Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Proof-theoretical studies on the bounded functional interpretation

Patrícia Engrácia

Doutoramento em Matemática
(Álgebra, Lógica e Fundamentos)

2009

Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Proof-theoretical studies on the bounded functional interpretation

Patrícia Engrácia

Tese orientada pelo
Professor Doutor Fernando Ferreira

Doutoramento em Matemática
(Álgebra, Lógica e Fundamentos)

2009

This PhD thesis was partially supported by POCI 2010 and FSE.

Ciência.Inovação 2010 Programa Operacional Ciência e Inovação 2010

MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Acknowledgements

First of all, I am grateful for the guidance, assistance and support of my supervisor, Professor Fernando Ferreira. I want to express my deepest appreciation for his dedication, patience and, most of all, for his care for details. I am also thankful for the time he granted me (from his busy schedule) for regular meetings and conversations. As a result of several of these meetings, we have, also, submitted for publication a recent joint work [EF].

I also want to express my appreciation to Paulo Oliva for being so accessible and helpful and for some discussions concerning the joint work with Fernando Ferreira mentioned above. I would like to thank my friend and colleague Jaime Gaspar for some conversations and for his well-written master thesis [Gas07] (which I used recurrently as a reference for some theoretical background).

I want to thank FCT for providing me a doctoral grant, SFRH/BD/17922/2004 (which lasted until September 2008) and also CMAF-FCT for a scholarship grant during the preparation of this thesis (2009).

Last but not least, I would like to thank Gonçalo for all his love, support and patience. I would also like to thank my parents and brother, as well as Gonçalo's family, for all their support and advices.

Patrícia Engrácia
Lisboa, March 23, 2009

Abstract

This dissertation studies the bounded functional interpretation of Ferreira and Oliva. The work follows two different directions. We start by focusing on the generalization of the bounded functional interpretation to second-order arithmetic (a.k.a. analysis). This is accomplished via bar recursion, a well-founded form of recursion. We carry out explicitly the bounded functional interpretation of the (non-intuitionistic) law of the double negation shift with bar recursive functionals of finite type. As a consequence, we show that full numerical comprehension has bounded functional interpretation in the classical case.

In the other direction, we extend the bounded functional interpretation with new base types, representing an abstract class of normed spaces. Some studies on the representation of the real numbers are carried out, as it is useful to have a representation which meshes well with the notion of majorizability. A majorizability theorem holds. We carry out the extension of the bounded functional interpretation to new base types and prove a soundness theorem with characteristic principles similar to the numerical case. We also extend the classical direct bounded functional interpretation of Peano arithmetic to new base types and prove the corresponding soundness theorem. The characteristic principles are also similar to the ones in the numerical case. In the classical setting, these prove that linear operators are automatically bounded and that Cauchy sequences (with a modulus of Cauchyness) of elements of the new base type do converge. Relying on the characteristic principles (and on a special form of choice), a logical version of the Baire category theorem of functional analysis is proved. As a consequence, we also prove logical versions of the Banach-Steinhaus and the open mapping theorems.

Keywords: bounded functional interpretation, majorizability, charateristic principles, bar recursion, collection.

Resumo

O objecto de estudo desta dissertação é a interpretação funcional limitada de Ferreira e Oliva. Este trabalho segue duas direcções distintas. Começamos por nos centrar na generalização da interpretação funcional limitada para a aritmética de segunda ordem (análise). Esta generalização obtém-se recorrendo a *bar recursion*, uma forma bem fundada de recursão. Efectuamos explicitamente a interpretação funcional limitada da lei (não intuicionista) de *double negation shift*, usando para tal funcionais de *bar recursion*. Consequentemente, mostramos que é possível interpretar, no caso clássico, compreensão numérica para fórmulas arbitrárias.

Por outro lado, estendemos a interpretação funcional limitada com novos tipos base, representando, nomeadamente, espaços normados. Para tal, foram elaborados alguns estudos sobre representações dos números reais, uma vez que é conveniente ter uma representação que se relacione bem com a noção de majoração. Prova-se que a teoria estendida é uma teoria de majoração. Estendemos a interpretação funcional limitada para novos tipos base e provamos o respectivo teorema de correcção. Os princípios característicos da interpretação estendida são semelhantes aos do caso numérico. Generalizamos também a interpretação funcional limitada directa da aritmética de Peano para novos tipos base e provamos o teorema de correcção. Os princípios característicos são também semelhantes aos do caso numérico. Com base nestes, toda a sequência de Cauchy (com módulo de convergência) de elementos do novo tipo base converge e todo o operador linear é automáticamente limitado. Como consequência dos princípios característicos, aliados a uma forma especial de escollha, prova-se uma versão lógica do teorema da categoria de Baire da análise funcional. Seguidamente, provamos versões lógicas dos teoremas de Banach-Steinhaus e da aplicação aberta.

Palavras-chave: interpretação functional limitada, majoração, princípios característicos, *bar recursion*, colecção.

Resumo Alargado

O objecto de estudo desta dissertação é a interpretação funcional limitada de Fernando Ferreira e Paulo Oliva. A noção de majoração desempenha um papel fundamental nesta nova interpretação. Ao utilizá-la como ferramenta para extracção de informação computacional de demonstrações em matemática, pretende-se obter majorantes para testemunhas existenciais, ao invés de testemunhas exactas. O estudo segue duas direcções diferentes. Por um lado, pretende-se generalizar a interpretação funcional limitada com funcionais de *bar-recursion* e por outro, com um novo tipo base, representando espaços normados.

Com estes objectivos em mente, começamos por descrever as aritméticas de Heyting, HA^ω, e de Peano, PA^ω, em todos os tipos finitos. Descrevemos também dois dos modelos destas aritméticas: a estrutura de todos os funcionais de teoria de conjuntos, $\mathcal{S}^\omega$, e a estrutura de todos os funcionais majorados, $\mathcal{M}^\omega$. A relação de majoração de $\mathcal{M}^\omega$ deve-se a Bezem:

$$
\begin{aligned}
&x \leq_0^* y := x \leq_0 y \\
&x \leq_{\rho\to\sigma}^* y := \forall u^\rho, v^\rho \ \left(u \leq_\rho^* v \to xu \leq_\sigma^* yv \wedge yu \leq_\sigma^* yv\right)
\end{aligned}
$$

onde ρ e σ são tipos finitos e $\leq_0$ é a desigualdade entre naturais (os objectos de tipo 0 correspondem aos números naturais). De seguida, fazemos uma abordagem à interpretação funcional de Gödel (também conhecida por interpretação Dialectica). Descrevemos a transformação de fórmulas que a define, assim como os seus princípios característicos (entende-se por princípios característicos, os princípios que se podem juntar à aritmética, garantindo a existência de um teorema de correcção). Apresentamos os correspondentes teoremas de correcção e caracterização, assim como teoremas de extracção e de conservação.

A interpretação funcional limitada assenta numa transformação de fórmulas diferente da interpretação Dialectica, recorrendo a uma versão intensional da relação de majoração de Bezem, denotada por $\trianglelefteq$:

$$
\begin{aligned}
&x \trianglelefteq_0 y \leftrightarrow x \leq_0 y \\
&x \trianglelefteq_{\rho\to\sigma} y \to \forall u^\rho, v^\rho \ \left(u \trianglelefteq_\rho u \to xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv\right) \\
&\frac{A_{bd} \wedge u \trianglelefteq_\rho v \to su \trianglelefteq_\sigma tv \wedge tu \trianglelefteq_\sigma tv}{A_{bd} \to s \trianglelefteq_{\rho\to\sigma} t}.
\end{aligned}
$$

onde ρ e σ são tipos finitos e A_{bd} uma fórmula limitada da linguagem (fórmula cujas quantificações são todas limitadas, i.e., da forma $\forall x \trianglelefteq t$ e $\exists x \trianglelefteq t$, sendo t um termo não contendo x). No caso da regra, u e v são variáveis que não aparecem na conclusão. Estas relações dizem-se intensionais por serem governadas, em parte, por uma regra. Estendemos a aritmética de Heyting com os novos símbolos relacionais e quantificadores limitados, $\mathsf{HA}^{\omega}_{\trianglelefteq}$. Apresentamos igualmente a transformação de fórmulas da interpretação funcional limitada e os seus princípios característicos, assim como os teoremas de correcção e caracterização. Ao contrário da interpretação funcional de Gödel, a interpretação funcional limitada obtém majorantes para testemunhas (independentes de certos parâmetros) que podem contradizer verdades de teoria de conjuntos. Por exemplo, refuta o axioma de extensionalidade. A interpretação functional limitada interpreta, ainda, princípios que não têm análogo na interpretação Dialéctica, como por exemplo, o lema fraco de König.

Discutimos ainda uma nova forma de escolha, que denotamos por tameAC

$$\mathsf{tameAC}: \qquad \tilde{\forall} f \exists g \trianglelefteq f \forall x \ (\exists y \trianglelefteq fx\ A_{bd}(x,y) \to A_{bd}(x, gx)),$$

(sendo A_{bd} uma fórmula limitada da linguagem de $\mathsf{HA}^{\omega}_{\trianglelefteq}$) a qual se pode juntar à teoria por ser auto-interpretável. Este princípio é de grande importância no trabalho desenvolvido.

As duas interpretações funcionais referidas acima interpretam a aritmética de Heyting em todos os tipos finitos. Como exemplo, interpretamos explicitamente o princípio (intuicionista) $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$ através destas interpretações funcionais. A aritmética de Peano interpreta-se em dois passos. Primeiro, é interpretada na aritmética de Heyting, através de uma tradução negativa, seguidamente, interpreta-se a aritmética de Heyting através de uma das interpretações funcionais. Recentemente, Ferreira introduziu uma interpretação funcional limitada directa da aritmética de Peano. Apresentamos a sua transformação de fórmulas, princípios característicos, assim como os respectivos teoremas de correcção e caracterização. Demonstramos também que, no caso clássico, os princípios característicos desta interpretação directa são equivalentes aos da interpretação funcional limitada da aritmética de Heyting, quando restritos a fórmulas limitadas.

Denomina-se por *flattening* a passagem das teorias intensionais $\mathsf{HA}^{\omega}_{\trianglelefteq}$ e $\mathsf{PA}^{\omega}_{\trianglelefteq}$ para HA^{ω} e PA^{ω}, obtida através da substituição de todas as ocorrências dos símbolos intensionais $\trianglelefteq$ pelos extensionais da majorização de Bezem, $\leq^*$.

Bar recursion é uma forma bem fundada de recursão que foi estendida a todos os tipos finitos por Clifford Spector. O seu uso permite estender a interpretação funcional limitada à aritmética de segunda ordem (análise). Para tal, seguimos o trabalho de Spector e interpretamos explicitamente (usando a interpretação funcional limitada) o princípio não intuicionista de *double negation shift*

$$\mathsf{DNS}: \qquad \forall n\ \neg\neg A(n) \to \neg\neg\forall n\ A(n)$$

(n é natural e A uma fórmula arbitrária da linguagem de $\mathsf{HA}^{\omega}_{\trianglelefteq}$), baseando-nos na intuição obtida pela interpretação funcional limitada do princípio $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$ (note--se que DNS é uma generalização deste princípio). De modo a interpretar a tradução negativa da compreensão numérica para todas as fórmulas

$$\mathsf{CA}^0: \qquad \exists f \forall n\ (f(n) = 0 \leftrightarrow A(n))$$

(n é natural, f é uma função de $\mathbb{N}$ para $\mathbb{N}$ e A é uma fórmula arbitrária da linguagem), seguimos os passos de Spector aquando da interpretação Dialectica da tradução negativa de CA^0. Nesta análise, a interpretação do DNS desempenha um papel fundamental. O mesmo se passa no nosso estudo: com base na interpretação funcional limitada do DNS (e também no princípio tameAC referido acima), interpretamos a tradução negativa da compreensão numérica.

Analogamente aos recursores usuais, também a *bar recursion* tem um princípio de indução associado, chamado *bar induction*. No final do capítulo dedicado à extensão com funcionais de *bar recursion*, provamos que a *bar induction* tem interpretação funcional limitada.

Todos estes resultados são verificados na teoria $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$, onde BR é o esquema de axiomas que regulam os funcionais de *bar recursion* e $\Delta_{\mathcal{M}^\omega}$ é o conjunto de todas as sentenças universais (com matrizes intensionalmente limitadas) cujo *flattening* é verdadeiro na estrutura $\mathcal{M}^\omega$. Enquanto alguns resultados dependem de uma forma não essencial de factos de $\Delta_{\mathcal{M}^\omega}$ e poderiam ter sido demonstrados em $\mathsf{HA}^\omega_\trianglelefteq$, outros parecem depender de uma maneira crucial de alguns factos de $\Delta_{\mathcal{M}^\omega}$ (por exemplo, para mostrar que os funcionais de *bar recursion* são majorizáveis no sentido intensional). Deste modo, o tratamento feito não é óptimo. No entanto, optámos por este pois facilita a compreensão da demonstração dos resultados e por, deste modo, não desviar a atenção dos resultados em si. Deixamos o tratamento óptimo para trabalho futuro. Acreditamos ser possível ver o princípio DNS como um caso particular de *bar induction*.

No que diz respeito à extensão com novos tipos base para classes abstractas de espaços, tomamos X como um novo tipo base para espaços normados. Cada objecto de tipo X é interpretado como um vector do espaço normado. A linguagem de $\mathsf{HA}^\omega_\trianglelefteq$ é estendida com variáveis e constantes de tipo X, nomeadamente com uma constante $\|.\|$ para a norma. As desigualdades intensionais são estendidas para o novo tipo

$$
\begin{array}{l}
x \trianglelefteq_0 y \leftrightarrow x \leq_0 y \\
x \trianglelefteq_{\mathsf{X}} y \rightarrow \|x\| \leq_{\mathbb{R}} (y)_{\mathbb{R}} \\
x \trianglelefteq_{\rho\to\sigma} y \rightarrow \forall u^\rho, v^\rho \ \ (u \trianglelefteq_\rho u \rightarrow xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv) \\
\dfrac{A_{bd} \wedge \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}}{A_{bd} \rightarrow s \trianglelefteq_{\mathsf{X}} t} \\
\dfrac{A_{bd} \wedge u \trianglelefteq_\rho v \rightarrow su \trianglelefteq_\sigma tv \wedge tu \trianglelefteq_\sigma tv}{A_{bd} \rightarrow s \trianglelefteq_{\rho\to\sigma} t}.
\end{array}
$$

Os majorantes destas desigualdades são de tipo aritmético, i.e., tipos finitos que não envolvam o novo tipo X. Em particular, objectos de tipo X são majorados por objectos de tipo 0 (naturais). $(y)_{\mathbb{R}}$ é o natural y representado como real e $\leq_{\mathbb{R}}$ é a desigualdade entre reais. Naturalmente, temos de trabalhar com uma representação adequada dos reais. Por questões técnicas, é conveniente usar uma representação que se relacione bem com a noção de majoração. Em particular, esta deve satisfazer a seguinte condição: queremos que exista $g : \mathbb{N} \to \mathbb{N}$ tal que se $f : \mathbb{N} \to \mathbb{N}$ representa um real no intervalo $[-n, n]$, então $fi \leq gn$ para todo o natural i. Por esta razão, usamos a representação binária com sinal, ao invés da representação mais usual através de sucessões de Cauchy de racionais.

Denotamos a extensão da aritmética de Heyting (com os símbolos intensionais) com novo tipo X por $\mathsf{HA}^{\omega,\mathsf{X}}_\trianglelefteq$ (semelhante para a aritmética de Peano). Esta é uma de majoração. Estendemos a interpretação funcional limitada para novos tipos base e provamos o respectivo teorema de correcção. Os princípios característicos desta interpretação estendida são semelhantes aos do caso numérico. Generalizamos também a interpretação funcional limitada directa da aritmética de Peano com novos tipos base e provamos o teorema de correcção. Os princípios característicos são também semelhantes aos do caso numérico.

Com base nos princípios característicos, provamos que toda a sucessão de Cauchy (com módulo de convergência) de elementos de tipo X converge e que todo o operador linear é automaticamente limitado. Os princípios característicos, juntamente com o princípio de escolha tameAC, levam a que se prove uma versão lógica do teorema da categoria de Baire da análise funcional. Este pode ser visto como uma forma local de

$$\mathsf{bC}_{bd}^{\omega,\mathsf{X}}: \qquad \forall x \trianglelefteq a \exists y\, A_{bd}(x,y) \to \tilde{\exists} b \forall x \trianglelefteq a \exists y \trianglelefteq b\, A_{bd}(x,y)$$

(A_{bd} é uma fórmula limitada da linguagem) generalizada para fórmulas universais e extensionais. $\mathsf{bC}_{bd}^{\omega,\mathsf{X}}$ diz-se princípio de colecção limitada e é um dos princípios característicos da interpretação funcional limitada clássica. Os teoremas de Banach-Steinhaus e da aplicação aberta demonstram-se a partir do teorema da categoria de Baire. No nosso estudo, baseando-nos neste princpio de "colecção local", que é a versão lógica do teorema de Baire, provamos versões lógicas dos teoremas de Banach-Steinhaus e da aplicação aberta, que podem ser vistos como formas de colecção (global) para certas fórmulas simultaneamente universais e extensionais.

Contents

1

Introduction

In the mid 90's, Ulrich Kohlenbach developed a successful program for extracting computational information from ordinary proofs in mathematics (even proofs using non-constructive properties like the weak König's lemma), known as Proof Mining. The main tool for performing these extractions is Kurt Gödel's functional interpretation [Göd58], also known as Dialectica interpretation. The notion of majorizability plays a prominent role as well. For an introduction to Proof Mining, see [KO03], [Koh07], [Koh08b] or Kohlenbach's recent book [Koh08a]. In fact, Kohlenbach uses an interpretation, introduced in [Koh96], obtained from Gödel's interpretation, the monotone functional interpretation. Using this interpretation, he has been defending a shift of attention from precise witnesses to the extraction of *bounds* from proofs of $\forall\exists$ sentences. It is this shift of attention that allows the analysis of certain non-constructive principles. It also enables the extraction of numerical bounds from proofs in classical analysis. For instance, in the monotone interpretation, further axioms may be added in the soundness theorem (for instance, weak König's lemma), in opposition to the Dialectica interpretation, where only universal sentences may be added.

In 2005, Fernando Ferreira and Paulo Oliva presented in [FO05] a new interpretation, the so-called bounded functional interpretation. As opposed to Kohlenbach's monotone interpretation, it is based on a new assignment of formulas. In the bounded functional interpretation, the notion of majorizability also plays a crucial role. The interpretation is carried out in a new setting, an intensional setting. Heyting arithmetic in all finite types is extended with new (intensional) relations $\trianglelefteq$, the intensional counterpart (in the sense that the relations are partly governed by a rule) of the strong majorizability relations, $\leq^*$, defined by Marc Bezem in [Bez85] (after the work of William Howard [How73]). The language also contains new quantifiers, known as bounded quantifiers. In the above mentioned paper, Bezem defines the structure $\mathcal{M}^\omega$ of the strongly majorizable functionals, which plays an important role in our studies. The bounded functional interpretation has similarities to the monotone functional interpretation, in the sense that it does not care for precise witnesses but only bounds for them. It can be used to prove similar results, first obtained via the monotone interpretation. The bounded functional interpretation is not set-theoretically faithful, since it "injects" uniformities (obtaining majorizing witnesses independent from certain parameters) which contradict certain set-theoretical truths. For instance, it refutes the axiom of extensionality. Furthermore, the bounded functional interpretation interprets new principles which have no analogue in the Dialectica or monotone interpretations. For instance, one can interpret a very general form of L. E. J. Brouwer's FAN theorem as well as certain non-intuitionistic principles like weak König's lemma or the lesser limited principle of omniscience. In fact, it also interprets classical inconsistent principles, relying on the so-called characteristic principles (principles that can be added to Heyting arithmetic and still have a soundness theorem). The bounded functional interpretation has, as characteristic principles, versions of the axiom of choice, the independence of premises principle and Markov's principle (the three characteristic principles of Gödel's Dialectica interpretation), plus majorizability axioms, a version of contra collection and a disjunction property (which implies the lesser limited principle of omniscience, mentioned above).

The aforementioned interpretations provide an interpretation of Heyting arithmetic. In order to interpret Peano arithmetic, we first interpret it into Heyting arithmetic using a negative translation. Afterwards, Heyting arithmetic is interpreted via one of the functional interpretations. In 1967, Joseph Shoenfield defined a direct interpretation of Peano arithmetic in his well-known textbook [Sho67]. In the style of Shoenfield, Ferreira presented recently a direct bounded interpretation of Peano arithmetic in all finite types [Fer09]. Similarly to the bounded functional interpretation, this new interpretation also "injects" uniformities into classical mathematics (making it set-theoretically unsound). This is a consequence of the characteristic principles.

This dissertation studies the bounded functional interpretation. The work follows two different directions, suggested in [Fer06]:

- Extend bounded interpretations with bar recursive functionals;
- Extend bounded interpretations with new base types.

Following a belief of Ferreira stated in [Fer08], we carry out a study on the generalization of the bounded functional interpretation to second-order arithmetic (a.k.a. analysis), relying on bar recursive functionals. In 1962, Clifford Spector used a well-founded recursion principle, known as *bar recursion*, to give a remarkable characterization of the provably recursive functionals of full second-order arithmetic [Spe62]. Spector extended the bar notions to all finite types. In order to achieve our goal, we follow his seminal work. First, we focus in the *double negation shift* principle

$$\mathsf{DNS}: \qquad \forall n\, \neg\neg A(n) \to \neg\neg \forall n\, A(n)$$

(n is a natural number and A is an arbitrary formula) and prove that it has bounded functional interpretation using bar recursive functionals in all finite types. The DNS principle is a generalization of the intuitionistic law $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$. In order to accomplish the DNS interpretation, we rely on the intuition obtained while carrying out the bounded functional interpretation of $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$. We follow Spector's steps in analyzing full numerical comprehension via the Dialectica interpretation. In Spector's analysis, the interpretation of DNS plays an instrumental role. This will also be the case in our study for the bounded functional interpretation: we get the interpretation of the negative translation of full numerical comprehension

$$\mathsf{CA}^0: \qquad \exists f \forall n \ (f(n) = 0 \leftrightarrow A(n)),$$

where n is a natural number, f is a function from $\mathbb{N}$ to $\mathbb{N}$ and A is an arbitrary formula, relying upon the bounded functional interpretation of DNS (in our treatment, we also need a special form of choice).

In analogy with the usual recursors and induction, bar recursion is a principle of definition with a corresponding principle of proof, known as *bar induction*. We finalize this topic with the bounded functional interpretation of bar induction. The argument of the proof is an adaptation of the one given by Howard in [How68], relying also on bar recursors.

All of these results are verified in the intensional Heyting arithmetic in all finite types extended with bar recursive functionals plus a particular set of universal sentences, denoted by $\Delta_{\mathcal{M}^\omega}$. This set contains *all* universal sentences with bounded intensional matrices (matrices with bounded quantifications only) whose "flattenings" hold in $\mathcal{M}^\omega$. By flattening, we mean the passageway from the intensional to the extensional formulas, obtained by replacing all occurrences of $\trianglelefteq$ by the strong majorizability relations $\leq^*$. Of course, in our treatment, we rely on $\Delta_{\mathcal{M}^\omega}$. However, we must point out that whereas some use of $\Delta_{\mathcal{M}^\omega}$ seem to be essential (e.g., for proving that the bar recursors are majorizable in the intensional sense), other are inessential and could be proved in $\mathsf{HA}^\omega_\trianglelefteq$. Therefore, our treatment is not optimal. Nevertheless, we chose this treatment because it eases the reading and avoids distractions from the results themselves.

Concerning the second goal, we extend both the bounded functional interpretation of Heyting arithmetic as well as the direct bounded interpretation of Peano arithmetic to new base types, namely a type for normed spaces (following Kohlenbach). Until recently, Proof Mining dealt with theorems involving concrete spaces, such as Polish and compact metric spaces, necessarily represented in an effective way. Under suitable representations, functional interpretations may be applied to results in ordinary mathematical analysis. Lately, this approach has been extended to classes of *abstract* spaces (such as normed, metric and hyperbolic spaces) by Kohlenbach in [Koh05]. The introduction of new base types has expanded the domain of applications and has given further insights into Proof Mining. In particular, uniform bounds with respect to parameters in metrically bounded spaces, not only in compact spaces, can sometimes be obtained.

For technical reasons, it is extremely useful to have an effective representation of the reals which meshes well with the notion of majorizability. Therefore, we adopt the signed-digit representation instead of the more usual representation via Cauchy sequences of rational numbers.

As a consequence of extending the bounded functional interpretations to new base types, some new results can be achieved. For instance, relying on the characteristic principles, we show that Cauchy sequences (of objects of the new type) with modulus of Cauchyness do converge and that linear operators are automatically bounded. Furthermore, we focus on the principle of bounded collection (one of the classical characteristic principles). In general, collection for universal matrices does not have a bounded interpretation (in fact, it is inconsistent). Nevertheless, we show that the Banach-Steinhaus (a.k.a uniform boundness) and the open mapping theorems of functional analysis can be seen as instances of collection for universal matrices that do have bounded functional interpretations. Both the Banach-Steinhaus and the open mapping theorems rely on the Baire category theorem. A logical version of this theorem can, in fact, be proved using the characteristic principles of the direct bounded functional interpretation plus a "tame" form of choice, tameAC, (a self-interpretable version of choice). It can be seen as a kind of local collection for universal extensional formulas. In the Banach-Steinhaus and the open mapping theorems, this kind of local collection can be lifted up to global collection using the linearity of the operators.

In the next chapter, we present the necessary background in order to understand the following chapters. We begin by describing the formal systems of arithmetic used in this work, namely, Heyting arithmetic, HA^ω, and Peano arithmetic, PA^ω, in all finite types. We also present the type structures $\mathcal{S}^\omega$ of all set-theoretical functionals as well as $\mathcal{M}^\omega$ of all strongly majorizable functionals. In order to focus on the latter, Bezem's strong majorizability $\leq^*$ and some of its properties are presented. We describe Gödel's functional interpretation and the bounded functional interpretation of Ferreira and Oliva and their main results, namely, the soundness and the characterization theorems. The characteristic principles are described in the process. Some results of extraction and conservation are presented as well. Before introducing the bounded functional interpretation, we focus on the intensional theories $\mathsf{HA}^\omega_\trianglelefteq$ and $\mathsf{PA}^\omega_\trianglelefteq$, obtained by adding new relation symbols $\trianglelefteq$ (one for each type) as well as bounded quantifiers. The relations $\trianglelefteq$ are partly governed by a rule, hence not all properties of $\leq^*$ are satisfied by the intensional relations. Some new results on $\trianglelefteq$ are proved. While presenting the bounded functional interpretation, we introduce a tame axiom of bounded choice, denoted by tameAC. Under the characteristic principles, this principle is equivalent to an universal sentence (with bounded intensional matrix) and is self-interpretable. This principle is extremely useful, since it gives precise witnesses. It will play an important role in the following chapters. In chapter 2, we also carry explicitly the Dialectica and the bounded functional interpretation of the principle $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$. The bounded functional interpretation of this principle gives some insight to the interpretation of the double negation shift. Finally, in both interpretations, Peano arithmetic in all finite types is interpreted via the negative translation of Kuroda. We also present the direct functional interpretation of Ferreira and its main results. We prove that its characteristic principles are classically equivalent to the bounded versions of the characteristic principles of the intuitionistic bounded functional interpretation.

In chapter 3, we present the bar recursive functionals and bar induction. We argue that intensional Heyting arithmetic extended with bar recursors, $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR}$, plus the set $\Delta_{\mathcal{M}^\omega}$ mentioned above is a majorizability theory. Therefore, it has a corresponding soundness theorem. Following the work of Spector, we carry out explicitly the bounded functional interpretation of the double negation shift principle in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ and, afterwards, interpret the negative translation of full numerical comprehension. Finally, we focus on the bounded functional interpretation of bar induction using bar recursors in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$. This is not carried out directly. We prove, instead, that bar induction is provable in $\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^{\omega}}$ plus its characteristic principles.

In chapter 4, we generalize both the bounded functional interpretation as well as its classical direct version to a new base type X, representing the abstract class of normed spaces. In order to describe the extended framework, we study two possible representations of the reals, the usual one via Cauchy sequences of rational numbers and the signed-digit one. We show that, in HA^{ω}, there is an effective translation between these two representations. We adopt the latter, since it meshes well with the notion of majorizability. In fact, it is useful to have a representation which satisfies the following majorizability condition: there exists $g : \mathbb{N} \to \mathbb{N}$ such that if $f : \mathbb{N} \to \mathbb{N}$ represents a real number in $[-n, n]$, then $f(i) \leq g(n)$ for all $i \in \mathbb{N}$. As opposed to the Cauchy sequence representation, the signed-digit one satisfies this condition. Given this representation of the reals, we continue by presenting the extended arithmetics $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ and $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq}$. The bounded functional interpretation and the classical direct bounded functional interpretation are extended to the new base type as well as their characteristic principles. The soundness and the characterization theorems are proved. As a consequence of the characteristic principles, we prove that every Cauchy sequence (of objects of type X) with modulus of Cauchyness converges in X. We show that $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{tameAC}$ plus the characteristic principles prove the Baire-like theorem, a logical version of the Baire category theorem of functional analysis. This logical version is a "kind" of local collection for universal extensional formulas. The proofs of the logical versions of the Banach-Steinhaus and the open mapping theorems rely on this "local collection". In the process, we define linear operators and prove that, under the characteristic principles, linear operators are automatically bounded and extensional.

2

Functional interpretations: Dialectica and bounded functional interpretations

In this chapter, we begin by describing the language of Heyting and Peano arithmetic in all finite types. Afterwards, we present Gödel's Dialectica interpretation extended to finite--type Heyting arithmetic and the recent bounded functional interpretation of Ferreira and Oliva. On the latter, we describe not only the usual bounded functional interpretation of Peano arithmetic via a negative translation, but also a direct (Shoenfield-like) one.

2.1 Intuitionistic and classical arithmetic in all finite types

Let T be the set of all finite types (with ground type 0). T is defined recursively by

$$i)\ 0 \in \mathsf{T};$$
$$ii)\ \text{if}\ \ \rho, \sigma \in \mathsf{T},\ \text{then}\ \ \rho \to \sigma \in \mathsf{T}.$$

Objects of type 0 represent natural numbers and objects of type $\rho \to \sigma$ represent (total) functions mapping objects of type ρ to objects of type σ. It is usual to denote type $0 \to 0$ by type 1 and type $(0 \to 0) \to 0$ by type 2. In general, type $n+1$ denote the type $n \to 0$. These are called *pure types*.

As usual, the language of Heyting arithmetic in all finite types, HA^ω is denoted by $\mathcal{L}^\omega$, which is a many-sorted language with variables $x^\rho, y^\rho, z^\rho, ...$ and quantifiers $\forall x^\rho, \exists x^\rho$ for all types $\rho \in \mathsf{T}$. It also contains a predicate relation $=_0$ (equality between natural numbers) and the following constants: zero 0^0, successor S^1, combinators $\Pi_{\rho,\sigma}$ of type $\rho \to (\sigma \to \rho)$ and $\Sigma_{\rho,\sigma,\tau}$ of type $\big(\rho \to (\sigma \to \tau)\big) \to \big((\rho \to \sigma) \to (\rho \to \tau)\big)$, as well as simultaneous recursors $\underline{\mathrm{R}}_{\underline{\rho}}$ of type $0 \to \big(\underline{\rho} \to \big((\underline{\rho} \to (0 \to \underline{\rho})\big) \to \underline{\rho})\big)\big)$. The only primitive predicate is $=_0$.

As usual, a tuple of terms $t_1, t_2, \cdots, t_k$ is denoted by $\underline{t}$. More precisely, $\underline{t}^{\underline{\rho}}$ is the abbreviation of a tuple (possibly empty) $t_1^{\rho_1}, t_2^{\rho_2}, \cdots, t_k^{\rho_k}$.

Constants and variables of type ρ are *terms* of type ρ. If $t^{\rho\to\sigma}$ and q^{ρ} are terms, then tq is a term of type σ.

If $t, s_1, s_2, ..., s_n$ are terms, we write $ts_1s_2...s_n$ to denote the resulting term $(((ts_1)s_2)...)s_n$, meaning we associate to the left.

Atomic formulas are formulas of the form $s =_0 t$ with s and t being terms of type 0. *Formulas* are constructed recursively as follows:

$i)$ atomic formulas are formulas;

$ii)$ if A, B are formulas, then $A \wedge B,\ A \vee B,\ A \to B,\ A \to \bot$ are formulas;

$iii)$ if A is a formula, then for all $\rho \in \mathsf{T}$, $\forall x^{\rho}\ A$ and $\exists x^{\rho}\ A$ are also formulas.

As usual, $\neg A$ abbreviates $A \to \bot$, where $\bot$ is $0 =_0 1$ and $A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$.

Equality between terms of higher types $=_{\rho\to\sigma}$ is defined by:

$$s =_{\rho\to\sigma} t \text{ is } \forall x^{\rho}\ (sx =_{\sigma} tx)$$

where s, t are terms of type $\rho \to \sigma$ and x^{ρ} is a variable which does not occur in s, t.

HA^{ω} is based on intuitionistic logic. Beside the axioms of intuitionistic logic, the theory also has the following axioms for equality, successor, combinators and recursors:

i) *equality axioms*:

$$\begin{array}{ll} & n =_0 n; \\ \mathsf{E}: & n =_0 m \wedge A[n/w] \to A[m/w] \end{array}$$

where A is an atomic formula of the language, w is a distinguished variable of A and $A[t/w]$ is obtained by replacing w by t;

ii) *successor axioms*:

$$\begin{array}{l} S(n) \neq_0 0; \\ S(n) =_0 S(m) \to n =_0 m; \end{array}$$

iii) *axioms for combinators and recursors*:

$$\begin{array}{ll} \mathsf{E}_{\Pi}: & A[\Pi xy/w] \leftrightarrow A[x/w]; \\ \mathsf{E}_{\Sigma}: & A[\Sigma xyz/w] \leftrightarrow A[xz(yz)/w]; \\ \mathsf{E}_{\underline{\mathrm{R}}}: & \left\{ \begin{array}{l} A[(\mathrm{R}_i)_{\underline{\rho}}0\underline{y}\underline{z}/w] \leftrightarrow A[y_i/w] \\ A[(\mathrm{R}_i)_{\underline{\rho}}(Sn)\underline{y}\underline{z})/w] \leftrightarrow A[z_i(\underline{\mathrm{R}}_{\underline{\rho}}n\underline{y}\underline{z})n)/w] \quad \text{for all } \ i = 1, ..., k, \end{array} \right. \end{array}$$

where A is an atomic formula of the language, n is a natural number, w is a distinguished variable, $\underline{\rho} = \rho_1, \rho_2, ..., \rho_k$, $\underline{y} = y_1, y_2, ..., y_k$ and $\underline{z} = z_1, z_2, ..., z_k$ with y_i of type ρ_i and z_i of type $\underline{\rho} \to (0 \to \rho_i)$;

iv) *induction schema*:

$$\mathsf{IA}: \qquad A(0) \wedge \forall n^0 \left(A(n) \to A(S(n))\right) \to \forall n^0\, A(n)$$

where A is a arbitrary formula of the language;

We only have reflexivity for equality, since it can be shown that it is symmetric and transitive. One can also show that axiom E, as well as axioms for combinators and recursors hold for every formula of the language.

In HA^ω, it can be defined the usual less or equal numerical relation $\leq_0$ as well as the usual term $\max_0^{0\to(0\to 0)}$, giving the maximum of two natural numbers (to ease the readability, we may write max instead of $\max_0$, whenever it is clear). The relation $\leq_0$ is reflexive, transitive and satisfies:

$$\begin{array}{ll} i) & n \leq_0 \max(n,m) \wedge m \leq_0 \max(n,m); \\ ii) & n' \leq_0 n \wedge m' \leq_0 m \to \max(n',m') \leq_0 \max(n,m). \end{array}$$

In fact, at this point, we can define a pointwise less or equal relation $\leq_\rho$ for each $\rho \in \mathsf{T}$, given by:

$$x \leq_{\rho\to\sigma} y = \forall u^\rho \left(xu \leq_\sigma yu\right).$$

In order to ease the reading, we may write $\leq$ instead of $\leq_0$ (or even instead of $\leq_\rho$), whenever it is clear.

One may also define a bounded minimization operator, denoted by μ, where $\mu n \leq_0 k.P(n)$ is the least natural less or equal to k such that the primitive recursive operator P holds. For details, see [Koh08a] and [Fer06].

Using the combinators, one can prove the following:

Theorem 1 (**Combinatorial completeness**). *Let t be a term of type σ with a distinguished variable x of type ρ. Then, we can construct a term q of type $\rho \to \sigma$, whose free variables are those of t except for x, such that*

$$\mathsf{HA}^\omega \vdash A[t[s/x]/w] \leftrightarrow A[qs/w],$$

where A is an atomic formula of the language $\mathcal{L}^\omega$ with a distinguished variable w of type σ and s is a term of type ρ.

This property also holds for every formula of the language. Term q is usually denoted by $\lambda x.t$ and, now, the theorem states that the term $t[s/x]$ may be substituted by $(\lambda x.t)s$ in any formula.

Using the recursors, one may construct a closed term for each description of a primitive recursive function, satisfying the respective defining conditions of the description. Hence, HA^ω contains all primitive recursive functions. In fact, it is also possible to define functions beyond the primitive recursive ones, using higher type recursors (e.g., the Ackermann function). For details, see [Tro73].

Proposition 1. *For each quantifier-free formula $A_{qf}(\underline{x})$, there exists a closed term t such that*

$$\mathsf{HA}^\omega \vdash A_{qf}(\underline{x}) \leftrightarrow t\underline{x} =_0 0.$$

Observe that in HA^ω, $n =_0 0 \vee n \neq_0 0$. Consequently,

Corollary 1. *For each quantifier-free formula A_{qf}, we have*

$$\mathsf{HA}^\omega \vdash A_{qf} \vee \neg A_{qf}.$$

The classical theory PA^ω extended to all finite types is obtained by adding to HA^ω the law of excluded middle $A \vee \neg A$ for arbitrary formulas A of the language $\mathcal{L}^\omega$.

To finish this section, we present two models of Peano arithmetic in all finite types, PA^ω. Of course, they are also be models of HA^ω. These models are the *full set-theoretical model*, $\mathcal{S}^\omega$, and the model of *majorizable functionals*, $\mathcal{M}^\omega$. We only focus in these two models since they are the only ones we will need from here on. Nevertheless, for more models, see [Koh08a] or [Fer06].

The type structure $\mathcal{S}^\omega$ of all set-theoretical functionals is defined inductively as

$$S_0 := \mathbb{N};$$
$$S_{\rho\to\sigma} := (S_\sigma)^{S_\rho};$$
$$\mathcal{S}^\omega := \langle S_\rho \rangle_{\rho \in \mathsf{T}}.$$

Note that $S_{\rho\to\sigma}$ is the set of all functions from S_ρ to S_σ. It is clear that $\mathcal{S}^\omega$ is a model of PA^ω. It is called the *standard* structure of finite type arithmetic.

The model of all strongly majorizable functionals, $\mathcal{M}^\omega$, was constructed by Bezem in [Bez85], using Bezem's strong majorizability relation, a variation of Howard's relation *maj*. It is usually denoted by *s-maj*, but we will denote it by $\leq^*$.

Bezem's strong majorizability relation is given by

$i)$ $x \leq^*_0 y := x \leq_0 y$;

$ii)$ $x \leq^*_{\rho\to\sigma} y := \forall u^\rho, v^\rho \; (u \leq^*_\rho v \to xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)$.

Lemma 1. HA^ω *proves that*

i) $x \leq^*_\rho y \to y \leq^*_\rho y$

ii) $x \leq^*_\rho y \wedge y \leq^*_\rho z \to x \leq^*_\rho z$

for each type $\rho \in \mathsf{T}$.

The type structure $\mathcal{M}^\omega$ of all strongly majorizable set-theoretical functionals is defined inductively by

$$M_0 := \mathbb{N};$$
$$M_{\rho\to\sigma} := \{x \in M_\sigma^{M_\rho} : \exists x^* \in M_\sigma^{M_\rho} \; x \leq^*_{\rho\to\sigma} x^*\};$$
$$\mathcal{M}^\omega := \langle M_\rho \rangle_{\rho \in \mathsf{T}}.$$

For every x, function from $\mathbb{N}$ to M_ρ, with $\rho = \rho_1 \to (\rho_2 \to (\rho_3 \to ...(\rho_k \to 0)))$, we can define the following (using only the recursor R_0):

Definition 1. *Let $x \in M_\rho^{\mathbb{N}}$ with $\rho = \rho_1 \to (\rho_2 \to (\rho_3 \to ...(\rho_k \to 0)))$. Then, we define*

$$x^M(n) = \lambda \underline{v}.\ \max_0\{xi\underline{v} : i \leq n\},$$

where $\underline{v} = v_1^{\rho_1}, ..., v_k^{\rho_k}$.

Lemma 2. *Let $x, x' \in M_\rho^{\mathbb{N}}$ ($\rho = \rho_1 \to (\rho_2 \to (\rho_3 \to ...(\rho_k \to 0)))$) be such that*

$$\forall n \in \mathbb{N}\ \big(xn \leq^*_\rho x'n\big).$$

Then

$$x \leq^*_{0\to\rho} (x')^M\ \wedge\ x^M \leq^*_{0\to\rho} (x')^M.$$

Moreover, x, x^M and $(x')^M$ are in $M_{0\to\rho}$.

Proposition 2. *For each type $\rho \in \mathsf{T}$, $M_\rho^{\mathbb{N}} = M_{0\to\rho}$.*

Although not all functionals are in $\mathcal{M}^\omega$, we have the following result due to Howard [How73]:

Theorem 2. *For each closed term t of type ρ, there exists a closed term q of the same type such that*

$$\mathsf{HA}^\omega \vdash t \leq^*_\rho q.$$

Theorem 3. *$\mathcal{M}^\omega$ is a model of PA^ω.*

For details on the relation $\leq^*$ and on the model $\mathcal{M}^\omega$, see [Koh08a].

2.2 Gödel's Dialectica interpretation

2.2.1 Gödel's Dialectica interpretation

In 1958, Gödel presented an interpretation of the first-order Heyting arithmetic into a quantifier-free theory with finite-type functionals [Göd58]. This article was published in the journal *Dialectica*, which gave name to the interpretation, *Gödel's Dialectica interpretation*, also known as *Gödel's functional interpretation.* In this section, we present Gödel's interpretation extended to HA^ω. For further deltails, see also [AF98].

The interpretation assigns to each formula $A(\underline{x})$ of the language $\mathcal{L}^\omega$ a formula A^D of the form $\exists \underline{a} \forall \underline{b}\ A_D(\underline{x}, \underline{a}, \underline{b})$, where A_D is a quantifier-free formula. As before, $\underline{x}, \underline{a}, \underline{b}$ are tuples of variables.

Definition 2. *To each formula A of the language $\mathcal{L}^\omega$, we assign formulas A^D and A_D, such that A^D is of the form $\exists \underline{a} \forall \underline{b}\ A_D(\underline{a}, \underline{b})$ with A_D a quantifier-free formula and $\underline{a}, \underline{b}$ tuples (possibly empty) of variables whose type depends on the structure of A. The free variables of A^D are those of A. A_D is given by*

1. *A^D and A_D are A if A is atomic.*

If the interpretations of A and B are given by $\exists \underline{a} \forall \underline{b}\ A_D(\underline{a}, \underline{b})$ and $\exists \underline{c} \forall \underline{d}\ B_B(\underline{c}, \underline{d})$, respectively, then

2. $(A \wedge B)^D$ *is* $\exists \underline{a}, \underline{c} \forall \underline{b}, \underline{d}\ \big(A_D(\underline{a}, \underline{b}) \wedge B_D(\underline{c}, \underline{d})\big)$*;*
3. $(A \vee B)^D$ *is* $\exists n^0, \underline{a}, \underline{c} \forall \underline{b}, \underline{d}\ \big((n =_0 0 \to A_D(\underline{a}, \underline{b}) \wedge (n \neq_0 0 \to B_D(\underline{c}, \underline{d}))\big)$*;*
4. $(A \to B)^D$ *is* $\exists \underline{f}, \underline{g} \forall \underline{a}, \underline{d}\ \big(A_D(\underline{a}, \underline{g}\underline{a}\underline{d}) \to B_D(\underline{f}\underline{a}, \underline{d})\big)$*;*
5. $(\forall x\ A(x))^D$ *is* $\exists \underline{f} \forall x, \underline{b}\ A_D(x, \underline{f}x, \underline{b})$*;*
6. $(\exists x\ A(x))^D$ *is* $\exists x, \underline{a} \forall \underline{b}\ A_D(x, \underline{a}, \underline{b})$*.*

For a motivation, see [Koh08a].

As a consequence, the interpretation of $\neg A$ is given by $\exists \underline{f} \forall \underline{a}\ \neg A_D(\underline{a}, \underline{f}\underline{a})$, when the interpretation of A is given by $\exists \underline{a} \forall \underline{b}\ A_D(\underline{a}, \underline{b})$.

In the following, we present the soundness theorem of this interpretation. There are three fundamental principles which can be joined to the theory:

1. *Axiom of choice* AC^ω

   $$\mathsf{AC}^{\rho,\sigma}: \quad \forall x^\rho \exists y^\sigma\ A(x,y) \to \exists f^{\rho\to\sigma} \forall x^\rho\ A(x, fx),$$

   where A is an arbitrary formula of the language;
2. *Independence of premises principle for universal antecedents* $\mathsf{IP}^\omega_\forall$

   $$\mathsf{IP}^\rho_\forall: \quad \big(\forall \underline{x}\ A_{qf}(\underline{x}) \to \exists y^\rho\ B(y)\big) \to \exists y^\rho\ \big(\forall \underline{x}\ A_{qf}(\underline{x}) \to B(y)\big),$$

   where A_{qf} is a quantifier-free formula and B is an arbitrary formula of the language;
3. *Markov's principle* M^ω

   $$\mathsf{M}^\rho: \quad \neg \forall x^\rho\ A_{qf}(x) \to \exists x^\rho\ \neg A_{qf}(x),$$

   where A_{qf} is a quantifier-free formula of the language.

Theorem 4 (Soundness). *Let $A(\underline{x})$ be a formula of the language $\mathcal{L}^\omega$ with functional interpretation given by $\exists \underline{a} \forall \underline{b}\ A_D(\underline{x}, \underline{a}, \underline{b})$, such that*

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash A(\underline{x}),$$

where Δ is a set of purely universal sentences. Then, there exist closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega + \Delta \vdash \forall \underline{b}\ A_D(\underline{x}, \underline{t}\underline{x}, \underline{b}).$$

In the following, we state that every formula is equivalent to its interpretation if the principles above are added to HA^ω:

Theorem 5 (**Characterization**). *For each formula A of $\mathcal{L}^\omega$, we have*

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash A \leftrightarrow A^D.$$

In [Fer09], Ferreira observes that the characterization theorem carries out more information than the one stated in it. In fact, it ensures that there are no missing principles besides AC^ω, $\mathsf{IP}^\omega_\forall$ and M^ω: assume, for instance, that there is another characteristic principle, P (different from AC^ω, $\mathsf{IP}^\omega_\forall$, M^ω). Clearly, $\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \mathrm{P} \vdash \mathrm{P}$. By the soundness theorem, we get $\mathsf{HA}^\omega \vdash \mathrm{P}^D$. But now, $\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega$ proves $\mathrm{P} \leftrightarrow \mathrm{P}^D$. Hence, $\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \mathrm{P}$, meaning that the principle P is superfluous.

Theorem 6 (**Main theorem on program extraction by Dialectica interpretation**). *Let $A_{qf}(\underline{x}, \underline{y})$ be a quantifier-free formula whose free variables are among $\underline{x}, \underline{y}$ and $B(\underline{x}, \underline{z})$ an arbitrary formula whose only free variables are $\underline{x}, \underline{z}$. If*

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash \forall \underline{x} \ \big(\forall \underline{y} \ A_{qf}(\underline{x}, \underline{y}) \to \exists \underline{z} \ B(\underline{x}, \underline{z})\big)$$

where Δ is a set of purely universal sentences, then we can extract closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash \forall \underline{x} \ \big(\forall \underline{y} \ A_{qf}(\underline{x}, \underline{y}) \to B(\underline{x}, \underline{tx})\big).$$

Moreover, if $\mathcal{S}^\omega \vDash \Delta$, then the conclusion holds in $\mathcal{S}^\omega$.

In particular, we have the following:

Theorem 7 (**Program extraction**). *Let $A_{qf}(\underline{x}, \underline{y})$ be a quantifier-free formula of $\mathcal{L}^\omega$ whose free variables are among $\underline{x}, \underline{y}$. If*

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash \forall \underline{x} \exists \underline{y} \ A_{qf}(\underline{x}, \underline{y}),$$

where Δ is a set of purely universal sentences of the language. Then, there can be extracted closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega + \Delta \vdash \forall \underline{x} \ A_{qf}(\underline{x}, \underline{tx}).$$

Moreover, if $\mathcal{S}^\omega \vDash \Delta$, then the conclusion holds in $\mathcal{S}^\omega$.

Theorem 8 (**Conservation**). *Let $A_{qf}(\underline{x}, \underline{y})$ be a quantifier-free formula of HA^ω whose free variables are among $\underline{x}$ and $\underline{y}$. If*

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \forall \underline{x} \exists \underline{y} \ A_{qf}(\underline{x}, \underline{y}),$$

then

$$\mathsf{HA}^\omega \vdash \forall \underline{x} \exists \underline{y} \ A_{qf}(\underline{x}, \underline{y}).$$

2.2.2 The Dialectica interpretation of $\neg\neg A \wedge \neg\neg B \rightarrow \neg\neg(A \wedge B)$

In this section, we focus on an example. We carry out explicitly the functional interpretation of $\neg\neg A \wedge \neg\neg B \rightarrow \neg\neg(A \wedge B)$, which is a theorem in HA^ω, then it must have a functional interpretation (for a good understanding of this example, see [Oli06]). A straightforward calculation shows that, in order to carry out this interpretation, we must produce terms $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2$ such that

$$\forall \varphi_1, \varphi_2, g_1, g_2 \ \big(\big(\neg\neg\ A_D(\varphi_1\Lambda_1, \Lambda_1(\varphi_1\Lambda_1) \wedge \neg\neg B_D(\varphi_2\Lambda_2, \Lambda_2(\varphi_2\Lambda_2)\big) \rightarrow \\ \rightarrow \neg\neg\big(A_D(\Gamma_1, g_1\Gamma_1\Gamma_2) \wedge B_D(\Gamma_2, g_2\Gamma_2\Gamma_2)\big)\big).$$

It suffices to solve

$$\begin{cases} \varphi_i\Lambda_i = \Gamma_i \\ \Lambda_i(\varphi_i\Lambda_i) = g_i\underline{\Gamma} \end{cases}$$

for $i \in \{1, 2\}$ and $\underline{\Gamma}$ being the tuple Γ_1, Γ_2. This seems to carry a circularity problem, since we need Λ_i to get Γ_i and we also need Γ_i to get Λ_i. Nevertheless, this can be solved by supposing we have Γ_1. Then, $\Lambda_2 = \lambda y.g_2\Gamma_1 y$ and $\Gamma_2 = \varphi_2\Lambda_2$. Now, we can construct $\Lambda_1 = \lambda x.g_1 x(\varphi_2(\lambda y.g_2 xy))$ and $\Gamma_1 = \varphi_1\Lambda_1$. Define

$$\Lambda_1 = \lambda x.g_1 x(\varphi_2(\lambda y.g_2 xy)).$$

Take Γ_1, Λ_2 and Γ_2 as defined above. These terms interpret $\neg\neg A \wedge \neg\neg B \rightarrow \neg\neg(A \rightarrow B)$.

2.2.3 Negative translation and Dialectica interpretation

There are several negative translations of the classical into the intuitionistic logic. These translations assign to each formula A a formula A'. The first negative translation was due to Gödel [Göd33] and Gentzen in 1933 and was refined later by S. Kuroda and others. In this work, we adopt Kuroda negative translation ([Kur51]):

Definition 3. *Let A be a formula of the language $\mathcal{L}^\omega$. The negative translation of A, denoted by A', is $\neg\neg A^\dagger$ with $A^\dagger$ defined recursively by*

1. *$A^\dagger$ is A if A is an atomic formula;*
2. *$(A\Box B)^\dagger$ is $A^\dagger \Box B^\dagger$, where $\Box \in \{\wedge, \vee, \rightarrow\}$;*
3. *$\big(\forall x^\rho\ A(x)\big)^\dagger$ is $\forall x^\rho\ \neg\neg\big(A(x)\big)^\dagger$;*
4. *$\big(\exists x^\rho\ A(x)\big)^\dagger$ is $\exists x^\rho\ \big(A(x)\big)^\dagger$.*

The negative translation A' of A is intuitionistically equivalent to a *negative formula* (negative formulas are the ones build up from negated atomic formulas solely by means of $\wedge, \rightarrow$ and $\forall$).

Proposition 3. *Let A be a formula of PA^ω. If $\mathsf{PA}^\omega \vdash A$, then $\mathsf{HA}^\omega \vdash A'$.*

In the following, we denote the axiom of choice for quantifier-free matrices by $\mathsf{AC}^\omega_{\mathrm{qf}}$.

Theorem 9. *Let A be a formula of PA^ω and let Δ be a set of purely universal sentences. If*

$$\mathsf{PA}^\omega + \mathsf{AC}^\omega_{qf} + \Delta \vdash A$$

then

$$\mathsf{HA}^\omega + \mathsf{AC}^\omega_{qf} + \mathsf{M}^\omega + \Delta \vdash A'.$$

In the previous theorem, if we have AC^ω instead of $\mathsf{AC}^\omega_{\mathrm{qf}}$, the result is not true .

Theorem 10 (Extraction and conservation). *Let $A_{qf}(\underline{x}, \underline{y})$ be a quantifier free formula in PA^ω whose free variables are among $\underline{x}$ and $\underline{y}$ and Δ be a set of universal sentences. Suppose that*

$$\mathsf{PA}^\omega + \mathsf{AC}^\omega_{qf} + \Delta \vdash \forall \underline{x} \exists \underline{y}\ A_{qf}(\underline{x}, \underline{y})$$

then, there is closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{PA}^\omega + \Delta \vdash \forall \underline{x}\ A_{qf}(\underline{x}, \underline{tx}).$$

In particular, if $\mathcal{S}^\omega \vDash \Delta$, then the conclusion holds in $\mathcal{S}^\omega$.

2.3 Bounded functional interpretation

In this section, we describe the bounded functional interpretation within the theory $\mathsf{HA}^\omega_\unlhd$. We follow closely [FO05], omitting all proofs.

2.3.1 The intensional theory $\mathsf{HA}^\omega_\unlhd$

$\mathsf{HA}^\omega_\unlhd$ is an extension of HA^ω with language $\mathcal{L}^\omega_\unlhd$. This language is the extension of $\mathcal{L}^\omega$, obtained by joining a new primitive binary relation symbol $\unlhd_\rho$ for each type ρ, where $\unlhd_\rho$ is the "intentional" counterpart of $\leq^*_\rho$. The terms of $\mathcal{L}^\omega_\unlhd$ are the ones in $\mathcal{L}^\omega$. The new atomic formulas of the language are of the form $s \unlhd_\rho t$ where s and t are terms of type ρ. In the language, there are also new quantifiers, *bounded quantifiers*, of the form $\forall x \unlhd t\ A(x)$ and $\exists x \unlhd t\ A(x)$ for terms t not containing x. Formulas in which every quantifier is bounded are called *bounded formulas*.

The theory $\mathsf{HA}^\omega_\unlhd$ in the language $\mathcal{L}^\omega_\unlhd$ is the extension of HA^ω which has the additional axioms $\mathsf{B}_\exists$, $\mathsf{B}_\forall$, M_1, M_2 and the rule $\mathsf{RL}_\unlhd$:

$$\begin{array}{ll} \mathsf{B}_\forall : & \forall x \unlhd t\ A(x) \leftrightarrow \forall x\ (x \unlhd t \to A(x)) \\ \mathsf{B}_\exists : & \exists x \unlhd t\ A(x) \leftrightarrow \exists x\ (x \unlhd t \wedge A(x)) \end{array}$$

where t is a term not containing x,

$$\begin{array}{ll} \mathsf{M}_1 : & x \unlhd_0 y \leftrightarrow x \leq_0 y \\ \mathsf{M}_2 : & x \unlhd_{\rho\to\sigma} y \to \forall u \unlhd_\rho v\ (xu \unlhd_\sigma yv \wedge yu \unlhd_\sigma yv) \\ \mathsf{RL}_\unlhd : & \dfrac{A_{bd} \wedge u \unlhd_\rho v \to su \unlhd_\sigma tv \wedge tu \unlhd_\sigma tv}{A_{bd} \to s \unlhd_{\rho\to\sigma} t} \end{array}$$

where A_{bd} is a bounded formula, s and t are terms in the language and u and v are variables that do not occur free in the conclusion.

Moreover, the induction axiom is extended to all formulas of $\mathcal{L}^{\omega}_{\trianglelefteq}$.

In the following, we state some results:

Lemma 3. HA^{ω} *proves*

i) $x \trianglelefteq y \to y \trianglelefteq y$;

ii) $x \trianglelefteq y \land y \trianglelefteq z \to x \trianglelefteq z$;

iii) $x \trianglelefteq_1 y \to x \leq_1^* y$.

Proposition 4. $\mathsf{HA}^{\omega}_{\trianglelefteq}$ *proves that the axioms* E, E_{Π}, E_{Σ} *and* $E_{\underline{R}}$ *generalize for every formula of* $\mathcal{L}^{\omega}_{\trianglelefteq}$.

Proposition 5. HA^{ω} *proves that* $\Pi \trianglelefteq \Pi$, $\Sigma \trianglelefteq \Sigma$ *and* $\underline{R} \trianglelefteq \underline{R}$.

Definition 4. *A theory* $\mathsf{T}_{\trianglelefteq}$ *with language* $\mathcal{L}_{\trianglelefteq}$ *is called a* majorizability theory *if for every constant* c^{ρ} *of* $\mathcal{L}_{\trianglelefteq}$ *there exists a closed term* t^{ρ} *such that* $\mathsf{T}_{\trianglelefteq} \vdash c \trianglelefteq_{\rho} t$.

Theorem 11. $\mathsf{HA}^{\omega}_{\trianglelefteq}$ *is a majorizability theory.*

In majorizability theories, it can be defined the maximum function for higher types by

$$\max_{\rho\to\sigma}(u,v) = \lambda z^{\rho}. \max_{\sigma}(uz, vz).$$

Recall that $\max_0$ is the usual maximum between natural numbers.

Lemma 4. $\mathsf{HA}^{\omega}_{\trianglelefteq}$ *proves*

i) $x \trianglelefteq x \land y \trianglelefteq y \to x \trianglelefteq \max(x,y) \land y \trianglelefteq \max(x,y)$;

ii) $\max \trianglelefteq \max$.

The notion of maximum of two objects can be generalized to the maximum of a set of objects. To do so, define $\max^{0\to((0\to\rho)\to\rho)}$, given recursively by:

$i)$ $\max_{i\leq 0} si = s0$;

$ii)$ $\max_{i\leq n+1} si = \max\big(\max_{i\leq n} si, s(n+1)\big)$.

Although these two maximum functions are different, we denote both of them by max to ease the readability.

Lemma 5. $\mathsf{HA}^{\omega}_{\trianglelefteq}$ *proves*

i) $\forall i \leq n\ \ (si \trianglelefteq ri) \to \max_{i\leq n} si \trianglelefteq \max_{i\leq n} ri$;

ii) $\forall i \leq n\ \ (si \trianglelefteq ri) \to \forall i \leq n\ \ (si \trianglelefteq \max_{k\leq i} rk)$;

iii) $\forall i \leq n \ (si \trianglelefteq ri) \rightarrow \forall j \leq n \forall i \leq j \ (\max_{k \leq i} sk \trianglelefteq \max_{k \leq j} rk)$

for all n^0 *and* s, r *of type* $0 \rightarrow \rho$.

Proof These results are easily obtained by induction on n, using the previous lemma. To be precise, *i*) and *iii*) are proved by induction on n, while *ii*) only depends on *i*) and on the previous lemma. □

The definition x^M for $x \in M_\rho^{\mathbb{N}}$ with $\rho = \rho_1 \rightarrow (\rho_2 \rightarrow (\rho_3 \rightarrow ...(\rho_k \rightarrow 0)))$, can now be generalized for functionals x of type $0 \rightarrow \rho$ with ρ an arbitrary type of T:

Definition 5. *Let* x *be of type* $0 \rightarrow \rho$, $\rho \in \mathsf{T}$. *Then, we define* $x^M n = \max_{i \leq n} xi$.

Observe that from the previous lemma, one can easily prove:

i) if s is monotone, then $s \trianglelefteq s^M$;

ii) $\forall i \leq n \ (si \trianglelefteq ri) \rightarrow \forall i, j \leq n \ \big(i \leq j \rightarrow si \trianglelefteq r^M j \wedge r^M i \trianglelefteq r^M j\big)$

iii) $\forall i \leq n \ (si \trianglelefteq ri) \rightarrow s^M n \trianglelefteq r^M n$

for arbitrary s and r of type $0 \rightarrow \rho$. Moreover, we can also prove

$$s \trianglelefteq r \rightarrow s \trianglelefteq r^M \wedge s^M \trianglelefteq r^M.$$

Lemma 6. *For every closed term* t *of* $\mathcal{L}^\omega_\trianglelefteq$ *there exists another closed term* q *of* $\mathcal{L}^\omega_\trianglelefteq$ *such that* $\mathsf{HA}^\omega_\trianglelefteq \vdash t \trianglelefteq q$.

Definition 6. *An open term* t *in* $\mathcal{L}^\omega_\trianglelefteq$ *with free variables* $\underline{w}$ *has a* majorant $\tilde{t}$ *with the same free variables if*

$$\mathsf{HA}^\omega_\trianglelefteq \vdash \lambda \underline{w}.t \trianglelefteq \lambda \underline{w}.\tilde{t}.$$

A term t *is called* monotone *if it is self-majorizing. A functional* f *is said to be monotone if* $f \trianglelefteq f$.

From the lemma above, it follows that every open term has a majorant.

We use the following abbreviations for monotone quantifications:

$\tilde{\forall} x \ A(x)$ is $\forall x \ \big(x \trianglelefteq x \rightarrow A(x)\big)$;
$\tilde{\exists} x \ A(x)$ is $\exists x \ \big(x \trianglelefteq x \wedge A(x)\big)$.

2.3.2 Bounded functional interpretation

The bounded functional interpretation is defined as follows:

Definition 7. *To each formula A of the language $\mathcal{L}^{\omega}_{\unlhd}$ we associate formulas A^B and A_B of $\mathcal{L}^{\omega}_{\unlhd}$. A_B is a bounded formula and A^B has the form $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{b},\underline{c})$ where $\underline{b}$ and $\underline{c}$ are tuples of variables (possibly empty) whose types depend on the structure of A.*

1. $(A_{bd})^B$ *and* $(A_{bd})_B$ *are* A_{bd} *for bounded formulas* A_{bd}.

If we already have A^B and B^B given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{b},\underline{c})$ and $\tilde{\exists}\underline{d}\tilde{\forall}\underline{e}\ B_B(\underline{d},\underline{e})$ respectively, then

2. $(A \wedge B)^B$ *is* $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}\ \big(A_B(\underline{b},\underline{c}) \wedge B_B(\underline{d},\underline{e})\big)$,

3. $(A \vee B)^B$ *is* $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}\ \big(\tilde{\forall}\underline{c}' \unlhd \underline{c}\ A_B(\underline{b},\underline{c}') \vee \tilde{\forall}\underline{e}' \unlhd \underline{e}\ B_B(\underline{d},\underline{e}'\big)$,

4. $(A \to B)^B$ *is* $\tilde{\exists}\underline{f},\underline{g}\tilde{\forall}\underline{b},\underline{e}\ \big(\tilde{\forall}\underline{c} \unlhd \underline{gbe}\ A_B(\underline{b},\underline{c}) \to B_B(\underline{fb},\underline{e})\big)$,

5. $\big(\forall x \unlhd t\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\forall x \unlhd t\ A_B(x,\underline{b},\underline{c})$,

6. $\big(\exists x \unlhd t\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\exists x \unlhd t\tilde{\forall}\underline{c}' \unlhd \underline{c}\ A_B(x,\underline{b},\underline{c}')$,

7. $\big(\forall x\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{f}\tilde{\forall}a,\underline{c}\forall x \unlhd a\ A_B(x,\underline{f}a,\underline{c})$,

8. $\big(\exists x\ A(x)\big)^B$ *is* $\tilde{\exists}a,\underline{b}\tilde{\forall}\underline{c}\exists x \unlhd a\tilde{\forall}\underline{c}' \unlhd \underline{c}\ A_B(x,\underline{b},\underline{c}')$.

We can see negation as a case of implication and obtain

$$(\neg A)^B \text{is}\ \ \tilde{\exists}\underline{f}\tilde{\forall}\underline{b}\neg\tilde{\forall}\underline{c} \unlhd \underline{fb}\ A_B(\underline{b},\underline{c}).$$

Lemma 7 (Monotonicity Lemma). *Let $A(x)$ be a formula of $\mathcal{L}^{\omega}_{\unlhd}$ and assume that $(A(x))^B$ is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(x,\underline{b},\underline{c})$. Then*

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\unlhd} \vdash \underline{b} \unlhd \underline{b}' \wedge \underline{c} \lhd \underline{c} \wedge A_B(x,\underline{b},\underline{c}) \to A_B(x,\underline{b}',\underline{c}).$$

It is now possible to prove the soundness theorem:

Theorem 12 (Soundness). *Let $A(\underline{z})$ be a formula of the language $\mathcal{L}^{\omega}_{\unlhd}$ with free variables $\underline{z}$. Assume that $(A(\underline{z}))^B$ is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{b},\underline{c})$ and that Δ is a set of universal sentences (with bounded intensional matrices). If*

$$\mathsf{HA}^{\omega}_{\unlhd} + \Delta \vdash A(\underline{z}),$$

then there exist monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^{\omega}_{\unlhd} + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \unlhd \underline{a}\tilde{\forall}\underline{c}\ A(\underline{z},\underline{ta},\underline{c}).$$

Similarly to the Dialectica interpretation, the bounded functional interpretation also interprets some principles beyond those of $\mathsf{HA}^{\omega}_{\unlhd}$. These are:

1. *Bounded choice principle* bAC^ω

$$\mathsf{bAC}^{\rho,\tau}: \quad \forall x^\rho \exists y^\tau A(x,y) \to \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq fb\, A(x,y),$$

where A is an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$;

2. *Bounded independence of premise principle* $\mathsf{bIP}^\omega_{\forall bd}$

$$\mathsf{bIP}^\rho_{\forall bd}: \quad \big(\forall \underline{x}\, A_{bd}(\underline{x}) \to \exists y^\rho\, B(y)\big) \to \tilde{\exists} b \big(\forall \underline{x}\, A_{bd}(\underline{x}) \to \exists y \trianglelefteq b\, B(y)\big),$$

where A_{bd} is a bounded formula and B is an arbitrary formula;

3. *Bounded Markov's principle* bMP^ω_{bd}

$$\mathsf{bMP}^\rho_{bd}: \quad \big(\forall y^\rho \forall \underline{x}\, A_{bd}(\underline{x}, y) \to B_{bd}\big) \to \tilde{\exists} b\, \big(\forall y \trianglelefteq b \forall \underline{x}\, A_{bd}(\underline{x}, y) \to B_{bd}\big),$$

where A_{bd} and B_{bd} are bounded formulas. When B_{bd} is $\perp$ it gives

$$\neg \forall y \forall \underline{x}\, A_{bd}(\underline{x}, y) \to \tilde{\exists} b \neg \forall y \trianglelefteq b \forall \underline{x}\, A_{bd}(\underline{x}, y)$$

which implies the particular version

$$\neg\neg \exists y\, A_{bd}(y) \to \tilde{\exists} b \neg\neg \exists y \trianglelefteq b\, A_{bd}(y);$$

4. *Bounded universal disjunction principle* $\mathsf{bUD}^\omega_{\forall bd}$

$$\mathsf{bUD}^{\underline{\rho},\underline{\tau}}_{\forall bd}: \quad \tilde{\forall} \underline{b}^{\underline{\rho}} \tilde{\forall} \underline{c}^{\underline{\tau}}\, \big(\forall \underline{x} \trianglelefteq \underline{b}\, A_{bd}(\underline{x}) \vee \forall \underline{y} \trianglelefteq \underline{c}\, B_{bd}(\underline{y})\big) \to \forall \underline{x}\, A_{bd}(\underline{x}) \vee \forall \underline{y}\, B_{bd}(\underline{y}),$$

where A_{bd} and B_{bd} are bounded formulas;

5. *Bounded contra collection principle* $\mathsf{bBCC}^\omega_{bd}$

$$\mathsf{bBCC}^{\rho,\underline{\tau}}_{bd}: \quad \tilde{\forall} c^\rho \big(\tilde{\forall} \underline{b}^{\underline{\tau}} \exists z \trianglelefteq c \forall \underline{y} \trianglelefteq \underline{b}\, A_{bd}(\underline{y}, z) \to \exists z \trianglelefteq c \forall \underline{y}\, A_{bd}(\underline{y}, z)\big),$$

where A_{bd} is a bounded formula;

6. *Majorizability axioms* MAJ^ω

$$\mathsf{MAJ}^\rho: \quad \forall x^\rho \exists y^\rho\, (x \trianglelefteq y).$$

In [Fer09], Ferreira refers about 5., that from the weaker statement, saying that for each monotone b, there exists $z \trianglelefteq c$ such that $\forall y \trianglelefteq b\ A_b(y,z)$, one gets a stronger statement: theres exists a z (in fact, $z \trianglelefteq c$) such that we have $A_b(y,z)$ *for all* y. This element z works uniformly for each b. In this sense, it may be regarded as an ideal element.

We denote by $\mathsf{P}^\omega[\trianglelefteq]$ the sum total of all the characteristic principles.

Proposition 6. $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{P}^\omega[\trianglelefteq]$ *proves the* Bounded collection principle bBC^ω*:*

$$\mathsf{bBC}^{\rho,\tau}: \quad \tilde{\forall} c^\rho \left(\forall z \trianglelefteq c^\rho \exists y^\tau\ A(y,z) \to \tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b\ A(y,z)\right)$$

where A is an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$.

Theorem 13 (**Soundness extended**)**.** *Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$ with free variables $\underline{z}$. If its bounded interpretation is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A(\underline{z},\underline{b},\underline{c})$ and*

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{P}^\omega[\trianglelefteq] + \Delta \vdash A(\underline{z}),$$

with Δ a set of universal sentences (with bounded intensional matrices), then there exist closed monotone terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega_\trianglelefteq + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A(\underline{z},\underline{ta},\underline{c}).$$

At this point, we point out an axiom, a special form of choice, which we call tame axiom of choice

$$\mathsf{tameAC}: \qquad \tilde{\forall} h \exists f \trianglelefteq h \forall x \left(\exists z \trianglelefteq hx\ A_{bd}(x,z) \to A_{bd}(z, fz)\right),$$

where A_{bd} is a bounded formula of the language $\mathcal{L}^\omega_\trianglelefteq$. Of course, this can be generalized for tuples of variables. Observe that this enables one to write a precise witness instead of bounds for it.

Proposition 7. $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{P}^\omega[\trianglelefteq]$ *proves that* tameAC *is equivalent to a purely universal statement (with bounded intensional matrix).*

Proof Let A_{bd} be any bounded formula of the language $\mathcal{L}^\omega_\trianglelefteq$ and assume

$$\tilde{\forall} h \exists f \trianglelefteq h \forall x \left(\exists z \trianglelefteq hx\ A_{bd}(x,z) \to A_{bd}(x, fx)\right).$$

Since $\left(\exists z \trianglelefteq hx\ A_{bd}(x,z) \to A_{bd}(x,fx)\right)$ is bounded, by the bounded contra collection principle, the latter is equivalent to

$$\tilde{\forall} a, h \exists f \trianglelefteq h \forall x \trianglelefteq a \left(\exists z \trianglelefteq hx\ A_{bd}(x,z) \to A_{bd}(x, fx)\right)$$

which is a purely universal statement (with intensional bounded matrix). $\square$

The principle tameAC can be added to theory $\mathsf{HA}^\omega_\trianglelefteq$ and still have a soundness theorem:

Proposition 8. *Let $A(\underline{z})$ be an arbitrary formula of the language $\mathcal{L}^\omega_\trianglelefteq$ whose bounded functional interpretation is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{b},\underline{c})$. If*

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{tameAC} + \mathsf{P}^\omega[\trianglelefteq] \vdash A(\underline{z})$$

then, there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{tameAC} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{ta},\underline{c}).$$

Proof The only thing to do is to prove that $\mathsf{HA}^\omega_{\trianglelefteq}$+$\mathsf{tameAC}$ proves the bounded functional interpretation of tameAC, which is given by:

$$\tilde{\forall} a, h \exists f \trianglelefteq h \tilde{\forall} a' \trianglelefteq a \forall x \trianglelefteq a' \left(\exists z \trianglelefteq hx\ A_{bd}(x,z) \to A_{bd}(x, fx) \right)$$

where A_{bd} is a bounded formula of the language. This is an immediate consequence of tameAC. □

Consequently,

Proposition 9. *Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}$ with free variables among $\underline{z}$ and assume its bounded functional interpretation is given by $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c}\ A_B(\underline{z}, \underline{b}, \underline{c})$. If*

$$\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{P}^\omega[\trianglelefteq] + \mathsf{tameAC} + \Delta \vdash A(\underline{z})$$

then there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^\omega_{\trianglelefteq} + \Delta \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c}\ A_B(\underline{z}, \underline{ta}, \underline{c}),$$

where Δ is a set of all *purely universal statements (with bounded intensional matrices).*

Proof Observe that $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{P}^\omega[\trianglelefteq] + \Delta \vdash \mathsf{tameAC}$ since in $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{P}^\omega[\trianglelefteq]$, tameAC is equivalent to a universal statement (with intensional bounded matrix). □

In analogy with the Dialectica interpretation, there are also characterization and extraction theorems:

Theorem 14 (**Characterization**). *Let A be an arbitrary formula of $\mathcal{L}^\omega_{\trianglelefteq}$, then*

$$\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{P}^\omega[\trianglelefteq] \vdash A \leftrightarrow (A)^B.$$

Theorem 15 (**Program extraction**). *Let $A_{bd}(x,y)$ be a bounded formula of $\mathcal{L}^\omega_{\trianglelefteq}$ whose only free variables are x and y. If*

$$\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{P}^\omega[\trianglelefteq] \vdash \forall x \exists y\ A_{bd}(x,y)$$

then there is a monotone closed term t of the language such that

$$\mathsf{HA}^\omega_{\trianglelefteq} \vdash \tilde{\forall} a \forall x \trianglelefteq a \exists y \trianglelefteq ta\ A_{bd}(x,y).$$

2.3.3 The bounded functional interpretation of $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$

We focus again in $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$. As before, this intuitionistic principle must have a bounded functional interpretation. In fact, the terms which interpret it via the Dialectica interpretation are the same used to the bounded functional interpretation.

Nevertheless, to verify that these terms are the ones to realize the interpretation is quite more difficult.

Take A and B arbitrary formulas of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$. Suppose A^B is $\tilde{\exists} a_1 \tilde{\forall} b_1\ A_B(a_1, b_1)$ and B^B is $\tilde{\exists} a_2 \tilde{\forall} b_2\ B_B(a_2, b_2)$. In fact, we should have written (possibly empty) tuples of variables in the previous quantifications. Nevertheless, we will omit them to ease the reading. A straightforward computation shows that we must produce monotone a_1^*, a_2^*, g_1^* and g_2^*, depending only on monotone f_1, f_2, ϕ_1 and ϕ_2, such that

$$\tilde{\forall} g_1 \trianglelefteq g_1^* \neg \tilde{\forall} a_1 \trianglelefteq \phi_1 g_1 \neg \tilde{\forall} b_1 \trianglelefteq g_1 a_1\ A_B(a_1, b_1) \tag{2.1}$$
$$\tilde{\forall} g_2 \trianglelefteq g_2^* \neg \tilde{\forall} a_2 \trianglelefteq \phi_2 g_2 \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2\ B_B(a_2, b_2) \tag{2.2}$$
$$\tilde{\forall} a_1 \trianglelefteq a_1^*, a_2 \trianglelefteq a_2^* \neg \tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2 \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2\ \big(A_B(a_1, b_1) \wedge B_B(a_2, b_2)\big) \tag{2.3}$$

lead to a contradiction.

Take

$$\begin{aligned} g_1^* &= \lambda x. f_1 x(\phi_2(\lambda y. f_2 x y)) \\ a_1^* &= \phi_1 g_1^* \\ g_2^* &= \lambda y. f_2 a_1^* y \\ a_2^* &= \phi_2 g_2^*, \end{aligned}$$

which, similarly to the Dialectica interpretation of the same principle, are the solution of

$$\begin{cases} a_i^* = \phi_i g_i^* \\ g_i^* a_i^* = f_i a_1^* a_2^*. \end{cases}$$

Since f_1, f_2, ϕ_1 and ϕ_2 are monotone, it follows that $g_1^*, g_2^*, a_1^*, a_2^*$ are also monotone (this relies mainly in the rule $\mathsf{RL}_{\trianglelefteq}$). Assume we have (2.1),(2.2) and (2.3) for $g_1^*, g_2^*, a_1^*, a_2^*$ defined above. We must reach a contradiction.

Take monotone a_1 with $a_1 \trianglelefteq a_1^*$ and define $g_2 = \lambda y. f_2 a_1 y$. Take, now, a monotone a_2 with $a_2 \trianglelefteq \phi_2 g_2$. Then $g_2 \trianglelefteq g_2^*$ and $a_2 \trianglelefteq a_2^*$. We get

$$\begin{aligned} &\tilde{\forall} b_1 \trianglelefteq g_1^* a_1\ A_B(a_1, b_1) \wedge \tilde{\forall} b_2 \trianglelefteq g_2 a_2\ B_B(a_2, b_2) \to \\ &\quad \to \tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2\ A_B(a_1, b_1) \wedge \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2\ B_B(a_2, b_2) \end{aligned}$$

because we have $f_1 a_1 a_2 \trianglelefteq g_1^* a_1$ by the definition of g_1^* and the fact that $a_2 \trianglelefteq \phi_2 g_2$. Note, also, that $g_2 a_2 = f_2 a_1 a_2$. By (2.3),

$$\tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2 \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2\ \big(A_B(a_1, b_1) \wedge B_B(a_2, b_2)\big) \to \bot.$$

Hence, we may conclude $\tilde{\forall} b_1 \trianglelefteq g_1^* a_1\ A_B(a_1, b_1) \to \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2\ B_B(a_2, b_2)$. Due to the arbitrariness of a_2, we get

$$\tilde{\forall} b_1 \trianglelefteq g_1^* a_1\ A_B(a_1, b_1) \to \tilde{\forall} a_2 \trianglelefteq \phi_2 g_2 \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2\ B_B(a_2, b_2).$$

By (2.2), $\neg \tilde{\forall} b_1 \trianglelefteq g_1^* a_1\ A_B(a_1, b_1)$. Due to the arbitrariness of a_1 and noticing that $a_1^* = \phi_1 g_1^*$, we conclude

$$\tilde{\forall} a_1 \trianglelefteq \phi_1 g_1^* \neg \tilde{\forall} b_1 \trianglelefteq g_1^* a_1\ A_B(a_1, b_1),$$

which contradicts (2.1), when taking g_1 as g_1^*.

This not so simple interpretation will give some insight to carry out the bounded functional interpretation of the *double negation shift* principle

$$\mathsf{DNS}: \quad \forall n^0\, \neg\neg A(n) \to \neg\neg \forall n^0\ A(n).$$

2.3.4 Negative translation and bounded functional interpretation

We begin by extending Kuroda's negative translation to bounded quantifications:

Definition 8. *Let A be an arbitrary formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then Kuroda's negative translation A' is $\neg\neg A^{\dagger}$. The translation from A in $A^{\dagger}$ maintains unchanged atomic formulas, conjunctions, disjunctions, implications and existential quantifications. The translation of an universal quantification inserts a double negation after the quantification. For bounded quantifications, it is defined as follows:*

i) $(\forall x \trianglelefteq t\ A(x))^{\dagger}$ *is* $\forall x \trianglelefteq t\ \neg\neg(A(x))^{\dagger}$*;*

ii) $(\exists x \trianglelefteq t\ A(x))^{\dagger}$ *is* $\exists x \trianglelefteq t\ (A(x))^{\dagger}$*.*

We denote by $\mathsf{P}^{\omega}_{bd}[\trianglelefteq]$ the modification of $\mathsf{P}^{\omega}[\trianglelefteq]$ in which bAC^{ω} is replaced by $\mathsf{bAC}^{\omega}_{bd}$, the restriction of bAC^{ω} to bounded matrices.

The following results are obtained:

Theorem 16. *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$. If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{bd}[\trianglelefteq] \vdash A$$

then

$$\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{bd}[\trianglelefteq] \vdash A'.$$

Theorem 17 (**Extraction and Conservation**). *Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$ with free variables $\underline{z}$. Suppose that $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ (A')_B(\underline{z},\underline{b},\underline{c})$ is the bounded functional interpretation of the negative translation of A. If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{bd}[\trianglelefteq] \vdash A(\underline{z})$$

then there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ (A')_B(\underline{z},\underline{ta},\underline{c}).$$

Although these results were proved in the extended theories $\mathsf{HA}^{\omega}_{\trianglelefteq}$ and $\mathsf{PA}^{\omega}_{\trianglelefteq}$, one can associate to each formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$ a corresponding formula of $\mathcal{L}^{\omega}$ by replacing each occurrence of the intensional relation $\trianglelefteq_{\rho}$ by the extensional one, $\leq^{*}_{\rho}$:

Definition 9. *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then, it is defined by recursion on A a corresponding formula A^{*} of $\mathcal{L}^{\omega}$:*

1. *If A is an atomic formula with no occurrence of $\trianglelefteq$, then A^{*} is A;*

2. $(t \trianglelefteq_{\rho} q)^{*}$ *is* $t \leq^{*}_{\rho} q$ *for all* $\rho \in \mathsf{T}$*;*

3. $(A \Box B)^{*}$ *is* $A^{*}\Box B^{*}$ *for* $\Box \in \{\wedge, \vee \rightarrow\}$*;*

4. $(\forall x\ A)^{*}$ *is* $\forall x\ A^{*}$*;*

5. $(\exists x\ A)^*$ *is* $\exists x\ A^*$*;*

6. $(\forall x \trianglelefteq_\rho t\ A)^*$ *is* $\forall x\ \big(x \leq^*_\rho t \to A^*\big)$ *for all* $\rho \in \mathsf{T}$*;*

7. $(\exists x \trianglelefteq_\rho t\ A)^*$ *is* $\exists x\ \big(x \leq^*_\rho t \wedge A^*\big)$ *for all* $\rho \in \mathsf{T}$*.*

A^* *is called the* flattening *of* A*.*

The following result is clear:

Theorem 18 (**Flattening**)**.** *Let* A *be an arbitrary formula of* $\mathcal{L}^\omega_\trianglelefteq$*. If*

$$\mathsf{HA}^\omega_\trianglelefteq \vdash A$$

then

$$\mathsf{HA}^\omega \vdash A^*.$$

As mentioned above, while presenting the soundness theorem, we can join tameAC to $\mathsf{HA}^\omega_\trianglelefteq$ and still have a soundness theorem. In fact, in order to $\mathcal{S}^\omega$ be a model of the flattening of the theory $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{tameAC}$, we must check that the flattening of tameAC is set theoretically true:

Proposition 10.

$$\mathcal{S}^\omega \vDash (\mathsf{tameAC})^*.$$

Moreover, $(\mathsf{tameAC})^*$ *also holds in* $\mathcal{M}^\omega$*.*

Proof Our aim is to prove that the flattening of the instances of tameAC are true in $\mathcal{S}^\omega$. Take A a bounded formula (in the extensional sense). We want to show that

$$\forall h\ \ (h \leq^* h \to \exists f \leq^* h \forall x\ \ (\exists z \leq^* h(x)\ A(x,z) \to A(x, f(x))))$$

holds in $\mathcal{S}^\omega$. Let us fix $h \in S_{\rho\to\sigma}$ such that $h \leq^* h$. It is clear that

$$\forall x\ \ (\exists z \leq^* h(x)\ A(x,z) \to \exists y \leq^* h(x)\ A(x,y))$$

is true. This can be written as

$$\forall x \exists y \leq^* h(x)\ \big(\exists z \leq^* h(x)\ A(x,z) \to A(x,y)\big).$$

By the axiom of choice in the real world $\mathcal{S}^\omega$, there exists f such that

$$\forall x\ \ (f(x) \leq^* h(x) \wedge (\exists z \leq^* h(x)\ A(x,z) \to A(x, f(x))))\,.$$

Clearly, $f \leq^* h$ and $\forall x\ \big(\exists z \leq^* h(x)\ A(x,z) \to A(x, f(x))\big)$, as desired. Moreover, if we fix $h \in M_{\rho\to\sigma}$, by the argument above, f is also in $M_{\rho\to\sigma}$. $\square$

2.3.5 Direct bounded functional interpretation of Peano arithmetic

In the previous section, Peano arithmetic is interpreted in two steps (similarly to Gödel's Dialectica interpretation). First, Peano arithmetic is interpreted into Heyting arithmetic by a negative translation, and second, Heyting arithmetic is interpreted by the bounded functional interpretation. In 1967, Shoenfield defined a direct functional interpretation of Peano arithmetic [Sho67]. Recently, Ferreira defined, in [Fer09], a direct bounded functional interpretation of Peano arithmetic, in the style of Shoenfield. This is presented in this section.

Let the language of (intensional) Peano arithmetic, $\mathsf{PA}^{\omega}_{\trianglelefteq}$, be $\mathcal{L}^{\omega}_{\trianglelefteq}$ restricted to the logical words $\neg, \vee, \forall$ and to the bounded quantifier $\forall x \trianglelefteq t$, since the other logical connectives are defined classically in the usual manner. As so, we will no longer consider axiom $\mathsf{B}_{\exists}$ in this direct interpretation.

Definition 10. *To each formula A of the language $\mathcal{L}^{\omega}$ we assign formulas A^U and A_U, such that A^U is of the form $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\ A_U(\underline{b},\underline{c})$ and A_U is bounded, according to*

1. *A^U and A_U are A for A atomic formula.*

If A and B have interpretations given by $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\ A_U(\underline{b},\underline{c})$ and $\tilde{\forall}\underline{d}\tilde{\exists}\underline{d}\ B_U(\underline{d},\underline{e})$, respectively, then the remaining cases are defined as follows:

2. *$(A \vee B)^U$ is $\tilde{\forall}\underline{b},\underline{d}\tilde{\exists}\underline{c},\underline{e}\ (A_U(\underline{b},\underline{c}) \vee B_U(\underline{d},\underline{e}))$;*
3. *$(\forall x\ A(x))^U$ is $\tilde{\forall}a\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq a\ A_U(x,\underline{b},\underline{c})$;*
4. *$(\neg A)^U$ is $\tilde{\forall}\underline{f}\tilde{\exists}\underline{b}\tilde{\exists}\underline{b}' \trianglelefteq \underline{b}\ \neg A_U(\underline{b}',\underline{f}\underline{b}')$;*
5. *$(\forall x \trianglelefteq t\ A(x))^U$ is $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq t\ A_U(x,\underline{b},\underline{c})$.*

Lemma 8. *Let A be a formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall}\underline{b}\tilde{\forall}\underline{c}\tilde{\forall}\underline{c}' \trianglelefteq \underline{c}\ \big(A_U(\underline{b},\underline{c}') \to A_U(\underline{b},\underline{c})\big).$$

Since $A \to B$ is defined as $\neg A \vee B$, the interpretation of $A \to B$ is given by

$$\tilde{\forall}\underline{f},\underline{d}\tilde{\exists}\underline{b},\underline{e}\ \big(\tilde{\forall}\underline{b}' \trianglelefteq \underline{b}\ A_U(\underline{b}',\underline{f}\underline{b}') \to B_U(\underline{d},\underline{e})\big).$$

Again, there are some principles that can be added to $\mathsf{PA}^{\omega}_{\trianglelefteq}$ and still have soundness. The characteristic principles are:

1. *Monotone bounded choice* $\mathsf{mAC}^{\omega}_{\mathrm{bd}}$

$$\mathsf{mAC}^{\rho,\sigma}_{bd}: \qquad \tilde{\forall}\underline{b}^{\rho}\tilde{\exists}\underline{c}^{\sigma}\ A_{bd}(\underline{b},\underline{c}) \to \tilde{\exists}\underline{f}\tilde{\forall}\underline{b}\tilde{\exists}\underline{c} \trianglelefteq \underline{f}\underline{b}\ A_{bd}(\underline{b},\underline{c}),$$

where A_{bd} is a bounded formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$;

2. *Bounded collection principle* $\mathsf{bC}^{\omega}_{bd}$

$$\mathsf{bC}^{\rho,\underline{\sigma}}: \qquad \forall z \trianglelefteq c^{\rho} \exists \underline{y}^{\underline{\sigma}}\ A_{bd}(\underline{y}, z) \to \tilde{\exists} \underline{b} \forall z \trianglelefteq c \exists \underline{y} \trianglelefteq \underline{b}\ A_{bd}(\underline{y}, z),$$

where A_{bd} is a bounded formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$;

3. *Majorizability axioms* MAJ^{ω}

$$\mathsf{MAJ}^{\rho}: \qquad \forall x^{\rho} \exists y\ (x \trianglelefteq y).$$

The set consisting on the three characteristic principles is denoted by $\mathsf{P}^{\omega}_{cl}[\trianglelefteq]$, where cl refers to the classical setting.

Recall that when interpreting Peano arithmetic via a negative translation, the characteristic principles we added to the theory were the ones in $\mathsf{P}^{\omega}_{bd}[\trianglelefteq]$. In fact, these principles are classically equivalent to the ones in $\mathsf{P}^{\omega}_{cl}[\trianglelefteq]$:

Proposition 11.

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \mathsf{P}^{\omega}_{cl}[\trianglelefteq] \leftrightarrow \mathsf{P}^{\omega}_{bd}[\trianglelefteq].$$

Proof In classical logic, $\mathsf{bC}^{\omega}_{bd}$ is equivalent to $\mathsf{bBCC}^{\omega}_{bd}$ and the principles $\mathsf{bIP}^{\omega}_{\forall bd}$, $\mathsf{bMP}^{\omega}_{bd}$ and $\mathsf{bUD}^{\omega}_{\forall bd}$ are straightforward consequences of MAJ^{ω} plus classical arguments. It remains to prove that $\mathsf{bAC}^{\omega}_{bd} \leftrightarrow \mathsf{mAC}^{\omega}_{bd}$ under $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$. To the left-to-right implication, assume $\tilde{\forall} b \tilde{\exists} c\ A_{bd}(b, c)$. By $\mathsf{bAC}^{\omega}_{bd}$, theres exists a monotone f satisfying $\tilde{\forall} a \tilde{\forall} b \trianglelefteq a \tilde{\exists} c \trianglelefteq fa\ A_{bd}(b, c)$. The conclusion follows from MAJ^{ω}. For the other implication, suppose we have $\forall x \exists y\ A_{bd}(x, y)$. In particular, $\tilde{\forall} b \forall x \trianglelefteq b \exists y\ A_{bd}(x, y)$. By bC^{ω} and $\mathsf{mAC}^{\omega}_{bd}$, one gets $\tilde{\exists} f \tilde{\forall} b \tilde{\exists} c \trianglelefteq fb \forall x \trianglelefteq b \exists y \trianglelefteq c\ A_{bd}(x, y)$, as desired. □

The theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$ with the characteristic principles is not set-theoretically sound. For instances, it refutes the weakest form of extensionality:

$$\forall \Phi^2 \forall \alpha^1, \beta^1\ (\forall k^0\ (\alpha k = \beta k) \to \Phi\alpha = \Phi\beta).$$

Nevertheless, $\mathsf{PA}^{\omega}_{\trianglelefteq}$ with the three principles is consistent:

Theorem 19 (Soundness). *Let $A(\underline{z})$ be a formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$ with free variables $\underline{z}$. Assume A^B is $\tilde{\forall} \underline{b} \tilde{\exists} \underline{c}\ A_U(\underline{z}, \underline{b}, \underline{c})$ and that Δ is a set of universal sentences (with bounded intensional matrices). If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{cl}[\trianglelefteq] + \Delta \vdash A(\underline{z})$$

then, there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \Delta \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{b}\ A_U(\underline{z}, \underline{b}, \underline{tab}).$$

Consequently,

Theorem 20 (**Conservation**). *Let $A_{bd}(x,y)$ be a bounded formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$, whose only variables are x and y. If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{cl}[\trianglelefteq] \vdash \forall x \exists y\ A_{bd}(x,y),$$

then

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} a \forall x \trianglelefteq a \exists y\ A_{bd}(x,y).$$

As before, in the presence of the characteristic principles, each formula of the language is equivalent to its interpretation:

Theorem 21 (**Characterization**). *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{cl}[\trianglelefteq] \vdash A \leftrightarrow A^{U}.$$

The passageway from the intensional theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$ to PA^{ω} is obtained from the next result:

Theorem 22 (**Flattening**). *Let A be an arbitrary formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$. If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash A$$

then

$$\mathsf{PA}^{\omega} \vdash A^{*}$$

where A^{*} is the flattening of A.

Moreover, A^{*} is true in $\mathcal{S}^{\omega}$ and in $\mathcal{M}^{\omega}$.

Although the soundness theorem guarantees that the theory $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}^{\omega}_{cl}[\trianglelefteq]$ is consistent, its "flattened" version PA^{ω} plus the flattening of the characteristic principles is *inconsistent.* For instance, in $\mathsf{PA}^{\omega} + (\mathsf{bC}^{\omega}_{bd})^{*}$ one proves

$$\exists m \forall \alpha \leq^{*}_{1} 1\ \big(\exists n\ (\alpha n \neq 0) \to \exists n \leq m\ (\alpha n \neq 0)\big)$$

which is clearly false.

3

Full numerical comprehension and bounded functional interpretation

In [Spe62], Spector extended the Dialectica interpretation to second-order arithmetic. This was achieved by means of a well-founded recursion, known as bar recursion. Spector extended it to all finite types. We begin by presenting bar recursive functionals and bar induction (the respective induction). Afterwards, we extend the bounded functional interpretation to bar recursors and prove that the negative translation of the schema of full numerical comprehension has bounded functional interpretation. To do so, bar recursive functionals play the main role in interpreting the double negation shift principle

$$\mathsf{DNS}: \qquad \forall n^0\, \neg\neg A(n) \to \neg\neg\forall n^0\, A(n).$$

We get the bounded functional interpretation of the negative translation of full numerical comprehension relying on the bounded functional interpretation of DNS (plus a special form of choice, tameAC, mentioned in the previous chapter). We also prove that bar induction has a bounded functional interpretation.

3.1 Bounded functional interpretation extended to bar recursors

In this section, we extend the language of $\mathsf{HA}^\omega_\trianglelefteq$ with new constants $B^{\underline{\rho},\underline{\sigma}}$, the *bar recursors*, and consider the following defining axioms $\mathsf{BR}_{\underline{\rho},\underline{\sigma}}$:

$$\begin{aligned}\forall \psi^{(0\to\underline{\rho})\to 0}, z^{\underline{\tau}_1}, u^{\underline{\tau}_2}, n^0, \underline{s}^{0\to\underline{\rho}} \forall i \leq_0 k\ \Big(\big(\psi\overline{\underline{s},n} <_0 n \to B_i^{\underline{\rho},\underline{\sigma}}\psi\underline{z}\underline{u}n\underline{s} =_{\sigma_i} z_i n\overline{\underline{s},n}\big) \wedge \\ \big(\psi\overline{\underline{s},n} \geq_0 n \to B_i^{\underline{\rho},\underline{\sigma}}\psi\underline{z}\underline{u}n\underline{s} =_{\sigma_i} u_i\, \big(\lambda x. B_i^{\underline{\rho},\underline{\sigma}}\psi\underline{z}\underline{u}(n+1)(\overline{\underline{s},n} * x)\big)\, n\overline{\underline{s},n}\big)\Big),\end{aligned}$$

where $\underline{\tau}_1 = (0 \to \big((0 \to \underline{\rho}) \to \underline{\sigma}\big)$, $\underline{\tau}_2 = (\underline{\rho} \to \underline{\sigma}) \to \big(0 \to \big((0 \to \underline{\rho}) \to \sigma\big)\big)$, $\underline{\rho}, \underline{\sigma}$ are tuples of $k+1$ entries and $(\overline{s,n})^{0\to\tau}$ and $(\overline{s,n} * x)^{0\to\tau}$ are defined by

$$\overline{s,n}\ k =_\tau \begin{cases} sk, & \text{if } \ k < n \\ 0, & \text{otherwise} \end{cases}$$

$$(\overline{s,n} * x)k =_\tau \begin{cases} sk, & \text{if } \ k < n \\ x, & \text{if } \ k = n \\ 0, & \text{otherwise.} \end{cases}$$

Note that whereas $s^{0\to\rho}$ denotes an infinite sequence of objects of type ρ, $\overline{s,n}$, although formally of type $0 \to \rho$ as well, stands for the initial subsequence of s with length n, $\langle s_0, s_1, \cdots, s_{n-1}, 0, 0, \cdots\rangle$, and $\overline{s,n} * x$ is the concatenation of the finite sequence $\overline{s,n}$ with x. Let BR denote the collection of all statements $\mathsf{BR}_{\underline{\rho},\underline{\sigma}}$ for $\underline{\rho},\underline{\sigma}$ tuples of types in T. For details, see [Spe62], [Oli06], [Koh08a] or [AF98].

In [Spe62], Spector also presents briefly *bar induction*. It is referred to be a generalization of Brouwer's bar theorem [Bro27] to higher types. While bar recursion is a principle of definition, bar induction is a corresponding principle of proof, in analogy with the usual recursion and induction. The scheme of bar induction applied to formulas P and Q is given by

$$\mathsf{BI}: \qquad \text{Hyp1} \wedge \text{Hyp2} \wedge \text{Hyp3} \wedge \text{Hyp4} \to Q(\overline{0}, 0),$$

where

$$\begin{aligned}
&\text{Hyp1}: && \forall s^{0\to\rho} \exists n^0\ P(\overline{s,n}, n)\\
&\text{Hyp2}: && \forall s^{0\to\rho}, n^0 \forall m \leq_0 n\ \big(P(\overline{s,m}, m) \to P(\overline{s,n}, n)\big)\\
&\text{Hyp3}: && \forall s^{0\to\rho}, n^0\ \big(P(\overline{s,n}, n) \to Q(\overline{s,n}, n)\big)\\
&\text{Hyp4}: && \forall s^{0\to\rho}, n^0\ \big(\forall x^\rho\ Q(\overline{s,n} * x, n+1) \to Q(\overline{s,n}, n)
\end{aligned}$$

and $\overline{0} = \lambda n^0.0^\rho$. The hypothesis Hyp1-Hyp4 also entail $Q(\overline{s,n}, n)$ for all $s^{0\to\rho}$ and n^0 (Hyp2 is essential to obtain this generalization). It is well-known that we can argue by bar induction in the structure of majorizable functionals $\mathcal{M}^\omega$ (see [Koh08a]):

Lemma 9. $\mathcal{M}^\omega \vDash \mathsf{BI}$.

Proof To prove that BI holds in $\mathcal{M}^\omega$, assume Hyp2, Hyp3, Hyp4 and $\neg Q(\overline{0}, 0)$ for P, Q and sequences $s \in M_{0\to\rho}$. Then, we claim $\neg$Hyp1. By the assumption Hyp4, $\exists x_0 \in M_\rho\ \neg Q(\langle x_0, 0, 0, ...\rangle, 1)$. Again, by Hyp4, $\exists x_1 \in M_\rho\ \neg Q(\langle x_0, x_1, 0, 0, ...\rangle, 2)$. Using dependent choice on the meta-level, we get $\tilde{s} \in M_{0\to\rho}$, such that $\forall n^0\ \neg Q(\overline{\tilde{s}, n}, n)$. By Hyp3, we get $\exists \tilde{s} \forall n\ \neg P(\overline{\tilde{s}, n}, n)$, which contradicts Hyp1. $\square$

In this chapter, we work within the theory $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$, where $\Delta_{\mathcal{M}^\omega}$ is the set of all universal sentences (with intensional bounded matrices) whose flattenings hold in the structure $\mathcal{M}^\omega$ of majorizable functionals. The proofs in this chapter rely on some facts of $\Delta_{\mathcal{M}^\omega}$. Although the statements of BR are universal and their flattenings are true in $\mathcal{M}^\omega$, we will use $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ instead of $\mathsf{HA}^\omega_{\trianglelefteq} + \Delta_{\mathcal{M}^\omega}$. This clearly indicates that our language contains the bar recursors functionals.

Theorem 23. $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ *is a majorizability theory.*

Proof It suffices to check that the bar recursive functionals have majorants (within the theory). Let B^* be given by

$$B^* \psi zuns = \max_{i \leq n} B^p \psi zuis$$

where

$$B^p\psi zuns = \begin{cases} zn\overline{s,n}^M & \text{if } \ \psi\overline{s,n}^M < n \\ \max\left(zn\overline{s,n}^M, u\left(\lambda x.B^p\psi zu(n+1)(\overline{s,n} * x)\right) n\overline{s,n}^M\right) & \text{otherwise} \end{cases}$$

Kohlenbach's recent book [Koh08a] is a good reference for the terminology. In there, it is proved that $\mathcal{M}^\omega \vDash B \leq^* B^*$. Hence, the sentence $B \trianglelefteq B^*$ is in $\Delta_{\mathcal{M}^\omega}$. □

Since $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ is a majorizability theory and the sentences of $\mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ are universal (and so, self-interpretable), we have:

Theorem 24 (**Soundness**). *Let $A(\underline{z})$ be a formula of the language of $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ with free variables $\underline{z}$ and assume that its bounded functional interpretation is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z}, \underline{b}, \underline{c})$. If*

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash A(\underline{z})$$

then, there are monotone closed terms $\underline{t}$ of appropriate type such that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z}, \underline{ta}, \underline{c}).$$

Moreover, $\mathcal{M}^\omega \vDash \tilde{\forall}\underline{a}\forall\underline{z} \leq^* \underline{a}\tilde{\forall}\underline{c}\ (A_B)^*(\underline{z}, \underline{ta}, \underline{c})$.

3.2 The bounded functional interpretation of the double negation shift principle

In this section, we carry out explicitly the bounded functional interpretation of DNS. In order to get some intuition on it, recall the bounded functional interpretation of

$$\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B).$$

As we have seen, this interpretation is not straightforward. We follow a similar argument to prove the following theorem:

Theorem 25. DNS *has a bounded functional interpretation in* $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$.

Proof Let $A(n)$ be an arbitrary formula of the language of $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ and suppose that $A^B(n)$ is given by $\tilde{\exists}a\tilde{\forall}b\ A_B(n, a, b)$. A straightforward calculation shows that to interpreted DNS, given monotone ϕ, ψ_1 and ψ_2, one must produce monotone n^*, f^* and g^* (depending only on ϕ, ψ_1 and ψ_2) such that

$$\begin{aligned} &\forall n \leq n^*\tilde{\forall}g \trianglelefteq g^*\neg\tilde{\forall}a \trianglelefteq \phi ng\neg\tilde{\forall}b \trianglelefteq ga\ A_B(n, a, b) \to \\ &\qquad \to \neg\tilde{\forall}f \trianglelefteq f^*\neg\forall n \leq \psi_1 f\tilde{\forall}b \trianglelefteq \psi_2 f\ A_B(n, fn, b) \end{aligned}$$

is provable in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ (to ease the reading, no underlying is used to represent tuples and we write $\leq$ instead of $\leq_0$). Since the above statement is universal, it suffices to show that its flattening

$$\begin{aligned} &\forall n \leq n^*\tilde{\forall}g \leq^* g^*\neg\tilde{\forall}a \leq^* \phi ng\neg\tilde{\forall}b \leq^* ga\ A_B(n, a, b) \to \\ &\qquad \to \neg\tilde{\forall}f \leq^* f^*\neg\forall n \leq \psi_1 f\tilde{\forall}b \leq^* \psi_2 f\ A_B(n, fn, b) \end{aligned}$$

is true in $\mathcal{M}^\omega$, given ϕ, ψ_1 and ψ_2 monotone in the flattened sense (e.g., $\phi \leq^* \phi$). Of course, in the flattened sense, $\tilde{\forall} x \; A(x)$ is the abbreviation of $\forall x \; (x \leq^* x \to A(x))$. As we will show, the latter statement holds in $\mathcal{M}^\omega$, then we can simplify it by replacing the negative universals by the appropriate existencials. Nevertheless, we will argue intuitionistically below since the intuitionistical reasoning is rather elegant (in spite of its complexity): the classical argument is shorter but less natural. Furthermore, the interpretation of the double negation shift also carries out through in weaker theories. For instance, the argument given below can be adapted to show that it holds for the theory $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_i$, where Δ_i is the set of all universal sentences (with intensional bounded matrices) whose flattenings are provable in $\mathsf{E\text{-}HA}^\omega + \mathsf{BR} + \mathsf{BI}$ ($\mathsf{E\text{-}HA}^\omega$ stands for the intensional Heyting arithmetic together with the axiom of full extensionality). Notice that $\Delta_i \subseteq \Delta_{\mathcal{M}^\omega}$.

From here onwards and until the end of this proof, we work with the extensional majorizability symbols $\leq^*$. The statements we prove below are meant to be proved in $\mathcal{M}^\omega$. Given ψ in $M_{(0\to\rho)\to\sigma}$ monotone, we define ψ' by $\psi' s = \psi s^M$.

Take monotone ϕ, ψ_1 and ψ_2. We define $B'ns$ according to the following clauses

$$B'ns := \begin{cases} \overline{s,k} & \text{if } k \leq n, \psi_1' \overline{s,k} < k \text{ and } \forall i < k \; \left(\psi_1' \overline{s,i} \geq i\right) \\ B'(n+1)(\overline{s,n} * c) & \text{otherwise} \end{cases}$$

where n is a natural number, $s \in M_\rho^{\mathbb{N}}$ and

$$\begin{aligned} c &= \phi n g_{\overline{s,n}} \\ g_{\overline{s,n}} &= \lambda x. \psi_2'(B'(n+1)(\overline{s,n} * x)). \end{aligned}$$

The value $B'ns$ is in $M_\rho^{\mathbb{N}}$. In fact, we should think of this value as a finite sequence of elements of M_ρ. It is clear that B' can be defined by bar recursion.

Before we define n^*, f^* and g^*, it is convenient to prove some properties of B'.

Lemma 10. *Take $n \in \mathbb{N}$ and $s \in M_\rho^{\mathbb{N}}$, then*

$$\forall i \leq n \; \left(\psi_1' \overline{s,i} \geq i\right) \to \forall i < n \; \left(\overline{s,n} \; i = B'n(\overline{s,n})i\right).$$

Proof We argue by bar induction. Take

$$\begin{aligned} &P(s,n) = \exists i \leq n \; \left(\psi_1' \overline{s,i} < i\right) \\ &Q(s,n) = \forall i \leq n \; \left(\psi_1' \overline{s,i} \geq i\right) \to \forall i < n \; \left(\overline{s,n} \; i = B'n(\overline{s,n})i\right). \end{aligned}$$

Let us see that we have Hyp1-Hyp4 of bar induction. As we know, Hyp1 holds in the structure of majorizable functionals $\mathcal{M}^\omega$. Hyp2 is trivial. Hyp3 follows from the intuitionistic axiom $\varphi \to (\neg\varphi \to \psi)$. Let us focus on Hyp4. Take arbitrary s and n and assume

$$\begin{aligned} \forall x \; \Big(\forall i \leq n+1 \; \left(\psi_1'(\overline{\overline{s,n} * x, i}) \geq i\right) \to \forall i < n+1 \; \big(\overline{\overline{s,n} * x, n+1} \; i = \\ = B'(n+1)(\overline{\overline{s,n} * x, n+1})i\big)\Big). \end{aligned} \tag{3.1}$$

From the statement above, we want to prove $Q(\overline{s,n},n)$. Assume $\forall i \leq n\ (\psi_1'\overline{s,i} \geq i)$. By definition of B', $B'n(\overline{s,n}) = B'(n+1)(\overline{s,n} * c)$ with c given by

$$c = \phi n(\lambda x.\psi_2'(B'(n+1)(\overline{s,n} * x))).$$

Either $\psi_1'(\overline{s,n} * c) < n+1$ or $\psi_1'(\overline{s,n} * c) \geq n+1$. If the first case occurs, then

$$B'(n+1)(\overline{s,n} * c) = \overline{s,n} * c$$

and also $B'n\overline{s,n} = \overline{s,n} * c$. From this, it follows that $\forall i < n\ \big(\overline{s,n}\ i = B'n(\overline{s,n})i\big)$. On the other hand, if $\psi_1'(\overline{s,n} * c) \geq n+1$, then, by (3.1), we get

$$\forall i < n+1\ \big((\overline{s,n} * c)i = B'(n+1)(\overline{s,n} * c)i\big).$$

Clearly, $\forall i < n\ \ (\overline{s,n}i = B'n(\overline{s,n})i)$. $\square$

Lemma 11. *If $n \in \mathbb{N}$ and $s, r \in M_\rho^{\mathbb{N}}$, then*

$$\forall i < n\ \ (si \leq^* ri) \to \forall j\ \ (B'nsj \leq^* B'nrj)\,.$$

Proof We argue by bar induction. Take P and Q given by

$$\begin{aligned} P(r,n) &= \exists k \leq n\ (\psi_1'(\overline{r,k}) < k) \\ Q(r,n) &= \forall s\ \ (\forall i < n\ (si \leq^* ri) \to \forall j\ \ (B'n\overline{s,n}j \leq^* B'nrj))\,. \end{aligned}$$

As in the lemma above, Hyp1 and Hyp2 hold. Let us focus on Hyp3. Assume we have $P(\overline{r,n},n)$. Take $s \in M_\rho^{\mathbb{N}}$ such that $\forall i < n\ (si \leq^* ri)$. Let k_0 be the least natural number such that $\psi_1'\overline{r,k_0} < k_0$. Note that $k_0 \leq n$. By the definition of B', $B'n\overline{r,n} = B'nr = \overline{r,k_0}$. Since $\overline{s,k_0}^M \leq^* \overline{r,k_0}^M$, by the monotonicity of ψ_1, $\psi_1'\overline{s,k_0} \leq^* \psi_1'\overline{r,k_0} < k_0$. Let k_1 be the least natural number such that $\psi_1'\overline{s,k_1} < k_1$. Note that $k_1 \leq k_0$. Then $B'n\overline{s,n} = B'ns = \overline{s,k_1}$. This entails $\forall j\ \ (B'n\overline{s,n}j \leq^* B'n\overline{r,n}j)$. So, $Q(\overline{r,n},n)$.

It remains to prove Hyp4, i.e., $\forall x\ Q(\overline{r,n} * x, n+1) \to Q(\overline{r,n},n)$. Suppose that $\forall x\ Q(\overline{r,n} * x, n+1)$. If $\exists k \leq n\ \big(\psi_1'\overline{r,k} < k\big)$, then by Hyp3, we get $Q(\overline{r,n},n)$. Otherwise, by definition of B', $B'n\overline{r,n} = B'(n+1)\,(\overline{r,n} * c)$, where $c = \phi n g_{\overline{r,n}}$ and $g_{\overline{r,n}} = \lambda x.\psi_2'\,(B'(n+1)\,(\overline{r,n} * x))$. Recall we want to prove $Q(\overline{r,n},n)$. Take $s \in M_\rho^{\mathbb{N}}$ such that $\forall i < n\ \ (si \leq^* ri)$. We claim that c is monotone (extensionally), i.e., $c \leq^* c$. It suffices to prove that $g_{\overline{r,n}}$ is monotone, i.e., $x \leq^* y \to g_{\overline{r,n}}x \leq^* g_{\overline{r,n}}y$. Take $x \leq^* y$. Clearly, $\forall i < n+1\ \ ((\overline{r,n} * x)\,i \leq^* (\overline{r,n} * y)\,i)$. Then, by the hypothesis of $\forall x\ Q(\overline{r,n} * x, n+1)$, we get

$$\forall j\ \big(B'(n+1)\,(\overline{r,n} * x)\,j \leq^* B'(n+1)\,(\overline{r,n} * y)\,j\big).$$

By the monotonicity of ψ_2, it follows that $g_{\overline{r,n}}x \leq^* g_{\overline{r,n}}y$. Hence c is monotone.

We aim to show that $\forall j\ \ (B'n\overline{s,n}j \leq^* B'n\overline{r,n}j)$. Now, two cases may occur: either $\forall k \leq n\ \big(\psi_1'\overline{s,k} \geq k\big)$ or $\exists k \leq n\ \big(\psi_1'\overline{s,k} < k\big)$. If the first case occurs,

$$B'n\overline{s,n} = B'(n+1)\,(\overline{s,n} * d)$$

where $d = \phi n g_{\overline{s,n}}$ and $g_{\overline{s,n}} = \lambda x.\psi_2' \left(B'(n+1)\left(\overline{s,n} * x\right)\right)$. We claim that $g_{\overline{s,n}} \leq^* g_{\overline{r,n}}$. Take $x \leq^* y$. Then $\forall i < n+1 \ \left(\left(\overline{s,n} * x\right) i \leq^* \left(\overline{r,n} * y\right) i\right)$. By $Q(\overline{r,n} * y, n+1)$ and the monotonicity of ψ_2, the claim follows. Then $d \leq^* c$. By $Q(\overline{r,n} * d, n+1)$, we get

$$\forall j \ \left(B'(n+1)\left(\overline{s,n} * d\right) j \leq^* B'(n+1)\left(\overline{r,n} * c\right)\right).$$

At this point, we only have to observe that $B'(n+1)\left(\overline{s,n} * d\right) = B'n\overline{s,n}$ and $B'(n+1)\left(\overline{r,n} * c\right) = B'n\overline{r,n}$.

Finally, if $\exists k \leq n \ \left(\psi_1'\overline{s,k} < k\right)$, take k_0 the least natural number such that $\psi_2'\overline{s,k_0} < k_0$. Note that $k_0 \leq n$. By definition of B', $B'n\overline{s,n} = \overline{s,k_0}$. Since $\forall k \leq n \ \left(\psi_1'\overline{r,k} \geq k\right)$, by the previous lemma, $\forall i < n \ \left(\overline{r,n}\, i = B'n\overline{r,n}i\right)$. Then

$$\forall j < k_0 \ \left(B'n\overline{s,n}j \leq^* B'n\overline{r,n}j\right).$$

This result also extends for $j \geq k_0$ since $B'n\overline{r,n}j = \left(\overline{r,n} * c\right) j$ is monotone (and then majorizes 0). $\square$

The following is an immediate consequence of the above lemma:

Corollary 2. *Take $n \in \mathbb{N}$ and $s, r \in M_\rho^{\mathbb{N}}$ such that $si \leq^* ri$ for all $i < n$. Then*

$$\lambda x.\psi_2' \left(B'(n+1)\left(\overline{s,n} * x\right)\right) \leq^* \lambda x.\psi_2' \left(B'(n+1)\left(\overline{r,n} * x\right)\right).$$

In particular, given $r \in M_\rho^{\mathbb{N}}$ such that ri is monotone for all $i < n$, then the functional $\lambda x.\psi_2' \left(B'(n+1)(\overline{r,n} * x)\right)$ is monotone.

In order to ease the readability, we write $\langle s_0, s_1, ..., s_{n-1}, 0, 0, ...\rangle$ to denote $s \in M_\rho^{\mathbb{N}}$ such that $si = 0$ for $i \geq n$.

Let us define recursively

$$\begin{aligned} g_0^* &= \lambda x.\psi_2'(B'1\langle x, 0, 0, ...\rangle) \\ a_0^* &= \phi 0 g_0^* \\ g_{i+1}^* &= \lambda x.\psi_2'(B'(i+2)\langle a_0^*, a_1^*, \cdots, a_i^*, x, 0, 0, ...\rangle) \\ a_{i+1}^* &= \phi(i+1) g_{i+1}^*. \end{aligned}$$

By the above corollary, it is clear by induction that the a_i^*'s and the g_i^*'s are monotone. Define

$$\begin{aligned} f^* &= \langle a_0^*, a_1^*, a_2^*, ...\rangle^M \\ n^* &= \psi_1 f^* \\ g^* &= \max_{i \leq n^*} g_i^*. \end{aligned}$$

The monotonicity of the a_i^*'s and the g_i^*'s ensure that f^* and g^* are also monotone. Observe that n^*, f^* and g^* depend only on ϕ, ψ_1 and ψ_2.

In the following, we prove that the statements

$$\forall n \leq n^* \tilde{\forall} g \leq^* g^* \neg \tilde{\forall} a \leq^* \phi n g \neg \tilde{\forall} b \leq^* g a \; A_B(n, a, b) \tag{3.2}$$
$$\tilde{\forall} f \leq^* f^* \neg \forall n \leq \psi_1 f \tilde{\forall} b \leq^* \psi_2 f \; A_B(n, fn, b) \tag{3.3}$$

entail a contradiction.

Before continuing, let us introduce the notion of *nice sequences* and prove some properties.

Definition 11. *A sequence of monotone elements $a_0, a_1, ..., a_n$ of M_ρ is* nice *if for each $0 \leq i \leq n$, $a_i \leq^* \phi i g_i$, where $g_i = \lambda x.\psi_2' \left(B'(i+1)\langle a_0, a_1, ..., a_{i-1}, x, 0, 0, ...\rangle\right)$.*

Note that each g_i depends only on $a_0, a_1, ..., a_{i-1}$.

Lemma 12. *Take $a_0, a_1, ..., a_n$ a nice sequence with associated functions $g_0, g_1, ..., g_n, g_{n+1}$. Then for all $i \leq n+1$, g_i is monotone, $g_i \leq^* g_i^*$ and, for $i \leq n$, $a_i \leq a_i^*$. Moreover, for $i \leq n^*$, $g_i \leq^* g^*$.*

Proof The result is an easy consequence of corollary 2, reasoning by complete induction on $i \leq n$. □

Lemma 13. *Take $a_0, a_1, ..., a_{n^*}$ a nice sequence with associated functions $g_0, g_1, ..., g_{n^*}$, g_{n^*+1}. Then $\forall n < n^* \; (g_{n+1} a_{n+1} \leq^* g_n a_n)$.*

Proof Let $n < n^*$. By definition, we have

$$g_n a_n = \psi_2'(B'(n+1)\langle a_0 \cdots, a_{n-1}, a_n, 0, 0, \cdots\rangle)$$
$$g_{n+1} a_{n+1} = \psi_2'(B'(n+2)\langle a_0 \cdots, a_n, a_{n+1}, 0, 0, \cdots\rangle).$$

We consider two cases. Suppose that exists $k \leq n$ such that $\psi_1'\langle a_0, \cdots, a_k, 0, 0, \cdots\rangle < k+1$. Let k_0 be the least natural number such that $\psi_1'\langle a_0, \cdots, a_{k_0}, 0, 0, \cdots\rangle < k_0 + 1$. Then, by definition of B',

$$B'(n+1)\langle a_0, \cdots, a_{n-1}, a_n, 0, 0, \cdots\rangle = \langle a_0, \cdots, a_{k_0}, 0, 0, \cdots\rangle$$
$$B'(n+2)\langle a_0, \cdots, a_n, a_{n+1}, 0, 0, \cdots\rangle = \langle a_0, \cdots, a_{k_0}, 0, 0, \cdots\rangle$$

Therefore, $g_{n+1} a_{n+1} = g_n a_n$. Note that $g_n a_n$ is monotone since $a_0, ..., a_{k_0}$ are monotone.

For the second case, suppose $\forall k \leq n \; (\psi_1'\langle a_0, \cdots, a_k, 0, 0, \cdots\rangle \geq k+1)$. Then

$$B'(n+1)\langle a_0, \cdots, a_n, 0, 0, \cdots\rangle = B'(n+2)\langle a_0, \cdots, a_n, c, 0, 0, \cdots\rangle,$$

where $c = \phi(n+1) g_{n+1}$. Since, $a_{n+1} \leq^* \phi(n+1) g_{n+1} = c$, then

$$\psi_2'\big(B'(n+2)\langle a_0, \cdots, a_n, a_{n+1}, 0, 0, \cdots\rangle\big) \leq^* \psi_2'\big(B'(n+2)\langle a_0, \cdots, a_n, c, 0, 0, \cdots\rangle\big),$$

as desired. $\square$

Given, $\overline{a} = a_0, a_1, ..., a_{n^*}$ a nice sequence, observe that $\psi_1\langle a_0, a_1, ..., a_{n^*}, 0, 0, ...\rangle^M \leq \psi_1 f^* = n^* < n^*+1$. Let k_0 the least natural number such that $\psi\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M < k_0 + 1$. Note that $k_0 \leq n^*$. Define $f_{\overline{a}}$ as $\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M$. Observe that $f_{\overline{a}} \leq^* f^*$. Then $\psi f_{\overline{a}} \leq n^*$.

Lemma 14. *Given $\overline{a} = a_0, a_1, ..., a_{n^*}$ a nice sequence with associated functions $g_0, g_1, ..., g_{n^*}$, g_{n^*+1}, define $f_{\overline{a}}$ as above. Then $\psi_2 f_{\overline{a}} \leq^* g_i a_i$ for all $i \leq n^*$.*

Proof By the previous lemma, it suffices to prove that $\psi_2 f_{\overline{a}} = g_{n^*} a_{n^*}$. Let $f_{\overline{a}}$ be given by $\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M$ with k_0 the least natural number satisfying

$$\psi_1\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M < k_0 + 1.$$

Note that $k_0 \leq n^*$. Then

$$B'(n^* + 1)\langle a_0, \cdots, a_{n^*}, 0, 0, ...\rangle = \langle a_0, \cdots, a_{k_0}, 0, 0, ...\rangle.$$

Now, the conclusion is straightforward: $\psi_2 f_{\overline{a}} = \psi_2' \left(B'(n^* + 1)\langle a_0, \cdots, a_{n^*}, 0, 0, ...\rangle\right) = g_{n^*} a_{n^*}$. $\square$

At this point, we can prove the following:

Lemma 15. *Assume (3.3) holds, and let $\overline{a} = a_0, a_1, ..., a_{n^*}$ be a nice sequence with associated functions $g_0, g_1, ..., g_{n^*}, g_{n^*+1}$. Define $f_{\overline{a}}$ as above. In this situation,*

$$\neg\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* g_n a_n \ A_B(n, a_n, b).$$

Proof Assume $\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* g_n a_n \ A_B(n, a_n, b)$. By the above lemma,

$$\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* \psi_2 f_{\overline{a}} \ A_B(n, a_n, b),$$

with $f_{\overline{a}}$ given by $\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M$, where k_0 is the least natural number satisfying $\psi\langle a_0, a_1, ..., a_{k_0}, 0, 0, ...\rangle^M < k_0+1$. Then $\psi f_{\overline{a}} \leq k_0$. If $n \leq \psi_1 f_{\overline{a}}$, clearly we have $a_n \leq^* f_{\overline{a}} n$. By monotonicity of A_B in the entry of a_n, we get

$$\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* \psi_2 f_{\overline{a}} \ A_B(n, f_{\overline{a}} n, b),$$

which contradicts (3.3). $\square$

We have showed that, under the hypothesis (3.3),

$$\tilde{\forall} a_0, a_1, ..., a_{n^*} \ \left(\forall n \leq n^* \ (a_n \leq^* \phi n g_n) \to \neg\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* g_n a_n \ A_B(n, a_n, b)\right).$$

This entails

$$\tilde{\forall} a_0, a_1, ..., a_{n^*} \ \neg\forall n \leq n^* \ \left(a_n \leq^* \phi n g_n \wedge \tilde{\forall} b \leq^* g_n a_n \ A_B(n, a_n, b)\right) \tag{3.4}$$

since $\psi_1 f_{\overline{a}} \leq n^*$.

Lemma 16. *Under the hypothesis* (3.2), *we have*

$$\neg\tilde{\forall} a_0, a_1, ..., a_n \ \neg\forall i \leq n \ \left(a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)\right)$$

for all $n \leq n^*$.

Proof We argue by induction on n. For $n = 0$, the conclusion comes from (3.2):

$$\neg\tilde{\forall} a \leq^* \phi 0 g_0 \neg\tilde{\forall} b \leq^* g_0 a \ A_B(0, a, b).$$

To prove the induction step, take the induction hypothesis:

$$\neg\tilde{\forall} a_0, a_1, ..., a_n \ \neg\forall i \leq n \ \left(a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)\right)$$

with $n < n^*$ and assume

$$\tilde{\forall} a_0, a_1, ..., a_{n+1} \ \neg\forall i \leq n+1 \ \left(a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)\right),$$

which is equivalent to

$$\begin{aligned}\tilde{\forall} a_0, a_1, ..., a_n \tilde{\forall} a_{n+1} \ \neg\Big(\forall i \leq n \ \left(a_i \leq^* \phi i g_i\right) \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)\Big) \wedge \\ a_{n+1} \leq^* \phi(n+1) g_{n+1} \wedge \tilde{\forall} b \leq^* g_{n+1} a_{n+1} \ A_B(n+1, a_{n+1}, b)\Big).\end{aligned}$$

By (3.2), if $a_0, a_1, ..., a_n$ is a nice sequence and g_{n+1} is its $(n+1)$th associated function, then

$$\neg\tilde{\forall} a \leq^* \phi(n+1) g_{n+1} \neg\tilde{\forall} b \leq^* g_{n+1} a \ A_B(n+1, a, b).$$

In other words,

$$\begin{aligned}\tilde{\forall} a_0, ..., a_n \ \big(\forall i \leq n \ (a_i \leq^* \phi i g_i) \rightarrow \neg\tilde{\forall} a_{n+1} \neg\big(a_{n+1} \leq^* \phi(n+1) g_{n+1} \wedge \\ \tilde{\forall} b \leq^* g_{n+1} a_{n+1} \ A_B(n+1, a_{n+1}, b)\big)\big).\end{aligned}$$

Applying the intuitionistic rule (see the lemma below)

$$\frac{\forall x \forall z \ \neg \left(H(x) \wedge A(x) \wedge B(x, z)\right) \quad \forall x \ \left(H(x) \rightarrow \neg\forall z \ \neg B(x, z)\right)}{\forall x \ \neg \left(H(x) \wedge A(x)\right)}$$

we get

$$\tilde{\forall} a_0, a_1, ..., a_n \ \neg\forall i \leq n \ \left(a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)\right).$$

The contradiction follows from the induction hypothesis. $\square$

Lemma 17.

$$\frac{\forall x \forall z\ \neg\,(H(x) \wedge A(x) \wedge B(x,z)) \qquad \forall x\ \ (H(x) \to \neg \forall z\ \neg B(x,z))}{\forall x\ \neg\,(H(x) \wedge A(x))}$$

is a theorem in HA^ω. *In fact, it is still true for bounded quantifications (in the flattened sense).*

Proof Assume $\forall x \forall z\ \neg\,(H(x) \wedge A(x) \wedge B(x,z))$, $\forall x\ \big(H(x) \to \neg \forall z\ \neg B(x,z)\big)$ and $H(x_0) \wedge A(x_0)$. By the first assumption, we get $\forall z\ \neg\,(H(x_0) \wedge A(x_0) \wedge B(x_0,z))$. This entails $\forall z\ \neg B(x_0,z)$ since we have assumed $H(x_0) \wedge A(x_0)$. By $\forall x\ \big(H(x) \to \neg \forall z\ \neg B(x,z)\big)$, we get $\neg \forall z\ \neg B(x_0,z)$, which leads to a contradiction. Therefore, we conclude $\neg(H(x_0) \wedge A(x_0))$ and also $\forall x\ \neg\,(H(x) \wedge A(x))$.

To prove it for bounded quantifications (in the flattened sense), we only need to make a tiny change to H: $H'(x) = x \leq^* x \wedge H(x)$. □

Under the hypothesis (3.2), by the lemma 16 with $n = n^*$, we get

$$\neg \tilde{\forall} a_0, a_1, ..., a_{n^*}\ \neg \forall n \leq n^*\ \big(a_n \leq^* \phi n g_n \wedge \tilde{\forall} b \leq^* g_n a_n\ A_B(n, a_n, b)$$

which contradicts (3.4). With this contradiction, we end the proof. □

Corollary 3.

$$\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \mathsf{DNS}.$$

Proof Let A be (a universal closure of) an instance of DNS. Then, by the previous theorem, $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash A^B$. The result follows from the characterization theorem. □

3.3 The bounded functional interpretation of full numerical comprehension

As mentioned before, Spector introduced bar recursive functionals in order to interpret the principle CA^0 of full numerical comprehension,

$$\mathsf{CA}^0: \quad \exists f^1 \forall n^0\ \big(f(n) =_0 0 \leftrightarrow A(n)\big),$$

where A is an arbitrary formula of the language of finite-order arithmetic (not containing f free). The interpretation is done in the classical setting via a negative translation (Kuroda) followed by the bounded functional interpretation of Heyting arithmetic. In this section, we show that

$$\mathsf{PA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \mathsf{CA}^0.$$

We begin by proving, within $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$, that CA^0 is a consequence of the principle

$$\mathsf{bAC}^{0,\omega}: \quad \forall n^0 \exists x\, A(n,x) \to \tilde{\exists} f^1 \forall n \exists x \trianglelefteq fn\, A(n,x)$$

for *arbitrary* formulas A and x of any type. This is achieved using the following lemma:

Lemma 18. $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{cl}[\trianglelefteq]$ *proves*

$$\tilde{\forall} f \left(\tilde{\forall} a \exists b \trianglelefteq fa A(a,b) \to \exists h \trianglelefteq f \tilde{\forall} a\; A(a,ha) \right),$$

where A is an arbitrary universal formula (with bounded intensional matrix).

Proof As we have seen before, under the bounded contra collection principle for bounded matrices, tameAC is equivalent to

$$\tilde{\forall} a, f \exists h \trianglelefteq f \forall x \trianglelefteq a\; (\exists y \trianglelefteq fx\; A(x,y) \to A(x,hx)),$$

whose flattening holds in $\mathcal{M}^\omega$ (recall the proof of $\mathcal{M}^\omega \vDash \mathsf{tameAC}$ in the previous chapter). Then $\mathsf{PA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{cl}[\trianglelefteq] \vdash \mathsf{tameAC}$.

Let $A(a,b)$ be given by $\forall z\; B_{bd}(a,b,z)$ with B_{bd} a bounded formula. Take f monotone and assume $\tilde{\forall} a \exists b \trianglelefteq fa \forall z\; B_{bd}(a,b,z)$. Of course, we have $\tilde{\forall} d \tilde{\forall} a \exists b \trianglelefteq fa \forall z \trianglelefteq d\; B_{bd}(a,b,z)$ and by tameAC, it follows

$$\tilde{\forall} d \exists h \trianglelefteq f \tilde{\forall} a \forall z \trianglelefteq d\; B_{bd}(a,ha,z)$$

and, therefore, $\tilde{\forall} c, d \exists h \trianglelefteq f \tilde{\forall} a \trianglelefteq c \forall z \trianglelefteq d\; B_{bd}(a,ha,z)$. By bounded (contra) collection, we get $\exists h \trianglelefteq f \tilde{\forall} a \forall z\; B_{bd}(a,ha,z)$. $\square$

Proposition 12.

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{cl}[\trianglelefteq] \vdash \mathsf{bAC}^{0,\omega} \to \mathsf{CA}^0.$$

Proof Observe that, in the classical setting, $\mathsf{AC}^{00} \to \mathsf{CA}^0$ is a well-known fact. Take $A(n^0)$ an arbitrary formula of the language of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR}$. Then, by classical logic $\forall n \exists k\; ((k = 0 \wedge A(n)) \vee (k \neq 0 \wedge \neg A(n))))$. By AC^{00}, there is f^1 which witnesses such k. Of course, we get $\forall n\; (fn = 0 \leftrightarrow A(n))$, as desired.

It remains to prove that $\mathsf{bAC}^{0,\omega} \to \mathsf{AC}^{00}$. Take an arbitrary formula $A(n^0, k^0)$ whose bounded functional interpretation is given by $\tilde{\exists} a \tilde{\forall} b\; A_B(n,k,a,b)$. Suppose we have $\forall n \exists k\; A(n,k)$. By characterization theorem, we have $\forall n \exists k \tilde{\exists} a \tilde{\forall} b\; A_B(n,k,a,b)$. By $\mathsf{bAC}^{0\sigma}$, we get

$$\tilde{\exists} f, g \forall n \exists k \leq fn \tilde{\exists} a \trianglelefteq gn \tilde{\forall} b\; A_B(n,k,a,b).$$

By the previous lemma, there are h and s so that $\forall n \tilde{\forall} b\; (sn \trianglelefteq sn \wedge\; A_B(n,hn,sn,b))$. In particular, we have that $\forall n \tilde{\exists} a \tilde{\forall} b\; A(n,hn,a,b)$ which is equivalent to $\exists h \forall n\; A(n,hn)$, as desired. $\square$

We have showed that, within $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$, we have $\mathsf{bAC}^{0\omega} \to \mathsf{CA}^0$. In order to achieve our goal, by Modus Ponens, it suffices to prove

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{cl}[\trianglelefteq] \vdash \mathrm{bAC}^{0,\omega}.$$

We argue that this result follows from

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \big(\mathsf{bAC}^{0,\omega}\big)'.$$

In fact, by the soundness theorem, we get $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash \Big(\big(\mathsf{bAC}^{0,\omega}\big)'\Big)^B$ (we must see each instance of $\mathsf{bAC}^{0,\omega}$ as given by its universal closure). By the characterization theorem of the bounded functional interpretation for the classical case, it follows that $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{cl}[\trianglelefteq] \vdash \mathrm{bAC}^{0,\omega}$.

Now, we finally prove:

Proposition 13.

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash (\mathsf{bAC}^{0,\omega})'.$$

Proof This relies on the adaptation of a well-known argument. The (Kuroda) negative translation of $\mathsf{bAC}^{0\sigma}$ is given by $\neg\neg\big(\forall n\neg\neg\exists x\, A^\dagger(n,x) \to \tilde{\exists} f \forall n \neg\neg\exists x \trianglelefteq fn\, A^\dagger(n,x)\big)$. Equivalently, we have $\forall n\neg\neg\exists x\, A^\dagger(n,x) \to \neg\neg\tilde{\exists} f\forall n\neg\neg\exists x \trianglelefteq fn\, A^\dagger(n,x)$. Assume $\forall n\neg\neg\exists x\, A^\dagger(n,x)$. In the previous section, we proved $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \mathsf{DNS}$. Applying DNS to the assumption, we obtain $\neg\neg\forall n\exists x\, A^\dagger(n,x)$. By the bounded choice principle, $\forall n\exists x\, A^\dagger(n,x) \to \tilde{\exists} f\forall n\exists x \trianglelefteq fn\, A^\dagger(n,x)$. Then, by intuitionistic logic,

$$\neg\neg\forall n\exists x\, A^\dagger(n,x) \to \neg\neg\tilde{\exists} f\forall n\exists x \trianglelefteq fn\, A^\dagger(n,x)$$

and, therefore $\neg\neg\tilde{\exists} f\forall n\neg\neg\exists x \trianglelefteq fn\, A^*(n,x)$. $\square$

3.4 The bounded functional interpretation of bar induction

In this section, we prove that bar induction has a bounded functional interpretation. In order to do so, we will need the following statements:

$$\tilde{\forall}\psi z u \forall n^0 \forall s^{0\to\rho}\forall r^{0\to\rho}\big(\forall i < n\ (si \trianglelefteq ri) \to (B^{\rho,\sigma})^p\psi zuns \trianglelefteq (B^{\rho,\sigma})^p\psi zunr\big) \tag{3.5}$$

with ψ, z and u of appropriate types. Kohlenbach proves in his recent book [Koh08a] that the flattenings of these statements hold in $\mathcal{M}^\omega$, hence they are in $\Delta_{\mathcal{M}^\omega}$.

Theorem 26.

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \mathsf{BI}$$

Proof The following proof is an adaptation of the proof made by Howard in [How68]. Take P and Q formulas of the language and assume that

$$\begin{array}{ll}
\mathrm{Hyp1}: & \forall s \exists n\ P(\overline{s,n}, n) \\
\mathrm{Hyp2}: & \forall s, n \forall m \leq n\ \big(P(\overline{s,m}, m) \to P(\overline{s,n}, n)\big) \\
\mathrm{Hyp3}: & \forall s, n\ \big(P(\overline{s,n}, n) \to Q(\overline{s,n}, n)\big) \\
\mathrm{Hyp4}: & \forall s, n\ \big(\forall x\ Q(\overline{s,n} * x, n+1) \to Q(\overline{s,n}, n)\big).
\end{array}$$

We want to prove $Q(\overline{0}, 0)$.

Notice that, in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq]$, P and Q are equivalent to their bounded functional interpretations. Suppose

$$\begin{aligned}
\big(P(s,n)\big)^B &= \tilde{\exists} a \tilde{\forall} b\ P_B(s,n,a,b) \\
\big(Q(s,n)\big)^B &= \tilde{\exists} c \tilde{\forall} d\ Q_B(s,n,c,d).
\end{aligned}$$

Therefore, from Hyp1, Hyp2, Hyp3 and Hyp4, it follows

$$\forall s \exists n \tilde{\exists} a \tilde{\forall} b\ P_B(\overline{s,n}, n, a, b) \tag{3.6}$$

$$\forall s, n \forall m \leq n\ \big(\tilde{\exists} a_1 \tilde{\forall} b_1\ P_B(\overline{s,m}, m, a_1, b_1) \to \tilde{\exists} a_2 \tilde{\forall} b_2\ P_B(\overline{s,n}, n, a_2, b_2)\big) \tag{3.7}$$

$$\forall s, n\ \big(\tilde{\exists} a \tilde{\forall} b\ P_B(\overline{s,n}, n, a, b) \to \tilde{\exists} c \tilde{\forall} d\ Q_B(\overline{s,n}, n, c, d)\big) \tag{3.8}$$

$$\forall s, n\ \big(\forall x \tilde{\exists} c_1 \tilde{\forall} d_1\ Q_B(\overline{s,n} * x, n+1, c_1, d_1) \to \tilde{\exists} c_2 \tilde{\forall} d_2\ Q_B(\overline{s,n}, n, c_2, d_2)\big). \tag{3.9}$$

Concerning (3.6), by bounded choice principle, there exists a monotone f such that

$$\tilde{\forall} s \forall s' \trianglelefteq s \exists n \leq fs \tilde{\exists} a \tilde{\forall} b\ P_B(\overline{s', n}, n, a, b)$$

and by (3.7), it follows $\tilde{\forall} s \forall s' \trianglelefteq s \tilde{\exists} a \tilde{\forall} b\ P_B(\overline{s', fs}, fs, a, b)$. Collection entails

$$\tilde{\forall} s \tilde{\exists} a \forall s' \trianglelefteq s \tilde{\exists} a' \trianglelefteq a \tilde{\forall} b\ P_B(\overline{s', fs}, fs, a', b)$$

and since P_B is monotone in the entry of a', we get $\tilde{\forall} s \tilde{\exists} a \forall s' \trianglelefteq s \tilde{\forall} b\ P_B(\overline{s', fs}, fs, a, b)$. By bounded axiom of choice, there exists a monotone g such that

$$\tilde{\forall} s \tilde{\exists} a \trianglelefteq gs \forall s' \trianglelefteq s \tilde{\forall} b\ P_B(\overline{s', fs}, fs, a, b)$$

and by monotonicity of P_B on the entry of a, it follows

$$\tilde{\forall} s \forall s' \trianglelefteq s \tilde{\forall} b\ P_B(\overline{s', fs}, fs, gs, b).$$

Now, let us focus in (3.7), which is equivalent to

$$\forall s, n \forall m \leq n \tilde{\forall} a_1\ \big(\tilde{\forall} b_1\ P_B(\overline{s,m}, m, a_1, b_1) \to \tilde{\exists} a_2 \tilde{\forall} b_2\ P_B(\overline{s,n}, n, a_2, b_2)\big).$$

By the bounded independence of premises $\mathsf{bIP}^\omega_{\tilde{\forall} bd}$,

$$\forall s, n \forall m \leq n \tilde{\forall} a_1 \tilde{\exists} a_2\ \big(\tilde{\forall} b_1\ P_B(\overline{s,m}, m, a_1, b_1) \to \tilde{\exists} a_2' \trianglelefteq a_2 \tilde{\forall} b_2\ P_B(\overline{s,n}, n, a_2', b_2)\big)$$

and by the bounded choice principle bAC^ω, there exists h, monotone, such that

$$\tilde{\forall} s,n,a_1 \forall m \leq n \forall s' \trianglelefteq s \tilde{\exists} a_2 \trianglelefteq hsna_1 \big(\tilde{\forall} b_1\ P_B(\overline{s',m},m,a_1,b_1) \to \\ \to \tilde{\exists} a_2' \trianglelefteq a_2 \tilde{\forall} b_2\ P_B(\overline{s',n},n,a_2',b_2)\big).$$

The latter implies

$$\tilde{\forall} s,n,a_1 \forall m \leq n \forall s' \trianglelefteq s\ \big(\tilde{\forall} b_1\ P_B(\overline{s',m},m,a_1,b_1) \to \tilde{\exists} a_2 \trianglelefteq hsna_1 \tilde{\forall} b_2\ P_B(\overline{s',n},n,a_2,b_2)\big)$$

and by monotonicity of P_B on entry of a_2, it follows

$$\tilde{\forall} s,n,a \forall m \leq n \forall s' \trianglelefteq s\ \big(\tilde{\forall} b\ P_B(\overline{s',m},m,a,b) \to \tilde{\forall} b\ P_B(\overline{s',n},n,hsna,b)\big).$$

Hyp3 is equivalent to $\forall s,n \tilde{\forall} a\ \big(\tilde{\forall} b\ P_B(\overline{s,n},n,a,b) \to \tilde{\exists} c \tilde{\forall} d\ Q_B(\overline{s,n},n,c,d)\big)$. As we did with Hyp2, by $\mathsf{bIP}^\omega_{\forall bd}$ and bAC^ω, there exists p, monotone, such that

$$\tilde{\forall} s,n,a \forall s' \trianglelefteq s\ \big(\tilde{\forall} b\ P_B(\overline{s',n},n,a,b) \to \tilde{\exists} c \trianglelefteq psna \tilde{\forall} d\ Q_B(\overline{s',n},n,c,d)\big).$$

Since Q_B is monotone in the entry of c, we get

$$\tilde{\forall} s,n,a \forall s' \trianglelefteq s\ \big(\tilde{\forall} b\ P_B(\overline{s',n},n,a,b) \to \tilde{\forall} d\ Q_B(\overline{s',n},n,psna,d)\big).$$

From (3.9), we get

$$\forall s,n\ \big(\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \tilde{\exists} c_1 \trianglelefteq fa \tilde{\forall} d_1\ Q_B(\overline{s,n} * x, n+1, c_1, d_1) \to \tilde{\exists} c_2 \tilde{\forall} d_2\ Q_B(\overline{s,n},c_2,d_2)\big),$$

which implies

$$\forall s,n \tilde{\forall} f\ \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d_1\ Q_B(\overline{s,n} * x, n+1, fa, d_1) \to \tilde{\exists} c_2 \tilde{\forall} d_2\ Q_B(\overline{s,n},c_2,d_2)\big).$$

The bounded independence of premises principle leads to

$$\forall s,n \tilde{\forall} f \tilde{\exists} c_2\ \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d_1\ Q_B(\overline{s,n} * x, n+1, fa, d_1) \to \tilde{\exists} c_2' \trianglelefteq c_2 \tilde{\forall} d_2\ Q_B(\overline{s,n},c_2',d_2)\big)$$

and by bounded choice principle, there exists a monotone ϕ such that

$$\tilde{\forall} s,n,f \forall s' \trianglelefteq s \tilde{\exists} c_2 \trianglelefteq \phi snf\ \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d_1\ Q_B(\overline{s',n} * x, n+1, fa, d_1) \to \\ \to \tilde{\exists} c_2' \trianglelefteq c_2 \tilde{\forall} d_2\ Q_B(\overline{s',n},c_2',d_2)\big).$$

The monotonicity of Q_B in the entry of c_2 entails

$$\tilde{\forall} s,n,f \forall s' \trianglelefteq s\ \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B(\overline{s',n} * x, n+1, fa, d) \to \tilde{\forall} d\ Q_B(\overline{s',n},\phi snf,d)\big).$$

At this point, from Hyp1-Hyp4, we have showed that there exist monotone f,g,h,p and ϕ such that

$$\tilde{\forall} s \forall s' \trianglelefteq s \tilde{\forall} b\ P_B(\overline{s',fs},fs,gs,b) \tag{3.10}$$

$$\tilde{\forall} s,n,a \forall m \leq n \forall s' \trianglelefteq s\ \big(\tilde{\forall} b\ P_B(\overline{s',m},m,a,b) \to \tilde{\forall} b\ P_B(\overline{s',n},n,hsna,b)\big) \tag{3.11}$$

$$\tilde{\forall} s,n,a \forall s' \trianglelefteq s\ \big(\tilde{\forall} b\ P_B(\overline{s',n},n,a,b) \to \tilde{\forall} d\ Q_B(\overline{s',n},n,psna,d)\big) \tag{3.12}$$

$$\tilde{\forall} s,n,f \forall s' \trianglelefteq s\ \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B(\overline{s',n} * x, n+1, fa, d) \to \\ \to \tilde{\forall} d\ Q_B(\overline{s',n},n,\phi snf,d)\big). \tag{3.13}$$

We want to prove $\tilde{\exists} w \tilde{\forall} d\ Q_B(\overline{0}, 0, w, d)$ which is equivalent to $Q(\overline{0}, 0)$ (by the characterization theorem).

In order to do so, let us define $Wsn := B^p fzuns$ (B^p is defined in the proof of theorem 23), with

$$\begin{aligned} zns &= psn\,(hsn\,(gs)) \\ ulns &= \phi snl. \end{aligned}$$

Observe that z and u are monotone, since f, g, p, h, ϕ are all monotone. By (3.5), for n and s such that $\forall i < n\ (si \trianglelefteq si))$, Wsn monotone. In particular $W\overline{0}0$ is monotone.

We claim that w given by $W\overline{0}0$ satisfies $\tilde{\forall} d\ Q(\overline{0}, 0, w, d)$. In order to do so, we prove the following results:

Lemma 19. $\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^{\omega}} + \mathsf{P}^{\omega}[\trianglelefteq]$ *proves that for all n^0 and s', s of type $0 \to \rho$ such that $\forall i < n\ (s'i \trianglelefteq si)$, if $f\left(\overline{s,n}^M\right) < n$, then*

$$\tilde{\forall} d\ Q_B\big(\overline{s',n}, n, W\overline{s,n}\, n, d\big).$$

Before proving the above lemma, notice that for all $n \in \mathbb{N}$ and $s, r \in M_{\rho}^{\mathbb{N}}$, if $\forall i < n\ \ (si \leq^* ri)$, then $\forall i\ \ (\overline{s,n}\, i \leq^* \overline{r,n}, i)$ is provable in $\mathcal{M}^{\omega}$. Consequently, in $\mathcal{M}^{\omega}$ we have $\overline{s,n} \leq^* \overline{r,n}^M \wedge \overline{s,n}^M \leq^* \overline{r,n}^M$. Therefore, $\Delta_{\mathcal{M}^{\omega}}$ contains the statements

$$\forall n^0 \forall s^{0\to\rho} \forall r^{0\to\rho}\ \ \big(\forall i < n\ \ (si \trianglelefteq ri) \to \big(\overline{s,n} \trianglelefteq \overline{r,n}^M \wedge \overline{s,n}^M \trianglelefteq \overline{r,n}^M\big)\big)\,.$$

Notice, also, that until the end of the proof, we may use facts, such as $A_{bd}(\overline{s,n} * sn) \leftrightarrow A_{bd}(\overline{s, n+1})$, with A_{bd} a bounded formula. Such a statement is in $\Delta_{\mathcal{M}^{\omega}}$.

Proof Take n, s and s' such that $\forall i < n\ \ (s'i \trianglelefteq si)$ and assume $f\left(\overline{s,n}^M\right) < n$. Then $\overline{s',n} \trianglelefteq \overline{s,n}^M$. By (3.10), we have $\tilde{\forall} b\ P_B\left(\overline{s',n, f\left(\overline{s,n}^M\right)}, f\left(\overline{s,n}^M\right), g\left(\overline{s,n}^M\right), b\right)$. Since $f\overline{s,n}^M < n$, then

$$\tilde{\forall} b\ P_B\left(\overline{s', f\left(\overline{s,n}^M\right)}, f\left(\overline{s,n}^M\right), g\left(\overline{s,n}^M\right), b\right)$$

and by (3.11), it follows $\tilde{\forall} b\ P_B\left(\overline{s',n}, n, h\left(\overline{s,n}^M\right) n\left(g\left(\overline{s,n}^M\right)\right), b\right)$. The latter and (3.12) entail

$$\tilde{\forall} d\ Q_B\left(\overline{s',n}, n, p\left(\overline{s,n}^M\right) n\left(h\left(\overline{s,n}^M\right) n\left(g\left(\overline{s,n}^M\right)\right)\right), d\right),$$

which is equivalent to $\tilde{\forall} d\ Q_B\big(\overline{s',n}, n, W\overline{s,n}\, n, d\big)$. ☐

Lemma 20. $\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^{\omega}} + \mathsf{P}^{\omega}[\trianglelefteq]$ *proves that for all n^0 and s', s of type $0 \to \rho$ such that $\forall i < n\ (s'i \trianglelefteq si)$, if $f\left(\overline{s,n}^M\right) \geq n$, then*

$$\tilde{\forall} a, d \forall x \trianglelefteq a\ Q_B\left(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a\right)(n+1), d\right) \to \tilde{\forall} d\ Q_B\left(\overline{s',n}, n, W\overline{s,n}\, n, d\right).$$

Proof Take n, s' and s such that $\forall i < n\ (s'i \trianglelefteq si)$. Then $\overline{s', n} \trianglelefteq \overline{s, n}^M$. Assume $f\left(\overline{s,n}^M\right) \geq n$. Define $\psi = \lambda x.W\left(\overline{s,n} * x\right)(n+1)$. By definition of W,

$$\psi = \lambda x.B^p fzu(n+1)(\overline{s,n} * x).$$

Using the rule $\mathsf{RL}_{\trianglelefteq}$ together with (3.5), it is straightforward to show that ψ is monotone, since f, z and u are monotone and $\forall i < n\ (si \trianglelefteq si)$. By (3.13), we get

$$\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B\big(\overline{s',n} * x, n+1, \psi a, d\big) \to \tilde{\forall} d\ Q_B\big(\overline{s',n}, n, \phi\left(\overline{s,n}^M\right) n\psi, d\big),$$

which is equivalent to

$$\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B\big(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a\right)(n+1), d\big) \to \\ \to \tilde{\forall} d\ Q_B\big(\overline{s',n}, n, \phi\left(\overline{s,n}^M\right) n\left(\lambda x.W\left(\overline{s,n} * x\right)(n+1)\right), d\big).$$

The latter entails

$$\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B\left(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a\right)(n+1), d\right) \to \tilde{\forall} d\ Q_B\left(\overline{s',n}, n, W\overline{s,n}\, n\right), d\big),$$

since Q_B is monotone in the entry of $W\overline{s,n}\,n$ and this is the maximum between $p\left(\overline{s,n}^M\right) n\left(h\left(\overline{s,n}^M\right) n\left(g\left(\overline{s,n}^M\right)\right)\right)$ and $\phi\left(\overline{s,n}^M\right) n\left(\lambda x.B^p fzu(n+1)(\overline{s,n} * x)\right)$. $\square$

By lemma19 and lemma20,

$$\forall n, s, s'\big(\forall i < n\ (s'i \trianglelefteq si) \to \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d\ Q_B\left(\overline{s',n} * x, n+1, W(\overline{s,n} * a)(n+1), d\right) \to \\ \to \tilde{\forall} d\ Q_B\big(\overline{s',n}, n, W\big(\overline{s,n}, n\big), d\big)\big)\big),$$

which is equivalent to

$$\forall n, s, s' \tilde{\forall} d\big(\forall i < n\ (s'i \trianglelefteq si) \to \big(\tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} d'\ Q_B\left(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a\right)(n+1), d'\right) \to \\ \to \tilde{\forall} d' \trianglelefteq d\ Q_B\left(\overline{s',n}, n, W\overline{s,n}\, n, d'\right)\big)\big).$$

The bounded Markov and the independence of premises principles imply

$$\forall n, s, s' \tilde{\forall} d \tilde{\exists} a, d'\big(\forall i < n\ (s'i \trianglelefteq si) \to \tilde{\exists} a' \trianglelefteq a, d'' \trianglelefteq d'\big(\tilde{\forall} a'' \trianglelefteq a' \forall x \trianglelefteq a'' \tilde{\forall} d''' \trianglelefteq d'' \\ Q_B\big(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a''\right)(n+1), d'''\big) \to \tilde{\forall} d' \trianglelefteq d\ Q_B\big(\overline{s',n}, n, W\overline{s,n}\, n, d'\big)\big)\big).$$

Equivalently, we have

$$\forall n, s, s' \tilde{\forall} d \tilde{\exists} a, d'\big(\forall i < n\ (s'i \trianglelefteq si) \to \big(\tilde{\forall} a' \trianglelefteq a \forall x \trianglelefteq a' \tilde{\forall} d'' \trianglelefteq d' \\ Q_B\big(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a'\right)(n+1), d''\big) \to \tilde{\forall} d' \trianglelefteq d\ Q_B\big(\overline{s',n}, n, W\overline{s,n}\, n, d'\big)\big)\big).$$

By the bounded choice principle, there exist monotone closed functionals u and v such that for n, for monotone s^*, s^{**}, d and for $s' \trianglelefteq s^*, s \trianglelefteq s^{**}$ such that $\forall i < n\ (s'i \trianglelefteq si)$, then

$$\tilde{\forall} a \trianglelefteq uns^*s^{**}d \forall x \trianglelefteq a \tilde{\forall} d' \trianglelefteq vns^*s^{**}d\ Q_B\big(\overline{s',n} * x, n+1, W\left(\overline{s,n} * a\right)(n+1), d'\big) \to \\ \to \tilde{\forall} d' \triangleleft d\ Q_B\big(\overline{s',n}, n, W\overline{s,n}\, n, d'\big) \tag{3.14}$$

In order to simplify the notation, let $D(s,n,r,d)$ denote $\forall d' \trianglelefteq d\, Q_B(s,n,Wrn,d')$ and define u' and v' by

$$\begin{aligned} u'nsd &= unssd \\ v'nsd &= vnssd. \end{aligned}$$

From (3.14), we have

$$\tilde{\forall} a \trianglelefteq u'ns^*d \forall x \trianglelefteq a\ D\big(\overline{s',n} * x, n+1, \overline{s,n} * a, v'ns^*d\big) \to D(\overline{s',n}, n, \overline{s,n}, d) \qquad (3.15)$$

for all monotone n, s^*, d and for $s' \trianglelefteq s^*,\ s \trianglelefteq s^*$ such that $\forall i < n\ (s'i \trianglelefteq si)$.

For all monotone d, define recursively $\langle a, b\rangle$ by

$$\begin{aligned} \langle a,b\rangle 0 &:= \langle u'0\overline{0}d, v'0\overline{0}d\rangle \\ \langle a,b\rangle(k+1) &:= \big\langle u'(k+1)({s_{k+1}}^M)b_k, v'(k+1)({s_{k+1}}^M)b_k\big\rangle, \end{aligned}$$

where

$$s_k i = \begin{cases} a_i & \text{if } \ i < k \\ 0 & \text{otherwise} \end{cases}$$

for $k > 0$.

Lemma 21. $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq]$ *proves that, given d, monotone, and a_k and b_k defined as above (depending on d), then a_k and b_k are monotone for all k.*

Proof We argue by induction on k. For $k = 0$, it is clear since $u', v', \overline{0}$ and d are monotone. As induction hypothesis, assume a_i and b_i are monotone for all $i < k$. Then, ${s_k}^M$ is monotone since $\forall i\ (s_k i \leq^* s_k i) \to {s_k}^M \leq^* {s_k}^M$ holds in $\mathcal{M}^\omega$ (implying that $s_k^M \trianglelefteq s_k^M$ is in $\Delta_{\mathcal{M}^\omega}$). Now, using the induction hypothesis, the conclusion comes easily since u' and v' are both monotone. $\square$

Now, we can prove the following:

Lemma 22. $\mathsf{HA}^\omega + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq]$ *proves that given d, monotone, and a_k and b_k defined as above, if*

$$(\forall i \leq k\ \ (c_i \trianglelefteq a_i \wedge x_i \trianglelefteq c_i)) \to D(\langle x_0, ..., x_k, 0, 0, ...\rangle, k+1, \langle c_0, ..., c_k, 0, 0, ...\rangle, b_k),$$

holds for all $c_0, ..., c_k, x_0, ..., x_k$ of the appropriate type, then, we have $D(\overline{0}, 0, \overline{0}, d)$, for all k^0.

Proof We argue by induction on k. For $k = 0$, it is straightforward by (3.15). For $k+1$, assume, as the induction hypothesis, that the implication stated in the lemma holds for k.

Suppose

$$\forall c_0,...,c_{k+1},x_0,...,x_{k+1}\ \big(\forall i\le k+1\ \ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)\to\\ \to D(\langle x_0,...,x_{k+1},0,0,...\rangle,k+2,\langle c_0,...,c_{k+1},0,0,...\rangle,b_{k+1})\big),$$

which is equivalent to

$$\forall c_0,...,c_k,x_0,...,x_k\forall c_{k+1}\forall x_{k+1}\ \big(\forall i\le k\ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)\to\big(c_{k+1}\trianglelefteq a_{k+1}\wedge x_{k+1}\trianglelefteq c_{k+1}\to\\ \to D(\overline{\langle x_0,...,x_k,0,0,...\rangle,k+1}*x_{k+1},k+2,\overline{\langle c_0,...,c_k,0,0,...\rangle,k+1}*c_{k+1},b_{k+1})\big)\big)$$

and to

$$\forall c_0,...,c_k,x_0,...,x_k\ \big(\forall i\le k\ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)\to\forall c_{k+1}\trianglelefteq a_{k+1}\forall x_{k+1}\trianglelefteq c_{k+1}\\ D(\overline{\langle x_0,...,x_k,0,0,...\rangle,k+1}*x_{k+1},k+2,\overline{\langle c_0,...,c_k,0,0,...\rangle,k+1}*c_{k+1},b_{k+1})\big).$$

Observe that $a_{k+1}=u'(k+1){s_{k+1}}^M b_k$, $b_{k+1}=v'(k+1){s_{k+1}}^M b_k$, $\langle x_0,...,x_k,0,0,...\rangle\trianglelefteq {s_{k+1}}^M$ and $\langle c_0,...,c_k,0,0,...\rangle\trianglelefteq {s_{k+1}}^M$ (by the hypothesis of $\forall i\le k\ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)$). By (3.15), it follows

$$\forall c_0,...,c_k,x_0,...,x_k\ \big(\forall i\le k\ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)\to\\ \to D(\overline{\langle x_0,...,x_k,0,0,...\rangle,k+1},k+1,\overline{\langle c_0,...,c_k,0,0,...\rangle,k+1},b_k)\big).$$

Equivalently,

$$\forall c_0,...,c_k,x_0,...,x_k\ \big(\forall i\le k\ (c_i\trianglelefteq a_i\wedge x_i\trianglelefteq c_i)\to\\ \to D(\langle x_0,...,x_k,0,0,...\rangle,k+1,\langle c_0,...,c_k,0,0,...\rangle,b_k)\big).$$

By the induction hypothesis, we get $D(\overline{0},0,\overline{0},d)$, as desired. $\square$

To finish the proof we will also need the following lemma, known as Kreisel's trick:

Lemma 23 (**Kreisel's trick**). $\mathsf{HA}^\omega+\mathsf{BR}+\Delta_{\mathcal{M}^\omega}+\mathsf{P}^\omega[\trianglelefteq]$ *proves that, given f of type $(0\to\rho)\to 0$, then we can define θ (depending on f), by bar recursion, such that*

$$\exists k\le\theta s\overline{0}0\ (f\overline{s,k}<k)$$

for all $s^{0\to\rho}$.

Proof Define θ as

$$\theta srn=\begin{cases}0 & \text{if } \exists k\le n\ \big(f\left(\overline{r,k}\right)<k\big)\\ 1+\theta s(\overline{r,n}*sn)(n+1) & \text{otherwise.}\end{cases}$$

θ is clearly obtained by bar recursion. Now, take an arbitrary s and define ϕi as $\phi i=\theta s(\overline{s,i})i$. Hence

$$\phi i=\begin{cases}0 & \text{if } \exists k\le i\ \big(f(\overline{s,k})<k\big)\\ 1+\phi(i+1) & \text{otherwise}\end{cases}$$

since $\overline{s,i} * si = \overline{s,i+1}$. Note that if $\phi i \neq 0$ and $j \leq i$, then $\phi j = 1 + \phi(j+1)$. By induction on j, it is straightforward to show that $\phi i \neq 0 \wedge j \leq i \to \phi 0 = j + \phi j$ since $j + 1 + \phi(j+1) = j + \phi j$. Choose $j = i$. Then, we get $\phi i \neq 0 \to \phi 0 = i + \phi i$ and taking $i = \phi 0$ leads to $\phi(\phi 0) \neq 0 \to \phi 0 = \phi 0 + \phi(\phi 0) \to \phi(\phi 0) = 0$. Thus $\phi(\phi 0) = 0$, which implies that $\exists k \leq \theta s \overline{0} 0 \ (f(\overline{s,k}) < k)$. □

Take d, monotone, and define a_k, b_k and s_k as above. Define s^* by $s^* = \lambda n.a_n$. Kreisel's trick implies that exists k such that $f\left(\overline{s^*,k}^M\right) < k$. Take $s' = \langle x_0, ..., x_{k-1}, 0, 0, ...\rangle$ and $s = \langle c_0, ..., c_{k-1}, 0, 0, ...\rangle$ such that $\forall i < k \ (x_i \trianglelefteq c_i \ \wedge \ c_i \trianglelefteq a_i)$. Therefore, $\overline{s',k} \trianglelefteq \overline{s,k}^M$. By (3.10), it follows that

$$\tilde{\forall} b \ P_B\left(\overline{\overline{s',k}, f\left(\overline{s,k}^M\right)}, f\left(\overline{s,k}^M\right), g\left(\overline{s,k}^M\right), b\right).$$

Since $\overline{s,k}^M \trianglelefteq \overline{s^*,k}^M$, then $f\overline{s,k}^M < k$. By (3.11),

$$\tilde{\forall} b \ P_B\left(\overline{s',k}, k, h\left(\overline{s,k}^M\right) k\left(g\left(\overline{s,k}^M\right)\right), b\right)$$

and (3.12) entails $\tilde{\forall} d' \ Q_B\left(\overline{s',k}, k, p\left(\overline{s,k}^M\right) k\left(h\left(\overline{s,k}^M\right) k\left(g\left(\overline{s,k}^M\right)\right)\right), d'\right)$, which is equivalent to $\tilde{\forall} d' \ Q_B\left(\overline{s',k}, k, W\overline{s,k}\, k, d'\right)$ and to $\tilde{\forall} d' \ D\left(\overline{s',k}, k, \overline{s,k}, d'\right)$. In particular, we have $D(\overline{s',k}, k, \overline{s,k}, b_{k-1})$. We have shown that

$$\forall i < k \ (c_i \trianglelefteq a_i \wedge x_i \trianglelefteq c_i) \to D(\langle x_0, ..., x_{k-1}, 0, 0, ...\rangle, k, \langle c_0, ..., c_{k-1}, 0, 0, ...\rangle, b_{k-1})$$

for all $c_0, ..., c_{k-1}, x_0, ..., x_{k-1}$. By lemma (22), it follows $D(\overline{0}, 0, \overline{0}, d)$. Hence, $\tilde{\forall} d \ D(\overline{0}, 0, \overline{0}, d)$, which entails $\tilde{\exists} w \tilde{\forall} d \ Q_B(\overline{0}, 0, w, d)$ with $w = W(\overline{0}, 0)$, as desired. □

Corollary 4. $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq]$ *proves that under* Hyp1-Hyp4 *of* BI*, we also have* $Q(\overline{s,n}, n)$ *for all* s *and* n.

Proof Assume Hyp1-Hyp4 for formulas P and Q. By the theorem above, it follows $Q(\overline{0}, 0)$. Take $s^{0\to\rho}$ and n^0 and define the following

$$\begin{aligned} P'(r,m) &:= P(\overline{s,n} * r, n+m) \\ Q'(r,m) &:= Q(\overline{s,n} * r, n+m), \end{aligned}$$

where

$$(\overline{s,n} * r)i = \begin{cases} si, & \text{if } i < n \\ r(i-n) & \text{otherwise.} \end{cases}$$

We claim that

$$\forall r \exists m \ P'(r,m). \tag{3.16}$$

by Hyp1, $\exists k\ P\left(\overline{\overline{s,n} * r,k},k\right)$. If $k \geq n$, take $m = n - k$. We get

$$P\left(\overline{\overline{s,n} * \overline{r,m},n+m},n+m\right),$$

which is equivalent to $P'\left(\overline{r,m},m\right)$. If $k < n$, Hyp2 ensures that $P\left(\overline{\overline{s,n} * r,m},m\right)$ for $m \geq k$, in particular, we have $P(\overline{s,n} * \overline{r,m},n+m)$.

$$\forall r,m \forall k \leq m\ \left(P'(\overline{r,k},k) \rightarrow P'(\overline{r,m},m)\right) \tag{3.17}$$

and

$$\forall r,n\ \left(P'(\overline{r,m},m) \rightarrow Q'(\overline{r,m},m)\right) \tag{3.18}$$

are trivial, by Hyp2 and Hyp3. Finally, we claim that

$$\forall r,m\ \left(\forall x\ Q'(\overline{r,m} * x,m+1) \rightarrow Q'(\overline{r,m},m)\right). \tag{3.19}$$

Take r,m and assume $\forall x\ Q'(\overline{r,m} * x,m+1)$, which is equivalent to

$$\forall x\ Q(\overline{s,n} * (\overline{r,m} * x),n+m+1)$$

and to $\forall x\ Q(\overline{\overline{s,n} * \overline{r,m},n+m} * x,n+m+1)$. By Hyp4, it follows

$$Q(\overline{\overline{r,n} * \overline{r,m},n+m},n+m).$$

Hence, we have $Q(\overline{s,n} * \overline{r,m},n+m)$ and $Q'(\overline{r,m},m)$. By (3.16)-(3.19) and the previous theorem, it follows $Q'(\overline{0},0)$, hence, we get $Q(\overline{s,n},n)$. $\square$

Observe that in the previous proof, we used sentences of the type $A(\overline{s,n} * sn) \leftrightarrow A(\overline{s,n+1})$ for arbitrary formulas (P and Q in the particular case). These sentences may not be in $\Delta_{\mathcal{M}^\omega}$ (we know that they are in the case of bounded formulas). Nevertheless, in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq]$, this equivalence for arbitrary formulas is an immediate consequence of the same equivalence for bounded formulas and the characterization theorem.

4

Bounded functional interpretations extended to new base types

Recently, Kohlenbach generalized the Dialectica and the monotone functional interpretations to classes of abstract spaces (normed, metric, hyperbolic, etc). This is accomplished by means of the introduction of new base types for abstract spaces. In this chapter, we generalize the bounded functional interpretation to new base types for normed spaces. We begin by presenting the extended framework. Afterwards, we generalize not only the bounded functional interpretation of Heyting arithmetic, but also the classic direct one of Peano arithmetic. At last, we present some applications in functional analysis.

4.1 The extended framework

As Kohlenbach presents in [Koh05], we extend the set of finite types to a new ground type and then extend the theories HA^ω and $\mathsf{HA}^\omega_{\trianglelefteq}$. Let X be a normed vector space over the reals and take it as a new ground type for the set of all finite types. Before going on describing the new set of all finite types with two base types, we have to make a digression on the representation of real numbers, since X is a normed space *over the reals*. In fact, we must determine how to represent them and how to describe their equality and inequality relations.

4.1.1 Representations of the real numbers

There are many classical constructions of the real numbers, such as Cauchy sequences of rational numbers, Dedekind cuts in the field of rationals, binary representations, signed--digit representations, and so on. Classically,, all these representations are equivalent, in the sense that they give rise to isomorphic structures.

In the following, we give a brief notion of the Cauchy sequence representation (for details see [Koh95]) and we explain the reasons to adopt another representation.

Cauchy sequence representation:

In the Cauchy sequence representation, real numbers are represented by Cauchy sequences of rational numbers with fixed Cauchy modulus 2^{-n}. In order to accomplish this, we begin by defining rational numbers. These are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers: $j(n,m)$ represents the rational number $\frac{\frac{n}{2}}{m+1}$ if n is even and $-\frac{\frac{n+1}{2}}{m+1}$ if n is odd. As so, each rational number can be represented as a code of a certain pair of natural numbers (not unique). The equality relation $=_{\mathbb{Q}}$ between the representants of rational numbers, the inequalities $<_{\mathbb{Q}}$, $\leq_{\mathbb{Q}}$ and the operators $+_{\mathbb{Q}}$ and $._{\mathbb{Q}}$ are defined in the usual way. To simplify the notation, we may write the rational numbers instead of their representations. Nevertheless we must always understand it as the representation.

Real numbers are represented as functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$(*) \qquad \forall n \ \left(|fn -_{\mathbb{Q}} f(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \tfrac{1}{2^{n+1}}\right).$$

Notice that f can be conceived as an infinite sequence of codes of rational numbers and therefore an infinite sequence of rationals. Every f verifying $(*)$ represents a Cauchy sequence of rationals with Cauchy modulus $\frac{1}{2^n}$.

So that each function f of type 1 represents a real number, f is chosen to code the real number given by the Cauchy sequence coded by $\widehat{f}$, where $\widehat{f}$ is defined as

$$\widehat{f}n := \begin{cases} fn & \text{if } \forall k <_0 n \ \left(|fk -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1}{2^{k+1}}\right) \\ fk & \text{for the least natural } k <_0 n \text{ such that } |fk -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} \frac{1}{2^{k+1}} \ . \end{cases}$$

Notice that for each type one f, $\widehat{f}$ satisfies $(*)$. Moreover, if f satisfies $(*)$, then $\forall n^0\ (fn =_0 \widehat{f}n)$. In this way, each function f codes a uniquely determined real number: the one given by the Cauchy sequence coded by $\widehat{f}$.

The equality and inequalities between real numbers represented as Cauchy sequences are defined as follows:

$$f^1 =_C g^1 \text{ is } \forall n \left(\left|\widehat{f}(n+1) -_{\mathbb{Q}} \widehat{g}(n+1)\right|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1}{2^n}\right)$$

$$f^1 <_C g^1 \text{ is } \exists n \left(\widehat{g}(n+1) -_{\mathbb{Q}} \widehat{f}(n+1) \geq_{\mathbb{Q}} \frac{1}{2^n}\right)$$

$$f^1 \leq_C g^1 \text{ is } \forall n \left(\widehat{f}(n+1) -_{\mathbb{Q}} \widehat{g}(n+1) <_{\mathbb{Q}} \frac{1}{2^n}\right).$$

If $f =_C g$, then f^1 and g^1 are representatives of the same real number.

Notice that none of the relations $=_C$, $<_C$ and $\leq_C$ is decidable. While $=_C$ and $\leq_C$ are Π^1_0 statements, the relation $<_C$ is a Σ^1_0 statement. The operations between real numbers are defined in the natural way, as well as the immersion of naturals and rationals into the reals.

The relations above have the usual properties. We will only show a few of them.

Lemma 24. HA^ω *proves*

i) $<_C$ *is transitive;*

ii) $\leq_C$ *is transitive;*

iii) $x <_C y \to x \leq_C y$ *for all* x^1, y^1*;*

iv) $x \leq_C y \to x <_C y + 1$ *for all* x^1, y^1*.*

Proof In order to simplify, it will be used $+$ and $-$ instead of $+_{\mathbb{Q}}$ and $-_{\mathbb{Q}}$.

i) Take x^1, y^1, z^1 such that $x <_C y \wedge y <_C z$. This means that there are natural numbers n_0 and m_0 which verify $\widehat{y}(n_0+1) - \widehat{x}(n_0+1) \geq_{\mathbb{Q}} \frac{1}{2^{n_0}}$ and $\widehat{z}(m_0+1) - \widehat{y}(m_0+1) \geq_{\mathbb{Q}} \frac{1}{2^{m_0}}$. We want to show that there exists n such that $\widehat{z}(n+1) - \widehat{x}(n+1) \geq_{\mathbb{Q}} \frac{1}{2^n}$. Take n as $\max_0(n_0, m_0)$ and assume, without loss of generality, that $n = m_0$. Then

$$\begin{aligned}\widehat{z}(n+1) - \widehat{x}(n+1) =_{\mathbb{Q}} \underbrace{\widehat{z}(n+1) - \widehat{y}(n+1)}_{\geq_{\mathbb{Q}} \frac{1}{2^n}} + \widehat{y}(n+1) - \widehat{y}(n) + \widehat{y}(n) - \ldots +\\ + \widehat{y}(n_0+2) - \widehat{y}(n_0+1) + \underbrace{\widehat{y}(n_0+1) - \widehat{x}(n_0+1)}_{\geq_{\mathbb{Q}} \frac{1}{2^{n_0}}} \geq_{\mathbb{Q}} \frac{1}{2^n}\end{aligned}$$

since

$$\begin{aligned}|\widehat{y}(n+1) - \widehat{y}(n) + \widehat{y}(n) - \widehat{y}(n-1) + \cdots + \widehat{y}(n_0+2) - \widehat{y}(n_0+1)|_{\mathbb{Q}} <_{\mathbb{Q}}\\ <_{\mathbb{Q}} \textstyle\sum_{k=n_0}^{n} \frac{1}{2^k} = \frac{1}{2^{n_0+1}} <_{\mathbb{Q}} \frac{1}{2^{n_0}}.\end{aligned}$$

ii) Take x^1, y^1 and z^1 such that $x \leq_C y$ and $y \leq_C z$. By definition of $\leq_C$, we have $\forall n^0 \left(\widehat{x}(n+1) -_{\mathbb{Q}} \widehat{y}(n+1) <_{\mathbb{Q}} \frac{1}{2^n}\right)$ and $\forall n^0 \left(\widehat{y}(n+1) -_{\mathbb{Q}} \widehat{z}(n+1) <_{\mathbb{Q}} \frac{1}{2^n}\right)$. Take an arbitrary n^0. We claim that

$$\widehat{x}(n+1) - \widehat{z}(n+1) <_{\mathbb{Q}} \frac{1}{2^n}.$$

Take m^0 such that $m \geq_0 n+1$. Then

$$\begin{aligned}\widehat{x}(n+1) - \widehat{z}(n+1) =_{\mathbb{Q}} \sum_{k=n+1}^{m} \underbrace{\Big(\widehat{x}(k) - \widehat{x}(k+1)\Big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} + \underbrace{\big(\widehat{x}(m+1) - \widehat{y}(m+1)\big)}_{<_{\mathbb{Q}} \frac{1}{2^m}} +\\ + \underbrace{\big(\widehat{y}(m+1) - \widehat{z}(m+1)\big)}_{<_{\mathbb{Q}} \frac{1}{2^m}} + \sum_{k=n+1}^{m} \underbrace{\Big(z(k+1) - z(k)\Big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} <_{\mathbb{Q}}\\ <_{\mathbb{Q}} \sum_{k=n+1}^{m} \frac{1}{2^k} + \frac{1}{2^{m-1}} =_{\mathbb{Q}} \frac{1}{2^{n+2}} - \frac{1}{2^{m+2}} + \frac{1}{2^{m-1}} <_{\mathbb{Q}}\\ <_{\mathbb{Q}} \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} =_{\mathbb{Q}} \frac{1}{2^n}.\end{aligned}$$

iii) Take x^1 and y^1 and assume $x <_C y$. By definition, there exists n_0 such that $\widehat{y}(n_0+1) -_{\mathbb{Q}} \widehat{x}(n_0+1) \geq_{\mathbb{Q}} \frac{1}{2^{n_0}}$. We claim that

$$\forall n^0 \ \left(\widehat{x}(n+1) -_{\mathbb{Q}} \widehat{y}(n+1) <_{\mathbb{Q}} \tfrac{1}{2^n}\right).$$

First, let us analyse the case $n <_0 n_0$. Take an arbitrary $n <_0 n_0$. Then

$$\begin{aligned}\widehat{x}(n+1) - \widehat{y}(n+1) =_{\mathbb{Q}} & \sum_{k=n+1}^{n_0} \underbrace{\Big(\widehat{x}(k) - \widehat{x}(k+1)\Big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} + \underbrace{\widehat{x}(n_0+1) - \widehat{y}(n_0+1)}_{\leq_{\mathbb{Q}} -\frac{1}{2^{n_0}}} + \\ & + \sum_{k=n+1}^{n_0} \underbrace{\big(\widehat{y}(k+1) - \widehat{y}(k)\big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} <_{\mathbb{Q}} \sum_{k=n+1}^{n_0} \frac{1}{2^k} - \frac{1}{2^{n_0}} =_{\mathbb{Q}} \\ =_{\mathbb{Q}} & \frac{1}{2^{n+2}} - \frac{1}{2^{n_0+1}} - \frac{1}{2^{n_0}} <_{\mathbb{Q}} \frac{1}{2^n}.\end{aligned}$$

For the second case, take $n \geq n_0$. We get

$$\begin{aligned}\widehat{x}(n+1) - \widehat{y}(n+1) =_{\mathbb{Q}} & \sum_{k=n_0+1}^{n} \underbrace{\big(\widehat{x}(k+1) - \widehat{x}(k)\big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} + \underbrace{\widehat{x}(n_0+1) - \widehat{y}(n_0+1)}_{\leq_{\mathbb{Q}} -\frac{1}{2^{n_0}}} + \\ & + \sum_{k=n_0+1}^{n} \underbrace{\big(\widehat{y}(k) - \widehat{y}(k+1)\big)}_{<_{\mathbb{Q}} \frac{1}{2^{k+1}}} <_{\mathbb{Q}} \sum_{k=n_0+1}^{n} \frac{1}{2^k} - \frac{1}{2^{n_0}} =_{\mathbb{Q}} \\ =_{\mathbb{Q}} & \frac{1}{2^{n_0+2}} - \frac{1}{2^{n+2}} - \frac{1}{2^{n_0}} =_{\mathbb{Q}} -\frac{3}{4}\frac{1}{2^{n_0}} - \frac{1}{2^{n+2}} <_{\mathbb{Q}} \frac{1}{2^n}.\end{aligned}$$

iv) Take x^1 and y^1. Assume $x \leq_C y$, i.e., $\forall n \left(\widehat{x}(n+1) -_{\mathbb{Q}} \widehat{y}(n+1) <_{\mathbb{Q}} \frac{1}{2^n}\right)$. We want to prove $\exists n_0 \left(\widehat{y+1}(n_0+1) -_{\mathbb{Q}} \widehat{x}(n_0+1) \geq_{\mathbb{Q}} \frac{1}{2^{n_0}}\right)$, which is equivalent to

$$\exists n_0 \ \left(\widehat{y}(n_0+1) -_{\mathbb{Q}} \widehat{x}(n_0+1) +_{\mathbb{Q}} 1 \geq_{\mathbb{Q}} \tfrac{1}{2^{n_0}}\right).$$

It suffices to choose $n_0 = 1$:

$$\widehat{y}(2) -_{\mathbb{Q}} \widehat{x}(2) +_{\mathbb{Q}} 1 >_{\mathbb{Q}} -\tfrac{1}{2} + 1 =_{\mathbb{Q}} \tfrac{1}{2} \geq_{\mathbb{Q}} \tfrac{1}{2}.$$

$\square$

This representation is very intuitive and easy to work with. Nevertheless, for technical reasons, it is useful to have an effective representation of the reals which meshes well with the notion of majorizability. In order to carry out efficiently the extension of the bounded functional interpretations to new base types, the representation must satisfy the following majorizability property: there exists a function g from $\mathbb{N}$ to $\mathbb{N}$ such that, if f

is the representation of a real number in $[-n, n]$, then for all $i \in \mathbb{N}$, $fi \leq gn$. This is not satisfied by the Cauchy sequence representation, since the representation of a rational number may be very large.

In the following, we present the signed digit representation (for details, see [Wei00]). As we will see, this representation satisfies the notion of majorizability described above.

Signed digit representation:

Real numbers are represented by tuples (n, α), where $n \in \mathbb{N}$ and $\alpha = \langle \alpha_1, \alpha_2, ... \rangle$ is a sequence of numbers in $\{0, 1, 2\}$. (n, α) represents the number $int\, n + \sum_{i=1}^{+\infty} (\alpha_i - 1)\frac{1}{2^i}$, where int is a type 1 function such that $int\, n$ is equal to $m \in \mathbb{N}$, as a rational, if $n =_0 2m$ and is equal to $-m$ (again, as a rational) if $n =_0 2m - 1$. Note that $int\, n$ represents an integer number. In fact, we see $int\, n$ not as a natural coding a rational, but as the rational itself. The sequence $(\alpha_n - 1)_n$ is a sequence in $\{-1, 0, 1\}$. Each of this sequences α represents the real number $\sum_{i=1}^{+\infty} (\alpha_i - 1)\frac{1}{2^i} \in [-1, 1]$. Note, however, that real numbers shall be represented by type one objects. To each f^1, we associate $\widetilde{f}$:

$$\widetilde{f}0 \;=\; f0$$

and for $n \geq 1$

$$\widetilde{f}n \;=\; \begin{cases} 0 & \text{if } fn = 0 \\ 1 & \text{if } fn \text{ is even} \\ 2 & \text{if } fn \text{ is odd.} \end{cases}$$

Whenever we need to work with real numbers, we shall use $\widetilde{f}$ instead of f. Each $\widetilde{f}$ represents an unique real number. Observe that $f^1 \in \mathbb{R}$ is an universal condition: $\forall i^0 \; (f(i+1) \in \{0, 1, 2\})$. Of course, for each f^1, $\widetilde{f} \in \mathbb{R}$.

Equality and inequality for this representation are given below:

$$f =_{\mathbb{R}} g \text{ is } \forall i \left(\left| int(f0) - int(g0) + \sum_{k=1}^{i+2} (\tilde{f}k - \tilde{g}k)2^{-k} \right|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1}{2^i} \right)$$

$$f <_{\mathbb{R}} g \text{ is } \exists i \left(int(f0) + \sum_{k=1}^{i+2} (\tilde{f}k)2^{-k} + \frac{1}{2^i} \leq_{\mathbb{Q}} int(g0) + \sum_{k=1}^{i+2} (\tilde{g}k)2^{-k} \right)$$

$$f \leq_{\mathbb{R}} g \text{ is } \forall i \left(int(f0) + \sum_{k=1}^{i+2} (\tilde{f}k)2^{-k} <_{\mathbb{Q}} int(g0) + \sum_{k=1}^{i+2} (\tilde{g}k)2^{-k} + \frac{1}{2^i} \right).$$

These relations have the right complexity for proof-theoretic studies: $<_{\mathbb{R}}$ is a Σ_1^0 statement and $=_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ are both Π_1^0 statements. We will see that these relations have the usual properties. In particular, any relation established between real numbers is independent from its representations.

In order to ease the reading, if n is a natural number, then we see n and $-n$ as elements of $\mathbb{R}$ and represent them by

$$
\begin{aligned}
(n)_{\mathbb{R}} &:= \langle 2n, 1, 1, ...\rangle \\
(-n)_{\mathbb{R}} &:= \langle 2n-1, 1, 1, 1, ...\rangle.
\end{aligned}
$$

Whenever it is clear that n is to be read as a real number, we simply write n instead of $(n)_{\mathbb{R}}$.

Proposition 14. HA^{ω} *proves that, given* $gn = 2n+3$, *then for each* f^1 *representing a real number in* $[-n, n]$ *with* $n \in \mathbb{N}$, *we have* $fi \leq gn$ *for all* $i \in \mathbb{N}$.

Proof Take f^1 representing a real number in $[-n, n]$. This means that $f \leq_{\mathbb{R}} n$ and $f \geq_{\mathbb{R}} -n$. In fact, it suffices to prove that $f0 \leq 2n+3$, since $fi \in \{0, 1, 2\}$ for all $i > 0$. For all i, we have

$$
\begin{aligned}
int(f0) + \textstyle\sum_{k=1}^{i+2}(fk-1)\frac{1}{2^k} &<_{\mathbb{Q}} n + \frac{1}{2^i} \\
int(f0) + \textstyle\sum_{k=1}^{i+2}(fk-1)\frac{1}{2^k} &>_{\mathbb{Q}} -n - \frac{1}{2^i},
\end{aligned}
$$

which imply

$$
\begin{aligned}
int(f0) &\leq_{\mathbb{Q}} int(f0) + \textstyle\sum_{k=1}^{i+2}(fk-1)\frac{1}{2^k} + 1 <_{\mathbb{Q}} n+2 \\
int(f0) &\geq_{\mathbb{Q}} int(f0) + \textstyle\sum_{k=1}^{i+2}(fk-1)\frac{1}{2^k} - 1 >_{\mathbb{Q}} -n-2.
\end{aligned}
$$

If $int(f0) \geq_{\mathbb{Q}} 0$, then $f0 =_0 2f(x0) <_0 2n+4$ (since $int(f0)$ is a positive integer, we look at it as a natural) and if $int(f0) <_{\mathbb{Q}} 0$, then $int(f0) =_{\mathbb{Q}} -k$ with $k \in \mathbb{N}$ and $f0 =_0 2k-1 <_0 2n+3$. Hence, $f \leq_1 gn$. □

This result gives the majorizability property. The following presents a majorizability condition between two different representations of the same real number. Notice that in the Cauchy sequence representation none of this properties is satisfied.

Lemma 25. HA^{ω} *proves that if* f *and* g *represent real numbers, then*

$$
f =_{\mathbb{R}} g \to g0 \leq f0 + 4 \wedge \forall i \ (i > 0 \to gi \leq 2).
$$

Proof Take f and g real numbers such that $f =_{\mathbb{R}} g$. We claim that

$$
g0 \leq f0 + 4 \wedge \forall i \ (i > 0 \to gi \leq 2).
$$

It suffices to show that $g0 \leq f0+4$, since the second condition is trivially satisfied. From $f =_{\mathbb{R}} g$, it follows

$$
\begin{aligned}
|int(g0) - int(f0)|_{\mathbb{Q}} \ \leq_{\mathbb{Q}} & \ \left| int(g0) - int(f0) + \sum_{k=1}^{i+2}(gk - fk)\frac{1}{2^k}\right|_{\mathbb{Q}} \\
& + \left|\sum_{k=1}^{i+2}(gk-fk)\frac{1}{2^k}\right|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1}{2^i} + 2 \qquad \text{for all } i.
\end{aligned}
$$

Hence, $|int(g0) - int(f0)|_{\mathbb{Q}} \leq_{\mathbb{Q}} 2$, which implies

$$int(g0) \leq_{\mathbb{Q}} int(f0) + 2 \wedge int(g0) \geq_{\mathbb{Q}} int(f0) - 2.$$

Suppose that $int(f0) \geq_{\mathbb{Q}} 0$ and $int(g0) \geq_{\mathbb{Q}} 0$. From this, we get

$$g0 = \ 2int(g0) \leq_0 2(int(f0) + 2) = 2\,int(f0) + 4 = f0 + 4.$$

If $int(g0) <_{\mathbb{Q}} 0$ and $int(f0) \geq_{\mathbb{Q}} 0$, which implies that $int(g0) =_{\mathbb{Q}} -k$ with $k \in \mathbb{N}$, $g0 = 2k - 1$ and $f0 =_{\mathbb{Q}} 2\,int(f0)$. We get

$$g0 = 2k - 1 \leq 2(-int(f0) + 2) - 1 = -f0 + 3 \leq f0 + 4.$$

The remaining cases are similar. □

Theorem 27. HA^ω *proves that there is an effective translation between the signed-digit representation and the Cauchy sequence representation. Furthermore, the arithmetic relations* $=, <$ *and* $\leq$ *are provably preserved by the translation.*

Proof First, we construct the translation from the signed-digit into the Cauchy sequence representation. Let (n, α) be a signed-digit representation of the real number $int\,n + \sum_{k=1}^{+\infty}(\alpha_k - 1)\frac{1}{2^k}$. Then we define the following Cauchy sequence

$$\left\langle int\,n + (\alpha_1 - 1)\tfrac{1}{2}, int\,n + (\alpha_1 - 1)\tfrac{1}{2} + (\alpha_2 - 1)\tfrac{1}{4}, ..., int\,n + \textstyle\sum_{i=1}^{k}(\alpha_i - 1)\frac{1}{2^i}, ...\right\rangle,$$

where int is the type 1 function defined above. As desired, the limit of this Cauchy sequence is $int\,n + \sum_{i=1}^{+\infty}(\alpha_i - 1)\frac{1}{2^i}$ and, indeed, it represents a real number. Let a_i be given by $int\,n + \sum_{k=1}^{i+1}(\alpha_k - 1)\frac{1}{2^k}$ for all $i \in \mathbb{N}$. Then

$$|a_{i+1} - a_i|_{\mathbb{Q}} =_{\mathbb{Q}} \left|a_i + (\alpha_{i+2} - 1)\tfrac{1}{2^{i+2}} - a_i\right|_{\mathbb{Q}} =_{\mathbb{Q}} \left|(\alpha_{i+2} - 1)\tfrac{1}{2^{i+2}}\right|_{\mathbb{Q}} <_{\mathbb{Q}} \tfrac{1}{2^{i+1}}.$$

Therefore, these two representations stand for the same real number.

The inverse translation is more complicated but the proof is straightforward. Take $\langle a_0, a_1, a_2, ...\rangle$ a Cauchy sequence representation of a real number. For each $i \in \mathbb{N}$, a_i can be decomposed as

$$\begin{aligned}
a_0 &=_{\mathbb{Q}} n + b_0, & |b_0| &<_{\mathbb{Q}} 1,\ n \in \mathbb{Z}\\
a_1 &=_{\mathbb{Q}} a_0 + b_1, & |b_1| &<_{\mathbb{Q}} \frac{1}{2}\\
&\vdots\\
a_{n+1} &=_{\mathbb{Q}} a_n + b_{n+1}, & |b_{n+1}| &<_{\mathbb{Q}} \frac{1}{2^{n+1}}.
\end{aligned}$$

We aim to obtain an integer number m and a sequence α in $\{-1, 0, 1\}$ such that $m + \sum_{k=1}^{+\infty}\alpha_k\frac{1}{2^k}$ is the real number represented by $\langle a_0, a_1, a_2, ...\rangle$. With m and α, we construct $(k, \alpha + 1)$, where $k \in \mathbb{N}$ is given by $2m$ if $m \geq_{\mathbb{Q}} 0$ and by $-2m - 1$ if $m <_{\mathbb{Q}} 0$.

The sequence $\alpha + 1$ is $\langle \alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1, ...\rangle$ in $\{0, 1, 2\}$. In the following, we work with m and α instead of $(k, \alpha + 1)$ to simplify the reading.

First, we determine the integer m. Let us sum b_0 with b_1. We want to write $b_0 + b_1$ as $\alpha_0 + x_0$ with $\alpha_0 \in \{-1, 0, 1\}$ and $|x_0| <_{\mathbb{Q}} \frac{1}{2}$, since $|b_0 + b_1| <_{\mathbb{Q}} \frac{3}{2}$. Then

$$\alpha_0 = \begin{cases} 1 & \text{if } b_0 + b_1 \geq_{\mathbb{Q}} \frac{1}{2} \\ 0 & \text{if } |b_0 + b_1| <_{\mathbb{Q}} \frac{1}{2} \\ -1 & \text{if } b_0 + b_1 \leq_{\mathbb{Q}} -\frac{3}{2} \end{cases}$$

Choose $m = n + \alpha_0$.

To determine α_1 (the first terms of the sequence α), we sum x_0 with b_2 and write it as $\alpha_1 \frac{1}{2} + x_1$ with

$$\alpha_1 = \begin{cases} 1 & \text{if } x_0 + b_2 \geq_{\mathbb{Q}} \frac{1}{4} \\ 0 & \text{if } |x_0 + b_2| <_{\mathbb{Q}} \frac{1}{4} \\ -1 & \text{if } x_0 + b_2 \leq_{\mathbb{Q}} -\frac{1}{4} \end{cases}$$

Since $|x_0 +_{\mathbb{Q}} b_2| < \frac{3}{4}$, we get $|x_1| <_{\mathbb{Q}} \frac{1}{4}$.

In general, if we want to compute the α_i with $i \geq_0 2$, we need to know x_{i-1} and sum it with b_{i+1}. α_i is determined by

$$\alpha_i = \begin{cases} 1 & \text{if } x_{i-1} + b_{i+1} \geq_{\mathbb{Q}} \frac{1}{2^{i+1}} \\ 0 & \text{if } |x_{i-1} + b_{i+1}| <_{\mathbb{Q}} \frac{1}{2^{i+1}} \\ -1 & \text{if } x_{i-1} + b_{i+1} \leq_{\mathbb{Q}} -\frac{1}{2^{i+1}} \end{cases}$$

and we write $x_{i-1} + b_{i+1} =_{\mathbb{Q}} \alpha_i \frac{1}{2^i} + x_i$ with $|x_i| <_{\mathbb{Q}} \frac{1}{2^{i+1}}$.

It remains to prove that $(k, \alpha+1)$ (intuitively (m, α)) represents the real number coded by $\langle a_0, a_1, a_2, ...\rangle$. It consists in proving that the limit of the sequence $(a_n)_n$ is equal to $m + \sum_{i=1}^{+\infty} \alpha_i 2^{-i}$. By construction, we have $\sum_{i=0}^{k} b_i = \sum_{i=0}^{k-1} \alpha_i \frac{1}{2^i} + x_{k-1}$, with $|x_{k-1}| <_{\mathbb{Q}} \frac{1}{2^k}$. Hence,

$$\lim_{k\to+\infty} a_k =_{\mathbb{Q}} \lim_{k\to+\infty} \left(n + \sum_{i=0}^{k} b_i\right) =_{\mathbb{Q}} m +_{\mathbb{Q}} \lim_{k\to+\infty} \sum_{i=1}^{k} \alpha_i \frac{1}{2^i} +_{\mathbb{Q}} \lim_{k\to+\infty} x_{k-1}.$$

Since $\lim_{k\to+\infty} x_k =_{\mathbb{Q}} 0$ (recall $\forall k^0 \; (|x_k| <_{\mathbb{Q}} \frac{1}{2^{k+1}})$, we get

$$\lim_{k\to+\infty} a_k =_{\mathbb{Q}} m +_{\mathbb{Q}} \sum_{i=1}^{+\infty} \alpha_i \frac{1}{2^i}$$

as desired.

So, we presented how to effectively translate from signed-digit to the Cauchy sequence representation and vice-versa. It remains to show that the translation between these representations preserve the equality and inequalities. We will see that the equality is preserved. Let (n, α) and (m, β) be the signed representation of the same real number. Hence, $(n, \alpha) =_{\mathbb{R}} (m, \beta)$:

$$\forall i \; \left(\left| int\, n - int\, m + \sum_{k=1}^{i+2} (\alpha_k - \beta_k) \frac{1}{2^k} \right|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1}{2^i} \right),$$

which is equivalent to

$$\forall i\ \big(\big|\left\langle int\, n+(\alpha_1-1)\tfrac{1}{2}, int\, n+(\alpha_1-1)\tfrac{1}{2}+(\alpha_2-1)\tfrac{1}{4},...\right\rangle(i+1)-$$
$$-\left\langle int\, m+(\beta_1-1)\tfrac{1}{2}, int\, m+(\beta_1-1)\tfrac{1}{2}+(\beta_2-1)\tfrac{1}{4},...\right\rangle(i+1)\big|_{\mathbb{Q}} <_{\mathbb{Q}} \tfrac{1}{2^i}\big).$$

By definition of $=_C$, we get

$$\left\langle int\, n+(\alpha_1-1)\tfrac{1}{2}, int\, n+(\alpha_1-1)\tfrac{1}{2}+(\alpha_2-1)\tfrac{1}{4},\cdots\right\rangle =_C$$
$$=_C \left\langle int\, m+(\beta_1-1)\tfrac{1}{2}, int\, m+(\beta_1-1)\tfrac{1}{2}+(\beta_2-1)\tfrac{1}{4},\cdots\right\rangle.$$

Now, take $(a_n)_n =_C (b_n)_n$. We write $(a_n)_n$ and $(b_n)_n$ as $(n+c_0, n+c_0+c_1, n+c_0+c_1+c_2,\cdots)$ and $(m+d_0, m+d_0+d_1, m+d_0+d_1+d_2,\cdots)$, respectively. If

$$(n+c_0, n+c_0+c_1, n+c_0+c_1+c_2,\cdots) =_C (m+d_0, m+d_0+d_1, m+d_0+d_1+d_2,\cdots),$$

then $\forall i\ \left(\left|n+\sum_{k=1}^{i+1} c_k - \left(m+\sum_{k=1}^{i+1} d_k\right)\right|_{\mathbb{Q}}\right) <_{\mathbb{Q}} \frac{1}{2^i}$. By construction, we can determine sequences $(\alpha_n)_n$ and $(\beta_n)_n$ such that

$$\textstyle\sum_{k=0}^{i+1} c_k =_{\mathbb{Q}} \sum_{k=0}^{i} \alpha_k \frac{1}{2^k} + x_i \quad \text{and} \quad \sum_{k=0}^{i+1} d_k =_{\mathbb{Q}} \sum_{k=0}^{i} \beta_k \frac{1}{2^k} + y_i,$$

for some $(x_n)_n$ and $(y_n)_n$ with $|x_i|, |y_i| <_{\mathbb{Q}} \frac{1}{2^{i+1}}$ for all $i \in \mathbb{N}$. Hence, for all i, we get

$$\left|n+\alpha_0-(m+\beta_0)+\textstyle\sum_{k=1}^{i+2}(\alpha_k-\beta_k)\frac{1}{2^k}\right|_{\mathbb{Q}} =_{\mathbb{Q}} \left|n+\sum_{k=0}^{i+2}\alpha_k\frac{1}{2^k}-\left(m+\sum_{k=0}^{i+2}\beta_k\frac{1}{2^k}\right)\right|_{\mathbb{Q}} <_{\mathbb{Q}}$$
$$<_{\mathbb{Q}} \left|n+\textstyle\sum_{k=0}^{i+3} c_k-\left(m+\sum_{k=0}^{i+3} d_k\right)\right|_{\mathbb{Q}} + |x_{i+2}|_{\mathbb{Q}} + |y_{i+2}|_{\mathbb{Q}} <_{\mathbb{Q}}$$
$$<_{\mathbb{Q}} \tfrac{1}{2^{i+2}} + \tfrac{1}{2^{i+3}} + \tfrac{1}{2^{i+3}} <_{\mathbb{Q}} \tfrac{1}{2^i}.$$

The later is equivalent to $(k_1, \alpha+1) =_{\mathbb{R}} (k_2, \beta+1)$, where $(k_1, \alpha+1)$ with $int(k_1) = n+\alpha_0$ and $\alpha = \langle\alpha_1, \alpha_2, ...\rangle$ is the translation of $(a_n)_n$ and $(k_2, \beta+1)$ with $int(k_2) = m+\beta_0$ and $\beta = \langle\beta_0, \beta_1, ...\rangle$ is translation of $(b_n)_n$ to the signed-digit representation. In a similar way, we prove that the less or equal relation is preserved by these translations. □

As shown above, given the signed-digit representation of a real number, there is an effective way to translate it to the Cauchy sequence representation and vice-versa, preserving $=, <$ and $\leq$. Hence, all the well-known properties of $=_C, <_C$ and $\leq_C$ are still satisfied by $=_{\mathbb{R}}, <_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$. We adopt the signed-digit representation. Nevertheless, it is easier to define the arithmetic operations and compute them with the Cauchy sequence representation. So, whenever it is necessary to perform computations with reals represented by the signed-digit representations, we translate them to the Cauchy sequence representation, make the desired computations and then, translate them back to the signed-digit representation. Reasoning in this way, we can easily prove some properties:

Lemma 26. HA^ω *proves, for all* n^0 *and* m^0*, that*

i) $(n)_{\mathbb{R}} + (m)_{\mathbb{R}} =_{\mathbb{R}} (n+m)_{\mathbb{R}}$

ii) $(n)_{\mathbb{R}}(m)_{\mathbb{R}} =_{\mathbb{R}} (nm)_{\mathbb{R}}$

iii) $|(int\, n)_{\mathbb{R}}|_{\mathbb{R}} \leq_{\mathbb{R}} (n)_{\mathbb{R}}$

iv) $|\widetilde{f} - (int(f0))_{\mathbb{R}}|_{\mathbb{R}} \leq_{\mathbb{R}} (1)_{\mathbb{R}}$ *for all* f^1.

Proof

i) We have $(n)_{\mathbb{R}} = \langle 2n, 1, 1, ...\rangle$ and $(m)_{\mathbb{R}} = \langle 2m, 1, 1, ...\rangle$ and represent them in Cauchy sequence representation: $(n)_C = \langle n, n, ...\rangle$ and $(m)_C = \langle m, m, ...\rangle$. Then, we sum $(n)_C$ with $(m)_C$ and obtain $(n+m)_C = \langle n+m, n+m, ...\rangle$, which we represent now in the signed-digit representation as $\langle 2(n+m), 1, 1, ...\rangle = (n+m)_{\mathbb{R}}$.

ii) This proof is very similar to the one above.

iii) We prove this in two steps. First, assume $int\, n \geq_{\mathbb{Q}} 0$. Then $int\, n =_{\mathbb{Q}} m$ with $n =_0 2m$. Hence $|(int\, n)_{\mathbb{R}}|_{\mathbb{R}} =_{\mathbb{R}} (int\, n)_{\mathbb{R}}$, since $|\langle int\, n, int\, n, int\, n, ...\rangle|_C$ is equal to $\langle int\, n, int\, n, int\, n, ...\rangle$. We want to see that $|(int\, n)_{\mathbb{R}}|_{\mathbb{R}} \leq_{\mathbb{R}} (n)_{\mathbb{R}}$, which is equivalent to $\langle 2m, 1, 1, ...\rangle \leq_{\mathbb{R}} \langle 2n, 1, 1, ...\rangle$. And this is clear, since $n = 2m$.

   Second, if $int\, n <_{\mathbb{Q}} 0$, then $int\, n =_{\mathbb{Q}} -m$ with $n = 2m - 1$. Then $|(int\, n)_{\mathbb{R}}|_{\mathbb{R}} =_{\mathbb{R}} |(-m)_{\mathbb{R}}| = \langle 2m, 1, 1, ...\rangle$. Hence, we want $\langle 2m, 1, 1, ...\rangle \leq_{\mathbb{R}} \langle 2n, 1, 1, ...\rangle$. This follows clearly from $2m = n + 1 \leq 2n$.

iv) This is proved for two cases: $\tilde{f} - (int(f0))_{\mathbb{R}} \geq_R 0$ and $\tilde{f} - (int(f0))_{\mathbb{R}} <_R 0$. However, since they are similar, so we only prove it for the first case. Assume that $\tilde{f} - (int(f0))_{\mathbb{R}} \geq_{\mathbb{R}} 0$. Then $|\tilde{f} - (int(f0))_{\mathbb{R}}|_{\mathbb{R}} =_{\mathbb{R}} \tilde{f} - (int(f0))_{\mathbb{R}}$. Let $\tilde{f}$ be$\langle f0, \tilde{f}1, \tilde{f}2, ...\rangle$, then $\tilde{f} - (int(f0))_{\mathbb{R}} = \langle f0, \tilde{f}1, \tilde{f}2, ...\rangle -_{\mathbb{R}} \langle f0, 1, 1, ...\rangle$. In the Cauchy sequence representation this is represented by

$$\left\langle int(f0) + \left(\tilde{f}1 - 1\right)\tfrac{1}{2}, int(f0) + \left(\tilde{f}1 - 1\right)\tfrac{1}{2} + \left(\tilde{f}2 - 1\right) + \tfrac{1}{4}, ... \right\rangle -_C$$

$$-_C \langle int(f0), int(f0), int(f0), ...\rangle =_C \left\langle \left(\tilde{f}1 - 1\right)\tfrac{1}{2}, \left(\tilde{f}1 - 1\right)\tfrac{1}{2} + \left(\tilde{f}2 - 1\right)\tfrac{1}{4}, ... \right\rangle.$$

   Then, the signed-digit representation of the result is $\langle 0, \tilde{f}1, \tilde{f}2, ...\rangle$. Let us check that $\langle 0, \tilde{f}1, \tilde{f}2, ...\rangle \leq_{\mathbb{R}} \langle 2, 1, 1, ...\rangle$:

$$\textstyle\sum_{k=1}^{i+2} \left(\tilde{f}k - 1\right)\frac{1}{2^k} <_{\mathbb{Q}} \sum_{k=1}^{+\infty} \frac{1}{2^k} =_{\mathbb{Q}} 1 <_{\mathbb{Q}} 1 + \frac{1}{2^i}$$

   for all $i \in \mathbb{N}$.

$\square$

4.1.2 The theory $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$

Part of this section follows closely the work of Kohlenbach (see, for instance, [Koh08a], [Koh05], [Koh08b] and [GK08]. Take a normed vector space over the reals and let X be a new ground type representing elements of the normed space. We denote the set of all finite types with ground types 0 and X as T^{X}. It is defined recursively by:

i) $0, \mathsf{X} \in \mathsf{T}^{\mathsf{X}}$
ii) $\rho, \sigma \in \mathsf{T}^{\mathsf{X}} \;\Rightarrow\; (\rho \to \sigma) \in \mathsf{T}^{\mathsf{X}}$.

T is still the set of all finite types with a unique ground type 0.

Let $\mathcal{L}^{\omega,\mathsf{X}}$ be the extension of the language $\mathcal{L}^{\omega}$ obtained by adding variables of type X, new constants

$$\begin{array}{l} 0_{\mathsf{X}} \text{ of type } \mathsf{X} \\ +_{\mathsf{X}} \text{ of type } \mathsf{X} \to (\mathsf{X} \to \mathsf{X}) \\ -_{\mathsf{X}} \text{ of type } \mathsf{X} \to \mathsf{X} \\ \cdot_{\mathsf{X}} \text{ of type } 1 \to (\mathsf{X} \to \mathsf{X}) \\ \|.\| \text{ of type } \mathsf{X} \to 1, \end{array}$$

new quantifiers $\forall x^{\rho}, \exists x^{\rho}$ for all $\rho \in \mathsf{T}^{\mathsf{X}}$ and by extending the constants Π, Σ and $\underline{R}$ in the natural way to the new types.

Using the new variables, we define an equality relation in X, denoted by $=_{\mathsf{X}}$:

$$x =_{\mathsf{X}} y \text{ is } \|x -_{\mathsf{X}} y\| =_{\mathbb{R}} 0_{\mathbb{R}}.$$

In fact, to be exact, we have to write $\|\widetilde{x -_{\mathsf{X}} y}\|$ since $\|x -_{\mathsf{X}} y\|$ *may not* be the representation of a real number. In the future, this will no longer be a problem, since we will add to the theory an axiom which guaranties that $\|x\|$ is *always* the representation of a real number. We use $\forall x^{\mathbb{R}}\ A(x)$ to abbreviate $\forall x^{1}\ A(\widetilde{x})$. With this abbreviation, there is no need to constantly remark that, when working with the reals, we must use $\widetilde{x}$ instead of x. Whenever it is clear, to ease the reading, we use $0, +, -$ instead of $0_{\mathsf{X}}, +_{\mathsf{X}}$ and $-_{\mathsf{X}}$. In the case of $\cdot_{\mathsf{X}}$, we may even not write the operation sign (and write αx instead of $\alpha \cdot_{\mathsf{X}} x$).

Let $\mathsf{HA}^{\omega,\mathsf{X}}$ be the extension of HA^{ω} to the language $\mathcal{L}^{\omega,\mathsf{X}}$, with the additional axioms for $+_{\mathsf{X}}$, $-_{\mathsf{X}}$, $\cdot_{\mathsf{X}}$ and $\|.\|$:

vector space axioms for $+_{\mathsf{X}}, -_{\mathsf{X}}, \cdot_{\mathsf{X}}$ and 0_{X} formulated with the equality $=_{\mathsf{X}}$, such as commutativity, associativity of $+_{\mathsf{X}}$ and distributivity of $+_{\mathsf{X}}$ with respect to $\cdot_{\mathsf{X}}$

$$\begin{array}{lll}
\mathsf{R} & : & \forall x^{\mathsf{X}}\ \big(\|x\| \in \mathbb{R}\big) \\
\mathsf{N1} & : & \forall x^{\mathsf{X}}\ \big(\|x -_{\mathsf{X}} x\| =_{\mathbb{R}} 0_{\mathbb{R}}\big) \\
\mathsf{N2} & : & \forall x^{\mathsf{X}}, y^{\mathsf{X}}\ \big(\|x -_{\mathsf{X}} y\| =_{\mathbb{R}} \|y -_{\mathsf{X}} x\|\big) \\
\mathsf{N3} & : & \forall x^{\mathsf{X}}, y^{\mathsf{X}}, z^{\mathsf{X}}\ \big(\|x -_{\mathsf{X}} z\| \leq_{\mathbb{R}} \|x -_{\mathsf{X}} y\| +_{\mathbb{R}} \|y -_{\mathsf{X}} z\|\big) \\
\mathsf{N4} & : & \forall \alpha^{1}, x^{\mathsf{X}}, y^{\mathsf{X}}\ \big(\|\alpha \cdot_{\mathsf{X}} x -_{\mathsf{X}} \alpha \cdot_{\mathsf{X}} y\| =_{\mathbb{R}} |\widetilde{\alpha}|_{\mathbb{R}} \cdot_{\mathbb{R}} \|x -_{\mathsf{X}} y\|\big) \\
\mathsf{N5} & : & \forall \alpha^{1}, \beta^{1}, x^{\mathsf{X}}\ \big(\|\alpha \cdot_{\mathsf{X}} x -_{\mathsf{X}} \beta \cdot_{\mathsf{X}} x\| =_{\mathbb{R}} |\widetilde{\alpha} -_{\mathbb{R}} \widetilde{\beta}| \cdot_{\mathbb{R}} \|x\|\big) \\
\mathsf{N6} & : & \forall x^{\mathsf{X}}, y^{\mathsf{X}}, u^{\mathsf{X}}, v^{\mathsf{X}}\ \big(\|(x +_{\mathsf{X}} y) -_{\mathsf{X}} (u +_{\mathsf{X}} v)\| \leq_{\mathbb{R}} \|x -_{\mathsf{X}} u\| +_{\mathbb{R}} \|y -_{\mathsf{X}} v\|\big) \\
\mathsf{N7} & : & \forall x^{\mathsf{X}}, y^{\mathsf{X}}\ \big(\|(-_{\mathsf{X}} x) -_{\mathsf{X}} (-_{\mathsf{X}} y)\| =_{\mathbb{R}} \|x -_{\mathsf{X}} y\|\big) \\
\mathsf{N8} & : & \forall x^{\mathsf{X}}, y^{\mathsf{X}}\ \big(\big|\|x\| -_{\mathbb{R}} \|y\|\big|_{\mathbb{R}} \leq_{\mathbb{R}} \|x -_{\mathsf{X}} y\|\big).
\end{array}$$

We use this unfamiliar set of axioms instead of the usual ones to ensure that the $+_X, -_X, \cdot_X$ and $\|.\|$ are extensional with respect to $=_X$.

In $\mathsf{N4}$ and $\mathsf{N5}$, we can write $\forall \alpha^{\mathbb{R}}, \beta^{\mathbb{R}}$ instead of $\forall \alpha^1, \beta^1$, with no need of writing $\widetilde{\alpha}$ to represent a real number. In this case, the axioms become easier to read. Nevertheless, recall that $\cdot_X$ is of the type $1 \to (X \to X)$, and then, the properties shall hold for all type 1 objects. Of course, whenever we are dealing with real numbers, α must be replaced by $\widetilde{\alpha}$, as happens in $\mathsf{N4}$ and $\mathsf{N5}$.

As aforementioned, the axiom R states that for any x of type X, $\|x\|$ represents a real number. The vector space axioms together with the axioms $\mathsf{N1}$ to $\mathsf{N8}$, prove that $(X, +_X, -_X, 0_X)$ is a linear space with a pseudo-norm $\|.\|$ and that $\|0_X\| =_{\mathbb{R}} 0_{\mathbb{R}}$. The only primitive predicate is $=_0$ still.

The axioms $\mathsf{N1}$-$\mathsf{N8}$ prove the reflexivity, symmetry and transitivity of $=_X$, besides of proving the $=_X$-extensionality of $+_X$, $-_X$, $\cdot_X$ and $\|.\|$:

$$\begin{aligned}
\mathsf{N3}, \mathsf{N4} \text{ and } \mathsf{N5} &\Rightarrow \forall \alpha^1, \beta^1, x^X, y^X \big(\alpha =_{\mathbb{R}} \beta \wedge x =_X y \to \alpha \cdot_X x =_X \beta \cdot_X y\big)\\
\mathsf{N6} &\Rightarrow \forall x^X, y^X, u^X, v^X \big(x =_X u \wedge y =_X v \to x +_X y =_X u +_X v\big)\\
\mathsf{N7} &\Rightarrow \forall x^X, y^X \big(x =_X y \to -_X x =_X -_X y\big)\\
\mathsf{N8} &\Rightarrow \forall x^X, y^X \big(x =_X y \to \|x\| =_X \|y\|\big).
\end{aligned}$$

Note however, that $x =_X y$ does not imply that $\|x\|(k) =_0 \|y\|(k)$, since the representation of a real number is not unique.

From the above, $\|.\|$ is a norm in the equivalence classes generated by $=_X$.

Since the theory $\mathsf{HA}^{\omega,X}$ is already presented, we proceed by extend it. In order to do so, we extend Bezem's strong majorizability for all types ρ in T^X. The following definition is due to Kohlenbach:

Definition 12. *For every $\rho \in \mathsf{T}^X$ we define inductively $\widehat{\rho} \in \mathsf{T}$:*

i) $\widehat{0} := 0$;

ii) $\widehat{X} := 0$;

iii) $\widehat{\rho \to \sigma} := \widehat{\rho} \to \widehat{\sigma}$.

We call arithmetic types *to the types in* T *and* mixed types *to all the other types if* T^X*.*

Clearly, if ρ is an arithmetic type, then $\widehat{\rho}$ is ρ and that for all types $\widehat{\widehat{\rho}}$ is $\widehat{\rho}$.

Bezem's strong majorizability relation extended to all types in T^X is given by:

Definition 13.

$$\begin{aligned}
&i) \quad n^0 \leq^*_0 m^0 := n \leq_0 m\\
&ii) \quad x^X \leq^*_X n^0 := \|x\| \leq_{\mathbb{R}} (n)_{\mathbb{R}}\\
&iii) \quad x^{\rho\to\sigma} \leq^*_{\rho\to\sigma} y^{\widehat{\rho\to\sigma}} := \forall u^{\rho}, v^{\widehat{\rho}} \big(u \leq^*_{\rho} v \to xu \leq^*_{\sigma} yv\big) \wedge \ \forall u^{\widehat{\rho}}, v^{\widehat{\rho}} \ \big(u \leq^*_{\widehat{\rho}} v \to yu \leq^*_{\widehat{\sigma}} yv\big).
\end{aligned}$$

Notice that the majorants are all of arithmetic type. In particular, if both objects are of arithmetic type, then the generalized $\leq^*$ coincides with usual Bezem's strong majorizability $\leq^*$. The properties of $\leq^*$ are also extended to all $\rho \in \mathsf{T}^{\mathsf{X}}$:

Lemma 27. $\mathsf{HA}^{\omega,\mathsf{X}}$ *proves*

i) $x \leq^*_\rho y \to y \leq^*_{\widehat{\rho}} y$

ii) $x \leq^*_\rho y \wedge y \leq^*_{\widehat{\rho}} z \to x \leq^*_\rho z.$

Proof In case $i)$, the result for types 0 and X follows from the reflexivity of $\leq_0$. For type $\rho \to \sigma$, it is straightforward from the definition of $\leq^*_{\rho\to\sigma}$. In case $ii)$, it is trivial for type 0. For type X, it follows from the transitivity of $\leq_{\mathbb{R}}$, since $n \leq_0 m \to (n)_{\mathbb{R}} \leq_{\mathbb{R}} (m)_{\mathbb{R}}$. For type $\rho \to \sigma$, we argue by induction on types. Assume $x \leq^*_{\rho\to\sigma} y$ and $y \leq^*_{\widehat{\rho\to\sigma}} z$. It suffices to prove that $\forall u^\rho, v^{\widehat{\rho}} \left(u \leq^*_\rho v \to xu \leq^*_\sigma zv\right)$, since $\forall u^{\widehat{\rho}}, v^{\widehat{\rho}} \left(u \leq^*_{\widehat{\rho}} v \to zu \leq^*_{\widehat{\sigma}} zv\right)$ follows from $y \leq^*_{\widehat{\rho\to\sigma}} z$. Take u^ρ and $v^{\widehat{\rho}}$ such that $u \leq^*_\rho v$. By $i)$, $u \leq^*_\rho v \to v \leq^*_{\widehat{\rho}} v$. Then $xu \leq^*_\sigma yv \wedge yv \leq^*_{\widehat{\sigma}} zv$. Using the induction hypothesis, it follows that $xu \leq^*_\sigma zv$. $\square$

Let $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ be the extension of the language $\mathcal{L}^{\omega,\mathsf{X}}$ obtained by adding new relational symbols $\trianglelefteq_\rho$ (between objects of type ρ and $\widehat{\rho}$) for every type $\rho \in \mathsf{T}^{\mathsf{X}}$ and new quantifiers $\forall x \trianglelefteq t$ and $\exists x \trianglelefteq t$ for terms t not containing x. The relation $\trianglelefteq_\rho$ is still the intensional counterpart of the generalized $\leq^*_\rho$. Of course, the new quantifiers are still called *bounded quantifiers* and formulas in which every quantifier is generalized bounded are called *bounded formulas.*

Definition 14. $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *is the extension of* $\mathsf{HA}^{\omega,\mathsf{X}}$ *with language* $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$*, obtained by adding the following axioms*

$\mathsf{B}_\forall$: $\forall x \trianglelefteq t A(x) \leftrightarrow \forall x \left(x \trianglelefteq t \to A(x)\right)$

$\mathsf{B}_\exists$: $\exists x \trianglelefteq t A(x) \leftrightarrow \exists x \left(x \trianglelefteq t \wedge A(x)\right)$

where t *is a term not containing* x*, and*

M_1 : $n \trianglelefteq_0 m \leftrightarrow n \leq_0 m$

M_2 : $x \trianglelefteq_{\mathsf{X}} n \to \|x\| \leq_{\mathbb{R}} (n)_{\mathbb{R}}$

M_3 : $x \trianglelefteq_{\rho\to\sigma} y \to \forall u^\rho, v^{\widehat{\rho}} \left(u \trianglelefteq_\rho v \to xu \trianglelefteq_\sigma yv\right) \wedge \forall u^{\widehat{\rho}}, v^{\widehat{\rho}} \left(u \trianglelefteq_{\widehat{\rho}} v \to yu \trianglelefteq_{\widehat{\sigma}} yv\right)$

and rules

$$\mathsf{RL}_1 : \quad \frac{A_{bd} \to \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}}{A_{bd} \to s \trianglelefteq_{\mathsf{X}} t}$$

$$\mathsf{RL}_2 : \quad \frac{A_{bd} \wedge u \trianglelefteq_\rho v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to su \trianglelefteq_\sigma tv \wedge tu' \trianglelefteq_{\widehat{\sigma}} tv'}{A_{bd} \to s \trianglelefteq_{\rho\to\sigma} t}$$

where s *and* t *are terms of* $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *and* u, v, u', v' *are variables not occurring free in the conclusion of* RL_2 *and* A_{bd} *is a bounded formula. Notice that in* RL_1*,* s *is of type* X *and* t *is of type 0 and in* RL_2*,* s *and* t *are of types* $\rho \to \sigma$ *and* $\widehat{\rho \to \sigma}$*, respectively.*

The axioms $\mathsf{B}_\forall$ and $\mathsf{B}_\exists$ are the generalization of the ones in $\mathsf{HA}^\omega_\trianglelefteq$. Since the converse of M_2 and M_3 do not have a generalized bounded interpretation, we use the rules RL_1 and RL_2 instead of the axioms.

Lemma 28. $\mathsf{HA}^{\omega,\mathsf{X}}_\trianglelefteq$ *proves*

i) $x \trianglelefteq y \to y \trianglelefteq y$

ii) $x \trianglelefteq y \wedge y \trianglelefteq z \to x \trianglelefteq z$

Proof This proof is an adaptation of the proof of lemma 27, since this one does not use the converse of M_2 and M_3, only its weakened versions, RL_1 and RL_2. ☐

As we have seen, $x \leq_\mathbb{R} y$ is of the form $\forall n^0\ A_{qf}(n,x,y)$ with $A_{qf}(n,x,y)$ given by $int(x0) + \sum_{k=1}^{n+2}(xk)2^{-k} <_\mathbb{Q} int(y0) + \sum_{k=1}^{n+2}(yk)2^{-k} + 2^{-n}$. Sometimes, it is useful to have an intensional inequality between real numbers:

Definition 15. *Take arbitrary* $x^\mathbb{R}$ *and* $y^\mathbb{R}$*. The relation* $\trianglelefteq_\mathbb{R}$ *is defined by*

$$x \trianglelefteq_\mathbb{R} y := p(x,y) \trianglelefteq_1 0,$$

where $p^{1\to(1\to1)}$ *is defined as*

$$p(x,y)n = \begin{cases} 0 & \text{if } A_{qf}(n,x,y) \\ 1 & \text{otherwise.} \end{cases}$$

As opposed to $x \leq_\mathbb{R} y$, $x \trianglelefteq_\mathbb{R} y$ is a quantifier-free statement.

Lemma 29. $\mathsf{HA}^{\omega,\mathsf{X}}_\trianglelefteq$ *proves that for all* $n^0, x^\mathbb{R}, y^\mathbb{R}$ *and* z^X*, we have*

i) $x <_\mathbb{R} y \to x \trianglelefteq_\mathbb{R} y$ *and* $x \trianglelefteq_\mathbb{R} y \to x \leq_\mathbb{R} y$*;*

ii) $\trianglelefteq_\mathbb{R}$ *is transitive;*

iii) $z \trianglelefteq_\mathsf{X} n \leftrightarrow \|z\| \trianglelefteq_\mathbb{R} (n)_\mathbb{R}$*;*

iv) $\|z\| \leq_\mathbb{R} y \wedge y \leq_\mathbb{R} (n)_\mathbb{R} \to z \trianglelefteq_\mathsf{X} n+1$.

It is worth to note that by *i)* and *iii)*, we also get $\|z\| <_\mathbb{R} n \to z \trianglelefteq_\mathsf{X} n$ and $z \trianglelefteq_\mathsf{X} n \to \|z\| \leq_\mathbb{R} n$.

Observe that we have $z \trianglelefteq_\mathsf{X} n \leftrightarrow \|z\| \trianglelefteq_\mathbb{R} (n)_\mathbb{R}$ for all z^X and n^0 and $\trianglelefteq_\mathbb{R}$ is easily related with $<_\mathbb{R}$ and $\leq_\mathbb{R}$. With this equivalence we may avoid the use of the rule RL_1 and the axiom M_2. Nevertheless, we may use M_2 and RL_1 whenever we need them.

Proof Along the proof, we will quantify over $\mathbb{R}$. Recall that we use $\forall x^\mathbb{R}\ A(x)$ to abbreviate $\forall x^1\ A(\widetilde{x})$. The properties must be proved for $\widetilde{x}$, nevertheless, to ease the reading, we omit this notation.

i) Take $x^{\mathbb{R}}, y^{\mathbb{R}}$. Of course, we have $x <_{\mathbb{R}} y \to x \leq_{\mathbb{R}} y$, which is equivalent to

$$\exists n_0 \ \left(int(x0) + \sum_{k=1}^{n_0+2} xk\ \tfrac{1}{2^k} + \tfrac{1}{2^{n_0}} \leq_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n_0+2} yk\ \tfrac{1}{2^k}\right) \to$$
$$\to \forall n \ \left(int(x0) + \sum_{k=1}^{n+2} xk\ \tfrac{1}{2^k} <_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n+2} yk\ \tfrac{1}{2^k} + \tfrac{1}{2^n}\right)$$

By intuitionistic logic,

$$\forall n_0 \forall n \ \left(int(x0) + \sum_{k=1}^{n_0+2} xk\ \tfrac{1}{2^k} + \tfrac{1}{2^{n_0}} \leq_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n_0+2} yk\ \tfrac{1}{2^k} \to\right.$$
$$\left.\to int(x0) + \sum_{k=1}^{n+2} xk\ \tfrac{1}{2^k} <_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n+2} yk\ \tfrac{1}{2^{-k}} + \tfrac{1}{2^n}\right).$$

In particular, we have

$$int(x0) + \sum_{k=1}^{n_0+2} xk\ \tfrac{1}{2^k} + \tfrac{1}{2^{n_0}} \leq_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n_0+2} yk\ \tfrac{1}{2^k} \to p(x,y)n \leq_0 0.$$

By RL_2,

$$int(x0) + \sum_{k=1}^{n_0+2} xk\ \tfrac{1}{2^k} + \tfrac{1}{2^{n_0}} \leq_{\mathbb{Q}} int(y0) + \sum_{k=1}^{n_0+2} yk\ \tfrac{1}{2^k} \to p(x,y) \trianglelefteq_1 0.$$

From the latter, we obtain $x <_{\mathbb{R}} y \to x \trianglelefteq_{\mathbb{R}} y$. The proof of that $x \trianglelefteq_{\mathbb{R}} y \to x \leq_{\mathbb{R}} y$, is straightforward.

ii) Take $x^{\mathbb{R}}, y^{\mathbb{R}}$ and $z^{\mathbb{R}}$ and assume that $x \trianglelefteq_{\mathbb{R}} y$ and $y \leq_{\mathbb{R}} z$. From i) together with the transitivity of $\leq_{\mathbb{R}}$, we get $x \leq_{\mathbb{R}} z$. Therefore, for all n^0, we get

$$int(x0) + \sum_{k=1}^{n+2} xk\ \tfrac{1}{2^k} <_{\mathbb{Q}} int(z0) + \sum_{k=1}^{n+2} zk\ \tfrac{1}{2^k} + \tfrac{1}{2^n}$$

i.e., $p(x,z)n \leq_0 0$. By the rule RL_2, we get $x \trianglelefteq_{\mathbb{R}} y \wedge y \trianglelefteq_{\mathbb{R}} z \to x \trianglelefteq_{\mathbb{R}} z$, as desired.

iii) Take n^0 and z^{X}. First, let us prove $z \trianglelefteq_{\mathsf{X}} n \to \|z\| \trianglelefteq_{\mathbb{R}} n$. Assume $z \trianglelefteq_{\mathsf{X}} n$. M_2 implies $\|z\| \leq_{\mathbb{R}} n$. Hence, for all m^0 we get

$$z \trianglelefteq_{\mathsf{X}} n \to int(\|z\|0) + \sum_{k=1}^{m+2} \|z\|k\ \tfrac{1}{2^k} <_{\mathbb{Q}} 2n + \sum_{k=1}^{m+2} \tfrac{1}{2^k} + \tfrac{1}{2^m},$$

which is equivalent to $z \trianglelefteq_{\mathsf{X}} n \to p(\|z\|, n)m \leq_0 0$. By RL_2, we get $z \trianglelefteq_{\mathsf{X}} n \to \|z\| \trianglelefteq_{\mathbb{R}} n$.
On the other hand, by i), we get $\|z\| \trianglelefteq_{\mathbb{R}} n \to \|z\| \leq_{\mathbb{R}} n$ and by RL_1, it follows that $\|z\| \trianglelefteq_{\mathbb{R}} n \to z \trianglelefteq_{\mathsf{X}} n$, since $\|z\| \trianglelefteq_{\mathbb{R}} n$ is quantifier-free.

iv) This result is a direct consequence of i) and iii). Take $n^0, y^{\mathbb{R}}$ and z^{X}. Then

$$\|z\| \leq_{\mathbb{R}} y \ \wedge \ y \leq_{\mathbb{R}} n \to \|z\| \leq_{\mathbb{R}} n.$$

Then $\|z\| <_{\mathbb{R}} n+1$ and, by i), $\|z\| \trianglelefteq_{\mathbb{R}} n+1$. By iii), we get $z \trianglelefteq_{\mathsf{X}} n+1$.

□

Lemma 30. *Let A be an arbitrary formula of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$. Then $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves*

$$\forall z^{\mathbb{R}} \exists n^0 \ \left(\forall x \trianglelefteq_{\mathsf{X}} n \ \left(\|x\| \leq_{\mathbb{R}} z \to A(x,z)\right) \ \leftrightarrow \ \forall x^{\mathsf{X}} \ \left(\|x\| \leq_{\mathbb{R}} z \to A(x,z)\right)\right).$$

Proof Take $z^{\mathbb{R}}$. There is m^0 such that $z \leq_{\mathbb{R}} m$. Define $n = m+1$. The right-to-left implication is trivial. To prove the direct implication, assume $\forall x \trianglelefteq_{\mathsf{X}} n \ \left(\|x\| \leq_{\mathbb{R}} z \to A(x,z)\right)$. Take an arbitrary x^X and assume $\|x\| \leq_{\mathbb{R}} z$. Then, since $z \leq_{\mathbb{R}} m$, from *iv*) of the previous lemma, we get $x \trianglelefteq_{\mathsf{X}} n$. Using the hypothesis, it follows $A(x,z)$, as desired. $\square$

Proposition 15. $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *proves that the axioms* E, E_{Π}, E_{Σ} *and* $\mathsf{E}_{\underline{R}}$ *generalize for every formula of* $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$.

Proof We present the proof for E and E_{Π}. For the other axioms, the argument is similar.

Since E generalizes for every formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$, it suffices to prove it to the new atomic formulas of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$: formulas of the form $s \trianglelefteq_{\rho} t$ where ρ is not an arithmetic type. We argue by induction on the type ρ. Let $s[w]$ be a term of type X and $t[w]$ a term of type 0 with a distinguished variable w of type 0 and assume $n =_0 m \wedge s[n/w] \trianglelefteq_{\mathsf{X}} t[n/w]$. By M_2, $n =_0 m \wedge \|s[n/w]\| \leq_{\mathbb{R}} t[n/w]$. And since $\|s[n/w]\| \leq_{\mathbb{R}} t[n/w]$ is a formula in $\mathcal{L}^{\omega,\mathsf{X}}$, we have $\|s[m/w]\| \leq_{\mathbb{R}} t[m/w]$. By RL_1, it follows $n =_0 m \wedge s[n/w] \trianglelefteq_{\mathsf{X}} t[n/w] \to s[m/w] \trianglelefteq_{\mathsf{X}} t[m/w]$.

For the induction step, take $s[w]$ and $t[w]$ terms of types $\rho \to \sigma$ and $\widehat{\rho} \to \widehat{\sigma}$, respectively, with a distinguished free variable w of type 0. And assume $n =_0 m \wedge s[n/w] \trianglelefteq_{\rho\to\sigma} t[n/w]$. Hence

$$\begin{aligned} &n =_0 m \wedge s[n/w] \trianglelefteq_{\rho\to\sigma} t[n/w] \wedge u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to \\ &\quad \to n =_0 m \wedge s[n/w]u \trianglelefteq_{\sigma} t[n(w]v \wedge t[n/w]u' \trianglelefteq_{\widehat{\sigma}} t[n/w]v' \end{aligned}$$

Using the induction hypothesis, we get

$$\begin{aligned} &n =_0 m \wedge s[n/w] \trianglelefteq_{\rho\to\sigma} t[n/w] \wedge u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to \\ &\quad \to s[m/w]u \trianglelefteq_{\sigma} t[m/w]v \wedge t[m/w]u' \trianglelefteq_{\widehat{\sigma}} t[m/w]v' \end{aligned}$$

and by RL_2, it follows $n =_0 m \wedge s[n/w] \trianglelefteq_{\rho\to\sigma} t[n/w] \to s[m/w] \trianglelefteq_{\rho\to\sigma} t[m/w]$.

For E_{Π}, it is also enough to prove the result for the new atomic formulas, formulas of the form $s \trianglelefteq_{\rho} t$. We argue by induction on ρ. For $\rho = \mathsf{X}$, take $s[w]$ a term of type X and $t[w]$ a term of type 0, with a distinguished free variable w. We claim that

$$s[\Pi xy/w] \trianglelefteq_{\mathsf{X}} t[\Pi xy/w] \leftrightarrow s[x/w] \trianglelefteq_{\mathsf{X}} t[x/w].$$

We only prove the direct implication, since the other way around is similar. Assume $s[\Pi xy/w] \trianglelefteq_{\mathsf{X}} t[\Pi xy/w]$. It implies $\|s[\Pi xy/w]\| \leq_{\mathbb{R}} (t[\Pi xy/w])_{\mathbb{R}}$, and since E_{Π} holds in $\mathsf{HA}^{\omega,\mathsf{X}}$, then, we get $\|s[x/w]\| \leq_{\mathbb{R}} t[x/w]$. The conclusion follows by RL_1.

For the induction step, take $s[w]$ of type $\rho \to \sigma$ and $t[w]$ of type $\widehat{\rho} \to \widehat{\sigma}$ with a distinguished free variable w. Then

$$\begin{aligned} &s[\Pi xy/w] \trianglelefteq_{\rho\to\sigma} t[\Pi xy/w] \wedge u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to \\ &\quad \to s[\Pi xy/w]u \trianglelefteq_{\sigma} t[\Pi xy/w]v \wedge t[\Pi xy/w]u' \trianglelefteq_{\widehat{\sigma}} t[\Pi xy/w]v' \end{aligned}$$

and by induction hypothesis applied to types σ and $\widehat{\sigma}$, we get

$$s[\Pi xy/w] \trianglelefteq_{\rho\to\sigma} t[\Pi xy/w] \wedge u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to s[x/w]u \trianglelefteq_{\sigma} t[x/w]v \wedge t[x/w]u' \trianglelefteq_{\widehat{\sigma}} t[x/w]v'.$$

By RL_2, we get $s[\Pi(x,y)/w] \trianglelefteq_{\rho\to\sigma} t[\Pi(x,y)/w] \to s[x/w] \trianglelefteq_{\rho\to\sigma} t[x/w]$. The converse implication is similar. $\square$

Proposition 16. $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *proves that* $\Pi_{\rho,\sigma} \trianglelefteq \Pi_{\widehat{\rho},\widehat{\sigma}}$, $\Sigma_{\rho,\sigma,\tau} \trianglelefteq \Sigma_{\widehat{\rho},\widehat{\sigma},\widehat{\tau}}$ *and* $\underline{R}_{\underline{\rho}} \trianglelefteq \left(\underline{R}_{\widehat{\underline{\rho}}}\right)^M$.

Proof The argument of the proof is similar in the two first cases, hence we only prove it for $\Pi_{\rho,\sigma}$ and $\underline{\mathrm{R}}_{\underline{\tau}}$ with $\rho, \sigma, \underline{\tau} \in \mathsf{T}^{\mathsf{X}}$.

To prove $\Pi \trianglelefteq \Pi$, notice that from

$$u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \wedge w \trianglelefteq_{\sigma} z \wedge w' \trianglelefteq_{\widehat{\sigma}} z' \to u \trianglelefteq_{\rho} v \wedge v \trianglelefteq_{\widehat{\rho}} v \wedge u' \trianglelefteq_{\widehat{\rho}} u'$$

and the previous proposition, we get

$$\begin{aligned} &u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \wedge w \trianglelefteq_{\sigma} z \wedge w' \trianglelefteq_{\widehat{\sigma}} z' \to \\ &\quad \to \Pi_{\rho,\sigma} uw \trianglelefteq_{\rho} \Pi_{\widehat{\rho},\widehat{\sigma}} vz \wedge \Pi_{\widehat{\rho},\widehat{\sigma}} vw' \trianglelefteq_{\widehat{\rho}} \Pi_{\widehat{\rho},\widehat{\sigma}} vz' \wedge \Pi_{\widehat{\rho},\widehat{\sigma}} u'w' \trianglelefteq_{\widehat{\rho}} \Pi_{\widehat{\rho},\widehat{\sigma}} u'z'. \end{aligned}$$

By applying RL_2 once, we get

$$u \trianglelefteq_{\rho} v \wedge u' \trianglelefteq_{\widehat{\rho}} v' \to \Pi_{\rho,\sigma} u \trianglelefteq_{\sigma\to\rho} \Pi_{\widehat{\rho},\widehat{\sigma}} v \wedge \Pi_{\widehat{\rho},\widehat{\sigma}} u' \trianglelefteq_{\widehat{\sigma}\to\widehat{\rho}} \Pi_{\widehat{\rho},\widehat{\sigma}} v'$$

and by applying it twice, it follows $\Pi_{\rho,\sigma} \trianglelefteq \Pi_{\widehat{\rho},\widehat{\sigma}}$.

For $\underline{\mathrm{R}}$, we begin by proving that, for all n^0, $\underline{\mathrm{R}}_{\underline{\rho}} n \trianglelefteq \underline{\mathrm{R}}_{\widehat{\underline{\rho}}} n$, with $\underline{\rho}$ being the tuple $\rho_1\rho_2...\rho_k$. We argue by induction on n^0. Take $i \leq_0 k$. For $n = 0$, notice that

$$\underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \wedge \underline{w} \trianglelefteq \underline{z} \wedge \underline{w}' \trianglelefteq \underline{z}' \to u_i \trianglelefteq v_i \wedge v_i \trianglelefteq v_i \wedge u_i' \trianglelefteq v_i' \wedge v_i' \trianglelefteq v_i'.$$

By the previous proposition, the latter is equivalent to

$$\begin{aligned} \underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \wedge \underline{w} \trianglelefteq \underline{z} \wedge \underline{w}' \trianglelefteq \underline{z}' \to (\mathrm{R}_i)_{\rho_i} 0\underline{u}\ \underline{w} \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}\ \underline{z} \wedge (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}\ \underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}\ \underline{z}' \wedge \\ \wedge (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{u}'\underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}'\underline{z}' \wedge (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}'\underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}'\underline{z}' \end{aligned}$$

and from this and RL_2, it follows

$$\underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \to (\mathrm{R}_i)_{\rho_i} 0\underline{u} \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v} \wedge (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{u}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i} 0\underline{v}'.$$

Again, by RL_2, we get $(\mathrm{R}_i)_{\rho_i}0 \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}0$. For the induction step, assume we have the induction hypothesis of $\underline{\mathrm{R}}_{\rho}n \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n$. Then

$$\underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \wedge \underline{w} \trianglelefteq \underline{z} \wedge \underline{w}' \trianglelefteq \underline{z}' \rightarrow \underline{\mathrm{R}}_{\rho}n\underline{u}\ \underline{w} \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{z} \wedge \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{w}' \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{z}' \wedge$$
$$\wedge \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{u}'\underline{w}' \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{z}' \wedge \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{w}' \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{z}'.$$

Thus, it follows

$$\underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \wedge \underline{w} \trianglelefteq \underline{z} \wedge \underline{w}' \trianglelefteq \underline{z}' \rightarrow w_i(\underline{\mathrm{R}}_{\rho}n\underline{u}\ \underline{w})n \trianglelefteq z_i(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{z})n \wedge$$
$$\wedge w_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{w}')n \trianglelefteq z_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}\ \underline{z}')n \wedge w_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{u}'\underline{w}')n \trianglelefteq z_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{z}')n \wedge$$
$$\wedge w_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{w}')n \trianglelefteq z_i'(\underline{\mathrm{R}}_{\widehat{\rho}}n\underline{v}'\underline{z}')n.$$

By the proposition above, the latter is equivalent to

$$\underline{u} \trianglelefteq \underline{v} \wedge \underline{u}' \trianglelefteq \underline{v}' \wedge \underline{w} \trianglelefteq \underline{z} \wedge \underline{w}' \trianglelefteq \underline{z}' \rightarrow (\mathrm{R}_i)_{\rho_i}(n+1)\underline{u}\ \underline{w} \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}\ \underline{z} \wedge$$
$$\wedge (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}\ \underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}\ \underline{z}' \wedge (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{u}'\underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}'\underline{z}' \wedge$$
$$\wedge (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}'\underline{w}' \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)\underline{v}'\underline{z}'.$$

By applying RL_2 twice, we get $(\mathrm{R}_i)_{\rho_i}(n+1) \trianglelefteq (\mathrm{R}_i)_{\widehat{\rho}_i}(n+1)$.

Hence, $\underline{\mathrm{R}}_{\rho}n \trianglelefteq \underline{\mathrm{R}}_{\widehat{\rho}}n$ for all n^0 and also $\underline{\mathrm{R}}_{\rho}n \trianglelefteq \left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M n$ since $\left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M n = \max_{i \leq n} \underline{\mathrm{R}}_{\widehat{\rho}}i$. Then

$$n \leq m \rightarrow \underline{\mathrm{R}}_{\rho}n \trianglelefteq \left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M n \wedge \left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M n \trianglelefteq \left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M m.$$

By RL_2, it follows $\underline{\mathrm{R}}_{\rho} \trianglelefteq \left(\underline{\mathrm{R}}_{\widehat{\rho}}\right)^M$, as desired. □

So far, we know that some constants of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ have majorants. We prove below that all constants have a majorant:

Proposition 17. $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *is a majorizability theory.*

Proof As we had seen above, Π, Σ and $\underline{\mathrm{R}}$ are majorizable. Hence, it suffices to prove that there exists closed terms majorizing the constants $0_{\mathsf{X}}, +_{\mathsf{X}}, -_{\mathsf{X}}, \cdot_{\mathsf{X}}$ and $\|.\|$. We have that $0_{\mathsf{X}} \trianglelefteq_{\mathsf{X}} 0^0$, since $\|0_{\mathsf{X}}\| =_{\mathbb{R}} 0^0$. This result is a consequence of RL_1.

For $+_{\mathsf{X}}$ we claim that $+_{\mathsf{X}} \trianglelefteq_{\mathsf{X}\to(\mathsf{X}\to\mathsf{X})} \lambda n^0, m^0.n+m$. Take $x \trianglelefteq_{\mathsf{X}} n, y \trianglelefteq_{\mathsf{X}} m, n \leq_0 n'$ and $m \leq_0 m'$. Then, by M_2 and $\mathsf{N6}$, we get $\|x +_{\mathsf{X}} y\| \leq_{\mathbb{R}} \|x\| + \|y\| \leq_{\mathbb{R}} n + m$. Then

$$x \trianglelefteq_{\mathsf{X}} n \wedge n \leq_0 n' \wedge y \trianglelefteq_{\mathsf{X}} m \wedge m \leq_0 m' \rightarrow \|x +_{\mathsf{X}} y\| \leq_{\mathbb{R}} n + m \wedge n + m \leq_0 n + m' \wedge$$
$$\wedge n + m \leq_0 n' + m' \wedge n' + m \leq_0 n' + m',$$

which implies

$$x \trianglelefteq_{\mathsf{X}} n \wedge n \leq_0 n' \wedge y \trianglelefteq_{\mathsf{X}} m \wedge m \leq_0 m' \rightarrow x +_{\mathsf{X}} y \trianglelefteq_{\mathsf{X}} n + m \wedge n + m \leq_0 n + m' \wedge$$
$$\wedge n + m \leq_0 n' + m' \wedge n' + m \leq_0 n' + m',$$

by the rule RL_1. Using RL_2 twice, we obtain $+_\mathsf{X} \trianglelefteq_{\mathsf{X}\to(\mathsf{X}\to\mathsf{X})} \lambda n, m.n+m$.

To prove that $-_\mathsf{X} \trianglelefteq_{\mathsf{X}\to\mathsf{X}} \lambda n^0.n$, notice that $x \trianglelefteq_\mathsf{X} n \to \|-_\mathsf{X} x\| \leq_\mathbb{R} n$, since $\|-_\mathsf{X} x\| =_\mathbb{R} \|x\|$. Then, by RL_1, $x \trianglelefteq_\mathsf{X} n \to -_\mathsf{X} x \trianglelefteq_\mathsf{X} n$ and, now, the conclusion follows by the rule RL_2.

For $\cdot_\mathsf{X}$, let us prove that $\cdot_\mathsf{X} \trianglelefteq_{1\to(\mathsf{X}\to\mathsf{X})} \lambda\alpha^1, n^0.(1+\alpha 0)n$. First, we claim that $\|\alpha \cdot_\mathsf{X} x\| \leq_\mathbb{R} (1+\alpha 0)\|x\|$ for all x^X:

$$\|\alpha \cdot_\mathsf{X} x\| =_\mathbb{R} |\widetilde{\alpha}|_\mathbb{R}\|x\| \leq_\mathbb{R} \big(|\widetilde{\alpha} - int(\alpha 0)|_\mathbb{R} + |int(\alpha 0)|_\mathbb{R}\big)\|x\| \leq_\mathbb{R} (1+\alpha 0)\|x\|.$$

The last inequality is due to the lemma 26. Take $\alpha \trianglelefteq_1 \beta, x \trianglelefteq_\mathsf{X} n$ and $n \leq_0 m$. Then $\|\alpha \cdot_\mathsf{X} x\| \leq_\mathbb{R} \big(1+\alpha 0)n$ and $\alpha 0 \leq_0 \beta 0$. Hence

$$\begin{aligned}\alpha \trianglelefteq_1 \beta \wedge x \trianglelefteq_\mathsf{X} n \wedge n \leq_0 m \to \|\alpha \cdot_\mathsf{X} x\| \leq_\mathbb{R} (1+\beta 0)n \wedge (1+\beta 0)n \leq_0 (1+\beta 0)m \wedge \\ \wedge(1+\alpha 0)n \leq_0 (1+\beta 0)m.\end{aligned}$$

By RL_1, it follows

$$\begin{aligned}\alpha \trianglelefteq_1 \beta \wedge x \trianglelefteq_\mathsf{X} n \wedge n \leq_0 m \to \alpha \cdot_\mathsf{X} x \trianglelefteq_\mathsf{X} (1+\beta 0)n \wedge (1+\beta 0)n \leq_0 (1+\beta 0)m \wedge \\ \wedge(1+\alpha 0)n \leq_0 (1+\beta 0)m\end{aligned}$$

and by using twice RL_2, we obtain $\cdot_\mathsf{X} \trianglelefteq \lambda\alpha^1, n^0.(1+\alpha 0)n$.

At last, we prove that $\|.\| \trianglelefteq_{\mathsf{X}\to 1} \lambda n^0, m^0.(2n+3)$. Take $x \trianglelefteq_\mathsf{X} n$, $n \leq_0 m$ and $i \leq_0 j$. Then $\|x\| \leq_\mathbb{R} n$ and by the proposition 14, we get $\|x\|(i) \leq_0 2n+3$. The conclusion is obtained by applying RL_2 twice. $\square$

Lemma 31. *For each closed term t^ρ of the language, there exists another closed term $q^{\widehat{\rho}}$ such that*

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash t \trianglelefteq_\rho q.$$

Proof Easy induction on the structure of terms. $\square$

The notion of *majorant of a term, monotone term* and *monotone functional* are naturally extended. Notice, however, that only terms or functionals of arithmetic type may be monotone. By the lemma above it is immediate that any term in $\mathsf{HA}^{\omega,\mathsf{X}}$ has a majorant.

4.2 Bounded functional interpretation extended to new base types

The theory $\mathsf{HA}^{\omega}_{\trianglelefteq}$ as been extended. The bounded functional interpretation will, as well, be extended to interpret the theory $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$. The clauses which define this interpretation extends naturally to the new types. Nevertheless, one shall pay attention to the monotone terms, since these must be of arithmetic type.

Definition 16. *To each formula A of the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ we associate formulas A^B and A_B of $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$. A_B is a bounded formula and A^B has the form $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{b},\underline{c})$ where $\underline{b}$ and $\underline{c}$ are (possibly empty) tuples of (arithmetic type) monotone terms.*

1. $(A_{bd})^B$ *and* $(A_{bd})_B$ *are* A_{bd} *for bounded formulas* A_{bd}.

If we already have A^B and B^B given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{b},\underline{c})$ and $\tilde{\exists}\underline{d}\tilde{\forall}\underline{e}\ B_B(\underline{d},\underline{e})$ respectively, then

2. $(A \wedge B)^B$ *is* $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}\ \big(A_B(\underline{b},\underline{c}) \wedge B_B(\underline{d},\underline{e})\big)$,
3. $(A \vee B)^B$ *is* $\tilde{\exists}\underline{b},\underline{d}\tilde{\forall}\underline{c},\underline{e}\ \big(\tilde{\forall}\underline{c}' \trianglelefteq \underline{c}\ A_B(\underline{b},\underline{c}') \vee \tilde{\forall}\underline{e}' \trianglelefteq \underline{e}\ B_B(\underline{d},\underline{e}')\big)$,
4. $(A \to B)^B$ *is* $\tilde{\exists}\underline{f},\underline{g}\tilde{\forall}\underline{b},\underline{e}\ \big(\tilde{\forall}\underline{c} \trianglelefteq \underline{gbe}\ A_B(\underline{b},\underline{c}) \to B_B(\underline{fb},\underline{e})\big)$,
5. $\big(\forall x \trianglelefteq t\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\forall x \trianglelefteq t\ A_B(x,\underline{b},\underline{c})$,
6. $\big(\exists x \trianglelefteq t\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\exists x \trianglelefteq t\tilde{\forall}\underline{c}' \trianglelefteq \underline{c}\ A_B(x,\underline{b},\underline{c}')$,
7. $\big(\forall x\ A(x)\big)^B$ *is* $\tilde{\exists}\underline{f}\tilde{\forall}a,\underline{c}\forall x \trianglelefteq a\ A_B(x,\underline{f}a,\underline{c})$,
8. $\big(\exists x\ A(x)\big)^B$ *is* $\tilde{\exists}a,\underline{b}\tilde{\forall}\underline{c}\exists x \trianglelefteq a\tilde{\forall}\underline{c}' \trianglelefteq \underline{c}\ A_B(x,\underline{b},\underline{c}')$.

By inspecting the clauses of the definition of the interpretation, one easily proves the following extension:

Lemma 32 (**Monotonicity Lemma**). *Let A^B be $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(x,\underline{b},\underline{c})$. Then*

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} \vdash \underline{b} \trianglelefteq \underline{b}' \wedge \underline{c} \triangleleft \underline{c} \wedge A_B(x,\underline{b},\underline{c}) \to A_B(x,\underline{b}',\underline{c}).$$

In the previous section, it was proved that $\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ is a majorizability theory. We are now ready to prove the soundness theorem:

Theorem 28 (**Soundness**). *Let $A(\underline{z})$ be a formula in the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ with free variables $\underline{z}$. Suppose that A^B is $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{b},\underline{c})$ and take Δ a set of universal (with bounded intensional matrices) sentences. If*

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \Delta \vdash A(\underline{z})$$

then there exist closed monotone terms $\underline{t}$ of appropriate type such that

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{ta},\underline{c}).$$

Proof We argue by induction on the length of the derivation of $A(\underline{z})$. In order to ease the readability, we do not use the underlining to represent tuples. First, we focus on the non--logical axioms and rules. All these axioms except $\mathsf{B}_\forall$ and $\mathsf{B}_\exists$ are universal statements, hence are sound. In the case of $\mathsf{B}_\forall$ and $\mathsf{B}_\exists$, we prove it in two steps, left-to-right and right-to-left implications.

$\mathsf{B}_\forall$: $\quad \forall x \trianglelefteq t\ A(x) \leftrightarrow \forall x\ \big(x \trianglelefteq t \to A(x)\big).$

Let A^B be $\tilde{\exists} b \tilde{\forall} c\ A_B(x,b,c)$. Then, we have:

$$\big(\forall x \trianglelefteq t\ A(x)\big)^B \text{ is } \tilde{\exists} b \tilde{\forall} c \forall x \trianglelefteq t\ A_B(x,b,c)$$
$$\big(\forall x\ \big(x \trianglelefteq t \to A(x)\big)\big)^B \text{ is } \tilde{\exists} f \tilde{\forall} a, c \forall x \trianglelefteq a\big(x \trianglelefteq t \to A_B(x, fa, c)\big).$$

The bounded functional interpretation for left-to-right implication is

$$\tilde{\exists} f, g \tilde{\forall} b, c, d\Big(\tilde{\forall} c' \trianglelefteq fbcd \forall x \trianglelefteq t\ A_B(x,b,c') \to \forall x \trianglelefteq c\big(x \trianglelefteq t \to A_B(x, gbc, d)\big)\Big).$$

Hence, we ask for closed monotone terms q and r such that

$$\tilde{\forall} a \forall z \trianglelefteq a \tilde{\forall} b, c, d\Big(\tilde{\forall} c' \trianglelefteq qbcd \forall x \trianglelefteq t\ A_B(x,b,c') \to \forall x \trianglelefteq c\big(x \trianglelefteq t \to A_B(x, rbc, d)\big)\Big),$$

where z is the tuple of free variables occurring in t. The terms q and r, given by $qbcd := d$ and $rbc := b$ do the job.
The right-to-left implication asks for monotone terms q, r and s such that

$$\tilde{\forall} a \forall z \trianglelefteq a \tilde{\forall} f, c\Big(\tilde{\forall} d \trianglelefteq qfc \tilde{\forall} c' \trianglelefteq rfc \forall x \trianglelefteq d\big(x \trianglelefteq t \to A_B(x, fd, c')\big) \to$$
$$\to \forall x \trianglelefteq t\big(A_B(x, sf, c)\big)\Big).$$

Clearly, $qfc := \tilde{t}[a/x]$, $rfc := c$ and $sf := f(\tilde{t}[a/z])$, where $\tilde{t}$ is a majorant for t, are monotone and do the job.

$\mathsf{B}_\exists$: $\quad \exists x \trianglelefteq t\ A(x) \leftrightarrow \exists x\ \big(x \trianglelefteq t \wedge A(x)\big).$
Assuming $(A(x))^B$ is given by $\tilde{\exists} b \tilde{\forall} c\ A_B(x,b,c)$, we have

$$\big(\exists x \trianglelefteq t\ A(x)\big)^B \text{ is } \tilde{\exists} b \tilde{\forall} c \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c\ A_B(x,b,c')$$
$$\big(\exists x \big(x \trianglelefteq t \wedge A(x)\big)\big)^B \text{ is } \tilde{\exists} a, b \tilde{\forall} c \exists x \trianglelefteq a \tilde{\forall} c' \trianglelefteq c\big(x \trianglelefteq t \wedge A_B(x,b,c')\big).$$

We assume z as the tuple of free variables of t. The left-to-right implication asks for monotone terms q, r and s such that

$$\tilde{\forall} a \forall z \trianglelefteq a \tilde{\forall} b, d\Big(\tilde{\forall} c \trianglelefteq qbd \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c\ A_B(x,b,c') \to$$
$$\to \exists x \trianglelefteq rb \tilde{\forall} d' \trianglelefteq d\big(x \trianglelefteq t \wedge A_B(x, sb, d')\big)\Big).$$

Take $qbd := d$, $rbd := \tilde{t}[a/z]$ and $sb := b$, where $\tilde{t}$ is a majorant for t. These terms do the job and are monotone. For the right-to-left implication we have to determine monotone terms q and r such that

$$\tilde{\forall} a \forall z \trianglelefteq a \tilde{\forall} b, c, d\Big(\tilde{\forall} c'' \trianglelefteq qbcd \exists x \trianglelefteq d \tilde{\forall} c' \trianglelefteq c''\big(x \trianglelefteq t \wedge A_B(x,b,c')\big) \to$$
$$\to \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c\ A_B(x, rbd, c')\Big).$$

Clearly, $qbcd := c$ and $rbd := b$ do the job.

Now, we analyse the two rules.

$$\mathsf{RL}_1: \qquad \frac{A_{bd}[z] \to \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}}{A_{bd}[z] \to s \trianglelefteq_{\mathsf{X}} t}.$$

Take z as the tuples of the free variables of A_{bd}. Assume that the premise has been derived in $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$. By the induction hypothesis,

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \tilde{\forall} a, b \forall z \trianglelefteq a \forall n^0 \trianglelefteq_0 b \ \left(A_{bd}[z] \to \|s\|(n+1) - t <_{\mathbb{Q}} \tfrac{1}{2^n}\right).$$

Since n is a natural number, it is clearly monotone. Hence, taking $b = n$ we obtain

$$\tilde{\forall} a \forall z \trianglelefteq a \forall n \ \left(A_{bd}[z] \to \|s\|(n+1) - t <_{\mathbb{Q}} \tfrac{1}{2^n}\right),$$

which implies $\tilde{\forall} a \forall z \trianglelefteq a\big(A_{bd}[z] \to \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}\big)$. Equivalently,

$$\forall a \forall z \ \big(A_{bd}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \to \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}\big).$$

By RL_1, it follows

$$\forall a \forall z \ \big(A_{bd}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \to s \trianglelefteq_{\mathsf{X}} t\big).$$

which is equivalent to $\tilde{\forall} a \forall z \trianglelefteq a \ \big(A_{bd}[z] \to s \trianglelefteq_{\mathsf{X}} t\big)$.

$$\mathsf{RL}_2: \qquad \frac{A_{bd}[z] \wedge u \trianglelefteq v \wedge u' \trianglelefteq v' \to s[z]u \trianglelefteq t[z]v \wedge t[z]u' \trianglelefteq t[z]v'}{A_{bd}[z] \to s[z] \trianglelefteq t[z]}$$

Take z as the tuple of the free variables of A_{bd}, s and t and that the premise has been derivable in $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$. Hence, by the induction hypothesis, $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves that

$$\begin{aligned}\forall z \trianglelefteq a \forall u \trianglelefteq b \forall v \trianglelefteq c \forall u' \trianglelefteq d \forall v' \trianglelefteq e \ \big(A_{bd}[z] \wedge u \trianglelefteq v \wedge u' \trianglelefteq v' \to \\ \to s[z]u \trianglelefteq t[z]v \wedge t[z]u' \trianglelefteq t[z]v'\big)\end{aligned}$$

for every monotone a, b, c, d and e.

Choose $b = c = v$ and $d = e = v'$. Then

$$\tilde{\forall} a \forall z \trianglelefteq a \ \big(A_{bd}[z] \wedge u \trianglelefteq v \wedge u' \trianglelefteq v' \to s[z]u \trianglelefteq t[z]v \wedge t[z]u' \trianglelefteq t[z]v'\big)$$

which is equivalent to

$$\forall a \forall z \ \big(A_{bd}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \wedge u \trianglelefteq v \wedge u' \trianglelefteq v' \to s[z]u \trianglelefteq t[z]v \wedge t[z]u' \trianglelefteq t[z]v'\big).$$

By RL_2, $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $\forall a \forall z \ \big(A_{bd}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \to s[z] \trianglelefteq t[z]\big)$, which is equivalent to $\tilde{\forall} a \forall z \trianglelefteq a \ \big(A_{bd}[z] \to s[z]u \trianglelefteq t[z]\big)$, as desired.

We continue by proving the theorem for the logical axioms. Until the end of the proof, take $(A(x))^B$, $(B(x))^B$ and $(C(x))^B$ as $\tilde{\exists} b \tilde{\forall} c\, A_B(x,b,c)$, $\tilde{\exists} d \tilde{\forall} e\, B_B(x,d,e)$ and $\tilde{\exists} u \tilde{\forall} v\, C_B(x,u,v)$, respectively.

1.

$$\frac{A \qquad A \to B}{B}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves A and $A \to B$. By the induction hypothesis, there are monotone terms t, q and r such that

$$\begin{array}{lcl} \mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} & \vdash & \tilde{\forall} c\; A_B(t,b) \\ \mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} & \vdash & \tilde{\forall} b, e\; \big(\tilde{\forall} c' \trianglelefteq qbe\; A_B(b,c) \to B_B(rb,e)\big). \end{array}$$

We want a monotone closed term s such that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \tilde{\forall} e\; B_B(s,e)$. Clearly, $s := rt$ is monotone and does the job.

2.

$$\frac{A \to B \qquad B \to C}{A \to C}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A \to B$ and $B \to C$. By the induction hypothesis, there are monotone terms t, q, r and s such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \;\vdash\; \tilde{\forall} b, e\; \big(\tilde{\forall} c \trianglelefteq tbe\; A_B(b,c) \to B_B(qb,e)\big) \tag{4.1}$$

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \;\vdash\; \tilde{\forall} d, v\; \big(\tilde{\forall} e \trianglelefteq rdv\; B_B(e,d) \to C_B(sd,v)\big). \tag{4.2}$$

We want to produce monotone closed terms p and l such that

$$\tilde{\forall} b, v\; \big(\tilde{\forall} c \trianglelefteq pbv\; A_B(b,c) \to C_B(lb,v)\big).$$

Let $pbv := tb(r(qb)v)$ and $lb := s(qb)$, clearly monotone. We fix monotone b and v and assume

$$\tilde{\forall} c \trianglelefteq tb(r(qb)v)\; A_B(b,c). \tag{4.3}$$

By (4.1), we get

$$\tilde{\forall} e \trianglelefteq r(qb)v\; \big(\tilde{\forall} c \trianglelefteq tb(r(qb)v)\; A_B(b,c) \to B_B(qb,e)\big) \tag{4.4}$$

and by (4.4) and (4.3), it follows

$$\tilde{\forall} e \trianglelefteq r(qb)v\; B_B(qb,e). \tag{4.5}$$

We conclude $C_B(s(qb),v)$ by (4.2) and (4.5).

3.1 $A \vee A \to A$.
We want to produce monotone closed terms t, q and r such that

$$\tilde{\forall} b, c, d\ \big(\tilde{\forall} c'' \trianglelefteq tbcd \tilde{\forall} c''' \trianglelefteq qbcd\ \big(\tilde{\forall} c' \trianglelefteq c''\ A_B(b, c') \vee \tilde{\forall} c' \trianglelefteq c'''\ A_B(d, c')\big) \to \\ \to A_B(rbd, c)\big).$$

Let $tbcd := c$, $qbcd := c$ and $rbd := \max(b, d)$. Recall that $\mathsf{HA}^{\omega}_{\trianglelefteq} \vdash \max \trianglelefteq \max$ and that r is only defined on arithmetic types. Hence, these terms are monotone. We fix monotone b, c and d and assume

$$\tilde{\forall} c'' \trianglelefteq c \tilde{\forall} c''' \trianglelefteq c\ \big(\tilde{\forall} c' \trianglelefteq c''\ A_B(b, c') \vee \tilde{\forall} c' \trianglelefteq c'''\ A_B(d, c')\big).$$

Then $\tilde{\forall} c' \trianglelefteq c\ \ (A_B(b, c') \vee A_B(d, c'))$ and also $A_B(b, c) \vee A_B(d, c)$. Since $b \trianglelefteq \max(b, d)$ and $d \trianglelefteq \max(b, d)$, by the monotonicity lemma we get $A_B(\max(b, d), c) \vee A_B(\max(b, d), c)$ which implies $A_B(\max(b, d), c)$.

3.2 $A \to A \wedge A$.
We have to produce monotone closed terms t, q and r such that

$$\tilde{\forall} b, c', c''\ \big(\tilde{\forall} c \trianglelefteq tbc'c''\ A_B(b, c) \to A_B(qb, c') \wedge A_B(rb, c'')\big).$$

By the monotonicity lemma it is clear that t, q and r given by $tbc'c'' := \max(c', c'')$, $qb := b$ and $rb := b$ do the job.

4.1 $A \to A \vee B$.
To interpret this axiom we need monotone terms t, q and r such that

$$\tilde{\forall} b, c, e\ \big(\tilde{\forall} c' \trianglelefteq tbce\ A_B(b, c') \to \tilde{\forall} c'' \trianglelefteq c\ A_B(qb, c'') \vee \tilde{\forall} e' \trianglelefteq e\ B_B(rb, e')\big).$$

Clearly $tbce := c$, $qb := b$ and $rb := b$ do the job.

4.2 $A \wedge B \to A$.
We need to produce monotone closed terms t, q and r such that

$$\tilde{\forall} b, c, d\ \big(\tilde{\forall} c' \trianglelefteq tbcd \tilde{\forall} e \trianglelefteq qbcd\ \big(A_B(b, c') \wedge B_B(d, e)\big) \to A_B(rbd, c)\big).$$

Take $tbcd := c$, $qbcd := b$ and $rbd := b$. By the monotonicity lemma, these terms do the job.

5.1 $A \vee B \to B \vee A$.
We want monotone closed terms t, q, r and s such that

$$\tilde{\forall} b, c, d, e\ \big(\tilde{\forall} c'' \trianglelefteq tbcde \tilde{\forall} e'' \trianglelefteq qbcde\ \big(\tilde{\forall} c' \trianglelefteq c''\ A_B(b, c') \vee \tilde{\forall} e' \trianglelefteq e''\ B_B(d, e')\big) \to \\ \to \tilde{\forall} e' \trianglelefteq e\ B_B(rbd, e') \vee \tilde{\forall} c' \trianglelefteq c\ A_B(sbd, c')\big).$$

The terms given by $tbcde := c$, $qbcde := e$, $rbd := d$ and $sbd := b$ do the job.

5.2 $A \wedge B \to B \wedge A$.
The proof is similar to the previous one.

6.

$$\frac{A \to B}{C \vee A \to C \vee B}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A \to B$. By the induction hypothesis, there exist monotone closed terms t and q such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \quad \vdash \quad \tilde{\forall} b, e \, \big(\tilde{\forall} c \trianglelefteq tbe \; A_B(b,c) \to B_B(qb,e)\big). \tag{4.6}$$

We aim to produce monotone closed terms r, s, p and l such that

$$\tilde{\forall} b, e, u, v \, \big(\tilde{\forall} v'' \trianglelefteq rbeuv \tilde{\forall} c \trianglelefteq sbeuv \, \big(\tilde{\forall} v' \trianglelefteq v'' \; C_B(u,v') \vee \tilde{\forall} c' \trianglelefteq c \; A_B(b,c')\big) \to \\ \to \tilde{\forall} v' \trianglelefteq v \; C_B(pbu, v') \vee \tilde{\forall} e' \trianglelefteq e \; B_B(lbu, e')\big).$$

Take $rbeuv := v$, $sbeuv := tbe$, $pbu := u$ and $lbu := qb$. These terms are monotone. Fix monotone b, e, u, v and assume

$$\tilde{\forall} e' \trianglelefteq e \, \Big(\tilde{\forall} v'' \trianglelefteq v \tilde{\forall} c \trianglelefteq tbe' \, \big(\tilde{\forall} v' \trianglelefteq v'' \; C_B(u,v') \vee \tilde{\forall} c' \trianglelefteq c \; A_B(b,c')\big)\Big).$$

Then $\tilde{\forall} e' \trianglelefteq e \, \big(\tilde{\forall} v' \trianglelefteq v \; C_B(u,v') \vee \tilde{\forall} c' \trianglelefteq t(b,e) \; A_B(b,c')\big)$. By the latter and (4.6), it follows $\tilde{\forall} v' \trianglelefteq v \; C_B(u,v') \vee \tilde{\forall} e' \trianglelefteq e \; B_B(q(b), e')$, as desired.

7.1

$$\frac{A \wedge B \to C}{A \to (B \to C)}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A \wedge B \to C$. By the induction hypothesis, there are monotone closed terms t, q and r such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \quad \vdash \quad \tilde{\forall} b, d, v \, \Big(\tilde{\forall} c \trianglelefteq tbdv \tilde{\forall} e \trianglelefteq qbdv \, \big(A_B(b,c) \wedge B_B(d,e)\big) \to C_B(rbd, v)\Big).$$

We want monotone closed terms s, p and l such that

$$\tilde{\forall} b, d, v \, \big(\tilde{\forall} c \trianglelefteq sbdv \; A_B(b,c) \to \big(\tilde{\forall} e \trianglelefteq pbdv \; B_B(d,e) \to C_B(rbd, v)\big)\big).$$

Clearly, s, p and l given by $sbdv := tbdv$, $pbdv := qbdv$ and $lbd := rbd$ are monotone and do the job.

7.2

$$\frac{A \to (B \to C)}{A \wedge B \to C}$$

This proof is similar to the previous one.

8. $\bot \to A$.
Trivial.

9.

$$\frac{A \to B(z)}{A \to \forall z B(z)}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A \to B(z)$. By the induction hypothesis, there are monotone terms t and q such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \quad \vdash \quad \tilde{\forall} a, b, e \forall z \trianglelefteq a \ \big(\tilde{\forall} c \trianglelefteq tabe\ A_B(b,c) \to B_B(z, qab, e)\big). \tag{4.7}$$

We want to produce monotone closed terms r and s such that

$$\tilde{\forall} a, b, e \ \big(\tilde{\forall} c \trianglelefteq rabe\ A_B(b,c) \to \forall z \trianglelefteq a\ B_B(z, sab, e)\big).$$

Take $rabe := tabe$ and $sabe := qab$. Fix monotone a, b and e and assume $\forall z \trianglelefteq a \tilde{\forall} c \trianglelefteq q(a,b,e)\ A_B(b,c)$. By the latter and (4.7), it follows $\forall z \trianglelefteq a B_B(z, rab, e)$.

10. $\forall x A(x) \to A(t)$.
Let z be the tuple of all the free variables of A and t. We want to produce monotone terms q, r and s such that

$$\tilde{\forall} a, c, f \forall z \trianglelefteq a \ \big(\tilde{\forall} b \trianglelefteq qacf \tilde{\forall} c' \trianglelefteq racf \forall x \trianglelefteq b\ A_B(x, z, fb, c') \to \\ \to A_B(t[z], z, saf, c)\big).$$

Take $qacf := \tilde{t}[a/z]$ where $\tilde{t}$ is a majorant for t (hence, monotone), $racf := c$ and $saf := f(\tilde{t}[a/z])$. Fix monotone a, c and f and assume

$$\tilde{\forall} b \trianglelefteq \tilde{t}[a/z] \tilde{\forall} c' \trianglelefteq c \forall x \trianglelefteq b\ A_B(x, z, fb, c').$$

Take b as $\tilde{t}[a/z]$. Then, $\forall x \trianglelefteq \tilde{t}[a/z]\ A_B(x, z, f(\tilde{t}[a/z]), c)$. Since $t(z) \trianglelefteq \tilde{t}[a/z]$, we get $A_B(t[z], z, f(\tilde{t}[a/z]), c)$.

11. $A(t) \to \exists x\ A(x)$.
Let z be the tuple of all the free variables of A and t. We want to produce monotone closed terms q, r and s such that

$$\tilde{\forall} a, b, c \forall z \trianglelefteq a \ \big(\tilde{\forall} c' \trianglelefteq qabc\ A_B(t[z], z, b, c') \to \exists x \trianglelefteq rab \tilde{\forall} c' \trianglelefteq c\ A_B(x, z, sab, c')\big).$$

It is clear that $qabc := c$, $rab := \tilde{t}[a/z]$, where $\tilde{t}$ is a majorant for t, and $sab = b$ do the job, since $t[z] \trianglelefteq \tilde{t}[a/z]$.

12.

$$\frac{A(z) \to B}{\exists z\ A(z) \to B}$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A(z) \to B$. By the induction hypothesis, there exist monotone terms t and q such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \quad \vdash \quad \tilde{\forall} a, b, e \forall z \trianglelefteq a \ \big(\tilde{\forall} c \trianglelefteq tabe\ A_B(z, b, c) \to B_B(qab, e)\big).$$

We want to produce monotone closed terms r and s such that

$$\tilde{\forall} a, b, e\ \big(\tilde{\forall} c \trianglelefteq rabe \exists z \trianglelefteq a \tilde{\forall} c' \trianglelefteq c\ A_B(z, b, c') \to B_B(sab, e)\big).$$

Clearly, r and s given by $rabe := tabe$ and $sab := qab$ are monotone and do the job.

To end the proof let us focus on the induction rule

$$\frac{A(0) \qquad \forall n^0\ \big(A(n) \to A(n+1)\big)}{\forall n^0\ A(n)}.$$

Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves $A(0)$ and $\forall n\ \big(A(n) \to A(n+1)\big)$. By the induction hypothesis, there are monotone closed terms t, p and q such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \ \vdash\ \tilde{\forall} c\ A_B(0, t, c) \tag{4.8}$$

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \ \vdash\ \tilde{\forall} a, b, e \forall n \trianglelefteq_0 a\ \big(\tilde{\forall} c \trianglelefteq pabe A_B(n, b, c) \to A_B(n+1, qab, e)\big). \tag{4.9}$$

We want to produce a monotone closed term r such that $\tilde{\forall} a, c \forall n \trianglelefteq_0 a\ A_B(n, ra, c)$. From (4.9) follows

$$\tilde{\forall} a, b \forall n \trianglelefteq_0 a\ \big(\tilde{\forall} c A_B(n, b, c) \to \forall c A_B(n+1, qab, e)\big). \tag{4.10}$$

Take ϕa as ψaa, where ψ is recursively defined as

$$\begin{cases} \psi 0 a = t \\ \psi(n+1)a = \max(\psi na, qa(\psi na)). \end{cases}$$

ψ has arithmetic type. One can show $\tilde{\forall} a, b \forall n \trianglelefteq_0 a\ A_B(n, \phi a, b)$ by induction: take a and $n \leq_0 a$ and assume $\tilde{\forall} c\ A_B(n, \psi na, c)$. By (4.10), we get $\tilde{\forall} c\ A_B(n+1, qa(\psi na), e)$, which implies

$$\tilde{\forall} c\ A_B(n+1, \psi(n+1)a, e, n+1),$$

by monotonicity. Hence, by the latter and (4.8), it follows $\forall n \trianglelefteq_0 a \tilde{\forall} c\ A_B(n, \psi na, c)$, and by monotonicity in the first entry, we obtain $\forall n \trianglelefteq_0 a \tilde{\forall} c\ A_B(n, \phi a, c)$, as desired. To end the proof, take r as $\lambda a.\phi a$. By construction, ψ is monotone in the second entry and it is easy to prove that $\forall a \forall n \leq m\ \psi na \trianglelefteq \psi ma$. Hence, ϕ is monotone. □

The characteristic principles in $\mathsf{P}^{\omega}[\trianglelefteq]$ are naturally extended to the new language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$. The set of the extended principles is denoted by $\mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$.

Proposition 18. $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ *proves* bBC^{ω}.

Proof Take c, monotone, and assume $\forall z \trianglelefteq c \exists y\ A(y, z)$ which is equivalent to $\forall z\ \ (z \trianglelefteq c \to \exists y\ A(y, z))$. By $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$, it follows $\forall z \tilde{\exists} b\ \ (z \trianglelefteq c \to \exists y \trianglelefteq b\ A(y, z))$ and by $\mathsf{bAC}^{\omega,\mathsf{X}}$, we get $\tilde{\exists} f \tilde{\forall} a \forall z \trianglelefteq a \tilde{\exists} b \trianglelefteq fa\ \big(z \trianglelefteq c \to \exists y \trianglelefteq b\ A(y, z)\big)$. Choose a and b as c and fc, respectively. Then $\tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b\ A(y, z)$. □

Theorem 29 (**Soundness extended**). *Let $A(\underline{z})$ be a formula of the language of $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ with free variable $\underline{z}$. Let $(A(\underline{z}))^B$ be $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{b},\underline{c})$ and suppose Δ is a set of universal (with bounded intensional matrices) sentences. If*

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] + \Delta \vdash A(\underline{z})$$

then there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{ta},\underline{c}).$$

Proof It suffices to prove that all the characteristic principles have bounded functional interpretation. To ease the reading, we will not underline tuples.

1. $\mathsf{bAC}^{\omega,\mathsf{X}}$.
   Let $(A(x,y))^B$ be $\tilde{\exists}b\tilde{\forall}c\ A_B(x,y,b,c)$. The interpretation of the left hand of $\mathsf{bAC}^{\omega,\mathsf{X}}$ is

   $$\tilde{\exists}f,g\tilde{\forall}c,d\forall x \trianglelefteq d\exists y \trianglelefteq fd\tilde{\forall}c' \trianglelefteq c\ A_B(x,y,gd,c').$$

   In order to simplify, we write $\forall x \trianglelefteq a\exists y \trianglelefteq fa\tilde{\forall}c' \trianglelefteq c\ A_B(x,y,b,c')$ as $B(a,b,c,f)$. Hence, the interpretation of the left hand becomes $\tilde{\exists}f,g\tilde{\forall}c,d\ B(d,gd,c,f)$, while the interpretation of the right hand is

   $$\tilde{\exists}f,g\tilde{\forall}c,d\tilde{\exists}f' \trianglelefteq f\tilde{\forall}c'' \trianglelefteq c\tilde{\forall}d' \trianglelefteq d\tilde{\forall}a \trianglelefteq d'\ B(a,gd',c'',f').$$

   We must produce monotone closed terms q,t,r and s such that

   $$\begin{aligned}&\tilde{\forall}b,c,f,g\tilde{\forall}c' \trianglelefteq tbcfg\tilde{\forall}b' \trianglelefteq qbcfg\big(B(b',gb',c',f) \to\\ &\quad\to \tilde{\exists}f' \trianglelefteq rfg\tilde{\forall}c'' \trianglelefteq c\tilde{\forall}b'' \trianglelefteq b\tilde{\forall}b''' \trianglelefteq b''\ B(b''',sf(gb''),c'',f')\big).\end{aligned}$$

   Take t,q,r and s given by $tbcfg := c$, $qbcfg := b$, $rfg := f$ and $sfa := a$ and take also monotone b,c,f,g. Assume $\tilde{\forall}c' \trianglelefteq c\tilde{\forall}b' \trianglelefteq b\ B(b',gb',c',f)$ and take $f' = f$. Then, using the transitivity of $\trianglelefteq$ and the monotonicity of B in the entry of gb', it follows

   $$\tilde{\exists}f' \trianglelefteq f\tilde{\forall}c'' \trianglelefteq c\tilde{\forall}b'' \trianglelefteq b\tilde{\forall}b''' \trianglelefteq b''\ B(b''',gb'',c'',f').$$

2. $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$.
   Let $(B(y))^B$ be $\tilde{\exists}c\tilde{\forall}d\ B_B(y,c,d)$. The bounded functional interpretation of the premise of $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$ is

   $$\tilde{\exists}b,c,f\tilde{\forall}d\ \big(\tilde{\forall}a \trianglelefteq fd\forall x \trianglelefteq a\ A_{bd}(x) \to \exists y \trianglelefteq b\tilde{\forall}d' \trianglelefteq d\ B_B(y,c,d')\big).$$

   Take $C(a,b,c,f)$ as $\tilde{\forall}d \trianglelefteq fa\forall x \trianglelefteq d\ A_{bd}(x) \to \exists y \trianglelefteq b\tilde{\forall}a' \trianglelefteq a\ B_B(y,c,a')$. Then, the interpretation above is $\tilde{\exists}b,c,f\tilde{\forall}d\ C(d,b,c,f)$, while the bounded functional interpretation of the conclusion of $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$ is

   $$\tilde{\exists}b,c,f\tilde{\forall}d\tilde{\exists}b' \trianglelefteq b\tilde{\forall}d'' \trianglelefteq d\ C(d'',b',c,f).$$

We want to produce monotone closed terms t, q, r and s such that

$$\tilde{\forall} b, c, d, f \ \big(\tilde{\forall} d' \trianglelefteq tbcdf \ C(d', b, c, f) \to \tilde{\exists} b' \trianglelefteq qbcdf \tilde{\forall} d' \trianglelefteq d \ C(d', b', rbcf, sbcf)\big).$$

Clearly, t, q, r, s given by $tbcdf := d$, $qbcdf := b$, $rbcf := c$ and $sbcf := f$ do the job.

3. bMP^{ρ}_{bd}.
The interpretation of the antecedent of $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$ is

$$\tilde{\exists} a, b \ \big(\tilde{\forall} a' \trianglelefteq a \tilde{\forall} b' \trianglelefteq b \forall y \trianglelefteq b' \forall x \trianglelefteq a' \ A_{bd}(x, y) \to B_{bd}\big)$$

while the consequent has the following interpretation

$$\tilde{\exists} a, b \tilde{\exists} b' \trianglelefteq b \big(\tilde{\forall} a' \trianglelefteq a \forall y \trianglelefteq b' \forall x \trianglelefteq a' \ A_{bd}(x, y) \to B_{bd}\big).$$

We want monotone closed terms t and q such that,

$$\begin{aligned} &\big(\tilde{\forall} a' \trianglelefteq a \tilde{\forall} b' \trianglelefteq b \forall y \trianglelefteq b' \forall x \trianglelefteq a' \ A_{bd}(x, y) \to B_{bd}\big) \to \\ &\qquad \to \tilde{\exists} b' \trianglelefteq tab \ \big(\tilde{\forall} a' \trianglelefteq qab \forall y \trianglelefteq b' \forall x \trianglelefteq a' \ A_{bd}(x, y) \to B_{bd}\big) \end{aligned}$$

for all monotone a and b. Obviously, take $tab := a$ and $qab := b$.

4. $\mathsf{bUD}^{\rho,\tau}_{\forall bd}$.
The interpretation of the premise is

$$\tilde{\forall} b, c \tilde{\forall} b' \trianglelefteq b \tilde{\forall} c' \trianglelefteq c \ \big(\forall x \trianglelefteq b' \ A_{bd}(x) \vee \forall y \trianglelefteq c' \ B_{bd}(y)\big),$$

while $\tilde{\forall} b, c \ \big(\forall x \trianglelefteq b \ A_{bd}(x) \vee \forall y \trianglelefteq c \ B_{bd}(y)\big)$ is the interpretation of the consequent of $\mathsf{bUD}^{\omega,\mathsf{X}}_{\forall bd}$. We must produce monotone closed terms t and q such that

$$\begin{aligned} \tilde{\forall} b, c \ \Big(\tilde{\forall} b' \trianglelefteq tbc \tilde{\forall} c' \trianglelefteq qbc \tilde{\forall} b'' \trianglelefteq b' \tilde{\forall} c'' \trianglelefteq c' \ \big(\forall x \trianglelefteq b'' \ A_{bd}(x) \vee \forall y \trianglelefteq c'' \ B_{bd}(y)\big) \to \\ \to \forall x \trianglelefteq b \ A_{bd}(x) \vee \forall y \trianglelefteq c \ B_{bd}(y)\Big). \end{aligned}$$

Clearly, $tbc := b$ and $qbc := c$ do the job.

5. $\mathsf{bBCC}^{\rho,\tau}_{bd}$.
We have to produce a monotone closed term t such that for all monotone b, c and c' such that $c' \trianglelefteq c$

$$\tilde{\forall} b' \trianglelefteq tbc \tilde{\forall} b'' \trianglelefteq b' \exists z \trianglelefteq c' \forall y \trianglelefteq b'' \ A_{bd}(y, z) \to \exists z \trianglelefteq c' \tilde{\forall} b' \trianglelefteq b \forall y \trianglelefteq b' \ A_{bd}(y, z).$$

Take $tbc := b$ and monotone b, c, c' such that $c' \trianglelefteq c$. Assume that

$$\tilde{\forall} b' \trianglelefteq b \tilde{\forall} b'' \trianglelefteq b' \exists z \trianglelefteq c' \forall y \trianglelefteq b'' \ A_{bd}(y, z).$$

By the transitivity of $\trianglelefteq$ we get $\tilde{\forall} b'' \trianglelefteq b \exists z \trianglelefteq c' \forall y \trianglelefteq b'' \ A_{bd}(y, z)$. Take $b'' := b$. Then, $\exists z \trianglelefteq c' \forall y \trianglelefteq b \ A_{bd}(y, z)$. Again, by transitivity, it follows

$$\exists z \trianglelefteq c' \forall b' \trianglelefteq b \forall y \trianglelefteq b' \ A_{bd}(y, z).$$

6. MAJ^ρ.
   The bounded functional interpretation of $\mathsf{MAJ}^{\omega,\mathsf{X}}$ is

$$\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \exists y \trianglelefteq fa \ (x \trianglelefteq y).$$

   Hence, we need to produce a monotone closed term t such that, for all monotone a, $\forall x \trianglelefteq a \exists y \trianglelefteq ta \ (x \trianglelefteq y)$. Clearly $ta := a$ does the job.

$\square$

The principle tameAC is also naturally extended. As before, under the characteristic principles, tameAC is equivalent to a purely universal statement. Moreover, it has a bounded functional interpretation in $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{tameAC}$.

Theorem 30. *Let $A(\underline{z})$ be a formula of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ with free variables $\underline{z}$ and bounded functional interpretation given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{b},\underline{c})$. If*

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] + \mathsf{tameAC} + \Delta \vdash A(\underline{z}),$$

where Δ is a set of all *purely universal statements, then there are monotone closed terms $\underline{t}$ of appropriate types such that*

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{ta},\underline{c}).$$

Proposition 19 (**Monotone axiom of choice**)**.** $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ *proves*

$$\Big(\tilde{\forall}\underline{a}\tilde{\forall}\underline{b}\tilde{\forall}\underline{b}' \trianglelefteq \underline{b}\ \ (A(\underline{a},\underline{b}') \to A(\underline{a},\underline{b})) \wedge \tilde{\forall}\underline{a}\tilde{\exists}\underline{b}\ A(\underline{a},\underline{b})\Big) \to \tilde{\exists}\underline{f}\tilde{\forall}\underline{a}\ A(\underline{a},\underline{fa}),$$

where A is an arbitrary formula of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$.

Proof To ease the reading, we do not underline the tuples. Assume the monotonicity property $\tilde{\forall}a\tilde{\forall}b\tilde{\forall}b' \trianglelefteq b\ \big(A(a,b') \to A(a,b)\big)$. $\tilde{\forall}a\tilde{\exists}b\ A(a,b)$ is equivalent to

$$\forall a\ \big(a \trianglelefteq a \to \exists b\ (b \trianglelefteq b \wedge A(a,b))\big).$$

By $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$, $\forall a\tilde{\exists}b\ \big(a \trianglelefteq a \to \exists b' \trianglelefteq b\ (b' \trianglelefteq b' \wedge A(a,b'))\big)$ and by the monotonicity property, we get $\forall a\tilde{\exists}b\ (a \trianglelefteq a \to A(a,b)$. By $\mathsf{bAC}^{\omega,\mathsf{X}}$, $\tilde{\exists}f\tilde{\forall}a\forall a' \trianglelefteq a\tilde{\exists}b \trianglelefteq f(a)\ \big(a' \trianglelefteq a' \to A(a',b)\big)$. Take $b := fa$ and $a' := a$. Then $\tilde{\exists}f\tilde{\forall}a\ A(a,f(a)$. $\square$

Theorem 31 (**Characterization**)**.** *For each formula A of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$, $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ proves $A \leftrightarrow A^B$.*

Proof We argue by induction on the logic structure of the formula. The conclusion is trivial for bounded formulas. To simplify the notation, no tuples will be underlined. We assume that A^B and B^B are given by $\tilde{\exists}b\tilde{\forall}c\ A_B(b,c)$ and $\tilde{\exists}d\tilde{\forall}e\ B_B(d,e)$, respectively.

1. $A \wedge B$.
Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ proves $A \leftrightarrow A^B$ and $B \leftrightarrow B^B$. We want to prove that $(A \wedge B) \leftrightarrow (A \wedge B)^B$, where $(A \wedge B)^B$ is given by $\tilde{\exists} b, d \tilde{\forall} c, e \left(A_B(b,c) \wedge B_B(d,e) \right)$. By the induction hypothesis, we have $A \wedge B \leftrightarrow A^B \wedge B^B$, which is equivalent to $\tilde{\exists} b \tilde{\forall} c\ A_B(b,c) \wedge \tilde{\exists} d \tilde{\forall} e\ B_B(d,e)$. Hence $(A \wedge B) \leftrightarrow \tilde{\exists} b, d \tilde{\forall} c, e \left(A_B(b,c) \wedge B_B(d,e) \right)$, as desired.

2. $A \vee B$.
Assume $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash A \leftrightarrow A^B$ and $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash B \leftrightarrow B^B$. To prove the left-to-right implication we only use intuitionist logic. By the induction hypothesis, we have $A \vee B \leftrightarrow A^B \vee B^B$, which is equivalent to

$$\tilde{\exists} b \tilde{\forall} c\ A_B(b,c) \vee \tilde{\exists} d \tilde{\forall} e\ B_B(d,e).$$

Hence, $\tilde{\exists} b, d \big(\tilde{\forall} c\ A_B(b,c) \vee \tilde{\forall} e\ B_B(d,e) \big)$, which implies

$$\tilde{\exists} b, d \left(\tilde{\forall} c \tilde{\forall} c' \trianglelefteq c\ A_B(b,c') \vee \tilde{\forall} e \tilde{\forall} e' \trianglelefteq e\ B_B(d,e') \right).$$

Equivalently, $\tilde{\exists} b, d \tilde{\forall} c, e \left(\tilde{\forall} c' \trianglelefteq c\ A_B(b,c') \vee \tilde{\forall} e' \trianglelefteq e\ B_B(d,e') \right)$. For the right-to-left implication, $\mathsf{bUD}^{\omega,\mathsf{X}}_{\forall bd}$ implies $(A \vee B)^B \to \tilde{\exists} b, d \left(\tilde{\forall} c\ A_B(b,c) \vee \tilde{\forall} e\ B_B(d,e) \right)$, which is equivalent to $\tilde{\exists} b \tilde{\forall} c\ A_B(b,c) \vee \tilde{\exists} d \tilde{\forall} e\ B_B(d,e)$.

3. $A \to B$.
Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ proves $A \leftrightarrow A^B$ and $B \leftrightarrow B^B$. We claim that $(A \to B) \leftrightarrow (A \to B)^B$. To the left-to-right implication, assume $A \to B$. This implies clearly that $A^B \to B^B$. Hence, $\tilde{\exists} b \tilde{\forall} c\ A_B(b,c) \to \tilde{\exists} d \tilde{\forall} e\ B_B(d,e)$, which implies, $\tilde{\forall} b \left(\tilde{\forall} c\ A_B(b,c) \to \tilde{\exists} d \tilde{\forall} e\ B_B(d,e) \right)$. By $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$ and the monotonicity lemma, we get $\tilde{\forall} b \tilde{\exists} d \left(\tilde{\forall} c\ A_B(b,c) \to \tilde{\forall} e\ B_B(d,e) \right)$, which is equivalent to $\tilde{\forall} b \tilde{\exists} d \tilde{\forall} e \left(\tilde{\forall} c\ A_B(b,c) \to B_B(d,e) \right)$. By $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$, we conclude

$$\tilde{\forall} b \tilde{\exists} d \tilde{\forall} e \tilde{\exists} c \left(\tilde{\forall} c' \trianglelefteq c\ A_B(b,c') \to B_B(d,e) \right).$$

By using twice the previous lemma, we obtain

$$\tilde{\exists} f, g \tilde{\forall} b, e \left(\tilde{\forall} c \trianglelefteq gbe\ A_B(b,c) \to B_B(fb,e) \right),$$

as desired.
To prove the right-to-left implication, assume $(A \to B)^B$ is given by

$$\tilde{\exists} f, g \tilde{\forall} b, e \left(\tilde{\forall} c \trianglelefteq gbe\ A_B(b,c) \to B_B(fb,e) \right) \tag{4.11}$$

and assume also A^B. Take b monotone. From A^B, it follows $\tilde{\forall} c\ A_B(b,c)$ and, consequently, $\tilde{\forall} c \trianglelefteq gbe\ A_B(b,c)$ for an arbitrary monotone e and g. By (4.11), we get $B_B(fb,e)$. Therefore, $\tilde{\exists} c \tilde{\forall} e\ B_B(c,e)$ for $c := fb$. Clearly, c is monotone. Thus,

$$(A \to B)^B \to (A^B \to B^B) \to (A \to B).$$

4. $\forall x\ A(x)$.
   Assume $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash\ A(x) \leftrightarrow (A(x))^B$ and that $(A(x))^B$ is given by $\tilde{\exists} f \tilde{\forall} a, c\ A_B(x, fa, c)$. First, we prove the left-to-right implication. By the induction hypothesis, we have $\forall x\ A(x) \leftrightarrow \forall x \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)$. By $\mathsf{bAC}^{\omega,\mathsf{X}}$ and the monotonicity property, it follows $\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \tilde{\forall} c\ A_B(x, fa, c)$, which is equivalent to $(\forall x\ A(x))^B$. For the right-to-left implication, assume $\tilde{\exists} f \tilde{\forall} a, c \forall x \trianglelefteq a\ A_B(x, fa, c)$. Hence, $\tilde{\forall} a \tilde{\exists} b \tilde{\forall} c \forall x \trianglelefteq a\ A_B(x, b, c)$, where b is given by fa. Then,

   $$\tilde{\forall} a \forall x \trianglelefteq a \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c).$$

   The conclusion follows by $\mathsf{MAJ}^{\omega,\mathsf{X}}$.

5. $\exists x\ A(x)$.
   Assume $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash A(x) \leftrightarrow (A(x))^B$. The bounded functional interpretation of $\exists x\ A(x)$ is $\tilde{\exists} a, b \tilde{\forall} c \exists x \trianglelefteq a \tilde{\forall} c' \trianglelefteq c\ A_B(x, b, c')$. In the left-to-right implication, the induction hypothesis implies $\exists x \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)$. By $\mathsf{MAJ}^{\omega,\mathsf{X}}$, we get

   $$\exists x\ \big(\exists a\ (x \trianglelefteq a) \wedge \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)\big).$$

   Since $x \trianglelefteq a$ ensures that a is monotone, we get $\exists x\ \big(\tilde{\exists} a\ (x \trianglelefteq a) \wedge \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)\big)$. Then, it is straightforward to get $(\exists x\ A(x))^B$. In the right-to-left implication, by $\mathsf{bBCC}^{\omega,\mathsf{X}}_{bd}$, we get $\tilde{\exists} a, b \exists x \trianglelefteq a \tilde{\forall} c\ A_B(x, b, c)$. $\exists x\ A(x)$ follows from intuitionistic logic and from the induction hypothesis.

6. $\forall x \trianglelefteq t\ A(x)$.
   Assume $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash A(x) \leftrightarrow (A(x))^B$. The bounded functional interpretation of $\forall x \trianglelefteq t\ A(x)$ is $\tilde{\exists} b \tilde{\forall} c \forall x \trianglelefteq t\ A(x, b, c)$. For the left-to-right implication, the induction hypothesis implies $\forall x \trianglelefteq t \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)$ and by $\mathsf{bBC}^{\omega,\mathsf{X}}$, we get

   $$\tilde{\exists} b \forall x \trianglelefteq t \tilde{\exists} b' \trianglelefteq b \tilde{\forall} c\ A_B(x, b', c).$$

   By the monotonicity property and intuitionistic logic, we get $(\forall x \trianglelefteq t\ A(x))^B$. Using only intuitionistic logic, the right-to-left implications is straightforward.

7. $\exists x \trianglelefteq t\ A(x)$.
   Assume that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ proves $A(x) \leftrightarrow \tilde{\exists} b \tilde{\forall} c\ A_B(x, b, c)$. The interpretation of $\exists x \trianglelefteq t\ A(x)$ is $\tilde{\exists} b \tilde{\forall} c \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c\ A(x, b, c')$. The left-to-right implication is straightforward, using the induction hypothesis and intuitionistic logic. In the right-to-left implication, assume $\tilde{\exists} b \tilde{\forall} c \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c\ A(x, b, c')$. By $\mathsf{bBCC}^{\omega,\mathsf{X}}_{bd}$, it follows $\tilde{\exists} b \exists x \trianglelefteq t \tilde{\forall} c\ A(x, b, c)$, and,by the induction hypothesis, we get $\exists x \trianglelefteq t\ A(x)$.

$\square$

As mentioned before, the characterization theorem ensures that there are no missing characteristic principles.

The following is a consequence of the soundness theorem:

Theorem 32 (**Program extraction**). *Let $A_{bd}(x,y)$ be a bounded formula of $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ whose only free variables are x and y. If*

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash \forall x \exists y\ A_{bd}(x,y)$$

then there is a monotone closed term t of the language such that

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} \vdash \tilde{\forall} a \forall x \trianglelefteq a \exists y \trianglelefteq ta\ A_{bd}(x,y).$$

Recall, that for x of type $0 \to \rho$, with $\rho \in \mathsf{T}$, we defined

$$x^M n = max_\rho\{xi : i \leq_0 n\}.$$

We extend this definition for types 0 and X:

Definition 17. *For each n^0 and x^{X}, we define $n^M := n$ and $x^M := \|x\|(0) + 1$.*

Then, we prove the following:

Lemma 33. *Take x^ρ with $\rho \in \{0, 1, \mathsf{X}\}$. Then $\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ proves $x \trianglelefteq x^M$.*

Proof The case of $\rho = 0$ is trivial and for $\rho = 1$, the result is a straightforward consequence of the definition of x^M. If $\rho = \mathsf{X}$, we want to prove that $x \trianglelefteq_{\mathsf{X}} \|x\|(0) + 1$. We claim that $\|x\| \leq_{\mathbb{R}} (\|x\|(0) + 1)_{\mathbb{R}}$. Recall that $int\,(\|x\|(0)) \leq_{\mathbb{Q}} \|x\|(0)$, since $\|x\|(0)$ is a natural number. Then $int\,(\|x\|(0)) + \sum_{k=1}^{i+2} (\|x\|(k) - 1)\, \frac{1}{2^k} <_{\mathbb{Q}} \|x\|(0) + 1 + \frac{1}{2^i}$ for all i^0. Therefore, $\|x\| \leq_{\mathbb{R}} (\|x\|(0) + 1)$. By RL_1, we get $x \trianglelefteq_{\mathsf{X}} \|x\|(0) + 1$. $\square$

As a consequence of the previous lemma and the program extraction theorem, we have

Theorem 33. *Let $A_{bd}(x,y)$ be a bounded formula in $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ containing only free variables x^ρ and y^σ, where ρ is restricted to the set $\{0, 1, \mathsf{X}\}$ and σ is an arbitrary type. If*

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash \forall x \exists y\ A_{bd}(x,y)$$

then there exists a monotone closed term t such that

$$\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}} \vdash \forall x \exists y \trianglelefteq tx\ A_{bd}(x,y).$$

Proof Take $A_{bd}(x,y)$ a bounded formula of $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ with x of type $\rho \in \{0, 1, \mathsf{X}\}$. By the program extraction theorem, there is a monotone closed term q such that $\tilde{\forall} a \forall x \trianglelefteq a \exists y \trianglelefteq qa\ A_{bd}(x,y)$ is provable in $\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}}$. By the lemma above, $x \trianglelefteq x^M$. Thus $\forall x \exists y \trianglelefteq q\left(x^M\right)\ A_{bd}(x,y)$. Then, choose t such that $tx := q\left(x^M\right)$. $\square$

Let $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ denote the classical version of $\mathsf{HA}_{\trianglelefteq}^{\omega,\mathsf{X}}$, i.e., $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ is Peano Arithmetic extend to the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$. To interpret the extended Peano arithmetic (with base types 0 and X), we proceed as usual. First, Peano arithmetic is interpreted into Heyting arithmetic via a negative translation and afterwards, Heyting arithmetic is interpreted by the extended bounded functional interpretation. We will use again Kuroda negative translation ($A \rightsquigarrow A'$) extended to bounded quantifiers.

Let $\mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ be the modification of $\mathsf{P}^{\omega,\mathsf{X}}[\trianglelefteq]$ obtained by replacing $\mathsf{bAC}^{\omega,\mathsf{X}}$ by $\mathsf{bAC}^{\omega,\mathsf{X}}_{bd}$, where $\mathsf{bAC}^{\omega,\mathsf{X}}_{bd}$ is the restriction of $\mathsf{bAC}^{\omega,\mathsf{X}}$ to bounded matrices. As well, $\mathsf{bBC}^{\omega,\mathsf{X}}$ restricted to bounded matrices is denoted by $\mathsf{bBC}^{\omega,\mathsf{X}}_{bd}$

Following the proof of proposition 18, then $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves $\mathsf{bBC}^{\omega,\mathsf{X}}_{bd}$.

Theorem 34. *If* $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq}+\mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ *proves* A*, then* $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}+\mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ *proves* A' *for an arbitrary formula* A *in the language* $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$*.*

Proof It suffices to prove the theorem for the axioms $\mathsf{B}_\forall$, $\mathsf{B}_\exists$ (since the remaining non--logical axioms are universal), for the rules RL_1, RL_2 and for the principles in $\mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$.

$(\mathsf{B}_\forall)'$ is given by $\neg\neg\left(\forall x \trianglelefteq t\ \neg\neg(A(x))^\dagger \leftrightarrow \forall x\ \neg\neg\left(x \trianglelefteq t \to (A(x))^\dagger\right)\right)$. From $\mathsf{B}_\forall$, we get $\forall x \trianglelefteq t\ \neg\neg(A(x))^\dagger \leftrightarrow \forall x\ \left(x \trianglelefteq t \to \neg\neg(A(x))^\dagger\right)$ and since $(\varphi \to \neg\neg\psi) \leftrightarrow \neg\neg(\varphi \to \psi)$ is intuitionistically true, then

$$\forall x \trianglelefteq t\ \neg\neg(A(x))^\dagger \leftrightarrow \forall x\ \neg\neg\big(x \trianglelefteq t \to (A(x))^\dagger\big).$$

The proof for $\mathsf{B}_\exists$ is even simpler. For the the rule RL_1, assume $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves $A_{bd} \to \|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}$. In order to ease the reading, let us write $\|s\| \leq_{\mathbb{R}} (t)_{\mathbb{R}}$ as $\forall n\ C(s,t,n)$, where C is decidable. By the induction hypothesis, $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves $\neg\neg\left((A_{bd})^\dagger \to \forall n\ \neg\neg C(s,t,n)\right)$, which is equivalent to $(A_{bd})^\dagger \to \neg\neg\forall n\ \neg\neg C(s,t,n)$. This implies $(A_{bd})^\dagger \to \forall n\ \neg\neg C(s,t,n)$. Equivalently, we have $(A_{bd})^\dagger \to \forall n\ C(s,t,n)$ since C is decidable. By RL_1, it follows $(A_{bd})^\dagger \to s \trianglelefteq_{\mathsf{X}} t$.

The proof for rule RL_2 is similar, hence we skip it.

Now, we focus on the principles of $\mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$.

1. $\mathsf{bAC}^{\omega,\mathsf{X}}_{bd}$.
   We claim that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves
   $$\neg\neg\left(\forall x\neg\neg\exists y\ (A_{bd}(x,y))^\dagger \to \tilde{\exists} f\tilde{\forall} b\neg\neg\forall x \trianglelefteq b\neg\neg\exists y \trianglelefteq f(b)\ (A_{bd}(x,y))^\dagger\right).$$
   Assume $\forall x\neg\neg\exists y\ (A_{bd}(x,y))^\dagger$. By $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$, we obtain $\forall x\tilde{\exists} a\neg\neg\exists y \trianglelefteq a\ (A_{bd}(x,y))^\dagger$, which, implies $\tilde{\exists} f\tilde{\forall} b\forall x \trianglelefteq b\tilde{\exists} a \trianglelefteq f(b)\neg\neg\exists y \trianglelefteq a\ (A_{bd}(x,y))^\dagger$ by $\mathsf{bAC}^{\omega,\mathsf{X}}_{bd}$. Since $\trianglelefteq$ is transitive and $\exists x\ \neg\neg A \leftrightarrow \neg\neg\exists x\ A$, it follows $\tilde{\exists} f\tilde{\forall} b\forall x \trianglelefteq b\neg\neg\exists y \trianglelefteq f(b)\ (A_{bd}(x,y))^\dagger$. Now, the conclusion is straightforward.

2. $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$.
   By $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$, it is clear that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves
   $$\neg\neg\left(\left(\forall x\ \neg\neg(A_{bd}(x))^\dagger \to \exists y\ (B(y))^\dagger\right) \to \tilde{\exists} b\ \left(\forall x\ \neg\neg(A_{bd}(x))^\dagger \to \exists y \trianglelefteq b\ (B(y))^\dagger\right)\right).$$

3. $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$.
   Similar to the one above, using $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$ instead of $\mathsf{bIP}^{\omega,\mathsf{X}}_{\forall bd}$.

4. $\mathsf{bUD}^{\omega,\mathsf{X}}_{\forall bd}$.
   We want to prove that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves the double negation of

$$\tilde{\forall} b \neg\neg \tilde{\forall} c \neg\neg \left(\forall x \trianglelefteq b \neg\neg(A_{bd}(x))^\dagger \vee \forall y \trianglelefteq c \neg\neg(B_{bd}(y))^\dagger\right) \rightarrow \\ \rightarrow \forall x \neg\neg(A_{bd}(x))^\dagger \vee \forall y \neg\neg (B_{bd}(y))^\dagger.$$

   It is a straightforward consequence of $\mathsf{bUD}^{\omega,\mathsf{X}}_{\forall bd}$, $\neg\neg(A \vee B) \rightarrow (\neg\neg A \vee \neg\neg B)$ and $\neg\neg\forall x\ A \rightarrow \forall x\ \neg\neg A$.

5. $\mathsf{bBCC}^{\omega,\mathsf{X}}_{bd}$.
   We claim that the double negation of

$$\tilde{\forall} c \neg\neg \left(\tilde{\forall} b \neg\neg\exists z \trianglelefteq c \forall y \trianglelefteq b \neg\neg(A_{bd}(y,z)^\dagger \rightarrow \exists z \trianglelefteq c \forall y \neg\neg(A_{bd}(y,z))^\dagger\right)$$

   is provable in $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$. Assume $\tilde{\forall} b \neg\neg\exists z \trianglelefteq c \forall y \trianglelefteq b \neg\neg(A_{bd}(y,z)^\dagger$. Then $\exists z \trianglelefteq c \forall y \neg\neg(A_{bd}(y,z))^\dagger$ is a straightforward consequence of $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$ and $\mathsf{bBCC}^{\omega,\mathsf{X}}_{bd}$.

6. $\mathsf{MAJ}^{\omega,\mathsf{X}}$.
   Trivial.

□

As a consequence of the previous and the soundness theorems, it follows:

Theorem 35. *Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ with negative translation $A'(\underline{z})$. The bounded functional interpretation of $A'(\underline{z})$ is given by $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ (A')_B(\underline{z},\underline{b},\underline{c})$. If*

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq] \vdash A(\underline{z}),$$

then there exist monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ (A')_B(\underline{z},\underline{b},\underline{c}).$$

Theorem 36 (Extraction and Conservation). *Let $A_{bd}(x,y)$ be a bounded formula in the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$, containing only free variables x^ρ and y^σ, where $\rho \in \{0,1,\mathsf{X}\}$ and σ is an arbitrary type. If*

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq] \vdash \forall x^\rho \exists y^\sigma\ A_{bd}(x,y)$$

then there exists a monotone closed term t of appropriate type such that

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \forall x \exists y \trianglelefteq tx\ A_{bd}(x,y).$$

Proof Assume $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$ proves $\forall x^{\rho} \exists y^{\sigma}\ A_{bd}(x,y)$. Then, in $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} + P^{\omega,\mathsf{X}}_{bd}[\trianglelefteq]$, we get $\forall x \neg\neg \exists y\ A_{bd}(x,y)$. Using $\mathsf{bMP}^{\omega,\mathsf{X}}_{bd}$ and the theorem 33, there is a monotone closed term t such that $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \forall x \tilde{\exists} b \trianglelefteq tx \neg\neg \exists y \trianglelefteq b\ A_{bd}(x,y)$. Hence $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves

$$\forall x \tilde{\exists} b \trianglelefteq tx \neg\neg \exists y \trianglelefteq b\ A_{bd}(x,y),$$

and, clearly, also proves $\forall x \exists y \trianglelefteq tx\ A_{bd}(x,y)$. □

To each formula A of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$, we associate its corresponding flattening, obtained by replacing all intensional symbols $\trianglelefteq$ by the extensional ones $\leq^*$. The flattened formula is denoted A^*.

Then, the following is clear:

Theorem 37 (**Flattening**). *Let A be an arbitrary formula of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$. If $\mathsf{HA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ proves A, then $\mathsf{HA}^{\omega,\mathsf{X}}$ proves A^*.*

There are two kind of models for $\mathsf{HA}^{\omega,\mathsf{X}}$. On one hand, such a model can be obtained by letting the variables range over the appropriate universe of the full set-theoretic type structure $\mathcal{S}^{\omega,\mathsf{X}}$ with $\mathbb{N}$ and $(X, \|.\|_X)$ as the universe for base types 0 and X, respectively. All objects of type X are interpreted as vectors in the normed space X. In particular, 0_{X} is interpreted by the zero vector of the normed space. The operations $+_{\mathsf{X}}$ and $-_{\mathsf{X}}$ are interpreted, respectively, as the addition in X and as the inverse of a vector with respect to $+$, while $\cdot_{\mathsf{X}}$ is interpreted as the operator which, given $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$, returns the scalar multiplication of unique real represented by $\widetilde{\alpha}$ by x. Finally, $\|.\|$ is interpreted by the function which associates to each vector $x \in X$, a specific representation (non-effective) of the real number $\|x\|_X$. For instance, $\|x\|_X$ can be represented by the sequence $\langle k, n_1, n_2, ...\rangle$ of integer numbers (we ignore the representation of integers by natural numbers), where k is the integer part of $\|x\|_X$ and $\langle n_1, n_2, ...\rangle$ is the binary representation (with no infinite sequence of 1's) of its decimal part. This corresponds to the canonical (also ineffective) representation $(.)_\circ$ of Kohlenbach [Koh08a].

On the other hand, a normed space $(\widehat{A}, \widehat{\|.\|})$ (necessarily separable) can be seen as the completion of the countable normed space $(A, \|.\|)$. The elements of this countable normed space are coded by natural numbers while a function $\|.\|_A$ of type $0 \to 1$ represents the pseudo-norm on $\mathbb{N}$: $\|n\|_A =_{\mathbb{R}} (\|x\|)_{\mathbb{R}}$, where $x \in A$ is represented by n and $(\|x\|)_{\mathbb{R}}$ is the representation of the real number $\|x\|$. For instance, the space $C([0,1])$ of all continuous functions $f : [0,1] \to \mathbb{R}$ together with the supremum norm $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ is the completion of $(X, \|.\|)$ where X is the set of all finite tuples of rationals numbers and $\|.\|$ is given by

$$\|\langle a_0, a_1, ..., a_n\rangle\| := \sup\{|a_0 + a_1 x + ... + a_n x^n| : x \in [0,1]\}.$$

The completion $(\widehat{A}, \widehat{\|.\|})$ of $(A, \|.\|)$ is now represented by the completion of $(\mathbb{N}, \|.\|_A)$: elements of the completion are represented as functions from $\mathbb{N}$ to $\mathbb{N}$ satisfying certain properties and the pseudo-norm $\|.\|_A$ is extended to a pseudo-norm $\widehat{\|.\|}_A$ on $\mathbb{N}^{\mathbb{N}}$. Actually, we may suppose that every element of $\mathbb{N}^{\mathbb{N}}$ is a representation. For details, see [Koh93] and [Koh08a].

The new kind of model of $\mathsf{HA}^{\omega,\mathsf{X}}$ can be obtained by letting the variables range over the appropriate universe of the structure $\mathcal{S}^{\omega,\mathsf{X}}$ where $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ are the universes for types 0 and X. Each object in X is interpreted as the representation (in $\mathbb{N}^{\mathbb{N}}$) of an element of $\widehat{A}$. The operations in X are interpreted by the operations in $\mathbb{N}^{\mathbb{N}}$ and $\|.\|^{\mathsf{X}\to 1}$ is interpreted by $\widehat{\|.\|_A}$ on $\mathbb{N}^{\mathbb{N}}$.

4.3 Bounded functional interpretation of Peano arithmetic extended to new base types

In this section, we extend the direct bounded functional interpretation of Peano arithmetic to new base types. As in the numerical case, the logical connectives are reduced to $\vee, \forall, \neg$. The other connectives and existential quantifiers are defined classically in the usual manner.

Definition 18. *To each formula A of the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$, we assign formulas A^U and A_U, such that A_U is bounded and A^U is of the form $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\ A_U(\underline{b},\underline{c})$, according to the following clauses*

1) if A is bounded, then A_U and A^U are A;

Take A^U and B^U as $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\ A_U(\underline{b},\underline{c})$ and $\tilde{\forall}\underline{d}\tilde{\exists}\underline{e}\ B_U(\underline{d},\underline{e})$, respectively. The remaining cases are described below

2) $(A \vee B)^U$ is $\tilde{\forall}\underline{b},\underline{d}\tilde{\exists}\underline{c},\underline{e}\ (A_U(\underline{b},\underline{c}) \vee B_U(\underline{d},\underline{e}))$;

3) $(\forall x\ A(x))^U$ is $\tilde{\forall}a\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq a\ A_U(x,\underline{b},\underline{c})$;

4) $(\neg A)^U$ is $\tilde{\forall}\underline{f}\tilde{\exists}\underline{b}\tilde{\exists}\underline{b}' \trianglelefteq \underline{b}\ (\neg A_U(\underline{b}',\underline{f}\underline{b}'))$;

5) $(\forall x \trianglelefteq t\ A)^U$ is $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq t\ A_U(x,\underline{b},\underline{c})$.

The defining clauses are exactly the same of the numerical case. However, one now must recall that the monotone terms are of arithmetic type. The monotonicity lemma also holds in the extended version:

Lemma 34 (Monotonicity Lemma). $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ *proves* $\tilde{\forall}\underline{b}\tilde{\forall}\underline{c}\tilde{\forall}\underline{c}' \trianglelefteq \underline{c}\ \big(A_U(\underline{b},\underline{c}') \to A_U(\underline{b},\underline{c})\big)$ *for every formula A of the language.*

Proof The proof is made by induction on the complexity of A. □

The three characteristic principles are naturally extended to new types. They are denoted by $\mathsf{mAC}_{bd}^{\omega,\mathsf{X}}$, $\mathsf{bC}_{bd}^{\omega,\mathsf{X}}$ and $\mathsf{MAJ}^{\omega,\mathsf{X}}$. The set containing the three characteristic principles is denoted by $\mathsf{P}_{cl}^{\omega,\mathsf{X}}[\trianglelefteq]$. Recall that is was proved before that $\mathsf{P}_{cl}^{\omega}[\trianglelefteq]$ is classically equivalent to $\mathsf{P}_{bd}^{\omega}[\trianglelefteq]$. Using a similar argument, one proves that $\mathsf{P}_{cl}^{\omega,\mathsf{X}}[\trianglelefteq]$ is classically equivalent to $\mathsf{P}_{bd}^{\omega,\mathsf{X}}[\trianglelefteq]$.

Theorem 38 (**Soundness**). *Take $A(\underline{z})$ an arbitrary formula of the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ with free variables $\underline{z}$. Let $A(\underline{z})^U$ be $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\ A_U(\underline{z},\underline{b},\underline{z})$. If*

$$\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \mathsf{P}_{cl}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are monotone closed terms $\underline{t}$ of appropriate types such that

$$\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}} \vdash \tilde{\forall}\underline{a},\underline{b}\forall\underline{z} \trianglelefteq \underline{a}\ A_U(\underline{z},\underline{b},\underline{tab}).$$

Proof We argue by induction on the length of the derivation of $A(\underline{z})$. To simplify the notation, we will not underline the tuples. As in the intuitionist case, all universal sentences are self interpreted. The proof follows closely the one given in [Fer09]. Hence, only some axioms and rules will be discussed. It relies in Shoenfield's axiomatization for classical logic.

1. $\neg A \vee A$
   Take z as the tuple of free variables of A and let A^U be given as $\tilde{\forall}b\tilde{\exists}c\ A_U(z,b,c)$. Then we look for closed monotone terms t and q such that

   $$\tilde{\forall}a,b,f\forall z \trianglelefteq a\ \Big(A_U(z,b,tabf) \vee \tilde{\exists}d \trianglelefteq qabf\ \neg A_U(z,d,fd)\Big).$$

   Clearly $t := \lambda a,b,f.fb$ and $q := \lambda a,b,f.b$ do the job. These terms are closed and obviously monotone.

2. $$\frac{A}{A \vee B}$$

   Take A and B formulas of the language with free variables z and bounded interpretations given by $\tilde{\forall}b\tilde{\exists}c\ A_U(z,b,c)$ and $\tilde{\forall}d\tilde{\exists}e\ B_U(z,d,e)$, respectively. Assume $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \mathsf{P}_{bd}^{\omega,\mathsf{X}}[\trianglelefteq] \vdash A(z)$. By the induction hypothesis, there exist monotone closed terms t such that $\mathsf{PA}^{\omega,\mathsf{X}} \vdash \tilde{\forall}a,b\forall z \trianglelefteq a\ A_U(z,b,tab)$. We look for monotone closed terms r and s such that $\tilde{\forall}a,b,d\forall z \trianglelefteq a\ \ (A_U(z,b,sabd) \vee B_U(z,d,rabd))$. Take $r := \lambda a,b,d.b$ and $s := \lambda a,b,d.tab$. They are closed, monotone and do the job.

3. $$\frac{A(0) \qquad \forall n^0\ (A(n) \to A(n+1))}{\forall n^0\ A(n)}$$

   This proof is similar to the one in the intuitionistic setting. Assume $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ proves $A(0)$ and $\forall n^0\ (A(n) \to A(n+1))$. Then, there are monotone closed terms t,p,q such that $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ proves both $\tilde{\forall}b\ A_U(0,b,tb)$ and

   $$\tilde{\forall}a,d,f\forall n \leq_0 a\ \big(\tilde{\exists}b \trianglelefteq padf\ \neg A_U(n,b,fb) \vee A_U(n+1,d,qadf)\big).$$

From the latter, we obtain $\tilde{\forall} a, f \forall n \leq_0 a \ \big(\tilde{\forall} b A_U(n, b, fb) \to \tilde{\forall} b A_U(n+1, b, qabf)\big)$, which, together with the induction rule, implies $\tilde{\forall} a, b \forall n \leq_0 a \ A_U(n, b, \phi ab)$, where $\phi ab := \psi aba$ and ψ is recursively defined as

$$\begin{aligned} &\psi 0ab = tb \\ &\psi(n+1)ab = \max\left(\psi nab, qab(\lambda b.\psi n, a, b)\right) \end{aligned}$$

ψ has arithmetic type. We want to produce monotone closed terms r such that $\tilde{\forall} a, b \forall n \trianglelefteq_0 a \ A_U(n, b, rab)$. It is enough to choose $r := \phi$. To prove the monotonicity of ϕ, we proceed as in the intuitionistic case.

4. $\mathsf{B}_\forall$.
Take A a formula of the language and t a term. Let A^U be $\tilde{\forall} b \tilde{\exists} c \ A_U(z, b, c)$, where z is the tuple of free variables of A and t. It is proved in two steps. First, we look at the left-to-right implication $\forall x \trianglelefteq t \ A(x, z) \to \forall x \ \big(x \trianglelefteq t \to A(x, z)\big)$. We want monotone closed terms r and s such that

$$\begin{aligned} \tilde{\forall} a, a', b, f \forall z \trianglelefteq a \big(&\tilde{\exists} c \trianglelefteq raa'bf \ \neg \forall x \trianglelefteq t \ A_U(x, z, c, fc) \vee \\ &\vee \forall x \trianglelefteq a' \ \ (\neg(x \trianglelefteq t) \vee A_U(x, z, b, saa'bf))\,\big). \end{aligned}$$

Take r and s given by $raa'bf := b$ and $saa'bf := fb$. We claim that these terms do the job. Take a, a', b, f monotone terms and $z \trianglelefteq a$. From the law of the excluded middle, it follows $\exists x \trianglelefteq t \ \neg A_U(x, z, b, fb) \vee \forall x \trianglelefteq t \ A_U(x, z, b, fb)$. Then, we get $\exists x \trianglelefteq t \ \neg A_U(x, z, b, fb) \vee \forall x \trianglelefteq t \ \ (x \trianglelefteq a' \to A_U(x, z, b, fb))$, which is equivalent to $\exists x \trianglelefteq t \ \neg A_U(b, fb, x, z) \vee \forall x \trianglelefteq a' \ \ (\neg(x \trianglelefteq t) \vee A_U(b, fb, x, z))$. Hence

$$\tilde{\exists} c \trianglelefteq b \ \big(\neg \forall x \trianglelefteq t \ A_U(c, fc, x, z)\big) \vee \forall x \trianglelefteq a' \ \big(\neg(x \trianglelefteq t) \vee A_U(b, fb, x, z)\big),$$

as desired.

Second, to interpret the right-to-left implication, we want to produce monotone closed terms p, q, r that for monotone a, b, f and $z \trianglelefteq a$ we have

$$\tilde{\exists} c \trianglelefteq pabf \tilde{\exists} d \trianglelefteq qabf \ \neg \forall x \trianglelefteq c \ \big(\neg(x \trianglelefteq t) \vee A_U(x, z, d, fcd)\big) \vee \forall x \trianglelefteq t \ A_U(x, z, b, rabf).$$

As was done above, one easily prove that the terms given by $pabf := \tilde{t}[a/z]$, $qabf := b, rabf := fb(\tilde{t}[a/z])$, with $\tilde{t}$ such that $t \trianglelefteq \tilde{t}$, do the job.

5. $\mathsf{mAC}^{\omega,\mathsf{X}}_{bd}$.
Let A_{bd} be a bounded formula of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ with free variables x, y, z. A simple calculation shows that the direct bounded functional interpretation of $\mathsf{mAC}^{\omega,\mathsf{X}}_{bd}$ is

$$\tilde{\forall} a, f, \varphi \tilde{\exists} b, h \forall z \trianglelefteq a \ \big(\tilde{\exists} b' \trianglelefteq b \ \neg A_{bd}(b', fb', z) \vee \tilde{\exists} h' \trianglelefteq h \tilde{\forall} b'' \trianglelefteq \varphi h' \tilde{\exists} c \trianglelefteq h'b'' \ A_{bd}(b'', c, z)\big).$$

Hence, we look for monotone closed terms t, q such that for monotone a, f, φ and $z \trianglelefteq a$ we have

$$\tilde{\exists} b \trianglelefteq taf\varphi \ \neg A_{bd}(z, b, fb) \vee \tilde{\exists} h \trianglelefteq qaf\varphi \tilde{\forall} b' \trianglelefteq \varphi h \tilde{\exists} c \trianglelefteq hb' \ A_{bd}(z, b', c).$$

Take $taf\varphi := \varphi f$ and $qaf\varphi := f$. It is straightforward to see that these terms do the job.

6. $\mathsf{bC}^{\omega,\mathsf{X}}_{bd}$.
   Take A_{bd} a bounded formula of the language. A simple calculation shows that we need to produce monotone closed terms t such that

$$\tilde{\forall} a, b \forall x \trianglelefteq a \left(\neg \forall z \trianglelefteq c \tilde{\exists} b' \trianglelefteq b \exists y \trianglelefteq b' \ A_{bd}(x,y,z) \vee \forall z \trianglelefteq c \exists y \trianglelefteq tab \ A_{bd}(x,y,z)\right).$$

   It suffices to define t as $t := \lambda a, b.b$.

7. $\mathsf{MAJ}^{\omega,\mathsf{X}}$.
   Since $(\mathsf{MAJ}^{\omega,\mathsf{X}})^U$ is given by $\tilde{\forall} a \tilde{\exists} b \forall x \trianglelefteq a \tilde{\exists} b' \trianglelefteq b \exists y \trianglelefteq b' \ (x \trianglelefteq y)$, then $t := \lambda a.a$ is such that $\tilde{\forall} a \forall x \trianglelefteq a \tilde{\exists} b \trianglelefteq ta \exists y \trianglelefteq b \ x \trianglelefteq y$ for all monotone a.

The proofs for the rules RL_1 and RL_2 are similar to the one in the intuitionistic case, thus we will not discuss them. □

Corollary 5. *Let A_{bd} be a bounded formula of $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$ whose only free variables are x and y. If*

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] \vdash \forall x \exists y \ A_{bd}(x,y),$$

then

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \tilde{\forall} a \forall x \trianglelefteq a \exists y \ A_{bd}(x,y).$$

Proof Assume $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] \vdash \forall x \exists y \ A_{bd}(x,y)$. Hence, by the Soundness Theorem, there exists a monotone closed term t such that

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} \vdash \tilde{\forall} a \forall x \trianglelefteq a \tilde{\exists} b \trianglelefteq ta \exists y \trianglelefteq b \ A_{bd}(x,y),$$

which implies $\tilde{\forall} a \forall x \trianglelefteq a \exists y \ A_{bd}(x,y)$. □

Theorem 39 (Characterization).

$$\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] \vdash A \leftrightarrow A^U$$

for any formula A of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$.

Proof We argue by induction on the complexity of A. If A is bounded, it is trivially done. In the case of negation, take A, a formula of the language such that $A \leftrightarrow A^U$, where A^U is given $\tilde{\forall} b \tilde{\exists} c \ A_U(b,c)$. We claim that $\neg A \leftrightarrow (\neg A)^U$ or, equivalently, $A \leftrightarrow \neg(\neg A)^U$, with $\neg(\neg A)^U$ given by $\tilde{\exists} f \tilde{\forall} b \tilde{\forall} b' \trianglelefteq b \ A_U(b', fb')$. By $\mathsf{MAJ}^{\omega,\mathsf{X}}$, it is equivalent to $\tilde{\exists} f \tilde{\forall} b \ A_U(b, fb)$ and using the monotonicity lemma, the latter is equivalent to $\tilde{\exists} f \tilde{\forall} b \tilde{\exists} c \trianglelefteq fb \ A_U(b,c)$. By $\mathsf{bC}^{\omega,\mathsf{X}}_{bd}$, we get A^U, which is equivalent to A by the induction hypothesis.

The disjunction is straightforward. Now, let us look at the universal quantification. We want to prove that $\forall x\ A(x) \leftrightarrow (\forall x\ A(x))^U$. Let $(A(x))^U$ be given by $\tilde{\forall} b \tilde{\exists} c\ A_U(x,b,c)$. Then, $(\forall x\ A(x))^U$ is $\tilde{\forall} a, b \tilde{\exists} c \forall x \trianglelefteq a\ A_U(x,b,c)$. From the monotonicity lemma and $\mathsf{bC}_{bd}^{\omega,\mathsf{X}}$, we get $\tilde{\forall} a, b \forall x \trianglelefteq a \tilde{\exists} c\ A_U(b,c,x)$, which is equivalent to $\forall x \tilde{\forall} b \tilde{\exists} c\ A_U(b,c,x)$, by $\mathsf{MAJ}^{\omega,\mathsf{X}}$. The conclusion follows from the induction hypothesis.

The case of the bounded quantification is similar. It is straightforward using only the monotonicity lemma and the bounded collection principle. $\square$

Each formula of the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$ has a corresponding flattening, obtained by replacing all occurrences of $\trianglelefteq$ by $\leq^*$. The following result is clear:

Theorem 40 (**Flattening**). *Let A be an arbitrary formula of the language $\mathcal{L}_{\trianglelefteq}^{\omega,\mathsf{X}}$. If $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}}$ proves A, then $\mathsf{PA}^{\omega,\mathsf{X}}$ proves A^*, where A^* is the flattening of A.*

As in the intuitionistic case, there are two kind of models for $\mathsf{PA}^{\omega,\mathsf{X}}$, obtained in the same way as the models for $\mathsf{HA}^{\omega,\mathsf{X}}$.

As a consequence of the direct bounded functional interpretation, we claim that every linear normed space is *weak-complete*:

Definition 19. *A Cauchy sequence $(x_n)_n$ has* modulus of Cauchy convergence *(also known as modulus of Cauchyness) if there exists $f^{0\to\rho}$ such that*

$$\forall k^0 \forall n, m \geq_0 fk\ \left(\|x_n - x_m\| <_{\mathbb{R}} \tfrac{1}{2^k}\right).$$

If such f exists, one may assume it is monotone.

A linear normed space X is said to be weak-complete *if every Cauchy sequence with modulus of Cauchy convergence converges in X.*

Proposition 20. $\mathsf{PA}_{\trianglelefteq}^{\omega,\mathsf{X}} + \mathsf{P}_{cl}^{\omega,\mathsf{X}}[\trianglelefteq]$ *proves that X is weak-complete.*

Proof Take $(x_n)_n$ be a Cauchy sequence with monotone modulus of convergence f. Then

$$\forall k \forall n, m \geq_0 fk\ \left(\|x_n - x_m\| <_{\mathbb{R}} \tfrac{1}{2^k}\right).$$

The latter is also true with $\trianglelefteq_{\mathbb{R}}$ instead of $<_{\mathbb{R}}$.

First, let us check that there exists N^0 such that $\|x_n\| <_{\mathbb{R}} N$ for all n^0. It is clear that $\forall n \geq f0\ (\|x_n - x_{f0}\| <_{\mathbb{R}} 1)$. Take N as a natural ($N \neq 0$) satisfying

$$1 + \max\{\|x_i - x_{f0}\| + \|x_{f0}\| : i < f0\} <_{\mathbb{R}} N.$$

Then

$$\|x_n\| \leq_{\mathbb{R}} \|x_n - x_{f0}\| + \|x_{f0}\| <_{\mathbb{R}} N.$$

At this point, we claim that

$$\forall r^0 \exists x \trianglelefteq_{\mathsf{X}} N \forall k \leq_0 r \forall m \leq_0 r\ \left(m \geq_0 fk \to \|x_m - x\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^k}\right).$$

Take r^0 and define x as x_{fr}. Since f is the modulus of convergence of $(x_n)_n$, we have that $\forall k \leq r \forall m \leq r \ \left(m \geq fk \to \|x_m - x\| \trianglelefteq_{\mathbb{R}} \frac{1}{2^k}\right)$, since $fr \geq fk$. As we had seen above, $\|x\| <_{\mathbb{R}} N$. By collection, we get

$$\exists x \trianglelefteq_{\mathsf{X}} N \forall k \forall m \geq fk \ \|x_m - x\| \leq_{\mathbb{R}} \frac{1}{2^k},$$

as desired. $\square$

Observe that every Cauchy sequence (with modulus of convergence) $(x_n)_n$ converges for certain x with $x = x_r$ for some r. Furthermore, if $x_n \trianglelefteq_{\mathsf{X}} N$ for all natural n, then $x \trianglelefteq_{\mathsf{X}} N$.

4.4 A logical view of the Banach-Steinhaus and the open mapping theorems

In the bounded functional interpretation of Peano arithmetic, one of the characteristic principles is a collection principle. In the classical case (as opposed to the intuitionistic case), collection must be restricted to bounded formulas. It is easy to provide a counter-example to collection for universal formulas. For instance, in the classical setting, we have that $\forall x \trianglelefteq_1 1 \exists n^0 \ (xn =_0 0 \vee \forall k^0 \ (xk \neq_0 0))$. If we had collection for universal matrices, from the latter, we would have

$$\exists m^0 \forall x \trianglelefteq_1 1 \exists n \leq_0 m \ (xn =_0 0 \vee \forall k \ (xk \neq_0 0)) \,.$$

Equivalently, $\exists m^0 \forall x \trianglelefteq_1 1 \ (\exists n \leq_0 m \ (xn =_0 0) \vee \forall k \ (xk \neq_0 0))$, which is, clearly, an absurd.

Nevertheless, in very particular situations, one has "collection" for universal formulas. As we will see, the Banach-Steinhaus and the open mapping theorems of functional analysis can be seen as collection of this sort.

In functional analysis, the Banach-Steinhaus and the open mapping theorems rely on the Baire category theorem. Our next result can be seen as a "logical" version of the Baire category theorem, in fact, a kind of local collection for extensional universal matrices.

Theorem 41 (**Baire-like Theorem**). *Let $A_{bd}(x^{\mathsf{X}}, n^0, k^0)$ be a bounded formula of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$. Then $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] + \mathsf{tameAC}$ proves the implication whose antecedent is $x =_{\mathsf{X}} y \wedge \forall k \ A_{bd}(x,n,k) \to \forall k \ A_{bd}(y,n,k)$ (i.e. $\forall k \ A_{bd}(x,n,k)$ is* extensional *with respect to x) and the consequent is given by*

$$\begin{aligned}&\forall x \trianglelefteq_{\mathsf{X}} 1 \exists n^0 \forall k^0 \ A_{bd}(x,n,k) \to \\ &\qquad \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r^0 \exists n^0 \forall x \trianglelefteq_{\mathsf{X}} 1 \ \left(\|x - z\| <_{\mathbb{R}} \frac{1}{2^r} \to \forall k^0 \ A_{bd}(x,n,k) \right).\end{aligned}$$

This result states that, within a specific ball with center z and radius r, the quantification $\forall\exists$ can be replaced by $\exists\forall$ for extensional universal matrices.

By the Baire-like theorem, if $\{x \in \mathsf{X} : \|x\| \leq_{\mathbb{R}} 1\}$ is contained in a countable union of closed sets, then at least one of the closed sets has non-empty interior.

For each n, let F_n be the set $\bigcap_k \{x \in \mathsf{X} : A_{bd}(x, n, k)\}$. By the extensionality of $\forall k\ A_{bd}(x, n, k)$, we have

$$\forall x^{\mathsf{X}} \forall y^{\mathsf{X}}\ (x =_{\mathsf{X}} y \wedge y \notin F_n \to x \notin F_n)$$

for all n. Take $y \in \mathsf{X} \backslash F_n$, i.e., $y \in \mathsf{X}$ such that $\exists k\ \neg A_{bd}(y, n, k)$. The latter entails

$$\forall x^{\mathsf{X}}\ (x =_{\mathsf{X}} y \to x \notin F_n)\,.$$

Consequently,

$$\forall x\ \left(\forall r\ \left(\|x - y\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r}\right) \to \exists k\ \neg A_{bd}(x, n, k)\right),$$

and, equivalently,

$$\forall x \exists r \exists k\ \left(\|x - y\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} \to \neg A_{bd}(x, n, k)\right).$$

By collection,

$$\forall N \exists r \exists k \forall x \trianglelefteq_{\mathsf{X}} N \exists r' \leq r \exists k' \leq k\ \left(\|x - y\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^{r'}} \to \neg A_{bd}(x, n, k')\right),$$

and also

$$\forall N \exists r \forall x \trianglelefteq_{\mathsf{X}} N\ \left(\|x - y\| <_{\mathbb{R}} \tfrac{1}{2^r} \to x \notin F_n\right),$$

meaning that F_n is closed for all n. Under this observation, the Baire-like theorem states that if $\{x^{\mathsf{X}} : \|x\| \leq_{\mathbb{R}} 1\} \subseteq \bigcup_n F_n$, then there exist n_0, $z \trianglelefteq_{\mathsf{X}} 1$ and r such that

$$\{x^{\mathsf{X}} : \|x\| \leq_{\mathbb{R}} 1 \wedge \|x - z\| <_{\mathbb{R}} \tfrac{1}{2^r}\} \subseteq F_{n_0},$$

i.e. there exists n_0 such that F_{n_0} has non-empty interior.

Proof Assume

$$\forall z \trianglelefteq_{\mathsf{X}} 1 \forall r^0, n^0 \exists x \trianglelefteq_{\mathsf{X}} 1 \exists k^0\ \left(\|x - z\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^{r+1}} \wedge \neg A_{bd}(x, n, k)\right).$$

By bounded collection and monotone choice, there exists a monotone f such that

$$\forall r, n \exists k \leq_0 frn \forall z \trianglelefteq_{\mathsf{X}} 1 \exists k' \leq_0 k \exists x \trianglelefteq_{\mathsf{X}} 1 \big(\|x - z\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^{r+1}} \wedge \neg A_{bd}(x, n, k')\big).$$

By classical logic and tameAC, there exists g (such that $g \trianglelefteq \lambda r^0, n^0, p^0.1^0$) and h (such that $h \trianglelefteq \lambda r^0, n^0, p^0.frn$) such that

$$\forall r, n \forall z \trianglelefteq_{\mathsf{X}} 1\ \left(\|grnz - z\| <_{\mathbb{R}} \tfrac{1}{2^r} \wedge \neg A_{bd}(grnz, n, hrnz)\right). \tag{4.12}$$

From the extensionality of $\forall k\ A_{bd}(x, n, k)$ with respect to x, we get

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \forall n\ (\neg \forall k\ A_{bd}(x, n, k) \to \forall y \trianglelefteq_{\mathsf{X}} 1\, (x =_{\mathsf{X}} y \to \neg \forall k' A_{bd}(y, n, k'))),$$

which is equivalent to

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \forall n\ \left(\exists k\ \neg A_{bd}(x, n, k) \to \forall y \trianglelefteq_{\mathsf{X}} 1 \big(\forall r\ \left(\|x - y\| \leq_{\mathbb{R}} \tfrac{1}{2^r}\right) \to \exists k'\ \neg A_{bd}(y, n, k)\big)\right).$$

By collection and classical logic, we get

$$\begin{aligned}\forall x \trianglelefteq_{\mathsf{X}} 1 \forall n, k \exists r', k'' \left(\neg A_{bd}(x,n,k) \rightarrow\right.\\ \left.\rightarrow \forall y \trianglelefteq_{\mathsf{X}} 1 \exists r \leq_0 r' \exists k' \leq_0 k'' \big(\|x-y\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} \rightarrow \neg A_{bd}(y,n,k')\big)\right),\end{aligned}$$

and, by bounded collection and monotone choice, there exist monotone f and l such that

$$\begin{aligned}\forall n, k \exists r'' \leq_0 fnk \exists k''' \leq_0 lnk \forall x \trianglelefteq_{\mathsf{X}} 1 \exists r' \leq_0 r'' \exists k'' \leq_0 k''' \left(\neg A_{bd}(x,n,k) \rightarrow\right.\\ \left.\rightarrow \forall y \trianglelefteq_{\mathsf{X}} 1 \exists r \leq_0 r' \exists k' \leq_0 k'' \left(\|x-y\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} \rightarrow \neg A_{bd}(y,n,k')\right)\right).\end{aligned}$$

By classical logic and tameAC, there exists f' (such that $f' \trianglelefteq \lambda n^0, k^0, p^0, q^0.lnk$) such that

$$\begin{aligned}\forall x \trianglelefteq_{\mathsf{X}} 1 \forall n, k \left(\neg A_{bd}(x,n,k) \rightarrow\right.\\ \left.\forall y \trianglelefteq_{\mathsf{X}} 1 \left(\|x-y\| \leq_{\mathbb{R}} \tfrac{1}{2^{fnk+1}} \rightarrow \neg A_{bd}(y,n,f'nkxy)\right)\right).\end{aligned} \tag{4.13}$$

By (4.12), for all n, r and $z \trianglelefteq_{\mathsf{X}} 1$, we have $\|grnz - z\| <_{\mathsf{X}} \frac{1}{2^r}$. Consequently, there exists k^0 such that $\frac{1}{2^k} <_{\mathbb{R}} \frac{1}{2^r} - \|grnz - z\|$:

$$\forall r, n \forall z \trianglelefteq_{\mathsf{X}} 1 \exists k \left(\tfrac{1}{2^k} \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} - \|grnz - z\|\right).$$

By monotone choice and bounded collection, there exists a monotone ϕ such that

$$\forall r, n \forall z \trianglelefteq_{\mathsf{X}} 1 \exists k \leq_0 \phi rn \left(\tfrac{1}{2^k} \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} - \|grnz - z\|\right)$$

and also

$$\forall r, n \forall z \trianglelefteq_{\mathsf{X}} 1 \left(\tfrac{1}{2^{\phi rn}} \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^r} - \|grnz - z\|\right). \tag{4.14}$$

The rest of the proof is an adaptation of the proof of the Baire category theorem of functional analysis. We define a sequence of nested balls (defined by their center x_n and radius r_n). Then, we prove that sequence $(x_n)_n$ converges for a point in the intersection of all balls.

Let us define $\psi n := \langle x_n, k_n, r_n \rangle$ by primitive recursion:

$$\begin{aligned}&\psi 0 := \langle g100, h100, \max\left(f0(h100)+1, 1, \phi 1\, 0\right)\rangle\\ &\psi(n+1) := \langle x_{n+1}, k_{n+1}, r_{n+1}\rangle,\end{aligned}$$

where

$$\begin{aligned}x_{n+1} &:= gr_n(n+1)x_n\\ k_{n+1} &:= hr_n(n+1)x_n\\ r_{n+1} &:= \max\left(f(n+1)k_{n+1}+1, n+2, \phi r_n(n+1)\right).\end{aligned}$$

Lemma 35. $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *proves that, given x_n, k_n, r_n defined by above, then*

$$\forall y \trianglelefteq_{\mathsf{X}} 1 \forall n^0 \left(\|y - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_n}} \rightarrow \forall i \leq_0 n\ \neg A_{bd}(y, i, f'ik_ix_iy)\right).$$

Proof

From (4.12), we have $\neg A_{bd}(x_n, n, k_n)$ for all n^0, and from (4.13), it follows

$$\forall n^0 \forall y \trianglelefteq_{\mathsf{X}} 1 \left(\|y - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{fnk_n+1}} \rightarrow \neg A_{bd}(y, n, f'nk_nx_ny) \right). \tag{4.15}$$

We claim that

$$\|y - x_{n+1}\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_{n+1}}} \rightarrow \|y - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_n}}.$$

Take $\|y - x_{n+1}\| \leq_{\mathbb{R}} \frac{1}{2^{r_{n+1}}}$. Then

$$\|y - x_n\| \leq_{\mathbb{R}} \|y - x_{n+1}\| + \|x_{n+1} - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_{n+1}}} + \|x_{n+1} - x_n\|.$$

By the definition of r_{n+1} and (4.14), we get

$$\frac{1}{2^{r_{n+1}}} \leq_{\mathbb{R}} \frac{1}{2^{\phi r_n(n+1)}} \leq_{\mathbb{R}} \frac{1}{2^{r_n}} - \|x_{n+1} - x_n\|.$$

Consequently, $\|y - x_n\| \leq_{\mathbb{R}} \frac{1}{2^{r_n}}$.

By induction, we get

$$\forall y^{\mathsf{X}} \forall n^0 \ \left(\|y - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_n}} \rightarrow \forall m \leq_0 n \ \|y - x_m\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_m}} \right).$$

Take $y \trianglelefteq_{\mathsf{X}} 1$ and n^0. By the latter and the definition of r_m, we conclude

$$\|y - x_n\| \leq_{\mathbb{R}} \tfrac{1}{2^{r_n}} \rightarrow \forall m \leq_0 n \ \left(\|y - x_m\| \leq_{\mathbb{R}} \tfrac{1}{2^{fmk_m+1}} \right).$$

Then, the conclusion follows by (4.15). □

Lemma 36. $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq}$ *proves that, given x_n and r_n defined by above, then $(x_n)_n$ is a convergent sequence. Moreover, if x is the limit of $(x_n)_n$, then $\forall n^0 \ \left(\|x - x_n\| \leq_{\mathbb{R}} \frac{1}{2^{r_n}} \right)$.*

Proof We claim that $(x_n)_n$ is Cauchy sequence with modulus of convergence $f = \lambda n.n$. Take $n, m \geq_0 k$. Without loss of generality, assume $m \leq_0 n$. Then

$$\|x_n - x_m\| \leq_{\mathbb{R}} \textstyle\sum_{i=m+1}^{n} \|x_i - x_{i-1}\| <_{\mathbb{R}} \sum_{i=m+1}^{n} \frac{1}{2^{r_{i-1}}},$$

by (4.12). Moreover, the definition of r_n implies

$$\|x_n - x_m\| <_{\mathbb{R}} \textstyle\sum_{i=m+1}^{n} \frac{1}{2^i} <_{\mathbb{R}} \frac{1}{2^m} \leq_{\mathbb{R}} \frac{1}{2^k}$$

as desired. Therefore, by proposition 20, the sequence converges for x (with $x = x_k$ for a certain k). Since $x_n \trianglelefteq_{\mathsf{X}} 1$ for any n^0, then $x \trianglelefteq_{\mathsf{X}} 1$. To end the proof, assume that exists n_0 such that $\|x - x_{n_0}\| >_{\mathbb{R}} \frac{1}{2^{r_{n_0}}}$. Then, there exists $a >_{\mathbb{R}} 0$ such that $\|x - x_{n_0}\| >_{\mathbb{R}} \frac{1}{2^{r_{n_0}}} + a$. It is easy to prove that $\|x - x_{n_0+1}\| >_{\mathbb{R}} \frac{1}{2^{r_{n_0+1}}} + a >_{\mathbb{R}} a$ and by induction we get $\forall n \geq_0 n_0 \ (\|x - x_n\| >_{\mathbb{R}} a)$, which contradicts the fact that $(x_n)_n$ converges to x. Therefore, $\forall n^0 \ \left(\|x - x_n\| \leq_{\mathbb{R}} \frac{1}{2^{r_n}} \right)$. □

By the two previous lemmas, we conclude

$$\exists x \trianglelefteq_{\mathsf{X}} 1 \forall n \forall i \leq_0 n \; \neg A_{bd}(x, i, f' i k_i x_i x).$$

Consequently, $\exists x \trianglelefteq_{\mathsf{X}} 1 \forall n^0 \exists k \; \neg A(x, n, k)$. We have showed

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \exists n^0 \forall k^0 \; A(x, n, k) \to \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r^0, n^0 \forall x \trianglelefteq_{\mathsf{X}} 1 \; \left(\|x - z\| \trianglelefteq_{\mathbb{R}} \tfrac{1}{2^{r+1}} \to \forall k^0 \; A(x, n, k)\right)$$

and this entails the thesis of the Baire-like theorem. $\square$

Corollary 6. *Let $A_{bd}(x^{\mathsf{X}}, n^0, y^{\mathsf{X}}, k^0)$ be a bounded formula of the language $\mathcal{L}^{\omega,\mathsf{X}}_{\trianglelefteq}$. Then* $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] + \mathsf{tameAC}$ *proves that the extensionality of $\forall k \; A_{bd}(x, n, y, k)$ with respect to x implies*

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \exists n^0 \exists y \trianglelefteq_{\mathsf{X}} n \forall k^0 \; A_{bd}(x, n, y, k) \to$$
$$\to \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r^0, n^0 \forall x \trianglelefteq_{\mathsf{X}} 1 \left(\|x - z\| <_{\mathbb{R}} \frac{1}{2^r} \to \exists y \trianglelefteq_{\mathsf{X}} n \forall k^0 \; A_{bd}(x, n, y, k)\right).$$

Proof Assume $\forall x \trianglelefteq_{\mathsf{X}} 1 \exists n \exists y \trianglelefteq_{\mathsf{X}} n \forall k \; A_{bd}(x, n, y, k)$. In particular,

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \exists n \forall k' \exists y \trianglelefteq_{\mathsf{X}} n \forall k \leq_0 k' \; A_{bd}(x, n, y, k).$$

We claim that the formula $\forall k' \exists y \trianglelefteq_{\mathsf{X}} n \forall k \leq_0 k' \; A_{bd}(x, n, y, k)$ is extensional with respect to x. Take $x =_{\mathsf{X}} x'$ and assume $\forall k' \exists y \trianglelefteq_{\mathsf{X}} n \forall k \leq_0 k' \; A_{bd}(x, n, y, k)$. By bounded collection, $\exists y \trianglelefteq_{\mathsf{X}} n \forall k \; A_{bd}(x, n, y, k)$, and, by the extensionality of $\forall k \; A_{bd}(x, n, y, k)$ with respect to x, we get

$$\exists y \trianglelefteq_{\mathsf{X}} n \forall k \; A_{bd}(x', n, y, k).$$

In particular, we have $\forall k' \exists y \trianglelefteq_{\mathsf{X}} n \forall k \leq_0 k' \; A_{bd}(x', n, y, k)$, as desired.

Applying the Baire-like theorem, there exists $z \trianglelefteq_{\mathsf{X}} 1$ and r^0 such that

$$\exists n \forall x \trianglelefteq_{\mathsf{X}} 1 \; \left(\|x - z\| <_{\mathbb{R}} \tfrac{1}{2^r} \to \forall k' \exists y \trianglelefteq_{\mathsf{X}} n \forall k \leq_0 k' \; A_{bd}(x, n, y, k)\right).$$

The conclusion follows from bounded collection. $\square$

Since the Banach-Steinhaus and the open mapping theorems are stated for linear operators, we define them and prove some properties.

Definition 20. *Let L be of type $\mathsf{X} \to \mathsf{X}$. L is called a* linear operator *if the following is verified:*

i) $L(x + y) =_{\mathsf{X}} Lx + Ly$ for all $x^{\mathsf{X}}, y^{\mathsf{X}}$;

ii) $L(\alpha x) =_{\mathsf{X}} \alpha(Lx)$, for all $\alpha^{\mathbb{R}}$ and x^{X}.

Proposition 21. *The theory $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq]$ proves that each linear operator is bounded, i.e., given $L^{\mathsf{X} \to \mathsf{X}}$, if L is a linear operator, then there exists $\alpha \in \mathbb{R}$ such that*

$$\forall x^{\mathsf{X}} \; (\|Lx\| \leq_{\mathbb{R}} \alpha \|x\|).$$

Proof Well, we have $\forall m^0 \forall x \trianglelefteq_{\mathsf{X}} m \exists n^0 \; (\|Lx\| \trianglelefteq_{\mathbb{R}} n)$. Collection entails

$$\forall m \exists n \forall x \trianglelefteq_{\mathsf{X}} m \; (\|Lx\| \trianglelefteq_{\mathbb{R}} n) .$$

In particular, there exists N such that $\forall x \trianglelefteq_{\mathsf{X}} 2 \; (\|L(x)\| \trianglelefteq_{\mathbb{R}} N)$. Take $x \neq_{\mathsf{X}} 0$. Then $\left\| \frac{1}{\|x\|} x \right\| <_{\mathbb{R}} 2 \to \left\| \frac{1}{\|x\|} x \right\| \trianglelefteq_{\mathbb{R}} 2$. By *iii*) of lemma 29, we get $\frac{1}{\|x\|} x \trianglelefteq_{\mathsf{X}} 2$. Then

$$\frac{1}{\|x\|} \|Lx\| =_{\mathbb{R}} \left\| L\left(\frac{1}{\|x\|} x\right) \right\| \leq_{\mathbb{R}} N,$$

since $\left\| L\big(\frac{1}{\|x\|} x\big) \right\| \trianglelefteq_{\mathbb{R}} N$. The latter entails $\|Lx\| \leq_{\mathbb{R}} N\|x\|$ for $x \neq_{\mathsf{X}} 0$.

It still remains to prove that $x =_{\mathsf{X}} 0 \to \|Lx\| =_{\mathbb{R}} 0$. Take $x =_{\mathsf{X}} 0$ and assume $\|Lx\| \neq_{\mathbb{R}} 0$. Then, there exists k^0 such that $\frac{1}{k} <_{\mathbb{R}} \|Lx\|$. Since $x =_{\mathsf{X}} 0$, we have $kNx =_{\mathsf{X}} 0$, with N the one used above. Again, by *iii*) of lemma 29, $\|kNx\| \trianglelefteq_{\mathbb{R}} 1 \leftrightarrow kNx \trianglelefteq_{\mathsf{X}} 1$. Then $kNx =_{\mathsf{X}} 0 \to kNx \trianglelefteq_{\mathsf{X}} 1$ and, as a consequence, it follows that $\|L(kNx)\| \leq_{\mathbb{R}} N$. This yields a contradiction: $\|Lx\| \leq_{\mathbb{R}} \frac{1}{k} <_{\mathbb{R}} \|Lx\|$. □

As a consequence, linear operators are extensional:

Corollary 7. $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq]$ *proves that if* $L^{\mathsf{X}\to\mathsf{X}}$ *is a linear operator, then* L *is extensional.*

Proof We want to prove that $x =_{\mathsf{X}} y \to Lx =_{\mathsf{X}} Ly$. By the linearity of L, there exists $\alpha \in \mathbb{R}$ such that $\forall x^{\mathsf{X}} (\|Lx\| \leq_{\mathbb{R}} \alpha\|x\|)$. Take $x^{\mathsf{X}}, y^{\mathsf{X}}$ such that $x =_{\mathsf{X}} y$. Then $x - y =_{\mathsf{X}} 0$, which implies that $\|x - y\| =_{\mathbb{R}} 0$ and consequently, $L(x - y) =_{\mathsf{X}} 0$.

Notice that the equality $L(x - y) =_{\mathsf{X}} Lx - Ly$ is not trivial, since extensionality has not been proved yet. Let us prove $L(x - y) =_{\mathsf{X}} L(x + (-y))$. From

$$(x-y)+(-(x+(-y))) =_{\mathsf{X}} (x-y)+(y+(-x)) =_{\mathsf{X}} x+((-y+y)+(-x)) =_{\mathsf{X}} x+(-x) =_{\mathsf{X}} 0,$$

we get, $L\big((x-y)+(-(x+(-y)))\big) =_{\mathsf{X}} 0$, which is equivalent to $L(x-y) =_{\mathsf{X}} L(x+(-y))$. Now, it is clear that $L(x - y) =_{\mathsf{X}} Lx - Ly$. □

In the following, the uniform boundness principle is presented as an instance of collection for universal formulas:

Theorem 42 (Banach-Steinhaus theorem). $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] + \mathsf{tameAC}$ *proves that for each family of linear operators* $(L_k)_{k^0}$*, with* L_k *of type* $\mathsf{X} \to \mathsf{X}$ *(for all k), then*

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \exists M \forall k \; (\|L_k x\| \leq_{\mathbb{R}} M) \to \exists M \forall x \trianglelefteq_{\mathsf{X}} 1 \forall k \; (\|L_k x\| \leq_{\mathbb{R}} M) .$$

Proof Assume that $L_k^{\mathsf{X}\to\mathsf{X}}$ is a linear operator for each k. Assume, as well,

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \exists M^0 \forall k^0 \; (\|L_k x\| \leq_{\mathbb{R}} M) .$$

For all k, $\|L_k x\| \leq_{\mathbb{R}} M$ is universal with bounded intensional matrix and is extensional with respect to x. Then, it is straightforward to check that $\forall k \ (\|L_k x\| \leq_{\mathbb{R}} M)$ is also extensional with respect to x. Therefore, by the Baire-like theorem

$$\exists M \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r \forall x \trianglelefteq_{\mathsf{X}} 1 \ \left(\|x - z <_{\mathbb{R}} \tfrac{1}{2^r} \to \forall k \ (\|L_k x\| \leq_{\mathbb{R}} M)\right),$$

meaning that there exists a ball with center in $z \trianglelefteq_{\mathsf{X}} 1$ and radius r^0 such that $\|L_k x\| \leq_{\mathbb{R}} M$ for x in the intersection of this ball with the unitary ball. Clearly, the unitary ball contains a smaller ball in which $\|L_k x\| \leq_{\mathbb{R}} M$ holds:

$$\exists M \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r \left(\tfrac{1}{2^r} + \|z\| <_{\mathbb{R}} 1 \wedge \forall x \trianglelefteq_{\mathsf{X}} 1 \ \left(\|x - z\| <_{\mathbb{R}} \tfrac{1}{2^r} \to \forall k \ (\|L_k x\| \leq_{\mathbb{R}} M)\right)\right).$$

Take M' as a natural such that $2^{r+1}(M + \|L_k z\|) \leq_{\mathbb{R}} M'$. We claim that

$$\forall x \trianglelefteq_{\mathsf{X}} 1 \forall k \ (\|L_k x\| \leq_{\mathbb{R}} M').$$

If $x =_{\mathsf{X}} 0$, it is trivial. Take an arbitrary $x \trianglelefteq_{\mathsf{X}} 1$ $(x \neq_{\mathsf{X}} 0)$ and define y as $\frac{1}{2^{r+1}} \frac{x}{\|x\|} + z$. Then

$$\|y\| =_{\mathbb{R}} \left\| \tfrac{1}{2^{r+1}} \tfrac{x}{\|x\|} + z \right\| \leq_{\mathbb{R}} \tfrac{1}{2^{r+1}} + \|z\| <_{\mathbb{R}} 1$$

and $\|y\| \trianglelefteq_{\mathbb{R}} 1$. By lemma 29, it follows $y \trianglelefteq_{\mathsf{X}} 1$. Moreover,

$$\|y - z\| =_{\mathbb{R}} \left\| \tfrac{1}{2^{r+1}} \tfrac{x}{\|x\|} \right\| =_{\mathbb{R}} \tfrac{1}{2^{r+1}} <_{\mathbb{R}} \tfrac{1}{2^r}.$$

Hence, $\|L_k y\| \leq_{\mathbb{R}} M$ for all k. Consequently, $\|L_k x\| \leq_{\mathbb{R}} M'$:

$$\|L_k x\| \leq_{\mathbb{R}} 2^{r+1}\|x\| \left(\|L_k y\| + \|L_k z\|\right) \leq_{\mathbb{R}} 2^{r+1}(M + \|L_k z\|).$$

$\square$

Theorem 43 (**Open-mapping theorem**). $\mathsf{PA}^{\omega,\mathsf{X}}_{\trianglelefteq} + \mathsf{P}^{\omega,\mathsf{X}}_{cl}[\trianglelefteq] + \mathsf{tameAC}$ *proves that for all linear operators* $L^{\mathsf{X}\to\mathsf{X}}$*, we have*

$$\forall y \trianglelefteq_{\mathsf{X}} 1 \exists x \ (Lx =_{\mathsf{X}} y) \to \exists M \forall y \trianglelefteq_{\mathsf{X}} 1 \exists x \trianglelefteq_{\mathsf{X}} M \ (Lx =_{\mathsf{X}} y).$$

When stated in this form, the open-mapping theorem is a form of collection for universal matrices.

Proof Take $L^{\mathsf{X}\to\mathsf{X}}$ and assume it is a linear operator and that $\forall y \trianglelefteq_{\mathsf{X}} 1 \exists x \ (Lx =_{\mathsf{X}} y)$. By $\mathsf{MAJ}^{\omega,\mathsf{X}}$, it follows $\forall y \trianglelefteq_{\mathsf{X}} 1 \exists M \exists x \trianglelefteq_{\mathsf{X}} M \ (Lx =_{\mathsf{X}} y)$, which is equivalent to

$$\forall y \trianglelefteq_{\mathsf{X}} 1 \exists M \exists x \trianglelefteq_{\mathsf{X}} M \forall k \ \left(\|Lx - y\| \leq_{\mathbb{R}} \tfrac{1}{2^k}\right).$$

The formula $\|Lx - y\| \leq_{\mathbb{R}} \frac{1}{2^k}$ is universal and is clearly extensional with respect to y. Then, it is straightforward to prove that $\forall k \ \left(\|Lx - y\| \leq_{\mathbb{R}} \frac{1}{2^k}\right)$ is also extensional with respect to y. By corollary 6, we obtain

$$\exists M \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r \forall y \trianglelefteq_{\mathsf{X}} 1 \ \left(\|y - z\| <_{\mathbb{R}} \tfrac{1}{2^r} \to \exists x \trianglelefteq_{\mathsf{X}} M \forall k \ \left(\|Lx = y\| \leq_{\mathbb{R}} \tfrac{1}{2^k}\right)\right).$$

Clearly, the latter holds in a smaller ball

$$\exists M \exists z \trianglelefteq_{\mathsf{X}} 1 \exists r \left(\tfrac{1}{2^r} + \|z\| <_{\mathbb{R}} 1 \wedge \forall y \trianglelefteq_{\mathsf{X}} 1 \ \left(\|y - z\| <_{\mathbb{R}} \tfrac{1}{2^r} \to \exists x \trianglelefteq_{\mathsf{X}} M \ (Lx =_{\mathsf{X}} y) \right) \right).$$

Since $z \trianglelefteq_{\mathsf{X}} 1$, there exists x_0 such that $Lx_0 =_{\mathsf{X}} z$. Take M' as a natural such that $2^{r+1}(M + \|x_0\|) <_{\mathbb{R}} M'$.

We claim that $\forall y \trianglelefteq_{\mathsf{X}} 1 \exists x' \trianglelefteq_{\mathsf{X}} M' \ (Lx' =_{\mathsf{X}} y)$. For $y =_{\mathsf{X}} 0$, it is trivial. Take $y \neq_{\mathsf{X}} 0$ such that $y \trianglelefteq_{\mathsf{X}} 1$ and define w as $\frac{1}{2^{r+1}} \frac{y}{\|y\|} + z$. Then

$$\|w\| =_{\mathbb{R}} \left\| \tfrac{1}{2^{r+1}} \tfrac{y}{\|y\|} + z \right\| \leq_{\mathbb{R}} \tfrac{1}{2^{r+1}} + \|z\| <_{\mathbb{R}} \tfrac{1}{2^r} + \|z\| <_{\mathbb{R}} 1$$

and $\|w\| \trianglelefteq_{\mathbb{R}} 1$. Hence, by lemma 29, $w \trianglelefteq_{\mathsf{X}} 1$. Also

$$\|w - z\| =_{\mathbb{R}} \left\| \tfrac{1}{2^{r+1}} \tfrac{y}{\|y\|} \right\| =_{\mathbb{R}} \tfrac{1}{2^{r+1}} <_{\mathbb{R}} \tfrac{1}{2^r}.$$

Then $\exists x \trianglelefteq_{\mathsf{X}} M \ (Lx =_{\mathsf{X}} w)$. Define x' as $2^{r+1}\|y\|(x - x_0)$. We have

$$\|x'\| =_{\mathbb{R}} 2^{r+1}\|y\|\|x - x_0\| \leq_{\mathbb{R}} 2^{r+1}(\|x\| + \|x_0\|) \leq_{\mathbb{R}} 2^{r+1}(M + \|x_0\|) <_{\mathbb{R}} M'.$$

Then $\|x'\| \trianglelefteq_{\mathbb{R}} M'$. By lemma 29, $x' \trianglelefteq_{\mathsf{X}} M'$. Moreover,

$$Lx' =_{\mathsf{X}} L\left((2^{r+1}\|y\|(x - x_0) \right) =_{\mathsf{X}} 2^{r+1}\|y\|(Lx - Lx_0) =_{\mathsf{X}} 2^{r+1}\|y\|(w - z) =_{\mathsf{X}} y.$$

$\square$

In both these logical versions of the Banach-Steinhaus and the open mapping theorems, the "local collection" for universal matrices in the Baire-like theorem is lifted to global collection. This is a consequence of the linearity of the operators, as occurs in the proofs in functional analysis. Observe that while the Banach-Steinhaus follows the usual textbook proof, the open mapping theorem does not, but relies instead in a collection principle.

5

Epilogue

We have extended the bounded functional interpretation to second-order arithmetic. All the results involved in this extension were verified in the theory $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$, where $\Delta_{\mathcal{M}^\omega}$ is the set of all universal sentences (with bounded intensional matrices) whose flattenings are true in $\mathcal{M}^\omega$. Of course, this treatment is not optimal, since we do not need all the sentences of $\Delta_{\mathcal{M}^\omega}$. We will leave a better treatment for future work. In fact, it is possible that the optimal treatment can be achieved by looking at the double negation shift from a new angle: we believe that it can be seen as a particular case of bar induction.

The extension of the bounded functional interpretation lead to some interesting results concerning some well-known theorems of functional analysis. It may be possible that a wider scope of theorems can be analysed using these tools. Moreover, some studies should be carried out with the goal of extending Spector's generalization to new base types.

These are interesting developments to consider for future work.

Bibliography

[AF98] J. Avigad and S. Feferman. Gödel's functional ("Dialectica") interpretation. In S. R. Buss, editor, *Handbook of Proof Theory*, volume 137 of *Studies in Logic and the Foundations of Mathematics*, pages 337–405. North Holland, Amsterdam, 1998.

[Bez85] M. Bezem. Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. *The Journal of Symbolic Logic*, 50:652–660, 1985.

[Bro27] L. E. J. Brouwer. Über definitionsbereiche von funktionen. *Math. Ann.*, 97(2):60–76, 1927.

[EF] P. Engrácia and F. Ferreira. The bounded functional interpretation of the double negation shift. Submitted.

[Fer06] F. Ferreira. Proof interpretations. In R. Kahle and I. Oitavem, editors, *Days in Logic '06 (Two tutorials)*, volume 38 of *Textos de Matemática*, pages 1–42. Departamento de Matemática da Universidade de Coimbra, 2006.

[Fer08] F. Ferreira. A most artistic package of a jumble of ideas. *dialectica*, 62:205–222, 2008. Special Issue: Gödel's *dialectica* Interpretation. Guest editor: Thomas Strahm.

[Fer09] F. Ferreira. Injecting uniformities into Peano arithmetic. *Annals of Pure and Applied Logic*, 157:122–129, 2009. Special Issue: Kurt Gödel Centenary Research Prize Fellowships. Editors: Sergei Artemov, Matthias Baaz and Harvey Friedman.

[FO05] F. Ferreira and P. Oliva. Bounded functional interpretation. *Annals of Pure and Applied Logic*, 135:73–112, 2005.

[Gas07] J. Gaspar. Em torno das interpretações funcionais da aritmética. Master's thesis, Universidade de Lisboa, Portugal, 2007. In Portuguese.

[GK08] P. Gerhardy and U. Kohlenbach. General logical metatheorems for functional analysis. *Transactions of the American Mathematical Society*, 360:2615–2660, 2008.

[Göd33] K. Gödel. Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines Mathematischen Kolloquiums*, 4:34–38, 1933.

[Göd58] K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *dialectica*, 12:280–287, 1958. Reprinted with an English translation in [Göd90].

[Göd90] K. Gödel. *Collected Works, Vol. II.* S. Feferman *et al.*, eds. Oxford University Press, Oxford, 1990.

[How68] W. A. Howard. Functional interpretation of bar induction by bar recursion. *Compositio Mathematica*, 20:107–124, 1968.

[How73] W. A. Howard. Hereditarily majorizable functionals of finite type. In A. S. Troelstra, editor, *Metamathematical investigation of intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*, pages 454–461. Springer, Berlin, 1973.

[KO03] U. Kohlenbach and P. Oliva. Proof mining: a systematic way of analysing proofs in mathematics. *Proceedings of the Steklov Institute of Mathematics*, 242:136–164, 2003.

[Koh93] U. Kohlenbach. Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation. *Annals of Pure and Applied Logic*, 64:27–94, 1993.

[Koh95] U. Kohlenbach. *Real growth in standard parts of analysis.* Habilitationsschrift, pp. xv+166, Frankfurt, 1995.

[Koh96] U. Kohlenbach. Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. *Archive for Mathematical Logic*, 36:31–71, 1996.

[Koh05] U. Kohlenbach. Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357:89–128, 2005.

[Koh07] U. Kohlenbach. Proof interpretations and the computational content of proofs in mathematics. *Bulletin of the EATCS*, 93:143–173, 2007.

[Koh08a] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics.* Springer Monographs in Mathematics. Springer, Berlin, 2008.

[Koh08b] U. Kohlenbach. Effective bounds from proofs in abstract functional analysis. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 223–258. Springer Verlag, 2008.

[Kur51] S. Kuroda. Intuitionistische Untersuchungen der formalistischen Logik. *Nagoya Mathematical Journal*, 3:35–47, 1951.

[Oli06] P. Oliva. Understanding and using Spector's bar recursive interpretation of classical analysis. In B. Löwe A. Beckmann, U. Berger and J. Tucker, editors, *Logical Approaches to Computational Barriers*, volume 3988 of *Lecture Notes in Computer Science*, pages 423–434. Springer, New York, 2006.

[Sho67] J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley Publishing Company, 1967. Republished in 2001 by AK Peters.

[Spe62] C. Spector. Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics. In F. D. E. Dekker, editor, *Recursive Function Theory*, volume 5, pages 1–27. American Mathematical Society, Providence, Rhode Island, 1962.

[Tro73] A. S. Troelstra (ed.). *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*. Springer, Berlin, 1973.

[Wei00] K. Weihrauch. *Computable Analysis*. Springer, Berlin, 2000.