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APPROXIMATION OF HYPERBOLIC CONSERVATION LAWS

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Abstract. In a first part, we study the zero diffusion-dispersion limit for a class of nonlinear hyperbolic and multi-dimensional conservation laws regularized in a fashion similar to to the Benjamin-Bona-Mahony-Burgers (BBMB) and Korteweg-deVries-Burgers (KdVB) equations. We establish the strong convergence toward classical entropy solutions by relying DiPerna's theory of entropy measure-valued solutions. Optimal conditions are determined for the balance between diffusion and dispersion coefficients. This allows us to propose criteria for the possible existence or non-existence of nonclassical solutions in the sense investigated by LeFloch. Our analysis distinguishes between several assumptions on the diffusion, the dispersion, and the flux-function and emphasize drastic differences between the BBMB and the KdVB models; distinct convergence behaviors are put in evidence and various energy-type arguments are discussed.

In the second part, we study the Riemann problem for nonlinear hyperbolic systems of conservation laws whose flux-function is solely Lipschitz continuous. Typical examples arise in the modelling of multi-phase flows and of elasto-plastic materials. To extend Lax's theory, the main difficulty is to handle possibly discontinuous wave speeds. We revisit certain fundamental notions such as the strict hyperbolicity, the genuine nonlinearity and the entropy inequalities. Our proofs rely on a generalized calculus for Lipschitz continuous mappings and the related Filippov's theory of ordinary differential equations with discontinuous coefficients. We identify here several new features arising in discontinuous solutions of the Riemann problem.

Key words. Hyperbolic conservation law, compressible fluid dynamics, multiphase flows, entropy, diffusive-dispersive regularization, Young measure, measure-valued solution, Riemann problem, shock wave, Lipschitz continuous flux.

Resumo. Numa primeira parte, estudamos o anulamento de limites difusivo--dispersivos para uma classe de leis de conservação hiperbólicas, não-lineares e multidimensionais, regularizadas de modo semelhante às equações de Benjamin-Bona-Mahony-Burgers (BBMB) e Korteweg-deVries-Burgers (KdVB). Prova-se a convergência forte para a solução entrópica clássica, apoiados na teoria de DiPerna das soluções a valores-medida. Determinamos condições optimais para o equilíbrio difusão-dispersão. Isso conduz-nos à proposta de critérios para a eventual existência ou inexistência das soluções não-clássicas investigadas por LeFloch. A nossa análise destrinça várias hipóteses sobre a difusão, a dispersão e a função de fluxo, enfatisando a existência de diferenças drásticas entre os modelos BBMB e KdVB; põem-se em evidência comportamentos limite distintos e discutem-se vários argumentos, de tipo, energia.

Na segunda parte, estudamos o problema de Riemann para sistemas hiperbólicos não-lineares de leis de conservação, cuja função fluxo é somente Lipschitz contínua. Exemplos típicos ocorrem na modelação de fluidos multifásicos ou de materiais elastoplásticos. Ao estendermos a teoria de Lax, a dificuldade essencial assenta no uso de velocidades de onda, possivelmente, descontínuas. Assim, reconsideramos algumas noções fundamentais como sejam as de hiperbolicidade estrita, de genuína não-linearidade e das desigualdades de entropia. As nossas demonstrações apoiam-se num cálculo diferencial generalizado, para funções Lipschitz contínuas, que correlacionamos com a teoria de Filippov para as equações diferenciais ordinárias de coeficientes descontínuos. Identificamos várias novas propriedades das soluções descontínuas do problema de Riemann.

Palavras chave. Leis de conservação hiperbólicas, dinâmica de fluidos compressíveis, fluidos multifásicos, entropia, regularização difusivo-dispersiva, medida de Young, solução a valores-medida, problema de Riemann, ondas de choque, fluxo Lipschitz contínuo.

Preface

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Part I

Introduction

Chapter 1

Resumo

1.1 O problema de Cauchy

O problema de Cauchy para os sistemas *hiperbólicos* não-lineares de $1^{\underline{a}}$ ordem homogéneos (sem *fontes*[1]), de *leis de conservação* de $u(x,t) \in \mathbb{R}^n$, tem a forma vectorial[2] e (multi) d-dimensional:

$$\partial_t u + \operatorname{div} \mathbf{f}(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times \left]0,+\infty\right[, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d, \tag{1.2}$$

onde $\mathbf{f} = (f_j)_{1\le j\le d}$, com coordenadas de *fluxo* $f_j : \mathbb{R}^n \to \mathbb{R}^n$, funções não--lineares.

1.1.1 Soluções descontínuas

Ou, no caso 1-dimensional:

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \qquad (x,t) \in \mathbb{R} \times \left]0,+\infty\right[, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

que se escreve em forma não-conservativa, com matriz jacobiana $\mathrm{D}f(u)$, como

$$\begin{aligned} \partial_t u + \mathrm{D}f(u)\,\partial_x u &= 0, \qquad (x,t) \in \mathbb{R} \times \left]0,+\infty\right[, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned}$$

[1] E, em meio homogéneo: não dependendo explicitamente de (x,t), espaço físico, só de $u(x,t)$, espaço de estados— por oposição a $\mathbf{f} = \mathbf{f}(x,t,u)$.

[2] Em vez de $u(x,t) \in \mathbb{R}^n$, mais geralmente num seu subconjunto de estados aberto e convexo. Usamos a notação $\operatorname{div} \mathbf{f}(u) := \sum_{j=1}^{d} \partial_{x_j} f_j(u)$, com f_j funções vectoriais.

Lembramos que *hiperbolicidade estrita* significa: a matriz real, $n \times n$, $\mathrm{D}f(u)$ tem n valores próprios distintos— as velocidades das linhas características[3].

Pela não-linearidade[4] de f, linhas características correspondentes a valores próprios diferentes intersectam-se, usualmente, em tempo finito, i.e., formam-se descontinuidades— *choques*.

Assim, e já que são soluções *globais* que procuramos, definimos solução-fraca do problema de Cauchy:

Definição 1.1.1. Se $u_0 \in L^q(\mathbb{R}^d)^n$ para algum $1 \le q \le \infty$, então $u \in L^\infty_{loc}\big([0,+\infty[; L^q(\mathbb{R}^d)\big)^n$ diz-se uma solução-fraca de (1.1)-(1.2) quando, para um qualquer vector-teste $\phi = (\phi_k)_{1\le k\le n} \in C_0^\infty\big(\mathbb{R}^d \times [0,+\infty[\big)^n$, se verifica[5]

$$(1.3)\qquad \int_0^{+\infty}\!\!\int_{\mathrm{R}^d} \partial_t\phi^T\, u + \sum_{j=1}^d \partial_{x_j}\phi^T\, f_j(u)\, dxdt + \int_{\mathrm{R}^d} \phi^T(x,0)\, u_0(x)\, dx = 0\,.$$

Em particular, (1.1) verifica-se no sentido das distribuições.

Integrando (1.1), com u solução regular, sobre $\Omega \subset \mathbb{R}^d$ domínio regular de bordo $\partial\Omega$ com normal exterior unitária ν, obtemos o sistema de n equações de equilíbrio $(t > 0)$[6]

$$\frac{d}{dt}\int_\Omega u(.,t)\, dx = -\int_{\partial\Omega} \mathbf{f}(u(.,t))\, \nu\, dS,$$

exprimindo a *conservação* das quantidades de densidades $u \in \mathbb{R}^n$ (cuja variação ao longo do tempo só acontece à custa dos *fluxos* $\mathbf{f}$ através de $\partial\Omega$).

Para u solução regular bilateralmente a uma superfície de descontinuidades atravessando Ω:

Seja $\mathcal{S}$, hipersuperfície suave e orientada de $\Omega\times]0,T[$, com $(n_x, n_t) = (\nu, -s)$ a normal exterior em $(x,t) \in \mathcal{S}$ (que, com $|\nu| = 1$, nos indica a propagação de $\mathcal{S}$ na direcção ν à velocidade s) e

$$\big[u(x,t)\big] = u_+(x,t) - u_-(x,t)\,, \qquad u_\pm(x,t) = \lim_{\epsilon\to 0^+} u\big((x,t) \pm \epsilon(n_x,n_t)\big),$$

o salto da descontinuidade de u em $(x,t) \in \mathcal{S}$. Usando (1.1) e (1.3) deduz-se

[3]Projecções das superfícies características, cf., e.g., John [22].

[4]Mas, independentemente de sua regularidade (bem como da do dado inicial).

[5]Com a notação de 'traço matricial', $\sum_{j=1}^d \partial_{x_j}\phi^T f_j(u) = \mathrm{tr}\Big(\mathrm{D}_x\,\phi^T\,\mathbf{f}(u)\Big)$, onde '$T$' indica a transposição.

[6]$\mathbf{f}$ é a matriz de colunas f_j.

Definição 1.1.2. Relação de Rankine-Hugoniot

$$s\,[u] = [\mathbf{f}(u)]\;\nu\,. \tag{1.4}$$

Soluções deste tipo, em que os pontos (x, t) de seu domínio não acumulam descontinuidades de diferentes superfícies, dir-se-ão seccionalmente regulares[7].

Apesar de, por um lado, esta classe não responder à questão da existência de solução[8] e de, por outro, (1.4) constituir uma restrição implícita em (1.3) às descontinuidades admissíveis (condição de *transmissão*), é fácil exemplificar (vd. Smoller [37] ou Godlewski-Raviart [17]) que, para funções seccionalmente regulares, (1.3) não terá solução única— o problema fundamental!

1.1.2 O problema físico

> "The umbilical cord that joins the theory of systems of hyperbolic conservation laws with continuum physics is still vital for the proper development of the subject and should not be severed.", Dafermos [11]

As leis de conservação hiperbólicas modelam muitos problemas em mecânica do contínuo, física, química, ... Se, na modelação, não se desprezarem os efeitos microscópicos dos mecanismos de difusão e/ou dispersão (e.g., condução térmica e capilaridade), então as equações ficam de tipo "parabólico".

De um modo geral, as soluções de equações hiperbólicas desenvolvem, em tempo finito, descontinuidades enquanto que as soluções de equações parabólicas permanecem regulares.

Assim, à simplificação na modelação[9] contrapõe-se a dificuldade matemática: as soluções-fracas (1.3) não são em geral únicas. Analisando o comportamento limite das equações parabólicas, vistas como aproximações das hiperbólicas, (e.g., o celebrado *"vanishing viscosity method"*), pretendemos seleccionar a solução-fraca *fisicamente relevante*.

Questão: mas, como caracterizar, *directamente*, tal solução-fraca-física?

[7] Claro, aqui o bom (...) espaço funcional, geral, é o das funções de variação limitada, BV, Vol'pert [42, 43].

[8] Embora de 1ª categoria, Dafermos [9], DiPerna [13].

[9] A discretização numérica das equações introduz igualmente dissipação, cf., como exemplo histórico, e.g., Lax[25, p. 608–611].

1.1.3 O problema histórico

Explicitamos alguns momentos da história dos sistemas hiperbólicos de leis de conservação que procuram responder aos problemas atrás levantados.

> "... une longue histoire du côté de la mécanique et de la physique mais... courte du côté des mathématiques.", Tartar [40]

> "Most basic equations of mathematical physics can be written as systems of conservation laws that have a convex extension. This is, for example, the case of the equations of Maxwellian electromagnetism, of elasticity, of the dynamics of compressible fluids in Eulerian form, and of magneto-fluid-dynamics, both nonrelativistic and relativistic.", Friedrichs-Lax [16]

A Entropia

Friedrichs-Lax notam que os exemplos físicos conhecidos têm associada uma *entropia convexa*: as *soluções clássicas* verificam mais *uma* equação conservativa, em $\eta(u)$,

$$\partial_t \eta(u) + \mathrm{div}_x\, q(u) = 0\,, \tag{1.5}$$

com $\eta, q_j : \mathbb{R}^n \to \mathbb{R}$, η função convexa, a entropia, e $q = (q_j)_{1\leq j\leq d}$, o fluxo de entropia. Em escrita não-conservativa[10]

$$\nabla^T\eta\, \partial_t u + \sum_{j=1}^{d} \nabla^T q_j\, \partial_{x_j} u = 0\,,$$

a qual resulta de (1.1), escrita também não-conservativamente

$$\partial_t u + \sum_{j=1}^{d} Df_j(u)\, \partial_{x_j} u = 0\,,$$

sse (sem convexidade) se verificar a condição de compatibilidade do par (η, q) com o fluxo $\mathbf{f}$ de (1.1):

Definição 1.1.3. (η, q) diz-se um par de *entropia-fluxo de entropia* para o sistema (1.1) quando

$$\nabla^T\eta\, Df_j = \nabla^T q_j\,, \qquad \forall 1 \leq j \leq d\,. \tag{1.6}$$

[10]matricial; ∇^T é o gradiente transposto.

Ora, esta condição, nas $d+1$ variáveis $\nabla\eta$ e Dq, é um sistema linear a $n \times d$ equações: sobredeterminado para $n > 2$. Em geral, impossível.

Portanto, a observação de Friedrichs-Lax é, *essencialmente*, de *natureza física!:* há informação, *necessária*, perdida[11] e que torna possível o sistema (1.6) para os exemplos físicos. Qual? Ou seja, é preciso reformular a questão "Em que condições existe uma nova lei de conservação, consequência das anteriores?"

Lembramos ainda, como condição necessária e suficiente— teste— de entropia, a simetria das matrizes $D^2\eta\, Df_j$, com $D^2\eta$ matriz hesseana. Derivando (1.6) em ordem a u_l obtemos

$$\nabla^T(\partial_{u_l}\eta)\ Df_j + \nabla^T\eta\, D(\partial_{u_l} f_j) = \nabla^T(\partial_{u_l} q_j)\ , \quad \forall 1 \le l \le n\,, \quad \forall 1 \le j \le d\,,$$

que, matricialmente, se escreve

$$D^2\eta\, Df_j + \left[\nabla^T\eta\ \partial_{u_k}\partial_{u_l} f_j\right]_{k,l=1}^{n} = D^2 q_j\,, \quad \forall 1 \le j \le d\,,$$

i.e., pela simetria em k, l (sem convexidade), as matrizes $D^2\eta\, Df_j$ são simétricas.

Assim, se a hesseana de η for não-singular, (1.1) fica equivalente, por multiplicação por $D^2\eta$, a um sistema simétrico.

Agora, resulta fácil que a existência de mais uma lei de conservação, a existência de um par de entropia ou a simetrizabilidade do sistema (1.1) se equivalem, sob a única hipótese da hesseana da entropia η ser não-singular. Além disso, entropia convexa equivale a simetrizabilidade à Friedrichs. E, neste caso, (1.1)-(1.2) é para soluções clássicas, localmente, um problema bem-posto.

Como consequência da simetrizabilidade de (1.1) obtemos a sua hiperbolicidade:

Definição 1.1.4. O sistema (1.1) diz-se hiperbólico se a matriz real, $n \times n$,

$$\nu\, \mathbf{Df}(u) := \sum_{j=1}^{d} \nu_j\, Df_j(u)\,,\ \text{com } |\nu| = 1,$$

tem valores próprios reais associados a uma base de $\mathbb{R}^n$ de vectores próprios. Quando a matriz tem n valores próprios distintos o sistema (1.1) diz-se estritamente hiperbólico.

[11] para além da convexidade!

A Entropia Convexa

Mas, para as soluções-fracas globais o que podemos dizer?

As soluções-fracas não verificam em geral a extensão do sistema (1.1) pela nova equação (1.5) porquanto uma solução seccionalmente regular deveria verificar a correspondente relação de Rankine-Hugoniot

$$s\big[\eta(u)\big] - \big[q(u)\big]\nu = 0,$$

que, pela não-linearidade de η, q e $\mathbf{f}$, será, em geral, incompatível com as anteriores (1.4). Na verdade, a equação reescreve-se, usando (1.6) e a fórmula de Taylor para η, q, $\mathbf{f}$, em $u = u_-$, como

$$\nabla^T\eta(u_-)\left(s\big[u\big] - \big[\mathbf{f}(u)\big]\nu\right) = o\big(\big|\big[u\big]\big|\big).$$

Volta-se pois ao problema físico.

Considera-se o sistema (1.1) perturbado

$$\partial_t u_\varepsilon + \operatorname{div}\,\mathbf{f}(u_\varepsilon) = \mathcal{P}\left(\varepsilon, u_\varepsilon\right), \qquad (x,t) \in \mathbb{R}^d \times]0, +\infty[, \tag{1.7}$$

e.g., por um termo de viscosidade destinado a se anular com os parâmetros $\varepsilon \to 0$. Cf. Kružkov [24], Friedrichs-Lax [16]— o método de evanescimento da viscosidade:

O programa é, assumindo a existência de soluções regulares u_ε e sob reminescência do 'integral de energia', multiplicar o sistema (1.7) por $\nabla^T\eta(u_\varepsilon)$, obtendo-se pela definição de entropia

$$\partial_t\eta(u_\varepsilon) + \operatorname{div}\,q(u_\varepsilon) = \nabla^T\eta(u_\varepsilon)\,\mathcal{P}\left(\varepsilon, u_\varepsilon\right)$$

e, com entropia convexa, o segundo membro ora obtido é majorado por um termo *negativo* $o(|\varepsilon|^\alpha)$ com $\alpha > 0$.

Então, sob hipótese das funções contínuas[12] $g : \mathbb{R}^n \to \mathbb{R}$ serem fracamente-contínuas, no sentido das distribuições

$$g(u_\varepsilon) \rightharpoonup g(u), \quad \text{quando} \quad \varepsilon \to 0,$$

deduz-se que o limite u é solução-fraca de (1.1) e verifica a desigualdade de entropia

$$\partial_t\eta(u) + \operatorname{div}\,q(u) \le 0, \tag{1.8}$$

ou seja, para soluções seccionalmente regulares não a igualdade, mas a desigualdade de Rankine-Hugoniot:

[12] de crescimento no infinito compatível com o espaço das soluções-fracas.

$$s\big[\eta(u)\big] - \big[q(u)\big]\nu \geq 0, \tag{1.9}$$

sempre no sentido das distribuições.

Kružkov [24] considerou a viscosidade $\mathcal{P}(\varepsilon, u_\varepsilon) = \varepsilon\Delta(u_\varepsilon)$, $\varepsilon > 0$, conforme à modelação: a difusão artificial $\varepsilon\Delta(u_\varepsilon)$ pretende simular a negligenciada aquando da modelação física,

$$\begin{aligned}\varepsilon\, \nabla^T\eta(u_\varepsilon)\, \Delta(u_\varepsilon) &= \varepsilon\Delta(\eta(u_\varepsilon)) - \varepsilon \sum_{j=1}^{d} \partial_{x_j} u_\varepsilon^T\, D^2\eta(u_\varepsilon)\, \partial_{x_j} u_\varepsilon \\ &\leq \varepsilon\Delta(\eta(u_\varepsilon)) = \varepsilon \operatorname{div}\left(\nabla\eta(u_\varepsilon)\right).\end{aligned}$$

Na dinâmica dos gases obtemos as equações de Euler, sob hipóteses simplificadoras, como o limite das equações de Navier-Stokes, onde a perturbação

$$\mathcal{P}(\varepsilon, u_\varepsilon) = \varepsilon \operatorname{div} P(u_\varepsilon, \nabla u_\varepsilon), \quad P_j(u_\varepsilon, \nabla u_\varepsilon) = \sum_{k=1}^{3} M_{jk}(u_\varepsilon)\, \partial_{x_k} u_\varepsilon,$$

com matrizes $M_{jk}(u_\varepsilon)$, $j, k = 1, 2, 3$, verificando boas propriedades de convexidade, vd. Godlewski-Raviart [17, p.44–46]:

$$\begin{aligned}\nabla^T\eta(u_\varepsilon)\, \mathcal{P}(\varepsilon, u_\varepsilon) &= \varepsilon \sum_{j,k=1}^{3} \nabla^T\eta(u_\varepsilon)\, \partial_{x_j}(M_{jk}(u_\varepsilon)\, \partial_{x_k} u_\varepsilon) \\ &= \varepsilon \sum_{j,k=1}^{3} \partial_{x_j}\big(\nabla^T\eta(u_\varepsilon)\, M_{jk}(u_\varepsilon)\, \partial_{x_k} u_\varepsilon\big) \\ &\qquad -\varepsilon \sum_{j,k=1}^{3} \partial_{x_j} u_\varepsilon^T\, D^2\eta(u_\varepsilon)\, M_{jk}(u_\varepsilon)\, \partial_{x_k} u_\varepsilon \\ &\leq \varepsilon \operatorname{div}\left(\nabla^T\eta(u_\varepsilon) \sum_{k=1}^{3} M_{jk}(u_\varepsilon)\, \partial_{x_k} u_\varepsilon\right)_{1\leq j\leq d}.\end{aligned}$$

Em particular, a segunda lei da termodinâmica é uma desigualdade de entropia (1.8).

Chegou-se assim à proposta de caracterização da "solução-fraca-física":

Definição 1.1.5. Uma solução-fraca do problema de Cauchy (1.1)-(1.2) verificando a desigualdade de entropia (1.8) para todas as entropias convexas diz-se uma solução entrópica.

Kružkov [24], implementando o programa acima, resolveu o problema de Cauchy para o caso escalar multidimensional. Analogamente, Lions-Perthame-Souganidis [30], para a dinâmica dos gases 1-dimensional. Em geral, a dificuldade assenta na (anunciada) escassez de entropias.

1.2 Perturbações difusivo-dispersivas

As perturbações atrás consideradas são difusivas. Nosso propósito: estudar, no caso escalar e multidimensional, aproximações difusivo-dispersivas das leis de conservação hiperbólicas.

Relevância: voltemos a '1.1.2 O problema físico', pág. 5. Consideraremos agora, também, os mecanismos de dispersão. As soluções *fisicamente relevantes* serão as anteriores, dizemos 'soluções-fracas entrópicas *clássicas*', e algumas novas, '*não-clássicas*', mas que ocorrem na prática, em mecânica dos sólidos, ciência dos materiais, ..., cf. Trukinovski [41] e LeFloch [28].

A nosso conhecimento, modelação física *realista* dos termos de difusão e dispersão é quase inexistente. Vulgo, os termos usados são lineares. Na subsecção 2.2.2 expomos a nossa estratégia de abordagem ao modelo difusivo-dispersivo abstracto

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div}(\mathcal{B}) + \delta \operatorname{div}(\mathcal{C}) . \tag{1.10}$$

De seguida, nos capítulos 3–5[13], estudamos os casos específicos de equações de Benjamin-Bona-Mahony-Burgers (BBMB) e Korteweg-deVries-Burgers (KdVB) generalizadas. Em particular, consideramos perturbações dispersivas não-lineares (funções homogéneas: razoáveis pelas hipóteses já necessárias, razoáveis por generalizarem os exemplos conhecidos; possíveis primeiros termos de um desenvolvimento assimptótico de uma perturbação geral).

O objectivo principal é a prova de convergência para a solução-fraca entrópica clássica, que obriga a um regime de predominância da difusão. A principal condição a impor diz respeito ao balanço δ/ε, que estando na fronteira do regime de convergência localizará então a região onde as soluções não-clássicas se podem formar, vd. LeFloch [28].

Tecnicamente, o tratamento das equações multidimensionais faz-se no quadro funcional das medidas de Young em L^p e das soluções a valores-medida, teoria fixada pelos contributos de Tartar-Schonbek-DiPerna-Szepessy ([39, 35, 14, 38]). Revêmo-la na subsecção 2.2.1.

1.2.1 Histórico

1-dimensional

Assinalamos o trabalho pioneiro de Schonbek [35]. A autora trata as equações KdVB e BBMB no caso 1-dimensional com difusão e dispersão lineares. Em particular, introduz as medidas de Young em L^p junto com a correspondente

[13] Versões livres de Correia-LeFloch [5, 6] para o cap. 3, [4] para o cap. 4 e no cap. 5 uma generalização do cap. 3.

extensão da teoria L^∞ de Tartar da compacidade por compensação aplicada às leis de conservação [39]. Prova a convergência, para soluções-fracas (não forçosamente entrópicas).

LeFloch-Natalini [29] desenvolvem outra abordagem, baseados no teorema de unicidade de soluções entrópicas a valores-medida de DiPerna [14] (especificando, numa generalização a L^p do resultado de DiPerna obtida por Szepessy [38]). Recuperam, para KdVB com difusão não-linear e dispersão linear, a convergência para as soluções-fracas entrópicas.

Hayes-LeFloch [18, 19] tratam então o caso limite do balanço entre a difusão e a dispersão na fronteira do regime de convergência, iniciando a discussão sobre as soluções não-clássicas, cf. LeFloch [28].

d-dimensional

Correia-LeFloch [5, 6] (cap. 3) tratam pela primeira vez o caso d-dimensional ($d > 1$) para a equação KdVB generalizada. Prova-se convergência para a solução-fraca entrópica, nos casos de difusão linear ou não-linear e dispersão linear.

Kondo-LeFoch [23] provam que o resultado anterior é óptimo para o caso da perturbação linear e com fluxo de crescimento no infinito quanto muito linear.

Em [4] (cap. 4) prova-se a convergência para a solução-fraca entrópica nas equações multidimensionais KdVB e BBMB generalizadas. A difusão é não-linear, a dispersão pode (pela primeira vez) ser linear ou não.

No capítulo 5, optimiza-se a generalização, quer da equação KdVB multidimensional, quer das hipóteses sobre a difusão não-linear e a dispersão linear ou não-linear.

1.2.2 Resultados

Se, na equação (1.10), fizermos $\varepsilon = 0$, obtemos uma equação tipo KdV. As suas soluções, enquanto $\delta \to 0$, tornam-se cada vez mais oscilantes: não convergem, Lax-Levermore [27]. Alternativamente, se fizermos $\delta = 0$, a equação (1.10) fica parabólica, semelhante à equação de Burgers ou à aproximação pseudo-viscosa de von Neumann e Richtmyer [44]. Agora as soluções aproximadas convergem-forte para a solução-fraca entrópica clássica, veja-se Marcati e Natalini [32].

Portanto, no caso geral, para se assegurar convergência, quando $\varepsilon, \delta \to 0$, precisamos estar num regime de predominância da difusão. Isto é garantido de dois modos, pelo balanço δ/ε e pela competição entre os crescimentos da difusão e da dispersão no infinito. Em particular, a convergência para soluções não-clássicas só ocorre para um equilíbrio δ/ε na fronteira da região de convergência.

O balanço difusão-dispersão constitui a principal condição a impor na prova de convergência nos nossos resultados. A abordagem assenta pois na procura e uso de métodos de energia gerais (e sob hipóteses minimais para a difusão e dispersão).

No capítulo 3, *essencialmente*, substituimos os argumentos 1-dimensionais de Schonbek e seguidos por LeFloch-Natalini, por novos. Podemos então tratar o caso multidimensional. Adicionalmente, suprimimos a interdependência entre o crescimento m do fluxo (no infinito) e o espaço L^q onde se prova a convergência. Podemos usar L^q arbitrariamente grande, só conforme ao dado inicial e $m \geq 1$ sem restrições. A difusão é linear ou não-linear e a dispersão é linear. Quanto ao balanço δ/ε, as nossas provas necessitam duma condição $\delta = o(\varepsilon^\gamma)$, $(\gamma > 0)$. Esperamos seja óptima, i.e., que a fronteira onde as soluções não-clássicas se podem formar corresponda a um equilíbrio $\delta = \mathcal{O}(\varepsilon^\gamma)$ (comprovado por Kondo-LeFoch [23] para o caso da perturbação linear e $m = 1$).

No capítulo 4 estudamos as equações KdVB e BBMB generalizadas e com difusão não-linear, dispersão linear ou não-linear. Procura-se compreender a competição entre os crescimentos (no infinito) da difusão e da dispersão (intervenientes no balanço δ/ε), tal como as diferenças entre os modelos (alternativos) KdVB e BBMB. No caso da equação BBMB aquela competição envolve ainda o crescimento m do fluxo (consequência de uma estimativa adicional), o que em particular fixa o espaço L^q onde a convergência é provada e restringe os resultados para valores de m tais que $1 \leq a \leq m \leq b$, onde a e b são função dos expoentes de crescimento da difusão e da dispersão. Como para a equação KdVB a situação é inversa, ter-se-á aqui uma diferença importante entre ambas as equações. Em particular, a troca, dita de Whitham, entre derivadas em tempo e em espaço nos termos dispersivos precisará ser devidamente avaliada. Quanto ao balanço δ/ε, ocorrem, para BBMB, alguns casos em que a nossa condição é agora $\delta = \mathcal{O}(\varepsilon^\gamma)$. Se a nossa estimativa for óptima, então, ou não existem, nestes casos, soluções não-clássicas, ou ter-se-á uma banda-fronteira onde elas se poderão formar (fenómeno novo).

No capítulo 5 o modelo KdVB tal como as respectivas hipóteses sobre a difusão e a dispersão são generalizados. Para simplificar, mantendo a "generalidade", assumimos a difusão não-linear, mas a dispersão é linear ou não-linear.

Todos os resultados anteriores são generalizados. Reobtemos uma condição $\delta = o(\varepsilon^\gamma)$, que esperamos seja óptima, e sob a qual a convergência se revela competição exclusiva entre o crescimento da difusão e da dispersão—

independente do fluxo. Isto é consequência do domínio da difusão não-linear. Em particular, os espaços de convergência podem ser escolhidos com q arbitrariamente grande, só sujeito ao dado inicial, e o crescimento do fluxo com expoente $m \geq 1$. (Pelo teorema de representação de Schonbek, "escolhidos" tal que $q > m$.)

1.3 O problema de Riemann

Vamo-nos agora dedicar ao problema de Riemann, Lax [26]. Ou seja, ao problema de Cauchy para o sistema 1-dimensional

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U}, \quad x \in \mathbb{R},\, t > 0, \tag{1.11}$$

com o dado inicial seccionalmente constante

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{1.12}$$

onde u_l, u_r pertencem a $\mathcal{U} := \mathcal{B}(u_*, \delta) \subset \mathbb{R}^N$, a bola de centro u_* e raio δ (suficientemente pequeno). Assumimos que a função $f : \mathcal{U} \to \mathbb{R}^N$ é Lipschitz contínua com matriz Jacobiana estritamente hiperbólica e genuinamente não-linear. Esta parte corresponde ao artigo de Correia-LeFloch-Thanh [7].

Em [26], Lax constrói a solução-fraca entrópica assumindo que o fluxo, f, é pelo menos de classe C^2. A dificuldade na extensão da teoria de Lax, ao caso do fluxo não-regular, coloca-se no uso das *velocidades de onda*, possivelmente, *descontínuas*. Em particular, precisamos generalizar as definições de *hiperbolicidade estrita*, de *não-linearidade genuína* e das *desigualdades de entropia*.

Tecnicamente, apoiamo-nos num cálculo diferencial generalizado para funções Lipschitz contínuas (que revêmos na secção 2.3). Uma derivada generalizada é um *conjunto de vectores* (amiúde não singular). Além disso, correlacionamos este cálculo com a teoria de Filippov [15] para as equações diferenciais ordinárias de coeficientes descontínuos, ver também Hörmander [20].

A modelação matemática de muitos problemas em dinâmica dos fluidos e ciência dos materiais conduz frequentemente aos sistemas hiperbólicos não-lineares de leis de conservação. As equações diferenciais parciais são "fechadas" por *relações constituintes* que modelam o comportamento do meio físico considerado, sendo o *fluxo* de cada lei de conservação descrito à custa das variáveis *conservativas*.

Ora, com frequência, as relações constituintes tomam formas diferentes em diferentes domínios das variáveis conservativas. Exemplos típicos ocorrem na modelação de fluidos multifásicos ou de materiais elastoplásticos. E.g., um material sólido pode comportar-se de modo diferente quando a sua densidade ultrapassa certo valor crítico. Por outro lado, as relações constituintes costumam ser obtidas por experimentação. Assim, os sistemas hiperbólicos de interesse prático terão fluxos que são funções, só, Lipschitz contínuas: perdem a regularidade habitualmente assumida na teoria matemática das leis de conservação.

Recordamos que o problema de Riemann tem um lugar fundamental nesta teoria. Em particular, revela informação importante acerca das soluções do problema de Cauchy geral para (1.11). O problema de Riemann é a base de muitas das aproximações numéricas (Godunov scheme, random choice method, front tracking algorithm, ...). Assim motivamos uma resolução *directa* do problema.

O propósito é o de identificar os novos fenómenos que ocorrem nas soluções descontínuas dos sistemas de leis de conservação com fluxo Lipschitz contínuo.

Na secção 6.2 trataremos o caso escalar, particularmente simples, mas interessante por exibir já o novo comportamento qualitativo das ondas de choque e de rarefacção associadas a velocidades de onda descontínuas.

A secção 6.3 contém a teoria geral de existência para sistemas do problema de Riemann (6.1)-(6.2). As soluções verificam uma generalização das *desigualdades de entropia* de Lax. Notamos, porém, que por falta de regularidade do fluxo, ainda que impondo as desigualdades de entropia, o problema de Riemann poderá não ter solução única.

Finalmente, na secção 6.4, estudamos um exemplo concreto que ocorre em dinâmica dos fluidos.

Para além das referências citadas socorremo-nos ainda da seguinte bibliografia: Courant-Friedrichs [8], Whitham [45], Majda [31], Hsiao [21], Tartar [40], Dafermos [12] e Serre [36].

Chapter 2

Introduction

2.1 A First Glance

We are concerned with nonlinear hyperbolic conservation laws. Namely, the Cauchy problem

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d, \tag{2.2}$$

where the unknown function $u = u(x,t) \in \mathbb{R}^N$ is scalar- or vector-valued and the flux $f : \mathbb{R}^N \to \mathbb{R}^d$ is a given function.

Nonlinear hyperbolic conservation laws arise in the modeling of many problems from continuum mechanics, physics, chemistry, etc. If small scale mechanisms of diffusion and dispersion are taken into account, e.g., heat conduction and capillarity in fluids, the equations become "parabolic". From a general standpoint, hyperbolic equations admit discontinuous solutions while parabolic equations have smooth solutions. Discontinuous solutions, understood in the generalized sense of the distribution theory, are usually non-unique. It is therefore fundamental to understand which solutions are selected by a specific zero diffusion-dispersion limit.

The *vanishing viscosity method* addressed this issue for multi-dimensional scalar conservation laws ($N = 1$ and arbitrary d in the Cauchy problem above) where solely a (linear) diffusion is considered, and has conducted to the definition of classical entropy weak solution: smooth solutions to (2.1) also satisfy an infinite list of additional conservation laws

$$\partial_t \eta(u) + \operatorname{div}\, q(u) = 0, \qquad q' = \eta' f', \tag{2.3}$$

where η is a convex function of u. For discontinuous solutions, Kružkov [24] shows that (2.3) should be replaced by the set of inequalities

$$\partial_t \eta(u) + \operatorname{div}\, q(u) \leq 0, \tag{2.4}$$

which must select physically meaningful discontinuous solutions. The condition (2.4) is called an entropy inequality; it is motivated by the second law of thermodynamics, in the context of gas dynamics.

By definition, an entropy weak solution of the Cauchy problem satisfies (2.1)-(2.2) in the sense of distributions, and additionally (2.4) for any entropy pair (η, q) with convex function η.

Notice that nonclassical solutions have relevant applications, e.g., in material science; see LeFloch [28].

Here, from chapter 3 to chapter 5[1], we consider the *zero diffusion-dispersion limit* for the multi-dimensional scalar conservation laws. Say, as illustration, our first case studied, [5, 6]:

We consider the approximation of (2.1)-(2.2) obtained by adding to the right-hand side of (2.1) a linear or nonlinear diffusion term, $b : \mathbb{R}^d \to \mathbb{R}^d$, plus a linear dispersion term, and approximating the initial data u_0 in (2.2) by $u_0^{\varepsilon,\delta}$, where ε, δ (> 0) are vanishing parameters

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div} \Big(\varepsilon\, b_j(\nabla u) + \delta\, \partial_{x_j}^2 u\Big)_{1 \le j \le d}, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{2.5}$$

$$u(x,0) = u_0^{\varepsilon,\delta}(x), \qquad x \in \mathbb{R}^d. \tag{2.6}$$

The main objective is to derive conditions under which, as ε and δ tend to zero, the solutions $u^{\varepsilon,\delta}$ still converge, in a strong topology, to the entropy weak solution of (2.1)-(2.2).

When $\varepsilon = 0$, equation (2.5) is a generalized version of the well-known Korteweg-deVries (KdV) equation, the solutions become more and more oscillatory as $\delta \to 0$. Approximate solutions do not converge, see Lax and Levermore [27]. When $\delta = 0$, (2.5) reduces to a nonlinear parabolic equation. Like for the Burgers equation or the pseudo-viscosity approximation of von Neumann and Richtmyer [44], the approximate solution converges strongly to the entropy weak solution, see Marcati and Natalini [32].

Therefore, to ensure the convergence of the zero diffusion-dispersion approximation (2.5)-(2.6), we must be in the dominant diffusion regime, that is diffusion overcomes dispersion. Indeed our main result establishes that, under rather broad assumptions, the solution of (2.5)-(2.6) tends to the entropy weak solution of (2.1)-(2.2) when $\varepsilon, \delta \to 0$ with $\delta = o(\varepsilon^\gamma)$, $\gamma > 0$. Then, nonclassical solutions should rely upon the frontier of that regime, given by an optimal δ/ε balance $\mathcal{O}(\varepsilon^\gamma)$, if the result is sharp. In particular, convergence results in this regime cannot be obtained by the measure-valued solutions approach that we apply. (The situation seems to be distinct for some cases

[1] As free versions of Correia and LeFloch [5, 6] for chap. 3, [4] for chap. 4 and an unpublished generalization of chap. 3 in chap. 5.

of the BBMB equation.) Another way diffusion can dominate is by growth competition. This will, possibly, also involve the flux growth and in turn force the use of an L^q space where convergence can be established. This also, leads to understand differences between the BBMB and the KdVB models (e.g., a different convergence behaviour tells us that we must be careful about "Whitham's" changes between time and space derivatives).

So, the emphasis is on general energy arguments.

Now, we review previous work on the subject restricted to one-dimensional equations ($N = 1$ and $d = 1$ in (2.1)-(2.2)).

The pioneer paper by Schonbek [35], where, in particular, the concept of L^p Young measures is introduced together with an extension of Tartar's compensated compactness method for conservation laws, treats the case of linear diffusion and linear dispersion for the so-called Benjamin-Bona-Mahony-Burgers (BBMB) and Korteweg-deVries-Burgers (KdVB) models. She proves convergence to (not necessarily entropy) weak solutions.

LeFloch and Natalini [29] developed another approach based on DiPerna's uniqueness theorem for entropy measure-valued solutions [14], specifically a generalization of DiPerna's result to L^p functions, due to Szepessy [38]. They manage to obtain convergence to classical entropy weak solutions of KdVB with linear dispersion and nonlinear diffusion. That method of proof was successful first in proving convergence of finite difference schemes. We refer to Szepessy ([38] and the references therein by Szepessy and co-authors) and Coquel and LeFloch [3].

Hayes and LeFloch [18, 19] treat the transitional case where both terms in KdVB, the diffusion and the dispersion, are in balance. This began the discussion around the nonclassical solutions, see LeFloch [28].

Correia and LeFloch [5, 6] solve for the first time a *multi-dimensional case*: for the generalized and multi-dimensional KdVB equation with linear or nonlinear diffusion and linear dispersion. Kondo and LeFoch [23] prove sharpness for the case of linear perturbations and a flux-function with at most linear growth at infinity.

In [4] we study, both, the BBMB and the KdVB multi-dimensional generalized equations with nonlinear diffusion and (linear or) also *nonlinear dispersion.*

Finally, in chapter 5 we strengthen the generalization of the KdVB model as well as the assumptions on the nonlinear diffusion and (general) linear or nonlinear dispersion.

In the next section 2.2 we review the functional setting (and main tool, on measure-valued solutions) that we use in our convergence proofs, and then, we explain our general strategy of approach to both models.

Let us now comment on the other approach we make of (2.1)-(2.2): for one-dimensional systems of conservation laws (N arbitrary and $d = 1$), as given by Correia-LeFloch-Thanh [7].

The mathematical modeling of many problems in fluid dynamics and material science often leads to nonlinear hyperbolic system of conservation laws. Such systems consist of nonlinear partial differential equations supplemented with constitutive relations describing the behaviour of the specific medium under consideration. The "flux" of each conservation law is expressed in terms of the "conservative" variables. Quite often in the applications, the constitutive relations have different forms in different ranges of values of the conservative variables. Typical examples are found in the modeling of multiphase flows and of elasto-plastic materials. A solid material, for instance, may have a different behaviour when its density exceeds some critical value. On the other hand, the constitutive relations must often be determined by experiments. In turn, the hyperbolic systems of interest in the applications admit flux-functions which are solely Lipschitz continuous and lack the differentiability property which is customarily assumed in the mathematical theory of conservation laws.

Our general objective is to identify new features arising in discontinuous solutions of system of conservation laws with Lipschitz continuous flux. Here, we will focus attention on the so-called Riemann problem (Lax [26]) for the strictly hyperbolic system

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U}, \quad x \in \mathbb{R},\, t > 0, \tag{2.7}$$

supplemented with the piecewise constant initial condition

$$u(x,0) = \begin{cases} u_l, & x < 0; \\ u_r, & x > 0. \end{cases} \tag{2.8}$$

We assume that the data u_l, u_r belong to $\mathcal{U} := \mathcal{B}(u_*, \delta) \subset \mathbb{R}^N$, the ball with center u_* and (small) radius δ. The function $f : \mathcal{U} \to \mathbb{R}^N$ is assumed to be Lipschitz continuous and the Jacobian matrix Df to be strictly hyperbolic. Each characteristic field of Df will be assumed to be genuinely nonlinear. (Since the flux is not smooth, these notions have to be reconsidered; see the beginning of section 6.3.)

Discontinuous solutions of (2.7) satisfying an entropy condition (required for uniqueness) will be sought. Recall that the Riemann problem plays a fundamental role within the theory of conservation laws and yields many interesting informations on general solutions of (2.7). It is the basis to develop a large class of numerical schemes (Godunov scheme, random choice method, front tracking algorithm, ...). By assuming f to be at least of class C^2 and δ sufficiently small, Lax [26] constructed the entropy solution of the Riemann problem (2.7)-(2.8). To extend Lax's theory to a Lipschitz continuous f, the difficulty is to handle possibly *discontinuous wave speeds.* We will rely here on a generalized calculus for Lipschitz continuous mappings (a brief review is presented in the section 2.3). A generalized derivative is a *set of vectors* rather than a single-valued function. We will also rely on the (related) theory developed earlier by Filippov [15] for ordinary differential equations with discontinuous coefficients, see also Hörmander [20].

2.2 Multi-Dimensional Scalar Equations

Here, we first review the L^p-Young measure functional setting established by Tartar-Schonbek-DiPerna-Szepessy ([39, 35, 14, 38]). In particular, we state the main tool that we use in our convergence proofs. Next, we explain the general strategy of our approach.

2.2.1 Entropy Measure-Valued Solutions

Here, we review basic material on Young measures and entropy measure-valued (e.m.-v.) solutions for conservation laws.

We begin with Schonbek's representation theorem, [35], for the Young measures associated with a sequence uniformly bounded in L^q, a generalization of the L^∞ setting first established by Tartar, [39].

Along this subsection, we suppose $1 < q < +\infty$ and $T \leq +\infty$ are fixed, Prob ($\mathbb{R}$) is the space of probability measures (non-negative measures with unit total mass).

Lemma 2.2.1. *Let $\{u_n\}$ be a bounded sequence in $L^\infty((0,T); L^q(\mathbb{R}^d))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}$ and a weakly-$\star$ measurable mapping $\nu : \mathbb{R}^d \times (0,T) \to$ Prob ($\mathbb{R}$) such that, for all functions $g \in C(\mathbb{R})$ satisfying*

$$g(u) = \mathcal{O}(|u|^m) \quad \text{as } |u| \to \infty, \qquad \textit{for some } m \in [0,q), \tag{2.9}$$

$\langle \nu_{(x,t)}, g\rangle$ belongs to $L^\infty((0,T); L^{q/m}_{loc}(\mathbb{R}^d))$ and the following limit representation holds

$$\lim_{n\to\infty} \iint_{\mathbb{R}^d \times (0,T)} g(\tilde{u}_n(x,t))\, \phi(x,t)\ dxdt \tag{2.10}$$

$$= \iint_{\mathbb{R}^d \times (0,T)} \langle \nu_{(x,t)}, g \rangle \ \phi(x,t) \ dxdt$$

for all $\phi \in L^1(\mathbb{R}^d \times (0,T)) \cap L^\infty(\mathbb{R}^d \times (0,T))$.

Conversely, given ν, *there exists a sequence* $\{u_n\}$ *satisfying the same conditions as above and such that* (2.10) *holds for any* g *satisfying* (2.9).

We use the notation $\langle \nu_{(x,t)}, g \rangle := \int_{\mathbb{R}} g(u)\, d\nu_{(x,t)}(u)$. Then, 'weak-$\star$ measurable' means that the real-valued function $\langle \nu_{(x,t)}, g \rangle$ is measurable with respect to (x,t) for each continuous g satisfying (2.9). The measure-valued function $\nu_{(\cdot)}$ is called a Young measure associated with the sequence $\{\tilde{u}_n\}$. As simple example we have the Dirac mass $\delta_{u(\cdot)}$ defined by

$$\langle \delta_{u(x,t)}, g \rangle = g(u(x,t)), \quad \text{for all } g \in C(\mathbb{R}) \text{ satisfying (2.9).}$$

The following result reveals the connection between the structure of ν and the strong convergence of the *subsequence*.

Lemma 2.2.2. *Suppose that* ν *is a Young measure associated with a sequence* $\{\tilde{u}_n\}$, *bounded in* $L^\infty((0,T); L^q(\mathbb{R}^d))$. *For* $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$, *the following statements are equivalent:*

(i) $\lim_{n\to\infty} \tilde{u}_n = u$ *in* $L^s((0,T); L^p_{\text{loc}}(\mathbb{R}^d))$, *for all* $s < \infty$ *and* $p \in [1,q)$;

(ii) $\nu_{(x,t)} = \delta_{u(x,t)}$, *for a.e.* $(x,t) \in \mathbb{R}^d \times (0,T)$.

Following DiPerna [14] and Szepessy [38] (for a generalization of DiPerna's result to L^p functions), we define a very weak notion of entropy solution to the hyperbolic first order Cauchy problem

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times [0, +\infty[\,, \tag{2.11}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{2.12}$$

Definition 2.2.1. Assume that $f \in C(\mathbb{R})^d$ satisfies the growth condition (2.9) and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. A Young measure ν associated with a bounded sequence $\{\tilde{u}_n\}$ in $L^\infty((0,T); L^q(\mathbb{R}^d))$ is called an entropy measure-valued (e.m.-v.) solution to (2.11)-(2.12) if

$$\partial_t \langle \nu_{(\cdot)}, |u-k| \rangle + \operatorname{div} \langle \nu_{(\cdot)}, \operatorname{sgn}(u-k)(f(u)-f(k)) \rangle \le 0, \tag{2.13}$$

for all $k \in \mathbb{R}$, in the sense of distributions on $\mathbb{R}^d \times (0,T)$; and

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \, \rangle \, dxds = 0, \tag{2.14}$$

for all compact set $K \subseteq \mathbb{R}^d$.

A function $u \in L^\infty((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d))$ is an entropy weak solution to (2.11)-(2.12) in the sense of Kružkov [24] and Volpert [42] if and only if the Dirac measure $\delta_{u(\cdot)}$ is an e.m.-v. solution. In the case $q = +\infty$, existence and uniqueness of such solutions were proved in [24]. The following results on e.m.-v. solutions were proved in [38]: Proposition 2.2.1 states that e.m.-v. solutions are actually Kružkov's solutions. Proposition 2.2.2 states that the problem has a unique solution in L^q.

Proposition 2.2.1. *Assume that f satisfies* (2.9) *and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Suppose that ν is an e.m.-v. solution to* (2.11)-(2.12). *Then there exists a function $u \in L^\infty((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d))$ such that*

$$\nu_{(x,t)} = \delta_{u(x,t)}, \qquad \text{for a.e. } (x,t) \in \mathbb{R}^d \times (0,T).$$

Proposition 2.2.2. *Assume that f satisfies* (2.9) *and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Then there exists a unique entropy solution $u \in L^\infty((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d))$ to* (2.11)-(2.12) *which, moreover, satisfies*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}, \qquad \text{for a.e. } t \in (0,T) \text{ and all } p \in [1,q].$$

The measure-valued mapping $\nu_{(x,t)} = \delta_{u(x,t)}$ is the unique e.m.-v. solution of the same problem.

Combining Propositions 2.2.1 and 2.2.2 and Lemma 2.2.2, we obtain the main convergence tool we will use. It is the result of the L^p-Young measure functional analysis setting as given by Tartar-Schonbek-DiPerna-Szepessy's theory.

Corollary 2.2.1. *Assume that f satisfies* (2.9) *and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $q > 1$. Let be $\{u_n\}$ a bounded sequence in $L^\infty((0,T); L^q(\mathbb{R}^d))$ with associated Young measure ν. If ν is an e.m.-v. solution to* (2.11)-(2.12), *then*

$$\lim_{n\to\infty} u_n = u \quad \text{in } L^s((0,T); L^p_{loc}(\mathbb{R}^d)), \quad \forall s < \infty,\ p \in [1,q),$$

$u \in L^\infty((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d))$ is the unique entropy solution to (2.11)-(2.12).

2.2.2 The Model

Consider an equation with abstract ε-diffusive and δ-dispersive terms; we omit the superscripts ε, δ in the solution notation $u^{\varepsilon,\delta}$:

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div}(\mathcal{B}) + \delta \operatorname{div}(\mathcal{C}). \tag{2.15}$$

Multiplying by $\eta'(u)$, where $\eta : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function and $q : \mathbb{R} \to \mathbb{R}^d$ is defined by $q_j' = \eta'\, f_j'$, $j = 1, \ldots, d$, we have

$$\begin{aligned}\partial_t \eta(u) + \text{div}\ q(u) &= \varepsilon\ \text{div}\,\big(\eta'(u)\ \mathcal{B}\big) - \varepsilon\, \eta''(u)\ \ \nabla u \cdot \mathcal{B} \\ &\quad + \delta\ \text{div}\,\big(\eta'(u)\ \mathcal{C}\big) - \delta\, \eta''(u)\ \ \nabla u \cdot \mathcal{C}.\end{aligned}$$

Integrating over $[0, t]$ and $\mathbb{R}^d$, assuming that u together with its space-derivatives are zero at infinity[2]:

$$\int_{\mathrm{R}^d} \eta\,(u(t)) - \eta\,(u_0)\ dx = -\int_{\mathrm{R}^d}\int_0^t \eta''(u)\ \ \nabla u \cdot (\varepsilon\,\mathcal{B} + \delta\,\mathcal{C})\ dsdx,$$

with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$, so $\eta'(u) = \text{sgn}(u)|u|^{\alpha}$ and $\eta''(u) = \alpha|u|^{\alpha-1}$, we deduce

Lemma 2.2.3. *Let $\alpha \geq 1$ be any real and suppose that $u_0 \in L^{\alpha+1}\big(\mathbb{R}^d\big)$. Any solution of* (2.15) *satisfies, for $t \in [0, T]$,*

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx &= \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx \\ &\quad - (\alpha+1)\,\alpha \int_{\mathrm{R}^d}\int_0^t |u|^{\alpha-1}\ \nabla u \cdot (\varepsilon\,\mathcal{B} + \delta\,\mathcal{C})\ dsdx.\end{aligned}$$

Proposition 2.2.3. *For any solution of* (2.15) *such that diffusion satisfy $\varepsilon\ \nabla u \cdot \mathcal{B} \geq 0$ and the dispersion $\mathcal{C}$ is arbitrary, if $u_0 \in L^2\big(\mathbb{R}^d\big) \cap L^q\big(\mathbb{R}^d\big)$ we have, with $2 \leq \alpha + 1 \leq q$ and $t \in [0, T]$,*

$$\delta \int_{\mathrm{R}^d}\int_0^t |u|^{\alpha-1}\ \nabla u \cdot \mathcal{C}\, dsdx \leq const.\, \|u_0\|^q_{L^q(\mathrm{R}^d)}.$$

If also $\delta\ \nabla u \cdot \mathcal{C} \geq 0$, then

$$\|u(t)\|_{L^{\alpha+1}(\mathrm{R}^d)} \leq const.\, \|u_0\|_{L^q(\mathrm{R}^d)};$$

$$|\,\delta\,| \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\ |\,\nabla u \cdot \mathcal{C}\,|\ dxds \leq const.\, \|u_0\|^q_{L^q(\mathrm{R}^d)}.$$

Our purpose is to bound u in $L^\infty\left((0,T); L^q\big(\mathbb{R}^d\big)\right)$. So, since the $\delta\ \nabla u \cdot \mathcal{C} \geq 0$ hypothesis is a unreasonable one in general, we need to prevent the dispersive δ-integral terms to go to $-\infty$. See the first case we studied, [5]:

[2]In such a way that divergence terms are conservative: with null space integrals. Tacitly, we suppose integrability of the default terms.

Take the simple linear dispersion $\mathcal{C} = \left(\partial^2_{x_j} u\right)_{1\le j\le d}$, therefore we have that $\nabla u \cdot \mathcal{C} = \sum_j \partial_{x_j} u\, \partial^2_{x_j} u$ has no sign.

Still, because $\nabla u \cdot \mathcal{C} = \frac{1}{2}\sum_j \partial_{x_j}\left(\partial_{x_j} u\right)^2$,

$$\delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1}\, \nabla u \cdot \mathcal{C}\, dxds \;=\; \delta/2 \int_0^t \int_{\mathbb{R}^d} \sum_j |u|^{\alpha-1}\, \partial_{x_j}\left(\partial_{x_j} u\right)^2 dxds,$$

from which, returning back to Lemma 2.2.3 with $\alpha = 1$, we obtain the first energy estimates:

Proposition 2.2.4. *For any solution of* (2.15) *with the trivial linear dispersion* $\mathcal{C} = \left(\partial^2_{x_j} u\right)_{1\le j\le d}$, *if* $u_0 \in L^2\left(\mathbb{R}^d\right)$ *and* $t \in [0,T]$, *we have*

$$\int_{\mathbb{R}^d} u(t)^2\, dx = \int_{\mathbb{R}^d} u_0^2\, dx - 2\,\varepsilon \int_{\mathbb{R}^d}\int_0^t \nabla u \cdot \mathcal{B}\, dsdx.$$

Assuming $\varepsilon > 0$ *and the diffusion hypothesis*

$$\exists r \ge 0,\; D > 0 : \quad \nabla u \cdot \mathcal{B} \ge D\, |\nabla u|^{r+1},$$

we have also

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^d)} &\le \|u_0\|_{L^2(\mathbb{R}^d)}; \\ 2\, D\, \varepsilon \int_0^t \int_{\mathbb{R}^d} |\nabla u|^{r+1}\, dxds &\le 2\,\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla u \cdot \mathcal{B}\, dxds \\ &\le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This is a consequence of the conservative structure of the δ-integrand. We generalize, as a version of our paper [5], to the linear case in subsections 5.3.1 and 5.4.1 with $a_{jl} \in \mathbb{R}$, $j, l = 1, \ldots, d$,

$$\mathcal{C} = \left(\sum_l a_{jl}\, \partial^2_{x_j x_l} u\right)_{1\le j\le d}.$$

In fact, we could have studied the general, "nondiagonal", linear case

$$\mathcal{C} = \left(\sum_{k,l} a_{jlk}\, \partial^2_{x_k x_l} u\right)_{1\le j\le d},$$

but it is a particular instance of the linear or nonlinear one:

$$\mathcal{C} = \left(\sum_k \partial_{x_k} c_{jk}\big(\nabla u\big) \right)_{1 \le j \le d},$$

which we analyse in subsections 5.3.2 and 5.4.2 subject to the hypothesis of existence of a potential $C = \big(C_j\big)_{1 \le j \le d}$ with jacobian[3] matrix $[c_{jk}]$. Trivial nonlinear examples are those of the form $c_{jk}\big(\nabla u\big) = c_{jk}\big(\partial_{x_j} u\big)$.

Concerning higher L^q energy estimates, we will see, subsection 5.4.2, it is a matter of competitive growths between $\nabla u \cdot \mathcal{B}$ and $\nabla u \cdot \mathcal{C}$.

2.3 One-Dimensional Systems

Finally, let us review a generalized calculus for Lipschitz continuous mappings, Clarke [1, 2] and Pourciau [34]. We will refer to Clarke [2].

2.3.1 Generalized Gradients

Let us recall here the notion of generalized gradients for Lipschitz continuous mappings and some fundamental results we will need. We follow closely the presentation in Clarke [2].

The ball in $\mathbb{R}^N$ with center u and radius r is denoted by $\mathcal{B}_N(u, r)$. By definition, given an open subset $\mathcal{U} \subset \mathbb{R}^N$, a vector-valued mapping

$$f : \mathcal{U} \to \mathbb{R}^M, \qquad f(u) = \big(f^1(u), f^2(u), \ldots, f^M(u)\big)$$

is k-Lipschitz continuous on the set $\mathcal{U}$ if

$$|f(u) - f(u')| \le k|u - u'|, \qquad u, u' \in \mathcal{U}. \tag{2.16}$$

It is k-Lipschitz continuous near some point u if, for some small $\epsilon > 0$ such that the ball $\mathcal{B}_N(u, \epsilon)$ is contained in $\mathcal{U}$, the function f is k-Lipschitz continuous on $\mathcal{B}_N(u, \epsilon)$. On the other hand, when f is Lipschitz continuous near some point u, by Rademacher's theorem it is differentiable almost everywhere (for the Lebesgue measure) on any neighbourhood of u on which f is Lipschitz continuous. We will denote by Ω_f the set of all the points at which f fails to

[3] As usually, the necessary condition on the potential for the linear case, $a_{jlk} = a_{jkl}$, $\forall j, l, k = 1, \ldots, d$, can be assumed without loss of generality: we symmetrize taking, with unchanged dispersion $\mathcal{C}$, $\tilde{a}_{jlk} = \tilde{a}_{jkl} = \frac{a_{jlk} + a_{jkl}}{2}$.

Still, of practical relevance, we can want potential by columns, instead rows, as div($\mathcal{C}$) is symmetric in j, k.

be differentiable. The notation $Df(v)$ will stand for the usual $M \times N$ matrix of partial derivatives which is well-defined whenever v is a point at which the partial derivatives exist. We are led to the following definition.

Definition 2.3.1. The *generalized Jacobian* $\partial f(u)$ of f at the point u is the convex hull of all $M \times N$ matrices Z obtained as limits of sequences of the form $Df(u_i)$, where $u_i \to u$ and $u_i \notin \Omega_f$. In other words, we set

$$\partial f(u) := \operatorname{co}\{\lim Df(u_i) \,/\;\; u_i \to u, u_i \notin \Omega_f\}, \tag{2.17}$$

where the notation "co" stands for the convex hull of a set.

When $M = 1$, given a real-valued function $f : \mathcal{U} \to \mathbb{R}$ which is Lipschitz continuous near some point $u \in \mathbb{R}^N$, the *generalized directional derivative* of f at u in the direction $v \in \mathbb{R}^N$ is denoted by $f^\circ(u; v)$ and defined by

$$f^\circ(u; v) := \limsup_{\substack{u' \to u \\ t \to 0+}} \frac{f(u' + tv) - f(u')}{t} \tag{2.18}$$

The *generalized gradient* of f at u is denoted by $\partial f(u)$ and defined by

$$\partial f(u) := \{w \in \mathbb{R}^N /\;\; f^\circ(u; v) \geq w \cdot v \quad \text{for all } v \in \mathbb{R}^N\}. \tag{2.19}$$

Some fundamental properties of generalized gradients are summarized below.

Proposition 2.3.1 ([2, Prop.2.6.2])**.** *Let* $f(u) = \big(f^1(u), f^2(u), \ldots, f^M(u)\big)$ *be a mapping which is Lipschitz continuous near some point* $u \in \mathbb{R}^N$*. Then the following statements hold:*

(a) $\partial f(u)$ *is a non-empty convex compact subset of* $\mathbb{R}^{M \times N}$.

(b) $\partial f(u)$ *is closed at* u*, that is, if* $u_i \to u$*,* $Z_i \in \partial f(u_i)$*,* $Z_i \to Z$*, then* $Z \in \partial f(u)$.

(c) $\partial f(u)$ *is upper semi-continuous at* u*, that is, for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $v \in \mathcal{B}_N(u, \delta)$

$$\partial f(v) \subset \partial f(u) + \epsilon \mathcal{B}_{M \times N},$$

where $\mathcal{B}_{M \times N}$ *is the unit ball with center* 0 *in the space of* $M \times N$*-matrices.*

(d) *If each component* f^i *is* k_i*-Lipschitz continuous at* u*, then* f *is* k*-Lipschitz continuous at* u *for some constant* k*, and* $\partial f(u) \subset k\overline{\mathcal{B}}_{M \times N}$*, where* $\overline{\mathcal{B}}_{M \times N}$ *is the closure of* $\mathcal{B}_{M \times N}$.

(e) $\partial f(u) \subset \partial f^1(u) \times \partial f^2(u) \times \ldots \times \partial f^M(u)$, *where the latter denotes the set of all matrices whose* i*-th row belongs to* $\partial f^i(u)$ *for each* i*. If* $M = 1$*, then* $\partial f(u) = \partial f^1(u)$ *(i.e., the generalized gradient and the generalized Jacobian coincide).*

In general, the generalized gradient is *not* lower semi-continuous. Recall that a set-valued function g with domain $\Omega \subset \mathbb{R}^N$ and taking values in $\mathbb{R}^M$ is said to be lower semi-continuous at a point $u \in \Omega$ if, for any open subset $\mathcal{U} \subset \Omega$ such that $\mathcal{U} \cap g(u) \neq \emptyset$, there exists $\eta > 0$ such that

$$g(v) \cap \mathcal{U} \neq \emptyset, \qquad v \in \mathcal{B}_N(u, \eta).$$

To ilustrate our claim, consider the real-valued function $h : \mathbb{R} \to \mathbb{R}$, $u \mapsto h(u) = |u|$. A simple calculation shows that

$$\partial h(u) = \begin{cases} \{-1\}, & u < 0, \\ [-1, 1], & u = 0, \\ \{1\}, & u > 0, \end{cases}$$

so that the generalized gradient ∂h is not lower semi-continuous at $u = 0$.

We now state some key results of the theory of Lipschitz continuous mappings, extending classical theorems which are well-known for smooth mappings.

Theorem 2.3.1 (Mean Value Theorem [2, Prop.2.6.5]). *Let* $f : \mathcal{U} \to \mathbb{R}^M$ *be Lipschitz continuous on an open convex set* $\mathcal{U} \subset \mathbb{R}^N$*, and let* u *and* v *some points in* $\mathcal{U}$*. Then, there exists a matrix* $A(u,v) \in \operatorname{co} \partial f([u,v])$ *(where* $[u,v]$ *stands for the straightline segment connecting* u *and* v*) such that*

$$f(v) - f(u) = A(u,v)\,(v-u)\,. \tag{2.20}$$

Theorem 2.3.2 (Chain rule formula [2, Cor.2.6.6]). *Let* $f : \mathbb{R}^N \to \mathbb{R}^M$ *be Lipschitz near* u *and let* $g : \mathbb{R}^M \to \mathbb{R}^K$ *be Lipschitz continuous near the point* $f(u)$*. Then, for any* $v \in \mathbb{R}^N$ *one has*

$$\partial(g \circ f)(u)v \subset \operatorname{co} \partial g(f(u))\, \partial f(u) v. \tag{2.21}$$

If g *is continuously differentiable near* $f(u)$*, then equality holds (and taking the convex hull is superfluous).*

Theorem 2.3.3 (Inverse mapping theorem [2, Th.7.1.1]). *Let f be Lipschitz continuous near a given point $u_0 \in \mathbb{R}^N$. If $\partial f(u_0)$ is non-singular, in the sense that every matrix of the generalized Jacobian $\partial f(u_0)$ is non-singular, then there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of u_0 and $f(u_0)$, respectively, and a unique Lipschitz function $g : \mathcal{V} \to \mathbb{R}^N$ such that*

$$g(f(u)) = u \quad \textit{for every } u \in \mathcal{U}$$

and

$$f(g(v)) = v \quad \textit{for every } v \in \mathcal{V}.$$

We will also need the implicit function theorem. Consider a mapping $h : \mathbb{R}^M \times \mathbb{R}^K \to \mathbb{R}^K$, together with the implicit equation

$$h(v, w) = 0 \quad \text{where}(v, w) \in \mathbb{R}^M \times \mathbb{R}^K. \tag{2.22}$$

Assume that h is Lipschitz continuous near the point $(v_0, w_0) \in \mathbb{R}^M \times \mathbb{R}^K$, and that (v_0, w_0) satisfies the equation (2.22). Denote $\pi_w \partial h(v_0, w_0)$ the projection in the w-direction, that is, the set of all $K \times K$ matrices A such that, for some $K \times M$ matrix B, the $K \times (K + M)$ matrix $(B\ A)$ belongs to $\partial h(v_0, w_0)$.

Theorem 2.3.4 (Implicit mapping theorem [2, Cor.7.1.1]). *Under the above notation and assumptions, suppose that each matrix of the set $\pi_w \partial h(v_0, w_0)$ is of maximal rank. Then, there exists a neighborhood $\mathcal{V}$ of v_0 and a unique Lipschitz continuous function $r : \mathcal{V} \to \mathbb{R}^K$ such that $r(v_0) = w_0$ and*

$$h(v, r(v)) = 0 \quad \textit{for every } v \in \mathcal{V}. \tag{2.23}$$

Part II

Multi-Dimensional Scalar Conservation Laws

Chapter 3

A First KdVB Equation[1]

Abstract. We consider a class of multi-dimensional conservation laws with vanishing linear or nonlinear diffusion and linear dispersion terms. Under a condition on the relative size of the diffusion and dispersion coefficients, we establish that the diffusive-dispersive solutions are uniformly bounded in a space L^p (p arbitrarily large) and converge to the classical entropy solution of the corresponding multi-dimensional hyperbolic conservation law. Previous results were restricted to one-dimensional equations and specific spaces L^p. Our proof is based on DiPerna's uniqueness theorem in the class of entropy measure-valued solutions.

3.1 Assumptions

Consider the Cauchy problem

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{3.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d, \tag{3.2}$$

where the unknown function $u = u(x,t)$ is scalar-valued and the flux $f : \mathbb{R} \to \mathbb{R}^d$ is a given function.

Let (3.1)-(3.2) approximated by adding to the right-hand side of (3.1) a linear or nonlinear diffusion, $b : \mathbb{R}^d \to \mathbb{R}^d$, plus a linear dispersion, and approximating the initial data u_0 in (3.2) by $u_0^{\varepsilon,\delta}$, where ε, δ (> 0) are vanishing parameters

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div} \Big(\varepsilon\, b_j(\nabla u) + \delta\, \partial^2_{x_j} u\Big)_{1 \le j \le d}, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{3.3}$$

[1]From Correia-LeFloch [5, 6]

(3.4) $$u(x,0) = u_0^{\varepsilon,\delta}(x), \qquad x \in \mathbb{R}^d.$$

Our main objective is to derive conditions under which, as ε and δ tend to zero, the solutions $u^{\varepsilon,\delta}$ converge in a strong topology to the entropy weak solution of (3.1)-(3.2).

Therefore, to ensure the convergence of the zero diffusion-dispersion approximation (3.3)-(3.4), it is necessary that diffusion dominate dispersion. The main result establishes that, under rather broad assumptions (see Theorems 3.2.1-3.2.3 below), the solutions of (3.3)-(3.4) tend to the entropy weak solution of (3.1)-(3.2) when $\varepsilon, \delta \to 0$ with $\delta << \varepsilon$.

For clarity, the main assumptions made in this paper are collected here. First concerning the flux function we shall assume

(H_1) for some $C_1 \geq 0$, $C_1' > 0$ and $m \geq 1$, $\quad |f'(u)| \leq C_1 + C_1' \, |u|^{m-1}$, for all $u \in \mathbb{R}$.

For the diffusion term, we fix $r \geq 0$ and assume

(H_2) for some $C_2, C_3 > 0$, $\quad C_2 \, |\lambda|^{r+1} \leq \lambda \cdot b(\lambda) \leq C_3 \, |\lambda|^{r+1}$, for all $\lambda \in \mathbb{R}^d$.

In the case $0 \leq r < 2$, we will need also

(H_3) $\mathrm{D}\, b(\lambda)$ is a positive definite matrix uniformly in $\lambda \in \mathbb{R}^d$.

We remark that the diffusion $b_j(\nabla u) = \partial_{x_j} u$ satisfies (H_3).

Throughout it is assumed $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, the initial data in (3.4) are smooth functions with compact support and are uniformly bounded in $L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $q > 2$. While in previous works [35, 29], a single value of q was treated, we can here handle arbitrary large values of q. For simplicity in the presentation, we will always consider exponents q of the form

$$q = 2 + n(r-1),$$

where $n \geq 0$ is any integer. Therefore, when the diffusion is superlinear, in the sense that (H_2) holds with $r > 1$, then arbitrary large values of q are obtained. Restricting attention to the diffusion-dominant regime we regard $\delta = \delta(\varepsilon)$ and we suppose that $u_0^{\varepsilon,\delta}$ approaches the initial condition u_0 of (3.2) in the sense that:

(3.5) $$\begin{gathered} \lim_{\varepsilon \to 0+} u_0^{\varepsilon,\delta} = u_0 \quad \text{in} \quad L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), \\ \|u_0^{\varepsilon,\delta}\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)}. \end{gathered}$$

3.2 Main Result

The following four convergence theorems concern a sequence $u^{\varepsilon,\delta}$ of smooth solutions to problem (3.3)-(3.4), defined on $\mathbb{R}^d \times [0,T]$ with a uniform T independent of ε, δ, and decaying rapidly at infinity.

First consider the hypothesis (H_2) with $r \geq 2$, that is the case of diffusions with (at least) quadratic growth.

Theorem 3.2.1. *Suppose that the flux f satisfies (H_1) with $m < q$ (which is always possible when $r > 1$ by choosing q large enough). Suppose that the diffusion b satisfies (H_2) with $r \geq 2$. If $\delta = o\big(\varepsilon^{\frac{3}{r+1}}\big)$, then the sequence $u^{\varepsilon,\delta}$ converges in $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, for all $s < \infty$ and $p < q$, to a function $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\right)$, which is the unique entropy solution to (3.1)-(3.2).*

Observe that m and q can be arbitrarily large in Theorem 3.2.1. To treat the case $r < 2$, we need the additional condition (H_3) on the diffusion. First for diffusion with linear growth ($r = 1$), we obtain a result in the space L^2 :

Theorem 3.2.2. *Suppose that f satisfies (H_1) with $m = 1$, and b satisfies (H_2)-(H_3) with $r = 1$. If $\delta = o\big(\varepsilon^2\big)$, then the sequence $u^{\varepsilon,\delta}$ converges in $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, for all $s < \infty$ and $p < 2$, to a function $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\right)$, which is the unique entropy solution to (3.1)-(3.2).*

In particular Theorem 3.2.2 covers the interesting case of a linear diffusion and a linear dispersion with an (at most) linear flux at infinity. The condition $\delta = o(\varepsilon^2)$ is *sharp*, since for $\delta = A\varepsilon^2$ (A fixed) the functions may converge to "nonclassical" entropy solutions; see Hayes-LeFloch [18, 19]. More generally, for general $r \geq 1$ we establish that:

Theorem 3.2.3. *Suppose that f satisfies (H_1) with $m \leq \frac{2r}{r+1} < q$, and b satisfies (H_2)-(H_3) for some $r \geq 1$. If $\delta = o\big(\varepsilon^{\frac{r+3}{r+1}}\big)$ then the sequence $u^{\varepsilon,\delta}$ converges in $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, for all $s < \infty$ and $p < q$, to a function $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\right)$, which is the unique entropy solution to (3.1)-(3.2).*

And, finally, also for $r = 1$ and $1 < m < 2$:

Theorem 3.2.4. *Suppose that f satisfies (H_1) with $1 < m < 2 < q$ and $C_1 = 0$, b satisfies (H_2) with $r = 1$ and the analogue of (H_3):*

(H_4) *for some* $C_4 > 0$ *and for all* $\lambda \in \mathbb{R}^d$ *and* $d \times d$ *matrix* Λ

$$C_4 \left|\mathrm{diag}(\Lambda)\right|^{\frac{2}{2-m}} \leq \Lambda.\, \mathrm{D}\, b(\lambda)\Lambda.$$

If $\delta = o\big(\varepsilon^{\max\{1, \frac{7-3m}{2}\}}\big)$, *then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, *for all* $s < \infty$ *and* $p < q$, *to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\right)$, *which is the unique entropy solution to* (3.1)-(3.2).

Our results can be extended to more general "diffusions" of the form $b(u, \nabla u, \mathrm{D}^2 u)$.

3.3 First Energy Estimates

The superscripts ε and δ are omitted in this section, except when emphasis is necessary. In the proof, we make frequent use of the following computation. Multiply (3.3) by $\eta'(u)$ where $\eta : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function and define $q : \mathbb{R} \to \mathbb{R}^d$ by $q_j' = \eta'\, f_j'$. We have

$$\begin{aligned}
\partial_t \eta(u) &= -\eta'(u)\ \mathrm{div}\, f(u) + \varepsilon \sum_j \partial_{x_j}(\eta'(u)\ b_j(\nabla u)) - \partial_{x_j}\eta'(u)\ b_j(\nabla u) \\
&\quad + \delta \sum_j \partial_{x_j}\Big(\eta'(u)\ \partial^2_{x_j} u\Big) - \partial_{x_j}\eta'(u)\ \partial^2_{x_j} u \\
&= -\,\mathrm{div}\ q(u) + \varepsilon \sum_j \partial_{x_j}(\eta'(u)\ b_j(\nabla u)) - \varepsilon\, \eta''(u) \sum_j \partial_{x_j} u\ b_j(\nabla u) \\
&\quad + \frac{\delta}{2} \sum_j 2\, \partial_{x_j}\Big(\eta'(u)\ \partial^2_{x_j} u\Big) - \eta''(u)\ \partial_{x_j}\big(\partial_{x_j} u\big)^2,
\end{aligned}$$

thus

$$\begin{aligned}
\text{(3.6)}\quad \partial_t \eta(u) + \mathrm{div}\ q(u) &= \varepsilon\ \mathrm{div}(\eta'(u)\ b(\nabla u)) - \varepsilon\, \eta''(u)\ \nabla u \cdot b(\nabla u) \\
&\quad - \frac{\delta}{2} \sum_j \eta''(u)\ \partial_{x_j}\big(\partial_{x_j} u\big)^2 + \delta \sum_j \partial_{x_j}\Big(\eta'(u)\ \partial^2_{x_j} u\Big).
\end{aligned}$$

When η is convex, the term containing $\eta''(u)$ has a favorable sign: the diffusion dissipates the entropy η. The last two terms in the right-hand side of (3.6) take also the form

$$\text{(3.7)}\qquad \frac{\delta}{2} \sum_j \eta'''(u)\ \big(\partial_{x_j} u\big)^3 - 3\, \partial_{x_j}\Big(\eta''(u)\ \big(\partial_{x_j} u\big)^2\Big) + 2\, \partial^2_{x_j}\big(\eta'(u)\ \partial_{x_j} u\big).$$

We begin by collecting fundamental energy estimates in several lemma.

Lemma 3.3.1. *Let $\alpha \geq 1$ be any real. Any solution of* (3.3) *satisfies, for $t \in [0,T]$,*

$$\int_{\mathrm{R}^d} \frac{|u(t)|^{\alpha+1}}{\alpha+1}\, dx \;+\; \alpha\, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\; \nabla u \cdot\, b(\nabla u)\, dxds = \int_{\mathrm{R}^d} \frac{|u_0|^{\alpha+1}}{\alpha+1}\, dx - \frac{\alpha}{2}\, \delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \sum_j \partial_{x_j}\big(\partial_{x_j} u\big)^2\; dxds. \tag{3.8}$$

For $\alpha \geq 2$, the last term in the above identity also equals

$$\frac{\alpha\,(\alpha-1)}{2}\, \delta \int_0^t \int_{\mathrm{R}^d} \mathrm{sgn}(u)\, |u|^{\alpha-2}\; \sum_j \big(\partial_{x_j} u\big)^3\; dxds. \tag{3.9}$$

Proof. Integrate (3.6) over the whole of $\mathbb{R}^d$ with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathrm{R}^d} \frac{|u|^{\alpha+1}}{\alpha+1}\, dx = -\alpha\, \varepsilon \int_{\mathrm{R}^d} |u|^{\alpha-1}\, \nabla u \cdot b(\nabla u)\, dx - \frac{\alpha\, \delta}{2} \int_{\mathrm{R}^d} \sum_j |u|^{\alpha-1} \partial_{x_j}\big(\partial_{x_j} u\big)^2\; dx,$$

which yields (3.8) after integration over $[0,t]$. One may use (3.7), instead, to obtain (3.9). □

Choosing $\alpha = 1$ in Lemma 3.3.1, we deduce immediately a uniform bound for u in $L^\infty((0,T); L^2(\mathbb{R}^d))$ together with a control for both $\nabla u \cdot b(\nabla u)$ in $L^1(\mathbb{R}^d \times (0,T))$ and ∇u in $L^{r+1}(\mathbb{R}^d \times (0,T))$.

Proposition 3.3.1. *For any solution of* (3.3) *and $t \in [0,T]$, we have*

$$\int_{\mathrm{R}^d} u(t)^2\, dx + 2\, \varepsilon \int_0^t \int_{\mathrm{R}^d} \nabla u \cdot b(\nabla u)\, dxds = \int_{\mathrm{R}^d} u_0^2\, dx \tag{3.10}$$

and, assuming (H_2),

$$\varepsilon \int_0^t \int_{\mathrm{R}^d} |\, \nabla u|^{r+1}\, dxds \leq C \int_{\mathrm{R}^d} u_0^2\, dx. \tag{3.11}$$

3.4 $\mathrm{L^q}$ Estimates

To derive additional a priori estimates, we use another value of α, motivated by controling the dispersive term in (3.9) with Hölder inequality, as follows:

$$\left|\int_0^t\int_{\mathbb{R}^d} \operatorname{sgn}(u)\,|u|^{\alpha-2}\sum_j\left(\partial_{x_j}u\right)^3\,dxds\right| \le \int_0^t\int_{\mathbb{R}^d} |u|^{\alpha-2}\,|\nabla u|^3\,dxds \tag{3.12}$$
$$\le \left[\int_0^t\int_{\mathbb{R}^d} |u|^{(\alpha-2)p}\,dxds\right]^{\frac{1}{p}}\left[\int_0^t\int_{\mathbb{R}^d} |\nabla u|^{3p'}\,dxds\right]^{\frac{1}{p'}}.$$

To take advantage of (3.11), we can choose $3\,p' = r+1$ provided $r \ge 2$. Then $p = \frac{r+1}{r-2}$, so $(\alpha-2)p = (r+1)\frac{\alpha-2}{r-2}$. Therefore it is rather natural to take the exponent $\alpha = r$ for the entropy, where r is given by the diffusion term. Thus we deduce from Lemma 3.3.1 a natural estimate for $|u(t)|^{r+1}$, involving the combination $\delta\,\varepsilon^{-\frac{3}{r+1}}$ of δ and ε.

Proposition 3.4.1. *Assume that (H_2) holds with $r \ge 2$ and $u_0 \in L^{r+1}(\mathbb{R}^d)$. For $t \in [0,T]$, we have*

$$\int_{\mathbb{R}^d} |u(t)|^{r+1}dx + (r+1)\,r\,\varepsilon\int_0^t\int_{\mathbb{R}^d} |u|^{r-1}\,\nabla u\cdot b(\nabla u)\,dxds \tag{3.13}$$
$$\le C_1(u_0)\left(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\,\max\left\{1,\left[t\,C_1(u_0)\left(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r-2}{3}}\right\}\right)$$
$$:= H_1\left(\delta\,\varepsilon^{-\frac{3}{r+1}}\right)$$

and

$$\varepsilon\int_0^t\int_{\mathbb{R}^d} |u|^{r-1}\,|\nabla u|^{r+1}\,dxds \le \frac{C}{(r+1)\,r}\,H_1\left(\delta\,\varepsilon^{-\frac{3}{r+1}}\right), \tag{3.14}$$

where $C > 0$ is some fixed constant and

$$C_1(u_0) := \max\left\{\|u_0\|^{r+1}_{L^{r+1}(\mathbb{R}^d)},\ \frac{(r+1)r(r-1)}{2}\left(C\|u_0\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{3}{r+1}}\right\}.$$

In particular Proposition 3.4.1 shows that, if $u_0 \in L^2\cap L^{r+1}$ and $\delta = \mathcal{O}\left(\epsilon^{\frac{3}{r+1}}\right)$, then $u(t) \in L^{r+1}$ uniformly for all $t \ge 0$.

To motivate the forthcoming derivation, let us consider the special case $r = 2$. Then (3.13) gives us an L^3 estimate. Returning to the original inequality (3.12), but now with the new value $\alpha = 3$, we now can estimate the dispersive term in (3.9) directly in view of the estimate (3.14). In this fashion, we deduce an L^4 estimate from Lemma 3.3.1. This argument can be continued inductively to reach any space L^q.

Actually Propositions 3.3.1 and 3.4.1 are the first two cases of a general result derived now. We define, for $n \geq 1$,

$$H_0\Big(\delta\, \varepsilon^{-\frac{3}{r+1}}\Big) = C_0(u_0) := \|u_0\|^2_{L^2(\mathbb{R}^d)};$$

$$H_n\Big(\delta\, \varepsilon^{-\frac{3}{r+1}}\Big) := C_n(u_0)\Big(1+\delta\, \varepsilon^{-\frac{3}{r+1}} \max\Big\{1, \Big[t\; C_n(u_0)\Big(1+\delta\, \varepsilon^{-\frac{3}{r+1}}\Big)\Big]^{\frac{r-2}{3}}\Big\}\Big); \tag{3.15}$$

$$C_n(u_0) := \max\Big\{ \|u_0\|^{n(r-1)+2}_{L^{n(r-1)+2}(\mathbb{R}^d)}, \frac{n(r-1)+2}{[(n-1)(r-1)+2]^{\frac{3}{r+1}}} \frac{n(r-1)+1}{[(n-1)(r-1)+1]^{\frac{3}{r+1}}} n\, \frac{r-1}{2}\Big(C\, H_{n-1}\Big(\delta\, \varepsilon^{-\frac{3}{r+1}}\Big)\Big)^{\frac{3}{r+1}}\Big\}.$$

Here $C > 0$ is some fixed constant. Note that H_n and C_n are uniformly bounded in ϵ, δ provided $u_0 \in L^2 \cap L^{n(r-1)+2}$ and $\delta = \mathcal{O}\big(\epsilon^{\frac{3}{r+1}}\big)$.

Proposition 3.4.2. *Assume that* (H_2) *holds with* $r \geq 2$ *and* $u_0 \in L^q(\mathbb{R}^d)$. *For* $t \in [0,T]$ *and* $n \geq 0$ *such that* $n(r-1)+2 \leq q$, *we have*

$$\int_{\mathbb{R}^d} |u(t)|^{n(r-1)+2}\; dx + \varepsilon\,(n(r-1)+2)(n(r-1)+1) \int_0^t\int_{\mathbb{R}^d} |u|^{n(r-1)}\; \nabla u\cdot b(\nabla u)\, dxds \leq H_n\Big(\delta\, \varepsilon^{-\frac{3}{r+1}}\Big), \tag{3.16}$$

$$\varepsilon \int_0^t\int_{\mathbb{R}^d} |u|^{n(r-1)}\, |\nabla u|^{r+1}\; dxds \tag{3.17}$$

$$\leq \frac{C}{(n(r-1)+2)(n(r-1)+1)} H_n\Big(\delta\, \varepsilon^{-\frac{3}{r+1}}\Big). \tag{3.18}$$

Proof of Propositions 3.4.1 and 3.4.2. Note first that (3.17) is an immediate consequence of (3.16) and the hypothesis (H_2). If $n = 0$, (3.16) coincides with (3.10) in Proposition 3.3.1. For $n = 1$, the estimate is Proposition 3.4.1.

To estimate the term in (3.9), with $\alpha = r$, we use (3.12):

$$\int_{\mathbb{R}^d} |u(t)|^{r+1}\, dx \;+\; (r+1)\, r\, \varepsilon \int_0^t\int_{\mathbb{R}^d} |u|^{r-1}\; \nabla u\cdot b(\nabla u)\, dxds \tag{3.19}$$

$$\leq \int_{\mathrm{R}^d} |u_0|^{r+1}\, dx + \frac{(r+1)r(r-1)}{2}\, \delta$$
$$\left[\int_0^t \int_{\mathrm{R}^d} |u|^{r+1}\, dxds\right]^{\frac{r-2}{r+1}} \left[\int_0^t \int_{\mathrm{R}^d} |\nabla u|^{r+1}\ dxds\right]^{\frac{3}{r+1}}.$$

By (H_2) the second term in the left hand side of (3.19) is positive. Integrate (3.19) over $[0,t]$ and use (3.11):

$$\begin{aligned} \|u\|^{r+1}_{L^{r+1}(\mathrm{R}^d\times(0,T))} \leq\ & t\ \|u_0\|^{r+1}_{L^{r+1}(\mathrm{R}^d)} \\ + \frac{(r+1)r(r-1)}{2}\ & t\ \left(C\, \|u_0\|^2_{L^2(\mathrm{R}^d)}\right)^{\frac{3}{r+1}}\ \delta\ \varepsilon^{-\frac{3}{r+1}}\ \|u\|^{r-2}_{L^{r+1}(\mathrm{R}^d\times(0,T))} \\ \leq\ & t\ C_1(u_0) \left(1 + \delta\ \varepsilon^{-\frac{3}{r+1}} \left(\|u\|^{r+1}_{L^{r+1}(\mathrm{R}^d\times(0,T))}\right)^{\frac{r-2}{r+1}}\right). \end{aligned}$$

Observe that the inequality

$$0 < X \leq K \left(1 + \Delta\, X^{\frac{\theta}{r+1}}\right),$$

where $0 \leq \theta < r+1$ and $K > 0$, implies

$$X \leq \max\left\{1,\ [K\,(1+\Delta)]^{\frac{r+1}{r+1-\Theta}}\right\}. \tag{3.20}$$

Thus we deduce

$$\|u\|^{r+1}_{L^{r+1}(\mathrm{R}^d\times(0,T))} \leq \max\left\{1,\ \left[t\ C_1(u_0)\left(1 + \delta\, \varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r+1}{3}}\right\}$$

and, returning to (3.19):

$$\begin{aligned} \int_{\mathrm{R}^d} |u(t)|^{r+1}\ dx + (r+1)\, r\, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{r-1}\, \nabla u \cdot b(\nabla u)\, dxds & \\ \leq\ C_1(u_0) \left(1 + \delta\, \varepsilon^{-\frac{3}{r+1}} \max\left\{1,\ \left[t\ C_1(u_0)\left(1 + \delta\, \varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r-2}{3}}\right\}\right) & \\ :=\ H_1\!\left(\delta\ \varepsilon^{-\frac{3}{r+1}}\right). & \end{aligned}$$

This completes the proof of (3.13).

This argument can be iterated. We return to the dispersive term and make an estimate similar to (3.12), but now having in view to apply (3.17), already established for $n = 1$:

$$\left|\int_0^t \int_{\mathrm{R}^d} \mathrm{sgn}(u)|u|^{\alpha-2} \sum_j \left(\partial_{x_j} u\right)^3\ dxds\right| \leq \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2}\ |\nabla u|^3\ dxds \tag{3.21}$$

$$\leq \left[\int_0^t \int_{\mathrm{R}^d} |u|^{(\alpha-2-\gamma)p}\, dxds\right]^{\frac{1}{p}} \left[\int_0^t \int_{\mathrm{R}^d} |u|^{\gamma p'}\, |\nabla u|^{3p'}\, dxds\right]^{\frac{1}{p'}},$$

where we choose $3p' = r+1$ and $\gamma p' = r-1$, so $(\alpha - 2 - \gamma)p = \left(\alpha - 2 - 3\,\frac{r-1}{r+1}\right)\frac{r+1}{r-2}$. Then (3.9) gives

$$\begin{aligned}
(3.22)\qquad & \int_{\mathrm{R}^d} |u(t)|^{\alpha+1} dx \,+\, (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\ \nabla u \cdot b(\nabla u)\, dxds \\
&\leq \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx + \frac{(\alpha+1)\,\alpha\,(\alpha-1)}{2\ [(r+1)\,r]^{\frac{3}{r+1}}} \left(C\,H_1\!\left(\delta\ \varepsilon^{-\frac{3}{r+1}}\right)\right)^{\frac{3}{r+1}} \\
&\qquad \delta\ \varepsilon^{-\frac{3}{r+1}} \left[\int_0^t \int_{\mathrm{R}^d} |u|^{(\alpha-2-\gamma)p}\, dxdt\right]^{\frac{r-2}{r+1}}.
\end{aligned}$$

We choose α so that $\alpha + 1 = (\alpha - 2 - \gamma)p$, i.e., $\alpha = 2r - 1$.
Integrating (3.22) over the interval $[0, t]$, we obtain

$$\begin{aligned}
&\|u\|^{2r}_{L^{2r}(\mathrm{R}^d\times(0,T))} \leq t\ \|u_0\|^{2r}_{L^{2r}(\mathrm{R}^d)} \\
&+ \frac{r\,(2r-1)\,(2r-2)}{[(r+1)\,r]^{\frac{3}{r+1}}}\, t\ \left(C\,H_1\!\left(\delta\ \varepsilon^{-\frac{3}{r+1}}\right)\right)^{\frac{3}{r+1}} \delta\ \varepsilon^{-\frac{3}{r+1}} \left(\|u\|^{2r}_{L^{2r}(\mathrm{R}^d\times(0,T))}\right)^{\frac{r-2}{r+1}} \\
&\qquad \leq t\ C_2(u_0)\left(1 + \delta\ \varepsilon^{-\frac{3}{r+1}}\ \left(\|u\|^{2r}_{L^{2r}(\mathrm{R}^d\times(0,T))}\right)^{\frac{r-2}{r+1}}\right),
\end{aligned}$$

with $C_2(u_0) := \max\left\{\|u_0\|^{2r}_{L^{2r}(\mathrm{R}^d)},\ \frac{r\,(2r-1)\,(2r-2)}{[(r+1)\,r]^{\frac{3}{r+1}}}\left(C\,H_1\!\left(\delta\ \varepsilon^{-\frac{3}{r+1}}\right)\right)^{\frac{3}{r+1}}\right\}$.
By (3.20), we obtain again

$$\|u\|^{2r}_{L^{2r}(\mathrm{R}^d\times(0,T))} \leq \max\left\{1,\ \left[t\ C_2(u_0)\ \left(1 + \delta\ \varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r+1}{3}}\right\}.$$

Then (3.22) gives

$$\begin{aligned}
&\int_{\mathrm{R}^d} |u(t)|^{2r}\ \ dx + 2r\,(2r-1)\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(r-1)}\ \nabla u \cdot b(\nabla u)\, dxds \\
&\qquad \leq\ C_2(u_0)\left(1 + \delta\ \varepsilon^{-\frac{3}{r+1}} \max\left\{1, \left[t\ C_2(u_0)\ \left(1 + \delta\ \varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r-2}{3}}\right\}\right) \\
&\qquad :=\ H_2\!\left(\delta\ \varepsilon^{-\frac{3}{r+1}}\right).
\end{aligned}$$

This proves (3.16) for $n = 2$. The general case follows by induction on n. □

We are now concerned with the case where the diffusion exponent in (H_2) satisfies $r < 2$. In this situation, we require the assumption (H_3), which for instance is satisfied by $b_j(\nabla u) = \partial_{x_j} u$.

Proposition 3.4.3. *Suppose that* (H_1)*–*(H_3) *hold with* m *and* r *such that* $m \le \frac{2r}{r+1}$ *and* $r \ge 1$*. For* $t \in [0,T]$*, we have*

$$\varepsilon^{\frac{r+3}{r+1}} \int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + \varepsilon^{\frac{2(r+2)}{r+1}} \int_0^T \int_{\mathbb{R}^d} \left|D^2 u\right|^2 \, dxdt \le C, \tag{3.23}$$

$$\int_{\mathbb{R}^d} |u(t)|^{2+\frac{r-1}{r}} \, dx + \varepsilon \int_0^T \int_{\mathbb{R}^d} |u|^{\frac{r-1}{r}} \, |\nabla u|^{r+1} \, dxdt \le C\left(1 + \delta^{\frac{r+1}{r}} \, \varepsilon^{-\frac{r+3}{r}}\right). \tag{3.24}$$

Proof. We differentiate (3.3) with respect to the space variable x:

$$\partial_t \nabla u + \operatorname{div}\left(f'(u).\, \nabla u\right) = \varepsilon \, \nabla \sum_j \partial_{x_j} \left(b_j(\nabla u)\right) + \delta \sum_j \partial^3_{x_j} \left(\nabla u\right).$$

Then we multiply by ∇u and integrate in $\mathbb{R}^d$. After integration by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx - \int_{\mathbb{R}^d} \Delta u \, f'(u) \cdot \nabla u \, dx$$
$$= -\varepsilon \int_{\mathbb{R}^d} \sum_k \nabla \partial_{x_k} u \cdot Db(\nabla u) \cdot \nabla \partial_{x_k} u \, dx - \frac{\delta}{2} \sum_j \int_{\mathbb{R}^d} \partial_{x_j} \left(\sum_k \left(\partial^2_{x_k x_j} u\right)^2 \right).$$

Thus, integrating on $[0,t]$ using (H_1) yields

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx &+ 2\,\varepsilon \int_{\mathbb{R}^d} \sum_k \nabla \partial_{x_k} u \cdot Db(\nabla u) \cdot \nabla \partial_{x_k} u \, dx \\
&\le \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + 2\, C_1 \int_0^t \int_{\mathbb{R}^d} \left|D^2 u\right| |u|^{m-1} \left|\nabla u\right| \, dxdt \\
&\le \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + \frac{C}{\varepsilon} \int_0^t \int_{\mathbb{R}^d} |u|^{2m-2} \left|\nabla u\right|^2 \, dxdt \\
&\quad + C_4 \, \varepsilon \int_0^t \int_{\mathbb{R}^d} \left|D^2 u\right|^2 \, dxdt,
\end{aligned}$$

and so, using (H_3),

$$\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + C_5 \, \varepsilon \int_0^t \int_{\mathbb{R}^d} \left|D^2 u\right|^2 \, dxdt$$

$$\leq \int_{\mathrm{R}^d} |\nabla u_0|^2 \; dx \; + \; \frac{C}{\varepsilon} \int_0^t \int_{\mathrm{R}^d} |u|^{2m-2} \, |\nabla u|^2 \; dxdt.$$

By Hölder inequality and for $m \leq \frac{r-1}{r+1}$,

$$\int_{\mathrm{R}^d} |\nabla u(t)|^2 \; dx \; + C_5 \, \varepsilon \int_0^t \int_{\mathrm{R}^d} \left|D^2 u\right|^2 \; dxdt \; \leq \int_{\mathrm{R}^d} |\nabla u_0|^2 \; dx$$
$$+ \, C \, \varepsilon^{-1} \left[\int_0^t \int_{\mathrm{R}^d} |\nabla u|^{r+1} \; dxdt\right]^{\frac{2}{r+1}} \; \left[\int_0^t \int_{\mathrm{R}^d} |u|^2 \; dxdt\right]^{\frac{r-1}{r+1}},$$

and now (3.23) follows from (3.10)-(3.11).
To prove (3.24) we use (3.8) for $\alpha \geq 1$:

$$\int_{\mathrm{R}^d} |u(t)|^{\alpha+1} \, dx \; + C \, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \, | \, \nabla u)|^{r+1} \, dxdt$$
$$\leq \int_{\mathrm{R}^d} |u_0|^{\alpha+1} \, dx \; + \; C' \, \delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \, | \, \nabla u| \, |D^2 u| \, dxdt.$$

We evaluate the last term using (H_2):

$$\delta \; \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \, | \, \nabla u| \, |D^2 u| \, dxdt$$
$$\leq \delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \left(\frac{C_2 \, \varepsilon}{(r+1) \, \delta} \, |\nabla u|^{r+1} \; + \; \frac{r}{r+1} \left(\frac{\delta}{C_2 \, \varepsilon}\right)^{\frac{1}{r}} \left|D^2 u\right|^{\frac{r+1}{r}} \right) \; dxdt$$
$$\leq \frac{\varepsilon}{2} \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \; |\nabla u|^{r+1} \; dxdt$$
$$+ \, C'' \, \delta^{\frac{r+1}{r}} \varepsilon^{-\frac{1}{r}} \; \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \left|D^2 u\right|^{\frac{r+1}{r}} \; dxdt.$$

So we have

$$\int_{\mathrm{R}^d} |u(t)|^{\alpha+1} \, dx + C \, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \, | \, \nabla u|^{r+1} \, dxdt$$
$$\leq \int_{\mathrm{R}^d} |u_0|^{\alpha+1} \, dx + C \, \delta^{\frac{r+1}{r}} \varepsilon^{-\frac{1}{r}} \; \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} \left|D^2 u\right|^{\frac{r+1}{r}} \; dxdt.$$

Taking $\alpha = 1 + \frac{r-1}{r}$, we deduce

$$\int_{\mathrm{R}^d} |u(t)|^{2+\frac{r-1}{r}} \; dx + C \, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\frac{r-1}{r}} \, | \, \nabla u|^{r+1} \, dxdt$$

$$\le \int_{\mathrm{R}^d} |u_0|^{2+\frac{r-1}{r}}\, dx + C\,\delta^{\frac{r+1}{r}}\varepsilon^{-\frac{1}{r}} \left[\int_0^t \int_{\mathrm{R}^d} |u|^2\, dxdt\right]^{\frac{r-1}{2r}} \left[\int_0^t \int_{\mathrm{R}^d} \left|D^2 u\right|^2\, dxdt\right]^{\frac{r+1}{2r}}.$$

The conclusion follows now easily. □

3.5 Convergence Proofs

Proof of Theorem 3.2.1. We first prove (2.13), based on the conservation law (3.7) with an arbitrary convex function, η, where we assume η', η'', η''' bounded functions on $\mathbb{R}$. We claim that there exists a bounded measure $\mu \le 0$ such that

$$\partial_t \eta(u) + \mathrm{div}\ q(u) \longrightarrow \mu \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \times (0,T)).$$

From (3.7), we obtain

$$\begin{aligned}\partial_t\eta(u) \,+\, \mathrm{div}\ q(u) &= \varepsilon\ \mathrm{div}(\eta'(u)\ b(\nabla u)) - \varepsilon\,\eta''(u)\ \ \nabla u \cdot b(\nabla u)\\ &+\ \frac{\delta}{2}\sum_j \eta'''(u)\ \left(\partial_{x_j} u\right)^3 - 3\,\partial_{x_j}\Big(\eta''(u)\ \left(\partial_{x_j} u\right)^2\Big) + 2\,\partial^2_{x_j}\big(\eta'(u)\ \partial_{x_j} u\big)\\ &:=\ \mu_1 + \mu_2 + \mu_3,\end{aligned}$$

with obvious notation. For each positive $\theta \in C_0^\infty(\mathrm{R}^d \times (0,T))$ we evaluate $\langle \mu_i, \theta\rangle$ for $i = 1,2,3$. To treat μ_1, we use Hölder inequality with the exponent $\frac{r+1}{r}$. In view of (H_2) and (3.11) of Proposition 3.3.1 and assumption (3.5), we get

$$\begin{aligned}|\langle \mu_1, \theta\rangle| &\le \varepsilon \int_0^T \int_{\mathrm{R}^d} \sum_j \left|\eta'(u)\, b_j(\nabla u)\ \partial_{x_j}\theta\right|\ dxdt\\ &\le C\,\varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla\theta|\ |b(\nabla u)|\ \ dxdt\end{aligned}$$

so

$$\begin{aligned}|\langle \mu_1, \theta\rangle| &\le C\,\varepsilon\, \|\,\nabla\theta\|_{L^{r+1}(\mathrm{R}^d\times(0,T))} \left[\iint_{\mathrm{supp}\,\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{r}{r+1}}\\ &\le C\,\varepsilon^{\frac{1}{r+1}}\, \|\,\nabla\theta\|_{L^{r+1}(\mathrm{R}^d\times(0,T))}.\end{aligned}$$

For μ_2, we use (H_2) and the convexity of η :

$$\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathrm{R}^d} \sum_j \theta \; \eta''(u) \;\; \nabla u \cdot b(\nabla u) \; dxdt \leq 0.$$

For μ_3, we use again Hölder inequality, as follows

$$\begin{aligned} |\langle \mu_3, \theta \rangle| \leq & \frac{\delta}{2} \int_0^T \int_{\mathrm{R}^d} \sum_j \Big| \theta \; \eta'''(u) \; \big(\partial_{x_j} u\big)^3 + 3\, \eta''(u) \left(\partial_{x_j} u\right)^2 \partial_{x_j} \theta \\ & + 2\, \eta'(u) \; \partial_{x_j} u \; \partial_{x_j}^2 \theta \Big| \; dxdt \\ \leq & \; C\, \delta \int_0^T \int_{\mathrm{R}^d} \theta \; |\nabla u|^3 \; dxdt + C\, \delta \int_0^T \int_{\mathrm{R}^d} \sum_j \left|\partial_{x_j} u\right|^2 \left|\partial_{x_j} \theta\right| \; dxdt \\ & + C\, \delta \int_0^T \int_{\mathrm{R}^d} \sum_j \left|\partial_{x_j} u\right| \; \left|\partial_{x_j}^2 \theta\right| \; dxdt \end{aligned}$$

so

$$\begin{aligned} |\langle \mu_3, \theta \rangle| \leq & \; C\, \delta \, \|\theta\|_{L^{\frac{r+1}{r-2}}(\mathrm{R}^d \times (0,T))} \; \left[\iint_{\operatorname{supp} \theta} |\nabla u|^{r+1} \; dxdt \right]^{\frac{3}{r+1}} \\ & + C\, \delta \, \| \, \nabla \theta \|_{L^{\frac{r+1}{r-1}}(\mathrm{R}^d \times (0,T))} \; \left[\iint_{\operatorname{supp} \theta} |\nabla u|^{r+1} \; dxdt \right]^{\frac{2}{r+1}} \\ & + C\, \delta \left[\int_0^T \int_{\mathrm{R}^d} \left(\sum_j \left| \partial_{x_j}^2 \theta \right| \right)^{\frac{r+1}{r}} dxdt \right]^{\frac{r}{r+1}} \; \left[\iint_{\operatorname{supp} \theta} |\nabla u|^{r+1} \; dxdt \right]^{\frac{1}{r+1}} \end{aligned}$$

therefore

$$|\langle \mu_3, \theta \rangle| \leq \; C\, \delta \left(\varepsilon^{-\frac{3}{r+1}} + \varepsilon^{-\frac{2}{r+1}} + \varepsilon^{-\frac{1}{r+1}} \right) \leq \; C\, \delta\, \varepsilon^{-\frac{3}{r+1}}.$$

Finally the condition $\delta = o\big(\varepsilon^{\frac{3}{r+1}}\big)$ is sufficient to imply the desired conclusion.

Using a standard regularization of $\operatorname{sgn}(u)$ and $|u-k|$ (for $k \in \mathbb{R}$), which fullfil the growth condition (2.9), we apply the limit representation (2.10) and conclude that ν satisfies (2.13).

To show (2.14) we follow DiPerna [14] and Szepessy [38]'s arguments. We have to check that, for each compact K of $\mathbb{R}^d$,

$$\lim_{t \to 0+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \, \rangle \;\; dxds$$

$$= \lim_{t\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s) - u_0(x)\right| \, dxds = 0.$$

By Jensen's inequality, where $m(K)$ stands for Lebesgue measure of K, we have

$$\frac{1}{t}\int_0^t\int_K |u^{\varepsilon,\delta}(x,s) - u_0(x)| \, dxds$$
$$\leq m(K)^{1/2}\left(\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s) - u_0(x)\right)^2 \, dxds\right)^{1/2}.$$

We will establish that

$$\lim_{t\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s) - u_0(x)\right)^2 \, dxds = 0.$$

Let $K_i \subset K_{i+1}$ ($i = 0, 1, \dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i\geq 0} K_i = \mathbb{R}^d$. We use the identity $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$:

$$\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(\cdot,s) - u_0\right)^2 \, dxds$$
$$\leq \frac{1}{t}\int_0^t\left(\int_{K_i} |u^{\varepsilon,\delta}(\cdot,s)|^2 \, dx - \int_{K_i} u_0^2 \, dx - 2\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s) - u_0\right) \, dx\right) ds$$
$$\leq \int_{\mathbb{R}^d\setminus K_i} u_0^2 \, dx + \frac{2}{t}\int_0^t\left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s) - u_0\right) \, dx\right| ds$$

for all $i = 0, 1, \dots$, where we used (3.10)-(3.5).

Since

$$\lim_{i\to\infty}\int_{\mathbb{R}^d\setminus K_i} u_0^2 \, dx = 0,$$

we only consider the last term above. Take $\{\theta_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty}\theta_n = u_0 \quad \text{in} \quad L^2(\mathbb{R}^d),$$

Cauchy-Schwarz inequality gives

$$\left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s) - u_0\right) \, dx\right| \leq \int_{K_i} |u_0 - \theta_n| \left|u^{\varepsilon,\delta}(\cdot,s) - u_0\right| \, dx$$
$$+ \left|\int_{K_i}\theta_n\left(u_0^{\varepsilon,\delta} - u_0\right) + \int_{K_i}\theta_n\left(u^{\varepsilon,\delta}(\cdot,s) - u_0^{\varepsilon,\delta}\right) \, dx\right|$$

$$\leq \|u_0-\theta_n\|_{L^2(\mathrm{R}^d)}\Big(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathrm{R}^d)}+\|u_0\|_{L^2(\mathrm{R}^d)}\Big)$$
$$+\|\theta_n\|_{L^2(\mathrm{R}^d)}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathrm{R}^d)}+\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|.$$

In view of (3.10) and (3.5)

$$\|u_0-\theta_n\|_{L^2(\mathrm{R}^d)}\Big(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathrm{R}^d)}\ +\ \|u_0\|_{L^2(\mathrm{R}^d)}\Big)$$
$$\leq 2\|u_0\|_{L^2(\mathrm{R}^d)}\|u_0-\theta_n\|_{L^2(\mathrm{R}^d)},$$

which tends to zero when $n\to\infty$, and since $\lim_{\varepsilon\to 0+}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathrm{R}^d)}=0$ by (3.5), it remains only to see that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|=0.$$

We have, by (3.3),

$$\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u\,dxd\tau\right|=\left|\int_0^s\int_{K_i}\theta_n\big(-\operatorname{div}f(u)+\varepsilon\operatorname{div}b-\delta\sum_j\partial^3_{x_j}u\big)\,dxd\tau\right|$$
$$=\left|\int_0^s\int_{K_i}\big(\nabla\theta_n\cdot f(u)-\varepsilon\ \nabla\theta_n\cdot b+\delta\sum_j\partial^3_{x_j}\theta_n\big)u\,dxd\tau\right|$$
$$:=\mu_1+\mu_2+\mu_3.$$

To deal with μ_1, we use Hölder inequality and (H_1)

$$\int_0^s\int_{K_i}|\nabla\theta_n|\ \ |f(u)|\ dx\,d\tau\leq C\int_0^s\int_{K_i}|\nabla\theta_n|\ dx\,d\tau$$
$$+C\left[\int_0^s\int_{K_i}|\nabla\theta_n|^{\frac{q}{q-m}}\ dx\,d\tau\right]^{\frac{q-m}{q}}\left[\int_0^s\int_{K_i}|u|^q\,dx\,d\tau\right]^{\frac{m}{q}}$$
$$\leq C\,s\,\|\nabla\theta_n\|_{L^1(\mathrm{R}^d)}+\ C\,s^{\frac{q}{q-m}}\,\|\nabla\theta_n\|_{L^{\frac{q}{q-m}}(\mathrm{R}^d)}\,\|u\|^m_{L^q(\mathrm{R}^d\times(0,T))}.$$

For μ_2, using (H_2) and once more Hölder inequality with (3.11) and (3.5), we get

$$\varepsilon\int_0^s\int_{K_i}\ |\nabla\theta_n|\,|b|\ dx\,d\tau\leq C_3\,\varepsilon\int_0^s\int_{K_i}|\nabla\theta_n|\,|\nabla u|^r\ dx\,d\tau$$
$$\leq C_3\,\varepsilon\left[\int_0^s\int_{K_i}|\nabla\theta_n|^{r+1}\ dxd\tau\right]^{\frac{1}{r+1}}\left[\int_0^s\int_{K_i}|\nabla u|^{r+1}\ dxd\tau\right]^{\frac{r}{r+1}}$$

$$\leq C\,\varepsilon^{1-\frac{r}{r+1}}\,s^{\frac{1}{r+1}}\,\|\,\nabla\theta_n\|_{L^{r+1}(\mathbb{R}^d)}.$$

Finally, for μ_3, we use Cauchy-Schwarz inequality with (3.10) and(3.5):

$$\begin{aligned}
&\delta\int_0^s\int_{K_i}\left|u\sum_j\partial^3_{x_j}\theta_n\right|dx\,d\tau\\
&\qquad\leq\delta\left[\int_0^s\int_{K_i}|u|^2\,dxd\tau\right]^{\frac12}\left[\int_0^s\int_{K_i}\left|\sum_j\partial^3_{x_j}\theta_n\right|^2dx\,d\tau\right]^{\frac12}\\
&\qquad\leq\delta\,s\,\|\,\nabla^3\theta_n\|_{L^2(\mathbb{R}^d)}\,\|u_0\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

thus

$$\begin{aligned}
\lim_{\varepsilon\to0+}\frac1t\int_0^t&\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|ds\leq\lim_{\varepsilon\to0+}\frac1t\left(\frac{C}{2}\,t^2\,\|\,\nabla\theta_n\|_{L^1(\mathbb{R}^d)}\right.\\
&+C\left(\frac{q}{q-m}+1\right)^{-1}t^{\frac{q}{q-m}+1}\,\|\,\nabla\theta_n\|_{L^{\frac{q}{q-m}}(\mathbb{R}^d)}\,\|u^{\varepsilon,\delta}\|^m_{L^q(\mathbb{R}^d\times(0,T))}\\
&+C\left(\frac{1}{r+1}+1\right)^{-1}t^{\frac{1}{r+1}+1}\,\varepsilon^{\frac{1}{r+1}}\,\|\,\nabla\theta_n\|_{L^{r+1}(\mathbb{R}^d)}\\
&\left.+\,\frac{\delta}{2}\,t^2\,\|\,\nabla^3\theta_n\|_{L^2(\mathbb{R}^d)}\,\|u_0\|_{L^2(\mathbb{R}^d)}\right)\\
&\leq\;C_n\left(t+t^{\frac{q}{q-m}}\lim_{\varepsilon\to0+}\|u^{\varepsilon,\delta}\|^m_{L^q(\mathbb{R}^d\times(0,T))}\right)\\
&\leq\;C_n\,t+C_{\varepsilon,\delta}\,t^{\frac{q}{q-m}},
\end{aligned}$$

where we have used (3.16) in Proposition 3.4.2. The desired conclusion when $t\to0+$ follows. □

Proof of Theorems 3.2.2, 3.2.3 and 3.2.4. In the previous proof, to establish (2.13) we started with the identity (3.7) and the condition $\delta=o\big(\varepsilon^{\frac{3}{r+1}}\big)$ as required, in particular to control the term in (3.7). We now keep the form (3.6) instead (3.7): the terms μ_1 and μ_2 introduced in the previous proof do not change. We only need discuss μ_3. It has now the form:

$$-\delta\sum_j\frac{\eta''(u)}{2}\partial_{x_j}(\partial_{x_j}u)^2+\delta\sum_j\partial_{x_j}(\eta'(u)\partial^2_{x_j}u).$$

The first term, concerning Theor. 3.2.2 and 3.2.3, is bounded as follows

$$\left|\delta\int_0^T\int_{\mathbb{R}^d}\sum_j\theta\,\eta''(u)\partial_{x_j}u\,\partial^2_{x_j}u\,dxdt\right|\;\leq C\,\delta\int_0^T\int_{\mathbb{R}^d}\theta\,|\nabla u|\,\left|D^2u\right|\,dxdt$$

$$\leq C\,\delta\ \int_0^T \int_{\mathrm{R}^d} \mu\left|D^2 u\right|^2 + \frac{1}{\mu}\left(\theta\left|\nabla u\right|\right)^2\, dxdt$$
$$\leq C\,\delta\ \left(\mu\,\varepsilon^{-2\frac{r+2}{r+1}} + \frac{1}{\mu}\,\varepsilon^{-\frac{2}{r+1}}\right)$$

using (3.23) and (3.11), and we take $\mu = \varepsilon$ and $\delta = o\big(\varepsilon^{\frac{r+3}{r+1}}\big)$.
The second term in μ_3 behaves better:

$$\delta\ \left|\int_0^T \int_{\mathrm{R}^d} \theta \sum_j \partial_{x_j}(\eta'(u)\partial^2_{x_j}u)\ \ dxdt\right| \leq\ \delta\left|\int_0^T \int_{\mathrm{R}^d} \sum_j \partial^2_{x_j}\theta\,\eta'(u)\partial_{x_j}u\,dxdt\right|$$
$$+\,\delta\left|\int_0^T \int_{\mathrm{R}^d} \sum_j \partial_{x_j}\theta\,\eta''(u)\left(\partial_{x_j}u\right)^2\ dxdt\right|$$
$$\leq\ C\,\delta\int_0^T \int_{\mathrm{R}^d} |\nabla u|\left|D^2\theta\right|\,dxdt + \,C\,\delta\int_0^T \int_{\mathrm{R}^d} |\nabla u|^2\,|\nabla\theta|\,dxdt$$
$$\leq\ C\,\delta\left[\int_0^T \int_{\mathrm{R}^d} |\nabla u|^{r+1}\ dxdt\right]^{\frac{1}{r+1}} +\ C\,\delta\left[\int_0^T \int_{\mathrm{R}^d} |\nabla u|^{r+1}\ dxdt\right]^{\frac{2}{r+1}}$$
$$\leq\ C\,\delta\,\varepsilon^{-\frac{2}{r+1}}.$$

This completes the proof of Theorems 3.2.2 and 3.2.3.
Concerning Theorem 3.2.4, we come back to the first term above, that we bound as:

$$\delta\left|\int_0^T \int_{\mathrm{R}^d} \sum_j \theta\,\eta''(u)\partial_{x_j}u\,\partial^2_{x_j}u\,dx\,dt\right| \leq C\,\delta\,\int_0^T \int_{\mathrm{R}^d} \mu\left(\theta\left|\mathrm{diag}(D^2u)\right|\right)^2$$
$$+\frac{1}{\mu}\,|\nabla u|^2\ dx\,dt \leq C\,\delta\,\left(\mu\,\varepsilon^{-3(2-m)} + \frac{1}{\mu}\,\varepsilon^{-1}\right)$$

that we can optimize taking $\mu = \varepsilon^{\frac{5-3m}{2}}$, $\delta = o(\varepsilon^{\frac{7-3m}{2}})$, and, because the second term is the same,

$$|\langle\mu_3,\theta\rangle| \leq\ C\,\delta\left(\varepsilon^{\frac{3m-7}{2}} +\ \varepsilon^{-1}\right) \leq\ C\,\delta\,\varepsilon^{\min\{-1,\frac{3m-7}{2}\}}.$$

Finally the condition $\delta = o(\varepsilon^{\max\{1,\frac{7-3m}{2}\}})$ is sufficient to imply the desired conclusion.

□

Chapter 4

BBMB and KdVB Equations[1]

Abstract. We analyse, in the setting of DiPerna's measure-valued solution theory, conditions under which solutions of multi-dimensional nonlinear BBMB- and KdVB-like equations converge to the classical entropy weak solution of a limit conservation law. The main conditions concern the balance between diffusion and dispersion, and lead us to guess the non-existence of nonclassical solutions or to locate the frontier where these can be formed. Unequal convergence behaviour for the BBMB and KdVB equations must emphasize limitations of "Whitham's" change between time and space derivatives.

4.1 Assumptions

We study here the convergence, as ε, δ tend to zero, of solutions for the multi-dimensional and generalized Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div} b(\nabla u) + \delta \operatorname{div} \partial_t\, c(\nabla u)\,,$$

and, changing the right-hand time-derivative by a derivative in space, the generalized Korteweg-deVries-Burgers (KdVB) equation

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div} b(\nabla u) + \delta \operatorname{div} \partial_{x_k}\, c(\nabla u)\,,$$

to the limit entropy weak solution of the conservation law

$$\partial_t u + \operatorname{div} f(u) = 0\,.$$

The diffusion function b will always be nonlinear, but the dispersion function c can be linear or *nonlinear*.

[1]From our paper [4]

As before, a dominant diffusion regime is natural. It is assured by two main ways. One concern optimal δ/ε balance. And, if we conjecture it is sharp, lead us to guess non-existence of nonclassical solutions or to locate the frontier where these solutions can be formed. The other concern the growth competition between the diffusion and the dispersion. In the case of the BBMB model, this involve also the flux growth, fixing the L^q space where convergence can be established. Inversely, for KdVB model we can handle, in general, a arbitrarily large L^q space. This put in evidence relevant differences between both the models. In particular, such unequal convergence behaviour shows that "Whitham's" change between time and space derivatives must be non-trivial.

Emphasis is on general energy arguments and understanding growth competition involving, possibly, the flux, the diffusion and the dispersion functions, so, e.g., we do not use specific dimension arguments.

Next, we put together all the hypothesis we need.

Let $u^{\varepsilon,\delta} : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, defined on an interval $[0,T]$ with a uniform T (independent of ε, δ), rapidly decaying at infinity, be smooth solutions to one of the initial value problems for the BBMB or the KdVB equations, accordingly ∂_ξ is ∂_t or ∂_{x_k}:

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div} b(\nabla u) + \delta \operatorname{div} \partial_\xi c(\nabla u)\,, \tag{4.1}$$

$$u(x,0) = u_0^{\varepsilon,\delta}(x), \tag{4.2}$$

where $u_0^{\varepsilon,\delta}$ is a convenient regularized approximation of the data $u_0 : \mathbb{R}^d \to \mathbb{R}$ for the perturbed conservation law

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times [0,+\infty[\,, \tag{4.3}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{4.4}$$

Throughout, it is assumed $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and that $u_0^{\varepsilon,\delta}$ are smooth functions with compact support, uniformly bounded in $L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $q \geq 2$. Restricting attention to the diffusion-dominant regime we regard $\delta = \delta(\varepsilon)$ and we suppose that $u_0^{\varepsilon,\delta}$ approaches the initial condition u_0 in the sense that

$$\begin{aligned} &\lim\nolimits_{\varepsilon\to 0^+} u_0^{\varepsilon,\delta} = u_0 \quad \text{in} \;\; L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), \\ &\|u_0^{\varepsilon,\delta}\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)}, \\ &\delta \,\|\, \nabla u_0^{\varepsilon,\delta}\|_{L^{\rho+1}(\mathbb{R}^d)} = o(\varepsilon^\gamma), \qquad \text{(BBMB eq.)} \end{aligned} \tag{4.5}$$

According to the L^p-Young measure setting, for the smooth flux, $f : \mathbb{R} \to \mathbb{R}^d$, we need to suppose a growth control:

(H_1) $\exists m \geq 1, \exists c_1 > 0 : \quad |f'(u)| \leq c_1 \, |u|^{m-1}, \quad \forall u \in \mathbb{R}.$

We generalize the usual linear functions or $|\nabla(u)|^{r-1}\nabla(u)$ examples of diffusion or dispersion as smooth gradients, $b = \nabla B$ and $c = \nabla C$, with positively homogeneous potentials $B, C : \mathbb{R}^d \to \mathbb{R}$ of $r+1$, $\rho+1 \geq 2$ degree. Define $c_2 := \max\limits_{|\lambda|=1} |\, b(\lambda)|$ and $c_3 := \max\limits_{|\lambda|=1} |\, c(\lambda)|$, we have

(H_2) $\exists r \geq 1, \exists c_2 > 0 : \quad |\, b(\lambda)\,| \leq c_2 \, |\lambda|^r, \quad \forall \lambda \in \mathbb{R}^d;$

(H_3) $\exists \rho \geq 1, \exists c_3 > 0 : \quad |\, c(\lambda)\,| \leq c_3 \, |\lambda|^\rho, \quad \forall \lambda \in \mathbb{R}^d.$

Define also $d_2 := \min\limits_{|\lambda|=1} B(\lambda)$, $d_3 := \min\limits_{|\lambda|=1} C(\lambda)$ and assume d_2, d_3 are positive, then (in that concern the diffusion b, it is the "usual" *diffusion hypothesis*)

(H_4) $\exists d_2 > 0 : \quad \lambda \cdot b(\lambda) \geq (r+1)\, d_2 \, |\lambda|^{r+1}, \;\; \forall \lambda \in \mathbb{R}^d;$

(H_5) $\exists d_3 > 0 : \quad C(\lambda) \geq d_3 \, |\lambda|^{\rho+1}, \;\; \forall \lambda \in \mathbb{R}^d.$

About potential C, we ask again the strict convexity hypothesis:

(H_6) $\exists d_1 > 0 : \forall \vec{v} \in \mathbb{R}^d, \quad \vec{v}^{\,t} \, D^2 C(\lambda)\, \vec{v} \geq d_1 \, |\lambda|^{\rho-1} \, |\, \vec{v}\,|^2, \;\; \forall \lambda \in \mathbb{R}^d.$

In the next chapter, we will see that, in very less restrictive hypothesis, the results we obtain here about KdVB are true to the more general equation:

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div} \big(\varepsilon \, b_j(u, \nabla u) + \delta \sum_k \partial_{x_k} c_{jk}(\nabla u)\big)_{1 \leq j \leq d}.$$

4.2 Main Results

We state here the convergence theorems we will next prove. The energy techniques we use ask, for a given dispersion growth of order $\rho \geq 1$, a diffusion growth of, at least, order $r = 2\rho + 1$ for the BBMB equation and $r = \rho + 1$ for the KdVB equation.

In the BBMB equation, besides the diffusion-dispersion growth competition, also the flux-function growth is fundamental to determine the $L^{\alpha+1}$ space where we prove strong convergence. In some cases the $L^{\alpha+1}$ space is specific according to the choice of the best δ/ε balance. This is in deep contrast with the KdVB equation where we can take arbitrarily large $L^{\alpha+1}$ spaces.

Also, because we suppose that the δ/ε balance we obtain, at least in some cases, is sharp, the strong convergence frontier must be differently located for the two equations.

These issues, that we have commented previously and we do not explore here, seem to provide an interesting comparison between the BBMB and KdVB models.

For the sake of simplicity in the next statements, we define

$$M_\rho := 2\,\frac{\rho+1}{\rho-1}(m-1); \quad M_r := 2\,\frac{r+1}{r-1}(m-1);$$

$$M_* := \frac{6(r+1)}{r+3+2\rho}; \quad M_s := 4+2\,\frac{r-(2\rho+1)}{r+1}.$$

Theorem 4.2.1 (BBMB). *Consider the Cauchy problem* (4.3)-(4.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and the flux* f *satisfies* (H_1) *for some known* m*. Let* $u^{\varepsilon,\delta}$ *be the solutions of the approaching problems* (4.1)-(4.2) *with diffusion and dispersion satisfying* (H_2), (H_4) *and* (H_3), (H_5), (H_6) *for* $1 \le \rho \le 3$ *and* $r \ge 2\rho+1$.

If $\delta = \mathcal{O}\big(\varepsilon^{\frac{\rho+1}{2}}\big)$, *(*$o(\varepsilon)$ *if* $\rho = 1$ *and* $r = 3$*), and* $q \ge \alpha+1$*, the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$*, for all* $s < \infty$ *and* $p < \alpha+1$*, to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^{\alpha+1}(\mathbb{R}^d)\right)$*, which is the unique entropy solution to* (4.3)-(4.4)*, where* $\alpha+1$ *is given by*

$$\alpha+1 = \begin{cases} M_\rho, & \text{if } \frac{5\rho-1}{2(\rho+1)} \le m < 2\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)}; \\ M_s, & \text{if } 2\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)} \le m < \frac{5r-1}{2(r+1)}. \end{cases}$$

Theorem 4.2.2 (BBMB). *Consider the Cauchy problem* (4.3)-(4.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and the flux* f *satisfies* (H_1) *for some known* m*. Let* $u^{\varepsilon,\delta}$ *be the solutions of the approaching problems* (4.1)-(4.2) *with diffusion and dispersion satisfying* (H_2), (H_4) *and* (H_3), (H_5), (H_6) *for* $\rho > 3$.

If $\delta = \mathcal{O}\big(\varepsilon^{\frac{\rho+1}{2}}\big)$ *and* $q \ge \alpha+1$*, then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$*, for all* $s < \infty$ *and* $p < \alpha+1$*, to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^{\alpha+1}(\mathbb{R}^d)\right)$*, which is the unique entropy solution to* (4.3)-(4.4)*, where* $\alpha+1 = M_\rho$ *if* $2\rho+1 \le r < \frac{4\rho^2-9\rho-1}{3(\rho-3)}$, $\frac{5\rho-1}{2(\rho+1)} \le m < 2\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)}$ *or* $r \ge \frac{4\rho^2-9\rho-1}{3(\rho-3)}$, $\frac{5\rho-1}{2(\rho+1)} \le m < \frac{5r-1}{2(r+1)}$*; and* $\alpha+1 = M_s$ *if* $2\rho+1 \le r < \frac{4\rho^2-9\rho-1}{3(\rho-3)}$, $2\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)} \le m < \frac{5r-1}{2(r+1)}$.

Theorem 4.2.3 (BBMB). *Consider the Cauchy problem* (4.3)-(4.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and the flux* f *satisfies* (H_1) *for some known* m*. Let* $u^{\varepsilon,\delta}$ *be the solutions of the approaching problems* (4.1)-(4.2) *with diffusion* *and dispersion satisfying* (H_2), (H_4) *and* (H_3), (H_5), (H_6) *for* $\rho \geq 1$ *and* $r \geq 2\rho + 1$.

If $\delta = \mathcal{O}\big(\varepsilon^{\frac{2\left(M_r - \frac{\rho+1}{r+1} M_*\right)}{M_* (M_r - 2)}}\big)$, $\frac{5r-1}{2(r+1)} \leq m < 2\frac{2r+\rho}{r+3+2\rho}$ *and* $q \geq \alpha + 1 = M_r$, *then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, *for all* $s < \infty$ *and* $p < \alpha + 1$, *to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^{\alpha+1}(\mathbb{R}^d)\right)$, *which is the unique entropy solution to* (4.3)-(4.4).

For each of these three results, conjecture that the $\delta = \mathcal{O}(\varepsilon^\gamma)$ hypothesis we make is sharp. Then, nonclassical solutions should not exist or a stripe-frontier is formed, a new phenomena. (In opposite way, if we have nonclassical solutions, then probably our result is not sharp.)

Theorem 4.2.4 (BBMB). *Consider the Cauchy problem* (4.3)-(4.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and the flux* f *satisfies* (H_1) *for some known* m. *Let* $u^{\varepsilon,\delta}$ *be the solutions of the approaching problems* (4.1)-(4.2) *with diffusion and dispersion satisfying* (H_2), (H_4) *and* (H_3), (H_5), (H_6) *for* $\rho \geq 1$ *and* $r \geq 2\rho + 1$.

If $\delta = o\big(\varepsilon^{\frac{1}{2}} + \frac{\rho+1}{r+1}\big)$ *and* $q \geq \alpha + 1$, *then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, *for all* $s < \infty$ *and* $p < \alpha + 1$, *to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^{\alpha+1}(\mathbb{R}^d)\right)$, *which is the unique entropy solution to* (4.3)-(4.4), *where* $\alpha + 1$ *is given by*

$$\alpha + 1 = \begin{cases} M_r, & \text{if} \quad 2\frac{2r+\rho}{r+3+2\rho} \leq m < 2\frac{2r^2 - r\rho + \rho}{(r+1)^2}; \\ M_s, & \text{if} \quad 2\frac{2r^2 - r\rho + \rho}{(r+1)^2} \leq m \leq 2 + 2\frac{r-\rho-1}{r+1}. \end{cases}$$

We remark that in all instances above, we have $\alpha + 1 > m$ and $\alpha + 1 > 2$.

Theorem 4.2.5 (KdVB). *Consider the Cauchy problem* (4.3)-(4.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and suppose that the flux* f *satisfies* (H_1) *with* $m < q$ *(which is always possible if* q *is large enough).*

Let $u^{\varepsilon,\delta}$ *be the solutions of the perturbed problem* (4.1)-(4.2) *with diffusion and dispersion satisfying* (H_2), (H_4) *and* (H_3) *such that* $r \geq \rho + 1$. *If* $\delta = o\big(\varepsilon^{\frac{\rho+2}{r+1}}\big)$, *then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\left((0,T); L^p_{loc}(\mathbb{R}^d)\right)$, *for all* $s < \infty$ *and* $p < q$, *to a function* $u \in L^\infty\left((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\right)$, *which is the unique entropy solution to* (4.3)-(4.4).

Note that this result agree with Theorem 3.2.1, p.33, where $\rho = 1$.

4.3 First Energy Estimates

Consider the equation (from now on, except if emphasis is necessary, the superscripts ε and δ are omitted)

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div} b(\nabla u) + \delta \operatorname{div} \partial_\xi c(\nabla u)\,. \tag{4.6}$$

Multiply by $\eta'(u)$. If $q' = \eta' f'$, $\nabla B = b$ and $\nabla C = c$, with homogeneous diffusion and dispersion potentials of degree $r+1$ and $\rho+1$,

$$\begin{aligned}\partial_t\, \eta(u) + \operatorname{div}\, q(u) &= \varepsilon \operatorname{div}\big(\eta'(u)\, b(\nabla u)\big) - \varepsilon\,(r+1)\,\eta''(u)\, B(\nabla u)\\ &\quad + \delta \operatorname{div}\big(\eta'(u)\,\partial_\xi\, c(\nabla u)\big) - \delta\,\rho\,\eta''(u)\,\partial_\xi\, C(\nabla u)\,.\end{aligned} \tag{4.7}$$

Then, with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$, integrate over $\mathbb{R}^d \times [0,t]$:

Lemma 4.3.1. *Let $\alpha \geq 1$ and $B, C : \mathbb{R}^d \to \mathbb{R}$ be diffusion and dispersion homogeneous potentials of degree $r+1$ and $\rho+1$. Each solution of* (4.6) *satisfies, for $t \in [0,T]$,*

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx &+ (\alpha+1)\,\alpha\,(r+1)\,\varepsilon \int_0^t\!\!\int_{\mathrm{R}^d} |u|^{\alpha-1}\, B(\nabla u)\, dxds\\ &+ (\alpha+1)\,\alpha\,\rho\,\delta \int_0^t\!\!\int_{\mathrm{R}^d} |u|^{\alpha-1}\,\partial_\xi\, C(\nabla u)\, dxds\\ &= \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx\,.\end{aligned} \tag{4.8}$$

For $\alpha \geq 2$, if ∂_ξ is ∂_t we have also

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx &+ (\alpha+1)\,\alpha\,(r+1)\,\varepsilon \int_0^t\!\!\int_{\mathrm{R}^d} |u|^{\alpha-1}\, B(\nabla u)\, dxds\\ &+ (\alpha+1)\,\alpha\,\rho\,\delta \int_{\mathrm{R}^d} |u(t)|^{\alpha-1}\, C(\nabla u(t))\, dx\\ &= \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx + (\alpha+1)\,\alpha\,\rho\,\delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\, C(\nabla u_0)\, dx\\ &\quad + (\alpha+1)\,\alpha\,(\alpha-1)\,\rho\,\delta \int_0^t\!\!\int_{\mathrm{R}^d} \operatorname{sgn}(u)\, |u|^{\alpha-2}\,\partial_t u\, C(\nabla u)\, dxds\,;\end{aligned} \tag{4.9}$$

and, if ∂_ξ is ∂_{x_k}

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx &+ (\alpha+1)\,\alpha\,(r+1)\,\varepsilon \int_0^t\!\!\int_{\mathrm{R}^d} |u|^{\alpha-1}\, B(\nabla u)\, dxds\\ &= \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx\\ &\quad + (\alpha+1)\,\alpha\,(\alpha-1)\,\rho\,\delta \int_0^t\!\!\int_{\mathrm{R}^d} \operatorname{sgn}(u)\, |u|^{\alpha-2}\,\partial_{x_k} u\, C(\nabla u)\, dxds.\end{aligned} \tag{4.10}$$

4.3.1 BBMB Equation

We obtain the BBMB first energy estimates from Lemma 4.3.1, with $\alpha = 1$ and $\partial_\xi = \partial_t$ in formula (4.8):

Proposition 4.3.1. *For any solution of* (4.6) *we have, in the conditions of* Lemma 4.3.1 *and by* (H_3)-(H_5),

$$\int_{\mathbb{R}^d} u(t)^2\,dx + 2\,d_2\,(r+1)\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |\,\nabla u\,|^{r+1}\,dxds + 2\,d_3\,\rho\,\delta \int_{\mathbb{R}^d} |\,\nabla u(t)\,|^{\rho+1}\,dx \le \|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}. \tag{4.11}$$

We want, also, to control the last term in (4.9): estimating $\partial_t u$. Multiply the equation (4.6) by $\varepsilon\,|\partial_t u|^\beta\,\partial_t u$,

$$\begin{aligned}\varepsilon\,|\partial_t u|^{\beta+2} + \varepsilon\,|\partial_t u|^\beta\,\partial_t u\,f'(u)\cdot\nabla u = \varepsilon^2\ &\mathrm{div}\,\big(|\partial_t u|^\beta\,\partial_t u\,b(\nabla u)\big)\\ &- (\beta+1)\,\varepsilon^2\,|\partial_t u|^\beta\,\partial_t\,B(\nabla u)\\ &+ \delta\,\varepsilon\ \mathrm{div}\,\big(|\partial_t u|^\beta\,\partial_t u\,\partial_t\,c(\nabla u)\big)\\ &- (\beta+1)\,\delta\,\varepsilon\,|\partial_t u|^\beta\ \nabla\partial_t u\,D^2C(\nabla u)\,\nabla\partial_t u,\end{aligned}$$

and integrate over $\mathbb{R}^d \times [0,t]$:

Lemma 4.3.2. *Let* $\beta \ge 0$, $B, C : \mathbb{R}^d \to \mathbb{R}$ *be the diffusion and dispersion homogeneous potentials. Each solution of* (4.6) *satisfies, for* $t \in [0,T]$,

$$\begin{aligned}\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^{\beta+2}\,dxds &+ (\beta+1)\,\varepsilon^2 \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^\beta\,\partial_t\,B(\nabla u)\,dxds\\ &+ (\beta+1)\,\delta\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^\beta\ \nabla\partial_t u\,D^2C(\nabla u)\,\nabla\partial_t u\,dxds\\ &= -\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^\beta\,\partial_t u\,f'(u)\cdot\nabla u\,dxds\,.\end{aligned}$$

Taking $\beta = 0$ and using Cauchy-Schwartz inequality, we obtain the

Proposition 4.3.2. *For any solution of* (4.6) *we have, in the conditions of* Lemma 4.3.2 *and by* (H_1), (H_2), (H_4) *and* (H_6),

$$\begin{aligned}\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^2\,dxds &+ 2\,d_2\,\varepsilon^2 \int_{\mathbb{R}^d} |\,\nabla u(t)\,|^{r+1}\,dx\\ &+ 2\,d_1\,\delta\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |\,\nabla\partial_t u\,|^2\,|\,\nabla u\,|^{\rho-1}\,dxds\\ &\le 2\,c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1} + c_1^2\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds\,.\end{aligned} \tag{4.12}$$

4.3.2 KdVB Equation

Once more, still with $\alpha = 1$ but $\partial_\xi = \partial_{x_k}$ in formula (4.8), Lemma 4.3.1, we deduce this way the KdVB first energy estimates:

Proposition 4.3.3. *For any solution of* (4.6), *with diffusion verifying* (H_4), *we have for* $t \in [0, T]$

$$\int_{\mathrm{R}^d} u(t)^2\, dx + 2\, d_2\, (r+1)\, \varepsilon \int_0^t \int_{\mathrm{R}^d} |\, \nabla u\, |^{r+1}\, dxds \le \|u_0\|_{L^2\left(\mathrm{R}^d\right)}. \tag{4.13}$$

4.4 $\mathbf{L^q}$ Estimates

We ask, here, for higher than the uniform L^2 a priori estimates that we get by Proposition 4.3.1 and Proposition 4.3.3.

4.4.1 BBMB Estimates

We obtain a lower bound for the left-hand side of (4.9) using (H_4) and (H_5) and an upper bound for the right-hand side using (H_3), Cauchy-Schwartz and Prop.4.3.2:

$$\begin{aligned}
&\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx + \; (\alpha+1)\, \alpha\, d_3\, \rho\, \delta \int_{\mathrm{R}^d} |u(t)|^{\alpha-1}\, |\, \nabla u(t)\, |^{\rho+1}\, dx \\
&\quad + (\alpha+1)\, \alpha\, d_2\, (r+1)\, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\, |\, \nabla u\, |^{r+1}\, dxds \\
&\le \|u_0\|_{\alpha+1}^{\alpha+1} \; + \; (\alpha+1)\, \alpha\, c_3\, \rho\, \delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\, |\, \nabla u_0\, |^{\rho+1}\, dx \\
&\quad + \varepsilon/2 \int_0^t \int_{\mathrm{R}^d} |\partial_t u|^2\, dxds + \; \left((\alpha+1)\, \alpha\, (\alpha-1)\, c_3\, \rho\, \delta\, \varepsilon^{-1}\right)^2 \\
&\qquad \varepsilon/2 \int_0^t \int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\, |\, \nabla u\, |^{2(\rho+1)}\, dxds \\
&\le \|u_0\|_{\alpha+1}^{\alpha+1} \; + \; (\alpha+1)\, \alpha\, c_3\, \rho\, \delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\, |\, \nabla u_0|^{\rho+1}\, dx \\
&\quad + c_2\, \varepsilon^2\, \|\, \nabla u_0\|_{r+1}^{r+1} + \; \left((\alpha+1)\, \alpha\, (\alpha-1)\, c_3\, \rho\right)^2 /2 \\
&\qquad \delta^2\, \varepsilon^{-1} \int_0^t \int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\, |\, \nabla u\, |^{2(\rho+1)}\, dxds \\
&\quad + c_1^2/2\, \varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(m-1)}\, |\, \nabla u\, |^2\, dxds\, .
\end{aligned} \tag{4.14}$$

To achieve our purpose, we will assimilate last two terms in the first member. This is done by the use of judicious Young's inequalities, solving the two terms cross-dependence and involving a competition between the parameters m, ρ and r.

The α-term.

Using the terms from the first member of (4.14) or those previously estimated in Proposition 4.3.1, we have:

$$\begin{aligned}\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathrm{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \leq\ & \delta^{2-\left(\frac{1}{p_3}+\frac{1}{p_4}\right)}\,\varepsilon^{-\left(1+\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & \left(\frac{\varepsilon}{p_1}\int_0^t\int_{\mathrm{R}^d}|\,\nabla u\,|^{r+1}\,dxds+\frac{\varepsilon}{p_2}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds\right. \\ & +\frac{\delta}{p_3}\int_0^t\int_{\mathrm{R}^d}|\,\nabla u\,|^{\rho+1}\,dxds+\frac{\delta}{p_4}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{\rho+1}\,dxds \\ & \left.+\frac{1}{p_5}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds\right),\end{aligned}$$

where

$$\frac{1}{p_1}+\cdots+\frac{1}{p_5}=1\,;\quad (r+1)\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+(\rho+1)\left(\frac{1}{p_3}+\frac{1}{p_4}\right)=2\,(\rho+1)\,;$$

$$\frac{\alpha-1}{p_2}+\frac{\alpha-1}{p_4}+\frac{\alpha+1}{p_5}=2\,(\alpha-2)\,;\qquad \text{and } \delta=\mathcal{O}(\varepsilon^\gamma) \text{ such that}$$

$$\gamma\left(2-\left(\frac{1}{p_3}+\frac{1}{p_4}\right)\right)=1+\left(\frac{1}{p_1}+\frac{1}{p_2}\right).$$

So, we have the system

$$\begin{aligned}&\frac{1}{p_1}+\frac{1}{p_2}=\frac{\rho+1}{r+1}\left(2-\left(\frac{1}{p_3}+\frac{1}{p_4}\right)\right);\\ &\frac{1}{p_5}=\frac{r-(2\rho+1)}{r+1}-\frac{r-\rho}{r+1}\left(\frac{1}{p_3}+\frac{1}{p_4}\right);\\ &\alpha+1=2\,\frac{3-2\,\frac{\rho+1}{r+1}+\frac{1}{p_1}+\frac{\rho+1}{r+1}\frac{1}{p_3}-\frac{r-\rho}{r+1}\frac{1}{p_4}}{1+\frac{1}{p_1}+\frac{1}{p_3}};\\ &\gamma=\frac{(r+3+2\rho)-(\rho+1)\left(\frac{1}{p_3}+\frac{1}{p_4}\right)}{(r+1)\left(2-\left(\frac{1}{p_3}+\frac{1}{p_4}\right)\right)}.\end{aligned}$$

A elementar, but long, tedious, analysis imposes the conclusions, as $r \geq 2\rho + 1$:

$\frac{1}{p_3} + \frac{1}{p_4} \in \left[0, \frac{r-(2\rho+1)}{r-\rho}\right]$, which correspond to increasing γ values from a $\min(\gamma) = \frac{1}{2} + \frac{\rho+1}{r+1}$ to the $\max(\gamma) = 1$. These, in some sense, agree with a, respectively, $\max(\alpha+1) = 4 + 2\,\frac{r-(2\rho+1)}{r+1}$ and the $\min(\alpha+1) = 3$. In fact, it will be relevant to know that $\alpha+1$ solutions associated to $\min(\gamma)$ are available between $\frac{6(r+1)}{r+3+2\rho}$ and the $\max(\alpha+1)$, but that (continuity) $\gamma < 1$ is equivalent to $\alpha + 1 > 3$.

The m-term.

Analysis here looks slightly different. This is because, e.g., $\rho + 1 \geq 2$:

$$\begin{aligned}\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds \leq\ & \delta^{-\left(\frac{1}{p_3}+\frac{1}{p_4}\right)}\,\varepsilon^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & \left(\frac{\varepsilon}{p_1} \int_0^t \int_{\mathrm{R}^d} |\,\nabla u\,|^{r+1}\,dxds + \frac{\varepsilon}{p_2} \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds\right. \\ & +\frac{\delta}{p_3} \int_0^t \int_{\mathrm{R}^d} |\,\nabla u\,|^{\rho+1}\,dxds + \frac{\delta}{p_4} \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{\rho+1}\,dxds \\ & \left. + \frac{1}{p_5} \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds\right),\end{aligned}$$

and we must have

$$\frac{1}{p_1} + \cdots + \frac{1}{p_5} = 1\,; \quad (r+1)\left(\frac{1}{p_1} + \frac{1}{p_2}\right) + (\rho+1)\left(\frac{1}{p_3} + \frac{1}{p_4}\right) = 2\,;$$

$$\frac{\alpha-1}{p_2} + \frac{\alpha-1}{p_4} + \frac{\alpha+1}{p_5} = 2\,(m-1)\,; \quad \text{and } \delta = \mathcal{O}\left(\varepsilon^{\beta}\right) \text{ such that}$$

$$\beta\left(\frac{1}{p_3} + \frac{1}{p_4}\right) = 1 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right).$$

The system is now, with $m > 1$, $\frac{1}{p_1} + \frac{1}{p_2} \in \left[0, \frac{2}{r+1}\right]$,

$$\frac{1}{p_3} + \frac{1}{p_4} = \frac{2}{\rho+1} - \frac{r+1}{\rho+1}\left(\frac{1}{p_1} + \frac{1}{p_2}\right);$$

$$\frac{1}{p_5} = \frac{\rho-1}{\rho+1} + \frac{r-\rho}{\rho+1}\left(\frac{1}{p_1} + \frac{1}{p_2}\right);$$

$$\alpha + 1 = 2\left(1 + \frac{m - \frac{2\rho}{\rho+1} - \frac{r-\rho}{\rho+1}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)}{1 - \frac{1}{p_1} - \frac{1}{p_3}}\right);$$

$$\beta\left(\frac{2}{\rho+1}-\frac{r+1}{\rho+1}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\right)=1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right).$$

For $\frac{1}{p_1}+\frac{1}{p_2}\in\left[0,\frac{2}{r+1}\right[$, β is a increasing function with $\min(\beta)=\frac{\rho+1}{2}\geq 1$ and $\sup(\beta)=+\infty$.

In fact, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{2}{r+1}$ (iff $\frac{1}{p_3}+\frac{1}{p_4}=0$) corresponds to the case where we don't use δ-terms in Young's inequality: if we are able to solve our problem in both these regimes (with or without β-constraints), the latter will be better than the former because of the best δ/ε balance, remember that $\max(\gamma)=1$, against $\min(\beta)=\frac{\rho+1}{2}\geq 1$.

Thus, we need to distinguish between several possibilities.

The Case $\mathbf{m=1}$. We have $m=1$ iff $\frac{1}{p_2}+\frac{1}{p_4}+\frac{1}{p_5}=0$ and then, necessarily, $\rho=1$ and $\frac{1}{p_1}=0$, $\frac{1}{p_3}=1$ and $\beta=1$, but $\alpha+1$ arbitrary.

So, when $m=1$, $\rho=1$ and $r\geq 3$ our problem has a solution in $L^{4+2\frac{r-3}{r+1}}$ if $\delta=\mathcal{O}(\varepsilon)$. When $m=1$ and $\rho>1$, it is a open problem.

Proposition 4.4.1. *Assume that (H_1)-(H_6) holds with $m=1$, $\rho=1$, $r\geq 3$ and $u_0\in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$, $\nabla u_0\in\left(L^{\rho+1}(\mathbb{R}^d)\cap L^{r+1}(\mathbb{R}^d)\right)^d$, for some $q\geq \alpha+1=4+2\frac{r-3}{r+1}$. We have for $t\in[0,T]$*

$$\begin{aligned}&\int_{\mathbb{R}^d}|u(t)|^{\alpha+1}\,dx+\ (\alpha+1)\,\alpha\,d_3\,\delta\int_{\mathbb{R}^d}|u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^2\,dx\\&\quad+(\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds\\&\quad\leq 3\,H(\delta,\varepsilon),\end{aligned}$$

with, for definiteness,

$$\begin{aligned}H(\delta,\varepsilon):=\ &\|u_0\|_{\alpha+1}^{\alpha+1}\ +\ (\alpha+1)\,\alpha\,c_3\,\delta\int_{\mathbb{R}^d}|u_0|^{\alpha-1}\,|\,\nabla u_0|^2\,dx\\&+c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1}+\left(\frac{c_1^2\,t\,(\delta^{-1}\,\varepsilon)}{4\,d_3}\ +\ \left(\frac{r-3}{r+1}\right)^{\frac{r-3}{4}}\right.\\&\frac{c_3^{\frac{r+1}{2}}\,t^{\frac{r-3}{4}}\left(\delta^2\,\varepsilon^{-\left(1+\frac{4}{r+1}\right)}\right)^{\frac{r+1}{4}}}{d_2\,(r+1)^2}\\&\left.\left[\left(4+2\frac{r-3}{r+1}\right)\left(3+2\frac{r-3}{r+1}\right)\left(2+2\frac{r-3}{r+1}\right)\right]^{\frac{r+1}{2}}\right)\end{aligned}$$

$$\left(\|u_0\|_2^2 + 2\,c_3\,\delta\,\|\,\nabla u_0\|_2^2\right),$$

where, if $\delta = \mathcal{O}\,(\varepsilon)$*, then* $H(\delta,\varepsilon) \leq const$ *.*

Proof. Because $m = 1$, we must have $\rho = 1$ and $\beta = 1$. Then, $r \geq 2\rho+1 = 3$, $\delta = \mathcal{O}\,(\varepsilon)$, and (4.14) writes

$$\begin{aligned}(4.15)\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\,dx +\ (\alpha+1)\,\alpha\,d_3\,\delta \int_{\mathrm{R}^d} |u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^2\,dx \\ +(\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon \int_0^t\int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds \\ \leq \|u_0\|_{\alpha+1}^{\alpha+1}\ +\ (\alpha+1)\,\alpha\,c_3\,\delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\,|\,\nabla u_0|^2\,dx \\ +c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1} +\ c_1^2/2\ \varepsilon \int_0^t\int_{\mathrm{R}^d} |\,\nabla u\,|^2\,dxds \\ +\frac{\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\right)^2}{2}\ \delta^2\,\varepsilon^{-1} \int_0^t\int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\,|\,\nabla u\,|^4\,dxds\,.\end{aligned}$$

By Proposition 4.3.1 with $\rho = 1$,

$$\begin{aligned}(4.16)\quad c_1^2/2\ \left(\delta^{-1}\,\varepsilon\right)\delta \int_0^t\int_{\mathrm{R}^d}\ |\,\nabla u\,|^2\,dxds \\ \leq \frac{c_1^2\,t\ (\delta^{-1}\,\varepsilon)}{4\,d_3}\ \left(\|u_0\|_2^2 + 2\,c_3\,\delta\,\|\,\nabla u_0\|_2^2\right);\end{aligned}$$

about the last term (the α-term, p. 57), taking in the Young's inequality $\frac{1}{p_3} + \frac{1}{p_4} = 0$, we guarantee the $\max(\alpha+1) = 4 + 2\frac{r-3}{r+1}$ and $\frac{1}{p_5} = \frac{r-3}{r+1}$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{4}{r+1}$: make $\frac{1}{p_2} = 0$, then

$$\begin{aligned}(4.17)\qquad \frac{\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\right)^2}{2}\,\delta^2\,\varepsilon^{-1} \int_0^t\int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\,|\,\nabla u\,|^4\,dxds \\ \leq \frac{1}{2\,t} \int_0^t\int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds + \Bigg[\frac{\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\right)^2}{2} \\ \left(\frac{p_5}{2\,t}\right)^{-\frac{1}{p_5}}\delta^2\,\varepsilon^{-\left(1+\frac{4}{r+1}\right)}\Bigg]^{p_1}\ \frac{\varepsilon}{p_1} \int_0^t\int_{\mathrm{R}^d} |\,\nabla u\,|^{r+1}\,dxds\,.\end{aligned}$$

Integrate (4.15) over $[0,t]$ and use (4.16) and (4.17) together with Proposition 4.3.1

$$\frac{1}{2}\int_0^t\int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds +\ (\alpha+1)\,\alpha\,d_3\,\delta \int_0^t\int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^2\,dxds$$

$$+(\alpha+1)\,\alpha\,d_2\,(r+1)\,t\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\leq t\,H(\delta,\varepsilon).$$

So (4.17) becomes

$$\frac{((\alpha+1)\,\alpha\,(\alpha-1)\,c_3)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathrm{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^4\,dxds\leq 2\,H(\delta,\varepsilon),$$

and finally (4.15) gives

$$\int_{\mathrm{R}^d}|u(t)|^{\alpha+1}\,dx+\;(\alpha+1)\,\alpha\,d_3\,\delta\int_{\mathrm{R}^d}|u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^2\,dx$$
$$+(\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\leq 3\,H(\delta,\varepsilon).$$

□

The Case m > 1, Without δ-terms. To remain concise, we will retain only the parameter critical points.

Here, we don't use δ-terms in Young's inequality. Then, for $\frac{1}{p_2}\in\left[0,\frac{2}{r+1}\right]$,

$$\frac{1}{p_5}=\frac{r-1}{r+1};\quad\frac{1}{p_1}=\frac{2}{r+1}-\frac{1}{p_2};\quad\alpha+1=2\,\frac{m-1+\frac{1}{p_2}}{\frac{r-1}{r+1}+\frac{1}{p_2}}\,.$$

Now, our problem is solvable if $\alpha+1$ belongs to $\left[3,4+2\frac{r-(2\rho+1)}{r+1}\right]$. As function of the variable $\frac{1}{p_2}$, $\alpha+1$ decreases between extreme values

$$\max(\alpha+1)=2\,\frac{r+1}{r-1}(m-1)\geq 3\quad\text{iff}\quad m\geq\frac{5r-1}{2(r+1)}\,,$$

$$\min(\alpha+1)=2\left(m-\frac{r-1}{r+1}\right)\leq 4+2\,\frac{r-(2\rho+1)}{r+1}\quad\text{iff}\quad m\leq 2+2\,\frac{r-\rho-1}{r+1}\,.$$

We ask, what extension of this interval agree with the minimal δ/ε balance $\gamma=\frac{1}{2}+\frac{\rho+1}{r+1}$?

$$\max(\alpha+1)\geq\frac{6(r+1)}{r+3+2\rho}\quad\text{iff}\quad m\geq 2\,\frac{2r+\rho}{r+3+2\rho}\,.$$

We ask, when best $\alpha+1$ is attained?

$$\max(\alpha+1) \geq 4 + 2\frac{r-(2\rho+1)}{r+1} \quad \text{iff} \quad m \geq 2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}\,.$$

We ask, what is the minimal γ for the $M := \max(\alpha+1) = 2\,\frac{r+1}{r-1}(m-1)$ corresponding to a given m?

$$\gamma = \begin{cases} \frac{2\left(M-\frac{\rho+1}{r+1}M_*\right)}{M_*\,(M-2)}\,, & \text{if } \frac{5r-1}{2(r+1)} \leq m \leq 2\,\frac{2r+\rho}{r+3+2\rho}\,, \\ \frac{1}{2}+\frac{\rho+1}{r+1}\,, & \text{if } \frac{2r+\rho}{r+3+2\rho} \leq m \leq 2+2\,\frac{r-\rho-1}{r+1}\,, \end{cases}$$

where $M_* := \frac{6(r+1)}{r+3+2\rho}$ is the (no)breaking value for $m = 2\,\frac{2r+\rho}{r+3+2\rho}$.

So, when $m \in \left[\frac{5r-1}{2(r+1)}, 2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}\right]$, $\alpha+1 = 2\,\frac{r+1}{r-1}(m-1)$ and we must take $\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}=0$, $\frac{1}{p_1}=\frac{2}{r+1}$, $\frac{1}{p_5}=\frac{r-1}{r+1}$. The m-term is bounded as:

$$(4.18)\quad c_1^2/2\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds \leq \frac{1}{4\,t}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds + \frac{c_1^{r+1}\,(r-1)^{\frac{r-1}{r+1}}\,t^{\frac{r-1}{2}}\,\varepsilon^{\frac{r-1}{2}}}{2^{\frac{(r-1)(r-3)}{2(r+1)}}\,(r+1)^{\frac{2r}{r+1}}}\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|\,\nabla u\,|^{r+1}\,dxds.$$

And, when $m \in \left[2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}, 2+2\,\frac{r-\rho-1}{r+1}\right]$, $\alpha+1 \equiv 4+2\frac{r-(2\rho+1)}{r+1}$. Hence, and because $\frac{1}{p_3}+\frac{1}{p_4}=0$,

$$\frac{1}{p_5} = \frac{r-1}{r+1}, \qquad \frac{1}{p_2} = \frac{r+1}{2\,(r-\rho)}\left(m-2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}\right),$$

$$\frac{1}{p_1} = \frac{r+1}{2\,(r-\rho)}\left(2\frac{2r-\rho}{r+1}-m\right).$$

The m-term is bounded in this case as:

$$(4.19)\quad c_1^2/2\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds \leq \frac{1}{4\,t}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds + \frac{(\alpha+1)\,\alpha\,d_2\,(r+1)}{2}\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds + \left(2\,c_1^2\left((\alpha+1)\,\alpha\,d_2\,(r+1)\,p_2\right)^{-\frac{1}{p_2}}\,p_5^{-\frac{1}{p_5}}\right)^{p_1}\frac{(t\,\varepsilon)^{p_1\,\frac{r-1}{r+1}}}{4\,p_1}\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|\,\nabla u\,|^{r+1}\,dxds.$$

We state and prove the next propositions:

Proposition 4.4.2. *Assume that* (H_1)-(H_6) *holds,* $\frac{5r-1}{2(r+1)} \leq m \leq 2\,\frac{2r+\rho}{r+3+2\rho}$, $\rho \geq 1$, $r \geq 2\rho+1$ *and* $u_0 \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$, $\nabla u_0 \in \left(L^{\rho+1}(\mathbb{R}^d) \cap L^{r+1}(\mathbb{R}^d)\right)^d$, *for some* $q \geq \alpha+1 = M$. *For* $t \in [0,T]$ *we have*

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\,dx + (\alpha+1)\,\alpha\,d_3\,\rho\,\delta \int_{\mathrm{R}^d} |u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^{\rho+1}\,dx \\ + (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds \\ \leq 3\,H(\delta,\varepsilon),\end{aligned}$$

where $M := 2\,\frac{r+1}{r-1}(m-1)$, $M_* := \frac{6(r+1)}{r+3+2\rho}$ *and, if* $\delta = \mathcal{O}\big(\varepsilon^{\frac{2\left(M-\frac{\rho+1}{r+1}M_*\right)}{M_*\,(M-2)}}\big)$, $H(\delta,\varepsilon) \leq const$. *Explicitly,*

$$\begin{aligned}H(\delta,\varepsilon) := \|u_0\|_{\alpha+1}^{\alpha+1} + (\alpha+1)\,\alpha\,c_3\,\rho\,\delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\,|\,\nabla u_0|^{\rho+1}\,dx \\ + c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1} + \Bigg(\frac{c_1^{r+1}\,(r-1)^{\frac{r-1}{r+1}}\,t^{\frac{r-1}{2}}\,\varepsilon^{\frac{r-1}{2}}}{2^{\frac{r^2-2r+5}{2(r+1)}}\,d_2\,(r+1)^{\frac{3r+1}{r+1}}} \\ + \frac{3(r-1)-(r+3+2\rho)(m-1)}{2\,d_3\,\rho\,(r-\rho)(m-1)}\,t + \frac{t^{\frac{p_1}{p_5}}}{2\,d_2(r+1)\,p_1} \\ \left[\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\right)^2 2^{-1+\frac{2}{p_5}}\,p_5^{-\frac{1}{p_5}}\,\delta^{2-\frac{1}{p_3}}\,\varepsilon^{-\left(1+\frac{1}{p_1}\right)}\right]^{p_1}\Bigg) \\ \left(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\right).\end{aligned}$$

Proof. We know that for m belonging to $\left[\frac{5r-1}{2(r+1)}, 2\,\frac{2r+\rho}{r+3+2\rho}\right]$, along as $\alpha+1 = 2\,\frac{r+1}{r-1}(m-1)$ we have $\min(\gamma) = \frac{2\left(M-\frac{\rho+1}{r+1}M_*\right)}{M_*\,(M-2)}$, which corresponds, p. 57, to $\frac{1}{p_2}+\frac{1}{p_4} = 0$ and $\frac{1}{p_3} = \frac{3(r-1)-(r+3+2\rho)(m-1)}{(r-\rho)(m-1)}$. Then $\frac{1}{p_1} = 3\,\frac{(\rho+1)((r+1)m-2r)}{(r+1)(r-\rho)(m-1)}$ and $\frac{1}{p_5} = \frac{2m}{m-1} - \frac{5r-1}{(r+1)(m-1)}$. Therefore we bound the α-term:

$$\begin{aligned}\frac{\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\right)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \\ \leq \frac{1}{4\,t}\int_0^t\int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds + \frac{\delta}{p_3}\int_0^t\int_{\mathrm{R}^d} |\,\nabla u\,|^{\rho+1}\,dxds \\ + \left[\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\right)^2 2^{-1+\frac{2}{p_5}}\,p_5^{-\frac{1}{p_5}}\,\delta^{2-\frac{1}{p_3}}\,\varepsilon^{-\left(1+\frac{1}{p_1}\right)}\right]^{p_1} \\ \frac{t^{\frac{p_1}{p_5}}}{p_1}\,\varepsilon\int_0^t\int_{\mathrm{R}^d} |\,\nabla u\,|^{r+1}\,dxds\,.\end{aligned}\tag{4.20}$$

Return to (4.14), which we integrate over $[0,t]$. Then control their right-hand side with (4.18), (4.20) and Prop.4.3.1:

$$(4.21)\ \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha+1}\,dxds+\ (\alpha+1)\,\alpha\,d_3\,\rho\,\delta\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{\rho+1}\,dxds$$
$$+\,(\alpha+1)\,\alpha\,d_2\,(r+1)\,t\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\leq t\,H(\delta,\varepsilon).$$

So, from (4.18), (4.20) and (4.21)

$$c_1^2/2\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds+\ \frac{((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho)^2}{2}$$
$$\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds\leq 2\,H(\delta,\varepsilon).$$

The conclusion follows from (4.14) and the $H(\delta,\varepsilon)$ definition. □

Proposition 4.4.3. *Assume that (H_1)-(H_6) holds with $2\,\frac{2r+\rho}{r+3+2\rho}\leq m\leq 2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}$ and $\rho\geq 1$, $r\geq 2\rho+1$ where $u_0\in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ together with $\nabla u_0\in\left(L^{\rho+1}(\mathbb{R}^d)\cap L^{r+1}(\mathbb{R}^d)\right)^d$, for some $q\geq\alpha+1=2\,\frac{r+1}{r-1}(m-1)$. For $t\in[0,T]$ we have*

$$\int_{\mathbb{R}^d}|u(t)|^{\alpha+1}\,dx+(\alpha+1)\,\alpha\,d_3\,\rho\,\delta\int_{\mathbb{R}^d}|u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^{\rho+1}\,dx$$
$$+\ (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\leq 3\,H(\delta,\varepsilon),$$

a constant bound, if $\delta=\mathcal{O}\left(\varepsilon^{\frac{1}{2}+\frac{\rho+1}{r+1}}\right)$. Explicitly:

$$H(\delta,\varepsilon):=\|u_0\|_{\alpha+1}^{\alpha+1}\ +\ (\alpha+1)\,\alpha\,c_3\,\rho\,\delta\int_{\mathbb{R}^d}|u_0|^{\alpha-1}\,|\,\nabla u_0|^{\rho+1}\,dx$$
$$+\,c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1}+\left(\frac{c_1^{r+1}\,(r-1)^{\frac{r-1}{r+1}}\,t^{\frac{r-1}{2}}\,\varepsilon^{\frac{r-1}{2}}}{2^{\frac{(r-1)(r-3)}{2(r+1)}}\,(r+1)^{\frac{2r}{r+1}}}\right.$$
$$+\left[\left((\alpha+1)\,\alpha\right)^{2-\frac{1}{p_2}}\left((\alpha-1)\,c_3\,\rho\right)^2\left(d_2\,(r+1)\,p_2\right)^{-\frac{1}{p_2}}p_5^{-\frac{1}{p_5}}\right.$$
$$\left.\left.\delta^2\,\varepsilon^{-\left(1+2\frac{\rho+1}{r+1}\right)}\right]^{p_1}\frac{(2\,t)^{\frac{p_1}{p_5}}}{2\,p_1}\right)\frac{\|u_0\|_2^2+2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}}{2\,d_2(r+1)}.$$

Proof. For $m \in \left[2\,\frac{2r+\rho}{r+3+2\rho}, 2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}\right]$, also $\alpha+1 = 2\,\frac{r+1}{r-1}(m-1)$, but, now, $\min(\gamma) = \frac{1}{2} + \frac{\rho+1}{r+1}$. Then, Young's inequality in p. 57 is done with $\frac{1}{p_3} + \frac{1}{p_4} = 0$,

$$\frac{1}{p_5} = \frac{r-(2\rho+1)}{r+1}, \qquad \frac{1}{p_1} = \frac{2\frac{2r^2-r\rho+\rho}{(r+1)^2} - m}{m - \frac{2r}{r+1}}, \qquad \frac{1}{p_2} = \frac{m - 2\frac{2r+\rho}{r+3+2\rho}}{\frac{(r+1)m-2r}{r+3+2\rho}} :$$

$$\begin{aligned}(4.22)\quad & \frac{((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \\ &\le \frac{1}{4\,t}\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha+1}\,dxds \\ &\quad + \frac{(\alpha+1)\,\alpha\,d_2\,(r+1)}{2}\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds \\ &\quad + \Big[\big((\alpha+1)\,\alpha\big)^{2-\frac{1}{p_2}}\,\big((\alpha-1)\,c_3\,\rho\big)^2\,\big(d_2\,(r+1)\,p_2\big)^{-\frac{1}{p_2}}\,p_5^{-\frac{1}{p_5}} \\ &\qquad \delta^2\,\varepsilon^{-\left(1+2\frac{\rho+1}{r+1}\right)}\Big]^{p_1}\,\frac{(2\,t)^{\frac{p_1}{p_5}}}{2\,p_1}\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|\,\nabla u\,|^{r+1}\,dxds\,.\end{aligned}$$

Integrate (4.14) over $[0,t]$ and bound it by the right-hand side using (4.18), (4.22), Prop.4.3.1, then, to conclude, proceed the same way as in the previous proof. □

Proposition 4.4.4. *Assume that* (H_1)*-*(H_6) *holds together with* $2\,\frac{2r^2-r\rho+\rho}{(r+1)^2} \le m \le 2 + 2\,\frac{r-\rho-1}{r+1}$*,* $\rho \ge 1$*,* $r \ge 2\rho+1$ *and* $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$*,* $\nabla u_0 \in \left(L^{\rho+1}(\mathbb{R}^d) \cap L^{r+1}(\mathbb{R}^d)\right)^d$*, for some* $q \ge \alpha+1 = 4 + 2\frac{r-(2\rho+1)}{r+1}$*. For* $t \in [0,T]$ *we have*

$$\begin{aligned}\int_{\mathbb{R}^d}|u(t)|^{\alpha+1}\,dx + (\alpha+1)\,\alpha\,d_3\,\rho\,\delta\int_{\mathbb{R}^d}|u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^{\rho+1}\,dx \\ + (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds \\ \le 3\,H(\delta,\varepsilon),\end{aligned}$$

which is constant, if $\delta = \mathcal{O}\left(\varepsilon^{\frac{1}{2}+\frac{\rho+1}{r+1}}\right)$*, and given by*

$$H(\delta,\varepsilon) := \|u_0\|_{\alpha+1}^{\alpha+1} + (\alpha+1)\,\alpha\,c_3\,\rho\,\delta\int_{\mathbb{R}^d}|u_0|^{\alpha-1}\,|\,\nabla u_0|^{\rho+1}\,dx$$

$$+ c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1}$$
$$+ \left(2\,c_1^2\,\big((\alpha+1)\,\alpha\,d_2\,(r+1)\,p_2\big)^{-\frac{1}{p_2}}\,p_5^{\,-\frac{1}{p_5}}\right)^{p_1}\,\frac{(t\,\varepsilon)^{p_1\,\frac{r-1}{r+1}}}{8\,d_2(r+1)\,p_1}$$
$$\big(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\big)\,.$$

Proof. For $m \in \left[2\,\frac{2r^2-r\rho+\rho}{(r+1)^2}, 2+2\,\frac{r-\rho-1}{r+1}\right]$, $\alpha+1 \equiv 4+2\frac{r-(2\rho+1)}{r+1}$ and $\min(\gamma) = \frac{1}{2}+\frac{\rho+1}{r+1}$. Bounding the α-term in p. 57 we have, if $r \neq 2\rho+1$:

$$\frac{1}{p_1}+\frac{1}{p_3}+\frac{1}{p_4}=0, \qquad \frac{1}{p_2}=2\frac{\rho+1}{r+1}, \qquad \frac{1}{p_5}=\frac{r-(2\rho+1)}{r+1}.$$

$$\frac{\big((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\big)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathrm{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \tag{4.23}$$
$$\leq \frac{1}{4\,t}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds$$
$$+ \Big[\big(2\,(\alpha+1)\,\alpha\big)^{1-2\frac{\rho+1}{r+1}}\,\big((\alpha-1)\,c_3\,\rho\big)^2\,\big(r-(2\rho+1)\big)^{\frac{1}{p_5}}$$
$$d_2^{-2\frac{\rho+1}{r+1}}\,(r+1)^{-\left(1+2\frac{\rho+1}{r+1}\right)}\,(\rho+1)^{2\frac{\rho+1}{r+1}}\,t^{\frac{1}{p_5}}\,\delta^2\,\varepsilon^{-\left(1+2\frac{\rho+1}{r+1}\right)}\Big]^{p_2}$$
$$\frac{(\alpha+1)\,\alpha\,d_2\,(r+1)}{4}\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds\,,$$

where we suppose, because $\delta = \mathcal{O}\left(\varepsilon^{\frac{1}{2}+\frac{\rho+1}{r+1}}\right)$ and we can take a constant sufficiently small, that

$$\Big[\big(2\,(\alpha+1)\,\alpha\big)^{1-2\frac{\rho+1}{r+1}}\,\big((\alpha-1)\,c_3\,\rho\big)^2\,\big(r-(2\rho+1)\big)^{\frac{1}{p_5}}$$
$$d_2^{-2\frac{\rho+1}{r+1}}\,(r+1)^{-\left(1+2\frac{\rho+1}{r+1}\right)}\,(\rho+1)^{2\frac{\rho+1}{r+1}}\,t^{\frac{1}{p_5}}\,\delta^2\,\varepsilon^{-\left(1+2\frac{\rho+1}{r+1}\right)}\Big]^{p_2} \leq 1.$$

Thus, once more, integrate (4.14) over $[0,t]$ and estimate the right-hand side using (4.19), (4.23) and Prop.4.3.1. To conclude, proceed as before.

In the case where $r = 2\rho+1$, since the first member of (4.23) is already the good term, it is an easy case. □

The Case m > 1, With δ-terms. In view of the precedent analysis (δ/ε balance only can became worse), the thing to do here is investigate if we can enlarge the set of solutions.

At first instance, we have also here two regimes: with or without ε-terms. It is easy verify that the ε-free regime, that is related to $\min(\beta) = \frac{\rho+1}{2}$, is enough. (The other is not able to increase our existence domain)

By hypothesis $\frac{1}{p_3} + \frac{1}{p_4} \neq 0$ and we confine ourselves to $\frac{1}{p_1} + \frac{1}{p_2} = 0$. So,

$$\frac{1}{p_3} + \frac{1}{p_4} = \frac{2}{\rho+1}, \qquad \frac{1}{p_5} = \frac{\rho-1}{\rho+1}, \qquad \alpha+1 = 2\frac{m-1-\frac{1}{p_4}}{\frac{\rho-1}{\rho+1}+\frac{1}{p_4}}.$$

And, if $\rho > 1$, we obtain for $\frac{1}{p_4} = 0$ the

$$\max(\alpha+1) = 2\,\frac{\rho+1}{\rho-1}\,(m-1) \geq 3 \quad \text{iff} \quad m \geq \frac{5\rho-1}{2(\rho+1)}.$$

Observe that $\frac{5\rho-1}{2(\rho+1)} < \frac{5r-1}{2(r+1)}$, but

$$\max(\alpha+1) = 4 + 2\frac{r-(2\rho+1)}{r+1} \quad \text{iff} \quad m = 2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)},$$

with $2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)} \geq \frac{5r-1}{2(r+1)}$ iff $\rho > 3$ and $r \geq \frac{4\rho^2-9\rho-1}{3(\rho-3)}$.

Because $\min(\alpha+1) = 2\,\left(m - \frac{r-1}{r+1}\right)$, that of the case before, we extend the interval of existence solely by the left side. Nevertheless, we assure connexion with the previous interval:

$$\min(\alpha+1) = 4 + 2\frac{r-(2\rho+1)}{r+1} \quad \text{iff} \quad m = 2 + 2\,\frac{r\rho-\rho^2-\rho-1}{(\rho+1)(r+1)},$$

where $2 + 2\,\frac{r\rho-\rho^2-\rho-1}{(\rho+1)(r+1)} > \frac{5r-1}{2(r+1)}$, always.

When $\rho = 1$, $m > 1$, for $\frac{1}{p_4} \in]0,1]$ we have

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_5} = 0, \qquad \frac{1}{p_3} = 1 - \frac{1}{p_4}, \qquad \alpha\ +1 = 2\left(1 + (m-1)\,p_4\right).$$

So, $\alpha+1$ decays from $+\infty$ to $2m$: for all $m > 1$, it crosses the optimum level $4 + 2\frac{r-3}{r+1}$, at $\frac{1}{p_4} = \frac{r+1}{2(r-1)}\,(m-1)$.

We summarize. With frozen $\min\beta = \frac{\rho+1}{2}$, we solve our problem for the $L^{\alpha+1}$ spaces: when $\rho = 1$ and $r \geq 3$, as

$$\alpha+1 \equiv 4 + 2\frac{r-3}{r+1}, \quad \text{for} \quad 1 < m < \frac{5r-1}{2(r+1)};$$

when $1 < \rho \leq 3$ or $\rho > 3$ with $r < \frac{4\rho^2-9\rho-1}{3(\rho-3)}$, as

$$\begin{cases} \alpha+1 = 2\,\frac{\rho+1}{\rho-1}\,(m-1)\,, & \text{if} \quad \frac{5\rho-1}{2(\rho+1)} \leq m \leq 2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)}\,, \\ \alpha+1 = 4+2^{\frac{r-(2\rho+1)}{r+1}}\,, & \text{if} \quad 2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)} \leq m < \frac{5r-1}{2(r+1)}\,; \end{cases}$$

and, when $\rho > 3$ with $r \geq \frac{4\rho^2-9\rho-1}{3(\rho-3)}$, as

$$\alpha+1 = 2\,\frac{\rho+1}{\rho-1}\,(m-1)\,, \quad \text{if} \quad \frac{5\rho-1}{2(\rho+1)} \leq m < \frac{5r-1}{2(r+1)}\,.$$

Proposition 4.4.5. *Assume that (H_1)-(H_6) holds with*

$$\begin{cases} 1 < \rho \leq 3 \ \text{or} \ \rho > 3,\ r < \frac{4\rho^2-9\rho-1}{3(\rho-3)}\,, \\ \frac{5\rho-1}{2(\rho+1)} \leq m \leq 2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)}\,, \end{cases} \quad \text{or} \quad \begin{cases} \rho > 3,\ r \geq \frac{4\rho^2-9\rho-1}{3(\rho-3)}\,, \\ \frac{5\rho-1}{2(\rho+1)} \leq m < \frac{5r-1}{2(r+1)}\,, \end{cases}$$

and $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $\nabla u_0 \in \left(L^{\rho+1}(\mathbb{R}^d) \cap L^{r+1}(\mathbb{R}^d)\right)^d$, for some $q \geq \alpha+1 = 2\,\frac{\rho+1}{\rho-1}\,(m-1)$. For $t \in [0,T]$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} |u(t)|^{\alpha+1}\,dx + (\alpha+1)\,\alpha\,d_3\,\rho\,\delta \int_{\mathbb{R}^d} |u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^{\rho+1}\,dx \\ + (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds \\ \leq 3\,H(\delta,\varepsilon), \end{aligned}$$

which is constant if $\delta = \mathcal{O}\big(\varepsilon^{\frac{\rho+1}{2}}\big)$, and given by

$$\begin{aligned} H(\delta,\varepsilon) := \|u_0\|_{\alpha+1}^{\alpha+1} \,+\, (\alpha+1)\,\alpha\,c_3\,\rho\,\delta \int_{\mathbb{R}^d} |u_0|^{\alpha-1}\,|\,\nabla u_0|^{\rho+1}\,dx \\ +\, c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1} + \Bigg(\big(2\,(\rho-1)\big)^{\frac{\rho-1}{2}}\,(\rho+1)^{\frac{3-\rho}{2}}\,\frac{c_1^{\rho+1}\,t^{\frac{\rho+1}{2}}}{2\,d_3\,\rho}\,\Big(\delta^{-1}\,\varepsilon^{\frac{\rho+1}{2}}\Big) \\ +\, \frac{t}{2\,d_3\,\rho\,p_3} \,+\, \frac{t^{\frac{p_1}{p_5}}}{2\,d_2\,(r+1)\,p_1}\,\Big[2^{-1+\frac{2}{p_5}}\,\big((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\big)^2\,p_5^{-\frac{1}{p_5}} \\ \delta^{2-\frac{1}{p_3}}\,\varepsilon^{-\left(1+\frac{1}{p_1}\right)}\Big]^{p_1}\Bigg)\,\big(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\big)\,. \end{aligned}$$

Proof. Bound the m-term by Young's inequality, p.58, where

$$\alpha+1 = 2\,\frac{\rho+1}{\rho-1}\,(m-1), \qquad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_4} = 0, \qquad \frac{1}{p_3} = \frac{2}{\rho+1}, \qquad \frac{1}{p_5} = \frac{\rho-1}{\rho+1},$$

and use Prop.4.3.1:

$$\begin{aligned}(4.24)\ & c_1^2/2\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds \\ & \leq \frac{1}{4\,t}\int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds + \,(2\,(\rho-1))^{\frac{\rho-1}{2}}\,\,(\rho+1)^{\frac{3-\rho}{2}} \\ & \frac{c_1^{\rho+1}\,t^{\frac{\rho+1}{2}}}{2\,d_3\,\rho}\left(\delta^{-1}\,\varepsilon^{\frac{\rho+1}{2}}\right)\left(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\right).\end{aligned}$$

Bounding the α-term, p. 57, the single constraint we have is $\alpha+1 = 2\,\frac{\rho+1}{\rho-1}\,(m-1)$, then, for $\frac{1}{p_3} \in \left[0, \frac{r-(2\rho+1)}{r-\rho}\right]$, we take

$$\frac{1}{p_2} + \frac{1}{p_4} = 0, \qquad \frac{1}{p_1} = \frac{\rho+1}{r+1}\left(2 - \frac{1}{p_3}\right), \qquad \frac{1}{p_5} = \frac{r-(2\rho+1)}{r+1} - \frac{r-\rho}{r+1}\,\frac{1}{p_3},$$

$$\begin{aligned}(4.25)\quad & \frac{((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t \int_{\mathrm{R}^d} |u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \\ & \leq \frac{1}{4\,t}\int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds + \frac{\delta}{p_3}\int_0^t \int_{\mathrm{R}^d} |\,\nabla u\,|^{\rho+1}\,dxds \\ & + \left[2^{-1+\frac{2}{p_5}}\,\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\right)^2 p_5^{-\frac{1}{p_5}}\,\delta^{2-\frac{1}{p_3}}\,\varepsilon^{-\left(1+\frac{1}{p_1}\right)}\right]^{p_1} \\ & \frac{t^{\frac{p_1}{p_5}}}{p_1}\,\varepsilon\int_0^t \int_{\mathrm{R}^d} |\,\nabla u\,|^{r+1}\,dxds\end{aligned}$$

$$\begin{aligned} & \leq \frac{1}{4\,t}\int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds + \left(\frac{t}{2\,d_3\,\rho\,p_3} + \frac{t^{\frac{p_1}{p_5}}}{2\,d_2\,(r+1)\,p_1}\right. \\ & \left.\left[2^{-1+\frac{2}{p_5}}\,\left((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\right)^2 p_5^{-\frac{1}{p_5}}\,\delta^{2-\frac{1}{p_3}}\,\varepsilon^{-\left(1+\frac{1}{p_1}\right)}\right]^{p_1}\right) \\ & \left(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\right)\,,\end{aligned}$$

where we have used Prop.4.3.1; and the singularity of $\frac{1}{p_5} = 0$ for $\frac{1}{p_3} = \frac{r-(2\rho+1)}{r-\rho}$ is harmless.

Proceed analogously to the previous proofs to conclude. □

Proposition 4.4.6. *Assume that* (H_1)-(H_6) *holds with* $1 \leq \rho \leq 3$, $r \geq 2\rho+1$ *or* $\rho > 3$, $r < \frac{4\rho^2-9\rho-1}{3(\rho-3)}$ *and* $2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)} \leq m < \frac{5r-1}{2(r+1)}$ *(but* $m \neq 1$ *if* $\rho = 1$*),* $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$*,* $\nabla u_0 \in \left(L^{\rho+1}(\mathbb{R}^d) \cap L^{r+1}(\mathbb{R}^d)\right)^d$*, for some* $q \geq \alpha + 1 = 4 + 2\frac{r-(2\rho+1)}{r+1}$*. For* $t \in [0, T]$ *we have*

$$\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\,dx + (\alpha+1)\,\alpha\,d_3\,\rho\,\delta \int_{\mathrm{R}^d} |u(t)|^{\alpha-1}\,|\,\nabla u(t)\,|^{\rho+1}\,dx$$
$$+ (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\leq 3\,H(\delta,\varepsilon),$$

which is constant if $\delta = \mathcal{O}\left(\varepsilon^{\frac{\rho+1}{2}}\right)$*, and given by*

$$H(\delta,\varepsilon) := \|u_0\|_{\alpha+1}^{\alpha+1} + (\alpha+1)\,\alpha\,c_3\,\rho\,\delta \int_{\mathrm{R}^d} |u_0|^{\alpha-1}\,|\,\nabla u_0|^{\rho+1}\,dx$$
$$+ c_2\,\varepsilon^2\,\|\,\nabla u_0\|_{r+1}^{r+1} + \left(2^{\frac{\rho-1}{\rho+1}}\,c_1^2\,\left((\alpha+1)\,\alpha\,d_3\,\rho\,p_4\right)^{-\frac{1}{p_4}}\,p_5^{-\frac{1}{p_5}}\,\delta^{-\frac{2}{\rho+1}}\,\varepsilon\right)^{p_3}$$
$$\frac{t_3^p}{4\,d_3\,p_3\,\rho}\,\left(\|u_0\|_2^2 + 2\,c_3\,\rho\,\delta\,\|\,\nabla u_0\|_{\rho+1}^{\rho+1}\right).$$

Proof. The m-term is bounded using Young's inequality in p.58, where

$$\alpha+1 = 4+2\,\frac{r-(2\rho+1)}{r+1}, \qquad \frac{1}{p_1}+\frac{1}{p_2} = 0, \qquad \frac{1}{p_3} = \frac{2}{\rho+1} - \frac{1}{p_4},$$
$$\frac{1}{p_5} = \frac{\rho-1}{\rho+1}, \qquad \frac{1}{p_4} = \frac{r+1}{2\,(r-\rho)}\left(m - 2\,\frac{2r\rho-\rho^2-r+2\rho}{(\rho+1)(r+1)}\right),$$

and with Prop.4.3.1. Once more, the singularity in the particular case of $\rho = 1$ is harmless. Without loss of generality:

$$c_1^2/2\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{2(m-1)}\,|\,\nabla u\,|^2\,dxds \leq \frac{1}{4\,t}\int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\,dxds$$
$$+ \frac{(\alpha+1)\,\alpha\,d_3\,\rho}{2\,t}\,\delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\,\nabla u\,|^{\rho+1}\,dxds$$
$$+ \left(2^{\frac{\rho-1}{\rho+1}}\,c_1^2\,\left((\alpha+1)\,\alpha\,d_3\,\rho\,p_4\right)^{-\frac{1}{p_4}}\,p_5^{-\frac{1}{p_5}}\,\delta^{-\frac{2}{\rho+1}}\,\varepsilon\right)^{p_3}$$
$$\frac{t^{p_3-1}}{2\,p_3}\,\delta \int_0^t \int_{\mathrm{R}^d} |\,\nabla u\,|^{\rho+1}\,dxds.$$

Bounding the α-term, p. 57, since $\alpha+1 = 4 + 2\,\frac{r-(2\rho+1)}{r+1}$:

$$\frac{1}{p_1}+\frac{1}{p_3}+\frac{1}{p_4}=0, \qquad \frac{1}{p_2}=2\,\frac{\rho+1}{r+1}, \qquad \frac{1}{p_5}=\frac{r-(2\rho+1)}{r+1}.$$

$$\begin{aligned}\frac{\big((\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\big)^2}{2}\,\delta^2\,\varepsilon^{-1}\int_0^t\int_{\mathrm{R}^d}|u|^{2(\alpha-2)}\,|\,\nabla u\,|^{2(\rho+1)}\,dxds \\ \leq \frac{1}{4\,t}\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds + \Big[\big((\alpha+1)\,\alpha\big)^{2\,\frac{r-\rho}{r+1}}\,\big((\alpha-1)\,c_3\,\rho\big)^2 \\ (\alpha-3)^{\frac{\alpha-3}{2}}\,\frac{\rho+1}{d_2\,(r+1)^2}^{\frac{\rho+1}{r+1}}\,t^{1-2\,\frac{\rho+1}{r+1}}\,\delta^2\,\varepsilon^{-\left(1+2\,\frac{\rho+1}{r+1}\right)}\Big]^{p_2} \\ (\alpha+1)\,\alpha\,d_2\,(r+1)\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u\,|^{r+1}\,dxds.\end{aligned}$$

Note that, because $\delta=\mathcal{O}\left(\varepsilon^{\frac{\rho+1}{2}}\right)$, the factor $\delta^2\,\varepsilon^{-\left(1+2\,\frac{\rho+1}{r+1}\right)}$ tends to zero, if $\rho>1$, or in the worse situation, for $\rho=1$, we can choose a sufficiently small constant such that

$$\begin{aligned}\big((\alpha+1)\,\alpha\big)^{2\,\frac{r-\rho}{r+1}}\,\big((\alpha-1)\,c_3\,\rho\big)^2(\alpha-3)^{\frac{\alpha-3}{2}}\,\frac{\rho+1}{d_2\,(r+1)^2}^{\frac{\rho+1}{r+1}} \\ T^{1-2\,\frac{\rho+1}{r+1}}\,\delta^2\,\varepsilon^{-\left(1+2\,\frac{\rho+1}{r+1}\right)}\leq 1/2.\end{aligned}$$

The conclusion follows as for the other proofs. □

4.4.2 KdVB Estimates

To derive higher L^q a priori estimates for the KdVB equation, consider (4.10) in Lemma 4.3.1: there we switch derivatives order in gradient degree. Thus, put diffusion and dispersion growths in competition.

We bound (4.10), at left using (H_4) and at right by (H_3):

$$\begin{aligned}(4.26)\quad &\int_{\mathrm{R}^d}|u(t)|^{\alpha+1}\,dx\;+\;(\alpha+1)\,\alpha\,(r+1)\,d_2\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u\,|^{r+1}\,dxds \\ &\leq \|u_0\|_{\alpha+1}^{\alpha+1}\;+\;(\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\,\delta\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-2}|\,\nabla u\,|^{\rho+2}\,dxds\,,\end{aligned}$$

integrate over $[0,t]$,

$$\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds+(\alpha+1)\,\alpha\,(r+1)\,d_2\,t\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u\,|^{r+1}\,dxds$$

$$\leq t\,\|u_0\|_{\alpha+1}^{\alpha+1} + (\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\,t\,\delta\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-2}|\,\nabla u\,|^{\rho+2}\,dxds\,.$$

Apply Young's inequality to the last term within Prop. 4.3.3:

$$(\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\,t\,\delta\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-2}|\,\nabla u\,|^{\rho+2}\,dxds$$
$$= \int_0^t\int_{\mathrm{R}^d}\left[p_2\,\frac{|u|^{\alpha+1}}{2}\right]^{\frac{1}{p_2}}\left[p_3\,\frac{(\alpha+1)\,\alpha\,d_2\,t\,\varepsilon\,|u|^{\alpha-1}|\,\nabla u\,|^{r+1}}{2}\right]^{\frac{1}{p_3}}$$
$$\Bigg[2^{p_1-2}\big((\alpha+1)\,\alpha\,t\big)^{1+\frac{p_1}{p_2}}\Bigg(c_3\,\rho\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}\,p_3^{-\frac{1}{p_3}}\,d_2^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}$$
$$\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\Bigg)^{p_1}\,2\,d_2\,(r+1)\,\varepsilon\,|\,\nabla u\,|^{r+1}\Bigg]^{\frac{1}{p_1}}\,dxds$$

$$\leq \frac{1}{2}\left(\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds\;+\;(\alpha+1)\,\alpha\,(r+1)\,d_2\right.$$
$$\left.t\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u\,|^{r+1}\,dxds\right)\;+\;\frac{1}{p_1}2^{p_1-2}\big((\alpha+1)\,\alpha\,t\big)^{1+\frac{p_1}{p_2}}$$
$$\left(c_3\,\rho\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}\,p_3^{-\frac{1}{p_3}}\,d_2^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\,\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\right)^{p_1}\|u_0\|_2^2,$$

where

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1,\quad \rho+2=\frac{r+1}{p_1}+\frac{r+1}{p_3},\quad \alpha-2=\frac{\alpha+1}{p_2}+\frac{\alpha-1}{p_3},$$

so that, we must have $r\geq\rho+1$ and

$$\frac{1}{p_2}=1-\frac{\rho+2}{r+1},\quad \frac{1}{p_3}=\frac{1}{\alpha-1}\left((\alpha+1)\,\frac{\rho+2}{r+1}-3\right),$$
$$\frac{1}{p_1}=\frac{1}{\alpha-1}\left(3-2\,\frac{\rho+2}{r+1}\right),\quad \frac{1}{p_1}+\frac{1}{p_3}=\frac{\rho+2}{r+1}.$$

Define

$$H_\alpha\left(\delta\;\varepsilon^{-\frac{\rho+2}{r+1}}\right)\;:=\frac{1}{p_1}2^{p_1-2}\,t^{\frac{p_1}{p_2}}\,\big((\alpha+1)\,\alpha\big)^{1+\frac{p_1}{p_2}}\Bigg(c_3\,\rho\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}\,p_3^{-\frac{1}{p_3}}$$
$$d_2^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\,\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\Bigg)^{p_1}\,\|u_0\|_{L^2(\mathrm{R}^d)}^2\,.$$

We conclude, both,

$$\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha+1}\,dxds+(\alpha+1)\,\alpha\,(r+1)\,d_2\,t\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}|\,\nabla u|^{r+1}\,dxds$$
$$\le 2t\left(\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^d)}+H_\alpha\Big(\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right)$$

and

$$(\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\,\delta\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-2}|\,\nabla u|^{\rho+2}\,dxds$$
$$\le\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^d)}+2H_\alpha\Big(\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\,.$$

So, we can finally come back to (4.26): we have then proved the following proposition which gives rise to an *arbitrarily large* L^q bound.

Proposition 4.4.7. *Assume that* (H_2), (H_3), (H_4) *holds with* $\rho\ge 1$, $r\ge\rho+1$ *and* $u_0\in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ *for some* $q\ge\alpha+1>3\,\frac{r+1}{\rho+2}$. *For* $t\in[0,T]$ *we have*

$$\int_{\mathbb{R}^d}|u(t)|^{\alpha+1}dx+(\alpha+1)\,\alpha\,(r+1)\,d_2\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-1}\,|\,\nabla u\,|^{r+1}\,dxds$$
$$\le 2\left(\|u_0\|^q_{L^q(\mathbb{R}^d)}+H_\alpha\Big(\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right);$$
$$(\alpha+1)\,\alpha\,(\alpha-1)\,c_3\,\rho\,\delta\int_0^t\int_{\mathbb{R}^d}|u|^{\alpha-2}|\,\nabla u|^{\rho+2}\,dxds$$
$$\le\|u_0\|^q_{L^q(\mathbb{R}^d)}+2H_\alpha\Big(\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\,.$$

Moreover, if $\delta=\mathcal{O}\left(\varepsilon^{\frac{\rho+2}{r+1}}\right)$, *then* $H_\alpha\Big(\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\le Const.$

4.5 Convergence Proofs

Lets reconsider the equation (4.7), with an arbitrary convex function η (where we assume η',η'',η''' bounded functions on $\mathbb{R}$),

$$\partial_t\,\eta(u)+\operatorname{div}\,q(u)=\varepsilon\,\operatorname{div}\big(\eta'(u)\,b(\nabla u)\big)-\,\varepsilon\,(r+1)\,\eta''(u)\,B(\nabla u)$$
$$+\,\delta\,\operatorname{div}\big(\eta'(u)\,\partial_\xi\,c(\nabla u)\big)-\,\delta\,\rho\,\eta''(u)\,\partial_\xi\,C(\nabla u).$$

We prove (2.13). As sufficient condition, we claim that there exists a bounded measure $\mu\le 0$ such that

$$\partial_t\eta(u)+\operatorname{div}\,q(u)\longrightarrow\mu\,,\qquad\text{in}\quad\mathcal{D}'(\mathbb{R}^d\times(0,T)).$$

We use the notation:

$$\mu_1 := \varepsilon \ \mathrm{div}(\eta'(u)\ b(\nabla u))\,;$$
$$\mu_2 := -\varepsilon\,(r+1)\,\eta''(u)\,B(\nabla u);$$
$$\mu_3 := \ \delta\ \mathrm{div}\left(\eta'(u)\,\partial_\xi\, c(\nabla u)\right);$$
$$\mu_4 := -\delta\,\rho\ \eta''(u)\,\partial_\xi\, C(\nabla u);$$

and, for each positive $\theta \in C_0^\infty(\mathbb{R}^d \times (0,T))$ we evaluate $\langle \mu_i, \theta\rangle$ for $i = 1,2,3,4$:

$$\begin{aligned} |\langle \mu_1, \theta\rangle| &\le \varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla\theta \cdot \eta'(u)\,b(\nabla u)|\ dxdt \\ &\le Const\ \varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla\theta|\ |\nabla u|^r\ dxdt \end{aligned}$$

in view of growth hypothesis (H_2). Use Hölder's inequality within Prop. 4.3.1 or 4.3.3 and assumption (4.5). We get

$$\begin{aligned} |\langle \mu_1, \theta\rangle| &\le Const\ \varepsilon^{\frac{1}{r+1}}\ \|\ \nabla\theta\|_{L^{(r+1)}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint |\nabla u|^{r+1}\ dxdt\right]^{\frac{r}{r+1}} \\ &\le C\ \varepsilon^{\frac{1}{r+1}}\ \|\ \nabla\theta\|_{L^{(r+1)}(\mathrm{R}^d\times(0,T))}\,. \end{aligned}$$

For μ_2, because $B(\nabla u) \ge 0$ and η is convex,

$$\langle \mu_2, \theta\rangle = -(r+1)\,\varepsilon \int_0^T \int_{\mathrm{R}^d} \theta\,\eta''(u)\,B(\nabla u)\ dxdt \le 0\,,$$

with, by Prop. 4.3.1 or 4.3.3 and assumption (4.5),

$$\begin{aligned} |\langle \mu_2, \theta\rangle| &\le Const\ \|\theta\|_{L^\infty(\mathrm{R}^d\times(0,T))}\,\varepsilon \iint |\nabla u|^{r+1}\ dxdt \\ &\le Const\ \|\theta\|_{L^\infty(\mathrm{R}^d\times(0,T))}. \end{aligned}$$

For μ_3, we have by hypothesis (H_3)

$$\begin{aligned} |\langle \mu_3, \theta\rangle| \le \delta &\int_0^T \int_{\mathrm{R}^d} |\nabla \partial_\xi\theta \cdot \eta'(u)\,c(\nabla u)|\ dxdt \\ &+ \delta \int_0^T \int_{\mathrm{R}^d} |\nabla\theta \cdot \eta''(u)\,\partial_\xi u\ c(\nabla u)|\ dxdt \\ \le\ & Const\ \delta \int_0^T \int_{\mathrm{R}^d} |\nabla\partial_\xi\theta|\ |\nabla u|^\rho\ dxdt \\ &+ Const\ \delta \int_0^T \int_{\mathrm{R}^d} |\nabla\theta|\ |\partial_\xi u|\ |\nabla u|^\rho\ dxdt. \end{aligned}$$

If $\partial_\xi = \partial_{x_k}$, by Hölder's inequalities

$$|\langle \mu_3, \theta \rangle| \leq \; Const \;\; \delta \, \varepsilon^{-\frac{\rho}{r+1}} \, \| \, \nabla \partial_{x_k} \theta \|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d \times (0,T))} \;\; \left[\varepsilon \iint |\nabla u|^{r+1} \; dxdt \right]^{\frac{\rho}{r+1}}$$
$$+ Const \;\; \delta \, \varepsilon^{-\frac{\rho+1}{r+1}} \, \| \, \nabla \theta \|_{L^{\frac{r+1}{r-\rho}}(\mathrm{R}^d \times (0,T))} \;\; \left[\varepsilon \iint |\nabla u|^{r+1} \; dxdt \right]^{\frac{\rho+1}{r+1}},$$

therefore, by Prop. 4.3.3 and assumption (4.5),

$$|\langle \mu_3, \theta \rangle| \leq \; C \, \delta \, \varepsilon^{-\frac{\rho+1}{r+1}} \left(\| \, \nabla \partial_{x_k} \theta \|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d \times (0,T))} + \| \, \nabla \theta \|_{L^{\frac{r+1}{r-\rho}}(\mathrm{R}^d \times (0,T))} \right).$$

And, if $\partial_\xi = \partial_t$, the term

$$\delta \int_0^T \int_{\mathrm{R}^d} |\nabla \theta| \; |\partial_t u| \; |\nabla u|^\rho \; dxdt \leq \delta \, \varepsilon^{-\left(\frac{1}{2} + \frac{\rho}{r+1}\right)} \| \, \nabla \theta \|_{L^{\frac{2(r+1)}{r+1-2\rho}}(\mathrm{R}^d \times (0,T))}$$
$$\left[\varepsilon \int_0^T \int_{\mathrm{R}^d} |\partial_t u|^2 \; dxdt \right]^{\frac{1}{2}} \left[\varepsilon \iint |\nabla u|^{r+1} \; dxdt \right]^{\frac{\rho}{r+1}},$$

and then use Prop. 4.3.1 and 4.3.2 with assumption (4.5) and each one of Prop. 4.4.1–4.4.6 to obtain:

$$|\langle \mu_3, \theta \rangle| \leq \; C \, \delta \, \varepsilon^{-\left(\frac{1}{2} + \frac{\rho}{r+1}\right)} \left(\| \, \nabla \partial_t \theta \|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d \times (0,T))} + \| \, \nabla \theta \|_{L^{\frac{2(r+1)}{r+1-2\rho}}(\mathrm{R}^d \times (0,T))} \right).$$

Finally, for μ_4,

$$|\langle \mu_4, \theta \rangle| \leq \rho \, \delta \int_0^T \int_{\mathrm{R}^d} |\partial_\xi \theta \, \eta''(u) \, C(\nabla u)| \; dxdt$$
$$+ \rho \, \delta \int_0^T \int_{\mathrm{R}^d} |\theta \, \eta'''(u) \, \partial_\xi u \; C(\nabla u)| \; dxdt$$

$$\leq \; Const \;\; \delta \int_0^T \int_{\mathrm{R}^d} |\partial_\xi \theta| \; |\nabla u|^{\rho+1} \; dxdt$$
$$+ Const \;\; \delta \int_0^T \int_{\mathrm{R}^d} |\theta| \; |\partial_\xi u| \; |\nabla u|^{\rho+1} \; dxdt.$$

If $\partial_\xi = \partial_{x_k}$, then

$$|\langle \mu_4, \theta \rangle| \leq \; Const \;\; \delta \, \varepsilon^{-\frac{\rho+1}{r+1}} \, \| \partial_{x_k} \theta \|_{L^{\frac{r+1}{r-\rho}}(\mathrm{R}^d \times (0,T))} \;\; \left[\varepsilon \iint |\nabla u|^{r+1} \; dxdt \right]^{\frac{\rho+1}{r+1}}$$

$$+\,Const\ \delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\,\|\theta\|_{L^{\frac{r+1}{r-\rho-1}}(\mathbb{R}^d\times(0,T))}\ \left[\varepsilon\iint|\nabla u|^{r+1}\,dxdt\right]^{\frac{\rho+2}{r+1}}$$

and, by Prop. 4.3.3 and assumption (4.5),

$$|\langle\mu_4,\theta\rangle|\le C\,\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\left(\|\partial_{x_k}\theta\|_{L^{\frac{r+1}{r-\rho}}(\mathbb{R}^d\times(0,T))}+\|\theta\|_{L^{\frac{r+1}{r-\rho-1}}(\mathbb{R}^d\times(0,T))}\right),$$

now, the condition $\delta=o(\varepsilon^{\frac{\rho+2}{r+1}})$ is sufficient to the conclusion.

If $\partial_\xi=\partial_t$, the last term

$$\delta\int_0^T\int_{\mathbb{R}^d}|\theta|\,|\partial_t u|\,|\nabla u|^{\rho+1}\,dxdt\le\delta\,\varepsilon^{-\left(\frac12+\frac{\rho+1}{r+1}\right)}\|\,\nabla\theta\|_{L^{\frac{2(r+1)}{r-2(\rho+1)}}(\mathbb{R}^d\times(0,T))}$$
$$\left[\varepsilon\int_0^T\int_{\mathbb{R}^d}|\partial_t u|^2\,dxdt\right]^{\frac12}\left[\varepsilon\iint|\nabla u|^{r+1}\,dxdt\right]^{\frac{\rho+1}{r+1}},$$

so, justifying as for μ_3, we have

$$|\langle\mu_4,\theta\rangle|\le C\,\delta\,\varepsilon^{-\left(\frac12+\frac{\rho+1}{r+1}\right)}\left(\|\partial_t\theta\|_{L^{\frac{r+1}{r-\rho}}(\mathbb{R}^d\times(0,T))}+\|\,\nabla\theta\|_{L^{\frac{2(r+1)}{r-2(\rho+1)}}(\mathbb{R}^d\times(0,T))}\right),$$

and condition $\delta=o(\varepsilon^{\frac12+\frac{\rho+1}{r+1}})$ is sufficient for the conclusion. Remark that, this is satisfied for all the cases, except for Prop. 4.4.1 (when $\rho=1$, $r=3$, $m=1$), Prop. 4.4.3, Prop. 4.4.4, Prop. 4.4.6 (when $\rho=1$, $r=3$, $1<m<\frac{5r-1}{2(r+1)}$). In fact, for the pointed out exceptions we asked $\delta=\mathcal{O}(\varepsilon^{\frac12+\frac{\rho+1}{r+1}})$, here, we need to restrict a *little* more.

Using a standard regularization of $\mathrm{sgn}(u)$ and $|u-k|$ (for $k\in\mathbb{R}$), which fullfils the growth condition (2.9) in the Young measure representation theorem, Lemma 2.2.1, p. 19, we apply the limit representation (2.10) and conclude that ν satisfies (2.13).

To show (2.14) we follow DiPerna [14] and Szepessy [38]'s arguments. We have to check that, for each compact K of $\mathbb{R}^d$,

$$\lim_{t\to0+}\frac1t\int_0^t\int_K\langle\nu_{(x,s)},|u-u_0(x)|\,\rangle\,dxds$$
$$=\lim_{t\to0+}\lim_{\varepsilon\to0+}\frac1t\int_0^t\int_K\left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\,dxds=0.$$

By Jensen's inequality, where $m(K)$ stands for Lebesgue measure of K,

$$\frac1t\int_0^t\int_K\left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\,dxds$$

$$\leq m(K)^{1/2}\left(\frac{1}{t}\int_0^t\int_K\left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\,dxds\right)^{1/2}.$$

We will establish that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\int_K\left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\,dxds=0.$$

Let $K_i \subset K_{i+1}$ ($i=0,1,\dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i\geq 0}K_i = \mathbb{R}^d$, use the identity $u^2 - u_0^2 - 2u_0(u-u_0) = (u-u_0)^2$:

$$\begin{aligned}
&\frac{1}{t}\int_0^t\int_K\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)^2\,dxds\\
&\leq \frac{1}{t}\int_0^t\left(\int_{K_i}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{K_i}u_0^2\,dx-2\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right)ds\\
&\leq \int_{\mathbb{R}^d\setminus K_i}u_0^2\,dx+\frac{2}{t}\int_0^t\left|\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right|\,ds,\quad \text{for all } i=0,1,\dots,
\end{aligned}$$

using Prop. 4.3.3 and assumption (4.5) in the KdVB equation case; and, in the case of the BBMB equation, Prop. 4.3.1 and also assumption (4.5): where we do not care with the missing term of Prop. 4.3.1, which tends to zero.

Since

$$\lim_{i\to\infty}\int_{\mathbb{R}^d\setminus K_i}u_0^2\,dx=0,$$

we need to consider only the second term.

Take $\{\theta_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty}\theta_n=u_0 \quad \text{in} \quad L^2(\mathbb{R}^d),$$

Cauchy-Schwarz inequality gives

$$\begin{aligned}
\left|\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right| &\leq \int_{K_i}|u_0-\theta_n|\left|u^{\varepsilon,\delta}(\cdot,s)-u_0\right|\,dx\\
&+\left|\int_{K_i}\theta_n\left(u_0^{\varepsilon,\delta}-u_0\right)+\int_{K_i}\theta_n\left(u^{\varepsilon,\delta}(\cdot,s)-u_0^{\varepsilon,\delta}\right)\,dx\right|\\
&\leq \|u_0-\theta_n\|_{L^2(\mathbb{R}^d)}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathbb{R}^d)}+\|u_0\|_{L^2(\mathbb{R}^d)}\right)\\
&\quad+\|\theta_n\|_{L^2(\mathbb{R}^d)}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathbb{R}^d)}+\left|\int_0^s\int_{K_i}\theta_n\,\partial_\tau u^{\varepsilon,\delta}\,dxd\tau\right|.
\end{aligned}$$

In view of Prop. 4.3.3 or 4.3.1 and (4.5),

$$\|u_0-\theta_n\|_{L^2(\mathrm{R}^d)}\Big(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathrm{R}^d)}+\|u_0\|_{L^2(\mathrm{R}^d)}\Big)\le Const\ \|u_0-\theta_n\|_{L^2(\mathrm{R}^d)}\,,$$

which tends to zero when $n\to\infty$ and since $\lim_{\varepsilon\to0+}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathrm{R}^d)}=0$, it remains to see that

$$\lim_{t\to0+}\lim_{\varepsilon\to0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_\tau u^{\varepsilon,\delta}\,dxd\tau\right|\,ds=0.$$

We have, by (4.1),

$$\begin{aligned}\left|\int_0^s\int_{K_i}\theta_n\,\partial_\tau u^{\varepsilon,\delta}\,dxd\tau\right| &= \Big|\int_0^s\int_{K_i}\theta_n\big(-\operatorname{div}f(u^{\varepsilon,\delta})+\varepsilon\ \operatorname{div}b\big(\nabla u^{\varepsilon,\delta}\big)\\ &\qquad\qquad +\delta\ \operatorname{div}\partial_\xi\, c\big(\nabla u^{\varepsilon,\delta}\big)\,dxd\tau\Big|\\ &\le \int_0^s\int_{K_i}\big|\nabla\theta_n\cdot f\big(u^{\varepsilon,\delta}\big)\big|\ dxd\tau\\ &\quad+\varepsilon\int_0^s\int_{K_i}\big|\nabla\theta_n\cdot b\big(\nabla u^{\varepsilon,\delta}\big)\big|\ dxd\tau\\ &\quad+\delta\left|\int_0^s\int_{K_i}\nabla\theta_n\cdot\partial_\xi\, c\big(\nabla u^{\varepsilon,\delta}\big)\,dxd\tau\right|\\ &:=\mu_1+\mu_2+\mu_3.\end{aligned}$$

To deal with μ_1, we use (H_1), Hölder's inequality, Prop. 4.4.1–4.4.7 and (4.5):

$$\begin{aligned}&\int_0^s\int_{K_i}|\nabla\theta_n|\,\big|f\big(u^{\varepsilon,\delta}\big)\big|\ dx\,d\tau\\ &\qquad\le c_1\left[\int_0^s\int_{K_i}|\nabla\theta_n|^{\frac{\alpha+1}{\alpha+1-m}}\ dx\,d\tau\right]^{\frac{\alpha+1-m}{\alpha+1}}\left[\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^{\alpha+1}\,dx\,d\tau\right]^{\frac{m}{\alpha+1}}\\ &\qquad\le C\,s\,\|\,\nabla\theta_n\|_{L^{\frac{\alpha+1}{\alpha+1-m}}(\mathrm{R}^d)}.\end{aligned}$$

For μ_2, using (H_2) and once more Hölder's inequality with Prop. 4.3.3 or 4.3.1 and (4.5), we get

$$\begin{aligned}&\varepsilon\int_0^s\int_{K_i}|\nabla\theta_n|\,\big|b\big(\nabla u^{\varepsilon,\delta}\big)\big|\ dx\,d\tau\\ &\qquad\le c_2\,\varepsilon\int_0^s\int_{K_i}|\nabla\theta_n|\,\big|\nabla u^{\varepsilon,\delta}\big|^r\ dx\,d\tau\\ &\qquad\le c_2\,\varepsilon^{1-\frac{r}{r+1}}\,s^{\frac{1}{r+1}}\,\|\,\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)}\left[\varepsilon\int_0^s\int_{K_i}\big|\nabla u^{\varepsilon,\delta}\big|^{r+1}\ dxd\tau\right]^{\frac{r}{r+1}}\end{aligned}$$

$$\leq C\,\varepsilon^{\frac{1}{r+1}}\,s^{\frac{1}{r+1}}\,\|\,\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)}.$$

Finally, for μ_3 we have to do a different analysis whether we work with the KdVB equation or the BBMB equation. Lets begin with the former: with (H_3), Hölder's inequality, Prop. 4.3.3 and (4.5), we have

$$\begin{aligned}
\delta\left|\int_0^s\int_{K_i}\nabla\partial_{x_k}\theta_n\cdot c\big(\,\nabla u^{\varepsilon,\delta}\big)\,dxd\tau\right| &\leq \delta\int_0^s\int_{K_i}|\nabla\partial_{x_k}\theta_n|\,\big|c\big(\nabla u^{\varepsilon,\delta}\big)\big|\,dxd\tau\\
&\leq c_3\,\delta\int_0^s\int_{K_i}|\nabla\partial_{x_k}\theta_n|\,\big|\nabla u^{\varepsilon,\delta}\big|^{\rho}\,dxd\tau\\
&\leq c_3\,\delta\,\varepsilon^{-\frac{\rho}{r+1}}s^{\frac{r+1-\rho}{r+1}}\,\|\nabla\partial_{x_k}\theta_n\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d)}\left[\varepsilon\int_0^s\int_{K_i}\big|\nabla u^{\varepsilon,\delta}\big|^{r+1}\,dxd\tau\right]^{\frac{\rho}{r+1}}\\
&\leq C\,\delta\,\varepsilon^{-\frac{\rho}{r+1}}s^{\frac{r+1-\rho}{r+1}}\,\|\nabla\partial_{x_k}\theta_n\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d)}.
\end{aligned}$$

Thus, since $\delta=o(\varepsilon^{\frac{\rho+2}{r+1}})$,

$$\begin{aligned}
&\lim_{\varepsilon\to0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_\tau u^{\varepsilon,\delta}\,dxd\tau\right|ds\\
&\quad\leq\lim_{\varepsilon\to0+}\frac{1}{t}\int_0^t\bigg(C\,s\,\|\,\nabla\theta_n\|_{L^{\frac{\alpha+1}{\alpha+1-m}}(\mathrm{R}^d)}+C\,\varepsilon^{\frac{1}{r+1}}\,s^{\frac{1}{r+1}}\,\|\,\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)}\\
&\qquad+C\,\delta\,\varepsilon^{-\frac{\rho}{r+1}}s^{\frac{r+1-\rho}{r+1}}\,\|\nabla\partial_{x_k}\theta_n\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d)}\bigg)\\
&\quad\leq C\,t\,\|\,\nabla\theta_n\|_{L^{\frac{\alpha+1}{\alpha+1-m}}(\mathrm{R}^d)},
\end{aligned}$$

and the desired conclusion follows as $t\to0^+$.

For the BBMB equation. Use (H_3), Hölder's inequality, Prop. 4.3.1 and (4.5):

$$\begin{aligned}
&\delta\left|\int_0^s\int_{K_i}\partial_\tau\big(\,\nabla\theta_n\cdot c\big(\,\nabla u^{\varepsilon,\delta}\big)\big)\,dxd\tau\right|\\
&\quad\leq c_3\;\delta\int_{K_i}|\nabla\theta_n|\,\big|\nabla u^{\varepsilon,\delta}(.,s)\big|^{\rho}\,dxd\tau\,+\,c_3\;\delta\int_{K_i}|\nabla\theta_n|\,\Big|\nabla u_0^{\varepsilon,\delta}\Big|^{\rho}\,dxd\tau\\
&\quad\leq c_3\;\delta^{\frac{1}{\rho+1}}\,\|\nabla\theta_n\|_{L^{\rho+1}(\mathrm{R}^d)}\left(\left[\delta\int_{K_i}\big|\nabla u^{\varepsilon,\delta}(.,s)\big|^{\rho+1}\,dxd\tau\right]^{\frac{\rho}{\rho+1}}\right.\\
&\qquad\left.+\left[\delta\int_{K_i}\Big|\nabla u_0^{\varepsilon,\delta}\Big|^{\rho+1}\,dxd\tau\right]^{\frac{\rho}{\rho+1}}\right)\\
&\quad\leq C\,\delta^{\frac{1}{\rho+1}}\,\|\nabla\theta_n\|_{L^{\rho+1}(\mathrm{R}^d)}\,.
\end{aligned}$$

So, because $\delta = o(\varepsilon^\gamma)$, $(\gamma > 0)$, we conclude, taking $t \to 0^+$ in

$$\lim_{\varepsilon \to 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n \, \partial_\tau u^{\varepsilon,\delta} \, dx d\tau \right| ds \leq C \, t \, \| \nabla \theta_n \|_{L^{\frac{\alpha+1}{\alpha+1-m}}(\mathbb{R}^d)}.$$

Chapter 5

A General KdVB Equation[1]

Abstract. Always in the setting of DiPerna's m.-v.-solution theory, we obtain general conditions under which the solution of multi-dimensional KdVB generalized equations converge to the classical entropy weak solutions of the limit conservation law. In particular, all the previously concerned KdVB results are generalized. And, the diffusion-dispersion relationship in the fixed framework is elucidated. We can handle arbitrarily large L^q and any flux-growth greater or equal to one.

5.1 Assumptions

We study here the limit behaviour, as ε, δ tend to zero, of solutions of the multi-dimensional KdVB-like equation

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}\left(\varepsilon\, b_j(u, \nabla u) + \delta \sum_k \partial_{x_k} c_{jk}(\nabla u)\right)_{1 \le j \le d}, \tag{5.1}$$

$$u(x,0) = u_0^{\varepsilon,\delta}(x), \tag{5.2}$$

a (vanishing diffusive-dispersive) perturbed of the conservation law

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad (x,t) \in \mathbb{R}^d \times [0, +\infty[\,, \tag{5.3}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{5.4}$$

This equation generalize the KdVB-like equations previously considered. We restrict ourselves to the case of the nonlinear diffusion but linear or nonlinear dispersion (which we generalize).

[1] Unpublished

The assumptions will be correspondently generalized. Also, with respect to the conclusions: the balance $\delta = o(\varepsilon^\gamma)$ agrees with the previous one and elucidates the growth competition between the diffusion and the dispersion; once more, we can handle an arbitrarily large L^q space and any $m \geq 1$.

Let $u^{\varepsilon,\delta} : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ be smooth solutions to the (5.1)-(5.2) initial value problem, defined on an interval $[0,T]$ with a uniform T (independent of ε, δ) and decaying rapidly at infinity; $u_0^{\varepsilon,\delta}$ is a convenient regularized approximation of the data (5.4), $u_0 : \mathbb{R}^d \to \mathbb{R}$.

Throughout it is assumed $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and the $u_0^{\varepsilon,\delta}$ are smooth functions with compact support and uniformly bounded in $L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $q \geq 2$. Restricting attention to the diffusion-dominant regime we regard $\delta = \delta(\varepsilon)$ and we suppose that $u_0^{\varepsilon,\delta}$ approaches the initial condition u_0 in the sense that

$$\begin{aligned} &\lim_{\varepsilon \to 0+} u_0^{\varepsilon,\delta} = u_0 \quad \text{in} \quad L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), \\ &\|u_0^{\varepsilon,\delta}\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{5.5}$$

According to the L^p-Young measure setting, we need to suppose

1) the vector-flux $f : \mathbb{R} \to \mathbb{R}^d$,

2) the vector-diffusion $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and

3) the matrix-dispersion $[c_{jk}] : \mathbb{R}^d \to \mathbb{R}^{d \times d}$,

are all smooth with a growth control at infinity:

(H_1) $\exists m \geq 1, \exists C_1 > 0 : \quad |f'(u)| \leq C_1\big(1 + |u|^{m-1}\big), \quad \forall u \in \mathbb{R}.$

(H_2) $\exists \mu, r \geq 0, \exists C_2 > 0 : \quad |\, b(u,\lambda)\, | \leq C_2\big(1 + |u|^\mu\, |\lambda|^r\big), \quad \forall u \in \mathbb{R}, \lambda \in \mathbb{R}^d.$

(H_3) $\exists \rho \geq 0, \exists C_3 > 0 : \quad \|\, [c_{jk}(\lambda)]\, \| \leq C_3\big(1 + |\lambda|^\rho\big), \quad \forall \lambda \in \mathbb{R}^d.$

Concerning the vector-diffusion b, given the fixed $\mu, r \geq 0$, we also assume a 'diffusion hypothesis'

(H_4) $\exists \varphi, \vartheta \in [0,1],\, D > 0 : \; D\, |u|^{\mu\varphi}\, |\lambda|^{r+1-\vartheta} \leq \lambda \cdot b(u,\lambda), \quad \forall u \in \mathbb{R}, \lambda \in \mathbb{R}^d,$

and, supposing $u_0 \in L^q(\mathbb{R}^d)$, the (H_2) and (H_4) compatibility conditions:

(H_5) $\quad \mu r(1 - \varphi) \leq (q - \mu)(1 - \vartheta).$

About the matrix-dispersion, we suppose it is a jacobian. And, finally, the "parabolic constraints":

(H_6) $\quad r - \vartheta \geq \rho + 1, \qquad \delta = o(\varepsilon^{\frac{\rho+2}{r+1-\vartheta}}).$

5.2 Main Result

We are concerned with the diffusion dominant regime where $r - \vartheta \geq \rho + 1$ (at least a quadratic diffusion growth for linear dispersion).

While in one-dimension a single value of q was treated, we can handle arbitrarily large values of q: the natural one as given by the initial data $u_0 \in L^q(\mathbb{R}^d)$, which must only be (by Schonbek's representation theorem) greater than m. Then, we don't need flux constraints anymore, neither by diffusion (growth) interaction, neither by diffusion-dispersion balance interference, cf. [35, 29].

So, convergence is a matter of pure diffusion-dispersion competition— flux independent—, with diffusion domination being quantified by the (parabolic) conditions $\delta = o(\varepsilon^{\frac{\rho+2}{r+1-\vartheta}})$ and $r - \vartheta \geq \rho + 1$.

In the appendix A we show that the condition $\delta = o\big(\varepsilon^{\frac{\rho+2}{r+1-\vartheta}}\big)$ is the better (in fact, the unique) we obtain using our technique. We hope it is *sharp*.

Theorem 5.2.1. *Consider the Cauchy problem* (5.3)-(5.4) *with initial data* $u_0 \in L^q(\mathbb{R}^d)$ *and suppose that the flux* f *satisfies* (H_1) *with* $m < q$ *(which is always possible if* q *is large enough).*

Let be $u^{\varepsilon,\delta}$ *the solutions of the perturbed problem* (5.1)-(5.2) *with diffusion and dispersion satisfying* (H_2), (H_4) *and* (H_3) *such that* $r - \vartheta \geq \rho + 1$. *If* $\delta = o(\varepsilon^{\frac{\rho+2}{r+1-\vartheta}})$, *then the sequence* $u^{\varepsilon,\delta}$ *converges in* $L^s\big((0,T); L^p_{loc}(\mathbb{R}^d)\big)$, *for all* $s < \infty$ *and* $p < q$, *to a function* $u \in L^\infty\big((0,T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\big)$, *which is the unique entropy solution to* (5.3)-(5.4).

5.3 First Energy Estimates

Except if emphasis is necessary, the superscripts ε and δ are omitted in this section.

We make repeated use of the following computation. Consider the equation

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}\big(\varepsilon\, b_j(u, \nabla u) + \delta\, \partial_{x_j} c_j(\nabla u)\big)_{1 \leq j \leq d}, \tag{5.6}$$

and multiply it by $\eta'(u)$, where $\eta : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function and $q : \mathbb{R} \to \mathbb{R}^d$ is defined by $q_j' = \eta'\, f_j'$, $j = 1, \ldots, d$, we have

$$\partial_t \eta(u) = -\sum_j \Big(q_j'(u)\,\partial_{x_j} u - \varepsilon\,\partial_{x_j}\big(\eta'(u)\, b_j(u, \nabla u)\big) + \varepsilon\,\eta''(u)\,\partial_{x_j} u\; b_j(u, \nabla u)$$
$$- \delta\,\partial_{x_j}\big(\eta'(u)\,\partial_{x_j} c_j(\nabla u)\big) + \delta\,\eta''(u)\,\partial_{x_j} u\; \partial_{x_j} c_j(\nabla u)\Big)$$

$$= -\operatorname{div} q(u) + \varepsilon \operatorname{div}\big(\eta'(u)\, b(u,\nabla u)\big) - \varepsilon\, \eta''(u)\ \nabla u \cdot b(u,\nabla u)$$
$$+ \delta \sum_j \partial_{x_j}\big(\eta'(u)\, \partial_{x_j} c_j(\nabla u)\big) - \delta\, \eta''(u) \sum_j \partial_{x_j} u\ \partial_{x_j} c_j(\nabla u).$$

5.3.1 Linear Dispersion[2]

If $c = \big(c_j(\nabla u)\big)_{1\le j\le d} : \mathbb{R}^d \to \mathbb{R}^d$ is a linear function with $[a_{jl}]$ matrix, then we achieve last equation as

$$\text{(5.7)}\qquad \partial_t \eta(u) + \operatorname{div} q(u) = \varepsilon \operatorname{div}\big(\eta'(u)\ b(u,\nabla u)\big) - \varepsilon\, \eta''(u)\ \nabla u \cdot b(u,\nabla u)$$
$$+ \delta \sum_{j,l} a_{jl}\, \partial_{x_j}\big(\eta'(u)\, \partial^2_{x_j x_l} u\big) - \delta\, \eta''(u)/2 \sum_{j,l} a_{jl}\, \partial_{x_l}\big(\partial_{x_j} u\big)^2 .$$

When η is convex, the ε-term containing $\eta''(u)$ has a favorable sign: the diffusion dissipates the entropy η, the remaining terms are almost conservative.

The δ-line in (5.7) takes also the interesting form with only almost conserved first order derivatives instead of second order ones:

$$\text{(5.8)}\qquad \delta/2 \sum_{j,l} a_{jl} \Big(\partial^2_{x_j x_l}\big(2\,\eta'(u)\, \partial_{x_j} u\big) - \partial_{x_j}\big(2\,\eta''(u)\, \partial_{x_l} u\ \partial_{x_j} u\big)$$
$$- \partial_{x_l}\Big(\eta''(u)\, \big(\partial_{x_j} u\big)^2\Big) + \eta'''(u)\, \partial_{x_l} u\, \big(\partial_{x_j} u\big)^2\Big).$$

We begin by collecting fundamental energy estimates. Integrate (5.7) equation over $[0,t]$ and then over the whole of $\mathbb{R}^d$, with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$:

$$\int_{\mathbb{R}^d} \frac{|u(t)|^{\alpha+1} - |u_0|^{\alpha+1}}{\alpha+1}\, dx = -\frac{\alpha}{2} \int_{\mathbb{R}^d}\int_0^t |u|^{\alpha-1}\big(2\,\varepsilon\, \nabla u \cdot b\big(u,\nabla u\big)$$
$$+ \delta \sum_{j,l} a_{jl}\, \partial_{x_l}\big(\partial_{x_j} u\big)^2\big)\, ds dx,$$

or, we may use (5.8) and replace the right-hand side by

$$-\frac{\alpha}{2} \int_{\mathbb{R}^d}\int_0^t 2\,\varepsilon\, |u|^{\alpha-1}\, \nabla u \cdot b\big(u,\nabla u\big)$$
$$- (\alpha-1)\, \delta\ \operatorname{sgn}(u)\, |u|^{\alpha-2} \sum_{j,l} a_{jl}\, \partial_{x_l} u\, \big(\partial_{x_j} u\big)^2\, ds dx,$$

which yields the

[2] The general linear case, "nondiagonal", will be treated within the Nonlinear Dispersion, p.85; see also subsection 2.2.2, p.21.

Lemma 5.3.1. *Let be $\alpha \geq 1$ any real and $u_0 \in L^{\alpha+1}\left(\mathbb{R}^d\right)$. Any solution of* (5.6) *with linear dispersion $c : \mathbb{R}^d \to \mathbb{R}^d$ of $[a_{jl}]$ matrix satisfies, for $t \in [0,T]$,*

$$\int_{\mathrm{R}^d} \frac{|u(t)|^{\alpha+1}}{\alpha+1}\, dx + \alpha\,\varepsilon \int_{\mathrm{R}^d}\int_0^t |u|^{\alpha-1}\, \nabla u \cdot b\big(u, \nabla u\big)\, dsdx = \int_{\mathrm{R}^d} \frac{|u_0|^{\alpha+1}}{\alpha+1}\, dx - \frac{\alpha}{2}\,\delta \int_{\mathrm{R}^d}\int_0^t |u|^{\alpha-1} \sum_{j,l} a_{jl}\, \partial_{x_l}\big(\partial_{x_j} u\big)^2\, dsdx. \tag{5.9}$$

For $\alpha \geq 2$, the last term in the above identity also equals

$$+\frac{\alpha\,(\alpha-1)}{2}\,\delta \int_{\mathrm{R}^d}\int_0^t \mathrm{sgn}(u)\, |u|^{\alpha-2} \sum_{j,l} a_{jl}\, \partial_{x_l} u\, \left(\partial_{x_j} u\right)^2 dsdx. \tag{5.10}$$

Choosing $\alpha = 1$ in (5.9), we deduce at once a first uniform bound for u in $L^\infty((0,T); L^2(\mathbb{R}^d))$ together with a control for both $\nabla u \cdot b(u, \nabla u)$ in $L^1(\mathbb{R}^d \times (0,T))$ and ∇u in $L^{r+1}(\mathbb{R}^d \times (0,T))$:

Proposition 5.3.1. *For any solution of* (5.6) *with linear dispersion and $u_0 \in L^2\left(\mathbb{R}^d\right)$, we have for $t \in [0,T]$:*

$$\int_{\mathrm{R}^d} u(t)^2\, dx + 2\,\varepsilon \int_{\mathrm{R}^d}\int_0^t \nabla u \cdot b(u, \nabla u)\, dsdx = \int_{\mathrm{R}^d} u_0^2\, dx. \tag{5.11}$$

Assuming $\varepsilon > 0$ and the diffusion hypothesis

$$\exists r \geq 0,\ D > 0 : \quad \nabla u \cdot b(u, \nabla u) \geq D\, |\nabla u|^{r+1},$$

then

$$\|u(t)\|_{L^2(\mathrm{R}^d)} \leq \|u_0\|_{L^2(\mathrm{R}^d)}; \tag{5.12}$$

$$2\,D\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |\nabla u|^{r+1}\, dxds \leq 2\,\varepsilon \int_0^t \int_{\mathrm{R}^d} \nabla u \cdot b(u, \nabla u)\, dxds \leq \|u_0\|^2_{L^2(\mathrm{R}^d)}. \tag{5.13}$$

5.3.2 Nonlinear Dispersion

If we want to consider the general linear[3] or nonlinear cases, the good structure of the KdVB generalized equation we know how to handle is

$$\partial_t u + \mathrm{div}\, f(u) = \mathrm{div}\, \Big(\varepsilon\, b_j(u, \nabla u) + \delta \sum_k \partial_{x_k}\, c_{jk}\big(\nabla u\big)\Big)_{1 \leq j \leq d}, \tag{5.14}$$

[3] Any linear case, subsection 2.2.2, p.21.

with $c_{jk} = \partial_k C_j$, $j,k = 1,\dots,d$, i.e., the matrix $[c_{jk}]$ is the jacobian matrix of a potential $C = \big(C_j\big)_{1\le j\le d}$. We have,

$$\begin{aligned}\partial_t\eta(u) + \operatorname{div}\, q(u) &= \varepsilon\, \operatorname{div}\big(\eta'(u)\, b(u,\nabla u)\big) - \varepsilon\, \eta''(u)\ \nabla u\cdot b(u,\nabla u)\\ &+ \delta\sum_{j,k}\partial_{x_j}\big(\eta'(u)\,\partial_{x_k}c_{jk}\big(\nabla u\big)\big) - \delta\sum_{j,k}\partial_{x_k}\big(\eta''(u)\,\partial_{x_j}u\ c_{jk}\big(\nabla u\big)\big)\\ &+ \delta\sum_{j,k}\eta'''(u)\,\partial_{x_k}u\ \partial_{x_j}u\ c_{jk}\big(\nabla u\big) + \delta\sum_{j}\eta''(u)\,\partial_{x_j}C_j\big(\nabla u\big),\end{aligned}$$

the δ-lines above takes also on the form

$$\begin{aligned}\delta\sum_{j,k}\Big(&\partial^2_{x_jx_k}\big(\eta'(u)\,c_{jk}\big(\nabla u\big)\big) - \partial_{x_j}\big(\eta''(u)\,\partial_{x_k}u\ c_{jk}\big(\nabla u\big)\big)\\ &- \partial_{x_k}\big(\eta''(u)\,\partial_{x_j}u\ c_{jk}\big(\nabla u\big)\big) + 1/d\ \partial_{x_j}\big(\eta''(u)\,C_j\big(\nabla u\big)\big)\\ &+ \eta'''(u)\,\partial_{x_j}u\,\big(\partial_{x_k}u\ c_{jk}\big(\nabla u\big) - 1/d\ C_j\big(\nabla u\big)\big)\Big)\,.\end{aligned}\tag{5.15}$$

Again, integrating both over $[0,t]$ and then over $\mathbb{R}^d$ with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$, we prove the

Lemma 5.3.2. *Let be $\alpha \ge 1$*[4] *any real and $u_0 \in L^{\alpha+1}\big(\mathbb{R}^d\big)$. Any solution of* (5.14) *with dispersion-matrix $[c_{jk}]$ having a potential, $\big(C_j\big)_{1\le j\le d}$, satisfies, for $t\in[0,T]$,*

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|u(t)|^{\alpha+1}}{\alpha+1}\,dx + \alpha\,\varepsilon\int_{\mathbb{R}^d}\int_0^t |u|^{\alpha-1}\ \nabla u\cdot b(u,\nabla u)\,dsdx &= \int_{\mathbb{R}^d}\frac{|u_0|^{\alpha+1}}{\alpha+1}\,dx\\ + \alpha\,\delta\int_{\mathbb{R}^d}\int_0^t (\alpha-1)\ \operatorname{sgn}(u)\,|u|^{\alpha-2}\sum_{j,k}\partial_{x_k}u\ \partial_{x_j}u\ c_{jk}\big(\nabla u\big)&\\ + |u|^{\alpha-1}\sum_j\partial_{x_j}C_j\big(\nabla u\big)\,dsdx.&\end{aligned}\tag{5.16}$$

For $\alpha\ge 2$, the δ-term in the above identity also equals

$$\begin{aligned}+\,\alpha\,(\alpha-1)\,\delta\int_{\mathbb{R}^d}\int_0^t \operatorname{sgn}(u)\,|u|^{\alpha-2}\sum_{j,k}\partial_{x_j}u\,\big(\partial_{x_k}u&\\ c_{jk}\big(\nabla u\big) - 1/d\ C_j\big(\nabla u\big)\big)\ dsdx.&\end{aligned}\tag{5.17}$$

So, once more if $\alpha = 1$, from (5.16) we obtain the first energy estimates:

[4] If $\alpha = 1$, then $\eta''' = 0$ and we can apply, formally, formula (5.16).

Proposition 5.3.2. *For any solution of* (5.14) *with dispersion-matrix* $[c_{jk}]$ *having a potential,* $\left(C_j\right)_{1\leq j\leq d}$*, and* $u_0 \in L^2\left(\mathbb{R}^d\right)$*, we have for* $t \in [0,T]$:

$$\int_{\mathrm{R}^d} u(t)^2\,dx + 2\,\varepsilon \int_{\mathrm{R}^d}\int_0^t \nabla u\cdot b(u,\nabla u)\,dsdx = \int_{\mathrm{R}^d} u_0^2\,dx. \tag{5.18}$$

Assuming $\varepsilon > 0$ *and* (H_4)*, then*

$$\|u(t)\|_{L^2(\mathrm{R}^d)} \leq \|u_0\|_{L^2(\mathrm{R}^d)}; \tag{5.19}$$

$$\begin{aligned} 2\,D\,\varepsilon \int_{\mathrm{R}^d}\int_0^t |u|^{\mu\varphi}|\nabla u|^{r+1-\vartheta}\,dxds &\leq 2\,\varepsilon\int_{\mathrm{R}^d}\int_0^t \nabla u\cdot b(u,\nabla u)\,dsdx \\ &\leq \|u_0\|^2_{L^2(\mathrm{R}^d)}. \end{aligned} \tag{5.20}$$

5.4 $\mathbf{L^q}$ Estimates

To derive higher L^q a priori estimates, we use $\alpha > 1$ in the previous lemmas. We aim to control the second version of the dispersive δ-term in previous lemmas (as they allow us to switch derivatives order with gradient degree) by the use of Hölder's inequalities. This works in view of the competitive growths as given by the actual form of diffusion and dispersion terms. We analyse only the case of clearly dominant diffusion growth $r - \vartheta \geq \rho + 1$.

To begin with, in the next subsection we motivate the use of the particular case of the linear dispersion ($\rho = 1$) and the simplified diffusion hypothesis (H_4) with $\mu = \vartheta = 0$.

5.4.1 Linear Dispersion

Let's dominate (5.10), Lemma 5.3.1, p.85, where $\left(c_j\left(\nabla u\right)\right)_{1\leq j\leq d}$ the linear dispersion becames a factor of the third order growth in $|\nabla u|$:

$$\begin{aligned} &\left|\int_0^t\int_{\mathrm{R}^d} \mathrm{sgn}(u)|u|^{\alpha-2}\sum_{j,l} a_{jl}\,\partial_{x_l}u\left(\partial_{x_j}u\right)^2\,dxds\right| \\ &\qquad \leq \|[a_{jl}]\| \int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-2}\,|\nabla u|^3\,dxds \\ &\qquad \leq \|[a_{jl}]\|\left[\int_0^t\int_{\mathrm{R}^d}|u|^{(\alpha-2)p}\,dxds\right]^{\frac{1}{p}}\left[\int_0^t\int_{\mathrm{R}^d}|\nabla u|^{3p'}\,dxds\right]^{\frac{1}{p'}}. \end{aligned} \tag{5.21}$$

To take advantage of (5.13), we can choose $3\,p' = r+1$ provided $r \geq 2$. In particular, we can't work the linear diffusion case this fashion.

If $3\,p' = r+1$, then $p = \frac{r+1}{r-2}$, so $(\alpha-2)p = (r+1)\frac{\alpha-2}{r-2}$. Therefore it is rather natural to take the exponent $\alpha = r$, the diffusion growth[5]. Thus we deduce from Lemma 5.3.1 a natural estimate for $|u(t)|^{r+1}$, involving the combination $\delta\,\varepsilon^{-\frac{3}{r+1}}$ of δ and ε.

Proposition 5.4.1. *Assume $u_0 \in L^2(\mathbb{R}^d) \cap L^{r+1}(\mathbb{R}^d)$, $\varepsilon > 0$ and holds the diffusion hypothesis, with $r \geq 2$:*

$$\exists r \geq 2,\ D > 0: \quad \nabla u \cdot b(u, \nabla u) \geq D\,|\nabla u|^{r+1}.$$

For $t \in [0,T]$, we have

$$\int_{\mathrm{R}^d} |u(t)|^{r+1} dx + (r+1)\,r\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{r-1}\ \nabla u \cdot b(u, \nabla u)\,dxds \leq H_1\Big(\delta\,\varepsilon^{-\frac{3}{r+1}}\Big)\,, \tag{5.22}$$

$$\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{r-1}\,|\nabla u|^{r+1}\ dxds \leq \frac{1}{(r+1)\,r\,D}\ H_1\Big(\delta\,\varepsilon^{-\frac{3}{r+1}}\Big)\,, \tag{5.23}$$

where

$$C_1(u_0) := \max\left\{ \|u_0\|^{r+1}_{L^{r+1}(\mathrm{R}^d)}\,,\ \frac{\|[a_{jl}]\|\,(r+1)\,r\,(r-1)}{2}\left(\frac{\|u_0\|^2_{L^2(\mathrm{R}^d)}}{2\,D}\right)^{\frac{3}{r+1}} \right\};$$

$$H_1\Big(\delta\,\varepsilon^{-\frac{3}{r+1}}\Big) := C_1(u_0)\,\Big(1 + \delta\,\varepsilon^{-\frac{3}{r+1}}\ \max\Big\{1, \Big[\,t\,C_1(u_0)\,\Big(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\Big)\Big]^{\frac{r-2}{3}}\Big\}\Big).$$

Proof. Note, (5.23) is an immediate consequence of (5.22) and the diffusion hypothesis. For (5.22), we use Lemma 5.3.1 with $\alpha = r \geq 2$, and we estimate the term in (5.10) using (5.21),

$$\begin{aligned}\int_{\mathrm{R}^d} |u(t)|^{r+1}\,dx &+ (r+1)\,r\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{r-1}\ \nabla u \cdot b(u,\nabla u)\,dxds \\ &\leq \int_{\mathrm{R}^d} |u_0|^{r+1}\,dx + \|[a_{jl}]\|\ \frac{(r+1)\,r\,(r-1)}{2}\ \delta\,\varepsilon^{-\frac{3}{r+1}} \\ &\qquad \left[\varepsilon \int_0^t \int_{\mathrm{R}^d} |\nabla u|^{r+1}\ dxds\right]^{\frac{3}{r+1}} \|u\|^{r-2}_{L^{r+1}(\mathrm{R}^d\times(0,t))}.\end{aligned} \tag{5.24}$$

[5] Better than for, the obvious choice, $\alpha = 2$— Appendix A.0.1, p.127.

By the diffusion hypothesis the second term in the left-hand side is positive, integrate over $[0,t]$ and use (5.13):

$$\begin{aligned}\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^d\times(0,t))} \leq\ & t\ \|u_0\|^{r+1}_{L^{r+1}(\mathbb{R}^d)} + \|[a_{jl}]\|\ \frac{(r+1)\,r\,(r-1)}{2} \\ & t\left(\frac{\|u_0\|^2_{L^2(\mathbb{R}^d)}}{2\,C_2}\right)^{\frac{3}{r+1}} \delta\ \varepsilon^{-\frac{3}{r+1}}\ \|u\|^{r-2}_{L^{r+1}(\mathbb{R}^d\times(0,t))} \\ \leq\ & t\ C_1(u_0)\left(1+\delta\ \varepsilon^{-\frac{3}{r+1}}\ \left(\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^d\times(0,t))}\right)^{\frac{r-2}{r+1}}\right).\end{aligned}$$

Observe that the inequality

$$0 < X \leq\ K\left(1+\Delta\,X^{\frac{\theta}{r+1}}\right),$$

where $0\leq\theta<r+1$ and $K,\Delta>0$, implies

$$X \leq\ \max\left\{1,\ [\,K\,(1+\Delta)\,]^{\frac{r+1}{r+1-\theta}}\right\}.$$

Thus we deduce

$$\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^d\times(0,t))} \leq\ \max\left\{1,\ \left[t\ C_1(u_0)\left(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r+1}{3}}\right\}$$

and, returning to (5.24):

$$\begin{aligned}&\int_{\mathbb{R}^d}|u(t)|^{r+1}\,dx + (r+1)\,r\,\varepsilon\int_0^t\int_{\mathbb{R}^d}|u|^{r-1}\,\nabla u\cdot b(u,\nabla u)\,dxds \\ &\qquad\leq C_1(u_0)\left(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\ \max\left\{1,\ \left[t\ C_1(u_0)\left(1+\delta\,\varepsilon^{-\frac{3}{r+1}}\right)\right]^{\frac{r-2}{3}}\right\}\right).\end{aligned}$$

This completes the proof of (5.22). □

In particular, Proposition 5.4.1 shows that, if $u_0\in L^2\cap L^{r+1}$ and $\delta=\mathcal{O}(\varepsilon^{\frac{3}{r+1}})$, then $u(t)\in L^{r+1}$ uniformly for all $t\in[0,T]$.

To motivate a forthcoming derivation, consider the special case of $r=2$. Then (5.22) gives[6] an L^3 estimate. Returning back to inequality (5.21), with the new value $\alpha=3$, we can now estimate the dispersive term in (5.10) directly in view of (5.23). In this fashion, we deduce an L^4 estimate from Lemma 5.3.1.

[6]In fact, we obtain (5.21) directly, without needing Hölder's inequality.

This argument was recursively continued to reach any space L^q, if $r \geq 2$, in our paper [5]. Actually Propositions 5.3.1 and 5.4.1 shall be in that argument the first two inductive cases of the general L^q result.

In fact, now, we improve to a nonlinear dispersion and without need of a stressing-tedious recursive argument[7] .

5.4.2 Nonlinear Dispersion

Consider now the nonlinear diffusion-dispersion equation (5.14) and equality (5.17) in Lemma 5.3.2, p. 86. Bound the right-hand side using the (H_3) hypothesis:

$$\int_{\mathrm{R}^d} |u(t)|^{\alpha+1}\, dx + (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\ \nabla u \cdot\ b(u, \nabla u)\, dxds$$
$$\leq \int_{\mathrm{R}^d} |u_0|^{\alpha+1}\, dx + (\alpha+1)\,\alpha\,(\alpha-1)\,\delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2} |\,\nabla u|$$
$$\Big(|\,[c_{jk}(\nabla u)]\,\nabla u\,| + |\big(C_j(\nabla u)\big)_{1\leq j\leq d}|\Big)\, dxds$$
$$\leq \|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathrm{R}^d)} + 2\,C_3\,(\alpha+1)\,\alpha\,(\alpha-1) \tag{5.25}$$
$$\delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2} |\,\nabla u|^{\rho+2}\, dxds\,,$$

now, bound the left-hand side, using the (H_4) hypothesis, and integrate over $[0,t]$:

$$\int_0^t \int_{\mathrm{R}^d} |u|^{\alpha+1}\, dxds + D\,(\alpha+1)\,\alpha\,t\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1} |\,\nabla u|^{r+1}\, dxds$$
$$\leq t\,\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathrm{R}^d)}$$
$$+2\,C_3\,(\alpha+1)\,\alpha\,(\alpha-1)\,t\,\delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2} |\,\nabla u|^{\rho+2}\, dxds\,.$$

Let us put diffusion and dispersion in competition: apply Young's inequality[8] to the last term and use (5.20) from Proposition 5.3.2:

$$2\,C_3\,(\alpha+1)\,\alpha\,(\alpha-1)\,t\,\delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2} |\,\nabla u|^{\rho+2}\, dxds$$
$$= \int_0^t \int_{\mathrm{R}^d} \left[p_2\,\frac{|u|^{\alpha+1}}{2}\right]^{\frac{1}{p_2}} \left[p_3\,\frac{D\,(\alpha+1)\,\alpha\,t\,\varepsilon\,|u|^{\alpha-1} |\,\nabla u|^{r+1}}{2}\right]^{\frac{1}{p_3}}$$

[7]See Appendix A, p.127.

[8]As learned in Appendix A, p.127.

$$\Bigg[2^{2(p_1-1)}\big((\alpha+1)\,\alpha\,t\big)^{1+\frac{p_1}{p_2}}\Bigg(C_3\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}p_3^{-\frac{1}{p_3}}\,D^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}$$
$$\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\Bigg)^{p_1}\,2\,D\,\varepsilon\,|\,\nabla u|^{r+1}\Bigg]^{\frac{1}{p_1}}\,dxds$$

$$\leq \frac{1}{2}\left(\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds+D\,(\alpha+1)\,\alpha\,t\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u|^{r+1}\,dxds\right)$$
$$+\frac{1}{p_1}2^{2(p_1-1)}\big((\alpha+1)\,\alpha\,t\big)^{1+\frac{p_1}{p_2}}\Bigg(C_3\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}p_3^{-\frac{1}{p_3}}D^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}$$
$$\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\Bigg)^{p_1}\,\|u_0\|^2_{L^2(\mathrm{R}^d)}$$

where

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1,\quad \rho+2=\frac{r+1}{p_1}+\frac{r+1}{p_3},\quad \alpha-2=\frac{\alpha+1}{p_2}+\frac{\alpha-1}{p_3},$$

so that

$$\frac{1}{p_1}+\frac{1}{p_3}=\frac{\rho+2}{r+1},\quad \frac{1}{p_2}=1-\frac{\rho+2}{r+1},$$
$$\frac{1}{p_3}=\frac{1}{\alpha-1}\left((\alpha+1)\,\frac{\rho+2}{r+1}-3\right),\quad \frac{1}{p_1}=\frac{1}{\alpha-1}\left(3-2\,\frac{\rho+2}{r+1}\right).$$

Define

$$H_\alpha\Big(\delta\;\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\;:=\frac{1}{p_1}2^{2(p_1-1)}\,t^{\frac{p_1}{p_2}}\,\big((\alpha+1)\,\alpha\big)^{1+\frac{p_1}{p_2}}\Bigg(C_3\,(\alpha-1)\,p_2^{-\frac{1}{p_2}}p_3^{-\frac{1}{p_3}}$$
$$D^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\,\delta\,\varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_3}\right)}\Bigg)^{p_1}\,\|u_0\|^2_{L^2(\mathrm{R}^d)}\,.$$

We deduce, both,

$$\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha+1}\,dxds+D\,(\alpha+1)\,\alpha\,t\,\varepsilon\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-1}|\,\nabla u|^{r+1}\,dxds$$
$$\leq 2t\left(\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathrm{R}^d)}+H_\alpha\Big(\delta\;\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right)$$

and

$$2\,C_3\,(\alpha+1)\,\alpha\,(\alpha-1)\,\delta\;\int_0^t\int_{\mathrm{R}^d}|u|^{\alpha-2}|\,\nabla u|^{\rho+2}\,dxds$$
$$\leq \|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathrm{R}^d)}+2H_\alpha\Big(\delta\;\varepsilon^{-\frac{\rho+2}{r+1}}\Big)\,.$$

So, we can finally come back to (5.25):

$$\int_{\mathbb{R}^d} |u(t)|^{\alpha+1}\, dx + (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1}\ \nabla u \cdot\ b(u,\nabla u)\, dxds \\ \leq 2\left(\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^d)} + H_\alpha\Big(\delta\ \varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right).$$

We have then proved at once[9] the following proposition which gives rise to an arbitrarily large L^q bound.

Proposition 5.4.2. *Assume that* (H_2), (H_3), (H_4) *holds with* $r \geq \rho+1$ *and* $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. *For* $t \in [0,T]$ *and* α *such that* $3\,\frac{r+1}{\rho+2} < \alpha+1 \leq q$ *we have*

$$\int_{\mathbb{R}^d} |u(t)|^{\alpha+1} dx + (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1}\ \nabla u \cdot b(u,\nabla u)\, dxds \\ \leq 2\left(\|u_0\|^{q}_{L^{q}(\mathbb{R}^d)} + H_\alpha\Big(\delta\ \varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right), \tag{5.26}$$

$$\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1}\ |\nabla u|^{r+1}\ dxds \leq \frac{2}{D\,(\alpha+1)\,\alpha} \Big(\|u_0\|^{q}_{L^{q}(\mathbb{R}^d)} \\ + H_\alpha\Big(\delta\ \varepsilon^{-\frac{\rho+2}{r+1}}\Big)\Big). \tag{5.27}$$

$$\delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-2}\ |\,\nabla u|^{\rho+2}\, dxds \leq \frac{1}{2\,C_3\,(\alpha+1)\,\alpha\,(\alpha-1)} \\ \left(\|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^d)} + 2H_\alpha\Big(\delta\ \varepsilon^{-\frac{\rho+2}{r+1}}\Big)\right). \tag{5.28}$$

5.5 Convergence Proof

Proof of Theorem. We first prove (2.13), based on the conservation law (5.15) with an arbitrary convex function η (where we assume η', η'', η''' bounded functions on $\mathbb{R}$). We claim that there exists a bounded measure $\mu \leq 0$ such that

$$\partial_t \eta(u) + \text{div}\ q(u) \longrightarrow \mu\,, \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \times (0,T)).$$

From (5.15), we obtain

$$\partial_t \eta(u) + \text{div}\ q(u) = \varepsilon\ \text{div}(\eta'(u)\ b(u,\nabla u)) - \varepsilon\,\eta''(u)\ \ \nabla u \cdot b(u,\nabla u) \\ + \delta \sum_{j,k} \eta'''(u)\,\partial_{x_j} u\,\big(\partial_{x_k} u\ c_{jk}(\nabla u) - 1/d\ C_j(\nabla u)\big)$$

[9]Without any recursion procedure: unlike our first approach in the linear case.

$$
\begin{aligned}
&- \delta \sum_{j,k} \partial_{x_k}\big(\eta''(u)\, \partial_{x_j} u \; c_{jk}\big(\nabla u\big)\big) + 1/d \; \partial_{x_j}\big(\eta''(u) \\
&\qquad\qquad C_j\big(\nabla u\big)\big) - \partial_{x_j}\big(\eta''(u)\, \partial_{x_k} u \; c_{jk}\big(\nabla u\big)\big) \\
&+ \delta \sum_{j,k} \partial^2_{x_j x_k}\big(\eta'(u)\, c_{jk}\big(\nabla u\big)\big)\,.
\end{aligned}
$$

We will use the notation:

$$
\begin{aligned}
\mu_1 &:= \varepsilon \; \mathrm{div}(\eta'(u)\; b(u,\nabla u)) \\
\mu_2 &:= -\varepsilon\, \eta''(u) \;\; \nabla u \cdot b(u,\nabla u) \\
\mu_3 &:= \; \delta \sum_{j,k} \eta'''(u)\, \partial_{x_j} u \left(\partial_{x_k} u \; c_{jk}\big(\nabla u\big) - 1/d \; C_j\big(\nabla u\big)\right) \\
&\quad - \delta \sum_{j,k} \partial_{x_k}\big(\eta''(u)\, \partial_{x_j} u \; c_{jk}\big(\nabla u\big)\big) + 1/d \; \partial_{x_j}\big(\eta''(u) \\
&\qquad\qquad C_j\big(\nabla u\big)\big) - \partial_{x_j}\big(\eta''(u)\, \partial_{x_k} u \; c_{jk}\big(\nabla u\big)\big) \\
&\quad + \delta \sum_{j,k} \partial^2_{x_j x_k}\big(\eta'(u)\, c_{jk}\big(\nabla u\big)\big)\,.
\end{aligned}
$$

For each positive $\theta \in C_0^\infty(\mathbb{R}^d \times (0,T))$ we evaluate $\langle \mu_i, \theta\rangle$ for $i = 1,2,3$:

$$
\begin{aligned}
|\langle \mu_1, \theta\rangle| &\leq \varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla\theta \cdot \eta'(u)\, b(u,\nabla u)| \; dxdt \\
&\leq C\,\varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla \theta| \; dxdt \; + C\,\varepsilon \int_0^T \int_{\mathrm{R}^d} |\nabla\theta| \; |u|^\mu \, |\nabla u|^r \, dxdt\,,
\end{aligned}
$$

in view of the growth hypothesis (H_2). So[10], using Hölder's inequality with the exponent $\frac{r+1-\vartheta}{r}$ within (5.20) of Proposition 5.3.2 and assumption (5.5), we get

$$
\begin{aligned}
|\langle \mu_1, \theta\rangle| &\leq C\,\varepsilon\, \|\, \nabla\theta\|_{L^1(\mathrm{R}^d \times (0,T))} + C\, \varepsilon^{\frac{1-\vartheta}{r+1}} \left[\iint_{\mathrm{supp}\,\theta} |\nabla\theta|^{\frac{r+1-\vartheta}{1-\vartheta}} \right. \\
&\qquad \left. |u|^{\mu\left(1+r\frac{1-\varphi}{1-\vartheta}\right)} \; dxdt\right]^{\frac{1-\vartheta}{r+1-\vartheta}} \left[\varepsilon \iint_{\mathrm{supp}\,\theta} |u|^{\mu\varphi} \; |\nabla u|^{r+1-\vartheta} \; dxdt\right]^{\frac{r}{r+1-\vartheta}} \\
&\leq C\,\varepsilon\, \|\, \nabla\theta\|_{L^1(\mathrm{R}^d \times (0,T))} + C\, \varepsilon^{\frac{1-\vartheta}{r+1}} \; \|\, \nabla\theta\|_{L^{\frac{r+1-\vartheta}{1-\vartheta}}(\mathrm{R}^d \times (0,T))}\,.
\end{aligned}
$$

For μ_2, because $\nabla u \cdot b(u,\nabla u) \geq 0$ and η is convex,

$$
\langle \mu_2, \theta\rangle = -\varepsilon \int_0^T \int_{\mathrm{R}^d} \theta \; \eta''(u) \;\; \nabla u \cdot b(u,\nabla u) \; dxdt \leq 0\,.
$$

[10] In the case where $\vartheta < 1$.

For μ_3, we have by (H_3) hypothesis

$$
\begin{aligned}
|\langle \mu_3, \theta\rangle| \leq \delta \int_0^T\int_{\mathrm{R}^d} \sum_{j,k} & \left|\eta'''(u)\ \theta\ \partial_{x_j}u\ \big(\partial_{x_k}u\ c_{jk}(\nabla u) - 1/d\ C_j(\nabla u)\big)\right| \\
& + \left|\eta''(u)\ \big(\partial_{x_k}\theta\ \partial_{x_j}u\ c_{jk}(\nabla u) + 1/d\ \partial_{x_j}\theta\ C_j(\nabla u)\right. \\
& \left. -\ \partial_{x_j}\theta\ \partial_{x_k}u\ c_{jk}(\nabla u)\big)\right| + \left|\eta'(u)\ \partial^2_{x_j x_k}\theta\ c_{jk}(\nabla u)\right|\ dxdt
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\,\delta \int_0^T\int_{\mathrm{R}^d} \theta\ |\nabla u|\ dxdt + C\,\delta \int_0^T\int_{\mathrm{R}^d} \theta\ |\nabla u|^2\ dxdt \\
&\quad + C\,\delta \int_0^T\int_{\mathrm{R}^d} \theta\ |\nabla u|^{\rho+2}\ dxdt \\
&\quad + C\,\delta \int_0^T\int_{\mathrm{R}^d} |\nabla\theta|\ dxdt + C\,\delta \int_0^T\int_{\mathrm{R}^d} |\nabla\theta|\,|\nabla u|\ dxdt \\
&\quad + C\,\delta \int_0^T\int_{\mathrm{R}^d} |\nabla\theta|\,|\nabla u|^{\rho+1}\ dxdt \\
&\quad + C\,\delta \int_0^T\int_{\mathrm{R}^d} \left|D^2\theta\right|\ dxdt + C\,\delta \int_0^T\int_{\mathrm{R}^d} \left|D^2\theta\right|\,|\nabla u|^{\rho}\ dxdt
\end{aligned}
$$

so, using again Hölder's inequality it follows:

$$
\begin{aligned}
|\langle \mu_3, \theta\rangle| \leq\ & C\ \delta\,\varepsilon^{-\frac{1}{r+1}}\ \|\theta\|_{L^{\frac{r+1}{r}}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{1}{r+1}} \\
& + C\ \delta\,\varepsilon^{-\frac{2}{r+1}}\ \|\theta\|_{L^{\frac{r+1}{r-1}}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{2}{r+1}}
\end{aligned}
$$

$$
\begin{aligned}
& + C\ \delta\,\varepsilon^{-\frac{\rho+2}{r+1}}\ \|\theta\|_{L^{\frac{r+1}{r-\rho-1}}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{\rho+2}{r+1}} \\
& + C\ \delta\,\|\ \nabla\theta\|_{L^1(\mathrm{R}^d\times(0,T))} \\
& + C\ \delta\,\varepsilon^{-\frac{1}{r+1}}\ \|\ \nabla\theta\|_{L^{\frac{r+1}{r}}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{1}{r+1}}
\end{aligned}
$$

$$
\begin{aligned}
& + C\ \delta\,\varepsilon^{-\frac{\rho}{r+1}}\ \|\ \nabla\theta\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\ dxdt\right]^{\frac{\rho}{r+1}} \\
& + C\ \delta\,\|D^2\theta\|_{L^1(\mathrm{R}^d\times(0,T))}
\end{aligned}
$$

$$+ C\ \delta\, \varepsilon^{-\frac{\rho+1}{r+1}} \|D^2\theta\|_{L^{\frac{r+1}{r-\rho}}(\mathbb{R}^d\times(0,T))} \left[\varepsilon \iint_{\operatorname{supp}\theta} |\nabla u|^{r+1}\, dxdt\right]^{\frac{\rho+1}{r+1}},$$

therefore, by (5.20) of Proposition 5.3.2 and assumption (5.5),

$$|\langle \mu_3, \theta\rangle| \leq\ C\,\delta\,\varepsilon^{-\frac{\rho+2}{r+1}}.$$

Finally the condition $\delta = o(\varepsilon^{\frac{\rho+2}{r+1}})$ is sufficient to the conclusion.

Using a standard regularization of $\operatorname{sgn}(u)$ and $|u-k|$ (for $k \in \mathbb{R}$), which satisfies the growth condition (2.9) in the Young measure representation theorem, Lemma 2.2.1, p. 19, we then apply the limit representation (2.10) and conclude that ν satisfies (2.13).

To show (2.14) we follow DiPerna [14] and Szepessy [38]'s arguments. We have to check that, for each compact K of $\mathbb{R}^d$,

$$\begin{aligned}\lim_{t\to 0+} \frac{1}{t}\int_0^t\int_K \langle \nu_{(x,s)}, |u-u_0(x)|\,\rangle\, dxds \\ = \lim_{t\to 0+}\lim_{\varepsilon\to 0+} \frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\, dxds = 0.\end{aligned}$$

By Jensen's inequality, where $m(K)$ stands for Lebesgue measure of K,

$$\begin{aligned}\frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\, dxds \\ \leq m(K)^{1/2}\left(\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\, dxds\right)^{1/2}.\end{aligned}$$

We will establish that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+} \frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\, dxds = 0.$$

Let $K_i \subset K_{i+1}$ ($i = 0, 1, \dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i\geq 0} K_i = \mathbb{R}^d$. We use the identity $u^2 - u_0^2 - 2u_0(u-u_0) = (u-u_0)^2$, inequality (5.5) and (5.18):

$$\begin{aligned}&\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)^2\, dxds \\ &\leq \frac{1}{t}\int_0^t \left(\int_{K_i} |u^{\varepsilon,\delta}(\cdot,s)|^2\, dx - \int_{K_i} u_0^2\, dx - 2\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\, dx\right) ds \\ &\leq \int_{\mathbb{R}^d\setminus K_i} u_0^2\, dx + \frac{2}{t}\int_0^t \left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\, dx\right| ds\end{aligned}$$

for all $i = 0, 1, \dots$

Since

$$\lim_{i\to\infty} \int_{\mathbb{R}^d \setminus K_i} u_0^2 \, dx = 0,$$

we only need to consider the second term.

Take $\{\theta_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \theta_n = u_0 \quad \text{in} \quad L^2(\mathbb{R}^d),$$

then the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\left| \int_{K_i} u_0 \left(u^{\varepsilon,\delta}(\cdot, s) - u_0 \right) dx \right| &\le \int_{K_i} |u_0 - \theta_n| \left| u^{\varepsilon,\delta}(\cdot, s) - u_0 \right| dx \\
&\quad + \left| \int_{K_i} \theta_n \left(u_0^{\varepsilon,\delta} - u_0 \right) + \int_{K_i} \theta_n \left(u^{\varepsilon,\delta}(\cdot, s) - u_0^{\varepsilon,\delta} \right) dx \right| \\
&\le \|u_0 - \theta_n\|_{L^2(\mathbb{R}^d)} \left(\|u^{\varepsilon,\delta}(\cdot, s)\|_{L^2(\mathbb{R}^d)} + \|u_0\|_{L^2(\mathbb{R}^d)} \right) \\
&\quad + \|\theta_n\|_{L^2(\mathbb{R}^d)} \|u_0^{\varepsilon,\delta} - u_0\|_{L^2(\mathbb{R}^d)} + \left| \int_0^s \int_{K_i} \theta_n \, \partial_s u^{\varepsilon,\delta} \, dx d\tau \right|.
\end{aligned}$$

In view of (5.18) and (5.5)

$$\begin{aligned}
\|u_0 - \theta_n\|_{L^2(\mathbb{R}^d)} &\left(\|u^{\varepsilon,\delta}(\cdot, s)\|_{L^2(\mathbb{R}^d)} \|u_0\|_{L^2(\mathbb{R}^d)} \right) \\
&\le 2 \|u_0\|_{L^2(\mathbb{R}^d)} \|u_0 - \theta_n\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which tends to zero when $n \to \infty$; since $\lim_{\varepsilon\to 0+} \|u_0^{\varepsilon,\delta} - u_0\|_{L^2(\mathbb{R}^d)} = 0$, it remains to see that

$$\lim_{t\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n \, \partial_s u^{\varepsilon,\delta} \, dx d\tau \right| ds = 0.$$

We have, by (5.14),

$$\begin{aligned}
\left| \int_0^s \int_{K_i} \theta_n \, \partial_s u^{\varepsilon,\delta} \, dx d\tau \right| &= \left| \int_0^s \int_{K_i} \theta_n \big(-\operatorname{div} f(u^{\varepsilon,\delta}) + \varepsilon \operatorname{div} b\big(u^{\varepsilon,\delta}, \nabla u^{\varepsilon,\delta}\big) \right. \\
&\qquad \left. + \delta \sum_{j,k} \partial^2_{x_j x_k} c_{jk}\big(\nabla u^{\varepsilon,\delta}\big) \, dx d\tau \right|
\end{aligned}$$

$$\le \int_0^s \int_{K_i} \left| \nabla \theta_n \cdot f\big(u^{\varepsilon,\delta}\big) \right| dx d\tau$$

$$
\begin{aligned}
&+\varepsilon \int_0^s \int_{K_i} \left|\nabla\theta_n \cdot b\big(u^{\varepsilon,\delta}, \nabla u^{\varepsilon,\delta}\big)\right| \, dx d\tau \\
&+\delta \int_0^s \int_{K_i} \sum_{j,k} \left|\partial^2_{x_j x_k} \theta_n \, c_{jk}\big(\nabla u^{\varepsilon,\delta}\big)\right| \, dx d\tau \\
:= {}& \mu_1 + \mu_2 + \mu_3.
\end{aligned}
$$

To deal with μ_1, we use (H_1) and Hölder's inequality within (5.26) and (5.5):

$$
\begin{aligned}
\int_0^s \int_{K_i} |\nabla\theta_n| \left|f\big(u^{\varepsilon,\delta}\big)\right| \, dx\, d\tau &\leq C_1 \int_0^s \int_{K_i} |\nabla\theta_n| \, dx\, d\tau \\
&\quad + C_1 \left[\int_0^s \int_{K_i} |\nabla\theta_n|^{\frac{q}{q-m}} \, dx\, d\tau\right]^{\frac{q-m}{q}} \left[\int_0^s \int_{K_i} |u^{\varepsilon,\delta}|^q \, dx\, d\tau\right]^{\frac{m}{q}} \\
&\leq C_1 \, s \, \|\nabla\theta_n\|_{L^1(\mathrm{R}^d)} + C\, s^{\frac{q-m}{q}} \, \|\nabla\theta_n\|_{L^{\frac{q}{q-m}}(\mathrm{R}^d)}.
\end{aligned}
$$

For μ_2, using (H_2) and once more Hölder's inequality with (5.20) and (5.5), we get

$$
\begin{aligned}
\varepsilon & \int_0^s \int_{K_i} |\nabla\theta_n| \left|b\big(u^{\varepsilon,\delta}, \nabla u^{\varepsilon,\delta}\big)\right| \, dx\, d\tau \\
&\leq C_2\, \varepsilon \int_0^s \int_{K_i} |\nabla\theta_n| \, dx\, d\tau + C_2\, \varepsilon \int_0^s \int_{K_i} |\nabla\theta_n| \left|\nabla u^{\varepsilon,\delta}\right|^r \, dx\, d\tau \\
&\leq C_2\, \varepsilon\, s \, \|\nabla\theta_n\|_{L^1(\mathrm{R}^d)} \\
&\quad + C_2\, \varepsilon^{1-\frac{r}{r+1}} \, s^{\frac{1}{r+1}} \, \|\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)} \left[\varepsilon \int_0^s \int_{K_i} \left|\nabla u^{\varepsilon,\delta}\right|^{r+1} \, dx d\tau\right]^{\frac{r}{r+1}} \\
&\leq C_2\, \varepsilon\, s \, \|\nabla\theta_n\|_{L^1(\mathrm{R}^d)} + C\, \varepsilon^{\frac{1}{r+1}} \, s^{\frac{1}{r+1}} \, \|\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)}.
\end{aligned}
$$

Finally for μ_3, with (H_3), Hölder's inequality, (5.20) and (5.5), we have

$$
\begin{aligned}
\delta \int_0^s \int_{K_i} \sum_{j,k} \left|\partial^2_{x_j x_k} \theta_n c_{jk}\big(\nabla u^{\varepsilon,\delta}\big)\right| \, dx d\tau &\leq \delta \int_0^s \int_{K_i} \left|D^2\theta_n\right| \left|c_{jk}\big(\nabla u^{\varepsilon,\delta}\big)\right| \, dx d\tau \\
&\leq C_3\, \delta \int_0^s \int_{K_i} \left|D^2\theta_n\right| \, dx d\tau + C_3\, \delta \int_0^s \int_{K_i} \left|D^2\theta_n\right| \left|\nabla u^{\varepsilon,\delta}\right|^\rho \, dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_3\, \delta\, s \, \left\|D^2\theta_n\right\|_{L^1(\mathrm{R}^d)} + C_3\, \delta\, \varepsilon^{-\frac{\rho}{r+1}} s^{\frac{r+1-\rho}{r+1}} \left\|D^2\theta_n\right\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d)} \\
&\qquad\qquad \left[\varepsilon \int_0^s \int_{K_i} \left|\nabla u^{\varepsilon,\delta}\right|^{r+1} \, dx d\tau\right]^{\frac{\rho}{r+1}} \\
&\leq C_3\, \delta\, s \, \left\|D^2\theta_n\right\|_{L^1(\mathrm{R}^d)} + C\, \delta\, \varepsilon^{-\frac{\rho}{r+1}} s^{\frac{r+1-\rho}{r+1}} \left\|D^2\theta_n\right\|_{L^{\frac{r+1}{r+1-\rho}}(\mathrm{R}^d)}.
\end{aligned}
$$

Thus, as yet $\delta = o(\varepsilon^{\frac{\rho+2}{r+1}})$,

$$\begin{aligned}
&\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|ds\\
&\qquad\le \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{C}{t}\Big((1+\varepsilon)\,t^2\,\|\nabla\theta_n\|_{L^1(\mathrm{R}^d)}+\,t^{\frac{q-m}{q}+1}\,\|\nabla\theta_n\|_{L^{\frac{q}{q-m}}(\mathrm{R}^d)}\\
&\qquad\qquad\qquad +t^{\frac{1}{r+1}+1}\,\varepsilon^{\frac{1}{r+1}}\,\|\nabla\theta_n\|_{L^{r+1}(\mathrm{R}^d)}+\,t^2\,\delta\,\left\|D^2\theta_n\right\|_{L^1(\mathrm{R}^d)}\Big)\\
&\qquad\le \lim_{t\to 0+}\lim_{\varepsilon\to 0+}C\,\Big((1+\varepsilon+\delta)\,t+t^{\frac{q}{q-m}}+t^{\frac{1}{r+1}}\,\varepsilon^{\frac{1}{r+1}}\Big),
\end{aligned}$$

the desired conclusion. □

Part III

One-Dimensional Hyperbolic Systems of Conservation Laws

Chapter 6

Hyperbolic Systems of Conservation Laws with Lipschitz Continuous Fluxes[1]

Abstract. For strictly hyperbolic systems of conservation laws with Lipschitz continuous flux-functions we generalize Lax's genuine nonlinearity condition and shock admissibility inequalities and we solve the Riemann problem when the left- and right-hand initial data are sufficiently close. Our approach is based on the concept of multivalued representatives of L^∞ functions and a generalized calculus for Lipschitz continuous mappings. Several interesting features arising with Lipschitz continuous flux-functions come to light from our analysis.

6.1 Assumptions

Our general objective is to identify new features arising in discontinuous solutions of systems of conservation laws with Lipschitz continuous flux. Here, we will focus attention on the so-called Riemann problem (Lax [26]) for the strictly hyperbolic system

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U}, \quad x \in \mathbb{R},\, t > 0, \tag{6.1}$$

supplemented with the piecewise constant initial condition

$$u(x,0) = \begin{cases} u_l, & x < 0; \\ u_r, & x > 0. \end{cases} \tag{6.2}$$

[1]From Correia-LeFloch-Thanh [7]

We assume that the data u_l, u_r belong to $\mathcal{U} := \mathcal{B}(u_*, \delta) \subset \mathbb{R}^N$, the ball with center u_* and (small) radius δ. The function $f : \mathcal{U} \to \mathbb{R}^N$ is assumed to be Lipschitz continuous and the Jacobian matrix to be strictly hyperbolic. Each characteristic field will be assumed to be genuinely nonlinear. (Since the flux is not smooth, these notions have to be reconsidered; see the begining of section 6.3 below.)

Discontinuous solutions of (6.1) satisfying an entropy condition (required for uniqueness) will be sought. Recall that the Riemann problem plays a fundamental role within the theory of conservation laws and yields many interesting informations on general solutions of (6.1). To extend Lax's theory to a Lipschitz continuous f, the difficulty is to handle possibly *discontinuous wave speeds*.

We will rely here on a generalized calculus for Lipschitz continuous mappings (a brief review is presented in the section 2.3). A generalized derivative is a *set of vectors* rather than a single value. We will also rely on the (related) theory developed earlier by Filippov [15] for ordinary differential equations with discontinuous coefficients, see also Hörmander [20].

An outline of the content of this chapter follows. Section 6.2 deals with the case of scalar conservation laws, wich is particularly straightforward but nevertheless of particular interest, as it allows us to exhibit the new qualitative behavior of shock waves and rarefaction waves associated with discontinuous wave speeds. Section 6.3 contains a general existence theory for the Riemann problem (6.1)-(6.2) for systems. Solutions satisfy a suitable generalization of Lax shock admissibility inequalities. Observe that the Riemann solution may be non-unique when the flux is not smooth, even when entropy inequalities are imposed. Finally, in Section 5, we investigate a specific example arising in fluid dynamics.

6.2 Scalar Conservation Laws

To begin with, in this section we consider the equation (6.1) when $N = 1$ and investigate the Riemann problem. Recall that we solely assume that the flux f belongs to $W^{1,\infty}(\mathbb{R})$. For such a function of a single variable one can set

$$f'_+(u) = \limsup_{\substack{v \to u \\ h \to 0+}} \frac{f(v+h) - f(v)}{h}, \qquad (6.3)$$

$$f'_-(u) = \liminf_{\substack{v \to u \\ h \to 0+}} \frac{f(v+h) - f(v)}{h}.$$

Proposition 6.2.1. *At every point $u \in \mathbb{R}$ we have*

$$\partial f(u) = \left[\, f'_-(u), f'_+(u) \,\right]. \tag{6.4}$$

Proof. First of all by the definition (2.18) we have

$$f'_+(u) = f^\circ(u;1)$$

and

$$\begin{aligned} f'_-(u) &= -\limsup_{\substack{v\to u\\ h\to 0+}} \left(-\frac{f(v+h)-f(v)}{h} \right) \\ &= -\limsup_{\substack{v\to u\\ h\to 0+}} \frac{(-f)(v+h)-(-f)(v)}{h} \\ &= -(-f)^\circ(u;1) = -f^\circ(u;-1). \end{aligned} \tag{6.5}$$

By definition, $w \in \partial f(u)$ if and only if

$$w \cdot v \le f^\circ(u;v), \quad v \in \mathbb{R}.$$

Since both sides of the last inequality are positively homogeneous of degree one, the condition reduces to

$$w \le f^\circ(u;1) \text{ and } -w \le f^\circ(u;-1).$$

From (6.5) we also easily deduce that

$$\begin{aligned} w &\le f^\circ(u;1) = f'_+(u), \\ w &\ge -f^\circ(u;-1) = f'_-(u). \end{aligned}$$

which completes the proof. □

The wave speed

$$\lambda(u) := f'(u)$$

solely belongs to $L^\infty(\mathbb{R})$. The associated shock speed defined by

$$\sigma(u,v) = \frac{f(v)-f(u)}{v-u} \tag{6.6}$$

is a Lipschitz function of its argument away from the diagonal $\{u = v\}$. Observe that given some state u_0 and for specific sequences $u, v \to u_0$ we may reach any value within the interval $\partial f(u_0)$.

We will generalize here Oleinik's construction of the solution of the Riemann problem (6.1)-(6.2) to the case of a Lipschitz continuous flux. To begin with, we will review the notion of generalized inverse of monotone mappings. Consider a function $h : [a, b] \to \mathbb{R}$ which is non-decreasing on a closed interval $[a, b] \subset \mathbb{R}$, i.e.,

$$y_0, y_1 \in [a, b], \quad y_0 \geq y_1 \Longrightarrow h(y_0) \geq h(y_1).$$

Then, the function h has locally bounded variation and its set of discontinuity points is at most countable. Moreover, at each discontinuity point y we can define left- and right-hand limits denoted $h_-(y)$ and $h_+(y)$, respectively. Since h is non-decreasing, there is no ambiguity between this notation and the one in (6.3). At points of continuity we have obviously that $h_-(y) = h_+(y) = h(y)$. The functions h_- and h_+ are the left- and right-continuous representatives of the function h. For each $\xi \in [h(a), h(b)]$ consider the set

$$G(\xi) := \{y \in [a, b] / \ h(y) = \xi\}. \tag{6.7}$$

We can distinguish between three cases: $G(\xi)$ may be a single point, or an interval $I \subset [a, b]$ with distinct endpoints, or the empty set. We state without proof (see [33]):

Lemma and Definition 6.2.1. *Let $h : [a, b] \to \mathbb{R}$ be a non-decreasing function. Its (non-decreasing) generalized inverse denoted by $h^{-1} : [h(a), h(b)] \to [a, b]$ is defined as follows at each $\xi \in [h(a), h(b)]$:*

(i) *If $G(\xi) = \{y\}$, then we set*

$$h^{-1}(\xi) = y.$$

(ii) *If $G(\xi)$ is an interval $I \subset [a, b]$ with distinct endpoints $y_0 < y_1$, then we can pick up any value*

$$h^{-1}(\xi) \in I,$$

for instance the lower bound y_0 of the interval I. In that case, ξ is a point of discontinuity of the function h, the set of such points ξ being of course at most countable.

(iii) *If $G(\xi) = \emptyset$, then there exists a unique value $y \in [a, b]$ such that $h_-(y) \leq \xi \leq h_+(y)$. Then we set*

$$h^{-1}(\xi) = y$$

and we have

$$h^{-1}(\xi) = y \quad \text{for all values} \quad \xi \in [h_-(y), h_+(y)].$$

The function $h^{-1}(\xi)$ is non-decreasing in ξ. Moreover, if h is strictly increasing, then its generalized inverse h^{-1} is continuous.

This notion is obviously consistent with the standard definition when h is invertible. Throughout the present paper, the inverse of a monotone function is always understood in the sense above.

Our main result in this section is the following one.

Theorem 6.2.1. *Consider a Lipschitz continuous flux-function f and some Riemann data u_l and u_r such that (for definiteness) $u_l < u_r$. Let*

$$\tilde{f} : [u_l, u_r] \to \mathbb{R}$$

be the (Lipschitz continuous) convex hull of f on the interval $[u_l, u_r]$. Consider also the generalized inverse of $\tilde{f}'$ in the sense of Definition 6.4

$$g := \left(\tilde{f}'\right)^{-1} : [\tilde{f}'_+(u_l), \tilde{f}'_+(u_r)] \to \mathbb{R}.$$

Then, the explicit formula

$$u(x,t) = \begin{cases} u_l, & x < t\,\tilde{f}'_+(u_l), \\ g(x/t), & t\,\tilde{f}'_+(u_l) < x < t\,\tilde{f}'_-(u_r), \\ u_r, & x > t\,\tilde{f}'_-(u_r), \end{cases} \tag{6.8}$$

defines a function with bounded variation which is the entropy solution of the Riemann problem (6.1)-(6.2) satisfying Oleinik's entropy inequalities.

Proof. Setting

$$v(\xi) := u(x,t), \quad \xi = x/t,$$

we must show that the Borel measure

$$\mu := -\xi\,\frac{dv}{d\xi} + \frac{d}{d\xi}\,(f(v)) = \left(-\xi + \hat{f}'(v)\right)\frac{dv}{d\xi} \tag{6.9}$$

vanishes identically, where $dv/d\xi$ is a measure and Vol'pert's superposition $\hat{f}'(v)$ is the function of bounded variation defined by

$$\begin{cases} f'_-(v(\xi)), & \text{at points of continuity of } v, \\ \int_0^1 f'\left(\; v_-(\xi) + (1 - \;)\, v_+(\xi)\right) d, & \text{at points of jump of} v. \end{cases}$$

Here, the representative f'_- is chosen for definiteness, only. See [33] for a justification of the above chain rule. Given an arbitrary Borel set B we can introduce the decomposition

$$\mu(B) = \mu(B_c) + \sum_m \mu(\{\xi_m\}), \qquad B = B_c \cup \{\xi_1, \xi_2, \ldots\},$$

in which v is continuous at every point of B_c and discontinuous at each $\xi_1, \xi_2, \ldots$ We can now deal with the set of points of continuity and of points of jump separately.

First of all, suppose that f is convex on the interval $[u_l, u_r]$, so that

$$\tilde{f}(u) = f(u), \quad u \in [u_l, u_r].$$

We distinguish between two situations. If v is continuous at some point ξ and that f is differentiable at $v(\xi)$, then we have by definition

$$\hat{f}'(v(\xi)) = f'(v(\xi)).$$

Since v is precisely the inverse of f' this yields

$$f'(v(\xi)) = \xi.$$

If now v is continuous at some point ξ but f is not differentiable at $v(\xi)$, i.e.,

$$f'_-(v(\xi)) < f'_+(v(\xi)),$$

then we have

$$v(f'_-(v(\xi))) = v(f'_+(v(\xi))).$$

Since v is monotone, v remains constant on the non-trivial interval

$$[f'_-(v(\xi)), f'_+(v(\xi))]$$

(which contains ξ). We conclude that the measure $dv/d\xi$ vanish identically in this interval. Collecting our conclusions in both cases, it follows that if B is a subset of the set of continuity points of v, then

$$\mu(B) = 0.$$

Next, let ξ be any point of discontinuity of v. We have

$$\mu(\{\xi\}) = -\xi\,(v_+(\xi) - v_-(\xi)) + f(v_+(\xi)) - f(v_-(\xi)).$$

Since f' is the inverse of v, f' must be constant on the interval $[v_-(\xi), v_+(\xi)]$, that is,

$$f'(u) = \xi, \quad u \in [v_-(\xi), v_+(\xi)].$$

Therefore, $w \mapsto f(w)$ is affine on this interval and is given by

$$f(w) = f(v_-(\xi)) + \xi(u - v_-(\xi)), \quad w \in [v_-(\xi), v_+(\xi)],$$

and in particular we obtain

$$\mu(\{\xi\}) = 0.$$

This completes the proof that (6.8) provides a solution of the scalar conservation law (6.1), at least when the flux f is assumed to be convex.

To treat the general case when f not need be convex let us set

$$\mathcal{A} := \{w / \tilde{f}(w) = f(w)\}.$$

Since both f and $\tilde{f}$ are continuous, the set $\mathcal{A}$ is closed and can be decomposed in a countable union of closed intervals, say $[a_n, b_n]$, $n = 1, 2, \cdots$. In each interval $[a_n, b_n]$ the function f is convex and our arguments in the first part of this proof show immediately that the formula (6.8) determine a weak solution of (6.1) if the initial data lie in $[a_n, b_n]$. The remaining set $\mathcal{A}^c$ is open and, therefore, can be decomposed into a countable union of open intervals (c_n, d_n), $n = 1, 2, \cdots$. Without loss of generality we can assume that $c_n, d_n \in \mathcal{A}$, so that

$$\tilde{f}'_-(c_n) = f'_-(c_n) \quad \text{and} \quad \tilde{f}'_+(d_n) = f'_+(d_n).$$

By definition, $\tilde{f}$ must be affine on the interval $[c_n, d_n]$. Thus, we get

$$\tilde{f}'_-(c_n) = f'_-(c_n) = \tilde{f}'_+(d_n) = f'_+(d_n) =: \lambda. \tag{6.10}$$

The conditions (6.10) imply that, at the point λ, the function v has a jump discontinuity and

$$v_-(\lambda) = c_n \quad \text{and} \quad v_+(\lambda) = d_n.$$

Then we have

$$\mu(\{\lambda\}) = -\lambda(v_+(\lambda) - v_-(\lambda)) + f(v_+(\lambda)) - f(v_-(\lambda)) = 0.$$

Therefore, if the initial data belong to the interval $[c_n, d_n]$, then λ is the unique point of discontinuity of v, and for $\xi \neq \lambda$, the function v is constant. This means that the function v (or, more precisely, $u = u(x,t)$) has a discontinuity propagating at the speed λ.

Finally, if the initial data take values in several distinct intervals, we can find a decomposition the formula (6.8) to reduce the problem to solutions with data belonging to a single interval.

To complete the proof, it remains to check that Oleinik's entropy inequalities hold at each discontinuity connecting some left-hand state u_- to a right-hand state u_+, that is,

$$\sigma(u_-, u_+) \leq \sigma(u_-, w), \quad w \in (u_-, u_+) \tag{6.11}$$

Consider the shock wave determined earlier from the conditions (6.10), with now

$$u_- = c_n, \quad u_+ = d_n, \quad \sigma(u_-, u_+) = \lambda.$$

Since $\tilde{f}$ is the convex hull of f and is distinct from f at each point of the interval (u_-, u_+), we have

$$\tilde{f}(w) < f(w), \quad w \in (u_-, u_+). \tag{6.12}$$

Thus, (6.12) yields for all $w \in (u_-, u_+)$

$$\begin{aligned} \sigma(u_-, w) &= \frac{f(w) - f(u_-)}{w - u_-} > \frac{\tilde{f}(w) - f(u_-)}{w - u_-} \\ &= \frac{\tilde{f}(w) - \tilde{f}(u_-)}{w - u_-} = \lambda = \sigma(u_-, u_+). \end{aligned}$$

The proof of Theorem (6.2.1) is complete. □

To illustrate some interesting features of the loss of regularity in the flux-function f, let us discuss an example. Suppose that, for some critical value $u_* \in \mathbb{R}$, the flus f is a smooth convex function in both intervals $u < u_*$, and $u > u_*$, but the speed $\lambda(u) = f'(u)$ is discontinuous at u_* with

$$\lambda_-(u_*) < \lambda_+(u_*),$$

so that the flux f is globally convex but solely Lipschitz continuous. Then, on one hand, a rarefaction connecting $u_l < u_*$ to $u_r > u_*$ contains a constant state

$$u(x,t) = \begin{cases} u_l, & x < t\,\lambda(u_l), \\ f'^{-1}(x/t), & t\,\lambda(u_l) < x < t\,\lambda_-(u_*), \\ u_*, & t\,\lambda_-(u_*) < x < t\,\lambda_+(u_*), \\ f'^{-1}(x/t), & t\,\lambda_+(u_*) < x < t\,\lambda(u_r), \\ u_r, & x > t\,\lambda(u_r), . \end{cases}$$

On the other hand, concerning shock waves, it is easy to see that the shock speed always has a limiting value if one data concides with u_* while the other approaches u_*, namely

$$\sigma(u_*, u_r) \to \lambda_-(u_*), \quad u_r \to u_*$$

and

$$\sigma(u_l, u_*) \to \lambda_+(u_*), \quad u_l \to u_*.$$

However, the speed $\sigma(u_l, u_r)$ has no limit when both $u_l, u_r \to u_*$ and instead we obtain

$$\liminf_{u_l, u_r \to u_*} \sigma(u_l, u_r) = \lambda_-(u_*),$$

and

$$\limsup_{u_l, u_r \to u_*} \sigma(u_l, u_r) = \lambda_+(u_*).$$

6.3 Riemann Problem for Systems

We now turn to general $N \times N$ systems (6.1) with Lipschitz continuous flux f and, following Lax's approach [26], we construct explicitly the entropy solution of the Riemann problem. As is usual, we restrict attention to self-similar solutions, $u(x,t) = u(y)$ with $y = x/t$ and rely on two fundamental families of solutions, the shock waves and the rarefaction waves.

Let us first introduce a notion of strict hyperbolicity for systems of conservation laws with non-smooth flux. Recall that all of the values u under consideration will remain in a ball $\mathcal{U} := \mathcal{B}(u_*, \delta_0)$ with sufficiently small radius δ_0. The system (6.1) is assumed to be strictly hyperbolic. We fix some $N \times N$ matrix A^* with real and distinct eigenvalues

$$\lambda_1^* < \ldots < \lambda_N^*$$

and corresponding basis of left- and right-eigenvectors l_j^* and r_j^*, $j = 1, \ldots, N$, respectively. After normalization we can have $|r_i^*| = 1$, $l_i^* \cdot r_j^* = 0$ if $i \neq j$ and $l_j^* \cdot r_j^* = 1$. We assume that the Jacobian matrix of the flux $f : \mathcal{U} \to \mathbb{R}^N$ remains close to A^*, i.e.,

$$\|Df(u) - A^*\| \le \eta \quad \text{for almost every } u \in \mathcal{B}(u_*, \delta_0), \tag{6.13}$$

where the constants δ_0 and η are sufficiently small and $\|B\|$ denotes the Euclidean norm of a matrix B. For η small enough, (6.13) implies that, for almost every $u \in \mathcal{B}(u_*, \delta_0)$, the matrix $Df(u)$ has N real and distinct eigenvalues

$$\lambda_1(u) < \ldots < \lambda_N(u)$$

and corresponding basis of left- and right-eigenvectors $l_j(u)$, $r_j(u)$, $j = 1, \ldots, N$, respectively. Moreover, for some uniform constant $C > 0$, (6.13) also implies for $j = 1, \ldots, N$ and for almost every $u \in \mathcal{B}(u_*, \delta_0)$

$$|\lambda_j(u) - \lambda_j^*| \le C\eta,$$

$$\begin{aligned} |l_j(u) - l_j^*| &\le C\eta, \\ |r_j(u) - r_j^*| &\le C\eta. \end{aligned} \tag{6.14}$$

Thanks to the definition of generalized Jacobian (see (2.19) in Section (2.3.1) and the property of convex hulls, the properties in (6.14) remain valid for the generalized Jacobian $\partial f(u)$, that is,

$$\|\overline{A} - A^*\| \le \eta \quad \text{for all } \overline{A} \in \partial f(u),\ u \in \mathcal{B}(u_*, \delta_0). \tag{6.15}$$

Let $\Lambda_j(u)$ the set of all j-eigenvalues of the matrices belonging to the set $\partial f(u)$. In view of (6.15), for each $\overline{\lambda}_j \in \Lambda_j(u)$ there exists a left-eigenvector $\bar{l}_j$ and a right-eigenvector $\overline{r}_j$ such that

$$\begin{aligned} |\overline{\lambda}_j - \lambda_j^*| &\le C\eta, \\ |\bar{l}_j - l_j^*| &\le C\eta, \\ |\overline{r}_j - r_j^*| &\le C\eta. \end{aligned} \tag{6.16}$$

The corresponding sets of "normalized" left- and right-eigenvectors will be denotes by $L_j(u)$ and $R_j(u)$, $j = 1, \ldots, N$, respectively:

$$\begin{aligned} |\bar{l}_j - l_j^*| &\le C\eta \quad \text{for all } \bar{l}_j \in L_j(u), \\ |\overline{r}_j - r_j^*| &\le C\eta \quad \text{for all } \overline{r}_j \in R_j(u). \end{aligned}$$

For $u \neq v$ we denote by $\Lambda_j(u, v)$ the set of j-eigenvalues $\overline{\lambda}_j$ of matrices $A(u, v) \in \mathrm{co}(\partial f([u, v]))$ satisfying

$$A(u, v)\,\big(v - u\big) = f(v) - f(u).$$

Second, we state a generalized notion of genuine nonlinearity for Lipschitz continuous flux-functions. Basically, we impose that characteristic speeds and wave speeds are monotone along wave curves. Precisely, for each $j = 1, \ldots, N$, each Lipschitz continuous curve $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto v(\epsilon) \in \mathcal{U}$ satisfying

$$|v'(\epsilon) - r_j^*| \le C\eta \quad \text{for allmost every } \epsilon \in (-\epsilon_0, \epsilon_0), \tag{6.17}$$

and each measurable selections $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \lambda(\epsilon), \sigma(\epsilon) \in \mathbb{R}$ satisfying

$$\sigma(\epsilon) \in \Lambda_j(v(0), v(\epsilon)), \quad \lambda(\epsilon) \in \Lambda_j(v(\epsilon)), \tag{6.18}$$

the functions $\lambda(\epsilon)$ and $\sigma(\epsilon)$ are (strictly) increasing. Moreover, for some uniform constant $m > 0$ and all $-\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon_0$, we have

$$\lambda(\epsilon_2) - \lambda(\epsilon_1) \ge m(\epsilon_2 - \epsilon_1). \tag{6.19}$$

This assumption represents a direct generalization of Lax's concept.

Finally, we assume the following regularity assumption on the flux along wave curves: for each Lipschitz continuous curve v satisfying (6.17)-(6.18), the function f is continuously differentiable at $v(\epsilon)$ for almost every $\epsilon \in (-\epsilon_0, \epsilon_0)$. For example, we will use later (when dealing with rarefaction waves) that the following chain rule holds

$$f(v(\epsilon))' = Df(v(\epsilon))\, v(\epsilon)' \quad \text{for almost every } \epsilon \in (-\epsilon_0, \epsilon_0).$$

We begin with the derivation of two classes of elementary solutions, which will be used next to solve the Riemann problem. A shock wave traveling at the speed σ

$$u(x,t) = \begin{cases} u_0, & x < \sigma t, \\ u, & x > \sigma t, \end{cases}$$

with $u_0, u \in \mathcal{U}$, must satisfy the Rankine-Hugoniot relations:

$$-\sigma\,(u - u_0) \,+\, f(u) - f(u_0) = 0. \tag{6.20}$$

The Hugoniot set of all states u connected to a fixed state u_0 decomposes into N curves, which must be firstly constrained with an entropy condition. Observe that, because the flux f is solely Lipschitz continuous, wave speeds are not defined as functions but rather as subsets of $\mathbb{R}$. Accordingly, we need a generalization of Lax shock admissibility inequalities, stated in (6.21) below.

Theorem 6.3.1. *Assume that the system (6.1) is strictly hyperbolic and genuinely nonlinear. For each $i = 1, \ldots, N$, there exist $\delta_1 < \delta_0$, $\varepsilon_1 > 0$, and a unique Lipschitz continuous mapping*

$$\phi_i : (-\varepsilon_1, 0] \times \mathcal{B}(u_*, \delta_1) \to \mathcal{B}(u_*, \delta_0),$$

and a unique bounded measurable mapping

$$\sigma_i : (-\varepsilon_1, 0] \times \mathcal{B}(u_*, \delta_1) \to \mathbb{R},$$

which is locally Lipschitz continuous on $(-\varepsilon_1, 0) \times \mathcal{B}(u_*, \delta_1)$, such that the following holds.

For every $\varepsilon \in (-\varepsilon_1, 0)$ and $u_0 \in \mathcal{B}(u_*, \delta_1)$ the left-hand state u_0 can be connected to the right-hand state $u := \phi_i(-\varepsilon; u_0$ by an i-shock wave with speed $\phi_i(-\varepsilon_1; u_0)$. That is, Rankine-Hugoniot relations (6.20) hold together with the following generalized Lax shock admissibility inequalities

$$\Lambda_i(u_0) \ni \sigma_i(0; u_0) > \sigma_i(\varepsilon; u_0) > \sigma_i\big(\varepsilon; \phi_i(\varepsilon; u_0)\big) \in \Lambda_i\big(\phi_i(\varepsilon; u_0)\big). \tag{6.21}$$

The function σ_i is increasing with respect to ε and

$$\begin{aligned} &\phi_i(0;u_0) = u_0, \\ &\partial\phi_i(0;u_0) \subset R_i(u_0), \\ &\sigma_i(0;u_0) \in \Lambda_i(u_0). \end{aligned} \tag{6.22}$$

Note in passing that the following Taylor-like expansion follows from Theorem 6.3.1

$$\phi_i(\varepsilon;u_0) \in u_0 + \varepsilon R_i(u_0) + o(\varepsilon)\,\mathcal{B}(0,1), \tag{6.23}$$

which determines the local behavior of the shock curve.

Proof. By the (generalized) mean-value theorem stated in Theorem 2.3.1, there exists a matrix-valued and measurable function

$$A(u_0,u) \in \mathrm{co}(\partial f([(u_0,u]))$$

such that

$$f(u) - f(u_0) = A(u_0,u)\,(u-u_0). \tag{6.24}$$

Hence, the Rankine-hugoniot relations (6.20) become

$$(-\sigma I + A(u_0,u))\;(u-u_0) = 0. \tag{6.25}$$

where I denotes the identity matrix.

Let us fix u_0. Thanks to (6.15), the averaging matrix $A(u_0,u)$ satisfies

$$\|A(u_0,u) - A^*\| \le \eta. \tag{6.26}$$

Let $\lambda_i(u_0,u)$ and $r_i(u_0,u)$, $i=1,\dots,N$ be the eigenvalues and right-eigenvectors of $A(u_0,u)$, respectively. The equations (6.25) take the following equivalent form: There exists $i=1,\dots,N$ and a real α such that

$$u - u_0 = \alpha r_i(u_0,u), \sigma = \lambda_i(u_0,u). \tag{6.27}$$

The main difficulty in order to solve (6.27) lies in the lack of regularity of the eigenvectors and eigenvalues of $A(u_0,u)$.

Consider (6.20) and multiply it successively by each left-eigenvector l_j^*:

$$-\sigma(u)\,l_j^*\cdot(u-u_0) + l_j^*\cdot\big(f(u)-f(u_0)\big) = 0, \quad j=1,\dots,N. \tag{6.28}$$

Fix some index i. The i-th equation in (6.28) determines the shock speed:

$$\sigma(u) = \frac{l_i^*\cdot(f(u)-f(u_0))}{l_i^*\cdot(u-u_0)} = \frac{l_i^*\cdot A(u_0,u)\,(u-u_0)}{l_i^*\cdot(u-u_0)}. \tag{6.29}$$

We are going to show that there exists a curve $\varepsilon \to \phi_i(\varepsilon; u_0)$ defined for small $|\varepsilon|$ such that along this curve, the shock speed

$$\sigma_i(\varepsilon; u_0) := \sigma(\phi_i(\varepsilon; u_0))$$

determined by (6.29) fulfils the system of N equations (6.28).

The formula (6.29) requires u to satisfy $\lambda_i^* \cdot (u - u_0) \neq 0$. For that reason, we restrict attention to the cone

$$C_{\gamma,i}(u_0) := \{u \in \mathcal{U} / \, |l_i^* \cdot (u - u_0)| > \gamma \, |u - u_0|\},$$

wher $\gamma \in [|l_i^*| \cdot \alpha, |l_i^*|)$ is a fixed constant, for some $\alpha \in (0, 1)$. Note that u_0 does not belong to this open cone. Note also that the Lipschitz regularity of the shock speed, as stated in the theorem, follows immediately.

Then, observe that the shock speed remains uniformly bounded in the cone $C_{\gamma,i}(u_0)$, namely

$$\begin{aligned}\sigma(u) &= \frac{l_i^* \cdot A^* \, (u - u_0)}{l_i^* \cdot (u - u_0)} + \frac{l_i^* \cdot (A(u - u_0) - A^*) \, (u - u_0)}{l_i^* \cdot (u - u_0)} \\ &= \lambda_i^* + \frac{l_i^* \cdot (A(u - u_0) - A^*) \, (u - u_0)}{l_i^* \cdot (u - u_0)}.\end{aligned}$$

In particular, we find

$$|\sigma(u) - \lambda_i^*| \leq \frac{|l_i^*|}{\gamma} \|A(u_0, u) - A^*\| \leq C\eta. \tag{6.30}$$

On the other hand, the shock speed is continuous on $C_{\gamma,i}(u_0)$. However, in general, it cannot be extended by continuity to $u = u_0$.

Plugging the expression (6.29) of the shock speed in the relations (6.28) yields for $j \neq i$:

$$\begin{aligned}F_j(u) := &- \frac{l_i^* \cdot (f(u) - f(u_0))}{l_i^* \cdot (u - u_0)} l_j^* \cdot (u - u_0) \\ &+ l_j^* \cdot (f(u) - f(u_0)) = 0.\end{aligned} \tag{6.31}$$

Since f is Lipschitz continuous and the the shock speed is bounded, the functions F_j are locally Lipschitz continuous on $C_{\gamma,i}(u_0)$. They are easily extended by continuity to $u = u_0$ by setting

$$F_j(u_0) = 0.$$

We now prove that the functions F_j are Lipschitz continuous up to the point u_0. To this end, it is sufficient to check that the gradients ∇F_j are uniformly bounded. We rewrite F_j in the form

$$F_j(u) = -\frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)} l_i^* \cdot (f(u) - f(u_0)) + l_j^* \cdot (f(u) - f(u_0)),$$

so that for almost every $u \in C_{\gamma,i}(u_0)$

$$\begin{aligned} \nabla F_j(u) = &- \frac{l_i^* \cdot (f(u) - f(u_0))}{l_i^* \cdot (u - u_0)} l_j^* \\ &+ \frac{l_j^* \cdot (u - u_0)}{(l_i^* \cdot (u - u_0))^2} l_i^* \cdot (f(u) - f(u_0))\, l_i^* \\ &- \frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)} l_i^* \cdot Df(u) + l_j^* \cdot Df(u). \end{aligned} \tag{6.32}$$

Since f is Lipschitz continuous and u belongs to the cone, every term in the right-hand side of the formula above is uniformly bounded.

Our objective now is to apply the implicit function theorem to the functions F_j. We claim that the $N-1$ vectors $\nabla F_j(u)$ are linearly independent in $\mathbb{R}^N$, uniformly for *almost every* $u \in \mathcal{U}$. We can rewrite the expression of the gradient as:

$$\begin{aligned} \nabla F_j(u) = &K_1 l_j^* + K_2(u)\, l_j^* + K_3(u)\, l_i^* \\ &+ K_4(u)\, l_i^* \cdot (Df(u) - A^*) + l_j^* \cdot (Df(u) - A^*) \end{aligned} \tag{6.33}$$

with

$$\begin{aligned} K_1 &= \lambda_j^* - \lambda_i^*, \\ K_2(u) &= -\frac{l_i^* \cdot (A(u_0, u) - A^*)\,(u - u_0)}{l_i^* \cdot (u - u_0)}, \\ K_3(u) &= \frac{l_j^* \cdot (u - u_0)}{(l_i^* \cdot (u - u_0))^2} l_i^* \cdot (A(u_0, u) - A^*)\,(u - u_0), \\ K_4(u) &= -\frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)}. \end{aligned}$$

We estimate these coefficients successively. Observe that K_1 is a constant independent of u. Next, using (6.26) and the fact that u belongs to the cone, we get for some constant $C' > 0$

$$|K_2(u)\, l_j^*| \le \frac{1}{\gamma} |l_i^*||l_j^*|\, \eta \le C'\, \eta.$$

Similarly, we obtain

$$|K_3(u)\, l_i^*| \le |K_3(u)||l_i^*| \le \left(\frac{|l_i^*|}{\gamma}\right)^2 |l_j^*|\, \eta \le C'\, \eta.$$

This proves that the second and third term in the right-hand side of (6.33) are of order η. The coefficient K_4 is of order 1 but, using (6.13), we have the estimate (for some constant $C' > 0$)

$$|K_4(u)\, l_i^* \cdot (Df(u) - A^*)\,| \le \frac{1}{\gamma}|l_i^*||l_j^*|\, \eta \le C'\, \eta$$

and, thus, the fourth term in the right-hand side of (6.33) is of order η as well. Finally, the last term satisfies

$$|l_j^* \cdot (Df(u) - A^*)\,| \le C'\, \eta.$$

It follows from the above estimates that for some uniform constant C'

$$|\,\nabla F_j(u) - K_1 l_j^*| \le C'\, \eta \quad \text{for almost every } u. \tag{6.34}$$

The functions F_j are defined within the cone only. Let $\tilde{F}_j$ be a Lipschitz continuous extension of F_j to the whole set $\mathcal{U}$ such that (6.34) still holds for the function $\tilde{F}_j$:

$$|\,\nabla \tilde{F}_j(u) - K_1 l_j^*| \le C'\, \eta \quad \text{for almost every } u.$$

Therefore, by the property of generalized gradients,

$$|\partial \tilde{F}_j(u) - K_1 l_j^*| \le C'\, \eta \quad \text{for every } u \in \mathcal{U}. \tag{6.35}$$

Since $\{l_j^*, j = 1, 2, \ldots, N\}$ is a basis, we can always assume that η is small enough so that (6.35) implies that the set made of the vector l_i^* and any selection of $N-1$ vectors in $\partial \tilde{F}_j(u)$, $j \ne i$, is a basis.

Consider the function $G = G(\varepsilon, w) \in \mathbb{R}^N$ defined for (ε, w) in a neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{R}^N$ by

$$\begin{aligned} G_i(\varepsilon, w) &:= l_i^* \cdot w, \\ G_j(\varepsilon, w) &:= \tilde{F}_j(u_0 + \varepsilon r_i^* + w) \quad \text{for } j \ne i. \end{aligned}$$

Differentiating with respect to w we get, for almost every (ε, w),

$$\begin{aligned} \partial_w G_i(\varepsilon, w) &= \{l_i^*\}, \\ \partial_w G_j(\varepsilon, w) &= \partial_u \tilde{F}_j(u_0 + \varepsilon r_i^* + w) \quad \text{for } j \ne i. \end{aligned}$$

Observe that

$$G(0,0)=0$$

and, as explained earlier,

$$\partial_w G(0,0)\subset \partial_w G_1(0,0)\times\partial_w G_2(0,0)\times\dots\partial_w G_N(0,0)$$

is of maximal rank. Applying the implicit function theorem (Theorem 2.3.4) to the function G, we see that there exist a $\varepsilon_1>0$ and a unique Lipschitz continuous function $w_i(.;u_0):(-\varepsilon_1,\varepsilon_1)\to\mathbb{R}^N$ such that $w_i(0;u_0)=0$ and

$$\begin{aligned}&\tilde{F}_j\left(u_0+\varepsilon r_i^*+w_i(\varepsilon;u_0)\right)=0 \quad \text{for } j\neq i,\\ &l_i^*\cdot w_i(\varepsilon;u_0)=0 \quad \text{for } \varepsilon\in(-\varepsilon_1,\varepsilon_1).\end{aligned} \tag{6.36}$$

Let us define

$$\begin{aligned}\phi_i(\varepsilon;u_0)&=u_0+\varepsilon r_i^*+w_i(\varepsilon;u_0),\\ \sigma_i(\varepsilon;u_0)&=\sigma(\phi_i(\varepsilon;u_0)).\end{aligned}$$

We need to show that these functions ϕ_i, σ_i are the ones for which we are searching. Taking the derivative in ε to the equations of (6.36) and applying the chain rule formula (2.21), we have

$$\begin{aligned}&0=l_i^*\cdot w_i'(\varepsilon;u_0),\\ &0=A_j\cdot(r_i^*+w_i'(\varepsilon;u_0)), \quad \text{for a.e. } \varepsilon\in(-\varepsilon_1,\varepsilon_1), \quad j\neq i,\end{aligned}$$

for some $A_j\in\partial\tilde{F}_j(u_0+\varepsilon r_i^*+w_i(\varepsilon;u_0))$. Observe that the vector A_j is close to $K_1 l_j^*$ in the sense that $\partial\tilde{F}_j(u_0+\varepsilon r_i^*+w_i(\varepsilon;u_0))$ fulfils the estimate (6.35). By writting

$$A_j=K_1 l_j^*+(A_j-K_1 l_j^*),$$

and substituting it into the last equality, after re-arranging the terms, we have

$$-K_1 l_j^*\cdot w_i'=(A_j-K_1 l_j^*)\cdot(r_i^*+w_i').$$

That yields

$$|K_1||l_j^*\cdot w_i'|\le|A_j-K_1 l_j^*|(|r_i^*|+|w_i'|)\le C'\,\eta(1+|w_i'|),$$

i.e.,

$$|l_j^*\cdot w_i'|\le\frac{C'\,\eta(1+|w_i'|)}{|K_1|}, \quad j\neq i.$$

Besides, w_i' can be expressed in terms of eigenvectors by, observe that $l_i^*\cdot w_i'=0$,

$$w_i'=\sum_{j\neq i}(l_j^*\cdot w_i')r_j^*.$$

Hence, we find

$$|w_i'| \leq \sum_{j\neq i} |l_j^* \cdot w_i'||r_j^*| \leq \sum_{j\neq i} \frac{C'\,\eta(1+|w_i'|)}{|K_1|} = \frac{N-1}{|K_1|} C'\,\eta(1+|w_i'|),$$

i.e.,

$$|w_i'| \leq \frac{\frac{N-1}{|K_1|} C'\,\eta}{1 - \frac{N-1}{|K_1|} C'\,\eta}.$$

Since it is not restrictive to require that

$$C \geq \frac{\frac{N-1}{|K_1|} C'}{1 - \frac{N-1}{|K_1|} C'\,\eta},$$

it follows that $Lip_\varepsilon(w_i) \leq C\,\eta$, and therefore

$$\begin{aligned} &|l_i^* \cdot (\phi_i(\varepsilon; u_0) - u_0)| - \gamma\,|\phi_i(\varepsilon; u_0) - u_0| = |\varepsilon| - \gamma\,|\varepsilon r_i^* - w_i(\varepsilon; u_0)| \\ &\quad > |\varepsilon| - \gamma(|\varepsilon| + Lip_\varepsilon(w_i)|\varepsilon|) > |\varepsilon| - \gamma|\varepsilon|\,(1 + C\,\eta) > 0, \end{aligned}$$

provided γ is chosen such that $\gamma < 1/(1 + C\,\eta)$, and thus

$$\phi_i(\varepsilon; u_0) \in C_{\gamma,i}.$$

This enable us to replace $\tilde{F}_j$ in (6.36) by F_j. Therefore, the i-Hugoniot curve $\phi_i(\varepsilon; u_0)$ is uniquely defined.

Let us next consider the relations (6.22). The first equality is obvious. Observe that

$$|\phi_i'(\varepsilon; u_0) - r_i^*| \leq Lip_\varepsilon(w_i) \leq C\,\eta \quad \text{for a.e. } \varepsilon \in (-\varepsilon_1, \varepsilon_1),$$

which implies

$$|\partial\phi_i(0; u_0) - r_i^*| \leq C\,\eta. \tag{6.37}$$

On the other hand, the upper semi-continuity property of generalized gradients (Proposition 2.3.1, item c)) show that given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|u - u_0| < \delta$

$$\partial f([u_0, u]) \subset \partial f(u_0) + \varepsilon\mathcal{B}(0, 1).$$

The right-hand side of the above inequality being convex we have

$$\mathrm{co}\,\partial f([u_0, u]) \subset \partial f(u_0) + \varepsilon\mathcal{B}(0, 1).$$

Since the eigenvalues and eigenvectors depend continuously upon their arguments, it follows from the last inclusion that, for any matrix $A(u_0, u) \in \operatorname{co} \partial f([u_0, u])$ with i-eigenvalue $\lambda_i(u_0, u)$ and i-eigenvector $\rho_i(u_0, u)$,

$$|\lambda_i(u_0, u) - \lambda_i(u_0)| \leq C'' \varepsilon,$$
$$|r_i(u_0, u) - r_i(u_0)| \leq C'' \varepsilon,$$

for some $C'' > 0$, $\lambda_i(u_0) \in \Lambda_i(u_0)$, and $r_i(u_0) \in R_i(u_0)$. Thus, we get

$$\begin{aligned} &\lambda_i(u_0, \phi_i(\varepsilon; u_0)) \to \lambda_i(u_0), \\ &r_i(u_0, \phi_i(\varepsilon; u_0)) \to r_i(u_0), \qquad \text{as } \varepsilon \to 0. \end{aligned} \tag{6.38}$$

Combining (6.27), (6.37) and (6.38), we obtain the second and third inclusions on (6.22).

We are left with checking the shock admissibility inequalities (6.21). As indicated above, we have

$$|\phi_i'(\varepsilon; u_0) - r_i^*| \leq C\,\eta \qquad \text{for a.e. } \varepsilon \in (-\varepsilon_1, \varepsilon_1).$$

Therefore, by our genuine nonlinearity assumption it follows that

$$\begin{aligned} &\sigma_i(\varepsilon, u_0) < \sigma_i(u_0)| \in \Lambda_i(u_0) \qquad \text{for all } -\varepsilon_1 < \varepsilon < 0, \\ &\sigma_i(\varepsilon, u_0) > \sigma_i(u_0)| \in \Lambda_i(u_0) \qquad \text{for all } 0 < \varepsilon < \varepsilon_1, \end{aligned}$$

so that the first inequality in (6.21) is satisfied and the part $\{\varepsilon > 0\}$ of the i-Hugoniot curve is excluded by violating (6.21). Considering the part of the i-Hugoniot curve "between" u_0 and $\phi_i(\varepsilon; u_0)$ as the Hugoniot curve issuing from $\phi_i(\varepsilon; u_0)$,

$$u(s) := \phi_i(\varepsilon; u_0) - (\varepsilon - s)\,\rho_i^* - w_i(\varepsilon - s; u_0), \quad \varepsilon \leq s \leq 0,$$

we find

$$u(0) = u_0, \quad u(\varepsilon) = \phi_i(\varepsilon; u_0),$$

and

$$u'(s) = \rho_i^* + w_i'(\varepsilon - s; u_0)$$

which satisfies the genuine nonlinearity assumption. The shock speed function $\sigma_i(s; \phi_i(\varepsilon; u_0))$ is increasing and, for $-\varepsilon_1 < \varepsilon < 0$,

$$\sigma_i(0; \phi_i(\varepsilon; u_0)) > \sigma_i(\varepsilon; \phi_i(\varepsilon; u_0)) \in \Lambda_i(\phi_i(\varepsilon; u_0)).$$

This establishes the second inequality in (6.21). The proof of Theorem 6.3.1 is completed. □

For each $i = 1, \dots, N$ the i-shock set $\mathcal{S}_i(u_0)$ is defined to be

$$\mathcal{S}_i(u_0) := \{\phi_i(\varepsilon; u_0) / \, \varepsilon \in (-\varepsilon_1, 0]\}.$$

Next, we search for self-similar, Lipschitz continuous solutions $u(x,t) = v(\xi)$, $\xi = x/t$ to (6.1) connecting a given left-hand state u_0 to some right-hand state u_1. A rarefaction wave $u(x,t) = v(\xi)$, $\xi = x/t$ satisfies the differential equation

$$-\xi \frac{dv}{d\xi}(\xi) + \frac{d}{d\xi} f(v(\xi)) = (-\xi I + Df(v(\xi))) \frac{dv}{d\xi}(\xi) = 0. \tag{6.39}$$

If (6.39) holds in the usual sense, then there exist right-eigenvector $r_i(v(\xi))$ and eigenvalues $\lambda_i(v(\xi))$ of $Df(v(\xi))$, and a scalar function $c(\xi)$ such that for all relevant values ξ:

$$\begin{aligned} &\frac{dv}{d\xi}(\xi) = c(\xi) r_i(v(\xi)), \\ &\xi = \lambda_i(v(\xi)). \end{aligned} \tag{6.40}$$

The function $\xi \to r_i(v(\xi))$ is L^∞ and continuous almost everywhere. Since the right-hand side of (6.40) may be discontinuous, we have to understand solutions of (6.40) in the sense of Filippov [15] and Dafermos [10].

Let us consider the following ordinary differential problem

$$\begin{aligned} &\frac{d\tilde{v}}{ds}(s; u_0) = r_i(\tilde{v}(s; u_0)), \quad \text{a.e. } s \in [0, \varepsilon_1), \\ &\tilde{v}(0; u_0) = u_0. \end{aligned} \tag{6.41}$$

For ε_1 sufficiently small, a solution of (6.41) in the sense of Filippov exists (see [15]). Precisely, there exists a Lipschitz continuous mapping $\tilde{v}(s; u_0)$, $s \in [0, \varepsilon_1)$ satisfying

$$\begin{aligned} &\frac{d\tilde{v}}{ds}(s; u_0) \in \bigcap_{\delta > 0} \overline{\text{co}}\, r_i \left(\tilde{v}(s; u_0) + \delta \mathcal{B}(0,1)\right) \quad \text{a.e. in } [0, \varepsilon_1), \\ &\tilde{v}(0; u_0) = u_0. \end{aligned}$$

The fact that r_i is continuous almost everywhere along the curve $\tilde{v}(.; u_0)$ yields

$$\bigcap_{\delta > 0} \overline{\text{co}}\, r_i \left(\tilde{v}(s; u_0) + \delta \mathcal{B}(0,1)\right) = \{r_i(\tilde{v}(s; u_0))\} \quad \text{a.e. in } [0, \varepsilon_1).$$

The last equality simply means that the function $\tilde{v}(.;u_0)$ is a solution of (6.41) in the usual sense as well. Thanks to the assumption of genuine non-linearity, the function $\lambda_i(\tilde{v}(s;u_0))$ is strictly increasing and admits a Lipschitz continuous inverse, denoted by

$$\begin{aligned}\psi : [\lambda(u_0), \lambda(\tilde{v}(\varepsilon_1;u_0))] &\to [0,\varepsilon_1]\\ \xi &\to s = \psi(\xi),\end{aligned}$$

which is increasind as well. We now claim that the function

$$v(\xi) := \tilde{v}(\psi(\xi);u_0), \quad \xi \in J := [\lambda(u_0), \lambda(\tilde{v}(\varepsilon_1;u_0))],$$

is a solution of (6.37). Clearly, v is Lipschitz continuous. Besides, let $\Omega_{\tilde{v}}$ be the set of all points at which $\tilde{v}$ fails to be differentiable, which has Lebesgue measure zero. Set

$$E = \{\xi \in J/\psi(\xi) \in \Omega_{\tilde{v}}\}.$$

By [33, Th.A.1] the measure $D\psi$ vanishes on E:

$$|D\psi|(E) = 0. \tag{6.42}$$

Therefore, (6.39) holds in the set E. For $\xi \in J \setminus E$ the function v satisfies

$$\begin{aligned}v'(\xi) &= \frac{d}{ds}\tilde{v}(\psi(\xi))\frac{d}{d\xi}\psi(\xi)\\ &= r_i(\tilde{v}(\psi(\xi)))\frac{d}{d\xi}\psi(\xi) = \frac{d}{d\xi}\psi(\xi) r_i(v(\xi)).\end{aligned}$$

From the above analysis we obtain the wave curve

$$\varepsilon \to \varphi_i(\varepsilon;u_0) := \tilde{v}(\varepsilon;u_0)$$

and arrive at the following conclusion.

Theorem 6.3.2. *Given $u_0 \in \mathcal{B}(u_*, \delta_0)$ and $i = 1, \ldots, N$, there exists a Lipschitz continuous curve $[0,\varepsilon_1) \ni \varepsilon \to \varphi_i(\varepsilon;u_0) \in \mathcal{B}(u_*,\delta_0)$ (defined over some small interval $[0,\varepsilon_1)$) such that the state u_0 can be connected to $\varphi_i(\varepsilon;u_0)$ from the right by a rarefaction wave.*

We define the i-rarefaction curve $\mathcal{R}_i(u_0)$ by

$$\mathcal{R}_i(u_0) := \{\varphi_i(\varepsilon;u_0)/\, \varepsilon \in [0,\varepsilon_1)\}.$$

The i-wave curve issuing from u_0 is

$$\mathcal{W}_i(u_0) := \mathcal{S}_i(u_0) \cup \mathcal{R}_i(u_0).$$

We are at the position to state the main result of this section.

Theorem 6.3.3. *There exist $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for every $u_0 \in \mathcal{B}(u_*, \delta_1)$ and $i = 1, \ldots, N$, there is a wave curve issuing from u_0*

$$\mathcal{W}_i(u_0) := \{\psi_i(\varepsilon^i; u_0) / \, \varepsilon^i \in (-\varepsilon_1, \varepsilon_1)\}.$$

Given data u_l, $u_r \in \mathcal{B}(u_*, \delta_1)$, the corresponding Riemann problem (6.1)-(6.2) admits a self-similar, piecewise Lipschitz continuous solution made of $N+1$ constant states

$$u_l = u_0, u_1, \ldots, u_N = u_r,$$

separated by elementary waves. The intermediate states satisfy $u_j \in \mathcal{W}_j(u_{j-1})$ with $u_j = \psi_j(\varepsilon^j, u_{j-1}) := \psi_j(\varepsilon^j)(u_{j-1})$ for some (wave strength) $\varepsilon^j \in (-\varepsilon_1, \varepsilon_1)$. The states u_{j-1} and u_j are connected by either a rarefaction wave if $\varepsilon^j \geq 0$ or by a shock satisfying the generalized Lax shock inequalities (6.21) if $\varepsilon^j < 0$.

Proof. Consider the mapping obtained by combining wave curves together $\varepsilon = (\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^N) \to \Psi(eps) = \psi_N(\varepsilon^N) \circ \psi_{N-1}(\varepsilon^{N-1}) \circ \ldots \circ \psi_1(\varepsilon^1)(u_l) - u_l$. It satisfies

$$\Psi(0) = 0.$$

Acording Theorems 6.3.1 and 6.3.2 we have

$$\psi_i(\varepsilon^i)(u) \in u + \varepsilon^i R_i(u) + o(\varepsilon^i)\mathcal{B}(0,1).$$

Hence, we get

$$\Psi(\varepsilon) \subset \sum_i \varepsilon^i R_i(v_i) + o(\varepsilon)\mathcal{B}(0,1),$$

where

$$\begin{aligned} &v_i = \psi_{i-1}(\varepsilon^{i-1}) \circ \ldots \circ \psi_1(\varepsilon^1)(u_l) \quad \text{for } i = 2, \ldots, N, \\ &v_1 = u_l. \end{aligned}$$

Thus we have

$$\partial\Psi(0) \subset \left(R_1(u_l), R_2(v_2), \ldots, R_N(v_N)\right). \tag{6.43}$$

The upper semi-continuity of the generalized gradient,

$$\partial f(v_i) \subset \partial f(u_l) + \varepsilon' \mathcal{B}(0,1) \quad \text{for } v_i \text{ near } u_l,$$

implies that R_i depends continuously on its argument upon small perturbation, i.e.,

$$R_i(v_i) \subset R_i(u_l) + \mathcal{O}(\varepsilon')\mathcal{B}(0,1).$$

We can assume that η and ε' are sufficiently small so that the last estimate and the hyperbolicity propertyt imply that any selection of the vector sets $R_i(v_i)$ is a basis of $\mathbb{R}^N$. Therefore, the matrix $\partial\Psi(0)$ shown by (6.43) is of maximal rank. Applying the inverse function theorem (Theorem 2.3.3) we conclude that, for $|u_r - u_l|$ sufficientlly small, there exists a unique vector $\varepsilon_0 = (\varepsilon_0^1, \varepsilon_0^2, \dots, \varepsilon_0^N)$ such that

$$\Psi(\varepsilon_0) = u_r - u_l.$$

In other words, we have

$$\psi_N(\varepsilon_0^N) \circ \psi_{N-1}(\varepsilon_0^{N-1}) \circ \ldots \circ \psi_1(\varepsilon_0^1) u_l = u_r,$$

which completes the proof of Theorem 6.3.3 □

6.4 A Model from Compressible Fluid Dynamics

In this last section we consider the Riemann Problem for the so-called p-system

$$\begin{aligned} u_t + p(v)_x &= 0, \\ v_t - u_x &= 0. \end{aligned} \tag{6.44}$$

Here $v > 0$ and u denote the specific volume and the velocity of the fluid, respectively. The pressure $p = p(v)$ is assumed to be smooth everywhere in $v > 0$ (say of class C^2) except at one point v_*. More precisely, we assume that

$$\begin{aligned} & p'_-(v_*) < p'_+(v_*), \quad p''(v_*^\pm) > 0, \\ & p'(v) < 0, \quad p''(v) > 0 \quad \text{for } v \neq v_*, \\ & \lim_{v\to 0^+} p(v) = +\infty, \qquad \lim_{v\to+\infty} p(v) = 0. \end{aligned} \tag{6.45}$$

These conditions are typical in models arising in fluid dynamics when the equation of state is defined by distinct formulas above and bellow some critical threshold. We set $U = (v, u)^T$ and $f(U) = (-u, p(v))^T$, so that (6.44) has the form (6.1) with U playing the role of u in (6.1). For $v \neq v_*$, the Jacobian matrix of the system is

$$Df(U) = \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \tag{6.46}$$

and the generalized Jacobian (in the sense of subsection 2.3.1) at the point (v_*, u) is

$$\partial f(v_*, u) = \begin{bmatrix} 0 & -1 \\ [p'_-(v_*), p'_-(v_*)] & 0 \end{bmatrix}. \tag{6.47}$$

Eigenvalues and eigenvectors are given by

$$\begin{aligned} &\lambda_1(v) \in \{-\sqrt{\overline{\lambda}}/\,\overline{\lambda} \in \partial p(v)\}, \quad \lambda_2(v) \in \{\sqrt{\overline{\lambda}}/\,\overline{\lambda} \in \partial p(v)\}, \\ &r_1(v) = (1, -\lambda_1(v))^T, \quad r_2(v) = (-1, \lambda_2(v))^T. \end{aligned} \tag{6.48}$$

The system (6.44) is strictly hyperbolic since

$$\lambda_1(v) < 0 < \lambda_2(v).$$

Furthermore, away from $v \neq v_*$ both characteristic fiels of the system are genuinely nonlinear since

$$\nabla\lambda_i(v) \cdot r_i(v) = \frac{p''(v)}{2\sqrt{-p'(v)}} > 0.$$

Finally, we set also

$$\begin{aligned} &\Omega_- := \{(v,u)/\, 0 < v < v_*\}, \quad \Omega_+ := \{(v,u)/\, v > v_*\}, \\ &\Omega_* := \{(v,u)/\, v = v_*\}, \end{aligned} \tag{6.49}$$

The first is decreasing while the second is increasing. We determine the rarefaction waves for the system (6.44) as follows. Let $U_0 = (v_0, u_0)$ be a fixed state. The rarefaction waves issued from U_0 are continuous solutions $U(\xi) = (v(\xi), u(\xi))$ (in each interval where $u(\xi) \notin \Omega_*$) to the problem

$$\begin{aligned} &\frac{d}{d\xi}U(\xi) = \alpha(\xi) r_i(v(\xi)), \quad \xi \geq \xi_0, \\ &\xi = \lambda_i(v(\xi)), \quad U(\xi_0) = U_0, \end{aligned} \tag{6.50}$$

where $i = 1$ or 2 and $\alpha = \alpha(\xi)$ is some real-valued function. Differentiating the relation $\xi = \lambda_i(v(\xi))$ away from the region Ω_* yields

$$\begin{aligned} 1 &= \nabla\lambda_i(v(\xi)) \cdot \frac{dv}{d\xi}(\xi) \\ &= \alpha(\xi)\, \nabla\lambda_i(v(\xi)) \cdot r_i(v(\xi)). \end{aligned} \tag{6.51}$$

Substituting (6.51) into (6.50) we obtain

$$v'(\xi) = (-1)^{i+1} \frac{2\sqrt{-p'(v)}}{p''(v)}, \quad u'(\xi) = \frac{2 - p'(v)}{p''(v)}.$$

Since $v'(\xi) \neq 0$ this system of ODE's enable us to write $u = u(v; U_0)$

$$\frac{du}{dv}(\xi) = (-1)^{i+1}\sqrt{-p'(v)}. \tag{6.52}$$

For $i = 1$ the condition $\lambda_1(v) > \lambda_1(v_0)$ yields $p'(v) > p'(v_0)$ and, therefore, $v > v_0$, since p' is strictly increasing by assumption. Hence, from (6.52) it follows that the 1-rarefaction curve is

$$\mathcal{R}_1(U_0) = \left\{ u(v; U_0) = u_0 + \int_{v_0}^{v} \sqrt{-p'(y)}\, dy, \quad v > v_0 \right\}. \tag{6.53}$$

Similarly, for $i = 2$ the 2-rarefaction curve is

$$\mathcal{R}_2(U_0) = \left\{ u(v; U_0) = u_0 - \int_{v_0}^{v} \sqrt{-p'(y)}\, dy, \quad v < v_0 \right\}. \tag{6.54}$$

For $U_1 \in \mathcal{R}_i(U_0)$ the i-rarefaction wave $\xi \to U(\xi)$ connecting U_0 to U_1 on the right is given by

$$U(\xi) = \begin{cases} U_0, & \xi \leq \lambda_i(v_0), \\ (v(\xi), u(v(\xi); u_0)), & \lambda_i(v_0) \leq xi \leq \lambda_i(v_1), \\ u_1 & \xi \geq \lambda_i(v_1). \end{cases} \tag{6.55}$$

It is solely a Lipschitz continuous function in the variable $\xi = x/t$. There may exist a new intermediate constant state, which is a direct consequence of the discontinuity in characteristic speed. The profile $v(\xi)$ in (6.55) is determined by inverting the relation $\xi = \lambda_i(v(\xi))$. For $i = 1$ one gets

$$v(\xi) = \begin{cases} (-p')^{-1}(\xi^2), & \xi < -\sqrt{-p'_-(v_*)} \quad \text{or} \\ & -\sqrt{-p'_+(v_*)} < \xi < -\sqrt{-p'(+\infty)}, \\ v_*, & -\sqrt{-p'_-(v_*)} \leq \xi \leq -\sqrt{-p'_+(v_*)}, \end{cases} \tag{6.56}$$

and, for $i = 2$,

$$v(\xi) = \begin{cases} (-p')^{-1}(\xi^2), & \sqrt{-p'(+\infty)} < \xi < \sqrt{-p'_+(v_*)} \quad \text{or} \\ & \xi > \sqrt{-p'_-(v_*)}, \\ v_*, & -\sqrt{-p'_+(v_*)} \leq \xi \leq \sqrt{-p'_-(v_*)} \end{cases} \tag{6.57}$$

We now summarize the above discussion.

Proposition 6.4.1. *For each $U_0 = (v_0, u_0)$ such that $v_0 > 0$ and for each $i = 1, 2$ the rarefaction curve $v \to u(v; U_0)$ issued from U_0, $\mathcal{R}_i(U_0)$, is globally* *defined by* (6.53) *and* (6.54). *For* $i = 1$ *this mapping is increasing and concave in* v *and for* $i = 2$ *it is decreasing and convex. Moreover, each mapping* $u(v; U_0)$ *is locally Lipschitz continuous in* $(v; U_0)$. *For each fixed* U_0 *it is of class* C^2 *in the variable* $v \neq v_*$, *but its derivative exhibits a jump at* $v = v_*$. *The same regularity holds true for* $u(v; U_0)$ *considered as a function of* v_0 *while keeping* v *and* u_0 *fixed.*

We turn to the investigation of shock waves of the system (6.44). That is, discontinuous solutions of (6.1) connecting two constant states $U_0 = (v_0, u_0)$ and $U = (v, u)$ at some speed s. Using the Rankine-Hugoniot condition and the generalized Lax shock inequalities ($i = 1, 2$)

$$\lambda_{i+}(v) < s < \lambda_{i-}(v_0), \tag{6.58}$$

and relying on the assumptions (6.45) and (6.49) we easily determine the shock curves:

$$\mathcal{S}_1(U_0) := \Big\{ u(v; U_0) = u_0 - \sqrt{- \left(p(v) - p(v_0)\right) \, (v - v_0)} \quad 0 < v < v_0 \Big\} \tag{6.59}$$

$$s = s_1(v; v_0) = - \sqrt{- \frac{p(v) - p(v_0)}{v - v_0}},$$

and

$$\mathcal{S}_2(U_0) := \Big\{ u(v; U_0) = u_0 + \sqrt{- \left(p(v) - p(v_0)\right) \, (v - v_0)}, \quad v > v_0 \Big\}, \tag{6.60}$$

$$s = s_2(v; v_0) = \sqrt{- \frac{p(v) - p(v_0)}{v - v_0}}.$$

We conclude that:

Proposition 6.4.2. *For each* $U_0 = (v_0, u_0)$ *(with* $v_0 > 0$*) and each* $i = 1, 2$ *the shock curve* $v \to u(v; U_0)$ *issued from* U_0, $\mathcal{S}_i(U_0)$, *is globally defined by* (6.59) *and* (6.60). *For* $i = 1$ *the mapping* $u(v; U_0)$ *is increasing and concave in the* v *variable and, for* $i = 2$, *is decreasing and convex. Moreover, each mapping* $u(v; U_0)$ *is locally Lipschitz continuous in* $(v; U_0)$. *For* U_0 *fixed it is of class* C^2 *in the variable* $v \neq v_*$, *but its derivative exhibits a jump at* $v = v_*$. *The shock speedd is a locally Lipschitz continuous function, which is of class* C^2 *at* $v \neq v_*$. *Finally, we have*

$$u(v_0; U_0) = u_0 \quad u'(v_0; U_0) = (-1)^{i+1} \sqrt{-p'_{\mp}(v_0)},$$
$$s_i(v_0; v_0) = (-1)^i \sqrt{-p'_{\mp}(v_0)}.$$

If, in addition to the assumption (6.45), the function p satisfies (for instance) $\int_1^\infty \sqrt{-p'(v)}\, dv = +\infty$, then the Riemann problem for the p-system admits a unique self-similar solution made of shock and rarefaction waves.

Appendix A

Hölder Inequalities

In subsection 5.4.1, based on Lemma 5.3.1 and a sensitive use of Hölder's inequality, we prove larger L^q bounds for the approximatted solutions $u^{\varepsilon,\delta}$ provided $\delta = \mathcal{O}\big(\varepsilon^{\frac{3}{r+1}}\big)$, $u_0^{\varepsilon,\delta} \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, and submitted to diffusion growth exponents $r \geq 2$.

Here, we compare the alternative Hölder inequalities.

Let us begin with

$$\begin{aligned}
\text{(A.1)} \quad \int_{\mathrm{R}^d} |u(t)|^{\alpha+1} dx + D\,(\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-1}\,|\nabla u|^{r+1}\; dxds \\
\leq \|u_0\|^{\alpha+1}_{L^{\alpha+1}(\mathrm{R}^d\times(0,t))} + \|a_{jl}\|\,\frac{(\alpha+1)\,\alpha\,(\alpha-1)}{2} \\
\delta\,\varepsilon^{-\frac{3}{r+1}} \left[\int_0^t \int_{\mathrm{R}^d} |u|^{(\alpha-2-\gamma)\frac{r+1}{r-2}}\, dxds\right]^{\frac{r-2}{r+1}} \\
\left[\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{\gamma\frac{r+1}{3}}\,|\nabla u|^{r+1}\; dxds\right]^{\frac{3}{r+1}},
\end{aligned}$$

as given by (5.10), Lemma 5.3.1. All the recursions we will make are step-based on the first energy estimates.

A.0.1 Fixed $|\nabla u|$

With $\gamma = 0$ above, we make (5.21), p.87. We want to profit of the full results in Proposition 5.3.1, i.e., of estimates (5.12)-(5.13).

We proceed recursively, without any changes in (5.21):

$$\int_{\mathrm{R}^d} |u(t)|^{\alpha_{n+1}+1}\, dx \leq \|u_0\|^{\alpha_{n+1}+1}_{L^{\alpha_{n+1}+1}(\mathrm{R}^d\times(0,t))} + \|a_{jl}\|\,\delta\,\varepsilon^{-\frac{3}{r+1}}$$

$$\frac{(\alpha_{n+1}+1)\alpha_{n+1}(\alpha_{n+1}-1)}{2}\left(\frac{\|u_0\|^2_{L^2(\mathrm{R}^d)}}{2D}\right)^{\frac{3}{r+1}}\left[\int_0^t\int_{\mathrm{R}^d}|u|^{(\alpha_{n+1}-2)\frac{r+1}{r-2}}\,dxds\right]^{\frac{r-2}{r+1}},$$

then, using the previous L^{α_n+1} estimate on u and fixing the estimate (5.13) for $|\nabla u|$:

$$\big(\alpha_{n+1}-2\big)\,\frac{r+1}{r-2}=\alpha_n+1\,,\qquad \alpha_0=1\,.$$

The solution

$$\alpha_n=(1-r)\left(\frac{r-2}{r+1}\right)^n+r\underset{n\to+\infty}{\longrightarrow}r^-$$

show us that $u\in L^{r+1}$ is the better estimate we can have.

In view of the last factor exponent $\frac{r-2}{r+1}<1$, we try $(\alpha-2)\,\frac{r+1}{r-2}=\alpha+1$. The solution $\alpha=r$ push us immeddiately to Proposition 5.4.1, p.88. And we learn that "to go on, only iterating too over $|\nabla u|$-estimates".

A.0.2 Iterated $|\nabla u|$

Here, we move γ: from the many possibilities, we need sellect a best.

Running to $+\infty$

We begin by using the "previous" estimates on, both, $|u|$ and $|\nabla u|$, let

$$\gamma\frac{r+1}{3}=\alpha_n-1\,,\quad (\alpha-2-\gamma)\frac{r+1}{r-2}=\alpha_n+1\,,\qquad\text{i.e.,}$$

$$\left(\alpha_{n+1}-2-3\,\frac{\alpha_n-1}{r+1}\right)\frac{r+1}{r-2}=\alpha_n+1\,,\qquad \alpha_0=1\,.$$

The solution is

$$\alpha_n=1+\frac{3}{r+1}(r-1)n\underset{n\to+\infty}{\longrightarrow}+\infty\,.$$

This justify our final assertion at the end of subsection 5.4.1. If $r\geq 2$, then we can reach any L^q estimate (q so large as we wish).

Now, we use the "previous" estimate on $|\nabla u|$, but the "next" one for $|u|$:

$$\left(\alpha_{n+1}-2-3\,\frac{\alpha_n-1}{r+1}\right)\frac{r+1}{r-2}=\alpha_{n+1}+1\,,\qquad \alpha_0=1\,,$$

has solution

$$\alpha_n=1+(r-1)n\underset{n\to+\infty}{\longrightarrow}+\infty\,,$$

growing up to $+\infty$ and slightly faster than the precedent.

The opposite possibility is to take the "previous" estimate on $|u|$ and the "next" for $|\nabla u|$:

$$\left(\alpha_{n+1} - 2 - 3\,\frac{\alpha_{n+1}-1}{r+1}\right)\frac{r+1}{r-2} = \alpha_n + 1\,, \qquad \alpha_0 = 1\,,$$

with solution

$$\alpha_n = 1 + \frac{3}{r-2}(r-1)n \underset{n\to+\infty}{\longrightarrow} +\infty\,.$$

It is the faster of the three. Also, suggest us an accurater analysis about $r = 2$: perhaps, we grow up to $+\infty$ in a most simpler way.

Last Cases

We run with the "previous" $|\nabla u|$, but a fixed $|u|$.

$$\left(\alpha_{n+1} - 2 - 3\,\frac{\alpha_n-1}{r+1}\right)\frac{r+1}{r-2} = \alpha_*\,, \qquad \alpha_0 = 1\,,$$

has solution

$$\alpha_n = \alpha_* + \frac{2r-1}{r-2} - \left(\alpha_* + \frac{r-1}{r-2}\right)\left(\frac{3}{r+1}\right)^n \underset{n\to+\infty}{\longrightarrow} \left(\alpha_* + \frac{2r-1}{r-2}\right)^- .$$

Or, with the "next" $|\nabla u|$ and a fixed $|u|$:

$$\left(\alpha - 2 - 3\,\frac{\alpha-1}{r+1}\right)\frac{r+1}{r-2} = \alpha_*\,.$$

In one step, we obtained the best bound above

$$\alpha = \alpha_* + \frac{2r-1}{r-2}.$$

Finally, the extreme case, with "next" $|u|$ and "next" $|\nabla u|$.

$$\left(\alpha - 2 - 3\,\frac{\alpha-1}{r+1}\right)\frac{r+1}{r-2} = \alpha + 1\,,$$

is not possible, because we must have $r = 1$, when $r \geq 2$.

A.0.3 Generalized Hölder Inequalities

If we consider all the possibilties together:

$$
\begin{aligned}
\text{(A.2)}\qquad \delta \int_0^t \int_{\mathrm{R}^d} |u|^{\alpha-2} |\nabla u|^3 \, dxds \leq \delta \varepsilon^{-\left(\frac{1}{p_1}+\frac{1}{p_4}\right)} \\
\left[\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{ap_1} |\nabla u|^{dp_1} \, dxds\right]^{\frac{1}{p_1}} \\
\left[\int_0^t \int_{\mathrm{R}^d} |u|^{bp_2} dxds\right]^{\frac{1}{p_2}} \left[\int_0^t \int_{\mathrm{R}^d} |u|^{cp_3} dxds\right]^{\frac{1}{p_3}} \\
\left[\varepsilon \int_0^t \int_{\mathrm{R}^d} |u|^{(\alpha-2-a-b-c)p_4} |\nabla u|^{(3-d)p_4} \, dxds\right]^{\frac{1}{p_4}},
\end{aligned}
$$

then it is easy to conclude how to obtain "$+\infty$", in a single step, and $\delta\varepsilon^{-\frac{\rho+2}{r+1}}$ gives the best balance we can have by this technique. A optimal one.

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