \leq z(x) \leq a(x).$$

Proof. The proof is similar to the one of proposition 4.2.5. □

Theorem 4.2.13. *Assuming (H2), for given lower and upper solutions α and β, with $\beta \leq \alpha$, the boundary value problem* (4.3)–(4.4) *has a solution $u(x) \in W^{4,1}(0, 2\pi)$.*

Proof. The proof is similar to the one of theorem 4.2.6. □

We can reach a similar conclusion, assuming the following hypothesis

(H2') there exist constants C, D with $C < 0$ and $D < 4C + 1/4$ such that

$$f\left(x, u_1, v_1\right) - f\left(x, u_2, v_2\right) \geq -C\left(u_1 - u_2\right) - D\left(v_1 - v_2\right) \tag{4.10}$$

for *a.e.* $x \in [0, 2\pi]$, $\beta(x) \leq u_1 \leq u_2 \leq \alpha(x)$, $v_1 \leq v_2$.

Remark 4.2.14. If $f(x, u, v)$ is a C^1 function in (u, v), the inequality in (H2') is equivalent to $\frac{\partial f}{\partial u} \leq -C$ and $\frac{\partial f}{\partial v} \leq -D$.

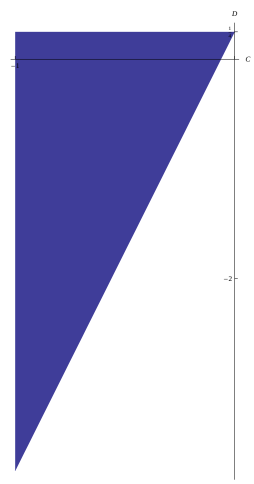


Figure 4.4: Admissible values C, D

Let $m < 0$ and $0 < M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM$, $D = m + M$)).

Defining $a(x)$ and $b(x)$ as above, we have the following result:

Proposition 4.2.15. *If f is a L^1-Carathéodory function satisfying (H2') for $\alpha(x), \beta(x)$ lower and upper solutions such that $\beta(x) \leq \alpha(x)$, then $b(x) \geq a(x)$.*

Proof. Setting $y(x) = b(x) - a(x)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $x_0 \in [0, 2\pi]$ such that $y(x_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 4.2.11, using instead the Maximum principle 4.2.2. □

Using the same arguments as in the previous case, we prove the following result:

Theorem 4.2.16. *Assuming (H2'), for given lower and upper solutions α and β, with $\alpha \leq \beta$, the boundary value problem* (4.3)–(4.4) *has a solution $u(x) \in W^{4,1}[0, 2\pi]$.*

4.3 Existence and localization for "simply supported" boundary conditions

We start by stating the following maximum principles that will allow us to conclude the pretended existence results.

Lemma 4.3.1 (Maximum Principle 3)**.** *Let $L < 1$ and $y \in W^{2,1}(0,\pi)$ such that*

$$y''(x) + Ly(x) \geq 0, \quad y(0) \leq 0, \ \ y(\pi) \leq 0.$$

Then $y(x) \leq 0$ for all $t \in [0,\pi]$. Moreover, if $y''(x) + Ly(x) \not\equiv 0$, then $y(x) < 0$ for all $t \in (0,\pi)$.

Lemma 4.3.2 (Maximum Principle 4)**.** *Let $L < 1$, $M \in \mathbb{R}$ and $y \in W^{2,1}(0,\pi)$ such that*

$$y''(x) + Ly^+(x) - My^-(x) \geq 0, \quad y(0) \leq 0, \ \ y(\pi) \leq 0,$$

where y^+, y^- are respectively the positive and negative parts of y. Then $y(x) \leq 0$ for all $x \in [0,\pi]$.

Consider the fourth order equation

$$u^{(4)}(x) = f\left(x, u(x), u''(x)\right), \quad x \in [0,\pi] \tag{4.11}$$

where f is a L^1 – Carathéodory function, and the boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0. \tag{4.12}$$

We say that $\alpha \in W^{4,1}[0,\pi]$ is a lower solution of the boundary value problem (4.11)–(4.12) if

$$\alpha^{(4)}(x) \leq f\left(x, \alpha(x), \alpha''(x)\right), \quad x \in [0,\pi]$$
$$\alpha(0) \leq 0, \quad \alpha(\pi) \leq 0, \quad \alpha''(0) \geq 0, \quad \alpha''(\pi) \geq 0.$$

A function $\beta \in W^{4,1}[0,\pi]$ satisfying the reversed inequalities is called an upper solution.

Let α, β be respectively a lower and an upper solution of (4.3)–(4.4), such that $\alpha(x) \leq \beta(x)$ for all $x \in [0,\pi]$.

In the following, let us assume the hypothesis

(H3) there exist constants C, D such that $C < 0$ or $D > 0$, with $D > -C - 1$, $D^2 > 4C$, and

$$f\left(x, u_2, v_2\right) - f\left(x, u_1, v_1\right) \geq -C\left(u_2 - u_1\right) + D\left(v_2 - v_1\right) \tag{4.13}$$

for *a.e.* $x \in [0, 2\pi]$, $\alpha(x) \leq u_1 \leq u_2 \leq \beta(x)$, $v_1 \leq v_2$.

Remark 4.3.3. If $f(x,u,v)$ is a C^1 function in (u,v), the inequality in (H3) is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $m < 0$ and $M < 1$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m+M)$).

Defining $a(x) = \alpha''(x) + m\alpha(x)$ and $b(x) = \beta''(x) + m\beta(x)$, we have the following result:

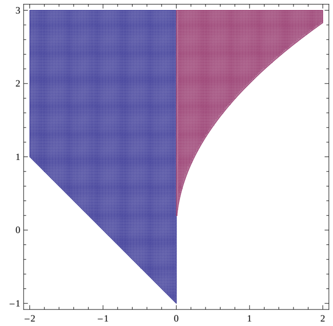


Figure 4.5: Admissible values C, D

Proposition 4.3.4. *If f is a L^1-Carathéodory function satisfying (H3) for $\alpha(x), \beta(x)$ lower and upper solutions such that $\alpha(x) \le \beta(x)$, then $b(x) \le a(x)$.*

Proof. Setting $y(x) = b(x) - a(x)$, then $y(0) \le 0$ and $y(\pi) \le 0$. Suppose towards a contradiction that there exists $x_0 \in [0, \pi]$ such that $y(x_0) > 0$. The result follows using similar arguments to the ones of Proposition 4.2.4 (using Maximum principle 4.3.1 instead). □

Let

$$p(x,z) = \begin{cases} b(x), & z < b(x) \\ z, & b(x) \le z \le a(x) \\ a(x), & z > a(x). \end{cases}$$

Consider the boundary value problem

$$u''(x) + mu(x) \equiv L_m\, u(x) = q(x), \quad u(0) = 0, \quad u(\pi) = 0,$$

with $q \in L^1[0,1]$. Since $m < 0$, the operator L_m is invertible, so that we can write its unique solution u as $u = L_m^{-1} q$.

Let us define $\tilde{a}(x)$ such that $\tilde{a}'' + m\tilde{a} = 0$, $\tilde{a}(0) = \alpha(0)$, $\tilde{a}(\pi) = \alpha(\pi)$, and $\tilde{b}(x)$ such that $\tilde{b}'' + m\tilde{b} = 0$, $\tilde{b}(0) = \beta(0)$, $\tilde{b}(\pi) = \beta(\pi)$. It is obvious that $\tilde{a}(x) \le 0$ and $\tilde{b}(x) \ge 0$ for all $x \in [0, \pi]$.

We have $\alpha(x) = L_m^{-1} a(x) + \tilde{a}(x)$ and $\beta(x) = L_m^{-1} b(x) + \tilde{b}(x)$, so, by the Maximum principle 4.3.1, for any function $z(x)$ we get

$$\alpha(x) \le L_m^{-1} p(x, z(x)) \le \beta(x).$$

Proceeding in a similar way as in the previous cases, we can reach an analogue conclusion:

Theorem 4.3.5. *Assuming (H3), for given lower and upper solutions α and β, with $\alpha \le \beta$, the boundary value problem* (4.11)–(4.12) *has a solution $u(x) \in W^{4,1}[0, \pi]$.*

Let us now consider an hypothesis somehow different from the ones considered above. Suppose that

(H4) there exist constants C, D with $C > 0$, $0 < D < 1$, and

$$f(x, u_2, v_2) - f(x, u_1, v_1) \ge C(u_2 - u_1) - D\,|v_2 - v_1| \tag{4.14}$$

for *a.e.* $x \in [0, 2\pi]$, $\alpha(x) \le u_1 \le u_2 \le \beta(x)$, $v_1, v_2 \in \mathbb{R}$.

Remark 4.3.6. If $f(x,u,v)$ is a C^1 function in (u,v), the inequality in (H4) is equivalent to $\frac{\partial f}{\partial u} \geq C$ and $\left|\frac{\partial f}{\partial v}\right| \leq D$.

Let $m < 0$ be such that $C + Dm - m^2 > 0$ and $D - m < 1$. Defining $a(x) = \alpha''(x) + m\alpha(x)$ and $b(x) = \beta''(x) + m\beta(x)$, we have the following result:

Proposition 4.3.7. *If f is a L^1-Carathéodory function satisfying (H4) for $\alpha(x), \beta(x)$ lower and upper solutions such that $\alpha(x) \leq \beta(x)$, then $b(x) \leq a(x)$.*

Proof. Setting $y(x) = b(x) - a(x)$, then $y(0) \leq 0$, $y(\pi) \leq 0$ and

$$\begin{aligned}
y''(x) =& \beta^{(4)}(x) - \alpha^{(4)}(x) + m(\beta''(x) - \alpha''(x)) + m^2(\beta(x) - \alpha(x)) - m^2(\beta(x) - \alpha(x)) \geq \\
\geq& f(x,\beta(x),b(x)-m\beta(x)) - f(x,\alpha(x),a(x)-m\alpha(x)) + my(x) - m^2(\beta(x)-\alpha(x)) \geq \\
\geq& C(\beta(x) - \alpha(x)) - D\,|y(x) - m(\beta(x) - \alpha(x))| + my(x) - m^2(\beta(x) - \alpha(x)) \geq \\
\geq& (C + Dm - m^2)(\beta(x) - \alpha(x)) - D\,|y(x)| + my(x).
\end{aligned}$$

In order to apply Maximum principle 4.3.2, we can rewrite the previous inequality as

$$y''(x) + (D - m)y^+(x) + (D + m)y^-(x) \geq 0$$

and conclude that $y(x) \leq 0$.

□

Proceding in a similar way as above, we can reach an analogue conclusion:

Theorem 4.3.8. *Assuming (H4), for given lower and upper solutions α and β, with $\alpha \leq \beta$, the boundary value problem* (4.11)–(4.12) *has a solution $u(x) \in W^{4,1}[0,\pi]$.*

4.4 Existence and localization for "clamped beam" boundary conditions

In this section we study the boundary value problem

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases} \tag{4.15}$$

We prove existence results using the method of lower and upper solutions. In order to apply this method, in the first three subsections we study the positivity of an auxiliar fourth order operator.

As it was said before, positivity results for this type of problems cannot be obtained as in the previous sections, because the usual technique of decomposing the operator into two second order operators does not work appropriately in this case. We found the answer by applying a very interesting result of J. Schröder in [49] concerning the oscillation properties of the solutions of a differential equation (subsection 4.4.2). The main result is given in the last subsection.

Positivity for the operator $u^{(4)} - m^4 u$

We begin by stating a general result of linear eigenvalue problems. Let X be a Banach space with an order cone X^+ having a nonempty interior and let $T : X \to X$ be a linear and completely continuous operator with spectral radius $r(T)$.

A cone X^+ defines the partial ordering in X given by $x \preceq y$ if and only if $y - x \in X^+$. We use the notation $x \prec y$ for $y - x \in X^+ \setminus \{0\}$ and $x \npreceq y$ for $y - x \notin X^+$, moreover $x \ll y$ means $y - x \in int(X^+)$. We will say that operator T is strongly positive if and only if $0 \preceq T(0)$ and the following property holds:

$$0 \prec x \qquad \text{implies} \qquad 0 \ll T\,x \quad \text{for all } x \in D(T). \tag{4.16}$$

Consider the equation

$$Tx = \lambda x, \quad x \succ 0$$

and the correspondent inhomogeneous equation

$$\lambda x - Tx = y, \quad y \succ 0. \tag{4.17}$$

In the sequel, we enunciate a classical result for the existence and uniqueness of solutions for equation (4.17), depending on the spectral radius of operator T.

Theorem 4.4.1. *[52, Corollary 7.27] For every $y \succ 0$, equation (4.17) has exactly one solution $x \succ 0$ if $\lambda > r(T)$ and no solution $x \succ 0$ if $\lambda \leq r(T)$.*

Moreover, given λ, $\mu \in \mathbb{R}$, the equation $\lambda\, x - T\, x = \mu\, y$, for $y \succ 0$, has a positive solution $x \succ 0$ if $sgn(\mu) = sgn(\lambda - r(T))$.

Remark 4.4.2. For the case $\mu = -1$, we have that if $0 < \lambda < r(T)$, then equation $Tx - \lambda x = y$ has a unique positive solution $x \succ 0$.

Let $m > 0$ be given, and consider the boundary value problem

$$\begin{cases} u^{(4)}(x) = m^4 u(x), & x \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases} \tag{4.18}$$

It is not difficult to verify that this problem has a nontrivial solution if and only if m solves the equation

$$\cos(m)\cosh(m) = 1. \tag{4.19}$$

Moreover the first positive root of equation (4.19) is $m_1 \approx 4,73004$.

Consider the boundary value problem (4.18) in the form

$$u = m^4\, T\, u,$$

where $T : C[0,1] \to C[0,1]$ is the operator that gives the unique solution u of $u^{(4)} = y$ satisfying the boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, i.e.,

$$T\, y(t) = \int_0^1 G_0(t,s)\, y(s)\, ds, \quad y \in C[0,1], \tag{4.20}$$

where

$$G_0(t,s) = -\frac{1}{6}\begin{cases} s^2\,(t-1)^2\,(s-3\,t+2\,s\,t), & \text{if} \quad 0 \le s \le t \le 1, \\ \\ t^2\,(s-1)^2\,(t-3\,s+2\,s\,t), & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

The regularity of this Green's function G_0 in $[0,1]\times[0,1]$ and its positivity in $(0,1)\times(0,1)$, imply that T is a linear, completely continuous and strongly positive operator.

It is obvious that the values $\lambda = \frac{1}{m^4}$, where $m>0$ solves (4.19), are the eigenvalues of operator T, and therefore, $r(T) = \frac{1}{{m_1}^4}$.

The theorem above allows us to conclude the following result:

Theorem 4.4.3. *Given $y \in C[0,1]$ a nontrivial function such that $y(x) \ge 0$ for all $x \in [0,1]$, the boundary value problem*

$$\begin{cases} u^{(4)}(x) - m^4\,u(x) = y(x), & \textit{for all } x \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0, \end{cases} \tag{4.21}$$

has a unique positive solution u if $0 \le m < m_1$ and no positive solution if $m \ge m_1$.

Remark 4.4.4. If we refer to an arbitrary interval $[a,b]$, we have that the maximum principle holds for $0 \le m < m_1/(b-a)$.

Positivity for the operator $u^{(4)} + m^4 u$

In this subsection we establish the range of values $m>0$ for which it is true that

$$\begin{cases} u^{(4)} + m^4 u \ge 0 \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{4.22}$$

implies $u \ge 0$.

We say that $[0,1]$ is an interval of non-oscillation for the differential equation $u^{(4)} + m^4 u = 0$ if no nontrivial solution of the equation $u^{(4)} + m^4 u = 0$ has more than three zeros in $[0,1]$ (the definition of interval of oscillation can be set as the opposite, that is, there exists u solving $u^{(4)} + m^4 u = 0$ with at least four zeros in the given interval).

In [49], Schröeder proved that if $[0,1]$ is an interval of non-oscillation for the differential equation $u^{(4)} + m^4 u = 0$, then (4.22) implies $u \ge 0$ in $[0,1]$. In the following, we will find for which values of m is $[0,1]$ an interval of non-oscillation.

It is clear that the solutions of the fourth order linear homogeneous equation

$$u^{(4)} + m^4 u = 0 \tag{4.23}$$

are given by the following expression

$$u(x) = e^{\frac{mx}{\sqrt{2}}}\left(A\cos\left(\frac{mx}{\sqrt{2}}\right) + B\sin\left(\frac{mx}{\sqrt{2}}\right)\right) + e^{-\frac{mx}{\sqrt{2}}}\left(C\cos\left(\frac{mx}{\sqrt{2}}\right) + D\sin\left(\frac{mx}{\sqrt{2}}\right)\right), \tag{4.24}$$

with $A, B, C, D \in \mathbb{R}$.

Since the equation is autonomous, if u is a solution of (4.23) for a given m and $u(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then $v(x) = u(x - x_0)$ is also a solution of (4.23) and $v(0) = 0$, so we can restrict ourselves to the solutions of (4.23) that vanish at $x = 0$, that is, we can take $C = -A$ in (4.24).

Lemma 4.4.5. *If* $[0,1]$ *is an interval of oscillation of* (4.23) *for a given* m_**, then it is also an interval of oscillation for all* $m > m_*$*.*

Proof. Suppose that u is a solution of $u^{(4)} + {m_*}^4 u = 0$ and $u(0) = u(a) = u(b) = u(c) = 0$ with $0 < a < b < c \leq 1$. Then $v(x) = u\left(\frac{m}{m_*}x\right)$ satisfies $v^{(4)} + m^4 v = 0$, $v\,(0) = v\left(\frac{m_*}{m}a\right) = v\left(\frac{m_*}{m}b\right) = v\left(\frac{m_*}{m}c\right) = 0$ and $\frac{m_*}{m}c < c \leq 1$. □

This lemma allows us to conclude that the set of values $m > 0$ for which $[0,1]$ is an interval of non-oscillation of (4.23) is an interval too. Such interval can be empty and it is bounded from above by $m = 3\pi\sqrt{2}$. This last property holds as a direct consequence of the previous lemma by using the fact that the function $e^{\frac{mx}{\sqrt{2}}} \sin\left(\frac{mx}{\sqrt{2}}\right)$ is a solution of equation (4.23) that vanishes four times in $[0,1]$ for $m = 3\pi\sqrt{2}$.

To characterise the values of positive m for which $[0,1]$ is a non-oscillation interval, we are interested in finding the infimum of the values m for which exists a solution of (4.23) with four zeros in $[0,1]$. The next lemma allows us to confine our search to the solutions of (4.23) that vanish at $x = 1$.

Lemma 4.4.6. *Consider* m *such that there exists a solution* $u(x)$ *of* (4.23) *such that* $u(0) = u(a) = u(b) = u(c) = 0$ *with* $0 < a < b < c < 1$*. Then* m *is not the smallest value for which* $[0,1]$ *is an interval of oscillation of* (4.23)*.*

Proof. Let $v(x) = \frac{u(c\,x)}{c}$. We have $v^{(4)} + (c\,m)^4 v = 0$, $v(0) = v\left(\frac{a}{c}\right) = v\left(\frac{b}{c}\right) = v(1) = 0$. Since $c\,m < m$, the result follows. □

Remark 4.4.7. Notice that in the proof of the previous result we have that $v'(0) = u'(0)$.

Taking $u(1) = 0$ and assuming that $\frac{m}{\sqrt{2}} = n\,\pi$ for some natural n, we deduce $A = 0$ and, in consequence, the expression (4.24) is reduced to

$$u(x) = e^{-n\,\pi\,x}\left(Be^{2n\pi x} + D\right)\,\sin n\pi x.$$

Clearly this function has at most three zeros in $[0,1]$ when $n = 1$. So, from Lemma 4.4.5, we deduce that interval $[0,1]$ is non-oscillatory for all $m \in (0, \sqrt{2}\pi]$.

Now, by choosing $D = -2\,B \neq 0$, we have that for $n = 2$ the previous function vanishes four times in $[0,1]$. Thus, by using Lemma 4.4.5 again, we know that $[0,1]$ is oscillatory for all $m \geq 2\,\sqrt{2}\,\pi$.

So if $\frac{m}{\sqrt{2}}$ is not a positive multiple of π, we restrict our investigation to the values $m \in (\sqrt{2}\,\pi, 2\,\sqrt{2}\,\pi)$. If it is the case, we deduce that, in equation (4.24), the following equality holds:

$$B = -e^{-\sqrt{2}m}\left(A\left(e^{\sqrt{2}m} - 1\right)\cot\left(\frac{m}{\sqrt{2}}\right) + D\right).$$

Now, define $m_0 \approx 5.553$ as the smaller positive solution of the equation

$$\tanh\frac{m}{\sqrt{2}} = \tan\frac{m}{\sqrt{2}}. \tag{4.25}$$

Let us now consider the set of solutions u such that $u'(0)=0$. In this case, the solutions u of (4.23) such that $u(0)=u(1)=u'(0)=0$ follow the expression

$$u(x,m,A)=2A\sin\left(\frac{mx}{\sqrt{2}}\right)\sinh\left(\frac{mx}{\sqrt{2}}\right)\left(\coth\left(\frac{m}{\sqrt{2}}\right)-\coth\left(\frac{mx}{\sqrt{2}}\right)+\cot\left(\frac{mx}{\sqrt{2}}\right)-\cot\left(\frac{m}{\sqrt{2}}\right)\right).$$

It is trivial to see that $\cot\left(\frac{m}{\sqrt{2}}\right)-\coth\left(\frac{m}{\sqrt{2}}\right)$ is a one-to-one map for $m\in(0,m_0]$, so if $0<m\le m_0$, we know that

$$\cot\left(\frac{mx}{\sqrt{2}}\right)-\coth\left(\frac{mx}{\sqrt{2}}\right)\neq\cot\left(\frac{m}{\sqrt{2}}\right)-\coth\left(\frac{m}{\sqrt{2}}\right)\quad\text{for all } x\in[0,1).$$

Since $\frac{mx}{\sqrt{2}}<2\pi$, the equation $\sin\left(\frac{mx}{\sqrt{2}}\right)=0$ cannot have more than one solution for $x\in(0,1)$ and consequently u does not vanish more than three times for $0<m\le m_0$.

Now consider the set of solutions u such that $u'(0)\neq0$. Given $u(x)$ such that $u'(0)\neq0$, then $v(x)=\frac{u(x)}{u'(0)}$ satisfies $v'(0)=1$ and $v(x)$ has exactly the same zeros of $u(x)$, so we can just refer to the case $u'(0)=1$. In consequence we study the functions given by the expression

$$\begin{aligned}u(x,m,A)=&\frac{e^{-\frac{mx}{\sqrt{2}}}}{\left(-1+e^{\sqrt{2}m}\right)m}\left[A\left(e^{\sqrt{2}m}-1\right)\left(e^{\sqrt{2}mx}-1\right)m\cos\left(\frac{mx}{\sqrt{2}}\right)+\right.\\
&\left.+\left(\left(e^{\sqrt{2}m}-e^{\sqrt{2}mx}\right)\left(\sqrt{2}-2Am\right)-A\left(e^{\sqrt{2}m}-1\right)\left(e^{\sqrt{2}mx}-1\right)m\cot\left(\frac{m}{\sqrt{2}}\right)\right)\sin\left(\frac{mx}{\sqrt{2}}\right)\right]=\\
=&\frac{2\sinh\left(\frac{mx}{\sqrt{2}}\right)\sin\left(\frac{mx}{\sqrt{2}}\right)}{m}\left[\frac{\sqrt{2}}{2}\left(\coth\left(\frac{mx}{\sqrt{2}}\right)-\coth\left(\frac{m}{\sqrt{2}}\right)\right)+\right.\\
&\left.+Am\left(\left(\cot\left(\frac{mx}{\sqrt{2}}\right)-\cot\left(\frac{m}{\sqrt{2}}\right)\right)-\left(\coth\left(\frac{mx}{\sqrt{2}}\right)-\coth\left(\frac{m}{\sqrt{2}}\right)\right)\right)\right]\end{aligned}\tag{4.26}$$

Now, we study this set of solutions. The obtained result is the following:

Proposition 4.4.8. *If $\sqrt{2}\pi<m\le m_0$, then the solutions* (4.26) *of* (4.23) *have at most three zeros.*

Proof. If $u(x,m,A)=0$ for some $x\in(0,1)$, then it is easy to write A as a function of m and x. Replacing A for $A(x,m)$ in the expression of the first derivative of $u(x,m,A)$ we get that the double zeros of $u(x,m,A)$ belonging to $(0,1)$ must satisfy the condition

$$\frac{\sinh\left(\frac{m(1-x)}{\sqrt{2}}\right)\sinh\left(\frac{mx}{\sqrt{2}}\right)}{\sinh\left(\frac{m}{\sqrt{2}}\right)}=\frac{\sin\left(\frac{m(1-x)}{\sqrt{2}}\right)\sin\left(\frac{mx}{\sqrt{2}}\right)}{\sin\left(\frac{m}{\sqrt{2}}\right)}.\tag{4.27}$$

In [11], Cabada, Cid and Sanchez proved that

$$\frac{\sin(m^*x)\sin(m^*(1-s))}{\sin(m^*)}<\frac{\sinh(m^*x)\sinh(m^*(1-s))}{\sinh(m^*)},\quad\forall x,s\in(0,1),\quad \pi<m^*\le k_0,$$

where k_0 is the smallest positive solution of the equation $\tan k=\tanh k$.

Taking $m^* = \frac{m}{\sqrt{2}}$ and $s = x$, we have that (4.27) cannot have any solution if $\sqrt{2}\pi < m \leq m_0 \equiv \sqrt{2}\, k_0$, and, consequently, there are no solutions $u(x, m, A)$ with a double zero in the interval $(0, 1)$.

Claim 4.4.9. Given $A < 0$ fixed, there exists $m_A > \sqrt{2}\pi$ close enough to $\sqrt{2}\pi$ such that $u(x, m, A) > 0$ for all $x \in (0, 1)$ and all $m \in (\sqrt{2}\pi, m_A)$.

To prove this, we will use the second expression in (4.26).

1. If $x \in (0, \frac{\sqrt{2}\pi}{2m})$, we have that $\cot\left(\frac{mx}{\sqrt{2}}\right) - \coth\left(\frac{mx}{\sqrt{2}}\right) < 0$ and $\sin\left(\frac{mx}{\sqrt{2}}\right) > 0$ for all $m \in (\sqrt{2}\pi, 2\sqrt{2}\pi)$. Moreover there is $\delta_1 > 0$ such that $\cot\left(\frac{m}{\sqrt{2}}\right) - \coth\left(\frac{m}{\sqrt{2}}\right) > 0$ for all $m \in (\sqrt{2}\pi, \sqrt{2}\pi + \delta_1)$. So the function u is positive for some values of $m > \sqrt{2}\pi$ close enough to $\sqrt{2}\pi$.

2. If $x \in (\frac{\sqrt{2}\pi}{2m}, \frac{\sqrt{2}\pi}{m})$, we have that $\sin\left(\frac{mx}{\sqrt{2}}\right) > 0$ for all $m \in (\sqrt{2}\pi, 2\sqrt{2}\pi)$ and there exists $\delta_2 > 0$ such that $\cot\left(\frac{mx}{\sqrt{2}}\right) - \cot\left(\frac{m}{\sqrt{2}}\right) \ll 0$ and $\coth\left(\frac{mx}{\sqrt{2}}\right) - \coth\left(\frac{m}{\sqrt{2}}\right) \approx 0$ for all $m \in (\sqrt{2}\pi, \sqrt{2}\pi + \delta_2)$. So again we have $u(x, m, A) > 0$ for m in such interval.

3. If $x \in (\frac{\sqrt{2}\pi}{m}, 1)$, by choosing m close enough to $\sqrt{2}\pi$ we can have the derivative of $-\cot\left(\frac{mx}{\sqrt{2}}\right)$ bounded from bellow in the given interval by a value as large as we want. Since $\left|\left(\coth\left(\frac{mx}{\sqrt{2}}\right)\right)'\right| < 1$ and $\sin\left(\frac{mx}{\sqrt{2}}\right) < 0$, it is easy to conclude that $u(x, m, A) > 0$ and the claim is proven.

Let us now now focus on the possible double zeros at $x = 1$. We have

$$u'(1, m, A) = \left(\sqrt{2}Am - 1\right) \frac{\sin\left(\frac{m}{\sqrt{2}}\right)}{\sinh\left(\frac{m}{\sqrt{2}}\right)} - \sqrt{2}Am \frac{\sinh\left(\frac{m}{\sqrt{2}}\right)}{\sin\left(\frac{m}{\sqrt{2}}\right)},$$

so, for $x = 1$ to be a double zero we must have

$$\sqrt{2}Am = \frac{\sin^2\left(\frac{m}{\sqrt{2}}\right)}{\sin^2\left(\frac{m}{\sqrt{2}}\right) - \sinh^2\left(\frac{m}{\sqrt{2}}\right)}. \tag{4.28}$$

Since $|\sin(x)| < |\sinh(x)|$ for all $x > 0$, for the previous equality to be true, we must have $A < 0$. In consequence, function $u(x, m, A)$ has no double zeros in $[0, 1]$ for all $A \geq 0$ and all $m \in (\sqrt{2}\pi, 2\sqrt{2}\pi)$. In such a case, by using the fact that the first derivative at $x = 0$ and $x = 1$ is positive, we have that function $u(x, m, A)$ has exactly three zeros in $[0, 1]$.

For the double zeros at $x = 1$ we can write

$$A(m) = \frac{\sin^2\left(\frac{m}{\sqrt{2}}\right)}{\sqrt{2}m\left(\sin^2\left(\frac{m}{\sqrt{2}}\right) - \sinh^2\left(\frac{m}{\sqrt{2}}\right)\right)}$$

and

$$u''(1,m,A(m)) = -\frac{\sqrt{2}m\left(\cosh\left(\frac{m}{\sqrt{2}}\right)\sin\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)\right)}{\sin^2\left(\frac{m}{\sqrt{2}}\right) - \sinh^2\left(\frac{m}{\sqrt{2}}\right)},$$

so the double zeros at $x = 1$ must have positive second derivative at $x = 1$ for $m < m_0$.

A careful analysis of the function on the right-hand side of the equality (4.28) allows us to conclude that given $A < 0$, there exist at most one value of $m \in (\sqrt{2}\pi, m_0]$ such that $u(x,m,A)$ has a double zero at $x = 1$.

So, fix $A < 0$ and denote

$$m_A = \inf\{m \in (\sqrt{2}\pi, m_0], \quad \text{such that } u(x,m,A) > 0 \quad \text{for all } x \in (0,1)\}.$$

If $m_A \geq m_0$, we have that function $u(x,m,A)$ is strictly positive in $(0,1)$. Otherwise, by increasing the value of m from m_A to m_0 (where the continuous dependence on the parameter m is obvious), there exists just one $m_1 \in [m_A, m_0]$ for which $u(1,m_1,A)$ has a double zero.

If $m < m_0$ we have that $u''(1,m_1,A) > 0$ and, in consequence, the double zero can only provide one extra zero and the solutions $u(x,m,A)$ cannot have more than three zeros in $[0,1]$.

When $m = m_0$ we have that $u(1,m_0,A) = u'(1,m_0,A) = u''(1,m_0,A) = 0$. Therefore, $u'''(1,m_0,A) \neq 0$ and the double zero only gives one extra zero as in the previous case. □

We can now state the following

Theorem 4.4.10. *If $m \in (0,m_0]$, then $[0,1]$ is an interval of non-oscillation for the differential equation $u^{(4)} + m^4 u = 0$.*

Proof. We have proven in Proposition 4.4.8 that $[0,1]$ is an interval of non-oscillation for all $m \in (\sqrt{2}\pi, m_0]$. From Lemma 4.4.5 we deduce that the same property holds for all $m \in (0, \sqrt{2}\pi]$. □

The next result shows us that the previous theorem is optimal.

Theorem 4.4.11. *If $m > m_0$ then there exists a solution of (4.23) with at least four zeros in $[0,1]$.*

Proof. Considering the expression (4.27) of the double zeros in $(0,1)$, let us define

$$f(x,m) = \frac{\sinh\left(\frac{m(1-x)}{\sqrt{2}}\right)\sinh\left(\frac{mx}{\sqrt{2}}\right)}{\sinh\left(\frac{m}{\sqrt{2}}\right)}, \quad g(x,m) = \frac{\sin\left(\frac{m(1-x)}{\sqrt{2}}\right)\sin\left(\frac{mx}{\sqrt{2}}\right)}{\sin\left(\frac{m}{\sqrt{2}}\right)}.$$

In the following we will prove that for $m > m_0$ there exists a solution $u(x,m,A)$ with a double zero for some $x \in (0,1)$ and that a small change of the value A provides two zeros in $(0,1)$ and, consequently, four zeros in $[0,1]$.

If $m_0 < m < 2\sqrt{2}\pi$, we have

$$f\left(\frac{1}{2},m\right) - g\left(\frac{1}{2},m\right) = \frac{1}{2}\left(\tanh\left(\frac{m}{2\sqrt{2}}\right) - \tan\left(\frac{m}{2\sqrt{2}}\right)\right) > 0.$$

On the other hand we have

$$f(0,m) - g(0,m) = f'(0,m) - g'(0,m) = 0$$

and

$$f''(0,m) - g''(0,m) = m^2 \left(\cot\left(\frac{m}{\sqrt{2}}\right) - \coth\left(\frac{m}{\sqrt{2}}\right)\right) < 0,$$

so there exists x_1 small enough such that $f(x_1,m) - g(x_1,m) < 0$.

Since f and g are continuous functions in the considered domain, for each $m_0 < m < 2\sqrt{2}\pi$ there exists $x_m \in (0, \frac{1}{2})$ such that $f(x_m,m) = g(x_m,m)$ and consequently there is $A_m < 0$ for which the solution $u(x,m,A_m)$ has a double zero in $(0,1)$.

Let us now see that with the same value of m, a small change of A must provide two zeros in $(0,1)$. For simplicity let us write the second expression in (4.26) in the compacted form

$$u(x,m,A) \equiv f_1(x,m)(f_2(x,m) + A\, f_3(x,m)).$$

Obviously

$$\frac{\partial u}{\partial A}(x,m,A) = f_1(x,m) f_3(x,m).$$

On the other hand, since $u(x_m,m,A_m) = 0$, we have that

$$f_1(x_m,m)\, f_2(x_m,m) = -A_m\, f_1(x_m,m)\, f_3(x_m,m).$$

Since $0 < x_m < \frac{1}{2}$, we have that $f_1(x_m,m) > 0$ and $f_2(x_m,m) > 0$. Therefore $A_m\, f_3(x_m,m) \neq 0$ and we can conclude that $\frac{\partial u}{\partial A}(x_m,m,A_m) \neq 0$. This means that a small change of A_m in one of the directions makes the solution break the $y = 0$ line, providing two zeros for the solution.

Now, from Lemma 4.4.5, we deduce that $[0,1]$ is an interval of oscillation for all $m > m_0$. □

Following the results given by Schröeder in [49] we can state the main result of this subsection:

Theorem 4.4.12. *If $0 < m \leq m_0$, then* (4.22) *implies $u \geq 0$.*

Remark 4.4.13. If we refer to an arbitrary interval $[a,b]$, we have that the anti - maximum principle is fulfilled for $0 \leq m \leq m_0/(b-a)$.

Let us now see that this last theorem is an equivalence. It is well known that the positivity of the operator is verified if and only if the associated Green's function is positive. For this we will use the Green's function (which is hard to deal with to prove positivity in this case, but relatively easy to prove non-positivity). Before to consider such function we use the following result that can be proven in the same way that [13, Theorem 3.1].

Lemma 4.4.14. *Suppose that there exist $\tilde{m} > 0$ and $u_{\tilde{m}} \ngeq 0$ in $[0,1]$ satisfying inequalities (4.22). Then for all $m > \tilde{m}$ there is $u_m \ngeq 0$ in $[0,1]$ fulfilling (4.22). In other words: if the related Green's function $G_{\tilde{m}}$ changes sign in $[0,1] \times [0,1]$ the same holds for all $m > \tilde{m}$.*

The Green's function is given by the following expression:

$$
\begin{aligned}
G_m(t,s) \;=\; & \frac{1}{4\sqrt{2}m^3}\Bigg\{2e^{\frac{m(t-s)}{\sqrt{2}}}\Bigg(\left[-1+e^{\sqrt{2}m(s-t)}\right)\cos\left(\frac{m(t-s)}{\sqrt{2}}\right) \\
& +\left(1+e^{\sqrt{2}m(s-t)}\right)\sin\left(\tfrac{m(t-s)}{\sqrt{2}}\right)\Bigg)\Bigg] \\
& -\frac{1}{\cos\left(\sqrt{2}m\right)+\cosh\left(\sqrt{2}m\right)-2}\Bigg[2e^{-\frac{m(3s+t-4)}{\sqrt{2}}}\left(-1+e^{\sqrt{2}mt}\right)\times \\
& \Bigg[\left(e^{\sqrt{2}m(s-2)}-e^{2\sqrt{2}m(s-1)}\right)\cos\left(\tfrac{m(s-2)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}m(s-2)}+e^{2\sqrt{2}m(s-1)}\right)\cos\left(\tfrac{ms}{\sqrt{2}}\right) \\
& +\left(e^{\sqrt{2}m(s-2)}-e^{\sqrt{2}m(s-1)}+e^{2\sqrt{2}m(s-1)}-e^{\sqrt{2}m(2s-3)}\right)\sin\left(\tfrac{ms}{\sqrt{2}}\right)\Bigg]\sin\left(\tfrac{mt}{\sqrt{2}}\right) \\
& -e^{-\frac{m(s+t)}{\sqrt{2}}}\Bigg(\left(1+e^{\sqrt{2}ms}\right)\cos\left(\tfrac{m(s-2)}{\sqrt{2}}\right)+\left(-2+e^{\sqrt{2}m}+e^{\sqrt{2}m(s-1)}-2e^{\sqrt{2}ms}\right)\cos\left(\tfrac{ms}{\sqrt{2}}\right) \\
& +\left(-1+e^{\sqrt{2}ms}\right)\sin\left(\tfrac{m(s-2)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m}-e^{\sqrt{2}m(s-1)}\right)\sin\left(\tfrac{ms}{\sqrt{2}}\right)\Bigg)\times \\
& \Bigg(\left(-1+e^{\sqrt{2}mt}\right)\cos\left(\tfrac{mt}{\sqrt{2}}\right)-\left(1+e^{\sqrt{2}mt}\right)\sin\left(\tfrac{mt}{\sqrt{2}}\right)\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

when $0 \le s \le t \le 1$, and

$$G_m(t,s) \equiv G_m(s,t) \text{ if } 0 \le t \le s \le 1.$$

From Theorem 4.4.12 we know that the Green's function G_m is nonnegative in $[0,1] \times [0,1]$ for all $m \in (0, m_0]$.

Theorem 4.4.15. *Function G_m changes sign for all $m > m_0$.*

Proof. Let us see that such function changes sign in $[0,1] \times [0,1]$ for all $m > m_0$, m close enough to m_0.

First note that for all $m > 0$ it is verified that

$$G_m\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\cos\left(\frac{m}{\sqrt{2}}\right)+\cosh\left(\frac{m}{\sqrt{2}}\right)-2}{2\sqrt{2}m^3\left(\sin\left(\frac{m}{\sqrt{2}}\right)+\sinh\left(\frac{m}{\sqrt{2}}\right)\right)},$$

which is strictly positive for all $m > m_0$.

On the other hand, one can verify that

$$G_m(0,s)=\frac{\partial}{\partial t}G_m(0,s)=0 \quad \text{for all } s \in [0,1].$$

Moreover

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}G_m(0,s) \;=\; & -\frac{e^{\frac{m(4-3s)}{\sqrt{2}}}}{\sqrt{2}m\left(\cos\left(\sqrt{2}m\right)+\cosh\left(\sqrt{2}m\right)-2\right)}\Bigg[\left(e^{\sqrt{2}m(s-2)}-e^{2\sqrt{2}m(s-1)}\right)\times \\
& \cos\left(\tfrac{m(s-2)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}m(s-2)}+e^{2\sqrt{2}m(s-1)}\right)\cos\left(\tfrac{ms}{\sqrt{2}}\right) \\
& +\left(e^{\sqrt{2}m(s-2)}-e^{\sqrt{2}m(s-1)}+e^{2\sqrt{2}m(s-1)}-e^{\sqrt{2}m(2s-3)}\right)\sin\left(\tfrac{ms}{\sqrt{2}}\right)\Bigg].
\end{aligned}
$$

Now, by defining

$$h(s) \equiv \frac{\partial^2}{\partial t^2} G_{m_0/s}(0,s),$$

we deduce that

$$h(1) = h'(1) = h''(1) = 0$$

and

$$\begin{aligned} h'''(1) &= \frac{e^{-\frac{3m_0}{\sqrt{2}}} m_0}{2\left(\cos\left(\sqrt{2}m_0\right) + \cosh\left(\sqrt{2}m\right) - 2\right)^2} \Big\{ 3\left(e^{3\sqrt{2}m_0}\left(\sqrt{2} - 2m_0\right) + 4\sqrt{2}e^{\sqrt{2}m_0} \right. \\ &\quad \left. - 4\sqrt{2}e^{2\sqrt{2}m_0} - 2m_0 - \sqrt{2}\right) \cos\left(\tfrac{m_0}{\sqrt{2}}\right) \\ &\quad + 3e^{\sqrt{2}m_0}\left(2m_0 + e^{\sqrt{2}m_0}\left(2m_0 + \sqrt{2}\right) - \sqrt{2}\right)\cos\left(\tfrac{3m_0}{\sqrt{2}}\right) \\ &\quad - \Big[e^{3\sqrt{2}m_0}\left(3\sqrt{2} - 2m_0\right) + 2m_0 + e^{\sqrt{2}m_0}\left(38m_0 - 9\sqrt{2}\right) - e^{2\sqrt{2}m_0}\left(38m_0 + 9\sqrt{2}\right) \\ &\quad + 2e^{\sqrt{2}m_0}\left(e^{\sqrt{2}m_0}\left(3\sqrt{2} - 2m_0\right) + 2m_0 + 3\sqrt{2}\right)\cos\left(\sqrt{2}m_0\right) + 3\sqrt{2}\Big) \sin\left(\tfrac{m_0}{\sqrt{2}}\right)\Big]\Big\} \\ &\approx 3.4412 \end{aligned}$$

Thus, we know that there is $\delta > 0$ such that $h(s) < 0$ for all $s \in (1-\delta, 1)$. In consequence, for all $\bar{m} > m_0$ close enough to m_0, there exist $\bar{s} \in (0,1)$ satisfying $\frac{\partial^2}{\partial t^2} G_{\bar{m}}(0,\bar{s}) < 0$ and we conclude that there is $\kappa > 0$ for which

$$G_{\bar{m}}(t,\bar{s}) < 0 \quad \text{for all } t \in (0,\kappa).$$

The result holds from Lemma 4.4.14. □

Non-homogeneous boundary conditions

In this subsection we prove the positivity for the operator $T_m = u^{(4)} + m^4 u$ in function spaces where the elements do not necessarily verify the boundary conditions in (4.22). First, we enunciate the following result (the proof follows from a direct computation):

Lemma 4.4.16. *Let h be a continuous function and $a, b, c, d \in \mathbb{R}$. Assume that the boundary value problem*

$$\begin{cases} u^{(4)} + m^4 u = h(x) \\ u(0) = a,\ u(1) = b,\ u'(0) = c,\ u'(1) = d \end{cases} \tag{4.29}$$

has only the trivial solution for $h \equiv 0$ and $a = b = c = d = 0$.

Then (4.29) has a unique solution given by

$$u(x) = \int_0^1 G_m(x,s)h(s)\,ds + a\,w_m(x) + b\,w_m(1-x) + c\,y_m(x) - d\,y_m(1-x),$$

where w_m and y_m are defined respectively as the unique solutions of

$$\begin{cases} u^{(4)} + m^4 u = 0 \\ u(0) = 1,\ u(1) = u'(0) = u'(1) = 0 \end{cases} \tag{4.30}$$

and

$$\begin{cases} u^{(4)} + m^4 u = 0 \\ u'(0) = 1,\ u(0) = u(1) = u'(1) = 0. \end{cases} \tag{4.31}$$

Theorem 4.4.17. *If* $0 < m < \sqrt{2}\pi$, *then*

$$\begin{cases} u^{(4)} + m^4 u \geq 0 \\ u(0) \geq 0,\ u(1) \geq 0,\ u'(0) = u'(1) = 0 \end{cases} \tag{4.32}$$

implies $u \geq 0$.

Proof. Note that $\sqrt{2}\pi < m_0$, so we only need to prove that w_m is positive for $x \in (0,1)$. The explicit expression of w_m is

$$\frac{1}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2}\left[\cos\left(\frac{mx}{\sqrt{2}}\right)\cosh\left(\frac{m(x-2)}{\sqrt{2}}\right) - \sin\left(\frac{mx}{\sqrt{2}}\right)\sinh\left(\frac{m(x-2)}{\sqrt{2}}\right) + \right.$$
$$\left. + \left(\cos\left(\frac{m(x-2)}{\sqrt{2}}\right) - 2\cos\left(\frac{mx}{\sqrt{2}}\right)\right)\cosh\left(\frac{mx}{\sqrt{2}}\right) + \sin\left(\frac{m(x-2)}{\sqrt{2}}\right)\sinh\left(\frac{mx}{\sqrt{2}}\right)\right] \tag{4.33}$$

and

$$w_m'(x) = \frac{\sqrt{2}m}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2}\left[\left(\cosh\left(\frac{mx}{\sqrt{2}}\right) - \cosh\left(\frac{m(x-2)}{\sqrt{2}}\right)\right)\sin\left(\frac{mx}{\sqrt{2}}\right) + \right.$$
$$\left. + \left(\cos\left(\frac{m(x-2)}{\sqrt{2}}\right) - \cos\left(\frac{mx}{\sqrt{2}}\right)\right)\sinh\left(\frac{mx}{\sqrt{2}}\right)\right]. \tag{4.34}$$

It is easy to see that $w_m'(x) < 0$ for $m < \sqrt{2}\pi$ which proves that w_m is positive. Computing the second derivative, we have

$$w_m''(1) = \frac{4m^2 \sin\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2},$$

so for $\sqrt{2}\pi < m < 2\sqrt{2}\pi$ we have $w_m(x) < 0$ for x close enough to 1 and therefore the result is sharp. □

Remark 4.4.18. We just mention without proof that if $u(0) = u(1)$, the previous result can be improved. We have the positivity for $m \leq m_0$, since $z_m(x) = w_m(x) + w_m(1-x)$ is nonnegative for $m \leq m_z$, where $m_z \approx 6,689$ is the least positive solution of equation

$$\tanh\left(\frac{m}{2\sqrt{2}}\right) = -\tan\left(\frac{m}{2\sqrt{2}}\right).$$

The following result concerns the positivity of T_m without the boundary conditions on u' being satisfied.

Theorem 4.4.19. *If* $0 < m \le m_0$, *then*

$$\begin{cases} u^{(4)} + m^4 u \ge 0 \\ u(0) = u(1) = 0,\ u'(0) \ge 0,\ u'(1) \le 0 \end{cases} \tag{4.35}$$

implies $u \ge 0$.

Proof. In this case, for simplicity, we will not present the long expression for $y_m(x)$, but let us remark that this solution is one of the solutions in (4.26) (we have $u(0) = u(1) = 0$ and $u'(0) = 1$). Computing the second derivative, we have that

$$y_m''(1) = \frac{2\sqrt{2}m\left(\cosh\left(\frac{m}{\sqrt{2}}\right)\sin\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right)\sinh\left(\frac{m}{\sqrt{2}}\right)\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2},$$

which is positive for $m < m_0$ and negative for $m > m_0$ close enough to m_0. Since there are no double zeros in $(0,1)$ for $m < m_0$ and for m close enough to 0 we obviously have $y_m(x) > 0$ for $x \in (0,1)$, we conclude that y_m is a positive function for $m < m_0$. The fact that $y_m''(1) < 0$ for $m > m_0$ implies that y_m takes negative values close enough to $x = 1$, which allows us to conclude that the result is sharp.

For the case $u'(0) = -u'(1) \ge 0$, even if $y_m(x) + y_m(1-x)$ is still positive for larger values of m, we cannot improve the conclusion since m_0 is the maximum value for which the Green function is positive. □

Corollary 4.4.20. *If* $0 < m < \sqrt{2}\pi$, *then*

$$\begin{cases} u^{(4)} + m^4 u \ge 0 \\ u(0) \ge 0,\ u(1) \ge 0,\ u'(0) \ge 0,\ u'(1) \le 0 \end{cases} \tag{4.36}$$

implies $u \ge 0$.

Main result

Now we prove the existence of solution for the boundary value problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases} \tag{4.37}$$

Definition 4.4.21. We say that $\alpha \in C^4[0,1]$ is a *lower solution* of (4.37) if

$$\begin{cases} \alpha^{(4)}(x) \le f(x, \alpha(x)), \\ \alpha(0) \le 0,\ \alpha(1) \le 0,\ \alpha'(0) \le 0,\ \alpha'(1) \ge 0. \end{cases} \tag{4.38}$$

We say that $\beta \in C^4[0,1]$ is an *upper solution* of (4.37) if β satisfies the reversed inequalities of the definition of lower solution.

Let us consider the following inequality that will appear later:

$$f(x, \alpha(x)) + k\alpha(x) \le f(x, u) + k\,u \le f(x, \beta(x)) + k\beta(x), \quad \alpha(x) \le u \le \beta(x). \tag{4.39}$$

We now state the main existence result:

Theorem 4.4.22. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and α, β are respectively a lower and an upper solution of* (4.37)*. If $\alpha \leq \beta$ and there exists $0 \leq k \leq 4\pi^4$ for which* (4.39) *holds, then there exists a solution $u(x)$ of* (4.37) *and $\alpha(x) \leq u(x) \leq \beta(x)$.*

Proof. Let $T_k\colon C[0,1] \to C[0,1]$ be the completely continuous operator that such $T_k h(x)$ is the unique solution of

$$\begin{cases} u^{(4)}(x) + k\,u(x) = f(x, h(x)) + k\,h(x), \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{4.40}$$

and consider the usual notation for the functional interval $[u, v] = \{w \in C[0,1]\colon u \leq w \leq v\}$. Given h_1, $h_2 \in C[0,1]$ with $h_1(x) \leq h_2(x)$, then, considering the two correspondent solution $u_i = T_k h_i$ for $i = 1$, 2 and defining $w = u_2 - u_1$, we have

$$\begin{cases} w^{(4)}(x) + k\,w(x) \geq 0, \\ w(0) = w'(0) = w(1) = w'(1) = 0. \end{cases} \tag{4.41}$$

Theorem 4.4.12 implies $w \geq 0$ and hence $u_2 \geq u_1$. Taking in consideration inequality (4.39) and corollary 4.4.20, we can easily check that $\alpha \leq T_k\alpha$, $\beta \geq T_k\beta$ and because T_k is nondecreasing, we also have $T_k[\alpha, \beta] \subset [\alpha, \beta]$. Since $[\alpha, \beta]$ is a convex, closed bounded nonempty set of $C[0,1]$, Schauder's fixed point Theorem implies the existence of a solution of (4.37) in $[\alpha, \beta]$. □

Remark 4.4.23. If we consider upper and lower solutions with extra conditions, we can improve the previous result using the correspondent results from last subsection:

(i) if $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$,

(ii) if $\alpha(0) = \alpha(1)$, $\beta(0) = \beta(1)$ and $\alpha'(0) = \alpha'(1) = \beta'(0) = \beta'(1) = 0$,

(iii) if $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$ and $\alpha'(0) = \alpha'(1) = \beta'(0) = \beta'(1) = 0$,

we can take $0 \leq k \leq {m_0}^4$.

Note that inequality (4.39) with $k < 0$ is always more restrictive than with $k = 0$, so the main theorems that we present here are only consequence of the results obtain in subsection 4.4.2. The positivity for the operator in subsection 4.4.1 was not used in the applications.

Final remarks

With the positivity results and the maximum principles that we obtained, it might be possible to search positive solutions using Krasnoselskii's fixed point Theorem in some appropriate cone, or some other fixed point theorem. Maybe that way it could be possible to introduce a dependence on intermediate derivatives in the nonlinearity. Variational methods may also be used, but that confines us to linear dependence on the second derivative (and no dependence on other intermediate derivatives). Even though, that would be a breakthrough.

For an approach more related to the beam deflection applications, it would be interesting to search for results that provide us solutions satisfying a priori bounds. As an example, we could search for conditions on the nonlinearity implying that solutions satisfy $\|u''\|_\infty \leq k$ for a given constant k. This could show us how to maximize the load of the beam and also to find an optimal distribution of that load under that specific constraint.

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