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Asymptotic Stability for Population Models and Neural Networks with Delays

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Tese Orientada pela
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Ao meu pai, à memória da minha mãe e à Salete.

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Abstract

In this thesis, the global asymptotic stability of solutions of several functional differential equations is addressed, with particular emphasis on the study of global stability of equilibrium points of population dynamics and neural network models.

First, for scalar retarded functional differential equations, we use weaker versions of the usual Yorke and 3/2-type conditions, to prove the global attractivity of the trivial solution. Afterwards, we establish new sufficient conditions for the global attractivity of the positive equilibrium of a general scalar delayed population model, and illustrate the situation applying these results to two food-limited population models with delays.

Second, for n-dimensional Lotka-Volterra systems with distributed delays, the local and global stability of a positive equilibrium, independently of the choice of the delay functions, is addressed assuming that instantaneous negative feedbacks are present.

Finally, we obtain the existence and global asymptotic stability of an equilibrium point of a general neural network model by imposing a condition of dominance of the nondelayed terms. The generality of the model allows us to study, as particular situations, the neural network models of Hopfield, Cohn-Grossberg, bidirectional associative memory, and static with S-type distributed delays.

In our proofs, we do not use Lyapunov functionals and our method applies to general delayed differential equations.

Keywords: Global asymptotic stability; local asymptotic stability; 3/2-type condition; Yorke condition; delayed population model; delayed neural network model.

Sumário

Nesta tese estuda-se a estabilidade global assimptótica de soluções de equações diferenciais funcionais que, pela generalidade com que são apresentadas, possuem uma vasta aplicabilidade em modelos de dinâmica de populações e em modelos de redes neuronais.

Numa primeira fase, para equações diferenciais funcionais escalares retardadas, assumem-se novas versões das condições de Yorke e tipo $3/2$ para provar a atractividade global da solução nula. Seguidamente, aplicam-se os resultados obtidos a um modelo geral de dinâmica de populações escalar com atrasos, obtendo-se condições suficientes para a atractividade global de um ponto de equilíbrio positivo, e ilustra-se a situação com o estudo de dois modelos conhecidos.

Numa segunda fase, para sistemas n-dimensionais de tipo Lotka-Volterra com atrasos distribuídos, estuda-se a estabilidade local e global de um ponto de equilíbrio positivo (caso exista) assumindo condições de dominância dos termos com atrasos pelos termos sem atrasos.

Por último, novamente assumindo condições de donimância, obtém-se a existência e estabilidade global assimptótica de um ponto de equilíbrio para um modelo geral de redes neuronais com atrasos. A generalidade do modelo estudado permite obter, como situações particulares, critérios de estabilidade global para modelos de redes neuronais de Hopfield, de Cohn-Grossberg, modelos de memória associativa bidireccional e modelos estáticos com atrasos distribuídos tipo-S.

De referir que as demonstrações apresentadas não envolvem o uso de funcionais de Lyapunov, o que permite obter critérios de estabilidade para equações diferenciais funcionais bastante gerais.

Palavras-chave: Estabilidade global assimptótica; estabilidade local assimptótica; condição tipo $3/2$; condição de Yorke; modelo populacional com atrasos; modelo de redes neuronais com atrasos;

Resumo

Nas diversas ciências, quer humanas quer exactas, cada vez mais as equações diferenciais constituem uma ferramenta chave no processo de modelação de realidades a estudar. A construção de um modelo é uma tentativa de descrever uma realidade que, na maioria das situações, não é mais do que uma aproximação para essa mesma realidade. Consequentemente, a procura de modelos cada vez mais realistas é uma constante preocupação em todas as ciências.

Em muitas aplicações, o processo de modelação é feito assumindo que o futuro estado do sistema em consideração é determinado apenas pelo presente; contudo, é reconhecido que em certas situações o desenvolvimento do sistema também depende do seu estado passado, sendo portanto pertinente a sua inclusão no modelo. Por exemplo, a construção de modelos para o crescimento populacional de espécies biológicas é necessariamente mais realista se levar em linha de conta o seu período de maturação. Equações diferenciais ordinárias e a maioria das equações diferenciais parciais não incorporam a sua dependência com o estado passado, enquanto que as equações diferenciais funcionais retardadas incorporam a seu estado passado, isto é, incluem atrasos.

O objectivo desta tese é o estudo da estabilidade global assimptótica de soluções de equações diferenciais funcionais que, pela generalidade com que são apresentadas, têm uma vasta aplicabilidade em modelos de dinâmica de populações e em modelos de redes neuronais. A tese está dividida em quatro capítulos: no Capítulo 1 apresentam-se alguns resultados básicos sobre estabilidade em equações diferenciais funcionais e efectua-se uma breve descrição do estado da arte; no Capítulo 2 o estudo centra-se na estabilidade global de soluções de equações diferenciais funcionais escalares; no Capítulo 3 estuda-se a estabilidade local e global assimptótica do equilíbrio positivo (caso exista) de um sistema n-dimensional com atrasos do tipo Lotka-Volterra; e no Capítulo 4 obtêm-se condições suficientes para a existência, unicidade e estabilidade global assimptótica de um equilíbrio para diversos modelos de redes neuronais com atrasos, quer distribuídos quer discretos.

Seguidamente apresentam-se, separadamente, cada um dos capítulos que cons-

tituem esta tese, salientando os principais pontos originais da investigação.

No primeiro capítulo, sendo um capítulo de preparação para a apresentação do trabalho de investigação realizado, apresentam-se algumas definições e alguns resultados básicos sobre estabilidade global de equações diferenciais funcionais. Faz-se ainda uma apresentação dos modelos a estudar, bem como das respectivas condições usadas no estudo da estabilidade presentes na literatura mais recente.

No Capítulo 2, obtêm-se condições suficientes para a estabilidade global atractiva da solução nula de uma equação diferencial funcional escalar na forma geral

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0, \tag{1}$$

onde $f : [0, +\infty) \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ é uma função contínua, com $C([-\tau, 0]; \mathbb{R})$ denotando o espaço das funções reais contínuas definidas em $[-\tau, 0]$, $\tau > 0$, com a norma do supremo e $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Para tal, assumem-se novas versões das condições de Yorke e tipo 3/2. As condições de tipo 3/2 impõem limites no tamanho do atraso $\tau > 0$, por forma a que o comportamento assimptótico das soluções de uma equação com atraso se assemelhe ao comportamento das soluções de equações diferenciais ordinárias.

Em [66], Yorke provou a estabilidade assimptótica da solução nula da equação escalar (1) introduzindo a conhecida condição de Yorke

$$-a\mathcal{M}(\varphi) \leq f(t, \varphi) \leq a\mathcal{M}(-\varphi), \qquad t \geq 0, \varphi \in C([-\tau, 0]; \mathbb{R}), \tag{2}$$

onde $a > 0$ e $\mathcal{M}(\varphi) := \max\{0, \sup_{\theta \in [-\tau, 0]} \varphi(\theta)\}$ é o funcional de Yorke, e assumindo a condição $a\tau < 3/2$. Posteriormente, em [64], a constante a foi substituída, na condição (2), por uma função contínua $\lambda(t) \geq 0$, sendo a condição de tipo 3/2 dada por $\sup_{t \geq \tau} \int_{t-\tau}^{t} \lambda(s)ds < 3/2$. Em [34] introduziu-se uma função racional $r(x) = \frac{-x}{1+bx}$, $b \geq 0$, surgindo a condição

$$ar(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq ar(-\mathcal{M}(-\varphi)), \qquad t \geq 0, \varphi \in C([-\tau, 0]; \mathbb{R}).$$

Dando seguimento ao trabalho desenvolvido em [14], nesta tese obtém-se a atractividade global da solução nula de (1) assumindo uma nova versão da condição de Yorke com a introdução de duas funções não negativas seccionalmente contínuas $\lambda_1(t), \lambda_2(t) \geq 0$ no lugar de a, isto é,

$$\lambda_1(t)r(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t)r(-\mathcal{M}(-\varphi)), \qquad t \geq 0, \varphi \in C([-\tau, 0]; \mathbb{R}),$$

e uma nova condição de tipo 3/2 dada pela desigualdade

$$\Gamma(\alpha_1, \alpha_2) \leq 1,$$

onde $\alpha_i := \sup_{t\geq\tau}\int_{t-\tau}^{t}\lambda_i(s)ds$, $i = 1, 2$, e

$$\Gamma(\alpha_1,\alpha_2) = \begin{cases} (\alpha_1 - 1/2)\alpha_2^2/2 & \text{if } \alpha_1 > 5/2 \\ \\ (\alpha_1 - 1/2)(\alpha_2 - 1/2), & \text{if } \alpha_1,\alpha_2 \leq 5/2 \\ \\ (\alpha_2 - 1/2)\alpha_1^2/2, & \text{if } \alpha_2 > 5/2 \end{cases} .$$

De notar que se $\alpha := \alpha_1 = \alpha_2$, então $\Gamma(\alpha_1, \alpha_2) \leq 1$ é equivalente a $\alpha \leq 3/2$.

Em biologia, diversos modelos escalares de dinâmica de populações têm a forma $\dot{y}(t) = y(t)g(t, y_t)$. Para estes modelos, do resultado obtido deduz-se um novo critério para a atractividade global de um ponto de equilíbrio positivo, no conjunto de todas as soluções positivas uma vez que só essas têm significado biológico. A finalizar o Capítulo 2, ilustra-se a situação com o estudo de dois modelos populacionais.

No Capítulo 3, estuda-se a estabilidade local e global atractiva de um ponto de equilíbrio positivo $x^* \in \mathbb{R}^n$ (caso exista) do modelo n-dimensional de tipo Lotka-Volterra com atrasos distribuídos

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij}\int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], i = 1,\dots,n, \qquad (3)$$

onde $b_i > 0$, $l_{ij} \in \mathbb{R}$, $\tau > 0$, $r_i(t)$ são funções contínuas positivas e $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ são funções de variação limitada normalizadas, i.e. $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1,\dots,n$.

Para o estudo destes sistemas não escalares, em vez de uma limitação no tamanho dos atrasos assume-se antes uma hipótese de dominância dos termos com atrasos pelos termos sem atrasos, isto é, assume-se que os sistemas possuem termos sem atrasos, $b_i x_i(t)$, que, pelo seu "peso", anulam o efeito dos termos com atrasos.

Seguindo o trabalho de investigação [12] desenvolvido para a situação escalar, considera-se o sistema linearizado (depois de um escalamento) em torno do equilíbrio x^* de (3) para a situação autónoma, isto é $r_i(t) \equiv 1$,

$$\dot{y}_i(t) = -\left[b_i y_i(t) + \sum_{j=1}^{n} l_{ij}\int_{-\tau}^{0} y_j(t+\theta)d\eta_{ij}(\theta)\right], \quad i = 1,\dots,n, \qquad (4)$$

cuja estabilidade determina a estabilidade local do equilíbrio x^* de (3). Assim, estudando as raízes da equação característica de (4), obtém-se uma condição necessária e suficiente para a estabilidade de (4), independentemente da escolha de $\tau > 0$ e das funções η_{ij}, estendendo simultaneamente o resultado obtido em [12] para a situação não escalar e o resultado em [24] para a situação n-dimensional com atrasos distribuídos. Mais concretamente, prova-se que (4) é exponencialmente assimptoticamente estável para qualquer $\tau > 0$ e qualquer conjunto de funções $\eta = (\eta_{ij})$ de variação limitada em $[-\tau, 0]$, com $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \dots, n$, tais que $\det M_\eta \neq 0$, se e só se $\hat{N}$ é uma M-matriz, onde $\hat{N} := diag(b_1, \dots, b_n) - [|l_{ij}|]$ e $M_\eta := diag(b_1, \dots, b_n) + [a_{ij}]$ com $a_{ij} = l_{ij}(\eta_{ij}(0) - \eta_{ij}(-\tau))$, $i, j = 1, \dots, n$. Seguidamente, assumindo condições um pouco mais restrictivas do que as obtidas para a situação linear, obtém-se a atractividade do equilíbrio x^* do sistema Lotka-Volterra (3).

No Capítulo 4 começa-se por estudar a estabilidade global assimptótica de um ponto de equilíbrio de um sistema n-dimensional geral de equações diferenciais retardadas não autónomo, $\dot{x}_i(t) = r_i(t)f_i(x_t)$, impondo novamente condições de dominância dos termos com atrasos pelos termos sem atrasos. Seguidamente, surgindo como uma generalização de diversos modelos de redes neuronais com atrasos, apresenta-se o modelo

$$\dot{x}_i(t) = -r_i(t)k_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \qquad t \geq 0, i = 1, \dots, n, \tag{5}$$

onde $r_i : [0, +\infty) \to (0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ e $f_i : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}$ são funções contínuas tais que, para cada $i = 1, \dots, n$, $r_i(t)$ é uniformemente limitada e $\int_0^{+\infty} r_i(t)dt = +\infty$, f_i é uma função de Lipschitz com constante l_i e existe $\beta_i > l_i$ tal que $(b_i(u) - b_i(v))/(u - v) \geq \beta_i$, $u, v \in \mathbb{R}$, $u \neq v$. Prova-se que, se $\beta_i > l_i$ para todo $i = 1, \dots, n$, então existe um único ponto de equilíbrio x^* de (5) que é globalmente assimptoticamente estável.

Dada a generalidade do modelo (5), este engloba, como subclasses, conhecidos modelos de redes neuronais de Hopfield, de Cohen-Grossberg, modelos de memória associativa bidireccional e modelos estáticos com atrasos distribuídos do tipo-S presentes na literatura recente, o que permite melhorar critérios para a existência e estabilidade global assimptótica de um equilíbrio para diversos modelos.

Por último, é importante referir que, contrariamente ao que é usual na literatura, a demonstração dos resultados de estabilidade global aqui apresentados não envolve o uso de funcionais de Lyapunov, o que permite obter resultados para sistemas mais gerais e consequentemente de maior aplicabilidade.

Contents

Introduction

Differential equations are essential modeling tools in many sciences. Although a model is a representation of a real process, both natural and manmade, sometimes the best one can get is a model of an approximation of that process. So, to look for better realistic models is always a challenge in all sciences.

In many applications, the modeling is done assuming that the future state of the system under consideration is determined solely by the present. But, in many sciences, such as biology, chemistry, physics, engineering, economics, it is known that many processes involve time delays, consequently it is often more realistic to include in the model some of the past history of the system. For example, animals must take time to digest their food before further activities and responses take place, hence it is not realistic to ignore times delay. The same happens when we formulate models for the growth of population species, since it is necessary to take into account their maturation period.

When a model does not incorporate a dependence on its past history, it generally consists of an *ordinary* or *partial differential equation.* Models incorporating past history, that is, that include delays, consist of *delay differential equations*, or *functional differential equations* (FDE's).

In the last decades, FDE's have attracted the attention of an increasing number of scientists due to their potential application as models in population dynamics ecology, epidemiology, disease evolution, neural networks, etc.. In this thesis, the research on global asymptotic stability of FDE's is mainly motivated by the extensive use of FDE's in dynamic population models, such as delayed logistic equations and Lotka-Volterra systems, and neural network models such as the Hopfield, Cohn-Grossberg, bidirectional associative memory, and static with S-type distributed delays models.

The objective of this thesis is to study the asymptotic behavior and stability of solutions of FDE's. The main goal is to obtain new criteria for stability of equilibrium points of FDE's, and apply them to several dynamic population and neural network models. The thesis is divided into four chapters: Chapter 1 contains basic stability results on FDE's and the state of the art; Chapter 2 focuses on the global stability of solutions of scalar FDE's; Chapter 3 is dedicated to the local and global stability analyses of multi-species Lotka-Volterra systems; and in Chapter 4 a new method is proposed to study the global stability of the steady state of several neural network models.

In 1837, Verhulst formulated the *growth logistic law* given by the scalar ordinary differential equation (ODE)

$$\dot{y}(t) = ay(t)\left(1 - \frac{1}{k}y(t)\right), \quad t \geq 0, \quad \dot{y} = \frac{dy}{dt}, \tag{6}$$

where $y(t)$ denotes the size of a population at time t and the constants $a > 0$ and $k > 0$ denote the growth rate and the carrying capacity of the ecosystem, respectively. In 1948, Hutchinson considered the delayed logistic equation

$$\dot{y}(t) = ay(t)\left(1 - \frac{1}{k}y(t-\tau)\right), \quad t \geq 0, \tag{7}$$

as a single species growth model with time delay, where $\tau \geq 0$ represents the maturation period of the species.

Nowadays, most of the scalar models in population dynamics have the form

$$\dot{y}(t) = y(t)g(t, y_t), \quad t \geq 0, \tag{8}$$

where $g : [0, +\infty) \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is the growth function, with $C([-\tau, 0]; \mathbb{R})$ the space of continuous real functions defined on $[-\tau, 0]$ with supremum norm, $\tau > 0$, and $y_t(\theta) := y(t + \theta)$, $\theta \in [-\tau, 0]$. See [19] and [28] to find a large number of scalar models of the form (8).

After a first chapter, where we introduce some notations and definitions on FDE's and we give some preliminary stability results, in Chapter 2 we study the global attractivity of the zero equilibrium of a general scalar FDE of the form, $\dot{x}(t) = f(t, x_t)$, for which we assume some original refinements of the Yorke and 3/2-type conditions present in the literature. Particular emphasis is given to the global stability of positive equilibria of differential equations with the form (8). At the end of Chapter 2 we give several new criteria for the global attractivity of two food-limited population models with delays.

The American biophysicist Alfred J. Lotka (1880-1949) and the Italian mathematician Vito Volterra (1860-1940) proposed, separately, an ecological model for two species, a predator (y) and a prey (x),

$$\begin{array}{l} \dot{x}(t) = x(t)(a - \alpha y(t)) \\ \\ \dot{y}(t) = y(t)(-b + \beta x(t)) \end{array}, \tag{9}$$

which describes their interaction. Here a, α, b, β are nonnegative constants, and $y(t)$ and $x(t)$ denote, respectively, the size of the predator population and the prey population at the time t. Naturally, by using time-delays this *Lotka-Volterra* model becomes more realistic. In 1928, Volterra investigated the following model with delays

$$\begin{aligned} \dot{x}(t) &= x(t)\left(a - cx(t) - \int_{-\tau}^{0} F_1(\theta)y(t+\theta)d\theta\right) \\ \dot{y}(t) &= y(t)\left(-b + dx(t) + \int_{-\tau}^{0} F_2(\theta)x(t+\theta)d\theta\right) \end{aligned}, \tag{10}$$

where all constants and functions are nonnegative. In biological terms, only positive solutions of Lotka-Volterra models are meaningful and it is particularly important to study the stability and attractivity of a positive equilibrium, if it exists. Since Volterra's work, many Lotka-Volterra type models with delays have arisen and there is an extensive literature dealing with theirs local and global stability - see e.g. the monographs of Gopalsamy [19], Kuang [28] and Smith [47].

In Chapter 3, we consider a n-dimensional Lotka-Volterra system with distributed delays and suppose that there is a positive equilibrium. The local asymptotic stability of the equilibrium point of the Lotka-Volterra system is given by the stability of linearized system. Assuming that instantaneous negative feedbacks are present, we obtain necessary and sufficient conditions, independently of the delays, for the asymptotic stability of the linearized system. Afterwards, the global asymptotic stability of the equilibrium of the Lotka-Volterra system with distributed delays is obtained assuming conditions of diagonal dominance of the instantaneous negative feedbacks over the competition terms. We emphasize that, in the literature, the usual approach to study the global stability of equilibria of FDE's relies on the use of Lyapunov functionals or Razumikhin methods. In general, constructing a Lyapunov functional for a concrete n-dimensional FDE is not an easy task. Frequently, a new Lyapunov functional for each model under consideration is required. Similarly as in [12] and [13] for the scalar situation, our techniques do not involve Lyapunov functionals. Our method applies to general Lotka-Volterra system, or even to broader frameworks, such as to general neural network models (NNM's).

In the last chapter, Chapter 4, we use the same techniques to study the global asymptotic stability of an equilibrium point of a general n-dimensional NNM with distributed delays.

Neural network models possess good potential applications in areas such as pattern recognition, signal and image processing, optimization (see [2], [7], [57], [63], and references therein). In optimization applications, it is required that the designed neural network converges to a unique and globally asymptotically stable equilibrium. Thus, it is important to establish sufficient conditions for systems to possess these dynamics.

In 1983, Cohen and Grossberg [9] proposed and studied the artificial neural network described by a system of ordinary differential equations

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \right], \quad i = 1, \ldots, n \tag{11}$$

and, in 1984, Hopfield [25] studied the particular situation of (11) with $k_i \equiv 1$,

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)), \quad i = 1, \ldots, n. \tag{12}$$

The finite switching speed of the amplifiers, communication time, and process of moving images led to the use of time-delays in models (11) and (12). Since then, several sufficient conditions have been obtained to ensure existence and global asymptotic stability of an equilibrium point of different generalizations of models (11) and (12) with delays.

Other NNM's have been studied, such as the static neural network model [42],

$$\dot{x}_i(t) = -x_i(t) + g_i \left(\sum_{j=1}^{n} a_{ij} x_j(t) + I_i \right), \quad i = 1, \ldots, n, \tag{13}$$

also with delays [58], and the bidirectional associative memory neural network [27],

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + I_i \\ \\ \dot{y}_i(t) = -y_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + J_i \end{cases}, \quad i = 1, \ldots, n \tag{14}$$

as well as some other generalizations with delays (see e.g. [1], [4], [60], [61]).

In Chapter 4, we first obtain the global asymptotic stability of the zero solution of a general n-dimensional delayed differential equation, $\dot{x}_i(t) = r_i(t) f_i(x_t)$, $i = 1, \cdots, n$, by imposing a condition of dominance of the nondelayed terms which cancels the delayed effect. Afterwards, using some properties of M-matrices, for a general NNM with delays we obtain a sufficient conditions for the existence of a unique equilibrium point, and for its global asymptotic stability. The results are applied to several well-Known NNM's such as all the above models (11), (12), (13), and (14).

The original results in this thesis, covered in Chapters 2 to 4, are taken from the articles [15], [16] and [39].

Chapter 1

Preliminary Results

In this chapter, we introduce functional differential equations (FDE's) and present an overview of the basic results on existence, uniqueness, and continuation of solutions for such equations. We also present the main results on stability of linear autonomous functional differential equations.

Afterwards, we introduce a class of FDE's which will be studied in Chapter 2, and give some important stability results found in recent literature.

Finally, we give some properties of M-matrices, which will be used for the study of stability of n-dimensional FDE's in Chapters 3 and 4.

1.1 Basic Results on Functional Differential Equations

The results in this section are taken from Chapter 2 of [23], where detailed proofs are given.

For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the n-dimensional real space with a norm $|\cdot|$, and for $a, b \in \mathbb{R}^n$, $a < b$, $C([a,b];\mathbb{R}^n)$ is the Banach space of continuous functions from $[a,b]$ to $\mathbb{R}^n$ with the norm of $\varphi \in C([a,b];\mathbb{R}^n)$ defined by $\|\varphi\| = \sup_{\theta\in[a,b]} |\varphi(\theta)|$. For $\tau > 0$ fixed, we denote $C_n := C([-\tau,0];\mathbb{R}^n)$. For $t_0 \in \mathbb{R}$, $\alpha \geq 0$, $x \in C([t_0-\tau, t_0+\alpha];\mathbb{R}^n)$ and $t \in [t_0, t_0+\alpha]$, we define $x_t \in C_n$ as

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-\tau, 0].$$

If D is a subset of $\mathbb{R} \times C_n$ and $f : D \to \mathbb{R}^n$ is a given function, then we define a *delay functional differential equation* as the relation

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0, \tag{1.1}$$

where "$\cdot$" represents the right-hand derivative.

Definition 1.1. *Let $t_0 \in \mathbb{R}$ and $\alpha > 0$.*

A function $x \in C([t_0 - \tau, t_0 + \alpha); \mathbb{R}^n)$ is said to be solution of (1.1) on $[t_0 - \tau, t_0 + \alpha)$ *if $(t, x_t) \in D$ and $x(t)$ satisfies (1.1) for all $t \in [t_0, t_0 + \alpha)$.*

For given $t_0 \in \mathbb{R}$ and $\varphi \in C_n$, we say that $x(t)$ is a *solution of* (1.1) *with initial value φ at t_0*, or simply a solution through (t_0, φ), if there is an $\alpha > 0$ such that $x(t)$ is a solution of (1.1) on $[t_0 - \tau, t_0 + \alpha)$ and $x_{t_0} = \varphi$. If the solution $x(t)$ of (1.1) through (t_0, φ) is unique, we write $x(t) = x(t, t_0, \varphi)$ and $x_t = x_t(t_0, \varphi)$. However, we often suppose that (t_0, φ) is fixed and we simply write $x(t)$ for $x(t, t_0, \varphi)$.

A point $x_0 \in \mathbb{R}^n$ is called an *equilibrium point*, or a *steady state solution*, of (1.1) if $f(t, \overline{x}_0) = 0$ for all $t \geq 0$, where $\overline{x}_0$ denotes the constant function in C_n, $\overline{x}_0(\theta) = x_0$, $\theta \in [-\tau, 0]$.

If $x(t)$ is a solution of equation (1.1) on an interval $[t_0 - \tau, \alpha)$, $\alpha > t_0$, we say that $\hat{x}(t)$ is a *continuation of* $x(t)$ if there is a $\beta > \alpha$ such that $\hat{x}(t)$ is defined on $[t_0 - \tau, \beta)$, coincides with $x(t)$ on $[t_0 - \tau, \alpha)$, and $\hat{x}(t)$ is a solution of equation (1.1) on $[t_0 - \tau, \beta)$. A solution $x(t)$ is *noncontinuable* if no such continuation exists.

Now, we give results on existence, uniqueness, and continuation of solutions of (1.1).

Theorem 1.1. *[23] Suppose D is an open subset in $\mathbb{R} \times C_n$, $f \in C(D; \mathbb{R}^n)$, and $(t_0, \varphi) \in D$.*

Then there is a solution $x(t)$ of (1.1) through (t_0, φ), defined on $[t_0 - \tau, t_0 + \alpha)$ for some $\alpha > 0$.

If $f(t, \varphi)$ is Lipschitizian in φ in each compact set in D, then the solution $x(t) = x(t, t_0, \varphi)$ of (1.1) on $[t_0 - \tau, t_0 + \alpha)$ is unique.

Theorem 1.2. *[23] Suppose $D \subseteq \mathbb{R} \times C_n$ is open and $f \in C(D; \mathbb{R}^n)$.*

If $x(t)$ is a noncontinuable solution of (1.1) on $[t_0 - \tau, b)$, $b > t_0$, then, for all compact sets $W \subseteq D$, there is $t_W \in (t_0, b)$ such that

$$(t, x_t) \notin W, \qquad t \in [t_W, b).$$

The equation (1.1) is said to be *autonomous* if $f(t, \varphi) \equiv f(\varphi)$ for all $(t, \varphi) \in D$, otherwise it is said to be *nonautonomous*. The equation (1.1) is said to be *linear* if $f(t, \varphi) = L(t, \varphi) + h(t)$, where $L(t, \varphi)$ is linear in φ, and it is said to be *linear autonomous* if $f(t, \varphi) = L(\varphi)$ with $L \in \mathcal{L}(C_n, \mathbb{R}^n)$, where $\mathcal{L}(C_n, \mathbb{R}^n)$ denotes the space of bounded linear operators from C_n to $\mathbb{R}^n$. Note that, by the Riesz representation theorem, there is an $n \times n$ matrix valued function η on $[-\tau, 0]$ of bounded variation such that

$$L(\varphi) = \int_{-\tau}^{0} \varphi(\theta) d\eta(\theta) := \int_{-\tau}^{0} [d\eta(\theta)] \varphi(\theta). \tag{1.2}$$

Now, we give special attention to linear autonomous retarded FDE's

$$\dot{x}(t) = Lx_t, \tag{1.3}$$

where L is a bounded linear mapping from C_n to $\mathbb{R}^n$. Thus, L has the form (1.2).

From Theorem 1.1, there is a unique solution of (1.3) with initial condition $\varphi \in C_n$ at zero, $x(\varphi) := x(t, 0, \varphi)$. A simple application of a Gronwall inequality and Theorem 1.2 imply that solutions $x(\varphi)$ are defined on $[-\tau, +\infty)$. Consequently, we can define the solution operator $T(t) : C_n \to C_n$, $t \geq 0$, by

$$T(t)\varphi = x_t(\varphi).$$

Furthermore, $(T(t))_{t\geq 0}$ is a C_0-semigroup of linear operators, with $T(t)$ compact for $t \geq \tau$. Its infinitesimal generator $A : D(A) \to C_n$ is given by

$$D(A) = \left\{ \varphi \in C_n : \dot{\varphi} \in C_n, \dot{\varphi}(0) = \int_{-\tau}^{0} [d\eta(\theta)]\varphi(\theta) \right\}$$

and

$$A\varphi = \dot{\varphi}. \tag{1.4}$$

We have the following result:

Theorem 1.3. *[23] If A is the infinitesimal generator of the C_0-semigroup of the solution operator $(T(t))_{t\geq 0}$, then $\sigma(A) = \sigma_P(A)$, and $\lambda \in \sigma(A)$ if and only if λ satisfies the characteristic equation*

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) := \lambda I - \int_{-\tau}^{0} e^{\lambda\theta}[d\eta(\theta)] = \lambda I - L(e^{\lambda \cdot} I). \tag{1.5}$$

For any λ in $\sigma(A)$, the generalized eigenspace $\mathcal{M}_\lambda(A)$ is finite dimensional and there is an integer k such that $\mathcal{M}_\lambda(A) = \mathcal{N}((\lambda I - A)^k)$, and we have the direct sum decomposition

$$C_n = \mathcal{N}((\lambda I - A)^k) \oplus Im((\lambda I - A)^k). \tag{1.6}$$

Note that $\lambda \in \sigma(A)$ if and only if there is $b \in \mathbb{C}^n$, $b \neq 0$, such that $\Delta(\lambda)b = 0$, which is equivalent to saying that $x(t) = e^{\lambda t}b$ is a solution on $\mathbb{R}$ of (1.3).

For $\lambda \in \sigma(A)$, from the above theorem, we know that $\mathcal{M}_\lambda(A)$ has finite dimension d_λ. Let $\Phi_\lambda = (\phi_1^\lambda, \ldots, \phi_{d_\lambda}^\lambda)$ be a basis for $\mathcal{M}_\lambda(A)$. Since $A\mathcal{M}_\lambda(A) \subseteq \mathcal{M}_\lambda(a)$, there is a $d_\lambda \times d_\lambda$ constant matrix B_λ such that

$$A\Phi_\lambda = \Phi_\lambda B_\lambda. \tag{1.7}$$

By (1.4), we have $\dot{\Phi}_\lambda = \Phi_\lambda B_\lambda$ and then

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{B_\lambda \theta}, \quad \theta \in [-\tau, 0].$$

From (1.7) together with a property of C_0-semigroups, we obtain $\frac{d}{dt}T(t)\Phi_\lambda = AT(t)\Phi_\lambda = T(t)A\Phi_\lambda = T(t)\Phi_\lambda B_\lambda$, and hence

$$T(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t}, \quad t \geq 0,$$

which implies

$$[T(t)\Phi_\lambda](\theta) = \Phi_\lambda(\theta)e^{B_\lambda t} = \Phi_\lambda(0)e^{B_\lambda(t+\theta)}, \quad \theta \in [-\tau, 0].$$

Clearly, the solution of (1.3) with initial condition $\Phi_\lambda a$ ($a \in \mathbb{C}^{d_\lambda}$) at $t_0 = 0$ is given by $x(t) = \Phi_\lambda(0)e^{B_\lambda t}a$, and is a solution of (1.3) on $\mathbb{R}$. This allows us to fully understand the behavior of the solutions of equation (1.3) with initial conditions on $\mathcal{M}_\lambda(A)$, that is, with the identification $\mathcal{M}_\lambda(A) \equiv \mathbb{C}^{d_\lambda}$, equation (1.3) restricted to $\mathcal{M}_\lambda(A)$ reads as the ODE $\dot{x} = B_\lambda x$. By a repeated application of the preceding process we obtain the following result:

Theorem 1.4. *[23] Suppose $\Lambda = \{\lambda_1, \ldots, \lambda_p\}$ is a finite subset of $\sigma(A)$ and let $\Phi_\Lambda = (\Phi_{\lambda_1}, \ldots, \Phi_{\lambda_p})$, $B_\Lambda = diag(B_{\lambda_1}, \ldots, B_{\lambda_p})$, where Φ_{λ_j} is a basis for $\mathcal{M}_{\lambda_j}(A)$ and B_{λ_j} is the matrix defined by $A\Phi_{\lambda_j} = \Phi_{\lambda_j}B_{\lambda_j}$, $j = 1, \ldots, p$.*

Define $N := \sum_{i=1}^p d_{\lambda_i}$. Then the only eigenvalue of B_{λ_j} is λ_j and, for each $a \in \mathbb{C}^N$, the solution $T(t)\Phi_\Lambda a$, with initial condition $\Phi_\Lambda a$ at $t_0 = 0$, is

$$x_t(\Phi_\Lambda a)(\theta) = [T(t)\Phi_\Lambda a](\theta) = \Phi_\Lambda(0)e^{B_\Lambda(t+\theta)}a, \quad -\tau \leq \theta \leq 0, \ t \in \mathbb{R}.$$

To obtain more information on the complementary space $Im((\lambda I - A)^k)$ in the decomposition (1.6), we define $C_n^* := C([0, \tau]; \mathbb{R}^{n^*})$, where $\mathbb{R}^{n^*}$ is the n-dimensional vector space of row vectors, and introduce the so-called *formal adjoint equation*

$$\dot{y}(r) = -\int_{-\tau}^{0} y(r - \theta)[d\eta(\theta)], \quad r \leq 0, \tag{1.8}$$

with $y(r) \in \mathbb{R}^{n^*}$. Notice that (1.8) must be solved backward and existence and uniqueness of solutions on $(-\infty, \tau]$ holds (letting $t = -r$ in (1.8), it becomes a retarded linear FDE). Consequently, we can also define the solution operator $T^*(r) : C_n^* \to C_n^*$, $r \leq 0$, by

$$T^*(r)\psi = y^r(\psi),$$

where $y^r(\psi)(\xi) = y(r+\xi)(\psi)$, $\xi \in [0, \tau]$. Letting $S(r) = T^*(-r)$, $r \geq 0$, $(S(r))_{r\geq 0}$ is a C_0-semigroup of linear operators on C_n^* and its infinitesimal generator $A^* : D(A^*) \to C_n^*$ is given by

$$A^*\psi = -\dot{\psi}$$

with

$$D(A^*) = \left\{ \varphi \in C_n^* : \dot{\psi} \in C_n^*, \dot{\psi}(0) = -\int_{-\tau}^{0} \psi(-\theta)[d\eta(\theta)] \right\}.$$

From Theorem 1.3 applied to $(S(r))_{r\geq 0}$, we conclude that $\sigma(A^*) = \sigma_P(A^*)$ and $\lambda \in \sigma(A^*)$ if and only if $y(t) = e^{-\lambda t}b$ is a solution of (1.8), where b is a nonzero row vector satisfying $b\Delta(\lambda) = 0$. Therefore, $\lambda \in \sigma(A^*)$ if and only if λ is a root of the characteristic equation (1.5), hence $\sigma(A^*) = \sigma_P(A^*) = \sigma(A)$. For any $\lambda \in \sigma(A^*)$, the generalized eigenspace $\mathcal{M}_\lambda(A^*)$ is finite dimensional, in fact $\dim(\mathcal{M}_\lambda(A^*)) = \dim(\mathcal{M}_\lambda(A))$. Furthermore, defining the *bilinear form* $(\cdot,\cdot) : C_n^* \times C_n \to \mathbb{R}$ by

$$(\psi,\varphi) := \psi(0)\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi, \qquad \psi \in C_n^*,\ \varphi \in C_n,$$

we have

$$(\psi, A\varphi) = (A^*\psi, \varphi), \qquad \psi \in C_n^*,\ \varphi \in C_n,$$

so A^* can be interpreted as a formal adjoint operator. The "alternative" theorem relative to the formal duality $(\cdot,\cdot)$ works well in this setting.

Lemma 1.5. *[23] Let $\lambda \in \sigma(A)$, $k \in \mathbb{N}$ and $\varphi \in C_n$. Then*

$$\varphi \in Im((\lambda I - A)^k) \text{ if and only if } (\psi,\varphi) = 0 \text{ for all } \psi \in \mathcal{N}((\lambda I - A^*)^k).$$

Now, fix $\lambda \in \sigma(A)$. As $\dim(\mathcal{M}_\lambda(A^*)) = \dim(\mathcal{M}_\lambda(A)) = d_\lambda$, let $\Psi_\lambda = col(\psi_1^\lambda, \ldots, \psi_{d_\lambda}^\lambda)$ and $\Phi_\lambda = (\phi_1^\lambda, \ldots, \phi_{d_\lambda}^\lambda)$ be bases for $\mathcal{M}_\lambda(A^*)$ and $\mathcal{M}_\lambda(A)$, respectively, and let $(\Psi_\lambda, \Phi_\lambda) := [(\psi_i^\lambda, \phi_j^\lambda)_{ij}]$. Then $(\Psi_\lambda, \Phi_\lambda)$ is a nonsingular matrix and thus may be taken as the identity by properly selecting ψ_i^λ or ϕ_i^λ, $i = 1, \ldots, d_\lambda$. Consider the decomposition (1.6) written as

$$C_n = P_\lambda \oplus Q_\lambda,$$

where $P_\lambda = \mathcal{M}_\lambda(A)$ and $Q_\lambda = Im((\lambda I - A)^k)$. Then, for any $\phi \in C_n$ we have $\phi = \phi^{P_\lambda} + \phi^{Q_\lambda}$, $\phi^{P_\lambda} \in P_\lambda$, $\phi^{Q_\lambda} \in Q_\lambda$, and

$$P_\lambda = \mathcal{M}_\lambda(A) = \{\phi \in C_n : \phi = \Phi_\lambda b, b \in \mathbb{C}^{d_\lambda}\},$$

$$Q_\lambda = \{\phi \in C_n : (\Psi_\lambda, \phi) = 0\}$$

and

$$\begin{aligned}(\Psi_\lambda, \phi) &= (\Psi_\lambda, \phi^{P_\lambda}) + (\Psi_\lambda, \phi^{Q_\lambda}) \\ &= (\Psi_\lambda, \Phi_\lambda b) \\ &= b,\end{aligned}$$

so that

$$\phi^{P_\lambda} = \Phi_\lambda b = \Phi_\lambda(\Psi_\lambda, \phi).$$

For $\mu, \lambda \in \sigma(A)$ with $\mu \neq \lambda$, we have $(\psi, \phi) = 0$ for all $\psi \in \mathcal{M}_\mu(A^*)$ and $\phi \in \mathcal{M}_\mu(A)$, hence the above detailed decomposition can easily be extended to $\Lambda = \{\lambda_1, \dots, \lambda_p\} \subseteq \sigma(A)$. Thus, we have the following result:

Theorem 1.6. *[23] Let $\Lambda = \{\lambda_1, \dots, \lambda_p\} \subseteq \sigma(A)$, and consider*

$$P_\Lambda := \bigoplus_{i=1}^{p} \mathcal{M}_{\lambda_i}(A) \quad \textit{and} \quad P_\Lambda^* := \bigoplus_{i=1}^{p} \mathcal{M}_{\lambda_i}(A^*).$$

Let Φ, Ψ be bases for P_Λ and P_Λ^*, respectively, such that $(\Psi, \Phi) = I_N$, where I_N is the $N \times N$ identity matrix and $N := \dim P_\Lambda = \dim P_\Lambda^*$. Then we have

$$C_n = P_\Lambda \oplus Q_\Lambda, \tag{1.9}$$

where

$$P_\Lambda = \{\phi \in C_n : \phi = \Phi b, b \in \mathbb{C}^N\},$$
$$Q_\Lambda = \{\phi \in C_n : (\Psi, \phi) = 0\},$$

and, for any $\phi \in C_n$, we have $\phi = \phi^{P_\Lambda} + \phi^{Q_\Lambda}$, $\phi^{P_\Lambda} \in P_\Lambda$, $\phi^{Q_\Lambda} \in Q_\Lambda$, with $\phi^{P_\Lambda} = \Phi(\Psi, \phi)$.

We shall refer to the decomposition (1.9) of C_n by saying that C_n *is decomposed by* Λ. From Theorem 1.4, on the generalized eigenspace P_Λ the equation (1.3) behaves as an ordinary differential equation. In particular, on P_Λ $T(t)$ is defined for $-\infty < t < +\infty$. Now, if we want to know the stability of a linear system (1.3), we need to have an estimate for the solutions on the complementary subspace Q_Λ.

1.2 Stability of Functional Differential Equations

In this section, we present some definitions and basic results on stability for FDE's.

First, we introduce several concepts of stability for solutions of equation (1.1). Consider (1.1) with $f : [\alpha, +\infty) \times C_n \to \mathbb{R}^n$, $\alpha \in \mathbb{R}$, continuous and satisfying enough additional smoothness conditions to ensure the uniqueness of the solution $x(t, t_0, \varphi)$ through $(t_0, \varphi) \in [\alpha + \tau, +\infty) \times C_n$ and that it is defined on $[t_0, +\infty)$.

Definition 1.2. *Suppose $f(t, 0) = 0$ for all $t \geq \alpha$, $\alpha \in \mathbb{R}$. We say that the solution $x \equiv 0$ of (1.1) is:*

(i) stable *if for any $t_0 \geq \alpha$, $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that, for all $\varphi \in C_n$,*

$$\|\varphi\| < \delta \Rightarrow \|x_t(t_0, \varphi)\| < \varepsilon, \quad \textit{for} \quad t \geq t_0;$$

(ii) uniformly stable *if for any* $\varepsilon > 0$*, there is a* $\delta = \delta(\varepsilon) > 0$ *such that, for all* $\varphi \in C_n$*,*

$$\|\varphi\| < \delta \Rightarrow \|x_t(t_0, \varphi)\| < \varepsilon, \quad \textit{for} \quad t \geq t_0 \geq \alpha;$$

(iii) (locally) asymptotically stable *if it is stable and for any* $t_0 \geq \alpha$*, there is* $b = b(t_0) > 0$ *such that, for all* $\varphi \in C_n$*,*

$$\|\varphi\| < b \Rightarrow |x(t, t_0, \varphi)| \to 0 \ \textit{as}\ t \to +\infty;$$

(iv) (locally) uniformly asymptotically stable *if it is uniformly stable and there is* $b > 0$ *such that, for every* $\eta > 0$*, there is a* $T = T(\eta) > 0$ *such that, for all* $\varphi \in C_n$*,*

$$\|\varphi\| < b \Rightarrow \|x_t(t_0, \varphi)\| < \eta, \quad \textit{for} \quad t_0 \geq \alpha \ \textit{and}\ t \geq t_0 + T;$$

(v) (locally) exponentially asymptotically stable *if there are* $b, k, \varepsilon > 0$ *such that, for* $\varphi \in C_n$*,*

$$\|\varphi\| \leq b \Rightarrow |x(t, t_0, \varphi)| \leq ke^{-\varepsilon(t-t_0)}, \quad \textit{for} \quad t \geq t_0 \geq \alpha;$$

(vi) globally attractive *if all solutions of (1.1) tend to zero as* $t \to +\infty$*;*

(vii) globally asymptotically stable *if it is stable and globally attractive;*

(viii) unstable *if it is not stable.*

If $y(t)$ is a solution of (1.1), then $y(t)$ is said to be *stable* if the solution $z(t) = 0$ of the equation

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t)$$

is stable. The other types of stability are defined similarly.

We remark that some authors (e.g. [28], pag. 149) use the term *globally asymptotically stable* to define a solution globally attractive.

Remark 1.1 If $f(t, \varphi)$ is linear in φ, then any solution $x(t)$ of (1.1) has the same stability of the zero solution, thus we often refer to the stability of the equation, instead of the stability of a solution.

Consider the linear autonomous system (1.3), the C_0-semigroup of solution operators $(T(t))_{t\geq 0}$ and its infinitesimal generator A. Since the operators $T(t)$ are compact for $t \geq \tau$, then for any real number β, there are only a finite number of λ in $\sigma(A)$ such that $Re\lambda \geq \beta$ (see [23] for details). Then, consider C_n decomposed by

$$\Lambda := \Lambda(\beta) = \{\lambda \in \sigma(A) : Re\lambda \geq \beta\}.$$

Theorem 1.7. *[23] For any $\beta \in \mathbb{R}$, let $\Lambda = \Lambda(\beta)$. If C_n is decomposed by Λ as $C_n = P_\Lambda \oplus Q_\Lambda$, then there exist positive constants γ and k such that*

$$\|T(t)\phi^{Q_\Lambda}\| \leq ke^{(\beta-\gamma)t}\|\phi^{Q_\Lambda}\|, \quad t \geq 0, \forall \phi^{Q_\Lambda} \in Q_\Lambda; \tag{1.10}$$

$$\|T(t)\phi^{P_\Lambda}\| \leq ke^{(\beta-\gamma)t}\|\phi^{P_\Lambda}\|, \quad t \leq 0, \forall \phi^{P_\Lambda} \in P_\Lambda. \tag{1.11}$$

An important corollary is obtained when $\beta = 0$ and $\Lambda(0) = \emptyset$.

Corollary 1.8. *If all of the roots of the characteristic equation (1.5) for (1.3) have negative real parts, then there exist positive constants k and γ such that*

$$\|T(t)\phi\| \leq ke^{-\gamma t}\|\phi\|, \quad \forall t \geq 0, \forall \phi \in C_n.$$

The above result says that, if all roots of characteristic equation of (1.3) have negative real parts, then system (1.3) is exponentially asymptotically stable. The reverse also holds, because of Theorem 1.4 and the known results about stability for linear ordinary differential equations.

The last result in this section concerns the stability of the zero solution of a perturbed autonomous FDE. Consider the FDE in C_n,

$$\dot{x}(t) = Lx_t + f(x_t), \tag{1.12}$$

where $L : C_n \to \mathbb{R}^n$ is a bounded linear operator and $f : C_n \to \mathbb{R}^n$ is a C^1 function with $f(0) = 0$ and $Df(0) = 0$. Thus, the linearization of (1.12) about $\phi = 0$ is

$$\dot{y}(t) = Ly_t. \tag{1.13}$$

The following result allows us to obtain the local exponential asymptotic stability of the zero solution of (1.12) from the exponential asymptotic stability of system (1.13).

Theorem 1.9. *Consider (1.12) with f in C^1, $f(0) = 0$ and $Df(0) = 0$. If (1.13) is exponentially asymptotically stable, then the zero solution of (1.12) is locally exponentially asymptotically stable. If $Re\lambda > 0$ for some λ satisfying the characteristic equation of (1.13), then the zero solution of (1.12) is unstable.*

Proof. See Theorem 2.4.2 in [28]. ∎

The above theorem allows us to deduce the local stability of a steady state x_0 of an autonomous FDE $\dot{x}(t) = f(x_t)$, with $f \in C^1$, if the linear equation $\dot{y}(t) = Df(x_0)y_t$ is hyperbolic, i.e., if

$$\{\beta \in \mathbb{R} : \det \Delta(\beta i) = 0\} = \emptyset.$$

1.3 3/2-Type Conditions for Scalar Functional Differential Equations

To study the behavior of solutions of delay differential equations, and in particular the stability of equilibria, one approach is to give conditions on the size of the delays and coefficients, such as the so-called 3/2-type conditions, so that the FDE is expected to behave similarly to an ordinary differential equation if the delays are sufficiently small. This is the setting initiated with the remarkable work of Wright [62], who studied the delayed logistic equation (7) and established the following global stability result:

Theorem 1.10. *[62] If $a\tau \leq 3/2$, then any positive solution $x(t)$ of the delayed logistic equation (7) converge to the positive equilibrium k as $t \to +\infty$.*

By the change $y(t) = -1 + \frac{x(t)}{k}$, we transfer the positive equilibrium to the origin and equation (7) can be written as

$$\dot{y}(t) = -(1 + y(t))ay(t - \tau), \quad t \geq 0. \tag{1.14}$$

The linearization of (1.14) about the trivial solution is the linear equation $\dot{y}(t) = -ay(t - \tau)$, and its characteristic equation is $\lambda + ae^{-\lambda\tau} = 0$. If $a\tau < \frac{\pi}{2}$ then all roots λ have negative real part. If $a\tau > \frac{\pi}{2}$ then there exists a root of the characteristic equation with positive real parte. Consequently, from Theorem 1.9, the positive equilibrium $x(t) \equiv k$ of (7) is locally exponentially asymptotically stable if $a\tau < \frac{\pi}{2}$, and unstable if $a\tau > \frac{\pi}{2}$. Wright [62] formulated the well-known Wright's conjecture, claiming that the above theorem is true if we assume $a\tau < \frac{\pi}{2}$, instead of $a\tau \leq \frac{3}{2}$.

In 1970, Yorke [66] considered a general scalar FDE

$$\dot{x}(t) = f(t, x_t), \quad t \geq 0, \tag{1.15}$$

where $f : [0, +\infty) \times C^\beta \to \mathbb{R}$ is continuous with $C^\beta = \{\phi \in C([-\tau, 0]; \mathbb{R}) : \|\phi\| < \beta\}$, $\beta > 0$, and introduced the so-called *Yorke condition*

$$-a\mathcal{M}(\varphi) \leq f(t, \varphi) \leq a\mathcal{M}(-\varphi), \quad t \geq 0, \varphi \in C := C([-\tau, 0]; \mathbb{R}), \tag{1.16}$$

where $a > 0$ and $\mathcal{M}(\varphi)$ is the Yorke functional

$$\mathcal{M}(\varphi) := \max\left\{0, \sup_{\theta \in [-\tau, 0]} \varphi(\theta)\right\}. \tag{1.17}$$

Yorke also introduced the following condition:

$$\begin{gathered}\text{for all sequences } t_n \to +\infty \text{ and } \varphi_n \in C, \|\varphi_n\| \leq \beta,\\ \text{if } \varphi_n \to c \neq 0, \text{ then } f(t_n, \varphi_n) \text{ does not converge to zero.}\end{gathered} \tag{1.18}$$

A solution $x(t)$ of (1.15) defined on $[-\tau, +\infty)$ is said to be *oscillatory* if it is not eventually zero and it has arbitrary large zeros, otherwise $x(t)$ is called *non-oscillatory.* Yorke [66] used the hypothesis (1.16) together with the restriction $a\tau < 3/2$ to deduce that all oscillatory solutions $x(t, \varphi)$ of (1.15) with sufficiently small initial condition φ (i.e. $\varphi \in C^\beta$ such that $\|\varphi\| \leq \frac{2\beta}{5}$) tends to zero as $t \to +\infty$, while the additional condition (1.18) is needed to force non-oscillatory solutions to converge to zero as $t \to +\infty$.

Later, Yoneyama [64] generalized the work of Yorke, replacing in the Yorke condition the constant a by a non-negative continuous function $\lambda(t)$ such that

$$\inf_{t \geq \tau} \int_{t-\tau}^{t} \lambda(s) ds > 0,$$

and established similar stability results for (1.15) under

$$-\lambda(t)\mathcal{M}(\varphi) \leq f(t, \varphi) \leq \lambda(t)\mathcal{M}(-\varphi), \quad t \geq 0, \varphi \in C, \tag{1.19}$$

and the new 3/2-type condition

$$\sup_{t \geq \tau} \int_{t-\tau}^{t} \lambda(s) ds < \frac{3}{2}. \tag{1.20}$$

Yoneyama [64] also proved that, for (1.15) with a piecewise continuous function f, the condition (1.20) is sharp, in the sense that there are functions f satisfying (1.19) with $\sup_{t \geq \tau} \int_{t-\tau}^{t} \lambda(s) ds = 3/2$, for which (1.15) has a nonzero periodic solution.

Since the work of Wright [62], there has been an extensive literature on 3/2-type conditions and on Yorke conditions, used to obtain several results about stability of solutions of scalar FDE's. For more discussions, we refer the reader to the books of Gopalsamy [19] and Kuang [28].

Some recent generalizations of the Yorke condition in [11], [14], [34] and [67] motivated our work presented in Chapter 2. In [14] we considered scalar FDE's of the form (1.15) with $f : [0, +\infty) \times C \to \mathbb{R}$ continuous and $C := C([-\tau, 0]; \mathbb{R})$ the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}$, $\tau > 0$, equipped with the sup norm $\|\varphi\| = \max_{\theta \in [-\tau, 0]} |\varphi(\theta)|$, and assumed the following hypotheses:

(h1) there exists a piecewise continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\sup_{t \geq \tau} \int_{t-\tau}^{t} \beta(s) ds < +\infty$, and such that for each $q \in \mathbb{R}$ there is $\eta(q) \in \mathbb{R}$ such that for $t \geq 0$ and $\varphi \in C$, $\varphi \geq q$, then

$$f(t, \varphi) \leq \beta(t)\eta(q);$$

(h2) if $w : [-\tau, +\infty) \to \mathbb{R}$ is continuous and $w_t \to c \neq 0$ in C as $t \to +\infty$, then $\int_0^{+\infty} f(s, w_s) ds$ diverges;

(h3) there are a piecewise continuous function $\lambda : [0, +\infty) \to [0, +\infty)$ and a constant $b \geq 0$ such that, for $r(x) := \frac{-x}{1+bx}$, $x > -1/b$, then

$$\lambda(t)r(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq \lambda(t)r(-\mathcal{M}(-\varphi)), \quad t \geq 0, \tag{1.21}$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b \in [-\infty, 0)$, and $\mathcal{M}(\varphi)$ is the Yorke functional defined in (1.17);

(h4) for $\lambda(t)$ as in **(h3)**, there is $T \geq \tau$ such that, for

$$\alpha := \alpha(T) = \sup_{t \geq T} \int_{t-\tau}^{t} \lambda(s) ds, \tag{1.22}$$

$\alpha \leq 3/2$ if $b > 0$, and $\alpha < 3/2$ if $b = 0$.

We remark that, if $b = 0$ we have $r(x) = -x$ and (1.21) coincide with the Yorke condition (1.19) introduced by Yoneyama. Moreover, in this situation, it is clear that **(h3)** and **(h4)** imply **(h1)**.

Hypotheses **(h1)**, **(h3)** are used to guarantee that all solutions of (1.15) are bounded (see [34]), and **(h2)** is used to prove that non-oscillatory solutions of (1.15) converge to zero as $t \to +\infty$, whereas **(h3)**, **(h4)** are used to prove the same for oscillatory solutions. In [14], the following result was proven:

Theorem 1.11. *[14] Assume* **(h1)-(h4)** *and consider* $\alpha := \sup_{t \geq T} \int_{t-\tau}^{t} \lambda(s) ds < 3/2$, *for some* $T \geq \tau$. *Then the zero solution of (1.15) is globally attractive.*

If $b > 0$ *and* $\lambda(t) > 0$ *for* t *large, the same result holds for* $\alpha = 3/2$.

The rational function $r(x)$ in (1.21) was first introduced by Liz et al. [34], with $\lambda(t) \equiv a$, and in [14] the situation of two different rational functions $r_1(x)$, $r_2(x)$ in the Yorke condition was also considered, $\lambda(t)r_1(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq \lambda(t)r_2(-\mathcal{M}(-\varphi))$, however under a constraint stronger than the 3/2-condition **(h4)**. Also, instead of introducing a rational function in (1.21), for a particular class of scalar FDE's, Muroya [38] considered a strictly decreasing function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and either $h(-\infty)$ or $h(+\infty)$ is finite. Recently, Zhang and Yan [67] considered two functions $\lambda_1(t)$, $\lambda_2(t)$ in (1.19), $-\lambda_1(t)\mathcal{M}(\varphi) \leq f(t, \varphi) \leq \lambda_2(t)\mathcal{M}(-\varphi)$, and the following 3/2-type condition

$$\min\{\alpha_1, \alpha_2\} \max\{\alpha_1^2, \alpha_2^2\} < (3/2)^3, \tag{1.23}$$

where $\alpha_i := \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s) ds$, $i = 1, 2$, for some $T \geq \tau$, to prove that the zero solution of (1.15) is globally asymptotically stable.

In Chapter 2, we unify several generalizations of the Yorke condition presented in [14], [34], [38], and [67], considering two functions $\lambda_1(t)$, $\lambda_2(t)$ in (1.21) and, for the case $b = 0$, placing a function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x)x < 0$ and $|h(x)| < |x|$, $x \neq 0$, instead of $r(x) = -x$. Thus we prove the global attractivity of the trivial solution of (1.15) under a generalized Yorke condition and a 3/2-condition weaker than (1.23).

In applications to biology, most of the scalar models in population dynamics have the form $\dot{y}(t) = y(t)g(t, y_t)$. Due to the biological interpretation, only positive solutions are meaningful. From our results, we shall deduce a new criterion for the global attractivity of the positive equilibrium (in the set of positive solutions). To illustrate the situation, we shall study two food-limited population models with delays, for which several criteria for the global attractivity of their equilibrium points are given.

Recently, some authors, [54] and [55] have extended Wright's study [62] to n-dimensional delayed differential systems. Tang and Zou [55] considered Lotka-Volterra systems with distributed delays of the form

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - \int_{-\tau_{ii}}^{0} x_i(t+\theta)d\eta_{ii}(\theta) \right. \\ \left. - \sum_{j\neq i}^{n} l_{ij} \int_{-\tau_{ij}}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], \qquad i = 1, \dots, n. \quad (1.24)$$

where $r_i \in C([0, +\infty); (0, +\infty))$, $l_{ij} \geq 0$, $\tau_{ij} \geq 0$ and η_{ij} are non-decreasing functions with variation 1, i.e. $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$, $i, j = 1, \dots, n$, and obtained several 3/2-type criteria for the global attractivity of the positive equilibrium of (1.24) by using a technique which extends to systems Wright's method [62] for scalar equations.

1.4 Nondelayed Diagonally Dominant Terms

In this section, we present a different line of investigation for studying the global stability of solutions of delayed differential equations.

In general, large delays induce instability of equilibria, oscillations and even existence of unbounded solutions. If the delays are small enough, they are expected to be negligible, so that a FDE should behave mainly like an ordinary differential equation. This is the line of investigation presented in last section, which has been especially fruitful in case of scalar equations since the works of Wright [62] and Yorke [66] with the so-called 3/2-type conditions and Yorke's condition.

However, in many situations, it is not realistic to assume that the delays are very small. An alternative setting to study stability of a delayed differential equation is to assume that it has non-delayed negative feedback terms which dominate, in some sense, the delay effect. This is the situation developed by e.g. Faria [12], Faria and Liz [13], Gyori [22], Seifert [46] for scalar equations, Campbell [3], Hofbauer and So [24], Kuang [29], [30], Kuang and Smith [31], Lu and Wang [35], for n-dimensional systems, and followed here in Chapters 3 and 4.

In Chapter 3, we consider linear FDE's in $\mathbb{R}^n$ with undelayed diagonal terms, given by

$$\dot{x}_i(t) = -\left[b_i x_i(t) + \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) \right], \quad i = 1, \dots, n, \tag{1.25}$$

and multiple species Lotka-Volterra type models of the form

$$\dot{x}_i(t) = r_i(t) x_i(t) \left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) \right], i = 1, \dots, n, \tag{1.26}$$

where $b_i, l_{ij} \in \mathbb{R}$, $\tau > 0$, $r_i(t)$ are positive continuous functions and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions, $i, j = 1, \dots, n$. The main purpose is to establish sufficient conditions of diagonal dominance of the instantaneous negative feedbacks over the matrix of all the delayed terms, so that the stability of a positive equilibrium of (1.26) follows independently of the choices of bounded functions η_{ij}.

In Chapter 4, we consider a general delayed differential system of the form

$$\dot{x}_i(t) = r_i(t) f_i(x_t), \quad t \geq 0, \ i = 1, \dots, n, \tag{1.27}$$

where $r_i : [0, +\infty) \to (0, +\infty)$ and $f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$ are continuous functions. The goal is to establish a general hypothesis over f, so that, independently of the delay $\tau > 0$, $x(t) \equiv 0$ is a globally asymptotically stable equilibrium. These results are applied to the study of a general neural network model written as

$$\dot{x}_i(t) = -r_i(t) k_i(x_i(t)) [b_i(x_i(t)) + f_i(x_t)], \quad t \geq 0, i = 1, \dots, n, \tag{1.28}$$

where $r_i : [0, +\infty) \to (0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : C_n \to \mathbb{R}$ are continuous functions, $i = 1, \dots, n$.

The work on Lotka-Volterra systems presented in Chapter 3 was strongly motivated by Faria [12], where scalar equations (1.25) and (1.26) were studied, and Hofbauer and So [24] and Campbell [3], who dealt with n-dimensional systems with discrete delays.

In [12], the scalar delayed logistic equation

$$\dot{x}(t) = r(t) x(t) [1 - b_0 x(t) - L_0(x_t)], \quad t \geq 0, \tag{1.29}$$

where $r : [0, +\infty) \to (0, +\infty)$ is a continuous function, $b_0 \in \mathbb{R}$, and $L_0 : C = C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is a linear bounded operator, was considered. The following criterion for the global asymptotic stability of its positive equilibrium was established:

Theorem 1.12. *[12] Consider equation (1.29), with $b_0 > 0$, $L_0 : C \to \mathbb{R}$ a bounded linear operator, and $r(t)$ uniformly bounded on $[0, +\infty)$ and such that $\int_0^{+\infty} r(t)dt = +\infty$. If*

$$b_0 + L_0(1) > 0, \qquad b_0 \geq \|L_0\|, \tag{1.30}$$

then the positive equilibrium $x^* = (b_0 + L_0(1))^{-1}$ of (1.29) is globally asymptotically stable (in the set of all positive solutions).

Note that the inequality $b_0 + L_0(1) > 0$ assures that equation (1.29) has a unique positive equilibrium, and the inequality $b_0 \geq \|L_0\|$ says that the term $b_0 x(t)$ "dominates" the delayed term $L_0(x_t)$.

Now, suppose that (1.29) is autonomous, i.e. $r(t) \equiv 1$. Then, its linearization about x^* has the form $\dot{x}(t) = -[b_0 x(t) + L_0(x_t)]$, and the following necessary and sufficient condition for the global stability of the linear equation was obtained in [12].

Theorem 1.13. *[12] Let $L : C \to \mathbb{R}$ be a linear bounded operator. Then the following conditions are equivalent:*

(i) *$L(1) > 0$ and L satisfies*

$$\text{for all } \varphi \in C \text{ such that } |\varphi(\theta)| < \varphi(0) \text{ for } \theta \in [-\tau, 0), \\ \text{then } L(\varphi) > 0; \tag{1.31}$$

(ii) *L has the form*

$$L(\varphi) = b_0\varphi(0) + l_0 \int_{-\tau}^{0} \varphi(\theta) d\eta(\theta), \qquad \varphi \in C, \tag{1.32}$$

with $b_0 > 0, l_0 \in \mathbb{R}$ and $\eta : [-\tau, 0] \to \mathbb{R}$ a normalized bounded variation function, $b_0 \geq |l_0|$ and $L(1) > 0$.

Under these conditions, the linear equation

$$\dot{x}(t) = -L(x_t) \tag{1.33}$$

is exponentially asymptotically stable.

It was also proved that if L has the form (1.32) with $b_0 < |l_0|$, then the characteristic equation for (1.33) has a root with positive real part. Consequently, for the autonomous situation, the criterion established in Theorem 1.12 is sharp.

The study of the nonscalar situation is far more complicated. Lu and Wang [35] studied the two-species Lotka-Volterra system without delayed intraspecific competitions given by

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t-\tau_{12})] \\ \dot{x}_2(t) &= x_2(t)[r_2 - a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t)] \end{aligned}, \tag{1.34}$$

assuming that there is a unique positive equilibrium $x^* = (x_1^*, x_2^*)$. As usually, only positive solutions were considered. They showed that x^* is globally asymptotically stable for all the choices of $\tau_{12}, \tau_{21} \geq 0$ if and only if the interaction matrix of the system $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ satisfies

$$a_{11} > 0,\ a_{22} > 0,\ -a_{12}a_{21} \leq a_{11}a_{22},\ \text{and } a_{12}a_{21} < a_{11}a_{22},$$

which is equivalent to saying that $\det M \neq 0$ and $\tilde{M} = \begin{bmatrix} a_{11} & -|a_{12}| \\ -|a_{21}| & a_{22} \end{bmatrix}$ is an M-matrix (see Section 1.5 for details). Hofbauer and So [24] extended this result to $n \geq 2$, as described below.

Considering the autonomous n-dimensional Lotka-Volterra system with discrete delays

$$\dot{x}_i(t) = x_i(t)\left[r_i - \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\right], \qquad i = 1, \ldots, n, \tag{1.35}$$

with $r_i > 0$, $a_{ij} \in \mathbb{R}$, $\tau_{ij} \geq 0$ for $1 \leq i, j \leq n$ and $\tau_{ii} = 0$ for $i = 1, \ldots, n$, Hofbauer and So [24] obtained necessary and sufficient conditions for the global asymptotic stability of a positive equilibrium x^* (if it exists), for all the choices of delays $\tau_{ij} \geq 0$ $i \neq j$.

Theorem 1.14. *[24] Suppose that there exists a positive equilibrium x^* of (1.35). Then x^* is globally asymptotically stable (for initial conditions $x_i(0) > 0$) for all $\tau_{ij} \geq 0$, $i \neq j$ and $\tau_{ii} = 0$ if and only if $a_{ii} > 0$, $\det M \neq 0$ and $\tilde{M}$ is an M-matrix, where $M = [a_{ij}]$ and $\tilde{M} = [\tilde{a}_{ij}]$ with $\tilde{a}_{ii} = a_{ii}$ and $\tilde{a}_{ij} = -|a_{ij}|$, $i \neq j$, $i, j = 1, \ldots, n$.*

For the linear system

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij}), \qquad i = 1, \cdot, n, \tag{1.36}$$

Hofbauer and So [24] showed the exponential asymptotic stability, independently of the choices of the delays with the same conditions.

Theorem 1.15. *[24] The linear system (1.36) is exponentially asymptotically stable for all choices of the delays $\tau_{ij} \geq 0$, $i \neq j$, $\tau_{ii} = 0$, $i, j = 1, \ldots, n$, if and only if $a_{ii} > 0$, $detM \neq 0$ and $\tilde{M}$ is an M-matrix.*

Later on, Campbell [3] proved the above theorem without the restriction $\tau_{ii} = 0$.

In Chapter 3, the goal is to extend both the results in [12] to n-dimensional equations, and the results in [24] to a general situation with distributed delays. First, for the linear system (1.25), we obtain necessary and sufficient conditions, independently of the choices of η_{ij}, $i, j = 1, \ldots, n$, for its exponential asymptotic stability. Afterwards, under conditions slightly stronger than the ones required for the linear stability, we prove the global asymptotic stability of the positive equilibrium x^* (if it exists) of (1.26).

1.5 M-Matrices

In some recent literature on global stability for n-dimensional systems of delayed differential equations, such as Lotka-Volterra and neural network models, the concept of M-matrix and its properties arise as an important tool (see e.g. [1], [3], [4], [7], [24], [53], [58]). Thus, in this section we give some concepts and results from matrix analysis, which are important for the investigation carried out in Chapter 3 and 4. For a complete study of M-matrices, we refer the reader to [17], Chapter 5.

A matrix $A = [a_{ij}]$ is said to be *non-negative*, $A \geq 0$, respectively *positive*, $A > 0$, if $a_{ij} \geq 0$, respectively $a_{ij} > 0$, for all i, j. If A and B are real matrices of equal dimension, then $A \geq B$ and $A > B$ mean $A - B \geq 0$ and $A - B > 0$, respectively. We also define $|A| := [|a_{ij}|]$. Similarly, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we say that $x > 0$ if $x_i > 0$ for $i = 1, \ldots, n$, and that $x \geq 0$ if $x_i \geq 0$ for $i = 1, \ldots, n$. For $x = (x_1, \ldots, x_n) > 0$, x^{-1} is the vector given by $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$.

A square matrix A is said to be *reducible* if it has the form

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}, \tag{1.37}$$

where A_1 and A_2 are square matrices of order at least 1 or if A can be transformed into the form (1.37) by a simultaneous permutation of rows and columns. A square matrix is said to be *irreducible* if it is not reducible.

For $n \in \mathbb{N}$, we denote by Z_n the set of all real square matrices of order n whose off-diagonal entries are all non-positive, i.e.,

$$Z_n = \{A = [a_{ij}] : a_{ij} \leq 0 \text{ for all } i \neq j \text{ and } i, j = 1, \ldots, n\}.$$

Definition 1.3. *For $n \in \mathbb{N}$, let $A \in Z_n$.*

The matrix A is said to be an M-matrix *if all eigenvalues of A have non-negative real part.*

The matrix A is said to be a non-singular M-matrix *if all eigenvalues of A have positive real part.*

Clearly, a non-singular M-matrix is an M-matrix.

The above definition agrees with the notation in [3] and [49]. Some authors (e.g. [38]) use the term "M-matrix" to denote a "non-singular M-matrix" as above defined, a situation the reader should be aware of, in order to avoid conceptual misunderstandings.

The following result is a fundamental theorem on non-singular M-matrices. It gives a list of properties which are important in the present work. For a complete list see [17].

Theorem 1.16. *[17] Let $A = [a_{ij}]$ be a matrix from Z_n. Then the following properties are equivalent:*

(i) *Every eigenvalue of A has a positive real part;*

(ii) *All principal minors of A are positive;*

(iii) *There exists a positive diagonal matrix $D = diag(d_1, \ldots, d_n)$ such that $AD = [a_{ij}d_j]$ satisfies the condition*

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}|d_j, \quad \text{for} \quad i = 1, \ldots, n;$$

(iv) *There exists a vector $x > 0$ such that $Ax > 0$;*

(v) *A is non-singular and $A^{-1} \geq 0$.*

For M-matrices, we have the following important theorem:

Theorem 1.17. *[17] Let $A = [a_{ij}]$ be a matrix from Z_n. Then the following properties are equivalent:*

(i) *Every eigenvalue of the matrix A has a non-negative real part;*

(ii) *All principal minors of A are non-negative;*

(iii) *$A + \epsilon I$ is a non-singular M-matrix for any $\epsilon > 0$.*

From the above theorems, we conclude that an M-matrix A is a non-singular M-matrix if and only if $\det A \neq 0$.

Now we present some properties which will be used later. The next two theorems are a direct consequence of Theorems 1.16 and 1.17.

Theorem 1.18. *[17] If A is a non-singular M-matrix, $B \in Z_n$, and $B \geq A$, then B is a non-singular M-matrix. If A is an M-matrix, $B \in Z_n$, and $B \geq A$, then B is an M-matrix.*

Theorem 1.19. *[17] Let $A \in Z_n$. If there exists a positive vector x such that $Ax \geq 0$, then A is an M-matrix.*

In general, the reverse of Theorem 1.19 is not true for $n \geq 2$. For example, the matrix $D = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ is an M-matrix but there is no $d = (d_1, d_2) > 0$ such that $Dd \geq 0$. However, we have the following result:

Theorem 1.20. *[17] Let A be an M-matrix. If A is irreducible, then there is a vector $x > 0$ such that $Ax \geq 0$.*

We now introduce a further concept. A square real matrix $A = [a_{ij}]$ of order n is said to be *diagonally dominant* if there exists $d = (d_1, \ldots, d_n) > 0$ such that

$$|a_{ii}|d_i > \sum_{j \neq i} |a_{ij}|d_j, \quad \text{for} \quad i = 1, \ldots, n.$$

Note that this is the definition given in [17]. Usually in the literature, the definition of diagonally dominant matrix is presented in a different form. From Theorem 1.16, if $A \in Z_n$ is a non-singular M-matrix, then A is diagonally dominant. As last, we have the following result:

Theorem 1.21. *[17] If A is a diagonally dominant matrix, then A is non-singular.*

Chapter 2

Global Attractivity for Scalar Functional Differential Equations

In this chapter we study the global attractivity of the trivial solution of a scalar functional differential equations of the general form $\dot{x}(t) = f(t, x_t)$ by refining the method of Yorke and the well-known 3/2-type conditions. The results are applied to establish sufficient conditions for the global attractivity of the positive equilibrium of scalar delayed population models of the form $\dot{x}(t) = x(t)f(t, x_t)$, and illustrated with the study of two food-limited population models with delay, for which several criteria for their global attractivity are given.

2.1 Notation and Definitions

Suppose $\tau > 0$ and let $C := C([-\tau, 0]; \mathbb{R})$ be the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}$, equipped with the sup norm $\|\varphi\| = \max_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$.

For $c \in \mathbb{R}$, we use c also to denote the constant function $\varphi(\theta) = c$, $\theta \in [-\tau, 0]$, in C. The set C is supposed to be partially ordered with

$$\varphi \geq \psi \quad \text{if and only if} \quad \varphi(\theta) \geq \psi(\theta),\ \theta \in [-\tau, 0].$$

Particularly, for $\varphi \in C$ and $c \in \mathbb{R}$, we say that $\varphi \geq c$ (respectively $\varphi \leq c$) if and only if $\varphi(\theta) \geq c$ (respectively $\varphi(\theta) \leq c$) for all $\theta \in [-\tau, 0]$.

A function $x : [a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, is said to be *oscillatory* if it is not eventually zero and it has arbitrarily large zeros, otherwise $x(t)$ is called *non-oscillatory*.

In this chapter, we consider the scalar FDE

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0, \tag{2.1}$$

where $f : [0,+\infty) \times C \to \mathbb{R}$ is a continuous function. As usual, for each $t \geq 0$, x_t denotes the function in C defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$. An equilibrium point E_* of (2.1) is said to be *globally attractive* if all solutions of equation (2.1) tend to E_*, as $t \to +\infty$.

2.2 Asymptotic Stability

In this section, we prove a main result on asymptotic stability of equilibrium points of general scalar FDE's (2.1). For f as in (2.1), we consider the next hypotheses:

(H1) there is a piecewise continuous function $\beta : [0,+\infty) \to [0,+\infty)$ with

$$\sup_{t \geq \tau} \int_{t-\tau}^{t} \beta(s)ds < +\infty,$$

and such that for each $q \in \mathbb{R}$ there is $\eta(q) \in \mathbb{R}$ such that for $t \geq 0$ and $\varphi \in C$, $\varphi \geq q$, then

$$f(t,\varphi) \leq \beta(t)\eta(q);$$

(H2) if $w : [-\tau,+\infty) \to \mathbb{R}$ is continuous and $w_t \to c \neq 0$ in C as $t \to +\infty$, then $\int_0^\infty f(s,w_s)ds$ diverges;

(H3) there exist piecewise continuous functions $\lambda_1, \lambda_2 : [0,+\infty) \to [0,+\infty)$ and a constant $b \geq 0$ such that, for $r(x) := \frac{-x}{1+bx}$, $x > -1/b$, then

$$\lambda_1(t)r(\mathcal{M}(\varphi)) \leq f(t,\varphi) \leq \lambda_2(t)r(-\mathcal{M}(-\varphi)), \quad t \geq 0, \tag{2.2}$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b \in [-\infty, 0)$, and $\mathcal{M}(\varphi) := \max\{0, \sup_{\theta\in[-\tau,0]} \varphi(\theta)\}$ is the Yorke's functional;

(H4) there is $T \geq \tau$ such that, for

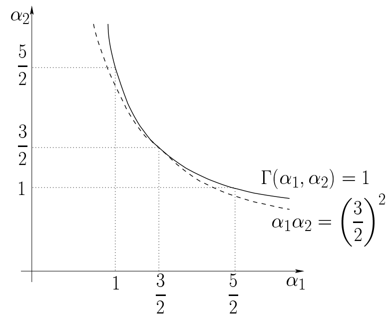
$$\alpha_i := \alpha_i(T) = \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s)ds, \quad i = 1,2, \tag{2.3}$$

we have

$$\Gamma(\alpha_1,\alpha_2) \leq 1, \tag{2.4}$$

where $\Gamma : (0,+\infty) \times (0,5/2) \cup (0,5/2) \times (0,+\infty) \to \mathbb{R}$ is defined by

$$\Gamma(\alpha_1,\alpha_2) = \begin{cases} (\alpha_1 - 1/2)\alpha_2^2/2 & \text{if } \alpha_1 > 5/2 \\ (\alpha_1 - 1/2)(\alpha_2 - 1/2), & \text{if } \alpha_1, \alpha_2 \leq 5/2 \\ (\alpha_2 - 1/2)\alpha_1^2/2, & \text{if } \alpha_2 > 5/2 \end{cases}. \tag{2.5}$$



For $t \geq 0$, $\varphi \in C$, note that **(H3)** implies that $f(t,\varphi) \leq 0$ if $\varphi \geq 0$ and $f(t,\varphi) \geq 0$ if $\varphi \leq 0$, and, together with **(H2)**, we conclude that $x = 0$ is the unique equilibrium of (1.1). It is also important to note that, if $b = 0$, then $r(x) = -x$ and taking $\beta(t) = \lambda_2(t)$ and $\eta(q) = |q|$ we conclude that **(H3)** and **(H4)** imply **(H1)**.

In comparison with the previous work [14], the major novelty here consists of considering two different functions $\lambda_1(t)$, $\lambda_2(t)$ in hypothesis **(H3)**. Consequently a new 3/2-type condition arises in hypothesis **(H4)**. In fact, **(H4)** is a generalization of **(h4)** since, for $\lambda_1(t) \equiv \lambda_2(t)$, we have $\alpha_1 = \alpha_2 := \alpha$ and $\Gamma(\alpha_1, \alpha_2) \leq 1$ reduces to $\alpha \leq 3/2$. Actually, the particular case of (2.2) with $b = 0$ was considered in [67], under the assumption (1.23), which is more restrictive than (2.4). We also remark that, as we shall see, $\Gamma(\alpha_1, \alpha_2) \leq 1$ is satisfied if

$$\alpha_1\alpha_2 \leq 9/4.$$

The goal here is to improve Theorem 1.11 by assuming the more general Yorke condition **(H3)** and 3/2-type condition **(H4)**. To achieve this, we deal separately with the cases of a rational function $r(x)$ in **(H3)** with $b = 0$ and $b > 0$. Furthermore, for the case $b = 0$, instead of **(H3)** we shall also consider a weaker hypothesis (see **(H3')** below), and generalize results in [11] and [36].

First, we take $b = 0$ in **(H3)**, so that $r(x) = -x$ for all $x \in \mathbb{R}$. For this situation, we replace the Yorke condition (2.2) by the following weaker condition:

(H3') there are piecewise continuous functions $\lambda_1, \lambda_2 : [0, +\infty) \to [0, +\infty)$ and $h : \mathbb{R} \to \mathbb{R}$, with h satisfying

$$h(x)x < 0 \quad \text{and} \quad |h(x)| < |x| \quad \text{for } x \neq 0, \tag{2.6}$$

such that

$$\lambda_1(t)h(\mathcal{M}(\varphi)) \leq f(t,\varphi) \leq \lambda_2(t)h(-\mathcal{M}(-\varphi)), \quad \text{for } t \geq 0, \varphi \in C. \tag{2.7}$$

Now we want to show the global attractivity of the zero solution of (2.1) under **(H2)**, **(H3')**, and **(H4)**. Next lemma shows that all non-oscillatory solutions tend to the equilibrium.

Lemma 2.1. *Assume* **(H3')** *and that* $\sup_{t\geq\tau}\int_{t-\tau}^{t}\lambda_i(s)ds$, $i=1,2$ *are finite. Then, all solutions of (2.1) are defined and bounded on* $[0,+\infty)$. *Moreover, if* **(H2)** *holds and* $x(t)$ *is a non-oscillatory solution of (2.1), then* $x(t)\to 0$ *as* $t\to+\infty$.

Proof. The first statement follows from the techniques in [65]. Assume now **(H2)**, and consider a non-oscillatory solution $x(t)$ of (2.1). If $x(t)>0$ for $t\geq t_1$, for some $t_1\geq 0$, then from **(H3')** we have $f(t,x_t)\leq 0$ for $t\geq t_1$, hence $x(t)$ is non-increasing on $[t_1,+\infty)$. Consequently, there is $c\geq 0$ such that $x(t)\to c$ as $t\to+\infty$. From the integral representation of solutions of (2.1) we have $x(t)=x(t_1)+\int_{t_1}^{t}f(s,x_s)ds$, $t\geq t_1$, and from **(H2)** we conclude that $c=0$. The case $x(t)<0$ for $t\geq t_2$, with t_2 large, is treated in a similar way. ∎

To deal with the oscillatory solutions, we need the following lemma.

Lemma 2.2. *Assume* **(H3')** *and that* $\sup_{t\geq\tau}\int_{t-\tau}^{t}\lambda_i(s)ds$, $i=1,2$ *are finite. Assume also that the function* h *in* **(H3')** *is non-increasing. Let* $x(t)$ *be an oscillatory solution of (2.1) and* $u,v\geq 0$ *be defined as*

$$u:=\limsup_{t\to+\infty}x(t),\qquad -v:=\liminf_{t\to+\infty}x(t). \tag{2.8}$$

Then, for any $T\geq\tau$ *and* $\alpha_i:=\alpha_i(T)=\sup_{t\geq T}\int_{t-\tau}^{t}\lambda_i(s)ds$, $i=1,2$, *we have*

$$u\leq h(-v)\max\{1/2,\alpha_2-1/2\},\qquad u\leq h(-v)\alpha_2^2/2 \tag{2.9}$$

and

$$-v\geq h(u)\max\{1/2,\alpha_1-1/2\},\qquad -v\geq h(u)\alpha_1^2/2. \tag{2.10}$$

Proof. Let $x(t)$ an oscillatory solution of (2.1). From Lemma 2.1, consider $u,v\geq 0$ defined by (2.8).

Fix $T\geq\tau$ and $\epsilon>0$. Then, there is $T_0\geq T$ such that

$$-v_\epsilon:=-(v+\epsilon)\leq x_t\leq u+\epsilon:=u_\epsilon,\qquad \text{for } t\geq T_0.$$

If $u=0$, clearly (2.9) holds. Otherwise, there is a sequence $(x(t_n))_{n\in\mathbb{N}}$ of local maxima such that $x(t_n)>0$, $t_n\to+\infty$, $t_n-2\tau\geq T_0$, and $x(t_n)\to u$ as $n\to+\infty$. We may assume that $x(t)<x(t_n)$ for $t_n-t>0$ small, for all $n\in\mathbb{N}$. As in Lemma 3.2 of [11], we deduce that, for each $n\in\mathbb{N}$, there is $\xi_n\in[t_n-\tau,t_n)$ such that

$x(\xi_n) = 0$ and $x(t) > 0$ for $t \in (\xi_n, t_n]$. Assume **(H3')**, with h non-increasing. Then, for $t \geq T_0$, we have $-x_t \leq v_\epsilon$, hence

$$\dot{x}(t) = f(t, x_t) \leq \lambda_2(t)h(-\mathcal{M}(-x_t)) \leq \lambda_2(t)h(-v_\epsilon), \tag{2.11}$$

and, for each $n \in \mathbb{N}$, we get

$$-x(t) \leq h(-v_\epsilon)\int_t^{\xi_n} \lambda_2(s)ds, \qquad t \in [T_0, \xi_n].$$

Consequently, for $t \in [\xi_n, t_n]$ and $\theta \in [-\tau, 0]$ we have $x(t+\theta) > 0$ if $t + \theta \in (\xi_n, t_n]$, and $x(t+\theta) \geq -h(-v_\epsilon)\int_{t-\tau}^{\xi_n} \lambda_2(s)ds$ if $t+\theta \leq \xi_n$. Therefore, $\mathcal{M}(-x_t) \leq h(-v_\epsilon)\int_{t-\tau}^{\xi_n} \lambda_2(s)ds$ and **(H3')** yields

$$\dot{x}(t) \leq \lambda_2(t)h\left(-h(-v_\epsilon)\int_{t-\tau}^{\xi_n} \lambda_2(s)ds\right) \leq \lambda_2(t)h(-v_\epsilon)\int_{t-\tau}^{\xi_n} \lambda_2(s)ds, \tag{2.12}$$

for $\xi_n \leq t \leq t_n$. From (2.11) and (2.12) we have

$$\dot{x}(t) \leq h(-v_\epsilon)\min\left\{\lambda_2(t), \lambda_2(t)\int_{t-\tau}^{\xi_n} \lambda_2(s)ds\right\}, \qquad \xi_n \leq t \leq t_n. \tag{2.13}$$

Let $\Lambda_n := \displaystyle\int_{\xi_n}^{t_n} \lambda_2(s)ds$. From (2.12),

$$\begin{aligned}
x(t_n) &= \int_{\xi_n}^{t_n} \dot{x}(t)dt \leq h(-v_\epsilon)\int_{\xi_n}^{t_n} \lambda_2(t)\left(\int_{t-\tau}^{\xi_n} \lambda_2(s)ds\right)dt \\
&= h(-v_\epsilon)\int_{\xi_n}^{t_n} \lambda_2(t)\left[\int_{t-\tau}^{t} \lambda_2(s)ds - \int_{\xi_n}^{t} \lambda_2(s)ds\right]dt \\
&\leq h(-v_\epsilon)\left(\alpha_2\int_{\xi_n}^{t_n} \lambda_2(t)dt - \int_{\xi_n}^{t_n} \lambda_2(t)\int_{\xi_n}^{t} \lambda_2(s)dsdt\right) \\
&\leq h(-v_\epsilon)[\alpha_2\Lambda_n - \Lambda_n^2/2].
\end{aligned} \tag{2.14}$$

Since $\Lambda_n \leq \alpha_2$ and the function $x \mapsto \alpha_2 x - x^2/2$ is increasing on $(-\infty, \alpha_2]$, we obtain

$$x(t_n) \leq h(-v_\epsilon)\alpha_2^2/2.$$

By letting $n \to +\infty$ and $\epsilon \to 0^+$, the above estimate leads to

$$u \leq h(-v)\alpha_2^2/2. \tag{2.15}$$

We now consider separately the cases $\Lambda_n \leq 1$ and $\Lambda_n > 1$, and adjust the arguments in [52].

If $\Lambda_n \le 1$, then $\Lambda_n \le \max(1, \alpha_2)$ and since $\alpha_2 x - x^2/2 \le \max(1,\alpha_2)x - x^2/2 \le \max(1, \alpha_2) - 1/2$ for $x \in (0, 1]$, from (2.14) we obtain

$$x(t_n) \le h(-v_\epsilon)(\max(1, \alpha_2) - 1/2) = h(-v_\epsilon) \max\{1/2, \alpha_2 - 1/2\}. \tag{2.16}$$

If $\Lambda_n > 1$, then there is $\eta_n \in (\xi_n, t_n)$ such that $\int_{\eta_n}^{t_n} \lambda_2(s)ds = 1$. From (2.13) we have

$$\begin{aligned}
x(t_n) &\le h(-v_\epsilon)\left\{\int_{\xi_n}^{\eta_n} \lambda_2(t)dt + \int_{\eta_n}^{t_n} \lambda_2(t)\left(\int_{t-\tau}^{\xi_n} \lambda_2(s)ds\right)dt\right\} \\
&= h(-v_\epsilon)\left\{\int_{\xi_n}^{\eta_n} \lambda_2(t)dt + \int_{\eta_n}^{t_n} \lambda_2(t)\left(\int_{t-\tau}^{\eta_n} \lambda_2(s)ds - \int_{\xi_n}^{\eta_n} \lambda_2(s)ds\right)dt\right\} \\
&= h(-v_\epsilon)\int_{\eta_n}^{t_n} \lambda_2(t)\left(\int_{t-\tau}^{\eta_n} \lambda_2(s)ds\right)dt \\
&= h(-v_\epsilon)\int_{\eta_n}^{t_n} \lambda_2(t)\left(\int_{t-\tau}^{t} \lambda_2(s)ds - \int_{\eta_n}^{t} \lambda_2(s)ds\right)dt \\
&\le h(-v_\epsilon)\left[\alpha_2 - \frac{1}{2}\left(\int_{\eta_n}^{t_n} \lambda_2(s)ds\right)^2\right] = h(-v_\epsilon)\left(\alpha_2 - \frac{1}{2}\right).
\end{aligned} \tag{2.17}$$

From (2.16) and (2.17), by letting $n \to +\infty$ and $\epsilon \to 0^+$, we conclude

$$u \le h(-v) \max\{1/2, \alpha_2 - 1/2\}. \tag{2.18}$$

Thus, from (2.15) and (2.18) we obtain (2.9).

The proof of the estimates in (2.10) follows using arguments similar to the ones above for the proof of (2.9), by considering a sequence $(x(s_n))_{n\in\mathbb{N}}$ of local minima of $x(t)$, and is omitted. ■

Theorem 2.3. *Assume* **(H2)**, **(H3')** *and* **(H4)**. *Then the zero solution of (2.1) is globally attractive.*

Proof. From Lemma 2.1, it is sufficient to consider the case of an oscillatory solution $x(t)$ of (2.1).

Let $x(t)$ be an oscillatory solution of (2.1). From Lemma 2.1, $x(t)$ is defined and bounded on $[-\tau, +\infty)$, so that we can define $u, v \in [0, +\infty)$ as in (2.8). Suppose that $u \ge v$ (the case $v \ge u$ is analogous). We have to show that $u = v = 0$.

For h as in **(H3')**, we define $\hat{h} : \mathbb{R} \to \mathbb{R}$ by $\hat{h}(x) = \max_{y\in[x,0]} h(y)$ if $x < 0$, $\hat{h}(0) = 0$ and $\hat{h}(x) = \min_{y\in[0,x]} h(y)$ if $x > 0$. Clearly, $\hat{h}$ is non-increasing, and

(2.6) and (2.7) are satisfied with h replaced by $\hat{h}$. Without loss of generality, in what follows we can assume that the function h in **(H3')** is non-increasing.

The first inequality in (2.10) and the fact that h is a non-increasing function satisfying (2.6), imply $h(-v) \leq -h(u)\max\{1/2, \alpha_1 - 1/2\}$ and from the first inequality in (2.9) we conclude that $u \leq -h(u)M(\alpha_1, \alpha_2)$, where

$$M(\alpha_1, \alpha_2) := \max\{1/2, \alpha_1 - 1/2\}\max\{1/2, \alpha_2 - 1/2\}. \tag{2.19}$$

Assume that $\alpha_1, \alpha_2 \leq 5/2$. If $\alpha_1 \leq 1$ or $\alpha_2 \leq 1$, then $M(\alpha_1, \alpha_2) \leq 1$, and if $1 \leq \alpha_1, \alpha_2 \leq 5/2$, then $M(\alpha_1, \alpha_2) = \Gamma(\alpha_1, \alpha_2) \leq 1$. Hence we conclude that $u \leq -h(u)$. If $u > 0$, from (2.6) we have $u < u$, which is a contradiction, therefore $u = 0$.

We now assume $\alpha_1 > 5/2$ (the situation $\alpha_2 > 5/2$ is analogous). From the second inequality in (2.9), the first in (2.10), and (2.6) we obtain

$$u \leq -h(u)(\alpha_1 - 1/2)\alpha_2^2/2 = -h(u)\Gamma(\alpha_1, \alpha_2) \leq -h(u),$$

and again we conclude that $u = 0$.

Since $u = 0$ and $0 \leq v \leq u$, thus also $v = 0$, and finally we conclude that $x(t) \to 0$ as $t \to +\infty$. ■

As a first corollary we have:

Corollary 2.4. *Assume* **(H2)**, **(H3')**, *and that for some* $T \geq \tau$ *and* $\alpha_i := \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s)ds$, $i = 1, 2$, *we have either*

$$\max\{1/2, \alpha_1 - 1/2\}\max\{1/2, \alpha_2 - 1/2\} \leq 1, \tag{2.20}$$

or

$$\alpha_1\alpha_2 \leq 9/4. \tag{2.21}$$

Then the zero solution of (2.1) is globally attractive.

Proof. From the above theorem, it is sufficient to prove that $\Gamma(\alpha_1, \alpha_2) \leq 1$ if either (2.20) or (2.21) holds.

Assuming (2.20), then $\alpha_1, \alpha_2 \leq 5/2$, and consequently $\Gamma(\alpha_1, \alpha_2) \leq M(\alpha_1, \alpha_2) \leq 1$, for $M(\alpha_1, \alpha_2)$ as in (2.19). Now assume that $\alpha_1\alpha_2 \leq 9/4$. For $\alpha_1, \alpha_2 \leq 5/2$, then necessarily (2.20) holds and we get $\Gamma(\alpha_1, \alpha_2) \leq 1$. In fact, if $\alpha_1, \alpha_2 \leq 3/2$ then $M(\alpha_1, \alpha_2) \leq 1$, so we may consider, e.g., the case $\alpha_1 > 3/2$ and $\alpha_2 < 3/2$. For $\alpha_1 \in (3/2, 5/2]$ and $\alpha_2 \leq 1$, we obtain $M(\alpha_1, \alpha_2) = (\alpha_1 - 1/2)/2 \leq 1$. If $\alpha_1 \in (3/2, 5/2]$ and $\alpha_2 > 1$, then $\alpha_2 \in (1, 9/(4\alpha_1)]$, and we get

$$\begin{aligned} M(\alpha_1, \alpha_2) &= \left(\alpha_1 - \frac{1}{2}\right)\left(\alpha_2 - \frac{1}{2}\right) \\ &\leq \frac{(2\alpha_1 - 1)(9 - 2\alpha_1)}{8\alpha_1} = -\frac{(2\alpha_1 - 3)^2}{8\alpha_1} + 1 < 1. \end{aligned}$$

Now, for $\alpha_1\alpha_2 \le 9/4$ with $\alpha_1 > 5/2$, we have $\alpha_2 \le 9/(4\alpha_1)$ and

$$\Gamma(\alpha_1,\alpha_2) - 1 = \left(\alpha_1 - \frac{1}{2}\right)\frac{\alpha_2^2}{2} - 1 \le \frac{1}{64\alpha_1^2}(-64\alpha_1^2 + 162\alpha_1 - 81) < 0.$$

Similarly, if $\alpha_1\alpha_2 \le 9/4$ with $\alpha_2 > 5/2$, we obtain $\Gamma(\alpha_1,\alpha_2) = (\alpha_2 - 1/2)\alpha_1^2/2 < 1$. ■

From the above proofs, it is clear that Theorem 2.3 holds if in **(H3')** one replaces $|h(x)| < |x|$ for $x \neq 0$ by $|h(x)| \le |x|$, provided that $\Gamma(\alpha_1,\alpha_2) < 1$. Hence, with $h(x) = -x$ in (2.7), a generalization of Yoneyama's classical result [65] is obtained as follows:

Corollary 2.5. *Assume* **(H2)** *and*

(H3*) *there are piecewise continuous functions* $\lambda_1, \lambda_2 : [0,+\infty) \to [0,+\infty)$ *such that*

$$-\lambda_1(t)\mathcal{M}(\varphi) \le f(t,\varphi) \le \lambda_2(t)\mathcal{M}(-\varphi), \quad \text{for } t \ge 0, \varphi \in C.$$

If in addition **(H4)** *holds with* $\Gamma(\alpha_1,\alpha_2) < 1$*, then the zero solution of (2.1) is globally attractive. In particular, this is the case if* $\alpha_1\alpha_2 \le 9/4$*, with* $(\alpha_1,\alpha_2) \neq (3/2,3/2)$*.*

Proof. For $\alpha_1,\alpha_2 > 0$ with $(\alpha_1,\alpha_2) \neq (3/2,3/2)$, then $\alpha_1\alpha_2 \le 9/4$ implies $\Gamma(\alpha_1,\alpha_2) < 1$, proving the last statement of the corollary. ■

Remark 2.1 We remark that hypothesis **(H3*)** reads as **(H3)** for the case $b = 0$. We also note that, if $\alpha_1\alpha_2 \le 9/4$, it is necessary to impose the restriction $(\alpha_1,\alpha_2) \neq (3/2,3/2)$. In fact, as already remarked, even for $\lambda_1(t) \equiv \lambda_2(t) \equiv \alpha$, with $\alpha > 0$, there are counter-examples for which $\alpha\tau = 3/2$ and the trivial solution is not globally attractive, showing that condition $\alpha\tau < 3/2$ is sharp (see, e.g. [64]).

In comparison with results in recent literature, we emphasize that Corollary 2.5 was obtained in [67] under the 3/2-type condition

$$\min\{\alpha_1,\alpha_2\}\max\{\alpha_1^2,\alpha_2^2\} < (3/2)^3,$$

which is clearly stronger than $\alpha_1\alpha_2 < 9/4$.

For the case of a scalar FDE with one discrete delay

$$\dot{x}(t) = f(t, x(t-\tau)), \quad t \ge 0, \tag{2.22}$$

the next criterion generalizes the result by Matsunaga et al. [37], where only the particular case of equation (2.1) with $f(t,x) = \lambda(t)h(x)$ was considered. In fact, for the particular equation $\dot{x}(t) = \lambda(t)h(x(t))$, with $\lambda : [0,+\infty) \to [0,+\infty)$ and $h : \mathbb{R} \to \mathbb{R}$ continuous functions and h non-increasing, in [37] the authors showed the global attractivity of its zero solution under assumption (2.6) and the usual 3/2-type condition, $\sup_{t\geq\tau} \int_{t-\tau}^{t} \lambda(s)ds \leq 3/2$.

Corollary 2.6. *Let $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that there are piecewise continuous functions $\lambda_1, \lambda_2 : [0,+\infty) \to [0,+\infty)$ and a continuous function $h : \mathbb{R} \to \mathbb{R}$, with h satisfying (2.6) and*

$$\lambda_1(t) \min\{0, h(x)\} \leq f(t,x) \leq \lambda_2(t) \max\{0, h(x)\}, \quad t \geq 0, x \in \mathbb{R}. \tag{2.23}$$

If in addition **(H2)** *and* **(H4)** *are satisfied, then the zero solution of (2.22) is globally attractive.*

In section 2.3, we shall apply these results to some scalar delayed differential equations used in population dynamics. Nevertheless, a simple illustration of Corollary 2.5 is shown by the following example.

Let $a, b : [0,+\infty) \to [0,+\infty)$ be continuous functions with $\int_0^{+\infty} a(t)dt = +\infty$ or $\int_0^{+\infty} b(t)dt = +\infty$, and consider the equation

$$\dot{x}(t) = -\max\{a(t)x(t), b(t)x(t-\tau)\}, \quad t \geq 0. \tag{2.24}$$

Defining $f(t,\varphi) = -\max\{a(t)\varphi(0), b(t)\varphi(-\tau)\}$, it is clear that **(H3*)** is satisfied with $\lambda_1(t) = \max\{a(t), b(t)\}$ and $\lambda_2(t) = \min\{a(t), b(t)\}$. Let $\alpha_i = \alpha_i(T)$ be as in **(H4)**. If $\Gamma(\alpha_1, \alpha_2) < 1$, from Corollary 2.5 we conclude that $x = 0$ is a global attractor of all solutions of (2.24).

Now, we consider the case of the rational function $r(x)$ in **(H3)** with $b > 0$.

By the time scaling $t \mapsto \tau t$ in (2.1), we may assume that the time delay is $\tau = 1$. Also, the scaling $x \mapsto bx$ allows us to reduce this situation to the case $b = 1$. Hence, without loss of generality, in the following lemmas we take $\tau = 1$ and $b = 1$, so that $C = C([-1,0];\mathbb{R})$ and

$$r(x) = -\frac{x}{1+x}, \quad x > -1.$$

We know that r is a decreasing function with $\lim_{x\to -1^+} r(x) = +\infty$ and $\lim_{x\to+\infty} r(x) = -1$.

In the case $b > 0$, the restriction $\alpha_1 \leq \alpha_2$ in **(H4)** will be imposed to deduce the global attractivity of the zero solution of (2.1). By the change of variables $x \mapsto y = -x$, we may as well consider a function $f(t,\varphi)$ for which $g(t,\varphi) := -f(t,-\varphi)$

satisfies **(H1)-(H4)**. Clearly, in this situation one should take the restriction $\alpha_2 \leq \alpha_1$ in **(H4)**. In some sense, the need for a restriction on the relative sizes of α_1, α_2 is natural, since the two different functions $\lambda_1(t)$, $\lambda_2(t)$, together with $r(x)$, are taken to impose a boundedness condition on f, with different types of bounds on the left- and right-hand sides of zero.

The following lemma assures the boundedness of solutions of (2.1) and shows that all non-oscillatory solutions tend to zero.

Lemma 2.7. *[14] Assume* **(H1)**, **(H3)** *and that* $\sup_{t\geq 1}\int_{t-1}^{t}\lambda_1(s)ds < +\infty$. *Then, all solutions of (2.1) are defined and bounded on* $[0, +\infty)$. *Moreover, if* **(H2)** *holds and* $x(t)$ *is a non-oscillatory solution of (2.1), then* $x(t) \to 0$ *as* $t \to +\infty$.

Proof. The result was proven in Lemma 2.1 in [14]. ■

Consequently, to get the global attractivity of the zero solution of (2.1), we only need to study the oscillatory solutions of (2.1).

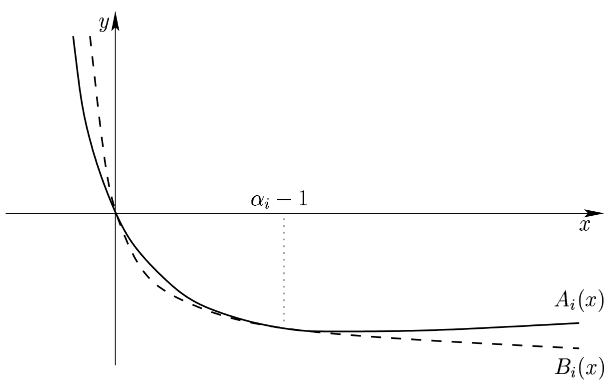
Following the work in [34], we define some auxiliary functions. For given $0 < \alpha_1 \leq \alpha_2$, we define the real functions $A_i : (-1, +\infty) \to \mathbb{R}$ and $B_i(-\frac{1}{\alpha_i+1}, +\infty) \to \mathbb{R}$, $i = 1, 2$, by

$$A_i(x) = x + \alpha_i r(x) + \frac{1}{r(x)}\int_x^0 r(t)dt, \quad \text{if } x \neq 0, x > -1, \quad A_i(0) = 0,$$

$$B_i(x) = \frac{1}{r(x)}\int_{-\alpha_i r(x)}^0 r(t)dt, \quad \text{if } x \neq 0, x > -\frac{1}{\alpha_i + 1}, \quad B_i(0) = 0.$$

Note that for $x \neq 0$ in domain of A_i, B_i, then

$$\begin{aligned} A_i(x) &= -1 + \alpha_i r(x) - \frac{1}{r(x)}\log(1 + x), \\ B_i(x) &= -\alpha_i - \frac{1}{r(x)}\log(1 - \alpha_i r(x)). \end{aligned} \tag{2.25}$$



The following properties can be easily checked and were given in [34]:

Lemma 2.8. *The functions A_i, B_i are differentiable, with $B_i'(x) < 0$ for all $x > -\frac{1}{\alpha_i+1}$ and $A_i'(x) < 0$ for $-1 < x < \alpha_i - 1$, $i = 1, 2$. Moreover, $A_i(\alpha_i - 1) = B_i(\alpha_i - 1)$, $A_i'(0) = \frac{1}{2} - \alpha_i$ and $A_i''(0) = 2\alpha_i - \frac{1}{3}$.*

From the above lemma, we conclude that B_i is decreasing on its domain and A_i is decreasing on $(-1, \alpha_i - 1)$, $i = 1, 2$.

For $\alpha_i > 1/2$, we consider also the auxiliary rational function

$$R_i(x) = A_i'(0)\frac{x}{1 - \frac{x}{\nu_i}}, \qquad x > \nu_i, \tag{2.26}$$

where $\nu_i := \frac{2A_i'(0)}{A_i''(0)} = -\frac{6\alpha_i - 3}{6\alpha_i - 1} < 0$. Note that $\nu_1 \geq \nu_2$ for $\alpha_1 \leq \alpha_2$.

Lemma 2.9. *For $\alpha_i > 1$, then $A_i(x) < R_i(x)$ for $x \in (\nu_i, 0)$ and $A_i(x) > R_i(x)$ for $x \in (0, \alpha_i - 1)$, $i = 1, 2$.*

Proof. See Lemma 3 in [34]. ∎

Lemma 2.10. *For $1 < \alpha_1 \leq \alpha_2$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where Γ is defined by (2.5), then $R_2(A_1(x)) \leq x$ for $0 \leq x < \alpha_1 - 1$.*

Proof. We have $R_1(\alpha_1 - 1) \geq \nu_1$ if and only if $(\alpha_1 - 3/2)(\alpha_1 - 1) \leq -\nu_1$. In particular, $R_1(\alpha_1 - 1) \geq \nu_1$ for $1 < \alpha_1 \leq 3/2$. From Lemma 2.9, and since R_1, R_2 are decreasing, we obtain

$$A_1(x) \geq R_1(x) > \nu_1 \geq \nu_2, \qquad 0 \leq x < \alpha_1 - 1,$$

thus also $R_2(A_1(x)) \leq R_2(R_1(x))$. Defining, for $0 \leq x < \alpha_1 - 1$,

$$\mathcal{R}(x) := R_2(R_1(x)) = \frac{-A_2'(0)(A_1'(0))^2 x}{-A_1'(0) + \left[\frac{A_1''(0)}{2} + \frac{A_2''(0)(A_1'(0))^2}{2A_2'(0)}\right] x},$$

we have $\mathcal{R}(x) = \frac{ax}{\beta + \gamma x}$, with $a = A_1'(0)A_2'(0)\nu_1\nu_2 > 0$, $\beta = \nu_1\nu_2 > 0$, and $\gamma = -(A_1'(0)\nu_1 + \nu_2) > 0$. Since

$$\mathcal{R}'(x) \leq \mathcal{R}'(0) = A_1'(0)A_2'(0) = \Gamma(\alpha_1, \alpha_2) \leq 1, \qquad x \geq 0,$$

we conclude that $R_2(A_1(x)) \leq \mathcal{R}(x) \leq x$, $0 \leq x < \alpha_1 - 1$. ∎

In the proof of the next lemma, we omit some trivial but long computations of derivatives of some polynomial functions, which are needed to study theirs signs on some interval. In any case, such computations can be easily checked with the help of a mathematical software.

Lemma 2.11. *For* $0 < \alpha_1 \le \alpha_2$ *such that* $\Gamma(\alpha_1, \alpha_2) = 1$*, then*

$$B_1(x) > \nu_2 \tag{2.27}$$

and

$$R_2(B_1(x)) \le x \quad \textit{for} \quad x \ge \max\{0, \alpha_1 - 1\}. \tag{2.28}$$

Proof. Fix $\alpha_1 \in (0, 3/2]$. For $\alpha_2 = \alpha_2(\alpha_1) > 0$ such that $\Gamma(\alpha_1, \alpha_2) = 1$, then $\nu_2 = \nu_2(\alpha_1) = -\frac{6}{6+\alpha_1^2}$ if $0 < \alpha_1 \le 1$, and $\nu_2 = \nu_2(\alpha_1) = -\frac{6}{2\alpha_1+5}$ if $1 < \alpha_1 \le 3/2$. On the other hand, since B_1 is decreasing, we have $B_1(x) > B_1(\infty) = -\alpha_1 + \log(\alpha_1 + 1)$ for $x > -\frac{1}{\alpha_1+1}$. Since the function $\alpha_1 \mapsto -\alpha_1 + \log(\alpha_1 + 1) - \nu_2(\alpha_1)$ is decreasing on $(0, 3/2]$ and positive at $\alpha_1 = 3/2$, we conclude that $B_1(x) > \nu_2$ for all $x > -\frac{1}{\alpha_1+1}$ and $0 < \alpha_1 \le 3/2$.

We now prove (2.28).

From Lemma 2.8 and 2.10, we have $A_1(\alpha_1 - 1) = B_1(\alpha_1 - 1)$ and we conclude that $R_2(B_1(\alpha_1 - 1)) \le \alpha_1 - 1$ if $\alpha_1 - 1 \ge 0$, thus the estimate (2.28) holds for $x = \max\{0, \alpha_1 - 1\}$. By using the definitions in (2.25) and (2.26), it is easy to see that $R_2(B_1(x)) \le x$ if and only if $F(x, \alpha_1) \le 0$, for F defined by

$$F(x, \alpha_1) = \left(1 + \frac{x}{\alpha_1(\frac{1}{2} - \alpha_2 + \frac{x}{\nu_2})}\right) \frac{\alpha_1 x}{1 + x} - \log\left(1 + \frac{\alpha_1 x}{1 + x}\right), \; x \ge \max\{0, \alpha_1 - 1\},$$

where $\alpha_2 = \alpha_2(\alpha_1)$ and $\nu_2 = \nu_2(\alpha_1)$ as above. Hence, to conclude (2.28), it is sufficient to show that $\frac{\partial F}{\partial x}(x, \alpha_1) \le 0$ for $x \ge \max\{0, \alpha_1 - 1\}$. We have

$$\frac{\partial F}{\partial x}(x, \alpha_1) = \frac{(ax^2 + bx + c)x}{4(1 + x)^2(\frac{1}{2} - \alpha_2 + \frac{x}{\nu_2})^2(1 + (1 + \alpha_1)x)},$$

where

$$a = a(\alpha_1) = (2 - 4\alpha_2 + 4/\nu_2)(1 + \alpha_1) + 4(\alpha_1/\nu_2)^2,$$

$$b = b(\alpha_1) = (2 - 4\alpha_2)(3 + 2\alpha_1) + 4(1 + \alpha_1^2 - 2\alpha_1^2\alpha_2)/\nu_2,$$

$$c = c(\alpha_1) = 4(1 - 2\alpha_2) + \alpha_1^2(1 - 4\alpha_2) + 4\alpha_1^2\alpha_2^2.$$

Case 1: $0 < \alpha_1 \le 1$. We have $c = 0$, $a = \frac{P_1(\alpha_1)}{9\alpha_1^2}$ and $b = \frac{2P_2(\alpha_1)}{\alpha_1^2}$, where

$$P_1(y) = y^6\left(y^2 + 12\right) + 6y^4(-y + 5) - 36\left(y^2 + 2\right)(y + 1),$$

$$P_2(y) = y^2\left(y^2 + 6\right) - 4(2y + 3).$$

By studying the signs of the derivatives of $P_1(y)$, $P_2(y)$, we can show that $P_1(y) < 0$, $P_2(y) < 0$ for $y \in (0,1)$, hence $a < 0$, $b < 0$, and consequently $\frac{\partial F}{\partial x}(x, \alpha_1) \leq 0$ for $x \geq 0$.

Case 2: $1 < \alpha_1 \leq 3/2$. In this case, we have $a = \frac{P_3(\alpha_1)}{9(2\alpha_1 - 1)}$, $b = \frac{2P_4(\alpha_1)}{3(2\alpha_1-1)}$ and $c = (\frac{4(\alpha_1-1)}{2\alpha_1-1})^2 > 0$, where

$$P_3(y) = 8y^5 + 36y^4 + 6y^3 - 97y^2 - 90y - 42, \qquad P_4(y) = 8y^3 + 16y^2 - 32y - 31.$$

Again, by studying the derivatives of $P_3(y)$, $P_4(y)$, we see that $a < 0$ and $b < 0$. To conclude that $\frac{\partial F}{\partial x}(x, \alpha_1) \leq 0$ for all $x > \alpha_1 - 1$, we need to show that $\alpha_1 - 1 \geq z_+(\alpha_1)$, where $z_+(\alpha_1) = \frac{b+\sqrt{b^2-4ac}}{2|a|}$ is the positive root of $ax^2 + bx + c$. But $\alpha_1 - 1 \geq z_+(\alpha_1)$ is equivalent to $P_5(\alpha_1) \leq 0$, where

$$P_5(y) = 16y^4(y+3) - 8y^2(11y+10) + 261y - 391.$$

By studying the sign of the derivatives of $P_5(y)$ and the position of its roots, one can see that $P_5(y) < 0$ for all $y \in (1, 3/2]$. This completes the proof. ■

Still following the ideas in [34], we now define $D_1 : [0, +\infty) \to \mathbb{R}$ by

$$D_1(x) = \begin{cases} A_1(x), & 0 \leq x < \alpha_1 - 1 \\ \\ B_1(x), & x \geq \max\{0, \alpha_1 - 1\} \end{cases},$$

so that $D_1 = B_1|_{[0,+\infty)}$ in the case $\alpha_1 \leq 1$. For $x \geq 0$, note that $x < \alpha_1 - 1$ is equivalent to $\alpha_1 r(x) < -x$. Since $\log x \geq x - 1$ for $x > 0$, from (2.25) we have

$$A_1(x) - B_1(x) \geq \alpha_1 - 1 + \alpha_1 r(x) + \frac{1}{r(x)} \left[\frac{1 - \alpha_1 r(x)}{1+x} - 1 \right] = 0, \; x > 0, \tag{2.29}$$

where the equality holds only if $x = \alpha_1 - 1$. For $0 < \alpha_1 \leq \alpha_2$ such that $\Gamma(\alpha_1, \alpha_2) = 1$, we therefore conclude that D_1 is continuous, decreasing and, from Lemmas 2.10 and 2.11,

$$R_2(D_1(x)) \leq x, \qquad x \geq 0. \tag{2.30}$$

A last preliminary lemma is established below.

Lemma 2.12. *Assume* **(H1)**,**(H3)** *with* $b > 0$, *and* **(H4)**. *Let* $x(t)$ *be an oscillatory solution of (2.1), and* $u, v \geq 0$ *be defined as*

$$u = \limsup_{t \to +\infty} x(t), \qquad -v = \liminf_{t \to +\infty} x(t). \tag{2.31}$$

Then we have

$$-v \geq B_1(u). \tag{2.32}$$

Moreover, if $\lambda_i(t) > 0$ *for* t *large and* $\alpha_i > 1$, $i = 1, 2$, *then*

$$-v \geq A_1(u) \text{ for } u < \alpha_1 - 1, \quad u \leq A_2(-v) \text{ for } v < 1. \tag{2.33}$$

Proof. From Lemma 2.7 $x(t)$ is bounded, hence we have $0 \leq u, v < +\infty$. Fix $\epsilon > 0$, and for T as in **(H4)** choose $T_0 \geq T$ such that

$$-v_\epsilon := -(v+\epsilon) \leq x(t) \leq u + \epsilon =: u_\epsilon, \quad t \geq T_0 - 2. \tag{2.34}$$

Now, consider a sequence $(x(s_n))_{n\in\mathbb{N}}$ of local minima $x(s_n) < 0$, $s_n \to +\infty$, $s_n - 2 \geq T_0$, and $x(s_n) \to -v$ as $n \to +\infty$. We may assume that s_n are chosen such that, for each $n \in \mathbb{N}$, $x(t) > x(s_n)$ for $s_n - t > 0$ small. As in the proof of Lemma 2.2 (see [11],[34]), for each $n \in \mathbb{N}$, we conclude that there exists $\eta_n \in [s_n - 1, s_n)$ such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n]$.

Define $\hat{\lambda}(t) = \alpha_1^{-1}\lambda_1(t)$. From (2.34) we have $\mathcal{M}(x_t) \leq u_\epsilon$ for $t \geq T_0 - 1$, where $\mathcal{M}$ is the Yorke's functional. Proceeding as in the proof of Theorem 2.7 of [14], using twice the first inequality in (2.2), we conclude that

$$x(s_n) \geq -\frac{1}{r(u_\epsilon)} \int_{\psi(s_n)}^{\psi(\eta_n)} r(s)ds,$$

where $\psi(t) = -\alpha_1 r(u_\epsilon)[1 - \int_{\eta_n}^t \hat{\lambda}(s)ds]$. Since $\psi(\eta_n) = -\alpha_1 r(u_\epsilon)$, $\psi(s_n) \geq 0$, and r is negative on $(0, +\infty)$, then

$$x(s_n) \geq -\frac{1}{r(u_\epsilon)} \int_0^{-\alpha_1 r(u_\epsilon)} r(s)ds = B_1(u_\epsilon).$$

By letting $n \to +\infty$ and $\epsilon \to 0^+$, we obtain the estimate (2.32).

Now, suppose that there exists $t_0 \geq T$ such that $\lambda_1(t) > 0$ for $t \geq t_0$. Arguing as in [14] and [34], consider the function $s_1 : [t_0, +\infty) \to [s_1(t_0), +\infty)$ defined by

$$s_1(t) = \frac{1}{\alpha_1}\int_0^t \lambda_1(s)ds, \quad t \geq t_0.$$

The function $s_1(t)$ is one-to-one and onto. Denoting by $t_1 = t_1(s)$ its inverse, we effect the change of variables $y(s) = x(t_1(s))$, $s \geq s_1(t_0)$. Equation (2.1) is transformed into an equation of the form

$$\dot{y}(s) = g_1(s, y_s), \quad s \geq s_1(t_0), \tag{2.35}$$

where g_1 satisfies the estimate (see [14], [34])

$$g_1(s, \varphi) \geq \alpha_1 r(\mathcal{M}(\varphi)), \quad s \geq s_1(t_0), \varphi \in C.$$

For $0 \leq u < \alpha_1 - 1$, then $\alpha_1 r(u) < -u$, and the estimate $-v \geq A_1(u)$ follows now from Lemma 4 of [34] applied to equation (2.35). Analogously, we consider the change $y(s) = x(t_2(s))$, where $t_2 = t_2(s)$ is the inverse of $s_2(t) = \frac{1}{\alpha_2}\int_0^t \lambda_2(s)ds$ for s large, leading to the equation $\dot{y}(s) = g_2(s, y_s)$, where g_2 satisfies

$$g_2(s,\varphi) \le \alpha_2 r(-\mathcal{M}(-\varphi)),$$

for s large and $\varphi \in C$ such that $\varphi > -1$. Note that $r(x)$ and $A_2(x)$ are defined only for $x > -1$ and we have $\alpha_2 r(-v) > v$ if $\alpha_2 > 1$ and $v < 1$. The proof of $u \le A_2(-v)$ is done in a similar way (see [14], [34] for more details) ■

Now we prove the main result for the situation $b > 0$ in (2.2).

Theorem 2.13. *Assume* **(H1)-(H4)**, *with $b > 0$ and $\lambda_i(t) > 0$ for t large, $i = 1, 2$. If $\alpha_1 \le \alpha_2$, then all solutions $x(t)$ of (2.1) are defined and bounded for $t \ge 0$ and satisfy $x(t) \to 0$ as $t \to +\infty$.*

Proof. As mentioned before, without loss of generality we can take $\tau = 1$ and $b = 1$. From Lemma 2.7, all non-oscillatory solutions tend to zero as $t \to +\infty$. Let $x(t)$ be an oscillatory solution of (2.1) and define $u, v \in \mathbb{R}_0^+$ as in (2.31). Replacing in **(H3)** α_2 by a constant $\hat{\alpha}_2 > \alpha_2$ if necessary, we may assume that $\Gamma(\alpha_1, \alpha_2) = 1$.

By the definition of D_1, if $u < \alpha_1 - 1$ then $D_1(u) = A_1(u)$, otherwise $D_1(u) = B_1(u)$, hence from (2.32) and (2.33) we have $-v \ge D_1(u)$. From (2.27) and (2.29) we get $-v \ge D_1(u) > \nu_2 > -1$. Since R_2 is decreasing, from (2.30) we now obtain

$$R_2(-v) \le R_2(D_1(u)) \le u. \tag{2.36}$$

If $v > 0$, then (2.33), (2.36), and Lemma 2.9 imply that

$$u \le A_2(-v) < R_2(-v) \le u,$$

which is a contradiction. Hence $v = 0$ and from (2.33) also $u = 0$. The proof is complete. ■

Corollary 2.14. *Assume* **(H1)-(H4)**, *with $b > 0$. If $\alpha_1 \le \alpha_2$ and $\Gamma(\alpha_1, \alpha_2) < 1$, then all solutions $x(t)$ of (2.1) are defined and bounded for $t \ge 0$ and satisfy $x(t) \to 0$ as $t \to +\infty$.*

Proof. If $\alpha_1 \le \alpha_2$ and $\Gamma(\alpha_1, \alpha_2) < 1$, we can find $\epsilon > 0$ such that **(H3)** and **(H4)** are fulfilled with $\lambda_i(t)$ replaced by $\hat{\lambda}_i(t) := \lambda_i(t) + \epsilon$, and the result is immediate from Theorem 2.13. ■

From Theorem 2.13 and Corollaries 2.5 and 2.14, we summarize the main results of this section as follows:

Theorem 2.15. *Assume* **(H1)**-**(H2)***, with* $\Gamma(\alpha_1, \alpha_2) < 1$ *for* Γ *as in (2.5). If* $b > 0$*, assume also that* $\alpha_1 \le \alpha_2$*. Then the zero solution of (2.1) is globally attractive. If* $b > 0$ *and* $\lambda_i(t) > 0$ *for* t *large,* $i = 1, 2$*, the same result holds for* $\Gamma(\alpha_1, \alpha_2) = 1$*.*

We recall that, as shown in proof of Corollary 2.4, we have $\Gamma(\alpha_1, \alpha_2) \le 1$ if either (2.20) or (2.21) holds; and that $\Gamma(\alpha_1, \alpha_2) < 1$ if (2.21) is satisfied with $(\alpha_1, \alpha_2) \ne (3/2, 3/2)$.

Remark 2.2 The present setting can be applied to equation (2.1) with time-dependent bounded discrete delays, $\dot{x}(t) = f_0(t, x(t - \tau_1(t)), \dots, x(t - \tau_n(t)))$, where $\tau_i : [0, +\infty) \to (0, +\infty)$ are continuous, $\tau_i(t) \le \tau$. In fact, for $f(t, \varphi) = f_0(t, \varphi(-\tau_1(t)), \dots, \varphi(-\tau_1(t))$ and $\tau(t) = \max\{\tau_i(t) : 1 \le i \le n\}$, $t \ge 0$, $\varphi \in C$, the results in this section are valid if we replace $\int_{t-\tau}^t \lambda_i(s)ds$ by $\int_{t-\tau(t)}^t \lambda_i(s)ds$, $i = 1, 2$, in hypothesis **(H4)**.

2.3 Scalar Population Models

Delayed functional differential equations are very useful in population dynamics modeling. Clearly, for single species, scalar equations are used. Consider the general scalar delayed population model of the form

$$\dot{y}(t) = y(t) f(t, y_t), \qquad t \ge 0, \tag{2.37}$$

where $f : [0, +\infty) \times C \to \mathbb{R}$ is a continuous function. Due to the biological interpretation of model (2.37), only positive solutions are of interest. Hence, we take admissible initial conditions

$$y_0 = \varphi, \quad \text{with} \quad \varphi \in C_0, \tag{2.38}$$

where C_α denotes the set

$$C_\alpha := \{\varphi \in C : \varphi(\theta) \ge \alpha \text{ for } \theta \in [-\tau, 0) \text{ and } \varphi(0) > \alpha\}, \qquad (\alpha \in \mathbb{R}). \tag{2.39}$$

Since $y(t, 0, \varphi) = \varphi(0) \exp\left(\int_0^t f(s, y_s)ds\right) > 0$, the solutions of initial value problems (2.37)-(2.38) are positive for $t > 0$ whenever they are defined.

Let $u(t)$ be a positive solution of (2.37) on $[-\tau, +\infty)$ whose stability we want to investigate ($u(t)$ could be a steady state or a periodic solution). The change $x(t) = y(t)/u(t) - 1$ transforms (2.37) into

$$\dot{x}(t) = (1 + x(t)) F(t, x_t), \qquad t \ge 0, \tag{2.40}$$

where $F(t, \varphi) = f(t, u_t(1 + \varphi)) - f(t, u_t)$, for which the set of admissible initial conditions is C_{-1}.

The goal is to apply the study in Section 2.2 to equations written in the form (2.40), improving recent stability results in the literature (see, e.g. [11], [14], [33], [34], [41], [51]).

For a given function $F : [0, +\infty) \times C_{-1} \to \mathbb{R}$ continuous, we assume hypotheses **(H1)**-**(H4)** restricted to C_{-1}, i.e., we suppose that **(H1)**-**(H4)** hold with $\varphi \in C$ replaced by $\varphi \in C_{-1}$. We note that if **(H3)** holds for $\varphi \in C_{-1}$ with $b < 1$, then $F(t, \varphi) \le \lambda_2(t) r(-1)$ for $t \ge 0$, $\varphi \in C_{-1}$, and consequently **(H1)** is fulfilled with $\beta(t) = \lambda_2(t)$ and $\eta(q) \equiv r(-1)$, $q \in \mathbb{R}$.

The following result gives us a global stability criterion for the zero solution of (2.40).

Theorem 2.16. *For $F : [0, +\infty) \times C_{-1} \to \mathbb{R}$ continuous, assume that hypotheses* **(H1)**-**(H4)** *with φ restricted to C_{-1} are satisfied. If $b \ne 1/2$, assume in addition that $\lambda_i(t) > 0$ for t large, $i = 1, 2$, and either*

(i) *$b > 1/2$ and $\alpha_1 \le \alpha_2$; or*

(ii) *$b < 1/2$ and $\alpha_2 \le \alpha_1$.*

Then, all solutions $x(t)$ of (2.40) with initial conditions in C_{-1} are defined for $t \ge 0$ and satisfy $x(t) \to 0$ as $t \to +\infty$.

Proof. We first suppose that $b \ge 1/2$. As in [14], the change of variables $y(t) = \log(1 + x(t))$, $t \ge 0$, transforms (2.40) into

$$\dot{y}(t) = f(t, y_t), \qquad t \ge 0, \tag{2.41}$$

where $f(t, \varphi) = F(t, e^{\varphi} - 1)$. For $\varphi \in C$, then $\psi = e^{\varphi} - 1 > -1$, i.e. $\psi \in C_{-1}$. Since F satisfies **(H3)** in the space C_{-1}, we have

$$\begin{aligned} &f(t, \varphi) \ge \lambda_1(t) r(\mathcal{M}(e^{\varphi} - 1)) = \lambda_1(t) r(e^{\mathcal{M}(\varphi)} - 1), \qquad t \ge 0, \varphi \in C; \\ &f(t, \varphi) \le \lambda_2(t) r(-\mathcal{M}(-e^{\varphi} + 1)) = \lambda_2(t) r(e^{-\mathcal{M}(-\varphi)} - 1), \\ &\qquad\qquad\qquad\qquad\qquad\qquad t \ge 0, \varphi \in C \text{ with } e^{\varphi} - 1 > -1/b. \end{aligned} \tag{2.42}$$

Case $b = 1/2$.

Define $h(x) = r(e^x - 1) = -2\left(1 - \frac{2}{e^x + 1}\right)$, $x \in \mathbb{R}$. Then h satisfies (2.6) and

$$\lambda_1(t) h(\mathcal{M}(\varphi)) \le f(t, \varphi) \le \lambda_2(t) h(-\mathcal{M}(-\varphi)), \qquad t \ge 0, \varphi \in C.$$

From Theorem 2.3, we conclude that the solutions $y(t)$ of (2.41) satisfy $y(t) \to 0$ as $t \to +\infty$.

Case $b > 1/2$.

Define $r_1(x) = \frac{-x}{1+(b-1/2)x}$. We can prove that

$$r(e^x - 1) \geq r_1(x) \quad \text{for all} \quad x \geq 0, \tag{2.43}$$
$$r(e^x - 1) \leq r_1(x) \quad \text{for all} \quad x \in (-1/(b-1/2), 0]. \tag{2.44}$$

In fact, (2.43) is equivalent to the inequality $w(x) := 1 + \frac{1}{2}x + e^x(\frac{1}{2}x - 1) \geq 0$, $x \geq 0$, which can be proven easily by studying the signs of $w'(x)$ and $w''(x)$. We can prove (2.44) analogously. Moreover, if $b > 1$, condition $x > -1/(b-1/2)$ implies $e^x - 1 > -1/b$. Hence, from (2.42), (2.43), and (2.44) we conclude that f satisfies **(H3)** with $r(x)$ replaced by $r_1(x)$. On the other hand, since F satisfies **(H1)** and **(H2)** for $\varphi \in C_{-1}$, it is clear that f satisfies **(H1)** and **(H2)** for $\varphi \in C$. Thus, for $\alpha_1 \leq \alpha_2$ in **(H4)**, from Theorem 2.13 we conclude that the zero solution is a global attractor of all solutions of (2.41).

Case $0 \leq b < 1/2$.

Again as in [14], the change of variables $z(t) = -\log(1 + x(t))$, $t \geq 0$, transforms the equation (2.40) into

$$\dot{z}(t) = g(t, z_t), \qquad t \geq 0, \tag{2.45}$$

where $g(t, \varphi) = -F(t, e^{-\varphi} - 1)$. We obtain

$$\begin{aligned} g(t,\varphi) &\leq \lambda_1(t)[-r(\mathcal{M}(e^{-\varphi} - 1))] = -\lambda_1(t) r(e^{\mathcal{M}(-\varphi)} - 1),\ t \geq 0, \varphi \in C; \\ g(t,\varphi) &\geq \lambda_2(t)[-r(-\mathcal{M}(-e^{-\varphi} + 1))] = -\lambda_2(t) r(e^{-\mathcal{M}(\varphi)} - 1),\ t \geq 0, \varphi \in C. \end{aligned} \tag{2.46}$$

Let $r_2(x) = \frac{-x}{1+(1/2-b)x}$. We now have $-r(e^{-x} - 1) \geq r_2(x)$ for $x \geq 0$ and $-r(e^{-x} - 1) \leq r_2(x)$ for $-1/(1/2 - b) < x \leq 0$, hence g satisfies **(H3)** restricted to C_{-1}, where (2.2) reads as

$$\lambda_2(t) r_2(\mathcal{M}(\varphi)) \leq g(t, \varphi) \leq \lambda_1(t) r_2(-\mathcal{M}(-\varphi)).$$

For $\alpha_2 \leq \alpha_1$ in **(H4)**, taking into account Theorem 2.13, we conclude that all solutions $z(t)$ of (2.45) satisfy $z(t) \to 0$ as $t \to +\infty$. ∎

Remark 2.3 If $b \neq 1/2$ and there are arbitrarily large zeros of $\lambda_1(t)$, $\lambda_2(t)$, from Theorem 2.14 we conclude that the statement in Theorem 2.16 is still valid if we further impose $\Gamma(\alpha_1, \alpha_2) < 1$.

Remark 2.4 Even in the situation $\lambda(t) := \lambda_1(t) = \lambda_2(t)$, $t \geq 0$, Theorem 2.16 slightly improves Theorem 3.2 in [14], where it was required the strict inequality $\alpha := \alpha_1 = \alpha_2 < 3/2$ if $b = 1/2$, instead of $\alpha \leq 3/2$. Therefore, all the criteria established in [14] for several population models can be improved at least for the case $b = 1/2$.

2.4 Examples

In this section, we apply the stability criterion given by Theorem 2.16 to two food-limited population models with delays. For each one, with different choices of functions $\lambda_1(t), \lambda_2(t)$, and $r(x)$ in (H3) we can obtain different stability criteria.

Example 2.1 We study the asymptotical behavior of positive solutions of the delay differential equation

$$\dot{N}(t) = \rho(t)N(t)\frac{K - \sum_{i=1}^n a_i N^p(t-\tau_i(t))}{K + \sum_{i=1}^n s_i(t)N^p(t-\tau_i(t))}, \quad t \geq 0, \tag{2.47}$$

where $n \in \mathbb{N}$, $a_i > 0$, $K > 0$, $p \geq 1$, and $\rho(t)$, $s_i(t)$, $\tau_i(t)$ are continuous functions with $0 \leq \tau_i(t) \leq \tau$, and $\rho(t), s_i(t) > 0$ for $t \geq 0$, $i = 1, \dots, n$. Equation (2.47) (with $n = 1$ or $n > 1$) has been studied by several authors (see [11], [14], [19], [21], [41], [50]).

We follow here the approach in [11]. For $a := \sum_{i=1}^n a_i$, let $1 + x(t) = (N(t)/N_*)^p$, where

$$N_* = \left(\frac{K}{a}\right)^{1/p},$$

is the unique positive equilibrium of (2.47), so that (2.47) becomes

$$\dot{x}(t) = -p\rho(t)(1+x(t))\frac{\sum_{i=1}^n a_i x(t-\tau_i(t))}{a + \sum_{i=1}^n s_i(t)[1 + x(t-\tau_i(t))]}, \quad t \geq 0. \tag{2.48}$$

This equation has the form (2.40), for F defined by

$$F(t,\varphi) = p\rho(t)f(t, \varphi(-\tau_1(t)), \dots, \varphi(-\tau_n(t))), \quad t \geq 0, \varphi \in C_{-1}, \tag{2.49}$$

with $f : [0, +\infty) \times [-1, +\infty)^n \to \mathbb{R}$ given by

$$f(t, x_1, \dots, x_n) = \frac{-\sum_{i=1}^n a_i x_i}{a + \sum_{i=1}^n s_i(t)(1 + x_i)}.$$

As a first criterion for the global attractivity of N_*, we have the following result:

Theorem 2.17. *Assume*

$$\int_0^{+\infty} \frac{\rho(t)}{1 + \sum_{i=1}^n s_i(t)} dt = +\infty, \tag{2.50}$$

and that $\Gamma(\alpha_1, \alpha_2) \leq 1$, *where* α_1, α_2 *are defined by*

$$\alpha_1 = \frac{p}{2}\sup_{t \geq T}\int_{t-\tau(t)}^t \frac{\rho(s)}{\underline{\sigma}(s)}ds, \quad \alpha_2 = p\sup_{t \geq T}\int_{t-\tau(t)}^t \frac{\rho(s)}{1 + \underline{\sigma}(s)}ds \tag{2.51}$$

for some $T > 0$ large, with

$$\underline{\sigma}(t) = \min\{1, \sigma(t)\} \quad \textit{for} \quad \sigma(t) = \min_{1\le i\le n}(s_i(t)/a_i),$$

and $\tau(t) = \max_{1\le i\le n} \tau_i(t)$ for $t \ge 0$.

Then, all solutions of (2.47) with initial conditions in C_0 tend to the positive equilibrium N_* as $t \to +\infty$. In particular, this result holds if in addition to (2.50) we have

$$p^2 \left(\int_{t-\tau(t)}^{t} \frac{\rho(s)}{\underline{\sigma}(s)} ds \right) \left(\int_{t-\tau(t)}^{t} \frac{\rho(s)}{1+\underline{\sigma}(s)} ds \right) \le 9/2 \quad \textit{for large } t \ge 0. \tag{2.52}$$

Proof. From (2.50), it follows that F satisfies **(H2)** restricted to C_{-1} (see [11], [14]). Set

$$r(x) = \frac{-x}{1+\frac{1}{2}x}, \qquad x \ge -1.$$

For given $t \ge 0$ and $\varphi \in C_{-1}$, denote $x_i := \varphi(-\tau_i(t))$ e $y := a^{-1}\sum_{i=1}^n a_i x_i$. Note that $y \ge -1$.

If $\mathcal{M}(-\varphi) = 0$ or $\sum_{i=1}^n a_i x_i \ge 0$, clearly $F(t,\varphi) \le 0$. Now, let $\mathcal{M}(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$. Then

$$f(t, x_1, \dots, x_n) \le \frac{-y}{1 + a^{-1}\underline{\sigma}(t)[a + \sum_{i=1}^n a_i x_i]} = \frac{-y}{1+\underline{\sigma}(t)(1+y)} \le \frac{r(y)}{1+\underline{\sigma}(t)}.$$

Since $y \ge -\mathcal{M}(-\varphi)$ and r is decreasing, we get

$$f(t, x_1, \dots, x_n) \le (1+\underline{\sigma}(t))^{-1} r(-\mathcal{M}(-\varphi)),$$

and hence the estimate

$$F(t,\varphi) \le (1+\underline{\sigma}(t))^{-1} p\rho(t) r(-\mathcal{M}(-\varphi)). \tag{2.53}$$

If $\mathcal{M}(\varphi) = 0$ or $\sum_{i=1}^n a_i x_i \le 0$, then $F(t,\varphi) \ge 0$. Suppose now that $\mathcal{M}(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$. Then, we have

$$f(t, x_1, \dots, x_n) \ge \frac{-y}{1 + a^{-1}\underline{\sigma}(t)[a + \sum_{i=1}^n a_i x_i]} = \frac{-y}{1+\underline{\sigma}(t)(1+y)} \ge \frac{r(y)}{2\underline{\sigma}(t)}$$

with $y \le \mathcal{M}(\varphi)$, hence

$$F(t,\varphi) \ge (2\underline{\sigma}(t))^{-1} p\rho(t) r(\mathcal{M}(\varphi)). \tag{2.54}$$

From (2.53) and (2.54), we conclude that $F : [0,+\infty) \times C_{-1} \to \mathbb{R}$ satisfies **(H3)** restricted to C_{-1} with $r(x)$ as above and $\lambda_1(t) = (2\underline{\sigma}(t))^{-1} p\rho(t)$, $\lambda_2(t) = (1 + \underline{\sigma}(t))^{-1} p\rho(t)$. Since the coefficient b in the rational function $r(x)$ is $b = 1/2 < 1$, then **(H3)** implies **(H1)**. For α_1, α_2 as in (2.51), hypothesis **(H4)** is satisfied. The conclusion follows from Theorem 2.16. ∎

Other criteria for the global attractivity of N_* are given below.

Theorem 2.18. *Assume (2.50) and*

$$\frac{p}{1+\sigma_0}\int_{t-\tau(t)}^{t}\rho(s)ds \le \frac{3}{2} \quad \textit{for large } t \ge 0, \tag{2.55}$$

where $\sigma_0 := \inf_{t\ge 0}\min_{1\le i\le n}(s_i(t)/a_i)$ *and* $\tau(t) = \max_{1\le i\le n}\tau_i(t)$ *for* $t \ge 0$*. Then all admissible solutions* $N(t)$ *of (2.47) satisfy* $N(t) \to N_*$ *as* $t \to +\infty$*.*

Proof. For σ_0 as above, set

$$r(x) = \frac{-x}{1+bx}, \qquad \text{where } b = \frac{\sigma_0}{1+\sigma_0}.$$

Again, for given $t \ge 0$ and $\varphi \in C_{-1}$, we consider $x_i := \varphi(-\tau_i(t))$ and $y := a^{-1}\sum_{i=1}^n a_i x_i$.

As in the above proof, only the cases $\mathcal{M}(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$, or $\mathcal{M}(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$ have to be addressed, since otherwise (2.2) is trivially satisfied.

Let $\mathcal{M}(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$. Then

$$f(t, x_1, \ldots, x_n) \le \frac{-y}{1 + a^{-1}\sigma_0[a + \sum_{i=1}^n a_i x_i]} = \frac{r(y)}{1+\sigma_0}.$$

Since $y \ge -\mathcal{M}(-\varphi)$ and r is decreasing, we get

$$f(t, x_1, \ldots, x_n) \le (1+\sigma_0)^{-1} r(-\mathcal{M}(-\varphi)),$$

and hence the estimate

$$F(t,\varphi) \le (1+\sigma_0)^{-1} p\rho(t) r(-\mathcal{M}(-\varphi)). \tag{2.56}$$

If $\mathcal{M}(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$, then we have

$$f(t, x_1, \ldots, x_n) \ge \frac{-y}{1 + a^{-1}\sigma_0[a + \sum_{i=1}^n a_i x_i]} = \frac{r(y)}{1+\sigma_0},$$

with $y \le \mathcal{M}(\varphi)$, hence

$$F(t,\varphi) \ge (1+\sigma_0)^{-1} p\rho(t) r(\mathcal{M}(\varphi)). \tag{2.57}$$

From (2.56) and (2.57), we conclude that $F : [0, +\infty) \times C_{-1} \to \mathbb{R}$ satisfies **(H3)**, restricted to C_{-1}, with $r(x)$ as above and $\lambda_1(t) = \lambda_2(t) = (1+\sigma_0)^{-1}p\rho(t)$. Since $b < 1$, then **(H3)** implies **(H1)**, and Theorem 2.16 yields the conclusion. ∎

Under additional conditions, different choices of $\lambda_1(t)$, $\lambda_2(t)$ in **(H3)** are possible, leading to better criteria.

Theorem 2.19. *Let* $\sigma(t) := \min_{1\le i\le n}(s_i(t)/a_i)$ *and* $\tau(t) := \max_{1\le i\le n}\tau_i(t)$ *for* $t \ge 0$. *In addition to (2.50), assume that one of the following conditions holds:*

(i) $\sigma^0 := \sup_{t\ge 0}\sigma(t) \le 1$ *and there is* $T \ge \tau$ *such that* $\Gamma(\alpha_1,\alpha_2) \le 1$, *where*

$$\alpha_1 = \frac{p\sigma^0}{1+\sigma^0}\sup_{t\ge T}\int_{t-\tau(t)}^{t}\frac{\rho(s)}{\sigma(s)}ds, \qquad \alpha_2 = p\sup_{t\ge T}\int_{t-\tau(t)}^{t}\frac{\rho(s)}{1+\sigma(s)}ds;$$

(ii) $\sigma_0 := \inf_{t\ge 0}\sigma(t) \ge 1$ *and there is* $T \ge \tau$ *such that* $\Gamma(\alpha_1,\alpha_2) \le 1$, *where*

$$\alpha_1 = p\sup_{t\ge T}\int_{t-\tau(t)}^{t}\frac{\rho(s)}{1+\sigma(s)}ds, \qquad \alpha_2 = \frac{p}{1+\sigma_0}\sup_{t\ge T}\int_{t-\tau(t)}^{t}\rho(s)ds.$$

Then, all positive solutions of (2.47) tend to the positive equilibrium N_* *as* $t \to +\infty$. *In particular, in both situations* **(i)** *and* **(ii)**, *this conclusion holds if (2.50) and* $\alpha_1\alpha_2 \le 9/4$.

Proof. For $0 < b < 1$, set

$$r_b(x) = \frac{-x}{1+bx}, \qquad \theta_b(t,x) = \frac{1+bx}{1+\sigma(t)(1+x)}, \qquad t \ge 0, x \ge -1.$$

Fix $\varphi \in C_{-1}$, $t \ge 0$, and denote $x_i := \varphi(-\tau_i(t))$, $y := a^{-1}\sum_{i=1}^n a_i x_i$. For $\mathcal{M}(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$, we have $-1 \le y \le 0$, and

$$f(t,x_1,\dots,x_n) \le \frac{-y}{1+\sigma(t)(1+y)} = r_b(y)\theta_b(t,y). \tag{2.58}$$

If $\mathcal{M}(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$, then $y \ge 0$ and

$$f(t,x_1,\dots,x_n) \ge \frac{-y}{1+\sigma(t)(1+y)} = r_b(y)\theta_b(t,y). \tag{2.59}$$

Note that $\sigma^0 \le 1$ if and only if $\sigma^0/(1+\sigma^0) \le 1/2$, and $\sigma_0 \ge 1$ if and only if $\sigma_0/(1+\sigma_0) \ge 1/2$. On the other hand, $\sup_{t\ge 0}\sigma(t)/(1+\sigma(t)) \le b$ implies that $y \mapsto \theta_b(t,y)$ is non-decreasing for all $t \ge 0$, and $\inf_{t\ge 0}\sigma(t)/(1+\sigma(t)) \ge b$ implies that $y \mapsto \theta_b(t,y)$ is non-increasing for all $t \ge 0$. For $\sigma^0 \le 1$, we choose $b = \sigma^0/(1+\sigma^0)$, and from (2.58) and (2.59) we therefore obtain

$$\lambda_1(t)r_b(\mathcal{M}(\varphi)) \le F(t,\varphi) \le \lambda_2(t)r_b(-\mathcal{M}(-\varphi)), \quad \text{for } t \ge 0, \varphi \in C_{-1}, \tag{2.60}$$

with $\lambda_1(t) = p\rho(t)\theta_b(t,+\infty)$ and $\lambda_2(t) = p\rho(t)\theta_b(t,0)$, i.e.,

$$\lambda_1(t) = \frac{p\sigma^0\rho(t)}{(1+\sigma^0)\sigma(t)}, \qquad \lambda_2(t) = \frac{p\rho(t)}{1+\sigma(t)}, \qquad t \ge 0.$$

In this case, $b \leq 1/2$ and $\lambda_1(t) \geq \lambda_2(t)$ for $t \geq 0$. For $\sigma_0 \geq 1$, choose $b = \sigma_0/(1+\sigma_0)$. Hence, (2.58) and (2.59) lead to (2.60), with

$$\lambda_1(t) = \frac{p\rho(t)}{1+\sigma(t)}, \qquad \lambda_2(t) = \frac{p\rho(t)}{1+\sigma_0}, \qquad t \geq 0.$$

For this situation, $b \geq 1/2$ and $\lambda_1(t) \leq \lambda_2(t)$ for $t \geq 0$. Invoking Theorem 2.16, the proof of the theorem is complete. ∎

We now related these results with known criteria established in the literature. In [11], Theorem 2.18 was proven with (2.55) replaced by $p\int_{t-\tau(t)}^{t} \rho(s)ds \leq \frac{3}{2}$ for large t. The more general case of equation (2.47) with possible unbounded delays was studied by Qian [41], who proved the global asymptotic stability of N_* assuming (2.50) and

$$\frac{p}{1+a^{-1}S_0} \sup_{t\geq\tau(t)} \int_{t-\tau(t)}^{t} \rho(s)ds \leq 1,$$

where $S_0 := \inf_{t\geq 0} \sum_{i=1}^{n} s_i(t)$. Clearly, $a^{-1}S(t) \geq \sigma(t)$. However, the above condition is stronger than (2.55) if

$$\frac{1+a^{-1}S_0}{1+\sigma_0} < \frac{3}{2}.$$

The case $n = 1$ of (2.47) reads as

$$\dot{N}(t) = \rho(t)N(t)\frac{K - aN^p(t-\tau(t))}{K + S(t)N^p(t-\tau(t))}, \qquad t \geq 0, \tag{2.61}$$

with $K > 0$, $p \geq 1$, and $\rho(t)$, $S(t)$, $\tau(t)$ are continuous and positive functions with $\tau(t) \leq \tau$. It has been studied by many authors (see [19], [21], [50] and references therein), since it has been proposed as an alternative to the delayed logistic equation (case $S(t) \equiv 0$ and $p = 1$) for a food-limited single population model. For (2.61), we have $\sigma(t) = a^{-1}S(t)$ and $\sigma_0 = a^{-1}\inf_{t\geq 0} S(t) = a^{-1}S_0$. With $a = 1$ and a single constant discrete delay $\tau(t) \equiv \tau$, So and Yu [50] established the uniform and asymptotic stability (but not the global attractivity) of the positive equilibrium N_* of (2.61) assuming (2.50) and

$$p \sup_{t\geq\tau} \int_{t-\tau}^{t} \frac{\rho(s)}{1+S(s)} ds < \frac{3}{2},$$

a condition less restrictive than (2.55). For (2.61), Theorem 2.18 was proven in [14] and [34], but the strict inequality was required in (2.55) if $S_0 := \inf_{t\geq 0} S(t) = a$, i.e., if $\sigma_0 = 1$.

Example 2.2 Consider the scalar FDE with one discrete delay proposed by Gopalsamy [19] and studied in [10] and [33],

$$\dot{N}(t) = \rho(t)N(t)\left[\frac{K - aN(t-\tau)}{K + \lambda(t)N(t-\tau)}\right]^{\alpha}, \qquad t \geq 0, \tag{2.62}$$

where $\rho, \lambda : [0, +\infty) \to (0, +\infty)$ are continuous, $a, K, \tau > 0$ and $\alpha \geq 1$ is the ratio of two odd integers. Note that for $\alpha = 1$ and $p = 1$, equations (2.61) and (2.62) coincide. As before, we only consider positive solutions, corresponding to initial conditions $\varphi \in C_0$. The unique positive equilibrium of (2.62) is $N_* = K/a$. As another illustration of Theorem 2.16, sufficient conditions for its global attractivity are established here, by arguing along the lines above for the study of the previous model (2.47).

Theorem 2.20. *Assume*

$$\int_0^{+\infty} \frac{\rho(s)}{(1+\lambda(s))^{\alpha}} ds = +\infty, \tag{2.63}$$

and that there is $T \geq \tau$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where

$$\alpha_1 = \frac{a^{\alpha}}{2} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{\underline{\lambda}(s)^{\alpha}} ds, \qquad \alpha_2 = \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{1 + a^{-1}\underline{\lambda}(s)} ds,$$

with $\underline{\lambda}(t) := \min\{a, \lambda(t)\}$, $t \geq 0$. Then $N_* = K/a$ is globally attractive (in the set of all positive solutions of (2.62)). In particular, this is the case if in addition to (2.63) we suppose that

$$a^{\alpha}\left(\int_{t-\tau}^{t} \frac{\rho(s)}{\underline{\lambda}(s)^{\alpha}} ds\right)\left(\int_{t-\tau}^{t} \frac{\rho(s)}{1 + a^{-1}\underline{\lambda}(s)} ds\right) \leq \frac{9}{2}, \qquad \textit{for large } t \geq 0.$$

Proof. Clearly, in (2.62) one may consider $a = 1$ by replacing K, $\lambda(t)$ by K/a, $\sigma(t) := \lambda(t)/a$, respectively. On the other hand, considering separately the cases $a \geq 1$ and $0 < a < 1$, one sees that (2.63) holds if and only if

$$\int_0^{+\infty} \frac{\rho(s)}{(1 + a^{-1}\lambda(s))^{\alpha}} ds = +\infty.$$

By replacing $\underline{\lambda}(t) := \min\{a, \lambda(t)\}$ by $\underline{\sigma}(t) = a^{-1}\underline{\lambda}(t) = \min\{1, \sigma(t)\}$, the study is therefore reduced to the case $a = 1$.

Let $a = 1$. After the change of variables $x(t) = \frac{N(t)}{K} - 1$, (2.62) becomes

$$\dot{x}(t) = -\rho(t)(1 + x(t))\left[\frac{x(t-\tau)}{1 + \sigma(t)(1 + x(t-\tau))}\right]^{\alpha}, \qquad t \geq 0. \tag{2.64}$$

This equation has the form (2.40), with $F(t,\varphi) = g(t,\varphi(-\tau))$, $t \geq 0$, $\varphi \in C_{-1}$, and g given by

$$g(t,x) = -\rho(t)\left[\frac{x}{1+\sigma(t)(1+x)}\right]^{\alpha}, \qquad t \geq 0, x \geq -1. \tag{2.65}$$

Condition (2.63) implies that F satisfies hypothesis **(H2)** restricted to C_{-1}. Now, define

$$r(x) = \frac{-x}{1+\frac{1}{2}x}, \qquad x \geq -1. \tag{2.66}$$

For $t \geq 0$ and $x \geq 0$, and since $-1 < r(x)/2 \leq 0$, we get

$$\begin{aligned} g(t,x) &\geq \rho(t)\left[\frac{-x}{1+\underline{\sigma}(t)(1+x)}\right]^{\alpha} \geq \frac{\rho(t)}{\underline{\sigma}(t)^{\alpha}}\left(\frac{-x}{2+x}\right)^{\alpha} \\ &= \frac{\rho(t)}{\underline{\sigma}(t)^{\alpha}}\left[\frac{r(x)}{2}\right]^{\alpha} \geq \frac{\rho(t)}{2\underline{\sigma}(t)^{\alpha}}r(x). \end{aligned}$$

For $t \geq 0$ and $-1 \leq x < 0$, and since $1+\underline{\sigma}(t)(1+x) \geq 1 \geq -x$, we obtain

$$g(t,x) \leq \rho(t)\left[\frac{-x}{1+\underline{\sigma}(t)(1+x)}\right]^{\alpha} \leq \rho(t)\frac{-x}{1+\underline{\sigma}(t)(1+x)} \leq \frac{\rho(t)}{1+\underline{\sigma}(t)}r(x).$$

Thus, F satisfies **(H3)** restricted to $\varphi \in C_{-1}$ with $r(x)$ as in (2.66), $\lambda_1(t) = \frac{\rho(t)}{2\underline{\sigma}(t)^{\alpha}}$, and $\lambda_2(t) = \frac{\rho(t)}{1+\underline{\sigma}(t)}$. ■

Remark 2.5 Liu [33] considered (2.62) with $K = a = 1$, and either $0 < \lambda(t) \leq 1$ for all $t \geq 0$, or $\lambda(t) \geq 1$ for all $t \geq 0$. With the notation above, these cases correspond to $\underline{\lambda}(t) \equiv \lambda(t)$, $\underline{\lambda}(t) \equiv a$ respectively. Liu proved the global attractivity of N_* assuming (2.63) and (for $K = a = 1$)

$$\limsup_{t\to+\infty}\int_{t-\tau}^{t}\frac{\rho(s)}{\lambda(s)^{\alpha}}ds \leq 3, \qquad \limsup_{t\to+\infty}\int_{t-\tau}^{t}\rho(s)ds \leq 3$$

if $\sup_{t\geq 0}\lambda(t) \leq 1$, $\inf_{t\geq 0}\lambda(t) \geq 1$, respectively. In this latter situation, Theorem 2.20 recovers the criterion in [33], whereas it improves it in the first case. The general situation, where $\lambda(t)$ has values smaller and greater than a (not addressed in [33]), was studied in [10] by effecting the change of variables $x(t) = (N(t)/N_*)^{\alpha} - 1$, so that (2.62) becomes (2.40) with

$$F(t,\varphi) = \alpha\rho(t)\left[\frac{1-(1+\varphi(-\tau))^{1/\alpha}}{1+\lambda(t)(1+\varphi(-\tau))^{1/\alpha}}\right]^{\alpha}, \qquad t \geq 0, \varphi \in C_{-1}. \tag{2.67}$$

In [10], the global attractivity of N_* was established under (2.63) and

$$\alpha \int_{t-\tau}^{t} \rho(s)ds \leq 3/2, \qquad \text{for large } t.$$

This result follows easily from our setting, since F defined by (2.67) satisfies **(H3)** in C_{-1}, with $r(x) = -x$ and $\lambda_1(t) = \lambda_2(t) = \rho(t)$, $t \geq 0$.

Other criteria for the global attractivity of N_* of (2.62) are given below.

Theorem 2.21. *Assume (2.63), and suppose that there is $T \geq \tau$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where*

$$\alpha_1 = \frac{1}{(\sigma_0)^{\alpha-1}(1+\sigma_0)} \sup_{t\geq T} \int_{t-\tau}^{t} \rho(s)ds, \qquad \alpha_2 = \frac{1}{(1+\sigma_0)} \sup_{t\geq T} \int_{t-\tau}^{t} \rho(s)ds,$$

and $\sigma_0 := a^{-1} \inf_{t\geq 0} \lambda(t)$. Then $N_* = K/a$ is globally atractive (in the set of all positive solutions of (2.62)). In particular, this is the case if in addition to (2.63) we suppose that

$$\int_{t-\tau}^{t} \rho(s)ds \leq \frac{3}{2}(\sigma_0)^{(\alpha-1)/2}(1+\sigma_0), \qquad \textit{for large } t \geq 0.$$

Proof. Arguing as above, in a similar way one proves that

$$g(t,x) \geq \lambda_1(t) r_b(x), \quad t \geq 0, x \geq 0,$$

$$g(t,x) \leq \lambda_2(t) r_b(x), \quad t \geq 0, -1 \leq x \leq 0,$$

where

$$\lambda_1(t) = \frac{\rho(t)}{(\sigma_0)^{\alpha-1}(1+\sigma_0)}, \qquad \lambda_2(t) = \frac{\rho(t)}{1+\sigma_0} \qquad \text{for } t \geq 0,$$

and

$$r_b(x) = \frac{-x}{1+bx}, \qquad \text{with } b = \frac{\sigma_0}{1+\sigma_0}.$$

If $\sigma_0 \leq 1$, then $b \leq 1/2$ and $\lambda_1(t) \geq \lambda_2(t)$, hence also $\alpha_1 \geq \alpha_2$; if $\sigma_0 \geq 1$, then $b \geq 1/2$ and $\lambda_1(t) \leq \lambda_2(t)$, thus $\alpha_1 \leq \alpha_2$. In both cases, Theorem 2.16 provides the conclusion. ∎

By using arguments similar to the ones used to prove Theorem 2.19, the above sufficient conditions for the global attractivity of N_* can still be weakened if either $0 < \lambda(t) \leq a$ for all $t \geq 0$, or $\lambda(t) \geq a$ for all $t \geq 0$. Clearly, the following result improves the work in [33], in both situations.

Theorem 2.22. *Assume (2.63). In addition, suppose that one of the following conditions holds:*

(i) *$\lambda(t) \geq a$ for all $t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, where, for some $T \geq \tau$ and $\sigma_0 := a^{-1} \inf_{t \geq 0} \lambda(t)$, α_1, α_2 are given by*

$$\begin{aligned}\alpha_1 &= a^{\alpha-1} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{(1+a^{-1}\lambda(s))\lambda(s)^{\alpha-1}} ds, \\ \alpha_2 &= \frac{1}{1+\sigma_0} \sup_{t \geq T} \int_{t-\tau}^{t} \rho(s) ds;\end{aligned} \tag{2.68}$$

(ii) *$\lambda(t) \leq a$ for all $t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, where, for some $T \geq \tau$ and $\sigma^0 := a^{-1} \sup_{t \geq 0} \lambda(t)$, α_1, α_2 are given by*

$$\alpha_1 = a^{\alpha} \frac{\sigma^0}{1+\sigma^0} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{\lambda(s)^{\alpha}} ds, \qquad \alpha_2 = \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{1+a^{-1}\lambda(s)} ds. \tag{2.69}$$

Then $N_* = K/a$ is globally attractive (in the set of all positive solutions of (2.62)). In particular, for both situations **(i)** *and* **(ii)***, this statement holds if (2.63) and $\alpha_1 \alpha_2 \leq 9/4$.*

Proof. Again we consider equation (2.64) obtained after scaling and translation of N_* to the origin, and reduce our study to the case $a = 1$ by considering $\sigma(t) := a^{-1}\lambda(t)$ instead of $\lambda(t)$. Let $F(t, \varphi) = g(t, \varphi(-\tau))$, $t \geq 0$, $\varphi \in C_{-1}$ for g as in (2.65).

Case 1: $\sigma(t) \geq 1$ for all $t \geq 0$. We have

$$\begin{aligned} g(t,x) &= \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{(1+\sigma(t))\sigma(t)^{-1} + x} \right)^{\alpha} \geq \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{(1+\sigma(t))\sigma(t)^{-1} + x} \right) \\ &\geq \frac{\rho(t)}{(1+\sigma(t))\sigma(t)^{\alpha-1}} \frac{-x}{1 + \frac{\sigma_0}{1+\sigma_0} x}, \qquad t \geq 0, x \geq 0. \end{aligned}$$

For $t \geq 0$ and $-1 \leq x \leq 0$, clearly $0 \leq -x \leq 1 + \sigma(t)(1+x)$, hence

$$g(t,x) \leq \rho(t) \frac{-x}{1+\sigma(t)(1+x)} \leq \frac{\rho(t)}{1+\sigma_0} \frac{-x}{1 + \frac{\sigma_0}{1+\sigma_0} x}, \qquad t \geq 0, -1 \leq x \leq 0.$$

We therefore conclude that F satisfies **(H3)** restricted to C_{-1}, where

$$\lambda_1(t) = \frac{\rho(t)}{(1+\sigma(t))\sigma(t)^{\alpha-1}}, \qquad \lambda_2(t) = \frac{\rho(t)}{1+\sigma_0}, \qquad t \geq 0,$$

and $r(x) = -\frac{x}{1+bx}$, $x \geq -1$, with

$$b := \frac{\sigma_0}{1+\sigma_0} \geq \frac{1}{2}.$$

In this situation, $\lambda_1(t) \le \lambda_2(t)$, thus $\alpha_1 \le \alpha_2$ for α_1, α_2 as in (2.68), and the conclusion follows from Theorem 2.16.

Case 2: $\sigma(t) \le 1$ for all $t \ge 0$. For $t \ge 0$ and $x \ge 0$, we have

$$\begin{aligned} g(t,x) &= \frac{\rho(t)}{\sigma(t)^\alpha}\left(\frac{-x}{\sigma(t)^{-1}+(1+x)}\right)^\alpha \ge \frac{\rho(t)}{\sigma(t)^\alpha}\left(\frac{-x}{(\sigma^0)^{-1}+(1+x)}\right)^\alpha \\ &\ge \frac{\rho(t)}{\sigma(t)^\alpha}\frac{-x}{(\sigma^0)^{-1}+(1+x)} = \frac{\rho(t)}{\sigma(t)^\alpha}\frac{\sigma^0}{1+\sigma^0}\frac{-x}{1+\frac{\sigma^0}{1+\sigma^0}x}. \end{aligned}$$

Let $t \ge 0$ and $-1 \le x \le 0$. Since $\alpha \ge 1$ and $1+\sigma(t)(1+x) \ge 1 \ge -x$, we have

$$g(t,x) \le \rho(t)\frac{-x}{1+\sigma(t)(1+x)} \le \frac{\rho(t)}{1+\sigma(t)}\frac{-x}{1+\frac{\sigma^0}{1+\sigma^0}x}.$$

This implies that F satisfies **(H3)** restricted to C_{-1}, where

$$\lambda_1(t) = \frac{\sigma^0}{1+\sigma^0}\frac{\rho(t)}{\sigma(t)^\alpha}, \qquad \lambda_2(t) = \frac{\rho(t)}{1+\sigma(t)}, \qquad t \ge 0,$$

and $r(x) = -\frac{x}{1+bx}$, $x \ge -1$, with

$$b := \frac{\sigma^0}{1+\sigma^0} \le \frac{1}{2}.$$

For α_1, α_2 as in (2.69), note that $\alpha_2 \le \alpha_1$. The result follows again by Theorem 2.16. ∎

Chapter 3

Stability for Lotka-Volterra Systems

In this chapter, we study the local and global stability of n-dimensional Lotka-Volterra systems with distributed delays and instantaneous negative feedbacks. For an introduction to such systems, see Section 1.4.

First, we obtain necessary and sufficient conditions, independent of the choice of the delays, for the exponential stability of an autonomous linear system of functional differential equations of the form $\dot{x}_i = -[b_i x_i(t) + L_i(x_t)]$, $i = 1, \dots, n$. It turns out that this system is the linearization about a positive equilibrium (if it exists) of a multiple species Lotka-Volterra type model. Afterwards, assuming there exists a positive equilibrium, we establish its global asymptotic stability under conditions slightly stronger than the ones required for the linear situation.

In Chapter 2, the global stability criteria were obtained by imposing constraints of the size of the delay, such as 3/2-type conditions. Here, to study the stability of n-dimensional systems, we assume that the so-called intraspecific competitions without delay $b_i x_i(t)$ dominate, in some sense, the delayed intraspecific competitions and interspecific interactions.

3.1 Notation and Definitions

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by $|x|_\infty$ or simply $|x|$ its supremum norm in $\mathbb{R}^n$, $|x|_\infty = \max\{|x_i| : i = 1, \dots, n\}$. If $d = (d_1, \dots, d_n) > 0$, we also consider the norm in $\mathbb{R}^n$ given by $|x|_d = \max\{d_i |x_i| : i = 1, \dots, n\}$. We use $\| \cdot \|_\infty$ (or simply $\| \cdot \|$), respectively $\| \cdot \|_d$, to denote the supremum norm in $C_n := C([-\tau, 0]; \mathbb{R}^n)$, $\tau > 0$, relative to the norm $| \cdot |_\infty$, respectively $| \cdot |_d$, in $\mathbb{R}^n$, that is, $\|\varphi\|_\infty = \max_{-\tau \le \theta \le 0} |\varphi(\theta)|_\infty$ and $\|\varphi\|_d = \max_{-\tau \le \theta \le 0} |\varphi(\theta)|_d$. For a bounded linear functional $L : C_n \to \mathbb{R}$, where C_n is equipped with the norm $\| \cdot \|_\infty$, respectively $\|\cdot\|_d$, we denote the usual operator norm by $\|\cdot\|$, respectively $\|\cdot\|_d$.

For $c \in \mathbb{R}^n$ $(n \geq 1)$, we use c to denote both the real vector and the constant function $\varphi(\theta) = c$ in C_n. For $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we denote by $a.\varphi$ the element in C_n, $a.\varphi := (a_1\varphi_1, \ldots, a_n\varphi_n)$. C_n is supposed to be partially ordered with

$$\varphi \geq \psi \text{ if and only if } \varphi_i(\theta) \geq \psi_i(\theta), \quad \theta \in [-\tau, 0], i = 1, \ldots, n.$$

In this chapter, we study the stability of a positive equilibrium of a multiple species Lotka-Volterra type model of the form

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij}\int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], \ i = 1, \ldots, n, \quad (3.1)$$

where $b_i, l_{ij} \in \mathbb{R}$, $r_i(t)$ are positive continuous functions and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions. In biological terms, only *positive* solutions of the Lotka-Volterra system (3.1) are meaningful. Therefore, we only consider solutions with initial conditions in $C_{\hat{0}} \subseteq C_n$, where

$$C_{\hat{0}} := \{(\varphi_1, \ldots, \varphi_n) \in C_n : \varphi_i(\theta) \geq 0, \text{ for } \theta \in [-\tau, 0), \varphi_i(0) > 0, i = 1, \ldots, n\}.$$

A positive equilibrium point x^* of (3.1) is said to be *globally asymptotically stable* (in the set of all positive solutions) if it is stable and is a global attractor of all positive solutions of (3.1).

A bounded linear operator $L : C_1 \to \mathbb{R}$ given by

$$L(\varphi) = l\int_{-\tau}^{0} \varphi(\theta)d\mu(\theta), \quad \varphi \in C_1,$$

for some $l \in \mathbb{R}$ and $\mu : [-\tau, 0] \to \mathbb{R}$ normalized bounded variation function, is said to be *monotone* (relative to the order in C_1 above defined) if μ is a non-decreasing function. If $l \geq 0$, respectively $l \leq 0$, then L is said to be *positive*, respectively *negative*; this means that $L(\varphi) \geq 0$ for all $\varphi \geq 0$, respectively $\varphi \leq 0$.

A real function $x : [0, +\infty) \to \mathbb{R}$ is said to be *eventually monotone* if there exists $t_0 > 0$ such that $x(t)$ is monotone on $[t_0, +\infty)$, otherwise it is said to be *not eventually monotone.*

3.2 Asymptotic Stability for Linear Functional Differential Equations

Let $C_n := C([-\tau, 0]; \mathbb{R}^n)$ be equipped with the supremum norm $\|\cdot\|_\infty$ or any equivalent norm. In the phase space C_n, consider an autonomous system of linear FDE's of the form

$$\dot{x}_i(t) = -[b_i x_i(t) + L_i(x_t)], \qquad i = 1, \ldots, n, \tag{3.2}$$

where $b_i \in \mathbb{R}$, $L_i : C_n \to \mathbb{R}$ are linear bounded operators, $i = 1, \ldots, n$, and, as usual, x_t denotes the function in C_n defined by $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$. Equivalently, one can write L_i as

$$L_i(\varphi) = \sum_{j=1}^{n} L_{ij}(\varphi_j), \qquad L_{ij}(\varphi_j) = l_{ij} \int_{-\tau}^{0} \varphi_j(\theta) d\eta_{ij}(\theta), \tag{3.3}$$

for $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_n$, for some $l_{ij} \in \mathbb{R}$ and some normalized functions of bounded variation η_{ij}, $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, so that (3.2) reads as

$$\dot{x}_i(t) = -\left[b_i x_i(t) + \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) \right], \qquad i = 1, \ldots, n. \tag{3.4}$$

Set $a_{ij} := L_{ij}(1)$. From (3.3), we have $a_{ij} = l_{ij}(\eta_{ij}(0) - \eta_{ij}(-\tau))$ and $|l_{ij}| = \|L_{ij}\|$. Let $B = diag(b_1, \ldots, b_n)$, $A = [a_{ij}]$, and $C = [l_{ij}]$, and define the matrices $M = B + A$ and $N = B + C$. In the sequel, consider also the matrices

$$\tilde{M} = B + \tilde{A}, \qquad \hat{N} = B + \hat{C},$$

where $\tilde{A} = [\tilde{a}_{ij}]$, $\hat{C} = [\hat{l}_{ij}]$, with $\tilde{a}_{ii} = a_{ii}$, $\tilde{a}_{ij} = -|a_{ij}|$ for $j \ne i$, and $\hat{l}_{ij} = -|l_{ij}|$ for $i, j = 1, \ldots, n$:

$$\tilde{M} = \begin{pmatrix} b_1 + a_{11} & -|a_{12}| & \cdots & -|a_{1n}| \\ & & \cdots & \\ -|a_{n1}| & -|a_{n2}| & \cdots & b_n + a_{nn} \end{pmatrix},$$
$$\hat{N} = \begin{pmatrix} b_1 - |l_{11}| & -|l_{12}| & \cdots & -|l_{1n}| \\ & & \cdots & \\ -|l_{n1}| & -|l_{n2}| & \cdots & b_n - |l_{nn}| \end{pmatrix}. \tag{3.5}$$

Note that all the off-diagonal entries of $\tilde{M}$ and $\hat{N}$ are non-positive, i.e. $\tilde{M}, \hat{N} \in Z_n$ (see Section 1.5).

For studying the stability of (3.2), we first translate an algebraic property of the matrix $\hat{N}$ into an analytical condition on the linear operators L_i.

Lemma 3.1. *For $d = (d_1, \ldots, d_n) > 0$, then*

$$\hat{N} d \ge 0 \quad \textit{if and only if} \quad \|L_i\|_{d^{-1}} \le d_i b_i, \; i = 1, \ldots, n.$$

Proof. Let $d = (d_1, \dots, n) > 0$. On the one hand, $\hat{N}d \geq 0$ is equivalent to

$$\sum_{j=1}^{n} d_j |l_{ij}| \leq d_i b_i, \qquad i = 1, \dots, n. \tag{3.6}$$

On the other hand, for L_i, L_{ij} as in (3.3), we have $\|L_{ij}\| = |l_{ij}|$ and

$$\|L_i\|_{d^{-1}} = \sum_{j=1}^{n} d_j |l_{ij}|.$$

■

Lemma 3.2. *Let $\tau > 0$, $b_i \in \mathbb{R}$ and $L_i : C_n \to \mathbb{R}$ be linear bounded operators, $i = 1, \dots, n$, such that*

(L1) *there is $d = (d_1, \dots, d_n) > 0$ such that $\|L_i\|_{d^{-1}} \leq d_i b_i$, $i = 1, \dots, n$.*

Then, all the characteristic roots λ of (3.2) have negative real parts, with the possible exception of $\lambda = 0$. If in addition $\det M \neq 0$*, then (3.2) is exponentially asymptotically stable.*

Proof. Write L_i as $L_i(\varphi) = \sum_{j=1}^{n} L_{ij}(\varphi_j)$, for $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$, with $L_{ij} : C_1 \to \mathbb{R}$ bounded linear operators, $i, j = 1, \dots, n$. The characteristic equation for (3.2) is

$$\det \Delta(\lambda) = 0, \qquad \text{for} \qquad \Delta(\lambda) = \lambda I + B + [(L_{ij}(e^{\lambda \cdot}))_{i,j=1}^{n}]. \tag{3.7}$$

Let $\lambda = \alpha + \beta i \neq 0$ be a root of (3.7), and consider $v \in \mathbb{C}^n$, $v \neq 0$, such that $\Delta(\lambda)v = 0$. For $d > 0$ as in **(L1)**, let $k \in \{1, \dots, n\}$ be such $|v|_{d^{-1}} = d_k^{-1}|v_k|$. We may suppose $v_k \in \mathbb{R}$, $v_k > 0$. We have

$$(\alpha + b_k)v_k = -ReL_k(e^{\lambda \cdot} v), \qquad \beta v_k = -ImL_k(e^{\lambda \cdot} v). \tag{3.8}$$

Suppose now that $\alpha \geq 0$. Since $\|L_k\|_{d^{-1}} \leq d_k b_k$, then

$$|L_k(e^{\lambda \cdot} v)| \leq d_k b_k \|e^{\lambda \cdot} v\|_{d^{-1}} \leq d_k b_k |v|_{d^{-1}} = b_k v_k,$$

hence

$$(ReL_k(e^{\lambda \cdot} v))^2 + (ImL_k(e^{\lambda \cdot} v))^2 \leq b_k^2 v_k^2. \tag{3.9}$$

If $ImL_k(e^{\lambda \cdot} v) = 0$, from (3.8) we have $\beta = 0$ and $\lambda = \alpha$, with

$$(\alpha + b_k)v_k = -L_k(e^{\alpha \cdot} v) \leq b_k v_k,$$

implying that $\alpha \le 0$, and therefore $\lambda = \alpha = 0$.

If $ImL_k(e^{\lambda\cdot}v) \neq 0$, from (3.8) and (3.9) we obtain

$$(\alpha + b_k)v_k = -ReL_k(e^{\lambda\cdot}v) < |L_k(e^{\lambda\cdot}v)| \le b_k v_k,$$

and we conclude that $\alpha < 0$, a contradiction. Thus, all the roots of (3.7) have negative real parts, with the possible exception of zero.

Finally, note that $\Delta(0) = B + A = M$. If $\det M \neq 0$, then $\lambda = 0$ is not a root of the characteristic equation (3.7). ■

Theorem 3.3. *Let $\tau > 0$, $b_i, l_{ij} \in \mathbb{R}$ and $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \dots, n$ be given. With the previous notation, suppose that $\det M \neq 0$ and $\hat{N}$ is an M-matrix. Then, (3.4) is exponentially asymptotically stable. Moreover, $b_i + a_{ii} > 0$, $i = 1, \dots, n$.*

Proof. Let $L_i(\varphi) = \sum_{j=1}^n L_{ij}(\varphi_j)$ be as in (3.3). We consider separately the cases of $\hat{N}$ irreducible and reducible.

Case 1.

If $\hat{N}$ is irreducible, then, from Theorem 1.20, there is $d = (d_1, \dots, d_n) > 0$ such that $\hat{N}d \ge 0$. In consequence of Lemma 3.1, hypothesis **(L1)** is satisfied, and the asymptotic stability of (3.4) follows from Lemma 3.2. From (3.6), we also have $b_i + a_{ii} \ge b_i - |l_{ii}| \ge 0$, $i = 1, \dots, n$, and if $b_i + a_{ii} = 0$ for some $i \in \{1, \dots, n\}$, then $0 = d_i(b_i - |l_{ii}|) = \sum_{j \neq i} d_j |l_{ij}|$, thus $l_{ij} = a_{ij} = 0$ for $1 \le j \le n$, $j \neq i$. This together with $b_i + a_{ii} = 0$ implies that the ith row of M is zero, which is not possible since $\det M \neq 0$.

Case 2.

If $\hat{N}$ is reducible, after a simultaneous permutation of rows and columns, which amounts to a permutation of the variables $x_1, \dots, x_n$ in (3.4), we may suppose that

$$\hat{N} = \begin{pmatrix} \hat{N}_{11} & \cdots & \hat{N}_{1l} \\ & \ddots & \\ 0 & \cdots & \hat{N}_{ll} \end{pmatrix}, \tag{3.10}$$

where $\hat{N}_{km}$ are $n_k \times n_m$ matrices, with $\hat{N}_{kk}$ irreducible or zero $n_k \times n_k$ blocks, $\sum_{k=1}^l n_k = n$. Accordingly to (3.10), we have

$$\begin{aligned} l_{ij} = 0, \text{ for } \quad & n_1 + \cdots + n_k + 1 \le i \le n_1 + \cdots + n_{k+1}, \\ & 1 \le j \le n_1 + \cdots + n_k, \quad 1 \le k \le l - 1. \end{aligned} \tag{3.11}$$

From (3.3) and (3.11), it follows that $[(L_{ij}(e^{\lambda\cdot}))_{i,j=1}^n]$ as well as the characteristic matrix $\Delta(\lambda)$ in (3.7) are also upper block triangular matrices. With the obvious notation, we write

$$\Delta(\lambda) = \lambda I + diag(B_1, \ldots, B_l) + \begin{pmatrix} \mathcal{L}_{11}(\lambda) & \cdots & \mathcal{L}_{1l}(\lambda) \\ & \ddots & \\ 0 & \cdots & \mathcal{L}_{ll}(\lambda) \end{pmatrix},$$

where $B_k = diag(b_{1+N(k)}, \ldots, b_{N(k+1)})$ for $N(k) = \sum_{m=1}^{k-1} n_m$ and $\mathcal{L}_{km}(\lambda)$ are $n_k \times n_m$ blocks.

Let $\lambda = \alpha + i\beta$ be a root of the characteristic equation (3.7). This means that $\det \Delta(\lambda) = 0$, or equivalently, $\det(\lambda I_{n_k} + B_k + \mathcal{L}_{kk}(\lambda)) = 0$ for some $k \in \{1, \ldots, l\}$ (where I_{n_k} is the identity matrix of dimension n_k).

If the block $\hat{N}_{kk}$ is irreducible, from Case 1 we conclude that $\alpha = Re\lambda < 0$. Now, suppose that $\hat{N}_{kk} = 0$ and $\alpha \geq 0$. Without loss of generality, we may assume that $k = 1$, so that

$$b_i = |l_{ii}|, \quad 1 \leq i \leq n_1 \quad \text{and} \quad l_{ij} = 0, \quad 1 \leq i, j \leq n_1, i \neq j.$$

The corresponding block $\lambda I_{n_1} + B_1 + \mathcal{L}_{11}(\lambda)$ of $\Delta(\lambda)$ is a diagonal matrix, with diagonal entries $\lambda + |l_{ii}| + L_{ii}(e^{\lambda \cdot})$, $1 \leq i \leq n_1$. Recall that

$$|L_{ii}(e^{\lambda \cdot})| = |l_{ii} \int_{-\tau}^{0} e^{\lambda\theta} d\eta_{ii}(\theta)| \leq |l_{ii}|.$$

If $\det(\lambda I_{n_1} + B_1 + \mathcal{L}_{11}(\lambda)) = 0$, then $\lambda + |l_{ii}| + L_{ii}(e^{\lambda \cdot}) = 0$ for some $i \in \{1, \ldots, n_1\}$, and in particular we get $\alpha \leq 0$. If $\alpha = 0$, then $|l_{ii}| + ReL_{ii}(e^{\lambda \cdot}) = 0$, implying that $\beta = -ImL_{ii}(e^{\lambda \cdot}) = 0$, which is a contradiction, since $\Delta(0) = M$ and $\det M \neq 0$ imply that $\lambda \neq 0$. We therefore conclude that (3.4) is exponentially asymptotically stable.

We show now that $b_i + a_{ii} > 0$, $i = 1, \ldots, n$, for a reducible matrix $\hat{N}$. Up to a permutation, $\hat{N}$ has the form (3.10). For irreducible diagonal blocks $\hat{N}_{kk}$, from Case 1 we derive that the diagonal entries $b_i + a_{ii}$ of M are positive. If the block $\hat{N}_{kk}$ is zero, then, for $1 + N(k) \leq i \leq N(k+1)$, we have $b_i = |l_{ii}|$ and the corresponding block M_{kk} of M is a diagonal matrix with $b_i + a_{ii}$ as diagonal entries. On the other hand, these diagonal entries $b_i + a_{ii}$ are non-zero, otherwise $\det M = 0$, hence they are positive. ∎

We have also shown that:

Corollary 3.4. *Let $\tau > 0$, $b_i, l_{ij} \in \mathbb{R}$ and $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \ldots, n$ be given.*

If $\hat{N}$ is an M-matrix, then all the roots λ of the characteristic equation (3.7) have negative real parts with the possible exception of $\lambda = 0$.

Remark 3.1 If $\hat{N}$ is an M-matrix, then $b_i - |l_{ii}| \geq 0$, $i = 1, \dots, n$. For $\hat{N}$ an irreducible M-matrix, one can even conclude that $b_i - |l_{ii}| > 0$, $i = 1, \dots, n$. In fact, under these assumptions, from Theorem 1.20 and Lemma 3.1 $\hat{N}$ satisfies **(L1)**; as in the proof of Theorem 3.3, $b_i - |l_{ii}| = 0$ implies now $l_{ij} = 0$ for $j = 1, \dots, n$, $j \neq i$, meaning that the ith-row of $\hat{N}$ is zero, which is not possible for an irreducible matrix.

Lemma 3.5. *Let $b_i > 0$, $l_{ij} \in \mathbb{R}$, $i, j = 1, \dots, n$ be given, and define N and $\hat{N}$ as above. If $\det N \neq 0$ and $\hat{N}$ is not an M-matriz, then there exist $\tau_{ij} \geq 0$ such that, for η_{ij} defined as the Heaviside functions $\eta_{ij}(\theta) = 0$ for $\theta \in [-\tau, -\tau_{ij}]$, $\eta_{ij}(\theta) = 1$ for $\theta \in (-\tau_{ij}, 0]$ and $\tau := \max\{\tau_{ij} : i, j = 1, \dots, n\}$, the characteristic equation for (3.4) has a root λ with $Re\lambda > 0$.*

Proof. The proof is given in Lemas 2.4 and 2.5 of [3] (see also [24]), and is omitted. ∎

Theorem 3.6. *Let $b_i > 0$, $l_{ij} \in \mathbb{R}$, $i, j = 1, \dots, n$, be given. Then, equation (3.4) is exponentially asymptotically stable for all the choices of $\tau > 0$ and sets of functions $\eta = (\eta_{ij}) \subseteq BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \dots, n$, and such that $\det M_\eta \neq 0$, if and only if $\hat{N}$ is an M-matrix. Here, M_η is defined by $M_\eta := B + [a_{ij}]$ for $a_{ij} = l_{ij}(\eta_{ij}(0) - \eta_{ij}(-\tau))$.*

Proof. For a given $\eta = (\eta_{ij}) \subseteq BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, then $M_\eta = \Delta(0)$, where $\det \Delta(\lambda) = 0$ is the characteristic equation (3.7), and hence $\det M_\eta \neq 0$ if and only if $\lambda = 0$ is not a root of (3.7). Also, for $\eta = (\eta_{ij})$ with η_{ij} as in the statement of Lemma 3.5, we have $M_\eta = N$. Now, the sufficiency is given by Theorem 3.3 and the necessity condition by Lemma 3.5. ∎

In applications, (3.2) often takes the form (3.4) with *non-decreasing* normalized bounded variation functions η_{ij}, i.e., L_{ij} are monotone operators. Clearly, in this case

$$\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1, \quad \|L_{ij}\| = |l_{ij}|, \quad a_{ij} = l_{ij}, \quad i, j = 1, \dots, n,$$

and in particular $M = N$. In this situation, the above theorem translates as:

Corollary 3.7. *Let $b_i > 0$, $l_{ij} \in \mathbb{R}$, $i, j = 1, \dots, n$, be given. Then, (3.4) is exponentially asymptotically stable for all the choices of $\tau > 0$ and non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1$, $i, j = 1, \dots, n$, if and only if $\det M \neq 0$ and $\hat{M}$ is an M-matrix. In particular, if $\det M \neq 0$ and $\hat{M}$ is an*

M-matrix, then the equation

$$\dot{x}_i(t) = -\left[b_i x_i(t) + \sum_{j=1}^{n} l_{ij} x_j(t-\tau_{ij})\right], \quad i = 1,\ldots,n \tag{3.12}$$

is exponentially asymptotically stable for all choices of discrete delays $\tau_{ij} \geq 0$, $i,j = 1,\ldots,n$.

Remark 3.2 Equation (3.12) was studied in [24] with the restriction $\tau_{ii} = 0$, and later in [3] without such constraint. With our notation, for (3.12) we have $M = N$, and $\tilde{M} = \hat{M}$ if all the diagonal delays are zero. In terms of the linear asymptotic stability, our Theorems 3.3 and 3.6 generalize the results in [3] and [24] to the situation with distributed delays. In fact, for (3.12) with $\tau_{ii} = 0$ Hofbauer and So [24] proved its asymptotic stability independently of the choices of delays $\tau_{ij} \geq 0$ if and only if $l_{ii} > 0$ $(1 \leq i \leq n)$, $\det M \neq 0$ and $\hat{M}$ is an M-matrix, while Campbell [3] proved the same result without the constraint $\tau_{ii} = 0$. We further note that So et al. [48] considered (3.12) for the "pure-delay-type" situation, i.e., with all $b_i = 0$. They established the asymptotic stability of (3.12) with $b_i = 0$ by imposing that $[\tilde{l}_{ij}]$, where $\tilde{l}_{ij} = -\frac{1+\frac{1}{9}l_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}{1-\frac{1}{9}l_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}|l_{ij}|$ for $j \neq i$, $\tilde{l}_{ii} = l_{ii}$, is a non-singular M-matrix, together with the 3/2-type condition $l_{ii}\tau_{ii} < 3/2$, $i = 1,\ldots,n$. For generalization of [48] to non-autonomous linear systems $\dot{x}_i(t) = -\sum_{j=1}^{n} l_{ij}(t)x_j(t-\tau_{ij}(t))$, $i = 1,\ldots,n$, see [49].

Example 3.1 Consider a *scalar* linear FDE on $C_1 = C([-\tau,0];\mathbb{R})$ of the form

$$\dot{x}(t) = -[b_0 x(t) + L_0(x_t)],$$

where $b_0 \in \mathbb{R}$ and $L_0 : C_1 \to \mathbb{R}$ is a linear bounded operator. We write $L_0(\varphi) = l_0 \int_{-\tau}^{0} \varphi(\theta)d\eta(\theta)$, for $|l_0| = \|L_0\|$ and some normalized bounded variation function $\eta : [-\tau,0] \to \mathbb{R}$. From Theorem 3.6, the following result is derived:

Corollary 3.8. *Let* $b_0, l_0 \in \mathbb{R}$ *be given. Then, the scalar linear FDE*

$$\dot{x}(t) = -\left[b_0 x(t) + l_0 \int_{-\tau}^{0} x(t+\theta)d\eta(\theta)\right] \tag{3.13}$$

is exponentially asymptotically stable for all choices of $\tau > 0$ *and* $\eta \in BV([-\tau,0];\mathbb{R})$ *with* $Var_{[-\tau,0]}\eta = 1$ *and* $b_0 + l_0 \int_{-\tau}^{0} d\eta(\theta) \neq 0$ *if and only if* $b_0 \geq |l_0|$.

Remark 3.3 The above result was established in [12], where the general case of a linear scalar FDE $\dot{x}(t) = -L(x_t)$, with $L : C_1 \to \mathbb{R}$ a linear bounded operator, was studied. Moreover, it was proven in [12] that if $L(1) > 0$ and L satisfies the hypothesis

(L1*) for all $\varphi \in C_1$ such that $|\varphi(\theta)| < \varphi(0)$ for $\theta \in [-\tau, 0)$, then $L(\varphi) > 0$,

then L has the form

$$L(\varphi) = b_0\varphi(0) + L_0(\varphi), \qquad \varphi \in C_1, \tag{3.14}$$

for some $b_0 > 0$ and (non-atomic at zero) linear bounded operator $L_0 : C_1 \to \mathbb{R}$, for which $b_0 \geq \|L_0\|$ and $b_0 + L_0(1) > 0$. The reverse is also true, see Theorem 1.13. In the next section, the relevance of assumption **(L1*)**, translated to the general framework of n-dimensional FDE's $\dot{x}(t) = f(t, x_t)$, will become clear.

Example 3.2 In biological models with two species, we have the situation $n = 2$. Assuming that $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are non-decreasing normalized functions, the equation (3.4) has the form

$$\begin{aligned} \dot{x}_1(t) &= -\left[b_1x_1(t) + a_{11}\int_{-\tau}^0 x_1(t+\theta)d\eta_{11}(\theta) + a_{12}\int_{-\tau}^0 x_2(t+\theta)d\eta_{12}(\theta)\right] \\ \dot{x}_2(t) &= -\left[b_2x_2(t) + a_{21}\int_{-\tau}^0 x_1(t+\theta)d\eta_{21}(\theta) + a_{22}\int_{-\tau}^0 x_2(t+\theta)d\eta_{22}(\theta)\right], \end{aligned} \tag{3.15}$$

with $M = N$ and

$$M = \begin{pmatrix} b_1 + a_{11} & a_{12} \\ a_{21} & b_2 + a_{22} \end{pmatrix}, \qquad \hat{M} = \begin{pmatrix} b_1 - |a_{11}| & -|a_{12}| \\ -|a_{21}| & b_2 - |a_{22}| \end{pmatrix}. \tag{3.16}$$

From Lemma 3.1, hypothesis **(L1)** is equivalent to saying that there is $d = (d_1, d_2) > 0$ such that $\hat{M}d \geq 0$. With some additional conditions, **(L1)** is equivalent to saying that $\hat{M}$ is an M-matrix.

Lemma 3.9. *Consider M, $\hat{M}$ as in (3.16), with $b_i \neq |a_{ii}|$, $i = 1, 2$. If $\det M \neq 0$ and $\hat{M}$ is an M-matrix, then there is $d = (d_1, d_2) > 0$ such that $\hat{M}d \geq 0$. Conversely, if there is $d = (d_1, d_2) > 0$ such that $\hat{M}d \geq 0$, then $\hat{M}$ is an M-matrix.*

Proof. If $\hat{M}$ is irreducible, the result follows from Theorem 1.20. If $\hat{M}$ is reducible, then either $a_{12} = 0$ or $a_{21} = 0$, and hence $\det \hat{M} = (b_1 - |a_{11}|)(b_2 - |a_{22}|) \neq 0$. Consequently, $\hat{M}$ is a non-singular M-matrix, thus there is $d = (d_1, d_2) > 0$ such that $\hat{M}d > 0$. The last statement follows from Theorem 1.19. ■

Since $b_i + a_{ii} > 0$ from Theorem 3.3, we note that in particular we have $b_i \neq |a_{ii}|$ if $a_{ii} \leq 0$, $i = 1, 2$. As a consequence of Corollary 3.7 and Lemma 3.9 we conclude the following:

Corollary 3.10. *Consider $b_i > 0$, $a_{ij} \in \mathbb{R}$ with $b_i \neq |a_{ii}|$, $i, j = 1, 2$. Then the system (3.15) is exponentially asymptotically stable for all choices of delays $\tau > 0$ and non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1$, $i = 1, 2$, if and only if* $\det M \neq 0$ *and there is $d = (d_1, d_2) > 0$ such that $\hat{M}d \geq 0$, for M, $\hat{M}$ as in (3.16).*

Example 3.3 Consider the following model for a ring of neurons with distributed delays

$$\dot{u}_i(t) = -b_i u_i(t) + \alpha_{ii} g_i(u_{t,i}) + \alpha_{i,i-1} g_{i-1}(u_{t,i-1}), \quad i = 1, \dots, n, \tag{3.17}$$

with the convention $i - 1 = n$ for $i = 1$, where $g_i : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ are smooth functions with $g_i(0) = 0$ and rescaled so that $g_i'(0)(1) = 1$, $i = 1, \dots, n$. The particular case of (3.17) with discrete delays,

$$\dot{u}_i(t) = -b_i u_i(t) + \alpha_{ii} g_i(u_i(t - \tau_i)) + \alpha_{i,i-1} g_{i-1}(u_{i-1}(t - \tau_{i-1})), \quad i = 1, \dots, n,$$

with $g_i : \mathbb{R} \to \mathbb{R}$, was studied in [3].

Next result generalizes Theorem 4.1 of [3] to the situation with distributed delays.

Theorem 3.11. *Suppose that $g_i : C_1 \to \mathbb{R}$ are C^1-functions such that $g_i(0) = 0$ and $g_i'(0)(1) = 1$. For $\gamma_i = \|g_i'(0)\|$, if*

$$\prod_{i=1}^{n} (b_i + \alpha_{ii}) > \prod_{i=1}^{n} \alpha_{i,i-1} \tag{3.18}$$

and

$$|\alpha_{ii}|\gamma_i \leq b_i, \ i = 1, \dots, n, \qquad \left| \prod_{i=1}^{n} \alpha_{i,i-1}\gamma_{i-1} \right| \leq \prod_{i=1}^{n} (b_i - |\alpha_{ii}|\gamma_i) \tag{3.19}$$

then the trivial equilibrium of (3.17) is locally asymptotically stable.

Proof. The linearized equation about zero has the form (3.2), with $L_{ii} = \alpha_{ii} g_i'(0)$, $L_{i,i-1} = \alpha_{i,i-1} g_{i-1}'(0)$ and $L_{ij} = 0$ for $j \neq i$, $j \neq i - 1$. From Theorem 3.3, $\det M \neq 0$ and $\hat{N}$ is an M-matrix imply the local asymptotic stability of the trivial solution of (3.17). Here, $M = B + A$, for $B = diag(b_1, \dots, b_n)$ and $A = [a_{ij}]$, where $a_{ij} = -\alpha_{ij}$ for $j = i, i - 1$ and $a_{ij} = 0$ for $j \neq i$, $j \neq i - 1$; and $\hat{N} = B - |C|$, for $C = [c_{ij}]$, with $c_{ij} = -\alpha_{ij}\gamma_j$ for $j = i, i - 1$, and zero otherwise. It is easy to check that (3.19) is equivalent to saying that $\hat{N}$ is an M-matrix. Together with (3.19), (3.18) means that $\det M \neq 0$. ∎

3.3 Global Stability for Lotka-Volterra Systems

The results in this section concern global stability for n species delayed Lotka-Volterra models.

We consider autonomous systems given by

$$\dot{x}_i(t) = r_i x_i(t)[1 - b_i x_i(t) - L_i(x_t)], \quad i = 1, \dots, n, \tag{3.20}$$

where $b_i \in \mathbb{R}$, $r_i > 0$ and $L_i : C_n \to \mathbb{R}$ are linear bounded operators. More generally, we shall also consider non-autonomous systems of FDE's of the form

$$\dot{x}_i(t) = r_i(t) x_i(t)[\alpha_i - b_i x_i(t) - L_i(x_t)], \quad i = 1, \dots, n,$$

where b_i, L_i are as in (3.20), $\alpha_i \in \mathbb{R}$, and $r_i : [0, +\infty) \to (0, +\infty)$ are continuous functions. For the sake of simplicity, we take $\alpha_i = 1$, $i = 1, \dots, n$, and write

$$\dot{x}_i(t) = r_i(t) x_i(t)[1 - b_i x_i(t) - L_i(x_t)], \quad i = 1, \dots, n, \tag{3.21}$$

As in Section 3.2, we write L_i as (3.3), for some $l_{ij} \in \mathbb{R}$ and $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]} \eta_{ij} = 1$, and denote $a_{ij} = L_{ij}(1)$, $i, j = 1, \dots, n$. Again, $B = diag(b_1, \dots, b_n)$, $M = B + [a_{ij}]$, $N = B + [l_{ij}]$ and $\tilde{M}$, $\hat{N}$ are as in (3.5).

In the sequel, for (3.21) the following hypotheses will be considered:

(L1) There is $d = (d_1, \dots, d_n) > 0$ such that $\|L_i\|_{d^{-1}} \le d_i b_i$, $i = 1, \dots, n$;

(L2) $\det \tilde{M} \neq 0$;

(L3) there is a vector $x^* = (x_1^*, \dots, x_n^*) > 0$ such that $Mx^* = [1, \dots, 1]^T$, i.e., x^* is a *positive* equilibrium of (3.21);

(L4) $r_i(t)$ is uniformly bounded on $[0, +\infty)$ and $\int_0^{+\infty} r_i(t)dt = +\infty$, $i = 1, \dots, n$.

If $x^* = (x_1^*, \dots, x_n^*)$ is a positive equilibrium of (3.21), for $y_i(t) = x_i(t) - x_i^*$ system (3.21) becomes

$$\dot{y}_i(t) = -r_i(t)(y_i(t) + x_i^*)[b_i y_i(t) + L_i(y_t)], \quad i = 1, \dots, n. \tag{3.22}$$

By biological reasons, we restrict our attention to *positive* solutions of (3.21). Therefore, we consider solutions with initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_{\hat{0}}, \tag{3.23}$$

for some $t_0 \ge 0$. Since a solution $x(t, t_0, \varphi)$ with initial condition in $C_{\hat{0}}$ at $t_0 \ge 0$ satisfies $x_i(t, t_0, \varphi) = x_i(t_0) exp\left(\int_{t_0}^t r_i(s)[1 - b_i x_i(s) - L_i(x_s)]ds\right) > 0$, then it is an *admissible solution*, in the sense that $x_t(t_0, \varphi) \in C_{\hat{0}}$, whenever it is defined.

Accordingly, if **(L3)** holds, the set of *admissible* initial conditions for (3.22) is the set $C_{-x^*} = C_{\hat{0}} - x^*$,

$$C_{-x^*} = \{\varphi \in C_n : \varphi_i(\theta) \geq -x_i^*, \text{ for } \theta \in [-\tau, 0), \varphi_i(0) > -x_i^*, i = 1, \dots, n\},$$

and the solutions $y(t, t_0, \varphi)$ of (3.22) with initial conditions $y_{t_0} = \varphi$, $\varphi \in C_{-x^*}$, are *admissible* solutions.

In this section, we study the global asymptotic stability of the positive equilibrium x^* of (3.20), or (3.21), if it exists. If in addition $\det M \neq 0$, then the positive equilibrium of (3.21) is unique. For (3.20), its local stability is deduced from Theorem 3.3:

Theorem 3.12. *Suppose that x^* is a positive equilibrium of the autonomous system (3.20). If $\det M \neq 0$ and $\hat{N}$ is an M-matrix, then x^* is locally asymptotically stable.*

Next, we prove some auxiliary results, for which it is convenient to write **(L1)** in a more suitable form. From Lemma 3.1, **(L1)** is equivalent to saying that there is $d = (d_1, \dots, d_n) > 0$ such that $\hat{N}d \geq 0$. Consequently, **(L1)** implies the inequalities

$$d_i(b_i + a_{ii}) \geq \sum_{j \neq i} d_j |a_{ij}|, \qquad i = 1, \dots, n. \tag{3.24}$$

From Theorem 1.19, **(L1)** also implies that $\hat{N}$ is an M-matrix. As we saw in Section 1.5, in general the reverse is not true for $n \geq 2$. On the other hand, since $\tilde{M} \geq \hat{N}$, if $\hat{N}$ is an M-matrix, the same happens to $\tilde{M}$; together with $\det \tilde{M} \neq 0$, this means that $\tilde{M}$ is a non-singular M-matrix, thus there is $c = (c_1, \dots, c_n) > 0$ such that $\tilde{M}c > 0$ (see Theorem 1.16). However, if **(L1)** and **(L2)** hold, one cannot conclude that $\tilde{M}d > 0$, for the same vector $d > 0$ as in **(L1)**. Also, from Theorem 1.21, if $\tilde{M}$ is a non-singular M-matrix, then $\det M \neq 0$; conversely, for any $n \geq 2$, we might have $\det M \neq 0$ and $\tilde{M}$ a singular M-matrix. In particular, we observe that, under **(L1)**-**(L3)**, x^* is the unique positive equilibrium of (3.20), or (3.21).

By effecting the change $z_i(t) = d_i^{-1} y_i(t)$, $i = 1, \dots, n$, where $d_1, \dots, d_n > 0$ are as in **(L1)**, (3.22) becomes

$$\dot{z}_i(t) = -r_i(t)(z_i(t) + d_i^{-1} x_i^*)[\hat{b}_i z_i(t) + \hat{L}_i(z_t)], \qquad i = 1, \dots, n, \tag{3.25}$$

with $\hat{b}_i = b_i d_i$, $\hat{a}_{ij} = a_{ij} d_j$, and $\hat{L}_i(\varphi) = L_i((d_j \varphi_j)_{j=1}^n) = \sum_{j=1}^n d_j L_{ij}(\varphi_j)$ for $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$.

With the previous notation, we get $\|\hat{L}_i\| = \|L_i\|_{d^{-1}}$. Consequently, if hypothesis **(L1)** holds for system (3.22), then for (3.25) we have

$$\|\hat{L}_i\| \leq \hat{b}_i.$$

Assuming **(L1)**, one may therefore assume without loss of generality that translating x^* to the origin and a scaling of the variables, (3.21) is transformed into (3.22), with $\|L_i\| \le b_i$, $i = 1, \ldots, n$.

A first lemma is stated in the more general framework of $\mathbb{R}^n$ with a norm $|\cdot|_d$, for some $d \in \mathbb{R}^n$, $d > 0$. Naturally, for FDE's in $\mathbb{R}^n$ for which a set $S \subseteq C_n = C([-\tau, 0]; \mathbb{R}^n)$ is chosen as the set of admissible initial conditions, a solution $y(t)$ with initial condition $y_{t_0} = \varphi \in S$ is said to be admissible if $y_t \in S$ for $t > t_0$ whenever y_t is defined.

Lemma 3.13. *Choose a set $S \subseteq C_n$ as the set of admissible initial conditions for*

$$\dot{y}(t) = f(t, y_t), \qquad t \ge t_0, \tag{3.26}$$

where $f : [t_0, +\infty) \times S \to \mathbb{R}^n$ continuous, $f = (f_1, \ldots, f_n)$. Let $\mathbb{R}^n$ be equipped with a norm $|\cdot|_d$, for some $d = (d_1, \ldots, d_n) > 0$, and assume that f satisfies the hypothesis

(L1*) *for all $t \ge t_0$ and $\varphi \in S$ such that $|\varphi(\theta)|_d < |\varphi(0)|_d$ for $\theta \in [-\tau, 0)$, then $\varphi_i(0) f_i(t, \varphi) < 0$, for some $i \in \{1, \ldots, n\}$ such that $|\varphi(0)|_d = d_i |\varphi_i(0)|$.*

Then, all admissible solutions of (3.26) are defined and bounded for $t \ge t_0$. Moreover, if $y(t) = y(t, t_0, \varphi)$ ($\varphi \in S$) is an admissible solution of (3.26) and $|y(t)|_d \le K$ for $t \in [t_0 - \tau, t_0]$, then $|y(t)|_d \le K$ for $t \ge t_0$.

Proof. Let $y(t)$ be an admissible solution of (3.26) on $[t_0 - \tau, a)$ for some $a > t_0$, with $|y(t)|_d \le K$ for $t \in [t_0 - \tau, t_0]$. Suppose that there is $t_1 > t_0$ such that $|y(t_1)|_d > K$, and define

$$T = \min \left\{ t \in [t_0, t_1] : \max_{s \in [t_0, t_1]} |y(s)|_d = |y(t)|_d \right\}.$$

We have $|y(T)|_d > K$ and

$$|y(t)|_d < |y(T)|_d \quad \text{for} \quad t \in [t_0, T).$$

Hence $|y_T(\theta)|_d = |y(T + \theta)|_d < |y(T)|_d$ for $-\tau \le \theta < 0$. By **(L1*)**, there is $i \in \{1, \ldots, n\}$ such that $|y(T)|_d = d_i |y_i(T)|$ and $y_i(T) f_i(t, y_T) < 0$ for all $t \ge t_0$. Suppose that $y_i(T) > 0$ (the situation $y_i(T) < 0$ is analogous). Since $d_i y_i(t) \le |y(t)|_d < d_i y_i(T)$ for $t_0 - \tau \le t < T$, then $\dot{y}_i(T) \ge 0$. On the other hand, from **(L1*)** and (3.26) we have $\dot{y}_i(T) = f_i(T, y_T) < 0$, a contradiction. This proves that $y(t)$ is extensible to $[t_0 - \tau, +\infty)$, and we have $|y(t)|_d \le K$ for all $t > t_0$. ∎

Theorem 3.14. *Let $x_i^* > 0$, $r_i(t) > 0$ for $t \geq 0$, $i = 1, \dots, n$, and $S = C_{-x^*}$. If $\det M \neq 0$ and* **(L1)** *holds, then (3.22) satisfies* **(L1*)** *on $[0, +\infty)$. In particular, all (admissible) solutions of (3.21) are defined and bounded on $[0, +\infty)$.*

Proof. As observed above, we may assume that equation (3.22) satisfies the condition $\|L_i\| \leq b_i$, $i = 1, \dots, n$. Equation (3.22) reads as (3.26), for $f_i(t, \varphi) = -r_i(t)(\varphi_i(0) + x_i^*)(b_i\varphi_i(0) + L_i(\varphi))$, $i = 1, \dots, n$. Let $t \geq 0$, $\varphi \in S = C_{-x^*}$ and suppose $|\varphi(\theta)|_\infty < |\varphi(0)|_\infty$ for $\theta \in [-\tau, 0)$. Set $K = |\varphi(0)|_\infty$. Consider the partition $I = I_1 \cup I_2 \cup I_3$ of $I := \{1, \dots, n\}$, where

$$I_1 = \{i \in I : \varphi_i(0) = K\},\ I_2 = \{i \in I : \varphi_i(0) = -K\},\ I_3 = \{i \in I : |\varphi_i(0)| < K\}.$$

Define

$$\begin{aligned}
-\gamma_1 &:= \min_{i \in I_1} \min_{\theta \in [-\tau, 0]} \varphi_i(\theta) > -K,\\
\gamma_2 &:= \max_{i \in I_2} \max_{\theta \in [-\tau, 0]} \varphi_i(\theta) < K,\\
\gamma_3 &:= \max_{i \in I_3} \max_{\theta \in [-\tau, 0]} |\varphi_i(\theta)| < K,
\end{aligned}$$

and $\varepsilon_0 = \min_{1 \leq k \leq 3}(K - \gamma_k)/2$. Consider

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n, \qquad \text{with} \qquad \varepsilon_i = \begin{cases} \varepsilon_0, & i \in I_1\\ -\varepsilon_0, & i \in I_2\\ 0, & i \in I_3 \end{cases}.$$

For $\# I_k = n_k$, $k = 1, 2, 3$, we may suppose that I is ordered in such a way that

$$I_1 = \{1, \dots, n_1\}, \qquad I_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \qquad I_3 = \{n_1 + n_2 + 1, \dots, n\},$$

so that ε reads as $\varepsilon = \varepsilon_0(1, \dots, 1, -1, \dots, -1, 0 \dots, 0)$, with the obvious notation for dots.

From the definition of ε_0, it is easy to check that $|\varphi_i(\theta) - \varepsilon_i| \leq K - \varepsilon_0$ for all $i \in I$, hence $\|\varphi - \varepsilon\|_\infty \leq K - \varepsilon_0$ and $|L_i(\varphi - \varepsilon)| \leq b_i(K - \varepsilon_0)$, $1 \leq i \leq n$.

For $i \in I_1$, from **(L1)** we have

$$\begin{aligned}
b_i\varphi_i(0) + L_i(\varphi) &= \varepsilon_0 b_i + (\varphi_i(0) - \varepsilon_0)b_i + L_i(\varphi - \varepsilon) + L_i(\varepsilon)\\
&\geq \varepsilon_0 b_i + L_i(\varepsilon)\\
&= \varepsilon_0 \left[(b_i + a_{ii}) + \sum_{j \in I_1, j \neq i} a_{ij} - \sum_{j \in I_2} a_{ij} \right].
\end{aligned} \tag{3.27}$$

Analogously, for $i \in I_2$ we obtain

$$\begin{aligned}
b_i\varphi_i(0) + L_i(\varphi) &= -\varepsilon_0 b_i + (\varphi_i(0) + \varepsilon_0)b_i + L_i(\varphi - \varepsilon) + L_i(\varepsilon)\\
&\leq -\varepsilon_0 b_i + L_i(\varepsilon)\\
&= \varepsilon_0 \left[-(b_i + a_{ii}) + \sum_{j \in I_1} a_{ij} - \sum_{j \in I_2, j \neq i} a_{ij} \right].
\end{aligned} \tag{3.28}$$

From (3.24), (3.27) and (3.28), we conclude that

$$\varphi_i(0)(b_i\varphi_i(0) + L_i(\varphi)) \geq 0, \quad i \in I_1 \cup I_2.$$

If there is $i \in I_1 \cup I_2$ such that $\varphi_i(0)(b_i\varphi_i(0) + L_i(\varphi)) > 0$, then **(L1*)** holds. If $\varphi_i(0)(b_i\varphi_i(0) + L_i(\varphi)) = 0$ for all $i \in I_1 \cup I_2$, from (3.24), (3.27) and (3.28) we deduce that

$$\sum_{j \in I_3} |a_{ij}| = 0, \quad i \in I_1 \cup I_2,$$

i.e., $a_{ij} = 0$ for all $i \in I_1 \cup I_2$, $j \in I_3$. (Note that this includes the case $I_3 = \emptyset$; however, $I_1 \cup I_2 \neq \emptyset$.) Hence, one can write

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \tag{3.29}$$

with M_{ij} matrices of dimensions $n_i \times n_j$, $i, j = 1, 2, 3$, and $M_{13} = 0$, $M_{23} = 0$. Again from (3.24), (3.27), (3.28), and the definition of the vector ε, we have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \eta = 0,$$

where $\varepsilon = (\eta, 0)$ and η is a $(n_1 + n_2) \times 1$ vector. But this is a contradiction since $\det M \neq 0$, and $M_{13} = 0$, $M_{23} = 0$ in (3.29) imply that

$$\det \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \neq 0.$$

■

After having established the boundedness of positive solutions of (3.21), we are in a position to prove the main result in this chapter, on the global attractivity of x^*.

Theorem 3.15. *Assume* **(L1)**-**(L4)**. *Then the positive equilibrium of (3.21) is globally asymptotically stable (in the set of all positive solutions).*

Proof. By translating x^* to the origin, (3.21) becomes (3.22). As already noticed, **(L1)** and **(L2)** imply that $\det M \neq 0$. From Theorem 3.14, all admissible solutions of (3.22) are defined and bounded on $[-\tau, +\infty)$, and the trivial equilibrium of (3.22) is uniformly stable (in the set $S = C_{-x^*}$ of all admissible solutions). It remains to prove that zero is globally attractive in S.

As in the proof of Theorem 3.14, after a scaling we may assume **(L1)** with $d = (1, \ldots, 1)$, i.e.,

$$\|L_i\| \leq b_i, \quad i \in I := \{1, \ldots, n\}.$$

Let $y(t) = (y_i(t))_{i=1}^n$ be an admissible solution to (3.22). Since $y(t)$ is defined and bounded for $t \geq 0$, set

$$-v_i = \liminf_{t\to+\infty} y_i(t), \qquad u_i = \limsup_{t\to+\infty} y_i(t), \qquad i \in I,$$

and

$$v = \max_{1\le i\le n} \{v_i\}, \qquad u = \max_{1\le i\le n} u_i.$$

with $u, v \in \mathbb{R}$ satisfying $-x_i^* \leq -v_i \leq u_i < +\infty$, $i \in I$.

It is sufficient to prove that $\max(u, v) = 0$. Assume e.g. that $|v| \leq u$, so that $\max(u, v) = u$. (The situation is analogous for $|u| \leq v$.)

Consider the decomposition of I, $I = I_1 \cup I_2 \cup I_3$, where

$$I_1 = \{i \in I : u_i = u\}, \qquad I_2 = \{i \in I : v_i = u, u_i < u\},$$

$$I_3 = \{i \in I : -u < -v_i \leq u_i < u\}.$$

Since $|v| \leq u$, then $I_1 \neq \emptyset$. Observe that the situation where one or both sets I_2, I_3 are empty is included in our setting. The proof is divided in several steps.

Claim 1. For each $i \in I_1 \cup I_2$, there is a sequence $(t_k^i)_{k\in\mathbb{N}}$ with $t_k^i \nearrow +\infty$, $b_i y_i(t_k^i) + L_i(y_{t_k^i}) \to 0$, and $y_i(t_k^i) \to u$ if $i \in I_1$, $y_i(t_k^i) \to -u$ if $i \in I_2$, as $k \to +\infty$.

To prove claim 1, for each $i \in I_1 \cup I_2$ we shall consider separately the cases of $y_i(t)$ eventually monotone and not eventually monotone.

Case 1. Assume that $y_i(t)$ is not eventually monotone.

Let $i \in I_1$, and consider $(t_k^i)_{k\in\mathbb{N}}$ with $t_k^i \nearrow +\infty$ as $k \to +\infty$, a sequence of local maximum points so that $y_i(t_k^i) \to u_i = u$. Clearly, $\dot{y}_i(t_k^i) = 0 = b_i y_i(t_k^i) + L_i(y_{t_k^i})$. For $i \in I_2$, the claim follows by considering a sequence of local minimum points $(t_k^i)_{k\in\mathbb{N}}$ with $t_k^i \nearrow +\infty$, $y_i(t_k^i) \to -v_i = -u$ as $k \to +\infty$.

Case 2. Assume that $y_i(t)$ is eventually monotone.

Let $i \in I_1 \cup I_2$. In this case, we have

$$\lim_{t\to+\infty} y_i(t) = u \text{ if } i \in I_1 \quad \text{ and } \quad \lim_{t\to+\infty} y_i(t) = -u \text{ if } i \in I_2, \tag{3.30}$$

and for t large, either $\dot{y}_i(t) \leq 0$ or $\dot{y}_i(t) \geq 0$.

If $\dot{y}_i(t) \geq 0$ for t large, then $b_i y_i(t) + L_i(y_t) \leq 0$, hence

$$\limsup_{t\to+\infty} (b_i y_i(t) + L_i(y_t)) := c \leq 0.$$

If $c < 0$, then there is $t_1 > 0$ such that $b_i y_i(t) + L_i(y_t) < c/2$ for $t \geq t_1$, implying that $\dot{y}_i(t) \geq -c r_i(t)(y_i(t) + x_i^*)/2$ and

$$y_i(t) + x_i^* \geq (y_i(t_1) + x_i^*) \exp\left(-\frac{c}{2}\int_{t_1}^t r_i(s)ds\right), \qquad t \geq t_1.$$

From **(L4)** and the above inequality, we obtain $y_i(t) \to +\infty$ as $t \to +\infty$, which is not possible. Thus $c = 0$, which proves the claim.

If $\dot{y}_i(t) \le 0$ for t large, in a similar way we get

$$\liminf_{t\to+\infty}(b_i y_i(t) + L_i(y_t)) := d \ge 0.$$

Suppose that $d > 0$. For any $\varepsilon > 0$, there is t_2 such that for $t \ge t_2$ we have $b_i y_i(t) + L_i(y_t) > d/2$ and $\|y_t\| \le u + \varepsilon$. Then, for $t \ge t_2$

$$0 < y_i(t) + x_i^* \le (y_i(t_2) + x_i^*)\exp\left(-\frac{d}{2}\int_{t_2}^t r_i(s)ds\right) \to 0, \quad \text{as} \quad t \to +\infty.$$

We therefore conclude that

$$\lim_{t\to+\infty} y_i(t) = -x_i^*. \tag{3.31}$$

Since we have assumed $u \ge 0$, (3.30) and (3.31) imply that $i \notin I_1$; and for $i \in I_2$, then $u = x_i^*$. But, for $t \ge t_2$,

$$0 < d/2 \le b_i y_i(t) + L_i(y_t) \le b_i y_i(t) + b_i(u+\varepsilon) \to b_i\varepsilon, \quad t \to +\infty.$$

Since $\varepsilon > 0$ is arbitrary, this is a contradiction. Hence $d = 0$, and claim 1 is proven.

Claim 2. For $i \in I_1 \cup I_2$, there is a sequence $(t_k^i)_{k\in\mathbb{N}}$, $t_k^i \nearrow +\infty$, such that $y_{t_k^i} \to \varphi^i = (\varphi_1^i, \dots, \varphi_n^i) \in C_n$ as $k \to +\infty$, with

$$\varphi_i^i(0) + L_i(\varphi^i) = 0, \quad \varphi_i^i(0) = \begin{cases} u & \text{if} \quad i \in I_1 \\ -u & \text{if} \quad i \in I_2 \end{cases}$$

and

$$-v_j \le \varphi_j^i(\theta) \le u_j, \quad 1 \le j \le n, -\tau \le \theta \le 0.$$

Suppose that $i \in I_1$ (the situation $i \in I_2$ is treated in an analogous way). From Claim 1, let $(t_k^i)_{k\in\mathbb{N}}$ be a sequence with $t_k^i \nearrow +\infty$, $b_i y_i(t_k^i) + L_i(y_{t_k^i}) \to 0$ and $y_i(t_k^i) \to u$ as $k \to +\infty$. Consider $\{y_{t_k^i} : k \in \mathbb{N}\} \subseteq C_n$, and fix $\varepsilon > 0$. Clearly $\{y_{t_k^i} : k \in \mathbb{N}\}$ is uniformly bounded with $\|y_{t_k^i}\| \le u + \varepsilon$ for $k \ge k_0$. On the other hand, from (3.22) and **(L4)** it follows that $\dot{y}(t)$ is uniformly bounded on $[0, +\infty)$, thus $\{y_{t_k^i} : k \in \mathbb{N}\} \subseteq C_n$ is bounded and equicontinuous. By Ascoli-Arzelà theorem, for a subsequence, still denoted by $(y_{t_k^i})$, we have $y_{t_k^i} \to \varphi^i$ for some $\varphi^i = (\varphi_1^i, \dots, \varphi_n^i) \in C_n$. By letting $k \to +\infty$ and $\varepsilon \to 0^+$, we conclude that φ^i satisfies all the requeriments in Claim 2.

In the remaining proof, sequences $(t_k^i)_{k\in\mathbb{N}}$ as in Claim 2 are supposed to be fixed, and φ^i denotes the limit in C_n of $(y_{t_k^i})$.

Observe that for $i \in I_1 \cup I_2$ and $j \in I_2 \cup I_3$, we have $\max_{\theta\in[-\tau,0]} \varphi_j^i(\theta) < u$. Now, define

$$J^i = \{j \in I_1 : \min_{\theta\in[-\tau,0]} \varphi_j^i(\theta) = -u, \max_{\theta\in[-\tau,0]} \varphi_j^i(\theta) = u\}, \quad i \in I_1 \cup I_2.$$

Claim 3. If $u > 0$, then $J^i = \emptyset$ for all $i \in I_1 \cup I_2$.

Let $u > 0$, and consider $i \in I_1$ and $j \in J^i$. Let $\theta_1, \theta_2 \in [-\tau, 0]$ be such

$$u = \varphi_j^i(\theta_1) = \lim_k y_j(t_k^i + \theta_1), \quad -u = \varphi_j^i(\theta_2) = \lim_k y_j(t_k^i + \theta_2).$$

Case 1. $\theta_2 < \theta_1$

Fix $\varepsilon > 0$ small. For some t_0, we have $\|y_t\| \le u_\varepsilon := u+\varepsilon$ for $t \ge t_0$, and from **(L1)** we obtain

$$\dot y_j(t) \le b_j r_j(t)(y_j(t) + x_j^*)(u_\varepsilon - y_j(t)).$$

By integrating over an interval $[s,t] \subseteq [t_0, +\infty)$, we obtain

$$(y_j(t) + x_j^*)(u_\varepsilon - y_j(s)) \le$$
$$\le (y_j(s) + x_j^*)(u_\varepsilon - y_j(t)) \exp\left((x_j^* + u_\varepsilon)b_j \int_s^t r_j(\sigma)d\sigma\right), \ t \ge s \ge t_0. \tag{3.32}$$

From **(L4)**, there is $\beta > 0$ such that $r_i(t) \le \beta$, $t \ge 0$. For $t = t_k^i + \theta_1$, $s = t_k^i + \theta_2$ in (3.32), by letting $k \to +\infty$ we conclude that

$$(u + x_j^*)(u_\varepsilon + u) \le (-u + x_j^*)(u_\varepsilon - u) \exp((x_j^* + u_\varepsilon)b_j\beta\tau).$$

Since $\varepsilon > 0$ is arbitrarily small, we conclude that $u = 0$, which contradicts our assumption.

Case 2. $\theta_1 < \theta_2$

For this situation, we first prove that $u < x_j^*$. Fix $\varepsilon > 0$ small. Then, $|b_j y_j(t) + L_j(y_t)| \le 2b_j u_\varepsilon$, for t large, and

$$\dot y_j(t) \ge -2b_j u_\varepsilon r_j(t)(y_j(t) + x_j^*),$$

leading to

$$(y_j(t) + x_j^*) \ge (y_j(s) + x_j^*) \exp\left(-2b_j u_\varepsilon \int_s^t r_j(\sigma)d\sigma\right), \quad t \ge s \ge t_0, \tag{3.33}$$

for some t_0 large. With $t = t_k^i + \theta_2$, $s = t_k^i + \theta_1$ in (3.33), by letting $k \to +\infty$ and $\varepsilon \to 0^+$, we get

$$(-u + x_j^*) \ge (u + x_j^*) \exp(-2b_j u\beta\tau) > 0,$$

and hence $u < x_j^*$.

Now, let $\varepsilon > 0$ be small so that $u_\varepsilon < x_j^*$. For $t \geq t_0$, we have

$$\dot{y}_j(t) \geq -b_j r_j(t)(y_j(t) + x_j^*)(u_\varepsilon + y_j(t)),$$

and integration over an interval $[s, t] \subseteq [t_0, +\infty)$ yields

$$\begin{aligned}&(y_j(t) + u_\varepsilon)(y_j(s) + x_j^*) \geq \\ &\quad \geq (y_j(s) + u_\varepsilon)(y_j(t) + x_j^*) \exp(-(x_j^* - u_\varepsilon) b_j \beta (t - s)), \quad t \geq s \geq t_0. \end{aligned} \tag{3.34}$$

From (3.34), with $t = t_k^i + \theta_2$, $s = t_k^i + \theta_1$, by letting $k \to +\infty$ and $\varepsilon \to 0^+$, we obtain

$$0 \geq 2u(x_j^* - u) \exp(-b_j(x_j^* - u)\beta\tau),$$

and therefore conclude that $u = 0$, which is a contradiction.

For $i \in I_2$, the proof of $J^i = \emptyset$ is similar.

Claim 4. $y(t) \to 0$ as $t \to +\infty$.

Recall that we are considering the case $|v| \leq u$. For the sake of contradiction, assume that $u > 0$.

Fix $i \in I_1 \cup I_2$, and choose $\varphi^i \in C_n$ as in Claim 2. Since $J^i = \emptyset$ from Claim 3, the definition of I_j, $j = 1, 2, 3$, leads to

$$\text{either} \quad \min_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) > -u \quad \text{or} \quad \max_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) < u, \quad j \in I.$$

Consider now the partition of I

$$I = I_1^i \cup I_2^i \cup I_3,$$

where I_3 is as above and

$$I_1^i = \{j \in I_1 \cup I_2 : \min_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) > -u\}, \; I_2^i = \{j \in I_1 \cup I_2 : \min_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) = -u\}.$$

Note that the set I_3 does not depend on i; also, $i \in I_1^i$ if $i \in I_1$ and $i \in I_2^i$ if $i \in I_2$.

We now adapt the procedure followed in the proof of Theorem 3.14. For $i \in I_1 \cup I_2$, define

$$\begin{aligned} -\gamma_1^i &= \min_{j \in I_1^i} \min_{-\tau \leq \theta \leq 0} \varphi_j^i(\theta) > -u, \\ \gamma_2^i &= \max_{j \in I_2^i} \max_{-\tau \leq \theta \leq 0} \varphi_j^i(\theta) < u, \\ \gamma_3^i &= \max_{j \in I_3} \max_{-\tau \leq \theta \leq 0} |\varphi_j^i(\theta)| < u, \end{aligned}$$

and $\varepsilon_0^i = \min_{1 \leq k \leq 3}(u - \gamma_k^i)/2$. Consider

$$e^i = (e_1^i, \ldots, e_n^i) \in \mathbb{R}^n, \quad \text{with} \quad e_j^i = \begin{cases} \varepsilon_0^i, & j \in I_1^i \\ -\varepsilon_0^i, & j \in I_2^i \\ 0, & j \in I_3 \end{cases}.$$

From the definition of ε_0^i, we have $\|\varphi^i - e^i\|_\infty \le u - \varepsilon_0^i$. For $i \in I_1$, from $\|L_i\| \le b_i$, and Claim 2, we get

$$\begin{aligned} 0 = b_i\varphi_i^i(0) + L_i(\varphi^i) &= \varepsilon_0^i b_i + (\varphi_i^i(0) - \varepsilon_0^i)b_i + L_i(\varphi^i - e^i) + L_i(e^i) \\ &\ge \varepsilon_0^i b_i + L_i(e^i) \\ &= \varepsilon_0^i \left[b_i + a_{ii} + \sum_{j\in I_1^i, j\neq i} a_{ij} - \sum_{j\in I_2^i} a_{ij} \right]. \end{aligned} \tag{3.35}$$

Analogously, for $i \in I_2$ we obtain

$$0 = b_i\varphi_i^i(0) + L_i(\varphi^i) \le \varepsilon_0^i \left[-(b_i + a_{ii}) + \sum_{j\in I_1^i} a_{ij} - \sum_{j\in I_2^i, j\neq i} a_{ij} \right]. \tag{3.36}$$

Now, from (3.6) (with $d_1 = \ldots = d_n = 1$), (3.35) and (3.36) we conclude that

$$\sum_{j\in I_3} |a_{ij}| = \sum_{j\in I_3} |l_{ij}| = 0, \qquad i \in I_1 \cup I_2,$$

or, equivalently,

$$a_{ij} = l_{ij} = 0 \quad \text{for } i \in I_1 \cup I_2,\ j \in I_3, \tag{3.37}$$

and

$$b_i = \sum_{j\in I} |a_{ij}| = \sum_{i\in I} |l_{ij}|, \qquad i \in I_1 \cup I_2. \tag{3.38}$$

At this stage, after a permutation of I, we may suppose that I is ordered in such way that

$$I_1 = \{1, \ldots, n_1\}, \qquad I_2 = \{n_1 + 1, \ldots, n_1 + n_2\},$$

$$I_3 = \{n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3\},$$

with $n_1+n_2+n_3 = n$. Recall that n_2, n_3 may be zero. According to this ordering, $\hat{N}$ has the form

$$\hat{N} = \left((\hat{N}_{ij})_{i,j=1}^3 \right)$$

where $\hat{N}_{ij}$ are $n_i \times n_j$ matrices, $i, j = 1, 2, 3$. If $I_3 \neq \emptyset$, from (3.37) we have $\hat{N}_{j3} = 0$ for $j = 1, 2$. Now, from (3.37)-(3.38) one writes M in the form (3.29) with $M_{13} = M_{23} = 0$, and concludes that

$$\tilde{M}_0\eta = 0, \qquad \text{where} \qquad \tilde{M}_0 = \begin{pmatrix} \tilde{M}_{11} & -|M_{12}| \\ -|M_{21}| & \tilde{M}_{22} \end{pmatrix},$$

where $\tilde{M}_{ii}$ are $n_i \times n_i$ matrices, $i = 1, 2$, and $\eta = (1, \dots, 1)$ is a $(n_1 + n_2)$-vector. This is not possible however, since $\det \tilde{M} \neq 0$ and $M_{13} = M_{23} = 0$ imply that $\det \tilde{M}_0 \neq 0$.

The above arguments show that $u = 0$, hence $v = 0$ as well. This ends the proof of the theorem. ∎

Remark 3.4 We remark that Tang and Zou [55] gave stability results for Lotka-Volterra systems of the form

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - \int_{-\tau_{ii}}^{0} x_i(t+\theta)d\eta_{ii}(\theta) \right.$$
$$\left. - \sum_{j\neq i}^{n} l_{ij} \int_{-\tau_{ij}}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], \ i = 1, \dots, n, \quad (3.39)$$

where $r_i(t)$ satisfy **(L4)**, η_{ij} are *non-decreasing* bounded normalized functions, and the constants l_{ij} are *non-negative.* In particular, in (3.39) all the operators L_{ij} are *positive.* In [55], the authors are primarily interested in the situation $\tau_{ii} > 0$, $i = 1, \dots, n$, where instantaneous negative feedbacks are absent, although the situation of zero diagonal delays is included in their setting. Several criteria for the global attractivity of the positive equilibrium of (3.39) (if it exists) are established, by imposing 3/2-type constraints on the diagonal delays τ_{ii}, and M-matrix-type conditions. Namely, for $M = [l_{ij}]$, where l_{ij}, $j \neq i$, are as in (3.39) and $l_{ii} = 1$, the following conditions are assumed in [55]: either (DD1) M satisfies $1 > \sum_{j\neq i} l_{ij}$, $i = 1, \dots, n$, or (DD2) $\hat{M}$ is a non-singular M-matrix.

Remark 3.5 In [47] pp 94-98, Smith considered the autonomous Lotka-Volterra competition system,

$$\dot{x}_i(t) = r_i x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], \ i = 1, \dots, n, \quad (3.40)$$

where $r_i > 0$, $b_i > 0$, and with all $l_{ij} \geq 0$ and η_{ij} normalized non-decreasing bounded variation functions - or, in other words, system (3.20) with all operators L_{ij} being positive, $i, j = 1, \dots, n$. For this situation, under the condition

$$\sum_{j=1}^{n} l_{ij} b_j^{-1} < 1, \qquad i = 1, \dots, n, \quad (3.41)$$

Smith proved the existence of a global attractive positive equilibrium of (3.40) (in the set of all positive solutions).

Since $l_{ij} \geq 0$, (3.41) implies that $\hat{N}d > 0$, for $d = (b_1^{-1}, \ldots, b_n^{-1})$, and, from Lemma 3.1, **(L1)** holds. In particular, $\hat{N}$ is a non-singular M-matrix and since $\tilde{M} \geq \hat{N}$, $\tilde{M}$ is also a non-singular M-matrix, so that $\det \tilde{M} \neq 0$. Therefore, Theorem 3.15 generalizes the criterion in [47].

In what follows, we give some consequences of the Theorem 3.15.

Corollary 3.16. *Assume* **(L2)**, **(L3)**, **(L4)** *and that $\hat{N}$ is an irreducible M-matrix. Then, the equilibrium x^* of (3.21) is globally asymptotically stable (in the set of all positive solutions).*

Proof. From Theorem 1.20 and Lemma 3.1, if $\hat{N}$ is irreducible, then $\hat{N}$ is an M-matrix if and only if **(L1)** holds. ∎

Corollary 3.17. *Assume* **(L3)**, **(L4)** *and that $\hat{N}$ is a non-singular M-matrix. Then, x^* is globally asymptotically stable (in the set of all positive solutions of (3.21)).*

Proof. If $\hat{N}$ is a non-singular M-matrix, then there is $d = (d_1, \ldots, d_n) > 0$ such that $\hat{N}d > 0$, so **(L1)** holds. Since $\tilde{M} \geq \hat{N}$, then $\tilde{M}$ is a non-singular M-matrix as well (see Theorem 1.18). ∎

Corollary 3.18. *Assume* **(L1)**, **(L3)**, **(L4)** *and that $a_{ii} > 0$ for $i = 1, \ldots, n$. Then x^* is globally asymptotically stable (in the set of all positive solutions of (3.21)).*

Proof. For $d = (d_1, \ldots, d_n) > 0$ as in **(L1)**, we have

$$d_i b_i \geq \sum_{j=1}^{n} d_j |a_{ij}|, \qquad i = 1, \ldots, n,$$

hence $\tilde{M}d \geq 2diag(a_{11}, \ldots, a_{nn})d > 0$. From Theorem 1.16, $\tilde{M}$ is a non-singular M-matrix. ∎

Corollary 3.19. *Consider equation (3.21), where L_i is written as in (3.3), and suppose that the operators L_{ij} are all negative, $i, j = 1, \ldots, n$.*

Assume **(L4)** *and that M is a non-singular M-matrix. Then there exists a positive equilibrium of (3.21), which is globally asymptotically stable (in the set of all positive solutions).*

Proof. The operators L_{ij} are all negative, thus they are given by (3.3), for non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$ and $l_{ij} \leq 0$, $i, j = 1, \ldots, n$. Consequently, we have $L_{ij}(1) = a_{ij} = l_{ij}$ and

$$M = N = \tilde{M} = \hat{N} = diag(b_1, \ldots, b_n) + [a_{ij}].$$

Since M is a non-singular M-matrix, hypotheses **(L1)** and **(L2)** are satisfied; moreover, by Theorem 1.16, $M^{-1} \geq 0$. Let $x^* = (x_1^*, \ldots, x_n^*)$ be the solution of $Mx = [1, \ldots, 1]^T$. Since $M^{-1} \geq 0$, then $x^* \geq 0$; and $x_i^* = 0$ if and only if all the entries of the ith-row of M^{-1} are zero, which is not possible. The conclusion follows now from Theorem 3.15. ∎

Observe that hypothesis **(L1)**, which for $n \geq 2$ is strictly stronger than having $\hat{N}$ an M-matrix, was used throughout the proof of Theorem 3.15. Also **(L1)** was essential to conclude that admissible solutions of (3.21) are bounded. For system (3.21), written as

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\right], \tag{3.42}$$

for $i = 1, \ldots, n$, it is interesting to investigate situations for which the criterion for the global asymptotic stability of the positive equilibrium x^* is sharp, in the sense that it coincides with the necessary and sufficient conditions, established in Section 3.2 for the situation $r_i(t) \equiv r_i > 0$, for the local asymptotic stability independently of τ and η_{ij} in (3.42). This is in general an open problem (in [16] a partial answer for the particular case of autonomous system was given). For the situation $n = 2$, we have

$$\begin{aligned}
\dot{x}_1(t) = x_1(t)\Big[r_1 - b_1 x_1(t) - a_{11} &\int_{-\tau}^{0} x_1(t+\theta)d\eta_{11}(\theta) \\
&- a_{12}\int_{-\tau}^{0} x_2(t+\theta)d\eta_{12}(\theta)\Big] \\
\dot{x}_2(t) = x_2(t)\Big[r_2 - b_2 x_2(t) - a_{21} &\int_{-\tau}^{0} x_1(t+\theta)d\eta_{21}(\theta) \\
&- a_{22}\int_{-\tau}^{0} x_2(t+\theta)d\eta_{22}(\theta)\Big].
\end{aligned} \tag{3.43}$$

where $a_{ij} \in \mathbb{R}$, $r_i, b_i > 0$ and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are non-decreasing normalized functions, $i = 1, 2$, with $b_i \neq |a_{ii}|$. Then, with the previous notation, $l_{ij} = a_{ij}$ and $M = N$. On one hand, from Lemmas 3.1 and 3.9, if $\det M \neq 0$, we conclude that $\hat{M}$ is an M-matrix if and only if **(L1)** holds. On the other hand, if $\hat{M}$ is an M-matrix, then $\tilde{M}$ is an M-matrix as well and, in this situation, if $\det \tilde{M} \neq 0$ then $\det M \neq 0$. Consequently, from Theorems 1.14 and 3.15 we have the following result:

Corollary 3.20. *Consider $n = 2$, $r_i, b_i > 0$, $a_{ij} \in \mathbb{R}$ with $b_i \neq |a_{ii}|$, $i = 1, 2$. Assume* **(L2)** *and* **(L3)**.

Then x^* is globally asymptotically stable (in the set of positive solutions) for all $\tau > 0$ and η_{ij} non-decreasing normalized functions if and only if $\hat{M}$ is an M-matrix.

We finalize this chapter with some applications.

Example 3.4 Consider the scalar delayed logistic equation

$$\dot{x}(t) = r(t)x(t)[1 - b_0 x(t) - L_0(x_t)], \qquad t \geq 0, \tag{3.44}$$

where $b_0 \in \mathbb{R}$, $r : [0, +\infty) \to (0, +\infty)$ is continuous and $L_0 : C_1 \to \mathbb{R}$ is a linear bounded operator. Note that for (3.44), **(L1)-(L3)** translate as

$$b_0 + L_0(1) > 0, \qquad b_0 \geq \|L_0\|. \tag{3.45}$$

Theorem 3.15 applied to the particular case $n = 1$ gives the following result:

Corollary 3.21. *For (3.44), suppose that* **(L4)** *and (3.45) are satisfied. Then the positive equilibrium $x^* = (b_0 + L_0(1))^{-1}$ of (3.44) is globally asymptotically stable (in the set of all admissible solutions).*

The above criterion was already established in [12]. Note that (3.45) is exactly the necessary and sufficient condition for the asymptotic stability of (3.13) in the statement of Corollary 3.8.

Example 3.5 Consider the following Lotka-Volterra system with distributed delays and symmetry:

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1 - a x_1(t) + \alpha \int_{-\tau}^{0} x_1(t+\theta)\, d\eta_{11}(\theta)\right. \\
&\qquad\qquad\qquad \left. + b_{12} \int_{-\tau}^{0} x_2(t+\theta)\, d\eta_{12}(\theta)\right] \\
\dot{x}_2(t) &= x_2(t)\left[r_2 - a x_2(t) + b_{21} \int_{-\tau}^{0} x_1(t+\theta)\, d\eta_{21}(\theta)\right. \\
&\qquad\qquad\qquad \left. + \alpha \int_{-\tau}^{0} x_2(t+\theta)\, d\eta_{22}(\theta)\right].
\end{aligned} \tag{3.46}$$

Here, $\tau, r_1, r_2, a, \alpha, b_{12}, b_{21}$ are constants, $\tau, r_1, r_2, a > 0$, and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are non-decreasing functions with $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$, $i, j = 1, 2$, and

$$\text{either} \quad b_{21} = -b_{12} \quad \text{or} \quad b_{21} = b_{12}.$$

The first situation models a predator-prey system (cf. [43], [44]), while the second one is used to describe a cooperative or competition model (cf. [45]).

Theorem 3.22. *Consider the predator-prey system with symmetry (3.46), where* $b_{21} = -b_{12} := \beta$. *If*

$$\max\left(\frac{r_2\beta}{r_1}, -\frac{r_1\beta}{r_2}\right) < a - \alpha, \tag{3.47}$$

then there exists a positive equilibrium $x^* = (x_1^*, x_2^*)$. *Additionally, if*

$$|\beta| < a - \alpha \quad \textit{and} \quad |\beta| \le a + \alpha, \tag{3.48}$$

then $x(t) \to x^*$ *as* $t \to +\infty$ *for every admissible solution* $x(t)$ *of (3.46).*

Proof. With $b_{21} = -b_{12} := \beta$, (3.47) is equivalent to saying that the equilibrium $x^* = (x_1^*, x_2^*)$,

$$x_1^* = \frac{r_1(a-\alpha) - r_2\beta}{(a-\alpha)^2 + \beta^2}, \qquad x_2^* = \frac{r_2(a-\alpha) + r_1\beta}{(a-\alpha)^2 + \beta^2},$$

is positive. Here $M = N = \begin{pmatrix} (a-\alpha)/r_1 & \beta/r_1 \\ -\beta/r_2 & (a-\alpha)/r_2 \end{pmatrix}$. With the previous notation, $\hat{M}$ is an M-matrix if and only if $|\alpha| + |\beta| \le a$; for this situation, this is equivalent to **(L1)**. And $\det \tilde{M} \neq 0$ means that $|\beta| \neq |a - \alpha|$. Under these circumstances, **(L1)**-**(L2)** translate as (3.48). ∎

We observe that the predator-prey situation $b_{21} = -b_{12} := \beta$ with discrete and distributed delays in (3.46) was addressed in [44] and [43], respectively, where the authors proved the global asymptotic stability of x^* (assuming its existence) under the weaker requirement

$$\sqrt{\alpha^2 + \beta^2} \le a.$$

However, in both papers, the following restrictive assumption in the symmetry was imposed:

$$\eta_{11} = \eta_{21} := \mu, \qquad \eta_{12} = \eta_{22} := \nu. \tag{3.49}$$

To be more precise, [44] studied the equation with discrete delays

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)[r_1 - ax_1(t) + \alpha x_1(t-\tau_1) - \beta x_2(t-\tau_2)]\\ \dot{x}_2(t) &= x_2(t)[r_2 - ax_2(t) + \beta x_1(t-\tau_1) + \alpha x_2(t-\tau_2)],\end{aligned}$$

whereas [43] dealt with the distributed delays situation

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1 - ax_1(t) + \alpha\int_{-\tau}^{0} x_1(t+\theta)d\mu(\theta) - \beta\int_{-\tau}^{0} x_2(t+\theta)d\nu(\theta)\right]\\ \dot{x}_2(t) &= x_2(t)\left[r_2 - ax_2(t) + \beta\int_{-\tau}^{0} x_1(t+\theta)d\mu(\theta) + \alpha\int_{-\tau}^{0} x_2(t+\theta)d\nu(\theta)\right],\end{aligned}$$

For a cooperative or competition model with symmetry, in a similar way we deduce:

Theorem 3.23. *Consider (3.46) with* $b_{21} = b_{12} := \beta$*, suppose that*

$$a - \alpha > \max\left(-\frac{r_2\beta}{r_1}, -\frac{r_1\beta}{r_2}\right),$$

and condition (3.48) is satisfied. Then, there exists a positive equilibrium $x^* = (x_1^*, x_2^*)$*, which is globally asymptotically stable.*

Theorem 3.23 was already obtained by Saito and Takeuchi [45], by using Lyapunov functionals. Here, we have used models (3.46) to illustrate the advantage of our approach, which enables us to obtain the global stability of general Lotka-Volterra type models (3.20), without having to construct specific Lyapunov functionals to each model under consideration, normally a rather difficult task. For the particular case of (3.46) with $b_{12} = \pm b_{21}$, from Theorems 3.7 and 3.15, one easily checks that the local and global stability of x^*, independently of the choices of the delay functions η_{ij}, coincide.

Chapter 4

Global Stability for Neural Network Models

In this chapter, our focus is the global asymptotic stability of steady states in various neural network models (NNM's). First, using the same techniques as in Chapter 3, we obtain the global asymptotic stability of the zero solution of n-dimensional delayed differential systems of the form $\dot{x}(t) = r_i(t)f_i(x_t)$, $i = 1, \dots, n$, by imposing a general condition of negative feedback effect, similar to the hypothesis **(L1*)** in Lemma 3.13. Afterwards, we establish sufficient conditions for the existence, uniqueness, and global asymptotic stability of an equilibrium point of the delayed system $\dot{x}_i(t) = -r_i(t)k_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)]$, $i = 1, \dots, n$, which is a generalization of the well known NNM's of Hopfield, Cohn-Grossberg, bidirectional associative memory, and static with S-type distributed delays. Finally, we use our results to improve several criteria for the existence and global attractivity of the equilibrium point of different types of NNM's.

4.1 Global Asymptotic Stability

Let $C_n := C([-\tau, 0]; \mathbb{R}^n)$ be equipped with the supremum norm $\|\cdot\|$ relative to the norm $|\cdot|$ in $\mathbb{R}^n$, where $|x| = \max_{1 \le i \le n} |x_i|$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. In the phase space C_n, consider a nonautonomous system of delayed differential equations of the form

$$\dot{x}_i(t) = r_i(t)f_i(x_t), \quad t \ge 0, \quad i = 1, \dots, n, \tag{4.1}$$

where $r_i : [0, +\infty) \to (0, +\infty)$ and $f_i : C_n \to \mathbb{R}$ are continuous functions, $i = 1, \dots, n$.

For (4.1) the following hypotheses will be considered:

(N1) **(i)** f_i is bounded on bounded sets of C_n, $i = 1, \dots, n$;

(ii) for all $\varphi \in C_n$ such that $\|\varphi\| = |\varphi(0)| > 0$, then $\varphi_i(0)f_i(\varphi) < 0$ for all $i \in \{1, \ldots, n\}$ such that $|\varphi_i(0)| = \|\varphi\|$;

(N2) $r_i(t)$ is uniformly bounded on $[0, +\infty)$ and $\int_0^{+\infty} r_i(t)dt = +\infty$, $i = 1, \ldots, n$.

Note that hypothesis **(N1)(ii)** is stronger than **(L1*)** in Lemma 3.13. If it holds, we deduce that all solutions of (4.1) are defined and bounded on $[0, +\infty)$. Moreover, **(N1)(ii)** implies that $x = 0$ is the unique equilibrium point of (4.1). Its global asymptotic stability is proved in the following result. We remark that the arguments used in the proof are similar to the ones in the proof of Theorem 3.15.

Theorem 4.1. *Assume* **(N1)**-**(N2)**. *Then the equilibrium $x = 0$ of (4.1) is globally asymptotically stable.*

Proof. Let $x(t) = (x_i(t))_{i=1}^n$ be a solution to (4.1). From Lemma 3.13, the zero solution is stable and $x(t)$ is defined and bounded on $[0, +\infty)$ and we set

$$-v_i = \liminf_{t\to+\infty} x_i(t), \qquad u_i = \limsup_{t\to+\infty} x_i(t), \qquad i \in I := \{1, \ldots, n\},$$

and

$$v = \max_{i\in I}\{v_i\}, \qquad u = \max_{i\in I}\{u_i\}.$$

Note that $u, v \in \mathbb{R}$ and $-v \leq u$.

It is sufficient to prove that $\max(u, v) = 0$. Assume e.g. that $|v| \leq u$, so that $\max(u, v) = u$. (The situation is analogous for $|u| \leq v$.)

Let $i \in I$ such that $u_i = u$ and fix $\varepsilon > 0$. There is $T = T(\varepsilon) > 0$ such that $\|x_t\| < u_\varepsilon := u + \varepsilon$ for $t \geq T$.

As in the proof of Theorem 3.15, first we prove that there is a sequence $(t_k)_{k\in\mathbb{N}}$ with

$$t_k \nearrow +\infty, \qquad x_i(t_k) \to u, \qquad \text{and} \qquad f_i(x_{t_k}) \to 0, \text{ as } k \to +\infty. \tag{4.2}$$

Case 1. Assume that $x_i(t)$ is eventually monotone. In this case, $\lim_{t\to+\infty} x_i(t) = u$ and, for t large, either $\dot{x}_i(t) \leq 0$ or $\dot{x}_i(t) \geq 0$. Assume e.g. that $\dot{x}_i(t) \leq 0$ for t large (the situation $\dot{x}_i(t) \geq 0$ is analogous). Then $f_i(x_t) \leq 0$ for t large, hence

$$\limsup_{t\to+\infty} f_i(x_t) = c \leq 0.$$

If $c < 0$, then there is $t_0 > 0$ such that $f_i(x_t) < c/2$ for $t \geq t_0$, implying that

$$x_i(t) \leq x_i(t_0) + \frac{c}{2}\int_{t_0}^t r_i(s)ds.$$

From **(N2)** and the above inequality, we obtain $x_i(t) \to -\infty$ as $t \to +\infty$, which is not possible. Thus $c = 0$, which proves (4.2).

Case 2. Assume that $x_i(t)$ is not eventually monotone. In this case there is a sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow +\infty$, $\dot{x}_i(t_k) = 0$ and $x_i(t_k) \to u$, as $k \to +\infty$. Then $f_i(x_{t_k}) = 0$ for all $k \in \mathbb{N}$, and (4.2) holds.

Now we have to show that $u = 0$, hence $v = 0$ as well.

For $t \geq T$, we have $\|x_t\| < u_\varepsilon$ and from **(N1)** and **(N2)** we conclude that there is $K > 0$ such that $|\dot{x}_j(t)| = |r_j(t)f_j(x_t)| < K$, $t \geq T, j \in I$. It follows that $x(t)$ and $\dot{x}(t)$ are uniformly bounded on $[0, +\infty)$, thus $\{x_{t_k} : k \in \mathbb{N}\} \subseteq C_n$ is bounded and equicontinuous. By Ascoli-Arzelà theorem, for a subsequence, still denoted by (x_{t_k}), we have $x_{t_k} \to \varphi$ for some $\varphi \in C_n$. Since $\|x_{t_k}\| \leq u_\varepsilon$ and $\varepsilon > 0$ is arbitrary, then $\|\varphi\| \leq u$. From (4.2), we get $\varphi_i(0) = u$ and $f_i(\varphi) = 0$. Clearly $\|\varphi\| = |\varphi_i(0)| = u$ and from hypothesis **(N1)(ii)** we conclude that $u = 0$, and the theorem is proven. ∎

In applications, NNM's often take the form

$$\dot{x}_i(t) = -r_i(t)k_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \quad t \geq 0, \quad i = 1, \dots, n, \tag{4.3}$$

where $r_i : [0, +\infty) \to (0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : C_n \to \mathbb{R}$ are continuous functions, $i = 1, \dots, n$.

In the sequel, for (4.3) the following hypotheses will be considered:

(A1) for each $i \in \{1, \dots, n\}$, there is $\beta_i > 0$ such that

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i, \quad \forall u, v \in \mathbb{R}, u \neq v;$$

(A2) $f_i : C_n \to \mathbb{R}$ is a Lipschitz function with constant l_i, $i = 1, \dots, n$.

Here, we give sufficient conditions for the existence, uniqueness and global asymptotic stability of an equilibrium point for system (4.3). To prove the existence and uniqueness of such equilibrium, we make use of arguments in recent literature [4], [7], [40] and [53]. First, we state the following lemma.

Lemma 4.2. *[18] If $H : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and injective function such that*

$$\lim_{|x|\to+\infty} |H(x)| = +\infty,$$

then H is a homeomorphism of $\mathbb{R}^n$.

Lemma 4.3. *Assume* **(A1)**, **(A2)** *and $\beta_i > l_i$ for $i = 1, \dots, n$. Then system (4.3) has a unique equilibrium point $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$.*

Proof. Define the continuous map

$$\begin{array}{rccl} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & (b_1(x_1)+f_1(x),\ldots,b_n(x_n)+f_n(x)), \quad x=(x_1,\ldots,x_n). \end{array}$$

First, we prove that H is injective. By way of contradiction, assume that there exist $x, y \in \mathbb{R}^n$, with $x \neq y$, such that $H(x) = H(y)$. It follows that $b_i(x_i) + f_i(x) = b_i(y_i) + f_i(y)$ for $i = 1, \ldots, n$, hence

$$|b_i(x_i) - b_i(y_i)| = |f_i(x) - f_i(y)|, \quad i = 1, \ldots, n,$$

and from the hypotheses we have

$$\beta_i|x_i - y_i| \leq l_i|x - y| < \beta_i|x - y|, \quad i = 1, \ldots, n,$$

which is a contradiction.

Now we prove that $\lim_{|x|\to+\infty} |H(x)| = +\infty$. Let $\gamma := \min_{1\leq i\leq n}(\beta_i - l_i) > 0$. For $x \in \mathbb{R}^n$ and $i_0 \in \{1, \ldots, n\}$ such that $|x_{i_0}| = |x|$, we have

$$\begin{aligned} |H(x)| &\geq |b_{i_0}(x_{i_0}) + f_{i_0}(x)| \\ &= |(b_{i_0}(x_{i_0}) - b_{i_0}(0)) + (f_{i_0}(x) - f_{i_0}(0)) + (b_{i_0}(0) + f_{i_0}(0))| \\ &\geq (\beta_{i_0} - l_{i_0})|x_{i_0}| - |b_{i_0}(0) + f_{i_0}(0)| \\ &\geq \gamma|x| - |b_{i_0}(0) + f_{i_0}(0)|, \end{aligned}$$

then $|H(x)| \to +\infty$, as $|x| \to +\infty$.

From the above lemma we conclude that H is a homeomorphism, hence there is a unique $x^* \in \mathbb{R}^n$ such that $H(x^*) = 0$, i.e., x^* is the unique equilibrium point of (4.3). ∎

Lemma 4.4. *Assume* **(A1)**, **(A2)** *and* $\beta_i > l_i$ *for* $i = 1, \ldots, n$. *Suppose that* $x^* = 0$ *is the equilibrium of (4.3). Then the function* $g = (g_1, \ldots, g_n) : C_n \to \mathbb{R}^n$ *defined by* $g_i(\varphi) = -k_i(\varphi_i(0))[b_i(\varphi_i(0)) + f_i(\varphi)]$, *satisfies* **(N1)**.

Proof. Clearly g satisfies **(N1)(i)**.

Let $\varphi \in C_n$ be such that $\|\varphi\| = |\varphi(0)| > 0$ and consider $i \in \{1, \ldots, n\}$ such that $|\varphi_i(0)| = \|\varphi\|$.

Since $x^* = 0$ is the equilibrium, then $b_j(0) + f_j(0) = 0$ for $j = 1, \ldots, n$. If $\varphi_i(0) > 0$, then $\|\varphi\| = \varphi_i(0)$ and from the hypotheses we conclude that

$$\begin{aligned} k_i(\varphi_i(0))(b_i(\varphi_i(0)) + f_i(\varphi)) &= k_i(\varphi_i(0))[(b_i(\varphi_i(0)) - b_i(0)) + (f_i(\varphi) - f_i(0))] \\ &\geq k_i(\varphi_i(0))(\beta_i - l_i)\|\varphi\| > 0. \end{aligned}$$

For the situation $\varphi_i(0) < 0$, analogously we conclude that $\varphi_i(0)g_i(\varphi) < 0$. ■

Assume that $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ is the equilibrium point of (4.3). By translating it to the origin by the change $\bar{x}(t) = x(t) - x^*$, (4.3) becomes

$$\dot{\bar{x}}_i(t) = -r_i(t)\bar{k}_i(\bar{x}_i(t))[\bar{b}_i(\bar{x}_i(t)) + \bar{f}_i(\bar{x}_t)], \quad t \geq 0, \quad i = 1, \dots, n, \tag{4.4}$$

with $\bar{k}_i(u) = k_i(u+x_i^*)$, $\bar{b}_i(u) = b_i(u+x_i^*) - b_i(x_i^*)$ and $\bar{f}_i(\varphi) = f_i(x^*+\varphi) - f_i(x^*)$. Clearly b_i and f_i satisfy **(A1)** and **(A2)** if and only if $\bar{b}_i$ and $\bar{f}_i$ satisfy **(A1)**, **(A2)**. From Lemmas 4.3 and 4.4, and Theorem 4.1, we have the following result:

Theorem 4.5. *Assume* **(A1)**, **(A2)**, *and* **(N2)**. *If $\beta_i > l_i$ for all $i \in \{1, \dots, n\}$, then system (4.3) has a unique equilibrium point which is globally asymptotically stable.*

4.2 Neural Network Models with Distributed Delays

In this section, we shall apply the study in the previous section to two different types of neural network models with distributed delays, improving recent stability results in the literature (see examples below).

4.2.1 Cohen-Grossberg Neural Network Models

Consider the following generalization of the Cohen-Grossberg model (11),

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right], \quad i = 1, \dots, n, \tag{4.5}$$

where $k_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_{ij} : C_1 \to \mathbb{R}$ are continuous functions, $i, j = 1, \dots, n$.

Remark 4.1 Model (4.5) generalizes several neural network models, which have been studied in [1], [3], [6], [20], [26], [32], [57] and [60].

For system (4.5), we assume **(A1)** and

(A3) $f_{ij} : C_1 \to \mathbb{R}$ is a Lipschitz function with constant l_{ij}, $i, j = 1, \dots, n$.

Define the square real matrices,

$$B = diag(\beta_1, \dots, \beta_n), \quad A = [l_{ij}] \quad \text{and} \quad N = B - A, \tag{4.6}$$

where $\beta_1, \dots, \beta_n$ are as in **(A1)**.

Theorem 4.6. *Assume* **(A1)** *and* **(A3)***. If N is a non-singular M-matrix, then there is a unique equilibrium point of (4.5), which is globally asymptotically stable.*

Proof. If N is a non-singular M-matrix, then (see Theorem 1.16) there is $d = (d_1, \dots, d_n) > 0$ such that $Nd > 0$, i.e.,

$$\beta_i d_i > \sum_{j=1}^{n} l_{ij} d_j, \qquad i = 1, \dots, n. \tag{4.7}$$

The change $y_i(t) = d_i^{-1} x_i(t)$ transforms (4.5) into

$$\dot{y}_i(t) = -k_i(d_i y_i(t)) d_i^{-1} \left[b_i(d_i y_i(t)) + \sum_{j=1}^{n} f_{ij}(d_j y_{j,t}) \right], \qquad i = 1, \dots, n. \tag{4.8}$$

Defining, for each $i \in \{1, \dots, n\}$,

$$\bar{f}_i(\varphi) = d_i^{-1} \sum_{j=1}^{n} f_{ij}(d_j \varphi_j), \qquad \varphi = (\varphi_1, \dots, \varphi_n) \in C_n,$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i u), \qquad \bar{k}_i = k_i(d_i u), \qquad u \in \mathbb{R},$$

system (4.8) has the form

$$\dot{y}_i(t) = -\bar{k}_i(y_i(t))[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t)], \qquad t \geq 0, \qquad i \in \{1, \dots, n\}. \tag{4.9}$$

For $\varphi, \psi \in C_n$ and $i \in \{1, \dots, n\}$, we have

$$|\bar{f}_i(\varphi) - \bar{f}_i(\psi)| = d_i^{-1} \left| \sum_{j=1}^{n} f_{ij}(d_j \varphi_j) - \sum_{j=1}^{n} f_{ij}(d_j \psi_j) \right| \leq \left(d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \right) \|\varphi - \psi\|,$$

thus $\bar{f}_i$ is a Lipschitz function with constant $l_i := d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j$, $i = 1, \dots, n$. Moreover, $\bar{b}_i$ satisfies **(A1)** with $\bar{\beta}_i = \beta_i$, and from (4.7) we have $\beta_i > l_i$, $i = 1, \dots, n$. The conclusion follows now from Theorem 4.5. ∎

A particular situation of the model (4.5) is the class of n-neuron Hopfield network with discrete delays

$$\dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t - \tau_{ij})), \qquad t \geq 0, \qquad i = 1, \dots, n, \tag{4.10}$$

where $\tau_{ij} \geq 0$, $b_i > 0$, $a_{ij} \in \mathbb{R}$ and $f_j : \mathbb{R} \to \mathbb{R}$ are C^1 such that $f_j(0) = 0$, $\lim_{u \to \pm\infty} f_j(u) = \pm 1$, $f_j'(u) > 0$, and $\sup_{u \in \mathbb{R}} f_j'(u) = f_j'(0) = 1$, $i, j = 1, \dots, n$ (see [3], [56]). These conditions imply that, for each $j \in \{1, \dots, n\}$, f_j is a

Lipschitz function with Lipschitz constant 1. The function $f_j(u) = \tanh(u)$, which is commonly used in the model (4.10), satisfies the above conditions.

Campbell [3] proved that (4.10) has a unique equilibrium which is global asymptotic stable if $\hat{M} = diag(b_1, \dots, b_n) - [|a_{ij}|]$ is a non-singular M-matrix. We emphasize however that Theorem 4.6 deals with the situation of distributed delays.

Example 4.1 Consider the Cohen-Grossberg NNM with discrete delays

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} \sum_{p=1}^{P} a_{ij}^{(p)} f_j(x_j(t - \tau_{ij}^{(p)})) + J_i \right], \qquad (4.11)$$

for $i = 1, \dots, n$, where $P \in \mathbb{N}$, $J_i, a_{ij}^{(p)} \in \mathbb{R}$, $\tau_{ij}^{(p)} \geq 0$, and $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $i, j = 1, \dots, n$, $p = 1, \dots, P$, recently studied in [7] and [59]. Let $\tau = \max\{\tau_{ij}^{(p)} : i, j = 1, \dots, n, p = 1, \dots, P\}$.

System (4.11) has the form (4.5) for $f_{ij}(\varphi) = -\sum_{p=1}^{P} a_{ij}^{(p)} f_j(\varphi(-\tau_{ij}^{(p)}))$, $\varphi \in C_1 = C([-\tau, 0], \mathbb{R})$. Since $f_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants l_i, f_{ij} is also a Lipschitz function, with Lipschitz constant $l_{ij} = \sum_{p=1}^{P} |a_{ij}^{(p)}| l_j$, for $i, j = 1, \dots, n$. Theorem 4.6 applied to system (4.11) gives the following result:

Corollary 4.7. *Assume* **(A1)** *and that $f_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant l_i, $i = 1, \dots, n$. If $N := B - A$, where $B = diag(\beta_1, \dots, \beta_n)$ and $A = [l_{ij}]$ with $l_{ij} = \sum_{p=1}^{P} |a_{ij}^{(p)}| l_j$, is a non-singular M-matrix, then there is a unique equilibrium point of (4.11), which is globally asymptotically stable.*

Remark 4.2 For system (4.11), the existence, uniqueness and global stability of an equilibrium point was already obtained by Y. Chen [7], but he assumed the following additional hypotheses:

(i) For each $i \in \{1, \dots, n\}$, there exist $\underline{k}_i, \overline{k}_i > 0$ such that

$$0 < \underline{k}_i \leq k_i(u) \leq \overline{k}_i, \quad \forall u \in \mathbb{R};$$

(ii) $\underline{N} := B\underline{K} - A\overline{K}$ is a non-singular M-matrix, where $\underline{K} = diag(\underline{k}_1, \dots, \underline{k}_n)$ and $\overline{K} = diag(\overline{k}_1, \dots, \overline{k}_n)$.

Note that, if **(i)** holds, then $\underline{N}$ is a non-singular M-matrix which implies that N is a non-singular M-matrix. However the reverse is not true. The above Corollary 4.7 improves strongly the criterion in [7].

4.2.2 Static Neural Network Models with S-Type Distributed Delays

Consider the following generalization of the static model (13),

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + f_i \left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i \right) \right], \quad (4.12)$$

for $i = 1, \ldots, n$, where $J_i, \omega_{ij} \in \mathbb{R}$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions, i.e., $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \ldots, n$. Assume the hypothesis:

(A4) $f_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant l_i, $i = 1, \ldots, n$.

For each $i \in \{1, \ldots, n\}$, the function defined by

$$\bar{f}_i(\varphi) = f_i \left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} \varphi_j(\theta) d\eta_{ij}(\theta) + J_i \right), \qquad \varphi = (\varphi_1, \ldots, \varphi_n) \in C_n$$

is a Lipschitz function with constant $l_i \sum_{j=1}^{n} |\omega_{ij}|$. Define the following square real matrices:

$$B = diag(\beta_1, \ldots, \beta_n) \quad \text{and} \quad M = B - [l_i |\omega_{ij}|]. \quad (4.13)$$

We have the following result:

Theorem 4.8. *Assume* **(A1)** *and* **(A4)***. If M is a non-singular M-matrix, then there is a unique equilibrium point of (4.12), which is globally asymptotically stable.*

Proof. The proof is analogous to the proof of Theorem 4.6, so it is omitted. ∎

Example 4.2 Consider the static neural network model with S-type distributed delay studied in [58]

$$\dot{x}_i(t) = -b_i(\lambda) x_i(t) + f_i \left(\sum_{j=1}^{n} \omega_{ij}(\lambda) \int_{-\tau(\lambda)}^{0} x_j(t+\theta) d\eta_{ij}(\lambda, \theta) + J_i(\lambda) \right), \quad (4.14)$$

$i = 1, \ldots, n$, where $\lambda \in \Lambda \subseteq \mathbb{R}$ is a real parameter, $f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $b_i, \tau : \Lambda \to [0, +\infty)$ and $J_i, \omega_{ij} : \Lambda \to \mathbb{R}$ are real functions with $0 \leq \tau(\lambda) \leq \tau$ for some $\tau > 0$, and, for each $\lambda \in \Lambda$, $\theta \mapsto \eta_{ij}(\lambda, \theta)$ are normalized bounded variation functions on $[-\tau(\lambda), 0]$, $i, j = 1, \ldots, n$.

Suppose that, for each $i, j = 1, \dots, n$, there exist $\underline{b}_i, \overline{\omega}_{ij} > 0$ such that,

$$0 < \underline{b}_i \leq b_i(\lambda), \quad \text{and} \quad |\omega_{ij}(\lambda)| \leq \overline{\omega}_{ij}, \quad \text{for all } \lambda \in \Lambda.$$

Assume that the functions f_i satisfy **(A4)**, and define the following square real matrices:

$$B(\lambda) = diag(b_1(\lambda), \dots, b_n(\lambda)), \quad M(\lambda) = B(\lambda) - [l_i|\omega_{ij}(\lambda)|], \quad \lambda \in \Lambda,$$

$$\underline{B} = diag(\underline{b}_1, \dots, \underline{b}_n) \quad \text{and} \quad \underline{M} = \underline{B} - [l_i\overline{\omega}_{ij}].$$

Definition 4.1. *System (4.14) is said to be globally asymptotically robust stable on Λ if, for each $\lambda \in \Lambda$, there is an equilibrium point of (4.14) which is globally asymptotically stable.*

The next result is an immediate consequence of Theorem 4.8.

Corollary 4.9. *Assume* **(A4)**. *If $\underline{M}$ is a non-singular M-matrix, then system (4.14) is globally asymptotically robust stable on Λ.*

Proof. Let $\lambda_0 \in \Lambda$. Since $\underline{M} \leq M(\lambda_0)$ and $\underline{M}$ is a non-singular M-matrix, then $M(\lambda_0)$ is a non-singular M-matrix as well (see Theorem 1.18), thus we have the result from Theorem 4.8. ∎

Remark 4.3 Besides the assumptions in Corollary 4.9, Wang and Wang [58] assumed that the maps $\lambda \mapsto b_i(\lambda)$ were bounded and that, for each $\lambda \in \Lambda$, $\theta \mapsto \eta_{ij}(\lambda, \theta)$ were non-decreasing normalized functions on $[-\tau(\lambda), 0]$. Thus the last result improves the main result in [58].

Remark 4.4 The results in this section also hold for non-autonomous models of the form (4.3), if the functions $r_i(t)$ satisfy **(N2)**.

4.3 Neural Network Models with Discrete Time-Varing Delays

Consider the following neural network model:

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(x_j(t - \tau_{ij}^{(p)}(t))) \right], \tag{4.15}$$

for $i = 1, \dots, n$, where $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, h_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ and $\tau_{ij}^{(p)} : [0, +\infty) \to [0, +\infty)$ are continuous functions, $h_{ij}^{(p)}$ are Lipschitz functions with constants $l_{ij}^{(p)}$, $\tau_{ij}^{(p)}$ are bounded and **(A1)** holds for b_i, $i, j = 1, \dots, n$, $p = 1, \dots, P$.

System (4.15) is a generalization of several neural network models with discrete time-varying delays [4], [5], [8]. It is important to note that the general setting of (4.15) allows us to consider as subclasses the bidirectional associative memory neural network models in [4] and [61].

Let $\tau \geq 0$ be such that $0 \leq \tau_{ij}^{(p)}(t) \leq \tau$ for all $t \geq 0$, $i, j \in \{1, \dots, n\}$ and $p \in \{1, \dots, P\}$, and define the square real matrices

$$B = diag(\beta_1, \dots, \beta_n) \qquad \text{and} \qquad N := B - [l_{ij}],$$

where $\beta_1, \dots, \beta_n$ are as in **(A1)** and $l_{ij} = \sum_{p=1}^{P} l_{ij}^{(p)}$.

Theorem 4.10. *Assume* **(A1)***,* $0 \leq \tau_{ij}^{(p)}(t) \leq \tau$ *and* $h_{ij}^{(p)}$ *are Lipschitz functions with constants* $l_{ij}^{(p)}$*,* $i, j \in \{1, \dots, n\}$*,* $p \in \{1, \dots, P\}$*.*

If N *is a non-singular M-matrix, then there is a unique equilibrium point of (4.15), which is globally asymptotically stable.*

Proof. Since N is a non-singular M-matrix, then (see Theorem 1.16) there is $d = (d_1, \dots, d_n) > 0$ such that $Nd > 0$, i.e.,

$$\beta_i > d_i^{-1} \left(\sum_{j=1}^{n} l_{ij} d_j \right), \qquad i \in I := \{1, \dots, n\}. \tag{4.16}$$

The change $z_i(t) = d_i^{-1} x_i(t)$ transforms (4.15) into

$$\dot{z}_i(t) = -\bar{k}_i(z_i(t))[\bar{b}_i(z_i(t)) + h_i(t, z_t)], \qquad i \in I, \qquad t \geq 0, \tag{4.17}$$

where

$$h_i(t, \varphi) = d_i^{-1} \left[\sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(d_j \varphi_j(-\tau_{ij}^{(p)}(t))) \right], \qquad t \geq 0, \varphi \in C_n, i \in I,$$

$$\bar{k}_i(u) = k_i(d_i u), \qquad \bar{b}_i(u) = d_i^{-1} b_i(d_i u), \qquad u \in \mathbb{R}, i \in I.$$

Note that $(\bar{b}_i(u) - \bar{b}_i(v))/(u - v) \geq \beta_i$ for $u, v \in \mathbb{R}$, $u \neq v$, i.e., condition **(A1)** is satisfied by the functions $\bar{b}_i(u)$, $i \in I$. For $\varphi, \psi \in C_n$ and $t \geq 0$ we have

$$|h_i(t, \varphi) - h_i(t, \psi)| \leq \left(d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \right) \|\varphi - \psi\|, \qquad i \in I,$$

that is, $h_i(t, \cdot)$ is a uniform Lipschitz function on C_n for all $t \geq 0$, with Lipschitz constant $l_i := d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j < \beta_i$.

Observe that system (4.17) has an equilibrium point $y^* = (y_1^*, \dots, y_n^*) \in \mathbb{R}^n$ if and only if $H(y^*) = 0$, where

$$H(y) = \left(\bar{b}_i(y_i) + d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(d_j y_j) \right)_{i=1}^{n}, \qquad y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

Arguing as in the proof of Lemma 4.3, we conclude that there is a unique point $y^* = (y_1^*, \ldots, y_n^*)$ such that $H(y^*) = 0$.

By translating the equilibrium to the origin by the change $y_i(t) = z_i(t) - y_i^*$, (4.17) becomes

$$\dot{y}_i(t) = g_i(t, y_t), \quad t \geq 0, i \in I, \tag{4.18}$$

where $g = (g_1, \ldots, g_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ is defined by

$$g_i(t, \varphi) = -\bar{k}_i(\varphi_i(0) + y_i^*)[\bar{b}_i(\varphi_i(0) + y_i^*) + h_i(t, \varphi + y^*)], \quad \varphi \in C_n, t \geq 0, i \in I.$$

Arguing as in the proof of Lemma 4.4, we conclude that g satisfies **(L1*)**, thus from Lemma 3.13 all solutions of (4.18) are defined and bounded on $[0, +\infty)$.

Let $y(t) = (y_i(t))_{i=1}^n$ be a solution of (4.18). Set

$$-v_i = \liminf_{t \to +\infty} y_i(t), \quad u_i = \limsup_{t \to +\infty} y_i(t), \quad i \in I,$$

and

$$v = \max_{i \in I}\{v_i\}, \quad u = \max_{i \in I}\{u_i\}.$$

Note that $u, v \in \mathbb{R}$ and $-v \leq u$.

It is sufficient to prove that $\max(u, v) = 0$. Assume e.g. that $|v| \leq u$, so that $\max(u, v) = u$. (The situation $|u| \leq v$ is analogous).

Fix $\epsilon > 0$ and let $T = T(\epsilon) > 0$ be such that $\|y_t\| < u_\epsilon := u + \epsilon$ for $t \geq T$. Let $i \in I$ such that $u_i = u$.

Arguing as in the proof of Theorem 4.1, we conclude that there is a sequence $(t_k)_{k \in \mathbb{N}}$ such that

$$t_k \nearrow +\infty, \quad y_i(t_k) \to u \quad \text{and} \quad g_i(t_k, y_{t_k}) \to 0, \quad \text{as} \quad k \to +\infty. \tag{4.19}$$

From our hypotheses, clearly we have g bounded on $[0, +\infty) \times K$ for all bounded sets $K \subseteq C_n$. Since $\|y_t\| < u_\epsilon$ for $t \geq T$, we have $(\dot{y}_j(t))_{j=1}^n$ bounded on $[0, +\infty)$. Hence $y(t)$ and $\dot{y}(t)$ are uniformly bounded on $[0, +\infty)$, thus $\{y_{t_k} : k \in \mathbb{N}\} \subseteq C_n$ is bounded and equicontinuous. By Ascoli-Arzelà theorem, for a subsequence, still denoted by (y_{t_k}), we have $y_{t_k} \to \varphi$ for some $\varphi \in C_n$. Since $\|y_{t_k}\| \leq u_\epsilon$ and $\epsilon > 0$ is arbitrary, then $\|\varphi\| \leq u$. Moreover, from (4.19) we get $\varphi_i(0) = u$.

Since the sequence $\left(\left(\tau_{ij}^{(p)}(t_k)\right)\right)_{k \in \mathbb{N}}$ in $\mathbb{R}^{Pn^2}$ is bounded, there is a subsequence of (t_k), still denoted by (t_k), which converges to a point $(\tau_{ij}^{(p)*}) \in [0, \tau]^{Pn^2}$. Thus

$$g_i(t_k, y_{t_k}) \to c_i \quad \text{as } k \to +\infty,$$

with

$$c_i := -\bar{k}_i(\varphi_i(0) + y_i^*)[\bar{b}_i(\varphi_i(0) + y_i^*) + \bar{h}_i(\varphi)],$$

where

$$\bar{h}_i(\varphi) := d_i^{-1}\left[\sum_{j=1}^{n}\sum_{p=1}^{P} h_{ij}^{(p)}(d_j(\varphi_j(-\tau_{ij}^{(p)*}) + y_j^*))\right].$$

Since y^* is the equilibrium point of (4.17), we have $\bar{b}_j(y_j^*) + \bar{h}_j(0) = 0$ for all $j \in I$.

If $\varphi_i(0) = u > 0$, then

$$\bar{b}_i(\varphi_i(0) + y_i^*) + \bar{h}_i(\varphi) = \bar{b}_i(\varphi_i(0) + y_i^*) - \bar{b}_i(y_i^*) + \bar{h}_i(\varphi) - \bar{h}_i(0)$$

$$\geq \beta_i\varphi_i(0) - d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j\|\varphi\| = \left(\beta_i - d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j\right)u > 0.$$

Since $\bar{k}_i(u + y_i^*) > 0$, we have $c_i \neq 0$, which contradicts (4.19). Hence $u = 0$ and then all solutions $y(t)$ of (4.18) verify $y(t) \to 0$ as $t \to +\infty$, that is, the equilibrium point of (4.15) is globally asymptotically stable. ■

Example 4.3 Consider the Cohen-Grossberg neural network model studied in [8]

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) - \sum_{j=1}^{n} a_{ij}f_j(x_j(t - \tau_{ij}(t))) + J_i\right], \quad (4.20)$$

$i = 1, \dots, n$, where $a_{ij}, c_{ij}, J_i \in \mathbb{R}$ and $\tau_{ij} : [0, +\infty) \to [0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i, g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $i, j = 1, \dots, n$, with τ_{ij} bounded.

Assume that the functions b_i satisfy **(A1)** and $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i and γ_i, $i = 1, \dots, n$. Define the square real matrices

$$B = diag(\beta_1, \dots, \beta_n) \quad \text{and} \quad N = B - [|c_{ij}|\gamma_j] - [|a_{ij}|\theta_j],$$

where $\beta_1, \dots, \beta_n$ are as in **(A1)**.

Clearly, (4.20) is a particular situation of (4.15). From Theorem 4.10 we have the following result:

Corollary 4.11. *Assume* **(A1)** *and that $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i and γ_i, $i = 1, \dots, n$.*

If N is a non-singular M-matrix, then there is an equilibrium point of (4.20), which is globally asymptotically stable.

Remark 4.5 Model (4.20) was studied in [8] and [53]. Chen and Rong [8] proved that all solutions of (4.20) converge exponentially to the equilibrium point with the additional hypotheses:

(i) $\tau_{ij}(t)$ are continuously differentiable functions with $\tau'_{ij}(t) \leq 1$ for all $t \geq 0$, $i, j = 1, \ldots, n$;

(ii) There are $\underline{k}_i, \overline{k}_i > 0$ such that

$$0 < \underline{k}_i \leq k_i(u) \leq \overline{k}_i, \qquad u \in \mathbb{R}, i = 1, \ldots, n.$$

Without condition **(i)** and assuming that there is $\underline{k}_i > 0$ such that $\underline{k}_i \leq k_i(u)$ for all $u \in \mathbb{R}$, $i = 1, \ldots, n$, instead of **(ii)**, Song and Cao [53] proved the exponential stability of (4.20). In a forthcoming paper, the exponential stability of the equilibrium of general models (4.5) and (4.15) will be addressed.

Example 4.4 Consider the Hopfield neural network model

$$\dot{x}_i(t) = -d_i(\lambda)x_i(t) + \sum_{j=1}^{n} c_{ij}(\lambda)g_j(x_j(t)) + \sum_{j=1}^{n} a_{ij}(\lambda)f_j(x_j(t - \tau_{ij}(t)))+ \\ +J_i(\lambda), \qquad i = 1, \ldots, n, \quad (4.21)$$

where $\lambda \in \Lambda \subseteq \mathbb{R}$ is a real parameter, $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are bounded continuous functions, $a_{ij}, c_{ij}, d_i, J_i : \Lambda \to \mathbb{R}$ are real functions and $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i, γ_i for $i, j = 1, \ldots, n$.

Note that, for each $\lambda \in \Lambda$, (4.21) looks like (4.20) when $k_i(u) \equiv 1$ and $b_i(u) = d_i(\lambda)u$ for $u \in \mathbb{R}$, $i = 1, \ldots, n$.

Assume that there are square real matrices $\bar{A} = [\bar{a}_{ij}] \geq 0$, $\bar{C} = [\bar{c}_{ij}] \geq 0$ and $\underline{D} = diag(\underline{d}_1, \ldots, \underline{d}_n)$, with $\underline{d}_i > 0$ for all $i \in \{1, \ldots, n\}$, such that, for each $\lambda \in \Lambda$,

$$|a_{ij}(\lambda)| \leq \bar{a}_{ij}, \qquad |c_{ij}(\lambda)| \leq \bar{c}_{ij}, \qquad \text{and} \qquad 0 < \underline{d}_i \leq d_i(\lambda), \qquad i, j = 1, \ldots, n.$$

For each $\lambda \in \Lambda$, define

$$D(\lambda) = diag(d_1(\lambda), \ldots, d_n(\lambda)), \qquad M(\lambda) = D(\lambda) - [|a_{ij}(\lambda)|\theta_j] - [|c_{ij}(\lambda)|\gamma_j] \qquad \text{and}$$

$$\underline{M} = \underline{D} - [\bar{a}_{ij}\theta_j] - [\bar{c}_{ij}\gamma_j].$$

From Theorem 4.10 we have the following result:

Corollary 4.12. *If $\underline{M}$ is a non-singular M-matrix, then system (4.21) is globally asymptotically robust stable on Λ.*

Proof. Let $\lambda_0 \in \Lambda$. Since $\underline{M} \leq M(\lambda_0)$ and $\underline{M}$ is a non-singular M-matrix, then (see Theorem 1.18) $M(\lambda_0)$ is also a non-singular M-matrix and the result follows from Theorem 4.10. ∎

Remark 4.6 In [26], the global asymptotic robust stability of the Hopfield model (4.21) with discrete independent delays $\tau_{ij}(t) \equiv \tau_{ij}$ was proved. Hence, our Corollary 4.12 is a generalization of the main result in [26].

It is important to note that the general setting of (4.15) allows us to consider as a subclass the bidirectional associative memory NNM with delays.

Example 4.5 Consider the following model:

$$\begin{cases} \dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + \displaystyle\sum_{p=1}^{P} g_i^{(p)}(x_i(t-\omega_i^{(p)}(t))) \right. \\ \qquad\qquad \left. - \displaystyle\sum_{j=1}^{m}\sum_{p=1}^{P} f_{ij}^{(p)}(y_j(t-\tau_{ij}^{(p)}(t))) \right], \quad i = 1,\ldots,n, \\ \dot{y}_j(t) = -h_j(y_j(t)) \left[a_j(y_j(t)) + \displaystyle\sum_{p=1}^{P} f_j^{(p)}(y_j(t-\rho_j^{(p)}(t))) \right. \\ \qquad\qquad \left. - \displaystyle\sum_{i=1}^{n}\sum_{p=1}^{P} g_{ji}^{(p)}(x_i(t-\sigma_{ji}^{(p)}(t))) \right], \quad j = 1,\ldots,m, \end{cases} \tag{4.22}$$

for $t \geq 0$ and $n, m, P \in \mathbb{N}$, where $k_i, h_j : \mathbb{R} \to (0,+\infty)$, $b_i, a_j, g_i^{(p)}, f_j^{(p)}, g_{ji}^{(p)}, f_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ are continuous functions and $\omega_i^{(p)}, \rho_j^{(p)}, \tau_{ij}^{(p)}, \sigma_{ji}^{(p)} : [0,+\infty) \to [0,+\infty)$ are bounded continuous functions, $i = 1,\ldots,n$, $j = 1,\ldots,m$ e $p = 1,\ldots,P$.

Arik [1] and Wang and Zou [60] studied the bidirectional associative memory neural network model with discrete delays described by

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\ \dot{y}_i(t) = -y_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_{ij})) + J_i \end{cases}, \quad i = 1,\ldots,n. \tag{4.23}$$

Wang and Zou [61] incorporated inhibitory self-connections terms into model

(4.23), and considered the following system

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + c_{ii} g_i(x_i(t-d_{ii})) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\ \\ \dot{y}_i(t) = -y_i(t) + l_{ii} f_i(y_i(t-m_{ii})) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_{ij})) + J_i \end{cases} \tag{4.24}$$

$i = 1, \ldots, n$. Recently, the following bidirectional associative memory neural network model with time-varying delays was considered in [4]:

$$\begin{cases} \dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{m} c_{ij} f_j(\lambda_j y_j(t-\tau_{ij}(t))) + I_i \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n, \\ \dot{y}_j(t) = -h_j(y_j(t)) \left[a_j(y_j(t)) - \sum_{i=1}^{n} d_{ji} g_i(\mu_i x_i(t-\sigma_{ji}(t))) + J_j \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, m, \end{cases} . \tag{4.25}$$

Model (4.22), here considered for the first time (as far as we know), arises as a generalization of all these models. Since (4.22) is a particular situation of (4.15), from Theorem 4.10 we have the following result:

Corollary 4.13. *Suppose that: a_j and b_i satisfy* **(A1)** *with constants α_j and β_i, respectively; $k_i(u) > 0$ and $h_j(u) > 0$ for all $u \in \mathbb{R}$; $f_j^{(p)}, g_i^{(p)}, f_{ij}^{(p)}, g_{ji}^{(p)}$ are Lipschitz functions with Lipschitz constants $\theta_j^{(p)}, \gamma_i^{(p)}, \theta_{ij}^{(p)}, \gamma_{ji}^{(p)}$ respectively; and $\omega_i^{(p)}, \rho_j^{(p)}, \tau_{ij}^{(p)}, \sigma_{ji}^{(p)}$ are bounded continuous functions, for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $p = 1, \ldots, P$.*

Define

$$N := \begin{bmatrix} B - G_d & -F \\ \\ -G & A - F_d \end{bmatrix}_{(n+m)\times(n+m)},$$

where

$$B = diag(\beta_1, \ldots, \beta_n), \quad A = diag(\alpha_1, \ldots, \alpha_m)$$

$$G_d = diag\left(\sum_{p=1}^{P} \gamma_1^{(p)}, \ldots, \sum_{p=1}^{P} \gamma_n^{(p)} \right), \quad F_d = diag\left(\sum_{p=1}^{P} \theta_1^{(p)}, \ldots, \sum_{p=1}^{P} \theta_m^{(p)} \right),$$

$$G = \left[\sum_{p=1}^{P} \gamma_{ji}^{(p)} \right]_{m\times n}, \quad F = \left[\sum_{p=1}^{P} \theta_{ij}^{(p)} \right]_{n\times m}.$$

If N is a non-singular M-matrix, then there is a unique equilibrium point of (4.22), which is globally asymptotically stable.

Remark 4.7 As remarked, (4.22) is a generalization of models (4.23), (4.24) and (4.25). With the same hypotheses of Corollary 4.13, the exponential stability of (4.23) and (4.24) was obtained in [1] and [61]. In [4], the same stability was obtained for system (4.25) with the additional hypotheses $k_i(u) \geq k_i > 0$ and $h_i(u) \geq h_i > 0$, $u \in \mathbb{R}$, $i = 1, \ldots, n$. As mentioned in Remark 4.5, the question of the exponential asymptotic stability for delayed neural networks will be addressed in the future, for the general framework of systems of the form (4.5) and (4.15).

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