EXISTENCE AND VARIATIONAL STABILITY OF SOLUTIONS OF KURZWEIL EQUATIONS ASSOCIATED WITH QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

Ph.D THESIS

BISHOP, SHEILA AMINA

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A Ph.D. Thesis

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Title

Declaration

I, **Bishop, Sheila Amina**, (Matric. Number: CUGP070184) declare that this research work was carried out by me under the supervision of Prof. E.O. Ayoola of the Department of Mathematics, University of Ibadan, Ibadan and Dr. P.O. Olanrewaju of the Department of Mathematics, Covenant University, Ota, Nigeria.

I attest that the thesis has not been presented either wholly or partly for the award of any degree elsewhere. All sources of data and scholarly information used in this thesis are duly acknowledged.

Signature—————————

Date—————————

Bishop, Sheila Amina

Certification

This is to certify that this research work was carried out by **Mrs. Bishop, Sheila Amina, (CUGP070184)** in the Department of Mathematics, School of Natural and Applied Sciences, College of Science and Technology, Covenant University, Ota, Nigeria.

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Dedication

With sincere gratitude to the Almighty God and to my Lord and Saviour Jesus Christ, I dedicate this work to my darling husband Mr. John Bishop and to my wonderful children David, Swalina and King.

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S. A. Bishop

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Abstract

The role of generalized ordinary differential equation (Kurzweil equation) in applying the technique of topological dynamics to the study of classical ordinary differential equation as outlined in [3, 4, 47, 51-58, 88-90] is a major motivation for studying this class of equations associated with the weak forms of the Lipschitzian quantum stochastic differential equations.

In this work, existence and uniqueness of solution of quantum stochastic differential equations associated with the Kurzweil equations under a more general Lipschitz condition were established. The results here generalize the results in the existing literatures thereby extending the class of equations for which the theory of quantum stochastic differential equation is applicable.

Existence of solution of quantum stochastic differential equation, enabled one to investigate and establish other qualitative properties of solution such as variational stability, variational attracting, variational asymptotic stability, converse variational stability and continuous dependence of solution on a parameter.

The results are established within the framework of the topological linear space of processes of finite variations. The theory of Kurzweil equations associated with quantum stochastic differential equation provides a basis for future application of the technique of topological dynamics to the study of quantum stochastic differential equation.

Contents

Chapter 1

Introduction

1.1 Background Of the study

Most differential equations (Ordinary, Stochastic, Quantum stochastic, etc.) do not have closed analytical solutions, that is solutions that can be written in closed forms. Only approximate solutions or numerical solutions can be obtained if such solutions exist. Before one ventures into the rigour of numerical computations, it is imperative to examine whether there exists a solution for the given differential equation, a unique solution probably for a given initial condition.

As with most ordinary differential equations, the qualitative properties of quantum stochastic differential equations (QSDEs) cannot be studied without knowing if the equation has a unique solution, because it would be needless studying the properties or behaviour of what does not exist. Therefore the study of the theory of the existence and uniqueness of solution is vital to both the analysis of qualitative properties of solutions and the numerical analysis.

There have been intensive research activities in the literature concerning the theoretical and numerical analysis of classical stochastic differential equations of the type

$$\begin{aligned} dX(t,w) &= H(X(t),t)dt + F(X(t),t)dQ(t) \\ X(t_0) &= X_0, \quad t \in [t_0,T] \end{aligned} \tag{1.1}$$

Equation (1.1) is understood in the integral form

$$X(t,w) = X_0 + \int_{t_0}^{t} (H(X(s),s)ds + F(X(s),s)dQ(s)) \quad (1.2)$$

where the first integral on the right hand side of (1.2) is in general a Lebesgue integral and the second integral is the Ito integral driven by a Martingale, in particular, a Brownian motion Q(t) and the coefficients H, F are sufficiently smooth deterministic ordinary functions defined on the space $I \times \mathbb{R}_n$, where $I = [t_0, T]$ and $n \in \mathbb{N}$.

The Ito integral cannot be interpreted as an ordinary Riemann-Steiltjes integral since Q is not differentiable in the ordinary sense. Equation (1.1) has found applications in diverse fields such as Stochastic Analysis, Engineering, Physics, Geology, Meteorology, Finance, AIDS/HIV epidemiology, medicine and other biomedical systems.

A noncommutative generalization of (1.1) is the following quantum stochastic differential equation (QSDE) introduced by Hudson and Parthasarathy [44]:

$$\begin{aligned} dX(t) &= E(X(t),t)d\wedge_\pi(t) + F(X(t),t)dA_g(t) \\ &\quad + G(X(t),t)dA_{f^+}(t) + H(X(t),t)dt), \ X(t_0) = X_0, \quad t \in I \qquad (1.3) \end{aligned}$$

In equation (1.3), the coefficients E, F, G, and H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes $\Lambda_\Pi, A_{f^+}, A_g$ and the Lebesgue measure t are defined. Equation (1.3) is understood in integral form as

$$\begin{aligned} X(t) &= X_0 + \int_{t_0}^{t} (E(X(s),s)d\wedge_\pi(s) + F(X(s),s)dA_g(s) \\ &\quad + G(X(s),s)dA_{f^+}(s) + H(X(s),s)ds), \ t \in I \qquad (1.4) \end{aligned}$$

Quantum stochastic differential equation arises from quantum theory which can be regarded as a theory of non-commutative probability (quantum probability) in which observables are represented by noncommuting, self-adjoint linear operators acting on dense domains of some Hilbert spaces.

It has been well established that the quantum stochastic differential equations introduced by Hudson and Parthasarathy [44] provide an essential tool in the theoretical description of physical systems, especially those arising in quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems. The time evolution in these models is given by a unitary cocycle that solves a Hudson-Parthasarathy quantum stochastic differential equation. In the sense of [22], these unitaries define a flow, which is a quantum Markov process that represents the Heisenberg time evolution of the observables of the physical system.

Several authors have studied how quantum stochastic models can be obtained as a limit of fundamental models in quantum field theory [1, 38, 42, 46]. This provides a sound justification for using quantum stochastic models to describe several physical systems.

So much work has been done on existence of solution of stochastic differential equations compared with quantum stochastic differential equation introduced above. However existing literatures [5, 6, 30, 46, 48, 60, 94, 95] show that the existence of solution for both classical stochastic differential equations and quantum stochastic differential equations are subject to the Lipschitz condition. This restrict the class under which the results are applicable.

In the work of [30], the Hudson and Parthasarathy [44] quantum stochastic calculus was employed to establish the equivalent form of quantum stochastic differential equation (1.3) given by

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &= P(X,t)(\eta,\xi) \\ X(t_0) &= X_0, t \in [t_0, T], \end{aligned} \tag{1.5}$$

where η, ξ lie in some dense subspaces of some Hilbert spaces which will be defined later. As explained in [3, 6, 30], the map $(X,t) \to P(X,t)(\eta,\xi)$ appearing in equation

(1.5) has the form

$$P(X,t)(\eta,\xi) = (\mu E)(X,t)(\eta,\xi) + (\gamma F)(X,t)(\eta,\xi) + (\sigma G)(X,t)(\eta,\xi) + H(X,t)(\eta,\xi) \quad (1.6)$$

where $\eta,\xi \in I\!D\underline{\otimes}I\!E$ is arbitrary, $(X,t) \in \tilde{\mathcal{A}} \times I$ and $H(X,t)(\eta,\xi) := \langle \eta, H(X,t)\xi \rangle$. The integral at the right hand side of equation (1.4) is the Hudson and Parthasarathy quantum stochastic integral introduced in [44].

The connection between Equation (1.2) and (1.4) is that (1.4) reduces to (1.2) by a suitable choice of parameters in a simple Fock space. Hence equation (1.3) is applicable to a wider class of real life problems than equation (1.1). In particular QSDE (1.3) often arise as mathematical models which describe among other things, quantum dynamical systems and several physical problems in quantum stochastic control theory and quantum stochastic evolutions [30, 31, 44, 46].

In comparison with equation (1.1), QSDE (1.3) has not enjoyed intensive research activities in respect of the investigation of the theoretical and numerical properties of solutions that live in certain infinite dimensional locally convex spaces. Equation (1.5) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side. There are other formulations of QSDE as developed in [5, 18, 38, 43, 44] but the nearest to the Ito stochastic calculus is the Hudson and Parthasarathy formulation [42]. Some recent investigations have been done concerning existence of solutions and their numerical approximations for equation (1.5) [6-8, 30, 33].

In [6], the equivalence of the Lipschitzian quantum stochastic differential equation (1.5) with the associated Kurzweil equation

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = DF(x,t)(\eta,\xi) \quad (1.7)$$

was established along with some numerical approximations. In arriving at these results, assumption of Lipschitz and Caratheodory conditions were imposed on the map

$(X, t) \rightarrow P(X, t)(\eta, \xi)$, these restrict the class of equations under which the results are applicable.

There is therefore the need to establish existence results that will depend on a more general Lipschitz condition. The aim of this research is to establish more general conditions that will guarantee the existence and uniqueness of solution of the Kurzweil equation(1.7) associated with QSDE (1.5) thereby generalizing the results in [6] and to investigate other qualitative properties of solution such as stability and continuous dependence of solution on a parameter. It is worth mentioning that this is the first time the qualitative properties (variational stability and continuous dependence on parameter) of solution of QSDE (1.5) will be considered.

The role of generalized ordinary differential equations in applying topological dyna mics to the study of ordinary differential equations as outlined in [3, 4, 6, 87] is the major motivation for studying this class of equations associated with the weak forms of quantum stochastic differential equations. This research is strongly motivated by the need to create a framework for the application of the technique of topological dynamics to the study of quantum stochastic differential equations as obtained in the case of ordinary differential equations [3, 4, 41, 47, 51-58, 79-87].

Therefore, the space of the associated Kurzweil equations (1.7) will then be a completion of the space of the equivalent non classical first order ordinary differential equation (1.5) as observed by [3]. The result on existence of solution will be applicable to a wider class of equations compared with the results in [6, 30]. Results on existence of solution enables one to investigate other properties of solution. Hence we shall investigate variational stability of solution, asymptotic variational stability of solution, variational attracting of solution, relationship between these concepts of stability, converse variational stability and continuous dependence of solution on parameters.

This work will consist of seven chapters. Section 1.1 of this chapter(1) will begin with general introduction. Sections 1.2 to 1.11 of this chapter 1, wiil consist of Ekhaguere [30] and Ayoola's [6-8] formulations and notations. Section 1.2 is devoted to some fundamental concepts and structures that are employed in subsequent chapters.

In section 1.3, a summary of the first and second fundamental formulae of quantum stochastic calculus due to Hudson and Parthasarathy [44] and their formulation of Boson quantum stochastic integration will be presented. Section 1.4 contains a description of some spaces of sesquilinear forms-valued maps.

Quantum stochastic differential equations are discussed in section 1.5 while section 1.6 contains a summary of some established results of Ekhaguere [30] giving the equivalent form of a quantum stochastic differential inclusion as a special case of quantum stochastic differential equations introduced above.

A summary of the results of Ayoola [6] concerning the concept of the Kurzweil equations associated with quantum stochastic differential equations within the frame work of Schwabik and Kurzweil [52-58, 80-87] formulations, will be presented in sections 1.7 to 1.10, . In this section the following will be discussed: the Kurzweil integrals associated with quantum stochastic processes, Kurzweil equations associated with quantum stochastic differential equations, a class of sesquilinear form-valued maps and lastly the equivalence of the quantum stochastic differential equation and the associated Kurzweil equation.

In chapter 2, a review some results on existence of solution of ordinary differential equations, classical Kurzweil equations, stochastic differential equations and quantum stochastic differential equations will be discussed. Also, some results on stability and continuous dependence of solution on parameters for ordinary differential equations and generalized differential ordinary equations will be considered. In chapter 3, the methods of establishing the main results will discussed.

The major contribution on existence and uniqueness of solution of Kurzweil equation associated with quantum stochastic differential equation(1.5) that satisfy a more general Lipschitz condition, will be established in chapter 4. Since the equivalence of equations (1.5) and (1.7) has been established in [6], the existence and uniqueness of solution of Kurzweil equation associated with quantum stochastic differential equation(1.5) will be established via its equivalent QSDE (1.3) so that the existence of solution of equation (1.5) will imply existence of solution of the associated Kurzweil equation (1.7).

Here the method of successive approximations in [18, 30] will be adopted. In chapter 5, all kinds of variational stability of solution will be studied and the

Lyapunov method will be employed to establish these results. The advantage of using Lyapunov's method is that it enables one to investigate variational stability without explicitly solving the differential equation.

In chapter 6, results on continuous dependence of solution on parameters will be established. Again, the method of convergence applied in [87] will be adoted to this present noncommutative quantum setting to establish the results in chapter 6. Chapter 7 will be devoted to summary, conclusion, outstanding contributions to knowledge, practical applications of QSDEs in real life and recommendations for further studies.

1.2 Fundamental Concepts and Notations

1.2.1 Notation.

Let D be an inner product space and H, the completion of D. We denote by $L^+(D, H)$ the set

$$\{X : D \to H : X \text{ is a linear map such that } DomX^* \supseteq D,$$

$$\text{where } X^* \text{ is the adjoint of } X\}$$

We remark that $L^+(D, H)$ is a linear space under the usual notions of addition and scalar multiplication of operators.

1.2.2 Definition.

(i) Let H be a Hilbert space. The Boson Fock space $\Gamma(H)$ determined by H is the Hilbert space direct sum

$$\Gamma(H) = \bigoplus_{n=0}^{\infty} H^{(n)}$$

where $H^{(0)} = \mathbb{C}$. For $n \geq 1$, $H^{(n)}$ is the subspace of the n-fold Hilbert space tensor product of H with itself comprising all symmetric tensors.

$$H^{(n)} = (H \otimes ... \otimes H)sym$$

(ii) For each $f \in H$, an element $e(f)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-1/2} \bigotimes^{n} f$$

is called an exponential vector or coherent vector in $\Gamma(H)$ corresponding to f. We remark here that the subspace E of $\Gamma(H)$ generated by the set of exponential vectors in $\Gamma(H)$ is dense in $\Gamma(H)$. Here $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is an n-fold tensor product of f with itself for $n \geq 1$. The element $e(0)$ in $\Gamma(H)$ is called the vacuum vector.

1.2.3 Remark. It is well known that the exponential vector $e(f)$ and the Boson

Fock space enjoy the following properties [42-44,65-67]

(i) Let $\varepsilon = \text{span}\{e(f) : f \in H\}$. Then ε is dense in $\Gamma(H)$.

(ii) $\forall f, g, \in H$, we have

$$\langle e(f), e(g) \rangle = \exp\langle f, g \rangle$$

(iii) The set $e(f) : f \in H$ is linearly independent in $\Gamma(H)$.

(iv) If H is a Hilbert space direct sum $H = H_1 \oplus H_2$, then the Fock space $\Gamma(H)$ factorizes as

$$\Gamma(H) = \Gamma(H_1) \otimes \Gamma(H_2).$$

For arbitrary $f_1 \in H_1, f_2 \in H_2$, an exponential vector in $\Gamma(H)$ is given by

$$e(f_1, f_2) = e(f_1) \otimes e(f_2)$$

(v) Since the exponential vectors are linearly independent, an operator with domain ε is well defined by specifying its action on $e(f), f \in H$.

1.2.4 Notation. In what follows, $I\!\!D$ is some inner product space with $\mathcal{R}$ as its completion, and γ is some fixed Hilbert space.

(i) For each $t \in \mathbb{R}_+$, we write $L^2_\gamma(\mathbb{R}_+)$ (Resp. $L^2_\gamma([0, t))$; resp. $L^2_\gamma([t, \infty))$), for the Hilbert space of square integrable, γ-valued maps on $\mathbb{R}_+ \equiv [0, \infty)$ (resp. $[0, t)$; resp. $[t, \infty)$).

(ii) The noncommutative stochastic processes which we shall discuss are densely defined linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$; the inner product of this complex Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$ and its norm by $\| \cdot \|$. For each $t > 0$, it is well known [43, 65-67, $1', 3'$] that the Hilbert space $L^2_\gamma(\mathbb{R}_+)$ can be decomposed into a direct sum

$$L^2_\gamma(\mathbb{R}_+) = L^2_\gamma([0,$$

t))$\oplus L^2_\gamma([t,\infty)) which leads to a factorization of the Fock space \Gamma(L^2_\gamma(\mathbb{R}_+))$ given by

$$\Gamma(L^2_\gamma(\mathbb{R}_+)) = \Gamma[L^2_\gamma([0,t)) \otimes L^2_\gamma([t,\infty))]$$

by Remark 1.2.3(iv).

(iii) Let $I\!E$,$I\!E_t$ and $I\!E^t, t > 0$, be the linear spaces generated by the exponential vectors in $\Gamma(L^2_\gamma(\mathbb{R}_+))$, $\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$, respectively,i.e.

$$I\!E = \text{span}\{e(f), f \in L^2_\gamma \mathbb{R}_+)\}, I\!E_t = \text{span}\{e(f), f \in L^2_\gamma([0,t))\}$$

and

$$I\!E^t = \text{span}\{e(f), f \in L^2_\gamma([t,\infty))\}.$$

Then we adopt the following spaces as in [7,8]:

(i) $\mathcal{A} \equiv L^+(I\!D \underline{\otimes} I\!E, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$,

(ii) $\mathcal{A}_t \equiv L^+(I\!D \underline{\otimes} I\!E_t,\ \mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))) \otimes 1^t$,

(iii) $\mathcal{A}^t \equiv 1_t \otimes L^+(I\!D \underline{\otimes} I\!E, \mathcal{R} \otimes \Gamma(L^2_\gamma([t,\infty)))),\ t > 0$,

where $\underline{\otimes}$ denotes algebraic tensor product and 1_t (resp. 1^t) denotes the identity map on

$$\mathcal{R} \otimes \Gamma(L^2_\gamma[0,t)), (\text{resp. } \Gamma(L^2_\gamma([t,\infty))), t > 0$$

. note that $\mathcal{A}^t$ and $\mathcal{A}_t, t > 0$, may be naturally identified with subspaces of $\mathcal{A}$.

1.2.5 Definition. For $\eta, \xi \in I\!D \underline{\otimes} I\!E$, we define $\|\cdot\|_{\eta\xi}$ on $\mathcal{A}$ by

$$\|x\|_{\eta\xi} = |\langle \eta, x\xi \rangle|, x \in \mathcal{A}.$$

Then $\{\|\cdot\|_{\eta\xi}, \eta, \xi \in I\!D \underline{\otimes} I\!E\}$ is a family of seminorms on $\mathcal{A}$; we write τ_w for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family.

1.2.6 Notation. We denote by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}^t$ the completions of the locally convex spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w), t > 0$, respectively. We remark that the net $\{\tilde{\mathcal{A}}_t : t \in \mathbb{R}_+\}$ furnishes a filtration of $\tilde{\mathcal{A}}$.

1.3 Boson Quantum Stochastic Integration

Before defining the quantum stochastic integral employed in the subsequent chapters,we present a number of important notations and definitions.

1.3.1 Definition.

Let $I \subseteq \mathbb{R}_+$.

(i) A map $X : I \to \tilde{\mathcal{A}}$ is called a stochastic process indexed by I.

(ii) A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. And we write $Ad(\tilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

(iii) A member X of $\mathrm{Ad}(\tilde{\mathcal{A}})$ is called

(a) weakly absolutely continuous if the map

$$t \longrightarrow \langle \eta, X(t)\xi \rangle, \quad t \in I$$

is absolutely continuous for arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$. We denote this subset of $\mathrm{Ad}(\tilde{\mathcal{A}})$ by $\mathrm{Ad}(\tilde{\mathcal{A}})_{wac}$.

(b) locally absolutely P-integrable if the map $\|x(\cdot)\|_{\eta\xi}$ is Lebesgue measurable and integrable on $[t_0, t) \subseteq I$ for each $t \in I$, $p \in (0, \infty)$ and arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $L^p_{loc}(\tilde{\mathcal{A}})$.

1.3.2 Definition. Let $B(\gamma)$ denote the Banach space of bounded endomorphisms of γ and let the spaces $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ be defined by $L^\infty_{\gamma,loc}(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \gamma |\ f$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$

$L^\infty_{B(\gamma),loc}(\mathbb{R}_+) = \{\pi : \mathbb{R}_+ \to B(\gamma) |\ \pi$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$

For $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define $\pi f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ by $(\pi f)(t) =$

$\pi(t)f(t), t \in \mathbb{R}_+$.

Also for $f \in L^2_\gamma(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define the operators $a(f), a^+(f)$ and $\lambda(\pi) \in L^+(I\!D, \Gamma(L^2_\gamma(\mathbb{R}_+)))$ as follows;

$$a(f)e(g) = \langle f, g\rangle_{L^2_\gamma(\mathbb{R}_+)} e(g)$$
$$a^+(f)e(g) = \frac{d}{d\sigma} e(g + \sigma f)|_{\sigma=0}$$
$$\lambda(\pi)e(g) = \frac{d}{d\sigma} e(e^{\sigma\pi} f)|_{\sigma=0}$$

for $g \in L^2_\gamma(\mathbb{R}_+)$.

1.3.3 Definition. The operators $a(f), a^+(f)$ and $\lambda(\pi)$ for arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ give rise to the operator-valued maps A_f, A^+_f and Λ_π defined by

$$A_f(t) \equiv a(f\chi_{[0,t)})$$
$$A^+_f(t) \equiv a^+(f\chi_{[0,t)})$$
$$\Lambda_\pi(t) \equiv \lambda(\pi\chi_{[0,t)})$$

$t \in \mathbb{R}_+$ where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$.

1.3.4 Remark. The operators $a(f), a^+(f)$, and $\lambda(\pi)$ are the annihilation, creation and gauge operators of quantum field theory.

The maps A_f, A^+_f, and Λ_π are the stochastic processes, called the annihilation,creation and gauge processes, respectively,when their values are identified with their ampliations on $\mathcal{R}\underline{\otimes}\Gamma(L^2_\gamma(\mathbb{R}_+))$; i.e. for any $r \in \{A_f, A^+_f, \Lambda_\pi\}$ and $\eta = c \otimes e(\alpha)$, with $\alpha \in L^2_\gamma(\mathbb{R}_+), c \in \mathcal{R}$, then $r(t)(c \otimes e(\alpha)) = r(t)c \otimes e(\alpha)$.

These are the stochastic integrators in the Hudson and Parthasarathy [42] formulation of the Boson quantum stochastic integration which we adopt in the sequel.

Next we give the definition of the stochastic integrals.

1.3.5 Definition. A stochastic process $p \in Ad(\tilde{\mathcal{A}})$ is called simple if there exists an increasing sequence $t_n, n = 0, 1, 2, ...$ with $t_0 = 0$ and $t_n \to \infty$ such that for each $n \geq 0$,

$$p(t) = p(t_n) \text{ and } t \in [t_n, t_{n+1})$$

1.3.6 Definition. Let $p, q, u, v \in Ad(\tilde{\mathcal{A}})$ be simple adapted stochastic processes and $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$. Then the family of operators $M = \{M(t) : t \geq 0\}$ in $Ad(\tilde{\mathcal{A}})$ defined by

$M(0) = 0$

$$\begin{aligned} M(t) &= M(t_n) + p(t_n)(\Lambda_\pi(t) - \Lambda_\pi(t_n)) + q(t_n)(A_f(t) - A_f(t_n)) \\ &\quad + u(t_n)(A^+_g(t) - A^+_g(t_n)) + v(t_n)(t - t_n), t_n < t < t_{n+1} \end{aligned} \tag{1.3.1}$$

is called the stochastic integral of p, q, u, v with respect to Λ_π, A_f, A^+_g and the Lebesgue measure t. It is denoted in integral form by

$$M(t) = \int_0^t (p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds)$$

and understood in differential form as

$M(0) = 0$

$dM(t) = p(t)d\Lambda_\pi(t) + q(t)dA_f(t) + u(t)dA^+_g(t) + v(t)dt,$

Next we present some results due to Hudson and Parthasarathy established in [7-9, 30].

1.3.1 Theorem. (a) Let p, q, u, v be simple adapted stochastic processes in $Ad(\tilde{\mathcal{A}})$ and let M be their stochastic integral. If $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$ with $\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta), c, d \in I\!\!D, \alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, and $t \geq 0$, then

$$\begin{aligned} \langle \eta, M(t)\xi \rangle &= \int_0^t \langle \eta, \{\langle \alpha(s), \pi(s)\beta(s)\rangle_\gamma p(s) \\ &\quad + \langle f(s), \beta(s)\rangle_\gamma q(s) + \langle \alpha(s), g(s)\rangle_\gamma u(s) + v(s)\}\xi\rangle ds \end{aligned} \tag{1.3.2}$$

(b) Assume that the following hold. For $j = 1, 2$

(i) p_j, q_j, u_j, v_j are simple adapted processes.

(ii) $f_j, g_j \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi_j \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$

(iii) $M_j(t) = \int_0^t (p_j(s)d\Lambda_{\pi_j}(s) + q_j(s)dA_{f_j}(s) + u_j(s)dA^+_{g_j}(s) + v_j(s)ds)$

Then for arbitrary $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$ such that $\eta = c \otimes e(\alpha), \eta = d \otimes e(\beta)$,
$\alpha, \beta \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$, we have

$$\begin{aligned}
\langle M_1(t)\eta, M_2(t)\xi\rangle &= \int_0^t \{\langle M_1(s)\eta, \{\langle \alpha(s), \pi_2(s)\beta(s)\rangle_\gamma p_2(s) \\
&\quad + \langle f_2(s), \beta(s)\rangle_\gamma q_2(s) + \langle \alpha(s), g_2(s)\rangle_\gamma u_2(s) + v_2(s)]\xi\rangle \\
&\quad + \langle [\langle \beta(s), \pi_1(s)\alpha(s)\rangle_\gamma p_1(s) + \langle f_1(s), \alpha(s)\rangle_\gamma q_1(s) \\
&\quad + \langle \beta(s), g_1(s)\rangle_\gamma u_1(s) + v_1(s)]\eta, M_2(s)\xi\rangle \\
&\quad + \langle \pi_1(s)\alpha(s)\otimes p_1(s)\eta + g_1(s)\otimes u_1(s)\eta, \\
&\quad \pi_2(s)\beta(s)\otimes p_2(s)\xi + g_2(s)\otimes u_2(s)\xi\rangle\} ds \qquad (1.3.3)
\end{aligned}$$

(c) Let $T > 0$ and $0 \leq t \leq T$. Then, under the hypothesis of item (a) above, there is a finite constant $K_{T,\xi}$ such that

$$\| M(t)\xi \|^2 \leq 6K^2_{T,\xi} \int_0^T e^{t-s}\{\|p(s)\xi\|^2 + \|q(s)\xi\|^2 + \|u(s)\xi\|^2 + \|v(s)\xi\|^2\}ds,$$
$$\xi \in I\!\!D\underline{\otimes}I\!\!E. \qquad (1.3.4)$$

(d) The results (a) - (c) above remain true if for each integrand $F \in \{p, q, u.v\}$ the map $t \longrightarrow F(t)\xi$ is measurable and satisfies

$$\int_0^t \|F(s)\xi\|^2 ds < \infty \ \ \forall \ \ t > 0 \ \text{ and } \ \forall \ \ \xi \in I\!\!D\underline{\otimes}I\!\!E.$$

1.3.7 Remark: Equations (1.3.2) and (1.3.3) are called the first and second fundamental formulae of quantum stochastic calculus. Equation (1.3.3) is essentially Ito's formula for simple integrands. Inequality (1.3.4) is a corollary of the second fundamental formula. Extension of the stochastic integral given by Definition (1.3.6) to the integrands in $L^2_{loc}(\tilde{\mathcal{A}})$ is not as straight forward as in the classical Ito case. Here we require estimates of the integral in the family of seminorms $\{\|.\|_{\eta\xi}, \eta, \xi \in I\!\!D\underline{\otimes}I\!\!E\}$ that generates the topology of $\tilde{\mathcal{A}}$, rather than an isometry. First, we present the following established results in [7-9].

1.3.2 Proposition. Let $p \in L^2_{loc}(\tilde{\mathcal{A}})$. Then there exists a sequence $p^{(n)}, n = 1, 2, ...$ of simple adapted processes such that for each $t > 0$,and for arbitrary $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$,

$$\lim_{n\to\infty} \int_0^t \|p(s) - p^{(n)}(s)\|^2_{\eta\xi} ds = 0 \tag{1.3.5}$$

1.3.3 Proposition. Assume that the following hold

(i) p, q, u, v are simple processes in $Ad(\tilde{\mathcal{A}})$.

(ii) $M(t) = \int_0^t (p(s)d\Lambda_\pi(s)+q(s)dA_f(s)+u(s)dA^+_g(s)+v(s)ds)$ for each $t \in [0, T], T > 0$.

For arbitrary $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$ with $\eta = c \otimes e(\alpha), \quad \xi = d \otimes e(\beta), \quad c, d \in I\!\!D, \quad \alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, let $K_{\eta\xi,T}$ be given by
$K_{\eta\xi,T} = \sup_{0\leq s\leq T}\max\{|\langle\alpha(s), \pi(s)\beta(s)\rangle|, \ |\langle f(s), \beta(s)\rangle|, |\langle\alpha(s), g(s)\rangle|, 1\}$
Then

$$\|M(t)\|_{\eta\xi} \leq K_{\eta,\xi,T} \int_0^T [\|p(s)\|_{\eta\xi} + \|q(s)\|_{\eta\xi} + \|u(s)\|_{\eta\xi} + \|v(s)\|_{\eta\xi}]ds \tag{1.3.6}$$

1.3.8 Remark. (Extension of Quantum Stochastic Integral)
Let p, q, u, v be elements of $L^2_{loc}(\tilde{\mathcal{A}})$. Then by Proposition 1.3.2 there exists simple adapted processes p_n, q_n, u_n, v_n which approximate p, q, u, v in $L^2_{loc}(\tilde{\mathcal{A}})$. We now set

$$M_n(t) = \int_0^t (p_n(s)d\Lambda_\pi(s) + q_n(s)dA_f(s) + u_n(s)dA^+_g(s) + v_n(s)ds)$$

Applying the inequality (1.3.6) to the difference $M_n(t) - M_m(t), m, n \in \mathbb{N}$, we have that the sequence$M_n(t)$ is a Cauchy sequence in $(\tilde{\mathcal{A}})$ and therefore converges to a limit in $(\tilde{\mathcal{A}})$ by the completeness of the locally convex space. The limit $M(t)$ is independent of the choice of approximating sequences and is defined to be the integral

$$M(t) = \int_0^t (p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds).$$

By employing the uniformity of the convergence on finite intervals, we may pass to the limit of approximations by simple processes, so that Theorem 1.3.1(a) for coefficients p, q, u, v belonging to $L^2_{loc}(\tilde{\mathcal{A}})$ remains valid.

Next, we present some concepts and definitions which are intended for reference purpose in this study. Such concepts are already contained in texts written by many authors [28, 48, 60, 78, 6′, 7′].

1.3.9 Notations.

(i) For each $\omega \in \Omega$ (Ω is a non-empty set), the map $t \to X(t, \omega)$ is called the corresponding sample path, realization or trajectory of the stochastic process.

(ii) Wiener Process, $W = \{W(t)\}_{t\geq 0}$: This is also called the Brownian motion in honour of R. Brown who in (1826-1827) observed the irregular motion of pollen particles in water and given by the notation $B = \{B(t)\}_{t\geq 0}$.

(iii) Martingales; Let $X(t) : t \in I$, $I = [0, \infty)$, be a stochastic process defined on a probability space (Ω, F, P) such that $E(|X(t)|) < \infty$ for all $t \geq 0$.
If $E(X_t \ / \ F_s) = X_s$, forall $t \geq s \geq 0$ where $\{F_t\}_{t\geq 0}$ is a filtration to which the process is adapted. Then $X(.)$ is called a martingale.

1.4 Spaces of Sesquilinear-form-Valued Maps

We shall employ certain spaces of maps whose values are sesquilinear forms on $I\!D\underline{\otimes}I\!E$. We have the following definitions and notations as in [7-9].

1.4.1 Notation. (i) We denote the space of sesquilinear forms on $I\!D\underline{\otimes}I\!E$ by sesq($I\!D\underline{\otimes}I\!E$). Thus,

sesq($I\!D\underline{\otimes}I\!E$) $= \{a : I\!D\underline{\otimes}I\!E \times I\!D\underline{\otimes}I\!E \to \mathbb{C}|$ the map $(\eta, \xi) \to a(\eta, \xi)$ is linear in ξ and conjugate linear in η, $\forall\ \eta, \xi \in I\!D\underline{\otimes}I\!E\}$

(ii) Let $I \subseteq \mathbb{R}_+$, we denote by $L^0(I, I\!D\underline{\otimes}I\!E)$ the set of all sesq($I\!D\underline{\otimes}I\!E$)$-$ valued maps on I. i, e. $L^0(I, I\!D\underline{\otimes}I\!E) = \{u : I \to sesq(I\!D\underline{\otimes}I\!E)\}$.

1.4.2 Remark. $L^0(I, I\!D\underline{\otimes}I\!E)$ acquires the structure of a linear space if the linear combination $\alpha u + \beta v, \alpha, \beta \in \mathbb{C}$, of u and v in $L^0(I, I\!D\underline{\otimes}I\!E)$ is defined by

$$(\alpha u + \beta v)(t)(\eta, \xi) = \alpha u(t)(\eta, \xi) + \beta v(t)(\eta, \xi),\ t \in I,\ \eta, \xi \in I\!D\underline{\otimes}I\!E.$$

We observe also that every $\tilde{\mathcal{A}}-$ valued map P on I is $L^0(I, I\!D\underline{\otimes}I\!E)$, since P may be identified with the map whose value at $t \in I$ is the sesquilinear form

$$(\eta, \xi) \to \langle \eta, P(t)\xi \rangle,\ \eta, \xi \in I\!D\underline{\otimes}I\!E$$

1.4.3 Definition. A member $z \in L^0(I, I\!D\underline{\otimes}I\!E)$ is:

(i) absolutely continuous if the map $t \to z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$.

(ii) of bounded variation if over all partition $\{t_j\}_{j=0}^n$ of I,

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^{n} |z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi)| \right) < \infty$$

(iii) of essentially bounded variation if z is equal almost everywhere to some member of $L^0(I, \mathbb{D}\underline{\otimes}\mathbb{E})$ of bounded variation.

(iv) A stochastic process $X : I \to \tilde{\mathcal{A}}$ is of bounded variation if

$$\sup\left(\sum |\langle \eta, X(t_j)\xi\rangle - \langle \eta, X(t_{j-1})\xi\rangle|\right) < \infty$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and where supremum is taken over all partitions $\{t_j\}_{j=1}^N$ of I.

1.4.4 Notation. We denote by $BV(\tilde{\mathcal{A}})$, the set of all stochastic processes of bounded variation on I.

1.4.5 Definition.

(i) For $X \in BV(\tilde{\mathcal{A}})$, define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$Var_{[a,b]}X_{\eta\xi} = \sup_{\tau}\left(\sum_{j=1}^{n} \|X(t_j) - X(t_{j-1})\|_{\eta\xi}\right)$$

where τ is the collection of all partition of the interval $[a, b] \subseteq I$. If $[a, b] = I$, we simply write $Var_I X_{\eta\xi} = Var_{\eta,\xi}X$. Then $\{Var_{\eta,\xi}X,\ \ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ is a family of seminorms which generates a locally convex topology on $BV(\tilde{\mathcal{A}})$.

1.4.6 Notation.

(i) We denote by $\overline{BV}(\tilde{\mathcal{A}})$; the completion of $BV(\tilde{\mathcal{A}})$ in the said topology.

(ii) For any member Z of $L^0(I, \mathbb{D}\underline{\otimes}\mathbb{E})$ of bounded variation, we write $Var_{[a,b]}Z_{\eta\xi}$ for its variation on $[a,b] \subseteq I$

(iii) for any arbitrary complex valued map $f : I \to \mathbb{C}$ of bounded variation we write

$$Var_{[a,b]}f = \sup_{\tau}\left(\sum_{j=1}^{n}|f(t_j) - f(t_{j-1})|\right),$$

For $[a,b] \subseteq I$, where τ is the collection of all partitions of $[a,b]$.

(iv) for $[a,b] \subseteq I$, write $Var_{[a,b]}f = Varf$.

1.5 Stochastic Differential Equations

We present Lipschitzian quantum stochastic differential equation in the framework of [30] formulation of Lipschitzian quantum stochastic differential inclusions.

The following notations and definitions will be required subsequent sections.

1.5.1 Definition. A stochastic process Φ will be called locally absolutely p - integrable if the map $t \to \|\Phi(t)\|_{\eta\xi}, t \in \mathbb{R}_+$, lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E$ and $p \in (0, \infty)$

1.5.2 Notation. For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+, L^2_{loc}(I \times \tilde{\mathcal{A}})$ denotes the set of maps $\Phi : I \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ such that the map $t \to \Phi(X(t), t)$ lies in $L^p_{loc}(\tilde{\mathcal{A}})$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$. In what follows, $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+), \pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+), 1$ is the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$. We introduce the process $A_f, A^+_g, \wedge_\pi$ and $s \to s1, s \in \mathbb{R}_+$ as the integrators.

1.5.3 Definition. Let $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})$ and (X_0, t_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then a relation of the form

$$\begin{aligned} X(t) &= X_0 + \int_{t_0}^t E(X(s), s) d \wedge_\pi (s) + F(X(s), s) dA_g(s) \\ &\quad + G(X(s), s) dA_{f^+}(s) + H(X(s), s) ds), \ t \in I \qquad (1.5.1) \end{aligned}$$

will be called a stochastic integral equation with coefficients E, F, G, H and initial data (X_0, t_0) if $X(t_0) = X_0$.

Equation (1.5.1) can be written in differential form as

$$\begin{aligned} dX(t) &= E(X(t), t) d \wedge_\pi (t) + F(X(t), t) dA_g(t) + G(X(t), t) dA_{f^+}(t) + H(X(t), t) dt \\ X(t_0) &= X_0, \text{almost all } \ t \in I \qquad (1.5.2) \end{aligned}$$

1.5.4 Definition: By a solution of equation (1.5.1), we mean a weakly absolutely continuous stochastic process $\phi \in L^2_{loc}(\tilde{\mathcal{A}})$ such that

$$\begin{aligned} d\phi(t) &= E(\phi(t),t)d\wedge_\pi(t) + F(\phi(t),t)dA_g(t) + G(\phi(t),t)dA_{f^+}(t) + H(\phi(t),t)dt \\ \phi(t_0) &= X_0, \text{almost all } \ t \in I \end{aligned}$$

1.5.5 Definition: Let $I \subseteq \mathbb{R}_+$

(i) A map $\Phi : I \times \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$ will be called Lipschitzian if for any $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there exists a function

$$K^\Phi_{\eta\xi} : I \longrightarrow (0, \infty)$$

lying in $L^1_{loc}(I)$ such that,

$$\|\Phi(x,t) - \Phi(y,t)\|_{\eta\xi} \leq K^\Phi_{\eta\xi}(t)\|x-y\|_{\eta\xi}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$. The functions $\{K^\Phi_{\eta\xi}(.) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ will be called Lipschitz functions for Φ; these are constants if Φ does not depend explicitly on t.

(ii) If for $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Phi_{\eta\xi}$ maps $I \times \tilde{\mathcal{A}}$ to $\mathbb{C}$, the complex field, then $\Phi_{\eta\xi}$ will be called Lipschitzian if

$$|\Phi_{\eta\xi}(x,t) - \Phi_{\eta\xi}(y,t)| \leq K^\Phi_{\eta\xi}(t)\|x-y\|_{\eta\xi}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$.

(iii) If Φ is a map from $I \times \tilde{\mathcal{A}}$ into the sesq$(\mathbb{D}\underline{\otimes}\mathbb{E})$ then for $(x,t) \in I \times \tilde{\mathcal{A}}$, the value of $\Phi(x,t)$ at $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, will be called Lipschitzian (resp. continuous) if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $(x,t) \longrightarrow \Phi(x,t)(\eta,\xi)$ from $I \times \tilde{\mathcal{A}}$ to $\mathbb{C}$ is Lipschitzian(resp.continuous).

1.6 Equivalent form of Quantum Stochastic Differential Equation

In this section, we present some established results in [30] concerning equivalent forms of quantum stochastic differential inclusions.

Except otherwise stated, E, F, G, H lie in $L^2_{loc}(I \times \tilde{\mathcal{A}})$ and (x_0, t_0) is some fixed point of $I \times \tilde{\mathcal{A}}$.

For $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E$, with $\eta = c \otimes e(\alpha)$ and $\xi = d \otimes e(\beta)$,

define $\mu_{\alpha\beta}, \gamma_\beta, \sigma_\alpha : I \longrightarrow \mathbb{C}$ by

$$\begin{aligned} \mu_{\alpha\beta} &= \langle \alpha(t), \beta(t) \rangle_\gamma \\ \gamma_\beta &= \langle f(t), \beta(t) \rangle_\gamma \\ \sigma_\alpha &= \langle \alpha(t), g(t) \rangle_\gamma, \quad t \in I \end{aligned}$$

To these functions, associate the maps $\mu E, \gamma F, \sigma G, P$ from $I \times \tilde{\mathcal{A}}$ into the set of sesquilinear forms on $I\!\!D\underline{\otimes} I\!\!E$ defined by

$$\begin{aligned} (\mu E)(x,t)(\eta,\xi) &= \langle \eta, \mu_{\alpha\beta}(t) E(x,t)\xi \rangle \\ (\gamma F)(x,t)(\eta,\xi) &= \langle \eta, \gamma_\alpha(t) F(x,t)\xi \rangle \\ (\sigma G)(x,t)(\eta,\xi) &= \langle \eta, \sigma_\beta(t) G(x,t)\xi \rangle \\ P(x,t)(\eta,\xi) &= (\mu E)(x,t)(\eta,\xi) + (\gamma F)(x,t)(\eta,\xi) + (\sigma G)(x,t)(\eta,\xi) \\ &\quad + H(x,t)(\eta,\xi) \end{aligned}$$

$\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E, (x,t) \in I \times \tilde{\mathcal{A}}$ where $H(x,t)(\eta,\xi) := \langle \eta, H(x,t)\xi \rangle$. P can also be written in the form

$$P(x,t)(\eta,\xi) = \langle \eta, P_{\alpha\beta}(x,t)\xi \rangle$$

where

$$P_{\alpha\beta} : I \times \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$$

is given by

$P_{\alpha\beta}(x,t) = \mu_{\alpha\beta}(t)E(x,t) + \gamma_\beta(t)F(x,t) + \sigma_\alpha(t)G(x,t) + H(x,t)$

for $(x,t) \in I \times \tilde{\mathcal{A}}$.

1.6.1 Proposition: Let E, F, G, H lie in $L^2_{loc}(I \times \tilde{\mathcal{A}})$. Then

(i) for arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$, and $X \in L^2_{loc}(\tilde{\mathcal{A}})$, the map $t \longrightarrow P(X(t),t)(\eta,\xi)$ lie in $L^1_{loc}(I)$.

(ii) the map P is

(a) Lipschitzian whenever E, F, G, H are Lipschitzian

(b) continuous whenever $\mu E, \gamma F, \sigma G$ and H are continuous.

1.6.2 Theorem. Let E, F, G, H lie in $L^2_{loc}(I \times \tilde{\mathcal{A}})$ and let (X_0, t_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then the stochastic integral equation

$$\begin{aligned} X(t) &= X_0 + \int_{t_0}^{t} (E(X(s),s)d\wedge_\pi(s) + F(X(s),s)dA_g(s) \\ &\quad + G(X(s),s)dA_{f^+}(s) + H(X(s),s)ds), \; t \in I \qquad (1.6.1) \end{aligned}$$

is equivalent to the initial value nonclassical ordinary differential equation

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi \rangle &= P(X,t)(\eta,\xi) \\ X(t_0) &= X_0, t \in [t_0, T], \qquad (1.6.2) \end{aligned}$$

for arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$ and almost all $[t_0, T] \subset I$.

1.6.3 Remark: Theorem 1.6.2 above has been established in [6]. The next theorem is a major result established in [30]. It concerns the existence and uniqueness of solution of Lipschitzian quantum stochastic differential equation(1.3) with the Lipschitz condition $W(t) = t$.

1.6.4 Theorem. Suppose that the coefficients E, F, G, H appearing in equation (1.3) are Lipschitzian and belong to $L^2_{loc}(I \times \tilde{\mathcal{A}})$. Then for any fixed point (X_0, t_0) of $I \times \tilde{\mathcal{A}}$, there exists a unique adapted and weakly absolutely continuous solution Φ of quantum stochastic differential equation (1.3) satisfying $\Phi(t_0) = X_0$.

1.7 Kurzweil Integrals Associated with Quantum Stochastic Processes

In this section, The origin of the Kurzweil equations associated with quantum stochastic processes within the frame work of [6, 87] is discussed. We first present some useful definitions and notations.

1.7.1 Definition:

(i) Let an interval $[a, b] \subset \mathbb{R}$ be given. A pair (τ, J) of a point $\tau \in \mathbb{R}$ and a compact interval $J \subset \mathbb{R}$ is called a tagged interval, τ is the tag of J.

(ii) A finite collection $\Delta = (\tau_j, J_j), j = 1, ..., k$ of tagged intervals is called a system in $[a, b]$ if $\tau_j \in J_j \subset [a, b]$, for every $j = 1, ..., k$ and the intervals J_j are nonoverlapping, that is $Int(J_i) \bigcap Int(J_j) = \phi$ for $i \neq j$ where $Int(J)$ denotes the interior of an interval J.

(iii) A system $\Delta = (\tau_j, J_j), j = 1, ..., k$ is called a partition of $[a, b]$ if

$$\bigcup_{j=1}^{k} J_j = [a, b]$$

(iv) Given a positive function $\delta : [a, b] \to (0, +\infty)$ called a gauge on $[a, b]$, a tagged interval (τ, J) with $\tau \in [a, b]$ is said to be δ-fine if

$$J \subset [\tau - \delta(\tau), \tau + \delta(\tau)].$$

(v) A system (in particular, a partition) $\triangle = \{(\tau_j; J_j),\ j = 1, \ldots, k\}$ is δ-fine if the point interval pair (τ_j, J_j) is δ-fine for every $j = 1, \ldots, k$.

Unless otherwise stated we shall let $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$ be arbitrary.
Assume that $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ is an $\tilde{\mathcal{A}}$-valued map of two variables $\tau, t \in [t_0, T]$.
We adopt the following as in [6].

1.7.2 Notation: Consider the family of complex valued functions $U(\tau, t)(\eta, \xi) := \langle \eta, U(\tau, t)\xi \rangle$ associated with the map U.

(i) We shall use the integral

$$\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$$

to denote the Kurzweil integral of $U(\tau, t)(\eta, \xi)$ and write

$$\text{(ii)}\ \ S(U, D)(\eta, \xi) = \sum_{j=1}^{k} [U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)].$$

For the Riemann-Kurzweil sum corresponding to the function $U(\tau, t)(\eta, \xi)$ and the partition

$$D : t_0 < \tau_1 < t_1 < \cdots < t_k = T \text{ of } [t_0, T] \subseteq \mathbb{R}_+.$$

(iii) If $f; [t_0, T] \to \tilde{\mathcal{A}}$ is a stochastic process, then for arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$, we set

$$U(\tau, t)(\eta, \xi) = \langle \eta, f(\tau)\xi \rangle$$

for $\tau, t \in [t_0, T]$ and write

$$\begin{aligned} S(U, D)(\eta, \xi) &= \sum_{j=1}^{k} [U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)] \\ &= \sum_{j=1}^{k} [\langle \eta, f(\tau_j)\xi \rangle (t_j - t_{j-1})] \end{aligned}$$

representing the classical Riemann sum for the function $f_{\eta\xi}(t) := \langle \eta, f(t)\xi \rangle$ and a given partition D of $[t_0, T]$ and we now write

$$\text{(iv)}\ \ \int_{t_0}^{T} \langle \eta, f(s)\xi \rangle ds = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t]$$

provided the Kurzweil integral $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ exists in this case. Hence

$$\int_{t_0}^{T} DU(\tau, t)(\eta, \xi) = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t] = \int_{t_0}^{T} f_{\eta\xi}(s) ds \qquad (1.7.1)$$

1.7.3 Remark: If $U : [t_0, T] \times [t_0, T] \to \mathbb{C}$ be such that U is Kurzweil integrable over $[t_0, T]$, then for $c \in [t_0, T]$, we have

$$\lim_{s \to c} \left[\int_{t_0}^{s} DU(\tau, t) - U(c, s) + U(c, c) \right] = \int_{t_0}^{c} DU(\tau, t) \tag{1.7.2}$$

For several properties enjoyed by Kurzweil integrals and the existence of at least one δ-fine partition D of $[t_0, T]$ for a given gauge δ.

1.8 Kurzweil Integrals associated with Quantum Stochastic Differential Equations

(i) Let the map $F : \tilde{\mathcal{A}} \times [t_0, T] \to sesq[\mathbb{D}\underline{\otimes}\mathbb{E}]$ be given as in equation (1.6). Then we refer to the equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF(X(\tau), t)(\eta, \xi);\ t \in [t_0, T] \tag{1.8.1}$$

as the Kurzweil equation associated with equation (1.6.2).

(ii) A map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is called a solution of equation (1.8.1) if

$$\langle \eta.\Phi(s_2)\xi\rangle - \langle \eta, \Phi(s_1)\xi\rangle = \int_{s_1}^{s_2} DF(\Phi(\tau), t)(\eta, \xi) \tag{1.8.2}$$

holds for every $s_1, s_2 \in [t_0, T]$ identically.

Equation (1.8.1) is understood in integral form (1.8.2) via its solution.

The following are immediate consequence of the above definitions, they are established results in [6].

1.8.1 Proposition. If a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of the Kurzweil equation (1.8.1) on $[t_0, T]$, then for every $u \in [t_0, T]$, the following holds

$$\langle \eta, \Phi(s)\xi\rangle = \langle \eta, \Phi(u)\xi\rangle + \int_{u}^{s} DF(\Phi(\tau), t)(\eta, \xi);\quad s \in [t_0, T] \tag{1.8.3}$$

Consequently if a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ satisfies the integral equation (1.8.3) for some $u \in [t_0, T]$ and all $s \in [t_0, T]$ then Φ is a solution of equation (1.8.1).

1.8.2 Proposition. If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of the Kurzweil equation (1.8.1) on $[t_0, T]$, then

$$\lim_{s\to\sigma}[\langle \eta, \Phi(s)\xi\rangle - F(\Phi(\sigma), s)(\eta, \xi) + F(\Phi(\sigma), \sigma)(\eta, \xi)] = \langle \eta, \Phi(\sigma)\xi\rangle \tag{1.8.4}$$

1.8.3 Remark: By virtue of Proposition 1.8.2, the following approximation holds: If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation (1.8.1), then for every $\sigma \in [t_0, T]$ and for arbitrary $\eta.\xi \in I\!D\underline{\otimes}I\!E$, the matrix element

$$\langle \eta, \Phi(s)\xi \rangle \approx \langle \eta, \Phi(\sigma)\xi \rangle + F(\Phi(s), s)(\eta, \xi) - F(\Phi(\sigma), \sigma)(\eta, \xi)$$

provided that s in $[t_0, T]$ is sufficiently close to σ.

The following section concerns the class of sesquilinear form - valued maps $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq[I\!D\underline{\otimes}I\!E]$ which are Kurzweil integrable.

1.9 Class of Kurzweil Integrable Sesquilinear form-valued Maps

1.9.1 Definition: For each $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E$, let $h_{\eta\xi} : [t_0, T] \to \mathbb{R}$ be a family of non-decreasing functions defined on $[t_0, T]$ and $W : [0, \infty) \to \mathbb{R}$ be a continuous and increasing function such that $W(0) = 0$. Then the map $F : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(I\!\!D\underline{\otimes} I\!\!E)$ is said to belong to the class $\mathbb{F}$ $(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E$ if for all $x, y \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$

(i) $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (1.9.1)

(ii) $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi) - F(y, t_2)(\eta, \xi) + F(y, t_1)(\eta, \xi)|$
$\leq W(\|x - y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (1.9.2)

Next, we present some established results in [6] and some results which are simple extensions of similar results in [87] to the present noncommutative quantum setting.

1.9.2 Theorem. Assume that the following conditions hold:

(i) the maps $U, U_m : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ are such that $(\tau, t) \to U_m(\tau, t)(\eta, \xi)$ are real valued and Kurzweil integrable over $[t_0, T]$ for each $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E \quad \forall\, m = 1, 2, \ldots$.

(ii) there is a gauge w on $[t_0, T]$ such that for every $\epsilon > 0$, there exists a map $p : [t_0, T] \to \mathbb{N}$ and a family of super additive interval functions $\Phi_{\eta\xi}$ on $[t_0, T]$ defined for closed intervals $J \subset [t_0, T]$ with $\Phi_{\eta\xi}([t_0, T]) < \epsilon$ such that for every $\tau \in [t_0, T]$

$$|U_m(\tau, J)(\eta, \xi) - U(\tau, J)(\eta, \xi)| < \Phi_{\eta\xi}(J)$$

provided that $m > p(\tau)$, and (τ, J) is an w-fine tagged interval with $\tau \in J \subseteq [t_0 T]$.

(iii) there exist real valued Kurzweil integrable functions $V_{\eta\xi}, W_{\eta\xi} : [t_0, T] \to \mathbb{R}$ and a gauge θ on $[t_0, T]$ such that for all $m \in \mathbb{N}$, $\tau \in [t_0, T]$,

$$V_{\eta\xi}(\tau, J) \leq U_m(\tau, J)(\eta, \xi) \leq W_{\eta\xi}(\tau, J).$$

for any θ-fine interval (τ, J), $\forall\, \eta, \xi \in I\!D\underline{\otimes}I\!E$.

Then the map $(\tau, t) \to U(\tau, t)(\eta, \xi)$ is Kurzweil integrable over $[t_0, T]$ and that

$$\lim_{m\to\infty} \int_{t_0}^{T} DU_m(\tau, t)(\eta, \xi) = \int_{t_0}^{T} DU(\tau, t)(\eta, \xi).$$

1.9.3 Lemma. Let $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ be Kurzweil integrable over $[t_0, T]$. Given $\epsilon > 0$ assume that

(i) the gauge δ on $[t_0, T]$ is such that

$$\left| \sum_{j=1}^{k} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi)] - \int_{t_0}^{T} DU(\tau, t)(\eta, \xi) \right| < \epsilon$$

for every δ - fine partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ of $[t_0, T]$.

(ii) $t_0 \leq \beta_1 \leq \alpha_1 \leq \gamma_1 \leq \beta_2 \leq \alpha_2 \leq \gamma_2 \leq \cdots \leq \beta_m \leq \alpha_m \leq \gamma_m \leq T$ represents a δ - fine system $(\alpha_j, [\beta_j, \gamma_j])$, $j = 1, 2, ..., m$, i.e. $\alpha_j \in [\beta_j, \gamma_j] \subset [\alpha_j - \delta(\alpha_j), \alpha_j + \delta(\alpha_j)]$, $j = 1, 2, ..., m$

then

$$\left| \sum_{j=1}^{m} [U(\alpha_j, \gamma_j)(\eta, \xi) - U(\alpha_j, \beta_j)(\eta, \xi)] - \int_{\beta_j}^{\gamma_j} DU(\tau, t)(\eta, \xi) \right| < \epsilon$$

Proof: The proofs are simple adaptation of arguments employed in in Lemma 1.13 [87] to the present noncommutative quantum setting.

The following theorems are extensions of theorems 1.14, 1.16 and 1.35 in [87] to this present noncommutative quantum setting.

1.9.4 Theorem. Assume that the following holds.

(i) the function $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ is given for which the integral $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ exists.

(ii) If $V_{\eta\xi} : [t_0, T] \times [t_0, T] \to \mathbb{R}$ is such that the integral $\int_{t_0}^{T} DV_{\eta\xi}(\tau, t)$ exists and there is a gauge θ on $[t_0, T]$ such that

$$\text{(iii)} \quad |t-\tau| \, .|U(\tau,t)(\eta,\xi) - U(\tau,\tau)(\eta,\xi)| \le (t-\tau).(V_{\eta,\xi}(\tau,t) - V_{\eta,\xi}(\tau,\tau))$$

for every $t \in [\tau - \theta(\tau), \tau + \theta(\tau)]$

then the inequality

$$\left| \int_{t_0}^{T} DU(\tau,t)(\eta,\xi) \right| \le \int_{t_0}^{T} DV_{\eta,\xi}(\tau,t) \qquad (1.9.3)$$

holds.

Proof: Assume that $\epsilon > 0$ is given. Since the integrals $\int_{t_0}^{T} DU(\tau,t)(\eta,\xi)$, $\int_{t_0}^{T} DV_{\eta,\xi}(\tau,t)$ exist, there is a gauge δ on $[a,b]$ with $\delta(s) \le \theta(s)$ for $s \in [t_0, T]$ such that for every δ-fine partition

$$D = \{a_0, \tau_1, \alpha_1, ..., \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[t_0, T]$ we have

$$\left| \sum_{j=1}^{k} [U(\tau_j, \alpha_j)(\eta,\xi) - U(\tau_j, \alpha_{j-1})(\eta,\xi)] - \int_{t_0}^{T} DU(\tau,t)(\eta,\xi) \right| < \epsilon \qquad (1.9.4)$$

From hypothesis (iii), we get

$$|U(\tau_i,\alpha_i)(\eta,\xi) - U(\tau_i,\tau_i)(\eta,\xi)| \le V_{\eta,\xi}(\tau_i,\alpha_i) - V_{\eta,\xi}(\tau_i,\tau_i)$$

when $\alpha_i > \tau_i$ and

$$|U(\tau_i,\alpha_i)(\eta,\xi) - U(\tau_i,\tau_i)(\eta,\xi)| \le V_{\eta,\xi}(\tau_i,\tau_i) - V_{\eta,\xi}(\tau_i,\alpha_i)$$

when $\alpha_i < \tau_i$. Hence for i=1,2,...,k we have

$$|U(\tau_i,\alpha_i)(\eta,\xi) - U(\tau_i,\alpha_{i-1})(\eta,\xi)| \le |U(\tau_i,\alpha_i)(\eta,\xi) - U(\tau_i,\tau_i)(\eta,\xi)|$$

$$+|U(\tau_i,\tau_i)(\eta,\xi) - U(\tau_i,\alpha_{i-1})(\eta,\xi)| \le V_{\eta,\xi}(\tau_i,\alpha_i) - V_{\eta,\xi}(\tau_i,\alpha_{i-1})$$

By (1.9.4) we get

$$\left|\int_{t_0}^{T} DU(\tau,t)(\eta,\xi)\right| \le \left|\sum_{j=1}^{k}[U(\tau_j,\alpha_j)(\eta,\xi)-U(\tau_j,\alpha_{j-1})(\eta,\xi)] - \int_{t_0}^{T} DU(\tau,t)(\eta,\xi)\right| +$$

$$\left|\sum_{j=1}^{k}[U(\tau_j,\alpha_j)(\eta,\xi)-U(\tau_j,\alpha_{j-1})(\eta,\xi)]\right|$$

$$< \epsilon + \sum_{j=1}^{k}[V_{\eta,\xi}(\tau_j,\alpha_j)-V_{\eta,\xi}(\tau_j,\alpha_{j-1}]$$

$$= \epsilon + \sum_{j=1}^{k}[V_{\eta,\xi}(\tau_j,\alpha_j)-V_{\eta,\xi}(\tau_j,\alpha_{j-1}] - \int_{t_0}^{T} DV_{\eta,\xi}(\tau,t) + \int_{t_0}^{T} DV_{\eta,\xi}(\tau,t)$$

$$< 2\epsilon + \int_{t_0}^{T} DV_{\eta,\xi}(\tau,t)$$

Since $\epsilon > 0$ was arbitrary, the inequality (1.9.3) is satisfied. This theorem gives an estimate of the integral $\int_{t_0}^{T} DU(\tau,t)(\eta,\xi)$ by another integral of a real valued function. (Since $\mathbb{C} \equiv \mathbb{R}^2$).

1.9.5 Theorem.

(i) Let $U : [t_0,T] \times [t_0,T] \to \tilde{\mathcal{A}}$ be such that $(\tau,t) \to U(\tau,t)(\eta,\xi)$ is Kurzweil integrable over $[t_0,c]$ for $c \in [t_0,T]$ and that the limit

$$\lim_{c\to T-}\left[\int_{t_0}^{c} DU(\tau,t)(\eta,\xi) - U(T,c)(\eta,\xi) + U(T,T)(\eta,\xi)\right] = I \tag{1.9.5}$$

exists for all $\eta,\xi \in I\!\!D\underline{\otimes}I\!\!E$. Then $\displaystyle\int_{t_0}^{T} DU(\tau,t)(\eta,\xi)$ exists and equals I.

(ii) Let $U : [t_0,T] \times [t_0,T] \to \tilde{\mathcal{A}}$ be such that $(\tau,t) \to U(\tau,t)(\eta,\xi)$ is Kurzweil integrable over $[t,T]$ and that the limit

$$\lim_{c\to 0^+}\left[\int_{c}^{T} DU(\tau,t)(\eta,\xi) - U(t_0,c)(\eta,\xi) + U(t_0,t_0)(\eta,\xi)\right] = I \tag{1.9.6}$$

exists for all $\eta,\xi \in I\!\!D\underline{\otimes}I\!\!E$. Then

$$\int_{t_0}^{T} DU(\tau,t)(\eta,\xi) \text{ exists and equals } I$$

(iii) Let $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ be such that $(\tau, t) \to U(\tau, t)(\eta, \xi)$ is Kurzweil integrable over $[t_0, T]$. Then for $c \in [t_0, T]$.

$$\lim_{s \to c} \left[\int_{t_0}^{c} DU(\tau, t)(\eta, \xi) - U(c, s)(\eta, \xi) + U(c, c)(\eta, \xi) \right] = \int_{t_0}^{c} DU(\tau, t)(\eta, \xi) \tag{1.9.7}$$

for all $\eta, \xi \in I\!D \underline{\otimes} I\!E$.

Proof: Assume that $\epsilon > 0$ is given by (i) for every $\epsilon > 0$ we can find a $B \in [t_0, T]$ such that for every $c \in [B, T)$ the inequality

$$\left| \int_{t_0}^{c} DU(\tau, t)(\eta, \xi) - U(T, c)(\eta, \xi) + U(\tau, T)(\eta, \xi) \right| < \epsilon \tag{1.9.8}$$

is satisfied.

Assume that $t_0 = c_0 < c_1 < \ldots$ is an increasing sequence $(c_p)_{p=1}^{\infty}$ of points $c_p \in [t_0, T)$ with $\lim_{p \to \infty} c_p = T$. By the assumption we have $u \in \ [t_0, c_p]$ for every $p = 1, 2, \ldots$ and therefore for every $p = 1, 2, \ldots,$ there exists a gauge W_p on $[t_0, T]$ such that $W_p : [t_0, c_p] \to (0, +\infty)$ and for any w_p-fine partition D of $[t_0, c_p]$ we have

$$\left| S(U, D)(\eta, \xi) - \int_{t_0}^{c_p} DU(\tau, t)(\eta, \xi) \right| < \frac{\epsilon}{2^{p+1}}, \; p = 1, 2, \ldots \tag{1.9.9}$$

For every $\tau \in [t_0, T]$ there is exactly one $p(\tau) = 1, 2, \ldots$ for which $\tau \in [c_{p(\tau)-1}, c_{p(\tau)}]$. Given $\tau \in [t_0, T]$ let us choose $\hat{w}(\tau) > 0$ such that $\hat{w}(\tau) \leq W_{p(\tau)}(\tau)$ and $[\tau - \hat{w}(\tau), \; \tau + \bar{w}(\tau)] \cap [t_0, T] \subset [t_0, c_{p(\tau)})$. Assume that $c \in [t_0, T]$ is given and that

$$\hat{D} = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-2}, \tau_{k-1}, \alpha_{k-1}\}$$

is a $\hat{w}$-fine partition of $[t_0, c]$. If $p(\tau_j) = p$ then $[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \hat{w}(\tau_j), \tau_j + \hat{w}(\tau_j)] \subset [t_0, c_p]$ and also $[\alpha_{j-1}, \alpha_j] \subset [\tau_j - w_p(\tau_j), \tau_j + w_p(\tau_j j]$ let

$$\sum_{j=1, p(\tau_j)=p}^{k-1} \left[U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi) - \int_{\alpha_{j-1}}^{\alpha_j} DU(\tau, t)(\eta, \xi) \right]$$

be the sum of those terms in the corresponding "total" sum

$$\sum_{j=1}^{k-1} \left[U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi) - \int_{\alpha_{j-1}}^{\alpha_j} DU(\tau, t)(\eta, \xi) \right]$$

for which the tags τ_j satisfy the relation $\tau_j \in [c_{p-1}, c_p)$. Since 1.9.4 holds, we obtain by Lemma (1.9.3)

$$\left| \sum_{j=1, p(\tau_j)=p}^{k-1} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j=1})(\eta, \xi) - \int_{\alpha_{j-1}}^{\alpha_j} DU(\tau, t)(\eta, \xi) \right| < \frac{\epsilon}{2^{p+1}}$$

and finally,

$$\begin{aligned}
&\left| \sum_{j=1}^{k-1} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi)] - \int_{t_0}^{c} DU(\tau, t)(\eta, \xi) \right| \\
&= \left| \sum_{j=1}^{k-1} \langle \eta, f(\tau_j)\xi \rangle (\tau_j - \alpha_{j-1}) - \int_{t_0}^{c} D[f_{\eta\xi}(\tau), t] \right| \\
&= \left| \sum_{j=1}^{k-1} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi) - \int_{\alpha_{j-1}}^{\alpha_j} DU(\tau, t)(\eta, \xi) \right| \\
&\leq \sum_{p=1}^{\infty} \left| \sum_{j=1, p(\tau_j)=p}^{k-1} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi)] - \int_{\alpha_{j-1}}^{\alpha_j} DU(\tau, t)(\eta, \xi) \right|
\end{aligned}$$

$$\leq \sum_{p=1}^{\infty} \frac{\epsilon}{2^{p+1}} = \epsilon$$

. Define now a gauge w on $[t_0, T]$ as follows.

For $\tau \in [t_0, T]$ set $0 < w(\tau) < \min\{T - \tau, \hat{w}(\tau)\}$ while

$$0 < w(\tau) < T - B$$

If $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is an arbitrary w-fine partition of $[t_0, T]$ then by the choice of the gauge w we have $\tau_k = \alpha_k = T$ and $\alpha_{k-1} \in [B, T]$. Using (1.9.3) we get

$|S(U, D)(\eta, \xi) - I| =$

$$= \left| \sum_{j=1}^{k-1} [U(\tau_j, \alpha_j)(\eta, \xi) - U(\tau_j, \alpha_{j-1})(\eta, \xi) - U(\tau_j, \alpha_k)(\eta, \xi) - U(\tau_j, \alpha_{k-1})(\eta, \xi) \right|$$

$$\leq \left|\sum_{j=1}^{k-1}[U(\tau_j,\alpha_j)(\eta,\xi) - U(\tau_j,\alpha_{j-1})(\eta,\xi) - \int_{t_0}^{\alpha_{k-1}} DU(\tau,t)(\eta,\xi) - I\right|$$

$$+ \left|\int_{t_0}^{\alpha_{k-1}} DU(\tau,t)(\eta,\xi) - U(T,\alpha_{k-1})(\eta,\xi) + U(T,T)(\eta,\xi) - I\right| <$$

$$< \epsilon + \left|\sum_{j=1}^{k-1}[U(\tau_j,\alpha_j)(\eta,\xi) - U(\tau_j,\alpha_{j-1})(\eta,\xi)] - \int_{t_0}^{\alpha_{k-1}} DU(\tau,t)(\eta,\xi)\right|$$

Since $\alpha_{k-1} < T$ and $\hat{D} = \{\alpha_0, \tau_1, \alpha_1, \cdots, \alpha_{k-2}, \tau_{k-1}, \alpha_{k-1}\}$, is a $\hat{w}$-fine partition of $[t_0, \alpha_{k-1}]$, the second term on the right hand side of the last inequality can be estimated by ϵ as shown above from which we obtain

$$|S(U,D)(\eta,\xi) - I| < 2\epsilon$$

and this inequality yields the existence of the integral

$$\int_{t_0}^{T} DU(\tau,t)(\eta,\xi)$$

as well as the equality

$$\int_{t_0}^{T} DU(\tau,t)(\eta,\xi) = I$$

Remark: To prove (iii), we let $c_0 \in [t_0, T]$ so that by (1.9.8)(iii) for every $\epsilon > 0$, we can find $a, B \in [c, T]$ such that for every $t_0 \in [B, T]$ the inequality

$$\left|\int_{c}^{T} DU(\tau,t)(\eta,\xi) + U(t_0,c)(\eta,\xi) - U(t_0,t_0)(\eta,\xi) - I\right| < \epsilon \tag{1.9.10}$$

is satisfied and in a similar way to the prove of (i) we obtain

$$\int_{t_0}^{T} DU(\tau,t)(\eta,\xi) = I$$

Proof of (iii) Let $\epsilon > 0$ be given and let w be a gauge on $[t_0, T]$ which corresponds to ϵ by the definition of the class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$, the inequality

$$\left|S(U,D)(\eta,\xi) - \int_{t_0}^{T} DU(\tau,t)(\eta,\xi)\right| < \epsilon$$

holds for every w-fine partition D of $[t_0, T]$. If $s \in [c - w(c), c + w(c)] \subset [t_0, T]$ then by Lemma 1.9.3, we get

$$\left| U(c,s)(\eta,\xi) - U(c,c)(\eta,\xi) - \int_c^s DU(\tau,t)(\eta,\xi) \right| < \epsilon$$

that is

$$\left| \int_{t_0}^s DU(\tau,t)(\eta,\xi) - U(c,s)(\eta,\xi) + U(c,c)(\eta,\xi) - \int_{t_0}^c DU(\tau,t)(\eta,\xi) \right|$$

$$= \left| \int_c^s DU(\tau,t)(\eta,\xi) - U(c,s)(\eta,\xi) + U(c,c)(\eta,\xi) \right| < \epsilon$$

which yields (iii). And the proof is completed.

Remark: Theorem 1.9.2 is a convergence result established in [6] while theorem 1.9.4 concerns some fundamental properties of the Kurzweil integral and the associated QSDE.
The next results are established in [6] and concerns the existence of the integral involved in the definition of the Kurzweil equation (1.8.1).

1.9.6 Theorem. Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0,T],\ h_{\eta\xi}, W)$, and $X : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subseteq [t_0,T]$ is the limit of a sequence $\{X_k\}_{k\in\mathbb{N}}$ of processes $X_k : [a,b] \to \tilde{\mathcal{A}}$ such that

$$\int_a^b DF(X_k(\tau),t)(\eta,\xi) \ \text{ exists for every } \ k \in \mathbb{N}.$$

Then the integral

$$\int_a^b DF(X(\tau),t)(\eta,\xi) \ \text{ exists and}$$

$$\int_s^b DF(X(\tau),t)(\eta,\xi) = \lim_{k\to\infty} \int_a^b DF(X_k(\tau),t)(\eta,\xi)$$

.

1.9.7 Theorem. Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0,T], h_{\eta\xi}, W)$ and that $X : [a,b] \to \tilde{\mathcal{A}}$ is the limit of a sequence of simple processes. Then the integral $\int_a^b DF(X(\tau),t)(\eta,\xi)$ exists for arbitrary $\eta,\xi \in I\!\!D\underline{\otimes}I\!\!E$.

1.9.8 Theorem. Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0,T], h_{\eta\xi}, W)$ and $X : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subseteq [t_0,T]$ is of bounded variation, then the integral

$$\int_a^b DF(X(\tau),t)(\eta,\xi) \quad \text{exists.}$$

The proofs of theorems 1.9.6, 1.9.7 and 1.9.8 are found in [6].

The following results are simple extensions of similar results in [87] to the present generalized noncommutative quantum setting.

1.9.9 Lemma: Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ if of class $\mathbb{F}\ (\tilde{\mathcal{A}} \times [t_0,T], h_{\eta\xi}, W)$. If $[a,b] \subseteq [t_0,T]$ and $X : [a,b] \to \tilde{\mathcal{A}}$ for all $x \in \tilde{\mathcal{A}}$ and if the integral

$$\int_a^b DF(X(\tau),t)(\eta,\xi)$$

exists, then for every $t_1, t_2 \in [a,b]$ the inequality

$$\left| \int_{t_1}^{t_2} DF(X(\tau),t)(\eta,\xi) \right| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \tag{1.9.11}$$

is satisfied.

Proof: Using (1.9.1) and (1.9.2)

$$|t-\tau| \cdot |F(x(\tau),t)(\eta,\xi) - F(x(\tau),\tau)(\eta,\xi)| \leq (t-\tau)(h_{\eta\xi}(t) - h_{\eta\xi}(\tau)$$

for any $\tau, t \in [a,b]$. The integral $\int_a^b dh_{\eta\xi}(t)$ exists and

$$\int_{s_1}^{s_2} dh_{\eta\xi}(t) = h_{\eta\xi}(s_2) - h_{\eta\xi}(s_1)$$

for every $s_1, s_2 \in [a, b]$.

Therefore (1.9.11) is an immediate consequence of Theorem 1.9.4.

1.9.10 Lemma: Assume that the map $(x, t) \to F(x, t)(\eta, \xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T],\ h_{\eta\xi}, W)$. If $[a, b] \subseteq [t_0, T]$ and $x : [a, b] \to \tilde{\mathcal{A}}$ is a solution of (1.7) then the inequality

$$\|x(t_2) - x(t_1)\|_{\eta\xi} \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \tag{1.9.12}$$

holds for every $t_1, t_2 \in [a, b]$.

Proof: The result follows directly from Lemma (1.9.9) if we take into account that by definition we have

$$|\langle \eta, x(s_2)\xi \rangle - \langle \eta, x(s_1)\xi \rangle| = \left| \int_{s_1}^{s_2} DF(x(\tau), t)(\eta, \xi) \right|$$

for every $s_1, s_2 \in [a, b]$.

$$= |F(x(\tau), s_2)(\eta, \xi) - F(x(\tau), s_1)(\eta, \xi)| \leq |h_{\eta\xi}(s_2) - h_{\eta\xi}(s_1)|$$

1.9.11 Corollary: Assume that the map $(x, t) \to F(x, t)(\eta, \xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_o, T], h_{\eta\xi}, W)$. If $[a, b] \subseteq [t_0, T]$ and $X : [a, b] \to \tilde{\mathcal{A}}$ is a solution of (1.7) then X is of bounded variation on $[a, b]$ and

$$Var X_{[a,b]} \leq h_{\eta\xi}(b) - h_{\eta\xi}(a) < +\infty \tag{1.9.13}$$

Moreover, every point in $[a, b]$ at which the function $h_{\eta\xi}$ is continuous is a continuity point of the solution $X : [a, b] \to \tilde{\mathcal{A}}$.

Proof: Let $a = S_0 < S_1 < \cdots < S_k = b$ be an arbitrary division of the interval $[a, b]$. By (1.9.12) we have

$$\sum_{j=1}^{k} \|x(s_j) - x(s_{j-1})\|_{\eta\xi} \leq \sum_{j=1}^{k} |h_{\eta\xi}(s_j) - h_{\eta\xi}(s_{j-1})| < h_{\eta\xi}(b) - h_{\eta\xi}(a)$$

passing to the supremum over all divisions of $[a, b]$ we obtain (1.9.13). The second statement is a consequence of the inequality (1.9.12).

In the next section, we present a summary of some established results of Ayoola [6] concerning the equivalence of the Kurzweil equation (1.7) and the associated Lipschitzian quantum stochastic differential equation

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle = P(X(t), t)(\eta, \xi)$$

where the map $(X, t) \longrightarrow P(X, t)(\eta, \xi)$ is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ and $W(t) = t$.

1.10 Equivalence of Kurzweil equation and the associated Lipschitzian Quantum Stochastic Differential Equation

1.10.1 Notation: The class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, denotes the class of sesquilinear form-valued maps which are Lipschitzian and satisfy the Caratheodory conditions.

1.10.2 Definition: A map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq[\mathbb{D}\underline{\otimes}\mathbb{E}]$ belongs to the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

(i) $P(x, .)(\eta, \xi)$ is measurable for each $x \in \tilde{\mathcal{A}}$

$\int_{t_0}^{t} M_{\eta\xi} ds < \infty$ and $|P(x, .)(\eta, \xi)| \leqslant M_{\eta\xi}(s), \quad (x, s) \in \tilde{\mathcal{A}} \times [t_0, T]$

(iii) There exists measurable functions $K_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ such that for each $t \in [t_0, T], \quad \int_{t_0}^{t} K_{\eta\xi} ds < \infty$, and

$$|P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi)| \leq K^p_{\eta\xi}(s) W(\|x - y\|_{\eta\xi})$$

For $(x, s), (y, s) \in \tilde{\mathcal{A}} \times [t_0, T]$ and where for (i) -(iii) $W(t) = t$. and

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s) ds + \int_{t_0}^{t} K_{\eta\xi}(s) ds$$

1.10.3 Definition: For $(X, t) \in \tilde{\mathcal{A}} \times [t_0, T]$ and P belonging to the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, we define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$,

$$F(X, t)(\eta, \xi) = \int_{t_0}^{t} P(X, s)(\eta, \xi) ds \tag{1.10.1}$$

The next result connects the two classes of maps defined above.

1.10.1 Theorem. Assume that for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is of class $C(\tilde{A} \times [t_0, T], W)$. Then for every $x, y \in \tilde{\mathcal{A}},\ t_1, t_2 \in [t_0, T]$, $F(x, t)(\eta, \xi)$ defined by (1.10.1) satisfies

(i) $|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \leq \int_{t_1}^{t_2} M_{\eta\xi}(s)ds$

(ii) $|F(x,t_2)(\eta,\xi) - F(x,t_1) + F(y,t_1) - F(y,t_2)|$

$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K^p_{\eta\xi}(s)ds$

(iii) The map $F(x,t)(\eta,\xi)$ belong to the class $\mathcal{F}(\tilde{\mathcal{A}} \times [t_0,T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, where

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K^p_{\eta\xi}(s)ds.$$

The next result establishes the equivalence of the Kurzweil equation and the associated QSDE

1.10.2 Theorem. If $X : [a,b] \longrightarrow \tilde{\mathcal{A}}, [a,b] \subseteq [t_0,T]$ is the limit of simple processes then

$$\int_a^b DF(X(\tau),t)(\eta,\xi) = \int_a^b P(X(s),s)(\eta,\xi)ds \tag{1.10.2}$$

The next result establishes the existence of solution for the Kurzweil equation associated with the Lipschitzian QSDE (1.5).

1.10.3 Theorem. A stochastic process $X : [t_0,T] \to \tilde{\mathcal{A}}$ is a solution of equation (1.5) if and only if X is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi \rangle = DF(X(\tau),t)(\eta,\xi) \tag{1.10.3}$$

on $[t_0,T]$, $t \in [t_0,T]$, and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$).

1.10.4 Remark: If X is a solution of (1.5) on $[t_0,T]$, then by the existence and uniqueness results established in [6, 30], X is adapted and weakly absolutely continuous and lie in $L^2_{loc}(\tilde{\mathcal{A}})$.

1.11 Statement of the Problem

The technique of topological dynamics can only be applied to the study of quantum stochastic differential equations when sufficient analytical properties of solution are established. As with most ordinary differential equations, we cannot study the qualitative properties of solution of quantum stochastic differential equations (QSDEs) without knowing if the equation actually has a unique solution. Therefore, the study of existence and uniqueness of solutions is vital to the analysis of qualitative properties of solutions of QSDEs. Results on existence of solution for the weak forms of QSDE is subject to the Lipschitz condition $W(t) = t$. This restrict the class under which the results are are applicable. There is therefore the need to establish existence of solution under a more general Lipschitz condition. These generalizations motivate the following questions:

(i) Suppose a quantum stochastic differential equation does not satisfy the Lipschitz condition $W(t) = t$?

(ii) How can we possibly establish that a solution exists for such an equation?

(iii) is the solution unique?

(iv) is the solution stable?

(v) how well does the solution behave when it depends continuously on a parameter?

1.12 Aims

The aim of this research is to establish a basis for the application of the technique of topological dynamics to the study of quantum stochatic differential equation as in classical ordinary differential equations.

1.13 Objectives

To achieve this aim, the following objectives are outlined:

(i) To establish existence and uniqueness of solution of the Kurzweil equation associated with quantum stochastic differential equations under a more general Lipschitz condition.

(ii) To investigate variational stability, variational attracting and variational asymptotic stability of solution.

(iii) To investigate converse variational stability of solution.

(iv) To investigate continuous dependence of solution on parameters of the quantum stochastic differential equation and the associated Kurzweil equations.

1.14 Justification

In view of the foregoing, this research is strongly motivated by the need to extend the solution space of quantum stochastic differential equation to a class of equations that satisfy a more general Lipschitz condition and to create a framework so that the technique of topological dynamics can be applied to the study of quantum stochastic differential equations introduced above.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, we shall review some important results on existence of solution, stability of solution and continuous dependence of solution on parameters by various authors. Some of these results are vital for the extension of results on existence of solution of quantum stochastic differential equations associated with the Kurzweil equation for a class of equations that do not necessarily satisfy the Lipschitz condition.

It is worth mentioning that to the best of our knowledge, from the literatures consulted, variational stability and continuous dependence of solution on parameters have not been considered within the context of Ayoola and Ekhaguere's [6-9, 30] formulations of quantum stochastic differential equations and inclusions introduced in chapter one.

The review shall essentially follow the sequence outlined below;

(2.2) Existence and Uniqueness of Solution of Ordinary Differential Equations (ODEs).

(2.3) Existence of Solution and Continuous Dependence of Solution on Parameters of Classical Kurzweil Equation.

(2.4) Existence and Uniqueness of Solution of SDEs and QSDEs.

(2.5) Stability of Solutions.

2.2 Existence of Solution of Ordinary Differential Equations

Most results on existence of solution revealed consistent use of the Lipschitz function to establish existence of solution of ordinary differential equations.
In the theory of ordinary differential equations [12, 21, 25, 27, 40, 41A, 41B, 55, 91, 100, 10′], Lipschitz continuity is the central condition of the Picard-Lindelof theorem which guarantees the existence and uniqueness of solution to an initial value problem. The method of successive approximation was used extensively in establishing existence result in the above references. Some of the proofs rely on transforming the differential equations, and applying the fixed point theorems [26, 49, 93, 97]. It was shown that the sequence of successive approximations converged and that the limit is the solution to the problem. The Gronwall's Lemma was used to establish uniqueness of solution.
The present approach to the concept of an ordinary differential equation goes back to C. Caratheodory [87]. Given an ordinary differential equation of the form

$$\dot{x} = f(x, t) \tag{2.2.1}$$

The starting point for Caratheodory's generalized approach to ordinary differential equations of the form (2.2.1) is the integral equation given by

$$x(t) = x(\alpha) + \int_{\alpha}^{t} f(x(s), s)ds \tag{2.2.2}$$

where the Lebesgue integral is involved in (2.2.2). The fundamental question of the existence of a solution of the ordinary differential equation (2.2.1) is treated by Caratheodory as the question of existence of solution of the integral equation (2.2.2) with the Lebesgue integral on the right hand side.
By the properties of the Lebesgue integral, a function $x : J \to \mathbb{R}^n$ satisfying (2.2.2) is necessarily absolutely continuous in its interval of definition because the indefinite

Lebesgue integral has this properties. Therefore it cannot be expected that a solution of (2.2.1) in the sense of Caratheodory possesses a derivative everywhere in its domain of definition. Generalized solutions to (2.2.1) are absolutely continuous functions for which their derivative exist almost everywhere with respect to the Lebesgue measure. Caratheodory's proof of existence of a solution to the initial value problem (2.2.1) makes use of successive approximations.

The local version of the existence theorem for a solution of (2.2.1) in the Caratheodory setting can be found in [27, 35, 57]. Within this context, the possibility of using Perron's concept of the nonabsolutely convergent integral in the integral equation (2.2.2) was also investigated. Hence it was established that when looking for a solution of (2.2.2), the Perron integral $\int_\alpha^t f(x(s), s)ds$ should first exist for every $t \in [\alpha, \alpha + \Omega]$ and therefore any function satisfying (2.2.2) behaves like the indefinite integral of a Perron integrable function.

In [41B] Henstock is following the approach of Caratheodory in deriving existence results for the integral equation (2.2.2). For the same reason as mentioned above (i.e. the case studied by Caratheodory with the Lebesgue integral). To establish existence of solution it was assumed that the function $f : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ is continuous, Perron integrable,and satisfy some other conditions. Henstock's conditions for the existence of a solution of (2.2.1), are almost the same as Caratheodory's conditions except for the condition that makes it possible to interchange the order of the limit and the integral.

Considering the results in [35] the notion of a solution of the ordinary differential equation (2.2.1) was weakened to the case of a function defined on a nondegenerate interval with x continuous on the given interval and differentiable almost everywhere. This concept is more general than the concept of Caratheodory, absolute continuity is not required for the solution. It was pointed out that if x is a function which satisfies

$$x(t) = x(\alpha) + (P) \int_\alpha^t f(x(s), s)ds$$

for $\alpha, t \in J$ then it is a solution of (2.2.1) in the above weakened sense. The fundamental existence and unicity results for such solutions of (2.2.1) are given in [35] for the case when f satisfies Henstock's conditions and the Lipschitz condition

$$\|f(x,t) - f(y,t)\| \leq L(t)\|x - y\|$$

locally in the domain of f with L integrable in the Lebesgue sense. The result in [35] concerning solutions in the weakened sense are reduced to the above mentioned case of solutions in the Perron-Henstock sense.

2.3 Existence of Solution and Continuous Dependence on Parameters of Classical Kurzweil Equations

In order to generalize certain results on continuous dependence of solutions of ordinary differential equations with respect to parameters, J. Kurzweil introduced what he called generalized ordinary differential equations (GODEs) for euclidean and Banach space-valued functions. Among other applications, the theory of GODEs has proved to be useful in investigating topological dynamics of ordinary differential equations.
The generalized differential equations were thoroughly studied in [47, 52, 53-55, 79-87, 84 - 86], in which Kurzweil obtained important new results on continuous dependence on a parameter for differential equations. The convergence effects for a sequence of ordinary differential equations with the sequence converging in the usual way to the Dirac function was established.
The methods of generalized differential equations were extended by Kurzweil also to the case of differential equations in a Banach space. Here new results concerning partial differential equations and some types of boundary value problems were obtained. This was quite a new phenomenon in the theory of differential equations. This contributions inspired many mathematicians working in the theory of partial differential

equations as observed by [47].

In many practical situations, the dynamical systems described by the differential equations contain external parameters as well as the dependent variable. In such a case it is necessary to investigate the existence of solution and the behaviour of such a solution with respect to the given parameters. Following the line of Kurzweil, the ideas of continuous dependence of solution on parameters was also developed by some authors who pointed out that in order to have continuous dependence on a parameter, a certain "*integralcontinuity*" of the right hand side of the differential equation is sufficient [51, 53].

However in [51], continuous dependence of solution on a parameter for the generalized ordinary differential (Kurzweil) equation was investigated. The starting point of Kurzweil is to consider continuous functions $f_k(x,t), \quad k = 0, 1, \ldots,$ where $x_k(t)$ denote the solution of the equation

$$\frac{dx}{dt} = f_k(x,t), \;\; x(0) = 0 \tag{3.3.1}$$

and $x_0(t)$ denotes the solution of the generalized form of equation (3.3.1) with k replaced by 0.

$$\frac{dx}{dt} = DF_0(x,t) \tag{3.3.2}$$

if there exists such a subsequence $\{k_j\}$ such that $x_{k_j}(t) \to x_0(t) \quad$ with $\; j \to \infty$. In such case the fact that $x_{k_j}(t) \to x_0(t)$ means that the solution $x_0(t)$ depends continuously on the parameter k. The main objective here, is to include in the theory of generalized ordinary differential equations the convergence effect of equations (3.3.1) with k replaced by 0 i.e.

$$\frac{dx}{dt} = 0 = f_0(x,t), x(0) = 0 \tag{3.3.3}$$

In [54] a theorem on continuous dependence of solution on a parameter for generalized ordinary differential equation defined in [51] was studied. The solutions are of bounded variation. Results on continuous dependence on a parameter are applied to classical differential equations with a disturbing term which approximates the Dirac function. However, only few related literatures revealed results on continuous dependence of solution on parameters [3, 4, 27, 40, 47, 53 - 55, 87].

The main motivation for studying continuous dependence of solution on parameters for this class of QSDEs, is the results due to [3, 4, 51, 54, 87] where the differential equation was used to obtain approximate results due to the convergence effect of the given differential equation. Again, Kurzweil in his seminal papers [51, 52], introduced the Kurzweil equations. He defined the generalized differential equations and proved an existence theorem for another class of generalized equations.

The approach of J. Kurzweil shows how the new approach to the general integration theory was growing up from the needs of ordinary differential equations. In particular, the presence of rapidly oscillating external forces was the main impulse for introducing a new concept of convergence into the theory of ordinary differential equations instead of the Lebesgue integrals or classical Riemann integral.

Kurzweil showed that generalized differential equations admit discontinuous functions as solutions. The solution whose existence is demonstrated has bounded variation and is continuous from the left. The approach in [51] differs from simple existence results for ordinary differential equations in that the existence domain for a solution of the generalized differential equation is an interval for the given initial condition. In arriving at the result in [51], the method of successive approximation was employed.

However, in [3, 4, 26, 87, 96] the fixed point method was employed to establish existence results. Schwabik [79 - 87] established results on existence of solution for a class of equations that do not necessarily satisfy the Lipschitz condition. Schwabik's conditions for existence of solution are very similar to Kurzweil's conditions and Artstein's conditions [3, 4, 51].

2.4 Existence of Solution of Stochastic Differential Equations and Quantum Stochastic Differential Equations

As with deterministic ordinary and partial differential equations, existence and uniqueness of solution of stochastic differential equations have been considered by many authors [48, 60-63, 71A, 71B, 94, 95, 101]. In [95] a class of stochastic differential equations with non-Lipschitz conditions was studied. A unique strong solution is obtained and the non confluence of the solutions of the stochastic differential equations was also established. The dependence with respect to initial values was also investigated. Here the solution of the SDE depends on the growth behavior of the non constant coefficients.

The result of [95] generalizes the classical Lipschitz condition for existence and uniqueness of solutions and also the linear growth conditions for the non-explosion of solutions [48, 60]. Again in [100], the existence of solution of SDE was established with an improvement on the conditions of [95]. As an application, a class of infinite dimensional stochastic differential equations over lattice fields were proved to have a unique solution.

F. Shizan [95] gave a survey on the recent developments in stochastic differential equations essentially in two parts; a study beyond Lipschitz conditions and isotropic flows corresponding to the critical Sobolev exponent. For the first case, he considered again two categories of situation:

(i) the coefficients verify local Lipschitz without global Lipschitz conditions;

(ii) the coefficients do not verify local Lipschitz conditions.

In [68], the basic tools for Bosonic Calculus were developed. A necesary and sufficient condition for the existence of a unitary evolution satisfying a quantum stochastic differential equation with bounded coefficients was obtained. This theory has many applications, such as in the dilation of dynamical semi groups, the construction of Q diffusion in the sense of [14] and modelling physical systems. The QSDE is unitarily equivalent to a symmetric boundary value problem (BVP) for the Schrodinger equation [38]. It was proved that the solution of the Hudson and Parthasarathy QSDE in the Fock space coincide with the solution of a symmetric (BVP) for the Schrodinger equation in the interaction representation generated by the energy operator of the environment.

In [16], it was shown that a stochastic differential equation of the form

$$dX_t = F(X_t, t)dW_t + G(X_t, t)dW_t + H(X_t, t)dt$$

has a unique solution in the L^2- space of the Clifford algebra for any initial condition provided that the coefficients F,G,H satisfy a Lipschitz condition with respect to changes in the initial condition, and in the coefficients F,G,H.

J. Martinlingsay and G. Adam [59A] established and proved the existence and uniqueness theorems for QSDE with nontrivial initial conditions for coefficients with completely bounded columns. Applications are given for the case of finite-dimensional initial space or, more generally, for coefficients satisfying a finite local condition. Necessary and sufficient conditions are obtained for a conjugate pair of quantum stochastic Cocycles on a finite dimension operator space to strongly satisfy such a QSDE. This gives an alternative appoach to quantum stochastic convolution cocycles on co-algebra.

The theory of quantum stochastic differential equations, which are non commutative generalizations of classical stochastic differential equations, have undergone rapid developments in recent times [15 - 18, 34, 43-46, 61, 68-70, 98]. The recent work done by Ekhaguere and Ayoola [6 - 9, 30, 31] have been of immense contribution to numerical solutions of Stochastic differential equations (SDEs) and quantum stochastic differential equations (QSDEs). Some underlying principles present in many of these papers, will be of immense contribution in extending the solution space of QSDEs to solutions that do not necessarily depend on the Lipschitz condition.

Ekhaguere [30] approach in the study of quantum stochastic differential inclusion (QSDI) within the frame work of Hudson and Parthasarathy formulation of quantum stochastic differential equations(QSDEs) is a major contribution in simplifying QSDEs (1.1) to the form (1.3). Results concerning the existence of solution of a Lipschitz QSDI and the relationship between the solutions of such an inclusion and those of its convexification were studied. This result also represents a generalization of the Gronwall Filippov existence theorem and the Filippov - Wazewski theorem for classical differential inclusions. A quantum stochastic differential equation is a special case of quantum stochastic differential inclusion.

In the paper [31], Ekhaguere studied quantum stochastic differential inclusions of hypermaximal monotone type, under very general conditions. Using a nice choice of the partitions of time interval, Ekhaguere introduced discrete schemes which approximate the quantum stochastic differential inclusions. Results of how the solutions of two such schemes compare was established alongside some proofs on uniform convergence of the sequence of approximating schemes. Lastly, existence of an evolution operator corresponding to each such inclusion was proved. However as mentioned earlier there have been corresponding developments in their numerical solutions. Unique and unitary analytic solutions of some of these equations are known to exist but are difficult to come by.

Ayoola in [6 - 8], has contributed immensely to the development of numerical schemes which is a major breakthrough in SDEs and QSDEs. Discrete schemes that approximate matrix elements of solution of the form (1.3) were established. This was accomplished by assuming some smoothness conditions on the map $t \to \langle \eta, X(t)\xi \rangle$, Lipschitz and continuity conditions on the map $(\eta, \xi) \to P(t, X)(\eta, \xi)$.

Questions of convergence and consistency in respect of discrete schemes that approximate matrix elements of solutions of QSDE (1.3) were also addressed.

The introduction of these schemes was facilitated by the differentiability of the matrix elements $\langle \eta, X(t)\xi \rangle$ of solution X of problem (1.3) since they have the advantage of being differentiable.

In [7], as an appendix, the existence and uniqueness of solution of equation (1.3) introduced in chapter one was established by supposing that the coefficients E,F,G,H appearing in (1.3) belong to $L^2_{loc}(I \times \tilde{\mathcal{A}})$ and are Lipschitzian. Also the map $(\eta, \xi) \to P(t,x)(\eta, \xi)$ is a sesquilinear form for fixed (t,x). The explicit form of this map is given by equation (1.6). Ayoola [8] also established the Lagrangian quadrature scheme. Although this scheme produced better results than the Euler scheme but subject to the Lipschitz condition and some other conditions.

However in [6], the equivalent form of an inclusion, which is a first order non classical initial value ordinary differential equation(1.5) was studied. The Kurzweil equation (1.7) associated with quantum stochastic differential equation (1.5) was introduced and studied. It was established that equations (1.5) and (1.7) are equivalent. And hence existence of solution of QSDE (1.5) imply existence of solution of the associated Kurzweil equation (1.7) and conversely. The results were used to obtain a reasonably high accurate approximate solution for QSDEs which is better than the Euler scheme and other multistep schemes considered in [7, 8].

This scheme is applicable to a wide class of equations that satisfy the Lipschitz and Caratheodory conditions. To the best of our knowledge, no other significant contributions have so far been reported in the literature concerning existence of solution for a class of quantum stochastic differential equation that satisfy a more general Lipschitz condition.

In analogy to ordinary differential equations, where existence of solution for a class of equation that fails to satisfy the Lipschitz condition were studied and established, in this thesis, we shall establish existence of solution for a class of quantum stochastic differential equation that satisfy a more general Lipschitz condition especially for those class of equations that will fail to satisfy the Lipschitz condition. Thereby, making the results in [6] a special case of the results established here.

2.5 Stability of Solution

The theory of qualitative properties of solutions of ordinary differential equations and generalized ordinary differential equations such as stability, convergence, boundedness, etc. have received series of attention in recent years [2, 13, 50, 59B, 63, 66, 67, 72, 73, 79, 83, 74 - 78].

Stability means insensitivity of the state of the system to small changes in the initial state or parameters of the system. For a stable system the trajectories which are close to each other at a specific instant should therefore, remain close to each other at all subsequent instants.

Lypunov [59B] introduced the concept of stability of a dynamical system. Two methods for dealing with stability problems were introduced. While the first method is of a special nature, the second method (direct method) has developed into an extra ordinary useful tool. The method is based on a real- valued Lyapunov function V, which can be viewed as a general distance from the origin. Lyapunov established sufficient conditions for stability. However, Perestjuk [71A, 71B] showed that Lyapunov's conditions are not only sufficient but necessary as well. Sufficient conditions for uniform stability in terms of a certain Lyapunov function were also formulated.

In [35], Lyapunov's second method was employed to establish integral and integral asymptotic stability of ordinary differential systems with respect to impulsive perturbations. The objective of this investigation was to obtain sufficient conditions for the integral and integral asymptotic stability of the trivial solution of the given equation. The proofs of these results crucially depend on almost everywhere differentiability of the function U, and this property is guaranteed because U is a function of bounded variation.

Again in [79, 81, 83, 92, 98, 99], the concept of variational stability was also introduced and studied. by H. Okamura. T. Vrkoč [99] considered Caratheodory equations and pointed out that Okamura's variational stability is equivalent to his concept of integral stability. There is an improvement of the results in [99] given in [23].

In the case of stochastic differential equations, it turns out that there are at least three different types of stochastic stability: stability in probability, almost sure stability and moment stability. Bucy recognized that stochastic Lyapunov function should have the super-martingale property and gave sufficient criteria for stability in probability and moment stability. Almost sure stability was considered by Has'minskii for linear stochastic differential equations. Stochastic stability has been one of the most active areas in stochastic analysis and many mathematicians have devoted their interest to it.

In [2, 13, 24, 48, 50, 56,], the Lyapunov's method was used to establish stability results for the given solution. Continuity and positive-definite conditions were assumed. However, in [7] the stability of the quantum stochastic differential equation was established using the simple process $(X_n(t))$ as an iterate to the initial conditions.

There are other methods of establishing stability of a stochastic differential equation as explained above but the closest and most applicable to our approach is the result due to [59B, 87] where a Lyapunov function is used with some conditions imposed on it. Aside from the Lyapunov method, there are also other methods such as the Lagrange method, but Lyapunov method has proved very effective in establishing stability results for most differential equations [41A, 41B, 74, 75, 83, 91]. In summary, Lyapunov stability of $y \equiv 0$ means that if a solution $y(t)$ starts near $y = 0$ it remains near $y = 0$ in the future $(t \geq 0)$; and Lyapunov asymptotic stability of $y \equiv 0$ means that, in addition $y(t) \to 0$ as $t \to \infty$ [40, 73, 100].

The contributions on existence and uniqueness of solution of Kurzweil equations and the associated QSDEs made by Ayoola, Ekaguere [6, 30] and contributions made by other authors cited above will be generalized here. The equations have been developed using Fock space stochastic calculus introduced by Hudson and Parthasarathy [44], and are non-commutative generalizations of classical stochastic differential equations [46, 68 - 70].

Chapter 3

Methodology

3.1 Introduction

In this chapter, we shall discuss the methods we used to establish the major results on existence of solution, variational stability and continuous dependence of solution on parameters. This chapter will consist of three sections. Section 3.1 shall concern the method of establishing our result on existence and uniqueness of solution of the Kurzweil equation (1.7) associated with QSDEs (1.5). In section 3.2, we shall discuss the method employed in establishing results on variational stability and lastly, we shall discuss the method of establishing results on continuous dependence of solution on parameters in section 3.3.

For existence of solution,we adopt the method of successive approximation employed in [30] to establish the main result on existence of solution. Since the equivalence of the quantum stochastic differential equation and the associated Kurzweil equation has been established in [6], we shall establish the existence of solution of equation (1.5) using the equivalent quantum stochastic differential equation (1.3). The existence of solution of the QSDE (1.5) will then imply the existence of solution of the associated Kurzweil equation (1.7).

The methods we shall discuss in sections 3.3 and 3.4 are extensions of the methods employed in [3, 4, 87] to the present noncommutative quantum setting. The discussion shall follow the outline bellow:

(3.2) Method of proof on Existence and Uniqueness of Solution.

(3.3) Method of proof on Variational Stability of Solution.

(3.4) Method of proof on Continuous Dependence on Parameters.

3.2 Method of proof on Existence and Uniqueness of Solution

We consider the map $(x, s) \to P(x, s)(\eta, \xi)$ which is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W), W(t) \neq t$. The existence of solution of the QSDE (1.3) will be established as follows:

by constructing a τ_w - Cauchy sequence $\{\Phi_n(t)\}_{n\geq 0}$ of the successive approximations of Φ with the property that the sequence $\{\frac{d}{d\tau}\langle \eta, \Phi_n(\tau)\xi\rangle\}_{n\geq 0}$ is also Cauchy in $\mathbb{C}$ for arbitrary $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$.

We Define

$$\begin{aligned}\Phi_{n+1}(t) &= X_0 + \int_{t_0}^{t}(E(\Phi_n(s), s)d\wedge_\pi(s) + F(\Phi_n(s), s)dA_g^+(s) \\ &\quad + G(\Phi_n(s), s)dA_f(s) + H(\Phi_n(t), t)ds).\end{aligned}$$

for $n \geq 0$ and establish the convergence of these successive approximations. This is possible since Φ_n is a cauchy sequence in $\tilde{\mathcal{A}}$ and must surely converge uniformly to some Φ in $\tilde{\mathcal{A}}$ since the space $\tilde{\mathcal{A}}$ is complete. Lastly we show that Φ satisfies the given quantum stochastic differential equation. This is also possible since the sequence of stochastic processes Φ_n are simple processes whose limit exists.

To establish uniqueness of solution, we adopt the most common technique to proving uniqueness. That is, assume that there exist two solutions say x and y that satisfy the given conditions, and then logically deducing their equality by applying the Gronwall's inequality. The solution of the QSDE (1.3) will then imply the solution of the Kurzweil equation (1.7) associated with the QSDE (1.5). This is also possible since the equivalence of equations (1.5) and (1.7) has been established in [6].

3.3 Method of proof on Variational Stability of Solution

In this section we discuss the method of establishing variational stability and asymptotic variational stability of solution of equation (1.7). Since equation (1.5) is equivalent to equation (1.7), the results will also hold for equation (1.5). Because it is difficult to explicitly write the solution to the given equation, we employ Lyapunov's method to establish results on all kinds of variational stability of the trivial solution $x \equiv 0$ of equation (1.7). Lyapunov's method enables one to investigate stability of solution without explicitly solving the differential equation by making use of a real-valued function called the Lyapunov's function that satisfies some conditions such as positive definite, continuity, etc.

In this case, we assume that the maps, $(X, t) \longrightarrow P(x, t)(\eta, \xi)$ and $(x, t) \longrightarrow F(x, t)(\eta, \xi)$ are of class $C(\tilde{\mathcal{A}} \times [t_0, T], W), W(t) \neq t$ and $F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ respectively. The stochastic processes considered here are also of bounded variation as in [13, 87].

Assume that for every initial value $x(t_0) = x_0 \in \tilde{\mathcal{A}}$, there exists a unique solution which is denoted by $x(t)$. Assume further that

$$F(0, t)(\eta, \xi) = 0 \quad \text{for all} \;\; t \geq t_0$$

So equation (1.7) introduced in section 1.1 has the solution $x \equiv 0$ corresponding to the initial value $x(t_0) = 0$. This solution is called the trivial solution.

To establish variational stability, it is only needful to assume that the function $V(x, t)(\eta, \xi)$ is real-valued, bounded and achieves its minimum at $x = 0$. The function $V(x, t)(\eta, \xi)$ has the guaranteed property that as the trajectory moves, the value of this function along the trajectories strictly decreases. Since $V(x, t)(\eta, \xi)$ is lower bounded by zero and is strictly decreasing, it must converge to a nonnegative limit as time goes to infinity. Indeed all the conditions imply that $V(x,t)(\eta,\xi) \to 0$ as $t \to \infty$. Since $x = 0$ is the only point in space where $V(x,t)(\eta,\xi)$ vanishes, we can conclude that $x(t)$ goes to the origin as time goes to infinity.

Once again we emphasize that the significance of Lyapunov's meyhod is that it allows stability of the system to be verified without explicitly solving the differential equation (1.7). Lyapunov's method, in effect, turns the question of determining stability into a search for a so-called Lyapunov function, a positive definite function of the state that decreases locally along trajectories. The type of theorems that prove existence of Lyapunov functions for every stable system are called converse Lyapunov theorems. This we shall establish by assuming knowledge of variational stability of the solution of equation (1.7).

3.4 Method of proof on Continuous Dependence on Parameters

For continuous dependence of solution on a parameter we adopt the method of convergence employed in [87] to our present noncommutative quantum setting. Since the solution of equation (1.5) belongs to an infinite dimensional locally convex space, a sequence $\{X_n\}$ in $\tilde{\mathcal{A}}$ converges to an element X in $\tilde{\mathcal{A}}$ if and only if the sequence $\{\langle \eta, X_n\xi \rangle\}$ converges to $\{\langle \eta, X\xi \rangle\}$ in the complex field for all $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$. Also since the stochastic process $x : [a, b] \to \tilde{\mathcal{A}}$ lie in $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$, by theorem 1.9.6 the integrals

$$\int_a^b DF(x(\tau), t)(\eta, \xi) \text{ and } \int_a^b P(x, t)(\eta, \xi) dt \ \text{ exist}$$

where

$$F(x, t)(\eta, \xi) = \int_a^b P(x, t)(\eta, \xi) dt$$

Again by theorem 1.9.6 we show that the integrals

$$\int_a^b DF_k(x(\tau), t)(\eta, \xi) \quad \text{and} \quad \int_a^b DF_0(x(\tau), t)(\eta, \xi)$$

exist and by taking the limit as $k \to \infty$ we can show that the two integrals are equal to each other. This is also possible since the stochastic process $x_1, x_2, \ldots, x_k$ are simple processes and lie in $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ and hence when $k \to \infty, F_k \to F_0$.

Chapter 4

Existence and Uniqueness of Solution of Kurzweil equations associated with Quantum Stochastic Differential Equations

4.1 Introduction

In this section, we establish existence and uniqueness of solution for the equivalent form of the quantum stochastic differential equation (1.5) introduced in chapter one. We consider the case of QSDE where the coefficients satisfy a more general Lipschitz condition of which the Lipschitz condition considered in [6] will be a special case of the results here.

On this occasion, we consider the case when the map $(x,t) \longrightarrow P(x,t)(\eta,\xi)$ is of class $C(\tilde{\mathcal{A}} \times [t_0,T], W)$ and $W(t) \neq t$. This generalizes the lipschitz case considered in [6] where $W(t) = t$. The results here consequently widens the process for which the theory of quantum stochastic differential equation is applicable.

We adopt the method of successive approximations as in [7, 30] to establish the results here. We also use the notations and definitions of the spaces and concepts presented in chapter 1.

4.2 Existence of Solution

We consider the following quantum stochastic differential equation introduced in chapter one.

$$\begin{aligned} dX(t) &= E(X(t),t)d\wedge_{\pi}(t) + F(X(t),t)dA_g(t) \\ &\quad + G(X(t),t)dA_{f^+}(t) + H(X(t),t)dt),\ X(t_0) = X_0, t \in I \qquad (4.2.1) \end{aligned}$$

We establish the existence and uniqueness of solution of equation (4.2.1) under the conditions of the following definition.

4.2.1 Definition A map $P : \tilde{\mathcal{A}} \times [t_0,T] \rightarrow sesq[I\!D\underline{\otimes}I\!E]$ is said to be of class $C(\tilde{\mathcal{A}} \times [t_0,T], W), W(t) \neq t$ if for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$

(i) $P(x,.)(\eta,\xi)$ is measurable for each $x \in \tilde{\mathcal{A}}$

(ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0,T] \rightarrow \mathbb{R}_+$ such that $\int_{t_0}^{t} M_{\eta\xi}(s)ds < \infty$ and $|P(x,.)(\eta,\xi)| \leqslant M_{\eta\xi}(s), \quad (x,s) \in \tilde{\mathcal{A}} \times [t_0,T]$

(iii) There exists measurable functions $K_{\eta\xi} : [t_0,T] \rightarrow \mathbb{R}_+$ such that for each $t \in [t_0,T], \quad \int_{t_0}^{t} K_{\eta\xi}(s)ds < \infty$, and

$$|P(x,s)(\eta,\xi) - P(y,s)(\eta,\xi)| \leq K^{p}_{\eta\xi}(s)W(\|x-y\|_{\eta\xi})$$

For $(x,s),(y,s) \in \tilde{\mathcal{A}} \times [t_0,T]$ and

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K_{\eta\xi}(s)ds$$

4.2.2 Definition For $(x,t) \in \tilde{\mathcal{A}} \times [t_0,T]$ and P belonging to class $C(\tilde{\mathcal{A}} \times [t_0,T], W)$, with $W(t) \neq t$, we define for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E\}$,

$$F(x,t)(\eta,\xi) = \int_{t_0}^{t} P(x,s)(\eta,\xi)ds \qquad (4.2.2)$$

Next we present and establish a major result.

4.2.1 Theorem

(i) Let $P(x,t)(\eta,\xi)$ be of class $C(\tilde{\mathcal{A}} \times [t_0,T], W), W(t) \neq t$.

(ii) Assume that the coefficients E, F, G, H appearing in equation (4.2.1) satisfy the general Lipschitz condition and belong to $L^2_{loc}(I \times \tilde{\mathcal{A}})$.

Then for any fixed point $(X_0, t_0) \in \tilde{\mathcal{A}} \times I$ there exists a unique adapted and weakly absolutely continuous solution Φ of the quantum stochastic differential equation (4.2.1) satisfying $\Phi(t_0) = X_0$.

Proof. We start by constructing a τ_w- Cauchy sequence $\{\Phi_n(t)\}_{n\geq 0}$ of successive approximations of Φ in $\tilde{\mathcal{A}}$. All through except otherwise stated $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ is arbitrary. Fix $T > t_0,\ t \in [t_0,T]$. Define $\Phi_0(t) = X_0$, and for $n \geq 0$

$$\begin{aligned}\Phi_{n+1}(t) &= X_0 + \int_{t_0}^{t} (E(\Phi_n(s),s)d\wedge_\pi(s) + F(\Phi_n(s),s)dA_g^+(s) \\ &\quad +G(\Phi_n(s),s)dA_f(s) + H(\Phi_n(s),s)ds).\end{aligned}$$

We let each $\Phi_n(t), n \geq 1$ define an adapted weakly absolutely continuous process in $L^2_{loc}(\tilde{\mathcal{A}})$.

By hypothesis, $E(X_0,s), F(X_0,s), G(X_0,s)$, and $H(X_0,s)$ belong to $\tilde{\mathcal{A}}_s$ for $s \in [t_0,T]$ and $E(X_0,.), F(X_0,.), G(X_0,.)$, and $H(X_0,.)$ lie in $L^2_{loc}(\tilde{\mathcal{A}})$. Therefore the quantum stochastic integral which defines $\Phi_1(t)$ exists for $t \in [t_0,T]$.

By equation (1.3.2), $\Phi_1(t)$ is weakly absolutely continuous and hence locally square integrable.

Assume now that $\Phi_n(t)$ is adapted and weakly absolutely continuous, then each $E(\Phi_n(s),s), F(\Phi_n(s),s), G(\Phi_n(s),s)$ and $H(\Phi_n(s),s)$ is adapted and lie in $L^2_{loc}(\tilde{\mathcal{A}})$. Thus $\Phi_{n+1}(t)$ is adapted and well defined.

Again by equation(1.3.2), $\Phi_{n+1}(t)$ is a weakly absolutely continuous process in $L^2_{loc}(\tilde{\mathcal{A}})$.

Hence we have proved our claim by induction. We consider the convergence of the successive approximations.

By equation (1.3.2) and the definition of the map P in section 1.6 we have,

$$\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} = | \langle \eta, (\Phi_{n+1}(t) - \Phi_n(t))\xi \rangle | =$$

$$= | \int_{t_0}^{t} (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi_{n-1}(s), s)(\eta, \xi))ds| \tag{4.2.3}$$

Remark. The hypothesis (ii) of proposition 1.6.1 holds in this case but with the general Lipschitz condition $W(t) \neq t$.

Hence since the coefficients E, F, G, H are Lipschitzian, by proposition 1.6.1(ii), the map $(x, t) \to P(X, t)(\eta, \xi)$ also satisfy the general Lipschitz condition with Lipschitz function $K^p_{\eta\xi} : [t_0, T] \to (0, \infty)$ lying in $L^1_{loc}([t_0, T])$ and is also of class $C(\tilde{\mathcal{A}} \times [t_0, T], W), W(t) \neq t$ i.e,

$$|P(x, t)(\eta, \xi) - P(y, t)(\eta, \xi)| \leq K^p_{\eta\xi}(t) W(\| x - y \|_{\eta\xi})$$

for all $\quad x, y \in \tilde{\mathcal{A}}, \quad t \in [t_0, T]$.

Hence substituting the last inequality in (4.2.3), we get

$$\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} \leq \int_{t_0}^{t} K^p_{\eta\xi}(s) W(\| \Phi_n(s) - \Phi_{n-1}(s) \|_{\eta\xi}) ds \tag{4.2.4}$$

Since the map $s \longrightarrow \|\Phi_1(s) - X_0\|_{\eta\xi}$ is continuous on $[t_0, T]$, we put

$$R_{\eta\xi} = \sup_{s \in [t_0, T]} \|\Phi_1(s) - X_0\|_{\eta\xi}, s \in [t_0, T]$$

this implies that $\|\Phi_1(s) - X_0\|_{\eta\xi} \leq R_{\eta\xi}$ and $\Rightarrow W(\|\Phi_1(s) - X_0\|_{\eta\xi}) \leq W(R_{\eta\xi})$ since $W(t) \neq t$.

Also let

$$M_{\eta\xi}(t) = \int_{t_0}^{t} K^p_{\eta\xi}(s) ds$$

From (4.2.4) we have

$$\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} \leq \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^n}{n!}, \quad n = 1, 2, ... \tag{4.2.5}$$

This we prove by induction as follows.

For $n = 1$, inequality (4.2.5) holds by considering (4.2.4). Assume that (4.2.5) holds for $n = k$

i.e.

$$\| \Phi_{k+1}(t) - \Phi_k(t) \|_{\eta\xi} \leq \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!}, n = 1, 2, ... \tag{4.2.6}$$

then by (4.2.4)

$$\|\Phi_{k+2}(t) - \Phi_{k+1}(t)\|_{\eta\xi} \leq \int_{t_0}^{t} K^p_{\eta\xi}(s) W(\|\Phi_{k+1}(s) - \Phi_k(s)\|_{\eta\xi}) ds$$

$$= \int_{t_0}^{t} K^p_{\eta\xi}(s) \frac{W(R_{\eta\xi})(M_{\eta\xi}(s))^k}{k!} ds$$

$$\leq \frac{W(R_{\eta\xi})}{k!} \int_{t_0}^{t} K^p_{\eta\xi}(s)(M_{\eta\xi}(s))^k ds \quad \text{by (4.2.6)}$$

By applying integration by parts, we obtain

$$\int_{t_0}^{t} K^p_{\eta\xi}(s)(M_{\eta\xi}(s))^k = \frac{(M_{\eta\xi}(t))^{k+1}}{k+1} \tag{4.2.7}$$

Therefore,

$$W\|\Phi_{k+2}(t) - \Phi_{k+1}(t)\|_{\eta\xi} \leq \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^{k+1}}{(k+1)!}$$

So that (4.2.5) holds for $n = k + 1$ and so holds for $n = 1, 2, 3, ...$

Therefore, for any $n > k$,

$$\| \Phi_{n+1}(t) - \Phi_{k+1}(t) \|_{\eta\xi} = \|\Sigma^n_{m=k+1}(\Phi_{m+1}(t) - \Phi_m(t)\|_{\eta\xi}$$

$$\leq \Sigma^n_{m=k+1}\|\Phi_{m+1}(t) - \Phi_m(t)\|_{\eta\xi}$$

$$\leq \Sigma^n_{m=k+1} \frac{W(R_{\eta\xi})(M_{\eta\xi}(T))^m}{m!}$$

It follows that $\Phi_n(t)$ is a cauchy sequence in $\tilde{\mathcal{A}}$ and converges uniformly to some $\Phi(t)$. Since $\Phi_n(t)$ is adapted and weakly absolutely continuous, the same is true of $\Phi(t)$. Next we show that $\Phi(t)$ satisfies the quantum stochastic differential equation (4.2.1).

Surely $\Phi(t_0) = X(t_0) = X_0$.

By equation (1.3.2)

$$\left\| \int_{t_0}^{t} [E(\Phi_n(s), s) d\wedge_\pi (s) + F(\Phi_n(s), s) dA_g^+(s) + G(\Phi_n(s), s) dA_f(s) + H(\Phi_n(s), s) ds] \right\|_{\eta\xi}$$

$$- \left\| \int_{t_0}^{t} [E(\Phi(s), s) d\wedge_\pi (s) + F(\Phi(s), s) dA_g^+(s) + G(\Phi(s), s) dA_f(s) + H(\Phi(s), s) ds] \right\|_{\eta\xi}$$

$$= \left| \int_{t_0}^{t} (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi(s), s)(\eta, \xi)) ds \right|$$

$$\leq \int_{t_0}^{t} K_{\eta\xi}^p(s) W(\| \Phi_n(s) - \Phi(s) \|_{\eta\xi}) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Since $\Phi_n(s) \longrightarrow \Phi(s)$ in $\tilde{\mathcal{A}}$ uniformly on $[t_0, T]$.

Thus

$$\begin{aligned}
\Phi(t) &= \lim_{n\to\infty} \Phi_{n+1}(t) \\
&= X_0 + \lim_{n\to\infty} (\int_{t_0}^{t} (E(\Phi_n(s), s) d\wedge_\pi (s) + F(\Phi_n(s), s) dA_g^+(s) \\
&+ G(\Phi_n(s), s) dA_f(s) + H(\Phi_n(s), s) ds) \\
&= X_0 + \int_{t_0}^{t} (E(\Phi(s), s) d\wedge_\pi (s) + F(\Phi(s), s) dA_g^+(s) \\
&+ G(\Phi(s), s) dA_f(s) + H(\Phi(s), s) ds).
\end{aligned}$$

That is $\Phi(t), t \in [t_0, T]$ is a solution of equation (4.2.1).

4.3 Uniqueness of Solution

Suppose that $Y(t), t \in [t_0, T]$ is another adapted weakly absolutely continuous solution with $Y(t_0) = X_0$. Then, by equation (1.3.2), we obtain again

$$\| \Phi(t) - Y(t) \|_{\eta\xi} = | \int_{t_0}^{t} (P(\Phi(s), s)(\eta, \xi) - P(Y(s), s)(\eta, \xi))ds|$$

$$\leq \int_{t_0}^{t} K_{\eta\xi}^{p}(s) W(\| \Phi(t) - Y(s) \|_{\eta\xi}) ds$$

Since the integral $\int_{t_0}^{t} K_{\eta\xi}^{p}(s)$ exists on $[t_0, T]$, it is also essentially bounded on the given interval. Hence, there exists a constant $C_{\eta\xi,t}$ such that

$$\text{ess sup} \;\; K_{\eta\xi}^{p}(s) = C_{\eta\xi,t}, \quad s \in [t_0, T].$$

Thus

$$\| \Phi(t) - Y(t) \|_{\eta\xi} \leq C_{\eta\xi,t} \int_{t_0}^{t} W(\| \Phi(s) - Y(s) \|_{\eta\xi}) ds$$

By the Gronwall's inequality, we conclude that $\Phi(t) = Y(t), \in [t_0, T]$. Hence the solution is unique.

4.3.1 Remark

The results on existence and uniqueness of solution of quantum stochastic differential equation established here implies existence of solution of the associated Kurzweil equation (1.7) for a class of equation that satisfy the general Lipschitz condition $W(t) \neq t$.

Chapter 5

Variational Stability of Kurzweil Equations associated with Quantum Stochastic Differential Equations

5.1 Introduction

Most differential equations, deterministic or stochastic, cannot be solved explicitly [37-40, 48, 63, 91, 94, 4′, 7′], nevertheless we can often deduce a lot of useful information by qualitative analysis about the behaviuor of their solutions from the functional form of their coefficients. The long term asymptotic behaviour and sensitivity of the solutions to small changes is of great interest. This is very important especially in measurement errors, initial values and many more.

In this section, we study variational stability of the unperturbed equation (1.5) and variational stability with respect to perturbations of the quantum stochastic differential equation (1.5) introduced in chapter one. Variational stability is a generalized concept which is suitable for the class of generalized nonclassical ordinary differential equations studied in chapter 4 because of the local finitness of the variation of a solution.

We employ the Kurzweil equation associated with this class of quantum stochastic differential equation(QSDE) to establish results on variational stability, variational attracting, variational asymptotic stability and converse variational stability.

It is important to mention here, that results on variational stability are not only restricted to the general case considered here but are applicable to the case studied in the literatures with the Lipschitz condition $W(t) = t$.

The next section will be divided into three sections 5.2, 5.3 and 5.4. Section 5.2 will be devoted to the concept of variational stability of the Kurzweil equation associated with QSDE. Here, results on variational stability, variational attracting, relationship between variational attracting and asymptotic variational stability will be established using their definitions and the converse method.

In section 5.3, we shall establish some auxiliary results which will be used to establish the main results on variational stability and asymptotic variational stability. We shall use Lyapunov's method to establish the major results on variational stability and asymptotic variational stability of the Kurzweil equations associated with QSDEs.

Lastly, in section 5.4, the converse of the theorems on variational stability and varaitional asymptotic stability will be discussed. It is worth mentioning that converse variational stability is more like a search for a Lyapunov's function [39, 52, 72, 87]. It gaurantees the existence of a Lyapunov function.

5.2 Concepts and Definitions of Variational Stability

We introduce the concept of variational stability of quantum stochastic differential equations driven by the Hudson - Parthasarathy integrators $\wedge_\pi(t), A_g^+(t), A_f(t)$

$$\begin{aligned} dX(t) &= E(X(t),t)d\wedge_\pi(t) + F(X(t),t)dA_g^+(t) \\ &\quad +G(X(t),t)dA_f(t) + H(X(t),t)dt \\ X(t_0) &= X_0, \ \ t \in [0,T] \end{aligned} \tag{5.2.1}$$

We shall consider the Kurzweil equation associated with the equivalent form of (5.2.1). As in the references [6, 7, 30] solutions of (5.2.1) are $\tilde{\mathcal{A}}$ - valued processes where $\tilde{\mathcal{A}}$ is a locally convex space defined previously. We adopt the definitions and notations of the following spaces defined in chapter one $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^2_{loc}(\tilde{\mathcal{A}})$ and $BV(\tilde{\mathcal{A}})$. For arbitrary $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$, the equivalent form of (5.2.1) is given by

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &= P(X(t),t)(\eta,\xi) \\ X(0) &= X_0, \quad t \in [0,T] \end{aligned} \tag{5.2.2}$$

where the map $(x,t) \to P(x,t)(\eta,\xi)$ is as defined by equation (1.6). We employ the associated Kurzweil equation introduced in chapter one given by

$$\begin{aligned} \frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle &= DF(X(\tau),t)(\eta,\xi) \\ X(0) &= X_0, \quad t \in [0,T], \end{aligned} \tag{5.2.3}$$

where

$$F(X,t)(\eta,\xi) = \int_0^t P(X,s)(\eta,\xi)ds \tag{5.2.4}$$

In chapter four, it has also been shown that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [0,T], h_{\eta\xi}, W)$ and the map $(x,t) \longrightarrow P(x,t)(\eta,\xi)$ is of class $C(\tilde{\mathcal{A}} \times [0,T], W)$ respectively.

Existence of solution has been established. Consequently, existence results enables one to investigate the variational stability of solution. This investigation is made possible since solutions of equation (5.2.3) (and therefore of (5.2.2)) are quantum stochastic processes of bounded variations. Variational stability deals with the measurement of the distance between solutions in the space of stochastic processes of bounded variation using the seminorms defined by their bounded variations.

Since the solution $X \in Ad(\tilde{\mathcal{A}})_{wac}$ of equation (5.2.3) are stochastic processes of bounded variations, we introduce and study the issue of variational stability of (5.2.3) in analogy to the case of generalized ordinary differential equation of classical type [87].

In addition to other assumptions, assume that the map $(x,t) \longrightarrow F(x,t)(\eta,\xi)$ satisfies

$$F(0,t_2)(\eta,\xi) - F(0,t_1)(\eta,\xi) = 0 \tag{5.2.5}$$

For every $t_1, t_2 \in [0,T]$ and for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$.

This assumption evidently implies that

$$\int_{s_1}^{s_2} DF(0,s)(\eta,\xi) = F(0,s_2)(\eta,\xi) - F(0,s_1)(\eta,\xi) = 0$$

$$= \int_{s_1}^{s_2} P(0,s)(\eta,\xi)ds = 0, \quad s_1, s_2 \in [0,T]$$

and therefore the trivial process given by $X(s) = 0$, for $s \in [0,T]$ is a solution of the Kurzweil equation (5.2.3).

Next we introduce some concepts of stability of the trivial solution $X(s) = 0, s \in [0,T]$ of equation (5.2.3).

5.2.1 Definition: The trivial solution $X \equiv 0$ of equation (5.2.3) is said to be variationally stable if for every $\epsilon > 0$, there exists $\delta(\eta,\xi,\epsilon) := \delta_{\eta\xi} > 0$ such that if $Y : [0,T] \to \tilde{\mathcal{A}}$ is a stochastic process lying in $Ad(\tilde{\mathcal{A}})_{\text{wac}} \cap BV(\tilde{\mathcal{A}})$ with

$$\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$$

and

$$Var\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi)\right) < \delta_{\eta\xi}$$

then we have

$$\|Y(t)\|_{\eta\xi} < \epsilon$$

For all $\quad t \in [0,T] \quad$ and for all $\quad \eta, \xi \in I\!D\underline{\otimes}I\!E$.

5.2.2 Definition: The trivial solution $X \equiv 0$ of equation (5.2.3) is said to be variationally attracting if there exists $\delta_0 > 0$ and for every $\epsilon > 0$, there exists $A = A(\epsilon), 0 \leq A(\epsilon) < T$ and $B(\eta, \xi, \epsilon) = B > 0$ such that if
$Y \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ with $\|Y(0)\|_{\eta\xi} < \delta_0$ and

$$Var\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi)\right) < B$$

then

$$\|Y(t)\|_{\eta\xi} < \epsilon, \quad \text{for all} \quad t \in [A, T]$$

5.2.3 Definition: The trivial solution $X \equiv 0$ of equation (5.2.3) is called variationally asymptotically stable if it is variationally stable and variationally attracting.
Together with (5.2.1) we consider the perturbed QSDE

$$\begin{aligned} dX(t) &= E(X(t), t)d\wedge_\pi(t) + F(X(t), t)dA_g^+(t) \\ &\quad +G(X(t), t)dA_f(t) + (H(X(t), t) + p(t))dt \\ X(t) &= X_0 \end{aligned} \qquad (5.2.6)$$

where $p \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ The perturbed equivalent form of (5.2.6) is given by

$$\begin{aligned} \frac{d}{dt}\langle n, X(t)\xi\rangle &= P(X(t), t)(\eta, \xi) + \langle \eta, p(t)\xi\rangle \\ X(0) &= X_0 \end{aligned} \qquad (5.2.7)$$

The Kurzweil equation associated with the perturbed QSDE (5.2.2) then becomes

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = D[F(X(\tau), t)(\eta, \xi) + Q(t)(\eta, \xi)] \qquad (5.2.8)$$

where $Q : [0, T] \to \tilde{\mathcal{A}}$ belongs to $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ as well.
We remark here that the map given by equation (5.2.7) is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ where

$$F(X, t)(\eta, \xi) = \int_{t_0}^t P(X, s)(\eta, \xi)ds$$

and

$$\langle \eta, p(t)\xi \rangle := Q(t)(\eta, \xi)$$

It follows that

$$|F(x,t_2)(\eta,\xi) + Q(t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - Q(t_1)(\eta,\xi)|$$

$$\leq |h_{\eta\xi}(t_2) + Var_{[0,t_2]}Q(t) - h_{\eta\xi}(t_1) - Var_{[0,t_1]}Q(t)|$$

$$\text{For} \quad x \in \tilde{\mathcal{A}}, \quad \text{and} \quad t_1, t_2 \in [0,T]$$

and

$$\begin{aligned}
&|F(x,t_2)(\eta,\xi) + Q(t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - Q(t_1)(\eta,\xi) \\
&-(F(y,t_2)(\eta,\xi) + Q(t_2)(\eta,\xi) - F(y,t_1)(\eta,\xi) - Q(t_1)(\eta,\xi))| \\
&\leq W(\|x-y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \\
&\leq W(\|x-y\|_{\eta\xi})|h_{\eta\xi}(t_2) + Var_{[0,t_2]}Q - h_{\eta\xi}(t_1) - Var_{[0,t_1]}Q|
\end{aligned}$$

and therefore the right hand side $F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)$ of equation (5.2.8) is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [0,T], \tilde{h}_{\eta\xi}, W)$ where

$$\tilde{h}_{\eta\xi}(t) = h_{\eta\xi}(t) + Var_{[0,t]}Q(t)(\eta,\xi),$$

and all fundamental results (e.g. the existence of solution) hold for equation (5.2.8) and hence (5.2.7).

5.2.4 Definition: The trivial solution $X \equiv 0$ of equation (5.2.3) is said to be variationally stable with respect to perturbations if for every $\epsilon > 0$ there exists $\delta = \delta_{\eta\xi} > 0$ such that if $\|Y_0\|_{\eta\xi} < \delta_{\eta\xi}$, $Y_0 \in \tilde{\mathcal{A}}$ and the stochastic process Q belongs to the set $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ such that

$$Var(Q(t)(\eta,\xi)) < \delta_{\eta\xi}, \text{ then } \|Y(t)\|_{\eta\xi} < \epsilon$$

for $t \in [0, T]$ where $Y(t)$ is a solution of (5.2.8) with $Y(0) = Y_0$.

5.2.5 Definition: The solution $X \equiv 0$ of (5.2.3) is called attracting with respect to perturbations if there exists $\delta_0 > 0$ and for every $\epsilon > 0$, there is a $A = A(\epsilon) \geq 0$ and $B(\eta, \xi, \epsilon) = B > 0$ such that if

$$\|Y_0\|_{\eta\xi} < \delta_0, \quad Y_0 \in \tilde{\mathcal{A}}$$

and $Q \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ satisfying $Var(Q(t)(\eta, \xi)) < B$,
then

$$\|Y(t)\|_{\eta\xi} < \epsilon,$$

for all $t \in [A, T]$, where $Y(t)$ is a solution of (5.2.8).

5.2.6 Definition: The trivial solution $X \equiv 0$ of equation (5.2.3) is called asymptotically stable with respect to perturbations if it is stable and attracting with respect to perturbations.

5.2.7 Notation: Denote by $BV(\tilde{\mathcal{A}}) \bigcap Ad(\tilde{\mathcal{A}})_{wac}$ the set of all adapted stochastic processes $\varphi : [0, T] \to \tilde{\mathcal{A}}$ that are weakly absolutely continuous and of bounded variation on $[t_0, T]$.

Remark: In analogy to the case of generalized ordinary differential equation [87], the concept of variational stability introduced in this section concerning QSDE(5.2.1) comes from the following idea.
If a certain stochastic process $Y : [0, T] \to \tilde{\mathcal{A}}$ is such that the initial value $Y(0)$ lies in some small neighbourhood of the trivial process $X \equiv 0$ in the locally convex space $\tilde{\mathcal{A}}$ and the variation of the complex valued function

$$\langle \eta, Y(s)\xi \rangle = \langle \eta, Y(0)\xi \rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi)$$

on $[0, T]$ is small enough for all $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$, then $Y(t)$ also lies in some small neighbourhood of the trivial process for all $t \in [0, T]$.

However, the stability with respect to perturbations is occasioned by the desirability that the solutions of the perturbed associated Kurzweil equation (5.2.8) be close to zero on the given interval $[0, T]$ whenever the initial value $Y(0)$ is close to zero and the variation of the perturbation term $Q(t)(\eta, \xi)$ of equation (5.2.8) is small enough. The next results shows the equivalence of these concepts of stability given by the above definitions.

5.2.1 Theorem:

(a) The trivial solution $X \equiv 0$ of the Kurzweil equation (5.2.3) associated with the equivalent form (5.2.2) of QSDE (5.2.1) is variationally stable if and only if it is stable with respect to perturbation.

(b) The trivial solution $X \equiv 0$ of (5.2.3) is variationally attracting if and only if it is attracting with respect to perturbations.

Proof (a)(i) Assume that the trivial solution of (5.2.3) is variationally stable.

For a given $\epsilon > 0$, let $\delta_{\eta\xi} = \delta_{\eta\xi}(\epsilon) > 0$ be given by Definition 5.2.1, assume that $Y_0 \in \tilde{\mathcal{A}}$ such that $\|Y_0\|_{\eta\xi} < \delta_{\eta\xi}$ and $Var(Q(t)(\eta, \xi)) < \delta_{\eta\xi}\ \forall\ \ \eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$ and $Y(t), t \in [0, T]$ is a solution of (5.2.8) satisfying $Y(0) = Y_0$. Since Y is a solution of (5.2.8) and hence of (5.2.7), then $Y \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ and satisfies for any $s_1, s_2 \in [0, T]$

$$\langle \eta, Y(s_2)\xi \rangle - \langle \eta, Y(s_1)\xi \rangle = \int_{s_1}^{s_2} DF(Y(\tau), t)(\eta, \xi) + Q(s_2)(\eta, \xi) - Q(s_1)(\eta, \xi)$$

Hence, by the additivity of the Kurzweil integrals

$$\langle \eta, Y(s_2)\xi \rangle - \int_0^{s_2} DF(Y(\tau), t)(\eta, \xi) - \langle \eta, Y(s_1)\xi \rangle$$

$$+ \int_0^{s_1} DF(Y(\tau), t)(\eta, \xi) = Q(s_2)(\eta, \xi) - Q(s_1)(\eta, \xi)$$

for any $s_1, s_2 \in [0,T]$.

Consequently,

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi)\right) = Var_{[0,T]}(Q(\tau)(\eta,\xi)) < \delta_{\eta\xi}$$

By the assumption of variational stability of the trivial solution, we have

$$\|Y(t)\|_{\eta\xi} < \epsilon, \quad t \in [0,T].$$

This implies that the trivial solution $x \equiv 0$ of (5.2.3) is stable with respect to perturbations.

(ii) Assume that the trivial solution of (5.2.3) is stable with respect to perturbations. For $\epsilon > 0$, let $\delta > 0$ be given by definition (5.2.4). Suppose that the process $Y : [0,T] \to \tilde{\mathcal{A}}$ lying in the set $Ad(e\mathcal{A})_{wac} \cap BV(\tilde{\mathcal{A}})$, is a solution of (5.2.8) such that $\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$ and

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi)\right) < \delta_{\eta\xi}$$

for $s_1, s_2 \in [0,T]$, we have

$$\begin{aligned}\langle \eta, Y(s_2)\xi\rangle - \langle \eta, Y(s_1)\xi\rangle &= \int_{s_1}^{s_2} DF(Y(\tau),t)(\eta,\xi) + \langle \eta, Y(s_2)\xi\rangle - \\ &\quad - \int_0^{s_2} DF(Y(\tau),t)(\eta,\xi) - \langle \eta, Y(s_1)\xi\rangle + \\ &\quad + \int_0^{s_1} DF(Y(\tau),t)(\eta,\xi) \\ &= \int_{s_1}^{s_2} DF(Y(\tau),t)(\eta,\xi) + Q(s_2)(\eta,\xi) - Q(s_1)(\eta,\xi) \qquad (5.2.9)\end{aligned}$$

where

$$Q(s)(\eta,\xi) = \langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi), \quad \text{for } s \in [0,T]$$

Since $Q \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ and (5.2.9) shows that the stochastic process Y is a solution of equation(5.2.8) on $[0,T]$ with this Q and $\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$. Moreover

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi)\right) < \delta_{\eta\xi}$$

Hence by the assumption of stability with respect to perturbations we get $\|Y(t)\|_{\eta\xi} < \epsilon$ for $t \in [0, T]$ and $X \equiv 0$ is variationally stable.

(b)(i) Assume that the trivial solution of (5.2.3) is variationally attracting. Then there exists a $\delta_0 > 0$ and for a given $\epsilon > 0$ also $A > 0$ and $B > 0$, by the Definition 5.2.2. If now $Y_0 \in \tilde{\mathcal{A}}$ is such that $\|Y(0)\|_{\eta\xi} < \delta_0$, Q belong to the set
$Y \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ where $VarQ_{[0,T]} < \delta_{\eta\xi}$ and $y(t)$ is a solution of (5.2.3) then

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi)\right) = Var(Q(t)(\eta, \xi)) < B$$

Hence by Definition 5.2.2 we have $\|Y(t)\|_{\eta\xi} < \epsilon \ \forall \ t \in [A, T]$ and $X \equiv 0$ is variationally attracting with respect to perturbations.

(ii) If $X \equiv 0$ is attracting with respect to perturbations, for $\epsilon > 0$, let $\delta_0 > 0$, $A = A(\epsilon) \geq 0$, $B = B(\epsilon) > 0$ be given by Definition 5.2.5 such that $\|Y(0)\|_{\eta\xi} < \delta_0$. Assume that $Y : [0, T] \to \tilde{\mathcal{A}}$ lies in the space
$Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$, such that $\|Y(0)\|_{\eta\xi} < \delta_0$, $Y(0) \in \tilde{\mathcal{A}}$ and

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi)\right) < B$$

$s \in [0, T]$, then for $s_1, s_2 \in [0, T]$, we have

$$\begin{aligned}
&\langle \eta. Y(s_2)\xi\rangle - \langle \eta, Y(s_1)\xi\rangle = \int_{s_1}^{s_2} DF(Y(\tau), t)(\eta, \xi) \\
&+\langle \eta, Y(s_2)\xi\rangle - \int_0^{s_2} DF(Y(\tau), t)(\eta, \xi) \\
&-\langle \eta, Y(s_1)\xi\rangle + \int_0^{s_1} DF(Y(\tau), t)(\eta, \xi) \\
&= \int_{s_1}^{s_2} DF(Y(\tau), t)(\eta, \xi) + Q(s_2)(\eta, \xi) - Q(s_1)(\eta, \xi)
\end{aligned}$$

Hence, we can set

$$Q(s)(\eta, \xi) = \langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi) \qquad (5.2.10)$$

for $s \in [0, T]$

So that from Definition 5.2.5, since Y is a solution of (5.2.8) on $[0, T]$ with this Q and

$\|Y(0)\|_{\eta\xi} < \delta_0$ such that

$Var(Q(t)(\eta, \xi)) < B$, then from (5.2.10) we get

$$Var(Q(s)(\eta, \xi)) < B \Rightarrow Var\left(\langle \eta, Y(s)\xi \rangle - \int_0^s DF(Y(\tau), t)(\eta, \xi)\right) < B$$

and by the assumption of attracting with respect to perturbation we get

$$\|Y(t)\|_{\eta\xi} < \epsilon \ \ \forall\, t \in [A, T]$$

and $X \equiv 0$ is variationally attracting.

The following result is a consequence of Theorem 5.2.1, Definition 5.2.3 and 5.2.6.

5.2.2 Theorem: The trivial solution $X \equiv 0$ of equation (5.2.3) is variationally asymptotically stable if and only if it is asymptotically stable with respect to perturbations.

5.3 Variational Stability and Asymptotic Variational Stability using the Lyapunov's Method

The following auxiliary results will be used to establish the main results in this section.

5.3.1 Proposition: Assume that $[a,b] \subset [0,T]$ and that there exists family of functions $f_{\eta\xi}, g_{\eta\xi} : [a,b] \to \mathbb{R}$ defined and continuous on $[a,b]$. If for every $\sigma \in [a,b]$ there exists $\partial(\sigma) > 0$ such that for every $\beta \in (0, \partial(\sigma))$ the inequality

$$f_{\eta\xi}(\sigma+\beta) - f_{\eta\xi}(\sigma) \leq g_{\eta\xi}(\sigma+\beta) - g_{\eta\xi}(\sigma)$$

holds, then

$$f_{\eta\xi}(s) - f_{\eta\xi}(a) \leq g_{\eta\xi}(s) - g_{\eta\xi}(a)$$

for all $s \in [a,b]$.

Proof. Let us denote

$$M_{\eta\xi} = \{s \in [a,b]; f_{\eta\xi}(\sigma) - f_{\eta\xi}(a) \leq g_{\eta\xi}(\sigma) - g_{\eta\xi}(a),\ \sigma \in [a,s] \subset [0,T]\}$$

and set $\zeta = \sup M_{\eta\xi}$. Since

$$f_{\eta\xi}(a+\beta) - f_{\eta\xi}(a) \leq g_{\eta\xi}(a+\beta) - g_{\eta\xi}(a)$$

for $\beta \in (0, \partial(a))$ and $\partial(a) > 0$, the set $M_{\eta\xi}$ is non-empty, $\zeta > a$ and

$$f_{\eta\xi}(s) - f_{\eta\xi}(a) \leq g_{\eta\xi}(s) - g_{\eta\xi}(a) \text{ for every } s < \zeta.$$

Using the continuity of $f_{\eta\xi}$ and $g_{\eta\xi}$ we have also that

$$f_{\eta\xi}(\zeta) - f_{\eta\xi}(a) \leq g_{\eta\xi}(\zeta) - g_{\eta\xi}(a).$$

If $\zeta < b$ then by assumption we have

$$f_{\eta\xi}(\zeta+\beta) - f_{\eta\xi}(\zeta) \leq g_{\eta\xi}(\zeta+\beta) - g_{\eta\xi}(\zeta).$$

for every $\beta \in (0, \partial(\zeta))$, $\partial(\zeta) > 0$ and therefore also

$$f_{\eta\xi}(\zeta + \beta) - f_{\eta\xi}(a) = f_{\eta\xi}(\zeta + \beta) - f_{\eta\xi}(\zeta) + f_{\eta\xi}(\zeta) + f_{\eta\xi}(a)$$

$$\leq g_{\eta\xi}(\zeta + \beta) - g_{\eta\xi}(\zeta) + g_{\eta\xi}(\zeta) - g_{\eta\xi}(a) = g_{\eta\xi}(\zeta + \beta) - g_{\eta\xi}(a)$$

This implies that $\zeta + \beta \in M_{\eta\xi}$ for $\beta \in (0, \partial(\zeta))$, i.e. $\zeta < \sup M_{\eta\xi}$ and this contradiction yields $\zeta = b$ and $M_{\eta\xi} = [a, b]$ and the proof is complete.

5.3.2 Lemma Since $\mathbb{C} \cong \mathbb{R}^2$ we assume the following:

(i) the map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ is real-valued such that for every $x \in \tilde{\mathcal{A}}$, the real-valued map $t \longrightarrow V(x,t)(\eta,\xi)$ is continuous on $[0,T]$.

$$\text{(ii)} \quad |V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq K\|x - y\|_{\eta\xi} \qquad (5.3.1)$$

for every $x, y \in \tilde{\mathcal{A}}, t \in [0,T]$ with a constant $K_{\eta\xi} := K > 0$.

(iii) there is a real valued map $\Phi_{\eta\xi} : \tilde{\mathcal{A}} \to \mathbb{R}$ such that for every solution $x : [0,T] \to \tilde{\mathcal{A}}$ of equation (5.2.3), we have

$$\limsup_{\beta \to 0} \frac{V(x(t+\beta), t+\beta)(\eta,\xi) - V(x(t),t)(\eta,\xi)}{\beta} \leq \Phi_{\eta\xi}(x(t)) \qquad (5.3.2)$$

for $t \in [0,T]$

(iv) If $Y : [0,t_1] \to \tilde{\mathcal{A}}$, $[0,t_1] \subset [0,T]$ belongs to $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$,

then the inequality

$$\begin{aligned} V(X(t_1),t_1)(\eta,\xi) \quad \leq \quad & V(X(0),0)(\eta,\xi) + KVar_{[0,t_1]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)(\eta,\xi)\right) \\ & + M_{\eta\xi}(t_1 - 0) \qquad (5.3.3) \end{aligned}$$

holds, where $M_{\eta\xi} = \sup_{t\in[0,t_1]} \Phi_{\eta\xi}(Y(t))$.

Proof. Let $y : [0,t_1] \to \tilde{\mathcal{A}}$ belong to $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ be given and let $\sigma \in [0,t_1] \subset [0,T]$ be an arbitrary point. Hence the real-valued function $V(y(t),t)(\eta,\xi)$ is continuous on $[0,t_1]$.

Assume that $x : [\sigma, \sigma+\beta_1(\sigma)] \subset [0,T] \to \tilde{\mathcal{A}}$ is a solution of (5.2.3) on the interval $[\sigma,\sigma+\beta_1(\sigma)]$, $\beta_1(\sigma) > 0$ with the initial condition $x(\sigma) = y(\sigma)$. The existence of such a solution is guaranteed by the existence results established in chapter 4. By the assumption (5.3.1) we then have

$$\begin{aligned} V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) &- V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi) \\ &\leq K\|y(\sigma+\beta)-x(\sigma+\beta)\|_{\eta\xi} \\ &= K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(x(\tau),t)(\eta,\xi)\right| \qquad (**) \end{aligned}$$

for every $\beta \in [0,\beta_1(\sigma)]$.

Remark the last inequality is obtained from the following

$$\begin{aligned} &K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,y(\sigma)\xi\rangle - \int_0^{\sigma+\beta} DF(x(\tau),t)(\eta,\xi)\right| \\ &= K\|y(\sigma+\beta)-y(\sigma)-x(\sigma+\beta)+x(\sigma)\|_{\eta,\xi}, \quad \text{where } x(\sigma)=y(\sigma) \end{aligned}$$

and

$$\int_\sigma^{\sigma+\beta} DF(x(\tau),t)(\eta,\xi) = x(\sigma+\beta)-x(\sigma)$$

By this inequality $(**)$ and by (5.3.2) we obtain

$$\begin{aligned} &V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi) \\ &= V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi) \\ &+V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi) \leq \\ &\leq K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(x(\tau),t)(\eta,\xi)\right| + \beta_{\eta\xi}\Phi(y(\beta)) \\ &\leq K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(x(\tau),t)(\eta,\xi)\right| + \beta M_{\eta\xi} + \beta\epsilon \end{aligned}$$

where $\epsilon > 0$ is arbitrary and $\beta \in (0, \beta_2(\sigma))$ with $\beta_2(\sigma) \leq \beta_1(\sigma)$,

$\beta_2(\sigma) > 0$ is sufficiently small.

Setting

$$\langle \eta, Q(s)\xi \rangle = \langle \eta, p(s)\xi \rangle = \langle \eta, y(s)\xi \rangle - \int_0^s DF(y(\tau), t)(\eta, \xi)$$

for $s \in [0, t_1]$.

As $(\eta, \xi) \longrightarrow Q(s)(\eta, \xi)$ is a sesquilinear form, there exists $Q : [0, t_1] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \bigcap BV(\tilde{\mathcal{A}})$, such that $Q(s)(\eta, \xi) = \langle \eta, Q(s)\xi \rangle$.

The last inequality can be used to derive the following estimates

$$\begin{aligned}
&V(y(\sigma + \beta), \sigma + \beta)(\eta, \xi) - V(x(\sigma), \sigma)(\eta, \xi) \\
&\leq K \left| \langle \eta, y(\sigma + \beta)\xi \rangle - \langle \eta, y(\sigma)\xi \rangle - \int_\sigma^{\sigma+\beta} DF(y(\tau), t)(\eta, \xi) \right| \\
&+K \left| \int_\sigma^{\sigma+\beta} D[F(y(\tau), t)(\eta, \xi) - F(x(\tau), t)(\eta, \xi)] \right| + \beta M_{\eta\xi} + \beta\epsilon \\
&\leq K |Q(\sigma + \beta)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \beta M_{\eta\xi} + \beta\epsilon \\
&+K \left| \int_\sigma^{\sigma+\beta} D[F(y(\tau), t)(\eta, \xi) - F(x(\tau), t)(\eta, \xi)] \right| \\
&\leq K \left(Var_{[0,\sigma+\beta]} Q(t)(\eta, \xi) - Var_{[0,\sigma]} Q(t)(\eta, \xi) \right) + \beta M_{\eta\xi} \\
&+\epsilon\beta + K \left| \int_\sigma^{\sigma+\beta} D[F(y(\tau), t)(\eta, \xi) - F(x(\tau), t)(\eta, \xi)] \right| \qquad (5.3.4)
\end{aligned}$$

for every $\beta \in (0, \beta_2(\sigma))$.

Considering the last term in (5.3.4), since the map $(x, t) \longrightarrow F(x, t)(\eta, \xi)$ is of class

$\mathbb{F}(\tilde{\mathcal{A}}\times[0,T],\ h_{\eta\xi},W)$ we obtain by Theorem 1.9.4 and Theorem 1.9.5 (iii) the estimate

$$\begin{aligned}
&\left|\int_{\sigma}^{\sigma+\beta} D[F(y(\tau),t)(\eta,\xi)-F(x(\tau),t)(\eta,\xi)]\right| \\
&\leq \int_{\sigma}^{\sigma+\beta} W(\|y(\tau)-x(\tau)\|_{\eta\xi})dh_{\eta\xi}(\tau) \\
&= \lim_{\alpha\to 0}\left[\int_{\sigma}^{\sigma+\alpha} W(\|y(\tau)-x(\tau)\|_{\eta\xi})dh_{\eta\xi}(\tau)+\int_{\sigma+\alpha}^{\sigma+\beta} W(\|y(\tau)-x(\tau)\|_{\eta\xi})dh_{\eta\xi}(\tau)\right] \\
&= W(\|y(\sigma)-x(\sigma)\|_{\eta\xi})(h_{\eta\xi}(\sigma)-h_{\eta\xi}(\sigma))+\lim_{\alpha\to 0}\int_{\sigma+\alpha}^{\sigma+\beta} W(\|y(\tau)-x(\tau)\|_{\eta\xi})dh_{\eta\xi}(\tau) \\
&= \lim_{\alpha\to 0}\int_{\sigma+\alpha}^{\sigma+\beta} W(\|y(\tau)-x(\tau)\|_{\eta\xi})dh_{\eta\xi}(\tau) \\
&\leq \sup_{s\in[\sigma,\sigma+\beta]} W(\|y(s)-x(s)\|_{\eta\xi})\lim_{\alpha\to 0}(h_{\eta\xi}(\sigma+\beta)-h_{\eta\xi}(\sigma+\alpha)) \\
&= \sup_{s\in[\sigma,\sigma+\beta]} W(\|y(s)-x(s)\|_{\eta\xi})(h_{\eta\xi}(\sigma+\beta)-h_{\eta\xi}(\sigma)), \qquad (5.3.5)
\end{aligned}$$

because $y(\sigma)=x(\sigma)$ and $W(\|y(\sigma)-x(\sigma)\|_{\eta\xi})=0$.

For $s\in[\sigma,\sigma+\beta_2(\sigma)]$ we have

$$\langle\eta,y(s)\xi\rangle-\langle\eta,x(s)\xi\rangle=\langle\eta,y(s)\xi\rangle-\langle\eta,y(\sigma)\xi\rangle-\int_{\sigma}^{s} DF(x(\tau),t)(\eta,\xi)$$

and therefore

$$\begin{aligned}
&\lim_{s\to\sigma_1}(\langle\eta,y(s)\xi\rangle-\langle\eta,x(s)\xi\rangle)=\langle\eta,y(\sigma_1)\xi\rangle-\langle\eta,y(\sigma)\xi\rangle- \\
&\lim_{s\to\sigma_1}(F(x(\sigma),s)(\eta,\xi)-F(x(\sigma),\sigma)(\eta,\xi)) \\
&=\langle\eta,y(\sigma_1)\xi\rangle-\langle\eta,y(\sigma)\xi\rangle-(F(x(\sigma),\sigma_1)(\eta,\xi)-F(x(\sigma),\sigma)(\eta,\xi)) \\
&=\langle\eta,Q(\sigma_1)\xi\rangle-\langle\eta,Q(\sigma)\xi\rangle,\quad \sigma_1>\sigma
\end{aligned}$$

and also

$$\lim_{s\to\sigma_1}\|y(s)-x(s)\|_{\eta\xi}=|Q(\sigma_1)(\eta,\xi)-Q(\sigma)(\eta,\xi)| \qquad (5.2.6)$$

For every $\epsilon>0$ we define

$$\alpha=\frac{\epsilon}{K(h_{\eta\xi}(t_1)-h_{\eta\xi}(0)+1)}>0 \qquad (5.3.7)$$

and assume that $r=r(\alpha)>0$ is such that $W(r)<\alpha$. Further, we choose

$$\gamma\in[0,\frac{r}{2}]\subset[0,T].$$

Since (5.3.6) holds, there is an $\beta_3(\sigma) \in (0, \beta_2(\sigma))$ such that

$$\|y(s) - x(s)\|_{\eta\xi} \leq |Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \gamma \tag{5.3.8}$$

for $s \in (\sigma, \sigma + \beta_3(\sigma))$ and also

$$W(\|y(s) - x(s)\|_{\eta\xi}) \leq W(|Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \gamma) \tag{5.3.9}$$

for $s \in (\sigma, \sigma + \beta_3(\sigma))$.

Setting:

$$N(\alpha) := N(\alpha, \eta, \xi) = \left\{\sigma_1 \in [0, T]; \;\; |Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| \geq \frac{r}{2}\right\}$$

since Q lies in $BV(\tilde{\mathcal{A}})$, the set $N(\alpha)$ is finite and we denote by $l(\alpha)$ the number of elements $N(\alpha)$.

If $\sigma \in [0, T] \backslash N(\alpha)$ and $s \in (\sigma, \sigma + \beta_3(\sigma))$ then by (5.3.9) we have

$$\begin{aligned} W(\|y(s) - x(s)\|_{\eta\xi}) &\leq W\left(\frac{r}{2} + \gamma\right) < W\left(\frac{r}{2} + \frac{r}{2}\right) \\ &= W(r) < \alpha \end{aligned}$$

and by (5.3.5) also

$$\left| \int_{\sigma}^{\sigma+\beta} D[F(y(\tau), t)(\eta, \xi) - F(x(\tau), t)(\eta, \xi)] \right| \leq \alpha(h_{\eta\xi}(\sigma + \beta) - h_{\eta\xi}(\sigma)) \tag{5.3.10}$$

whenever $\beta \in (0, \beta_3(\sigma))$.

If $\sigma \in [0, T] \cap N(\alpha)$ then there exists $\beta_4(\sigma) \in (0, \beta_3(\sigma))$ such that for $\beta \in (0, \beta_4(\sigma))$ we set

$$\begin{aligned} & h_{\eta\xi}(\beta + \sigma) - h_{\eta\xi}(\sigma_1) = |h_{\eta\xi}(\sigma + \beta) - h_{\eta\xi}(\sigma_1)| \\ & < \frac{\alpha}{(l(\alpha) + 1) W(|Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \gamma)} \end{aligned}$$

$\sigma_1 \in [0, T]$, $\sigma_1 > \sigma > 0$.

Hence (5.3.5) and (5.3.9) yield

$$\begin{aligned} & \left| \int_{\sigma}^{\sigma+\beta} D[F(y(\tau), t)(\eta, \xi) - F(x(\tau), t)(\eta, \xi)] \right| \\ & \leq W(|Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \gamma) \frac{\alpha}{(l(\alpha) + 1) W(|Q(\sigma_1)(\eta, \xi) - Q(\sigma)(\eta, \xi)| + \gamma)} \\ & = \frac{\alpha}{l(\alpha) + 1} \end{aligned} \tag{5.3.11}$$

for every $\beta \in [\sigma, \sigma + \beta_4(\sigma)]$.

Since the function $h_{\eta\xi,\alpha} : [0,T] \to \mathbb{R}$ is nondecreasing and continuous on $[0,T]$, we set

$$Var_{[t_1,t_2]} h_{\eta\xi,\alpha}(t) = h_{\eta\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1) = \frac{\alpha}{l(\alpha)+1} l(\alpha) < \alpha \tag{5.3.12}$$

for every $t_1, t_2 \in [0,T]$ and from (5.3.7) and (5.3.12) we have

$$h_{\eta\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1) < \alpha[h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1) + 1] = \frac{\epsilon}{K} \tag{5.3.13}$$

and by (5.3.10), (5.3.11) and by the definition of $h_{\eta\xi,\alpha}$ we obtain the inequality

$$\left| \int_{\sigma}^{\sigma+\beta} DF(y(\tau),t)(\eta,\xi) - DF(x(\tau),t)(\eta,\xi) \right| \leq$$

$$\leq |h_{\eta\xi,\alpha}(\sigma+\beta) - h_{\eta\xi,\alpha}(\sigma)|$$

for $\beta \in [0, \partial(\sigma)]$ and (5.3.4) gives

$$V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi) \leq K\left(Var_{[0,\sigma+\beta]}Q(\sigma+\beta)(\eta,\xi) - Var_{[0,\sigma]}Q(\sigma)(\eta,\xi)\right)$$

$$+\beta M_{\eta\xi} + \beta\epsilon + K(h_{\eta\xi,\alpha}(\sigma+\beta) - h_{\eta\xi,\alpha}(\sigma)) = g_{\eta\xi}(\sigma+\beta) - g_{\eta\xi}(\sigma) \tag{5.3.14}$$

for all $\sigma \in [0,T]$ and $\beta \in [0, \partial(\sigma)]$

where

$$g_{\eta\xi}(t) = KVar_{[0,T]}Q(t)(\eta,\xi) + M_{\eta\xi}(t) + \epsilon(t) + Kh_{\eta\xi,\alpha}(t), \; t \in [0,T]$$

The function $g_{\eta\xi}$ is of bounded variation on $[0,T]$ and continuous on $[0,T]$.

From Proposition 5.2.3 and (5.3.13) we obtain by (5.3.14) the inequality

$$V(y(t_2),t_2)(\eta,\xi) - V(y(t_1),t_1)(\eta,\xi) \leq g_{\eta\xi}(t_2) - g_{\eta\xi}(t_1)$$

$$= K\, Var_{[t_1,t_2]}Q(t)(\eta,\xi) + M_{\eta\xi}(t_2 - t_1) + \epsilon(t_2 - t_1) + K(h_{\eta\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1))$$

$$< K\, Var_{[t_1,t_2]}Q(t)(\eta,\xi) + M_{\eta\xi}(t_2 - t_1) + \epsilon(t_2 - t_1) + \epsilon$$

for $t_1, t_2 \in [0.T]$, since $\epsilon > 0$ can be arbitrary, we obtain from this inequality the result in (5.3.3) and the proof is completed.

The following definition will be used in the next theorem.

5.3.1 Definition: The real valued map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ is said to be positive definite if

(i) There exists a continuous nondecreasing function $b : [0,\infty) \longrightarrow \mathbb{R}$ such that $b(0) = 0$ and

(ii) $V(x,t)(\eta,\xi) \geq b(\|x\|_{\eta\xi})$ for all $(x,t) \in \tilde{\mathcal{A}} \times [0,T]$

(iii) $V(0,t)(\eta,\xi) = 0$, for all $(x,t) \in \tilde{\mathcal{A}} \times [0,T]$

The next theorems are the Lyapunov type theorems on the variational stability of solution of equation (5.2.3). As usual, $\eta,\xi \in I\!\!D \underline{\otimes} I\!\!E$ are arbitrary.

5.3.3 Theorem Suppose that the following conditions hold:

(i) the real valued map $t \longrightarrow V(x,t)(\eta,\xi)$ is continuous on $[0,T]$ for every $x \in \tilde{\mathcal{A}}$.

(ii) the map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ is positive definite in the sense of definition (5.3.1) above.

(iii) $V(0,t)(\eta,\xi) = 0$ and $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq K\|x-y\|_{\eta\xi}\ \forall\ x,y \in \tilde{\mathcal{A}}$, $K_{\eta\xi} := K > 0$ being a constant.

(iv) the map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ is non-increasing along every solution $x(t)$ of equation (5.2.3)

then, the trivial solution $X \equiv 0$ of (5.2.3) is variationally stable.

Proof: Since we assumed that the map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ is non-increasing whenever $x : [0,T] \rightarrow \tilde{\mathcal{A}}$, is a solution of (5.2.3) we have from equation (5.3.2) in Lemma 5.3.2

$$\limsup_{\beta\to 0}\frac{V(x(t+\beta),t+\beta)(\eta,\xi)-V(x(t),t)(\eta,\xi)}{\beta}\leq 0 \tag{5.3.15}$$

for $t\in[0,T]$.

To establish the theorem, we shall show that the conditions of variational stability according to definition (5.2.1) are fulfilled under these assumptions.

by lemma 5.3.2 the map $(x,t)\longrightarrow V(x,t)(\eta,\xi)$, satisfies the following.

(i) Let $\epsilon>0$ and let $y:[0,t_1]\to\tilde{\mathcal{A}}$ lie in $Ad(\tilde{\mathcal{A}})_{wac}\cap BV(\tilde{\mathcal{A}})$ be given.

Then we have

$$\limsup_{\beta\to 0}\frac{V(x(t+\beta),t+\beta)(\eta,\xi)-V(x(t),t)(\eta,\xi)}{\beta}\leq 0$$

for every $t\in[0,T]$.

by replacing $\Phi_{\eta\xi}x(t)$ in (5.3.2) with $\Phi_{\eta\xi}x(t)\equiv 0$.

(ii) Again since the map $(x,t)\longrightarrow V(x,t)(\eta,\xi)$, is continuous, we obtain the relation

$$|V(x,t)(\eta,\xi)-V(y,t)(\eta,\xi)|\leq K_{\eta\xi}\|x-y\|_{\eta\xi}$$

for every $x,y\in\tilde{\mathcal{A}},t\in[0,T]$ with a constant $K>0$.

Hence we obtain by (5.3.3) in Lemma 5.3.2, (iii) in definition (5.3.1) and hypothesis (iii) the inequality

$$\begin{aligned}V(y(r),r)(\eta,\xi)\leq\quad & V(y(0),0)(\eta,\xi)+K\,Var_{[0,r]}\left(\langle\eta,y(s)\xi\rangle-\int_0^s DF(y(\tau),t)(\eta,\xi)\right)\\ &\leq K\|y(0)\|_{\eta\xi}+K\,Var_{[0,r]}\left(\langle\eta,y(s)\xi\rangle-\int_0^s DF(y(\tau),t)(\eta,\xi)\right)\end{aligned} \tag{5.3.16}$$

which holds for every $r\in[0,t_1]\subset[0,T]$, $s\in[0,T]$.

Setting $\alpha(\epsilon)=\inf_{r\leq\epsilon}b(r)$. Then $\alpha(\epsilon)>0$ for $\epsilon>0$ and $\lim_{\epsilon\to 0}\alpha(\epsilon)=0$.

Further, choose $\delta_{\eta\xi}>0$ such that $2K\delta_{\eta\xi}<\alpha(\epsilon)$.

If in this situation the function y is such that

$$\|y(0)\|_{\eta\xi}<\delta_{\eta\xi}$$

and

$$Var_{[0,t_1]}\left(\langle \eta, y(s)\xi\rangle - \int_0^s DF(y(\tau),t)(\eta,\xi)\right) < \delta_{\eta\xi}$$

then by (5.3.16) we obtain the inequality

$$V(y(r),r)(\eta,\xi) \leq 2K\delta_{\eta\xi} \tag{5.3.17}$$

provided $r \in [0,t_1]$.

If there exists a $\hat{t} \in [0,t_1]$ such that $\|y(\hat{t})\|_{\eta\xi} \geq \epsilon$ then by(ii) of definition (5.3.1) we get the inequality

$$V(y(\hat{t}),\hat{t})(\eta,\xi) \geq b(\|y(\hat{t})\|_{\eta\xi}) \geq \inf_{r\leq\epsilon} b(r) = \alpha(\epsilon)$$

which contradicts (5.3.17). Hence $\|y(t)\|_{\eta\xi} < \epsilon$ for all $t \in [0,t_1]$ and by Definition (5.2.1) the solution $X \equiv 0$ of equation (5.2.3) is variationally stable.

5.3.4 Theorem: Suppose that the following conditions hold:

(i) the map $(x,t) \longrightarrow V(x,t)(\eta,\xi)$ satisfy the hypothesis of Theorem 5.3.3.

(ii) $\limsup_{\beta\to 0} \frac{V(x(t+\beta),t+\beta)(\eta,\xi)-V(x(t),t)(\eta,\xi)}{\beta} \leq \Phi_{\eta\xi}(x(t)$ holds for every solution $x \in \tilde{\mathcal{A}}$ of equation (5.2.3)

(iii) $\Phi_{\eta\xi} : \tilde{\mathcal{A}} \to \mathbb{R}$ is continuous with $\Phi_{\eta\xi}(0) = 0$, $\Phi_{\eta\xi}(x) > 0$ for $x \neq 0$.

Then the trivial solution $X \equiv 0$ of (5.2.3) is variationally asymptotically stable.

Proof: From hypothesis (ii) above, the map $V(x,t)(\eta,\xi)$ is non-increasing along every solution $X(t)$ of (5.2.3) and therefore by Theorem 5.3.3 the trivial solution $X \equiv 0$ of (5.2.3) is variationally stable. By Definition (5.2.3) it remains to show that the solution $X \equiv 0$ of equation (5.2.3) is variationally attracting in the sense of Definition (5.2.2). From the variational stability of the trivial solution $X \equiv 0$ of equation (5.2.3) there is a $\delta_0 > 0$ such that if $y : [0,T] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ and such that $\|y(0)\|_{\eta\xi} < \delta_0$,

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(y(\tau),t)(\eta,\xi)\right) < \delta_0,$$

then set $\|y(t)\|_{\eta\xi} < a, \quad a > 0$ for $t \in [0,T]$, i.e. $y : [0,T] \to \tilde{\mathcal{A}}$ is continuous on $[0,T]$. Let $\epsilon > 0$ be arbitrary. From the variational stability of the trivial solution we obtain that there is a $\delta_{\eta\xi}(\epsilon) > 0$ such that for every $y : [0,T] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ on $[0,T]$ and such that

$$\|y(0)\|_{\eta\xi} < \delta_{\eta\xi}(\epsilon) \tag{5.3.18}$$

and

$$Var_{[0,T]}\left(\langle \eta, y(s)\xi\rangle - \int_0^s DF(y(\tau),t)(\eta,\xi)\right) < \delta(\epsilon), \tag{5.3.19}$$

we have

$$\|y(t)\|_{\eta\xi} < \epsilon \tag{5.3.20}$$

for $t \in [0,T]$.

Again set $B(\epsilon) = \min(\delta_{\eta\xi}(0), \delta_{\eta\xi}(\epsilon))$ and

$$A(\epsilon) = -K\frac{\delta_0 + B(\epsilon)}{M_{\eta\xi}} > 0$$

where

$$M_{\eta\xi} = \sup\{-\Phi_{\eta\xi}(x); B(\epsilon) \le \|x\|_{\eta\xi} < \epsilon\} = -\inf\{\Phi_{\eta\xi}(x); B(\epsilon) \le \|x\|_{\eta\xi} < \epsilon\} < 0$$

and assume that

$$y : [0,T] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$$

and such that

$$\|y(t)\|_{\eta\xi} < \delta_0,$$

$$Var_{[0,T]}\left(\langle \eta, y(s)\xi\rangle - \int_0^s DF(y(t),t)(\eta,\xi)\right) < B(\epsilon) \tag{5.3.21}$$

Assume that $0 < A(\epsilon) < T$. We show that there exists a $t^* \in [0, A] \subset [0, T]$ such that $\|y(t^*)\|_{\eta\xi} < B(\epsilon)$. Assume the contrary i.e., $\|y(s)\|_{\eta\xi} \geq B(\epsilon)$ for every $s \in [0, A]$. Lemma 5.3.2 yields

$$\begin{aligned}
&V(y(A), A)(\eta, \xi) - V(y(0), 0)(\eta, \xi) \\
&\leq KVar_{[O,A]}\left(\langle \eta, y(s)\xi\rangle - \int_0^s DF(y(t), t)(\eta, \xi)\right) + M_{\eta\xi}A(\epsilon) < \\
&< KB(\epsilon) + M_{\eta\xi}\frac{-K(\delta_0 + B(\epsilon))}{M_{\eta\xi}} = -K\delta_0.
\end{aligned}$$

Hence,

$$V(y(A), A)(\eta, \xi)) \leq V(y(0), 0)(\eta, \xi) - K\delta_0 \leq K\|y(0)\|_{\eta\xi} - K\delta_0 < K\delta_0 - K\delta_0 = 0$$

and this contradicts the inequality

$$V(y(A), A)(\eta, \xi) \geq b(\|y(A)\|_{\eta\xi}) \geq b(B(\epsilon) > 0.$$

Hence necessarily there is a $t^* \in [0, A]$ such that

$$\|y(t^*)\|_{\eta\xi} < B(\epsilon)$$

and by (5.3.21) we have $\|y(t)\|_{\eta\xi} < \epsilon$ for $t \in [t^*, T] \subset [0, T]$ because (5.3.18) and (5.3.19) hold in view of the choice of $B(\epsilon)$ and (5.3.20) is satisfied for the case $t^* = 0$. Consequently, also $\|y(t)\|_{\eta\xi} < \epsilon$ for $t \in [0, T]$, $T > A$, because $t^* \in [0, A]$ and therefore the trivial solution $x \equiv 0$ is a variationally attracting solution of (5.2.3).

Therefore, by Definition 5.2.3, the trivial solution of (5.2.3) is variationally asymptotically stable and thus the result is established.

5.4 Converse Variational Stability Theorems

This section is devoted to the converse of the stability results, namely Theorems 5.3.3 and 5.3.4. The main goal here is to show that the variational stability and asymptotic variational stability imply the existence of Lyapunov functions with the properties described in Theorems 5.3.3 and 5.3.4. First we establish some auxiliary results. We introduce a modified notion of the variation of a stochastic process to suit the concept of converse variational stability.

5.4.1 Definition Assume that $\Phi : [a, b] \to \tilde{\mathcal{A}}$ is a given stochastic process. For a given decomposition

$$D : a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$$

of the interval $[a, b] \subseteq [0, T]$ and for every $\lambda \geq 0$ define

$$u_\lambda(\Phi, D, \eta, \xi) = \sum_{j=1}^{k} e^{-\lambda(b-\alpha_{j-1})} \|\phi(\alpha_j) - \Phi(\alpha_{j-1})\|_{\eta\xi}$$

and set

$$e_\lambda Var_{[a,b]}\Phi_{\eta\xi} = \sup_D u_\lambda(\Phi, D, \eta, \xi)$$

where the supremum is taken over all decompositions D of the interval $[a, b]$.

5.4.2 Definition The number $e_\lambda Var_{[a,b]}\Phi_{\eta\xi}$ is called the e_λ-variation of the map $t \to \langle \eta, \Phi(t)\xi \rangle$ over the interval $[a, b]$.

5.4.3 Notation Denote by $BV(\tilde{\mathcal{A}}) \bigcap Ad(\tilde{\mathcal{A}})_{wac} := A$ the set of all adapted stochastic processes $\varphi : [0, T] \to \tilde{\mathcal{A}}$ that are weakly absolutely continuous and of bounded variation on $[t_0, T]$.

5.4.1 Lemma: If $-\infty < a < b < +\infty$ and $\Phi : [a,b] \to \tilde{\mathcal{A}}$ is a stochastic process,then for every $\lambda \geq 0$ we have

$$e^{-\lambda(b-a)} Var_{[a,b]}\Phi_{\eta\xi} \leq e_\lambda Var_{[a,b]}\Phi_{\eta\xi} \leq Var_{[a,b]}\Phi_{\eta\xi} \tag{5.4.1}$$

If $a \leq c \leq b$, $\lambda \geq 0$ then the identity

$$e_\lambda Var_{[a,b]}\Phi_{\eta\xi} = e^{-\lambda(b-c)} e_\lambda Var_{[a,c]}\Phi_{\eta\xi} + e_\lambda Var_{[c,b]}\Phi_{\eta\xi} \tag{5.4.2}$$

holds.

Proof. For every $\lambda \geq 0$ and every decomposition D of $[a,b]$ we have

$$e^{-\lambda(b-a)} \leq e^{-\lambda(b-\alpha_{j-1})} \leq e^0 = 1 \;\text{ for }\; j = 1, 2, \ldots, k$$

Therefore

$$\begin{aligned} &e^{-\lambda(b-a)} u_0(\Phi, D, \eta, \xi) \leq u_\lambda(\Phi, D, \eta, \xi) \\ &\leq u_0(\Phi, D, \eta, \xi) = \sum_{j=1}^{k} |\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1})(\eta, \xi)| \end{aligned}$$

and passing to the supremum over all finite decomposition D of $[a,b]$ we obtain the inequality (5.4.1)

$$e^{-\lambda(b-a)} Var_{[a,b]}\Phi_{\eta\xi} \leq e_\lambda Var_{[a,b]}\Phi_{\eta\xi} \leq Var_{[a,b]}\Phi_{\eta\xi}$$

. The second statement can be established by restricting ourselves to the case of decomposition D which contain the point c as a node, i.e.

$$D : a = \alpha_0 < \alpha_l < \cdots < \alpha_{l-1} < \alpha_l = c < \alpha_{l+1} < \cdots < \alpha_k = b$$

then

$$
\begin{aligned}
u_\lambda(\Phi, D)(\eta, \xi) &= \sum_{j=1}^{k} e^{-\lambda(b-\alpha_{j-1})} |\langle \eta, \Phi(\alpha_j)\xi\rangle - \langle \eta, \Phi(\alpha_{j-1})\xi\rangle| \\
&= \sum_{j=1}^{l} e^{-\lambda(b-\alpha_{j-1})} |\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1})(\eta, \xi)| \\
&\quad + \sum_{j=l+1}^{k} e^{\lambda(b-\alpha_{j-1})} |\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1})(\eta, \xi)| \\
&= e^{-\lambda(b-c)} \sum_{j=1}^{l} e^{-\lambda(c-\alpha_{j-1})} |\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1})(\eta, \xi)| \\
&\quad + \sum_{j=l+1}^{k} e^{\lambda(b-\alpha_{j-1})} |\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1})(\eta, \xi)| \\
&= e^{-\lambda(b-c)} u_\lambda(\Phi, D_1, \eta, \xi) + u_\lambda(\Phi, D_2, \eta, \xi) \qquad (5.4.3)
\end{aligned}
$$

where

$$D_1 : a = \alpha_0 < \alpha_1 < \cdots < \alpha_{l-1} < \alpha_l = c$$

and

$$D_2 : c = \alpha_l < \alpha_{l+1} < \cdots < \alpha_k = b$$

are decompositions of $[a, c]$ and $[c, b]$, respectively. On the other hand, any two such decompositions D_1 and D_2 form a decomposition D of the interval $[a, b]$.
The equality

$$e_\lambda Var_{[a,b]} \Phi_{\eta\xi} = e^{-\lambda(b-c)} e_\lambda Var_{[a,c]} \Phi_{\eta\xi} + e_\lambda Var_{[c,b]} \Phi_{\eta\xi}$$

now easily follows from (5.4.3) when we pass the corresponding suprema.

5.4.2 Corollary: Assume that the following hold.
(i)If $a \leq c \leq b$ and $\lambda \geq 0$ then

$$e_\lambda Var_{[a,c]} \Phi_{\eta\xi} \leq e_\lambda Var_{[a,b]} \Phi_{\eta\xi} \qquad (5.4.4)$$

(ii) Let $\varphi(0) = 0, \quad \varphi(t) = x$ and set $\sup_{s\in[a,t]} \|\varphi(s)\|_{\eta\xi} < a$ for $a > 0,\ t > 0,\ \varphi \in A$.

(iii) For $\lambda \geq 0$, $s \geq 0$ and $x \in \tilde{\mathcal{A}}$ set

$$V(\lambda, \eta, \xi)(x, s) := V_\lambda(x, s)(\eta, \xi) = \inf_{\varphi\in A} \left\{ e_\lambda Var_{[0,s]} \left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) \right) \right\},$$

if $s > 0$ and

$$V_\lambda(x, s)(\eta, \xi) := \|x\|_{\eta\xi} \quad \text{if} \quad s = 0 \qquad (5.4.5).$$

Note that the definition of $V_\lambda(x, s)(\eta, \xi)$ makes sense because for $\varphi \in A$ the integral $\int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)$ is a function of bounded variation in the variable σ and therefore the function

$$\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)$$

is of bounded variation on $[0, s]$ as well and the e_λ-variation of this function is bounded.

The trivial process $\varphi \equiv 0$ evidently belongs to A for $x = 0$ and therefore we have

$$V_\lambda(0, s)(\eta, \xi) = 0 \qquad (5.4.6)$$

for every $s \geq 0$ and $\lambda \geq 0$ because

$$\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) = 0$$

for $\sigma > 0$.

Since

$$e_\lambda Var_{[0,s]} \left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) \right) \geq 0$$

for every $\varphi \in A$, we have by the definition (5.4.5) also the inequality

$$V_\lambda(x, s)(\eta, \xi) \geq 0 \qquad (5.4.7)$$

for every $s \geq 0$ and $x \in \tilde{\mathcal{A}}$.

5.4.3 Lemma: For $x, y \in \tilde{\mathcal{A}}$, $s \in [0, T]$ and $\lambda \geq 0$ the inequality

$$|V_\lambda(x, s)(\eta, \xi) - V_\lambda(y, s)(\eta, \xi)| \leq \|x - y\|_{\eta\xi} \qquad (5.4.8)$$

holds.

Proof. Assume that $s > 0$ and $0 < \beta < s$.

Let $\varphi \in A$ be arbitrary. Let $\varphi_\beta(\sigma) = \varphi(\sigma)$ for $\sigma \in [0, s-\beta]$, and set

$$\varphi_\beta(\sigma) = \varphi(\sigma-\beta) + \frac{1}{\beta}\left(y - \varphi(\sigma-\beta)\right)(\sigma - s + \beta)$$

for $\sigma \in [s-\beta, s]$.

The process φ_β coincides with φ on $[0, s-\beta]$ and is linear with $\varphi_\beta(s) = y$ on $[s-\beta, s]$.

By definition $\varphi_\beta \in A$ and by (5.4.2) from Lemma 5.4.1 we obtain

$$\begin{aligned}
V_\lambda(y,s)(\eta,\xi) &\leq e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi_\beta(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&= e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + e_\lambda Var_{[s-\beta,s]}\left(\langle \eta, \varphi_\beta(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_\beta(\tau),t)(\eta,\xi)\right) \\
&\leq e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + Var_{[s-\beta,s]}\left(\langle \eta, \varphi_\beta(\sigma)\xi\rangle\right) + Var_{[s-\beta,s]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\leq e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + \|y - \varphi(s-\beta)\|_{\eta\xi} + h_{\eta\xi}(s) - h_{\eta\xi}(s-\beta), \varphi(s) = y.
\end{aligned}$$

Since for every $\beta > 0$ we have

$$e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$- e_\lambda Var_{[s-\beta,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\leq e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

by (5.4.6), we obtain for every $\beta > 0$ the inequality

$$V_\lambda(y,s)(\eta,\xi) \le e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$
$$+\|y-\varphi(s-\beta)\|_{\eta\xi} + h_{\eta\xi}(s) - h_{\eta\xi}(s-\beta)$$

The function $h_{\eta\xi}$ is assumed continuous on $[0,T]$ and the stochastic process φ is such that $t \to \langle \eta, \varphi(\tau)\xi\rangle$ is continuous on $[0,T]$ and therefore we have

$$\lim_{\tau\to s}\langle \eta, \varphi(\tau)\xi\rangle = \langle \eta, \varphi(s)\xi\rangle = \langle \eta, x\xi\rangle;$$

moreover the last inequality is valid for every $\beta > 0$ and consequently we can pass to the limit $\beta \to 0$ in order to obtain

$$V_\lambda(y,s)(\eta,\xi) \le e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) + \|x-y\|_{\eta\xi}$$

for every $\varphi \in A$. Taking the infimum for all $\varphi \in A$ on the right hand side of the last inequality we arrive at

$$V_\lambda(y,s)(\eta,\xi) \le V_\lambda(x,s)(\eta,\xi) + \|x-y\|_{\eta\xi} \tag{5.4.9}$$

Since this reasoning is fully symmetric with respect to x and y we similarly obtain also

$$V_\lambda(x,s)(\eta,\xi) \le V_\lambda(y,s)(\eta,\xi) + \|x-y\|_{\eta\xi}$$

and this together with (5.4.9) yield (5.4.8) for $s > 0$.
If $s = 0$, then we have by definition

$$|V_\lambda(y,0)(\eta,\xi) - V_\lambda(x,0)(\eta,\xi)| = |\|y\|_{\eta\xi} - \|x\|_{\eta\xi}| \le \|x-y\|_{\eta\xi}$$

this proves the Lemma.

5.4.4 Corollary: Since $V_\lambda(0,s)(\eta,\xi) = 0$ for every $s \ge 0$, we have by (5.4.6) and (5.4.8)

$$0 \le V_\lambda(x,s)(\eta,\xi) \le \|x\|_{\eta\xi} \tag{5.4.10}$$

5.4.5 Lemma: For $y \in \tilde{\mathcal{A}}$, $s, r \in [0,T]$ and $\lambda \geq 0$, the inequality

$$|V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \leq \left(1 - e^{-\lambda|r-s|}\right) a + |h_{\eta\xi}(r) - h_{\eta\xi}(s)| \tag{5.4.11}$$

holds.

Proof. Suppose that $0 \leq s \leq r$ and $\varphi \in A$ is given. Set $\|y\|_{\eta\xi} \leq a$. Then by Lemma 5.4.1 we have

$$e_\lambda Var_{[0,r]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-\lambda(r-s)} e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$+ e_\lambda Var_{[s,r]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda(r-s)} V_\lambda(\varphi(s),s)(\eta,\xi) + e_\lambda Var_{[s,r]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda(r-s)}\left[V_\lambda\left(\varphi(s),s)\right)(\eta,\xi) + Var_{[s,r]}\left(\varphi_{\eta\xi}(\sigma_1)\right) - Var_{[s,r]}\left(\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)\right)\right]$$

$$\geq e^{-\lambda(r-s)}\left[V_\lambda(\varphi(s),s)(\eta,\xi) + \|y - \varphi(s)\|_{\eta\xi} + (h_{\eta\xi}(r) - h_{\eta\xi}(s))\right]$$

$$\geq e^{-\lambda(r-s)}\left[V_\lambda(y,s)(\eta,\xi) + (h_{\eta\xi}(r) - h_{\eta\xi}(s))\right] \tag{5.4.12}$$

The inequality (5.4.9) from Lemma 5.4.3 leads to

$$V_\lambda(\varphi(s),s)(\eta,\xi) + \|y - \varphi(s)\|_{\eta\xi} \geq V_\lambda(y,s)(\eta,\xi)$$

Taking the infimum over $\varphi \in A$ on the left hand side of (5.4.12) we have

$$V_\lambda(y,r)(\eta,\xi) \geq e^{-\lambda(r-s)}\left[V_\lambda(y,s)(\eta,\xi) + (h_{\eta\xi}(r) - h_{\eta\xi}(s)\right]$$

$$\geq e^{-\lambda(r-s)} V_\lambda(y,s)(\eta,\xi) + (h_{\eta\xi}(r) - h_{\eta\xi}(s)) \tag{5.4.13}$$

Now let $\varphi \in A$ be arbitrary. We define

$$\varphi^*(\sigma)(\eta,\xi) = \varphi(\sigma)(\eta,\xi) \quad \text{for } \sigma \in [0,s]$$

and

$$\varphi^*(\sigma)(\eta,\xi) = y(\eta,\xi) \quad \text{for} \quad \sigma \in [s,r].$$

We then have $\varphi^*(s)(\eta,\xi) = \varphi(s)(\eta,\xi) = y(\eta,\xi) := y$, $\varphi^* \in A$ and by (5.4.1), (5.4.6) we obtain

$$\begin{aligned}
V_\lambda(y,r)(\eta,\xi) &\leq e_\lambda Var_{[0,r]}\left(\varphi^*(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right) \\
&= e^{-\lambda(r-s)} e_\lambda Var_{[0,s]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + e_\lambda Var_{[s,r]}\left(\varphi^*(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right) \\
&\leq e^{-\lambda(r-s)} e_\lambda Var_{[0,s]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + Var_{[s,r]}\varphi^*(\sigma) + Var_{[s,r]}\int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi) \\
&\leq e^{-\lambda(r-s)} e_\lambda Var_{[0,s]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) \\
&\quad + h_{\eta\xi}(r) - h_{\eta\xi}(s).
\end{aligned}$$

Taking the infimum over all $\varphi \in A$ on the right hand side of this inequality we obtain

$$V_\lambda(y,r)(\eta,\xi) \leq e^{-\lambda(r-s)} V_\lambda(y,s)(\eta,\xi) + (h_{\eta\xi}(r) - h_{\eta\xi}(s))$$

Together with (5.4.13) we have

$$|V_\lambda(y,r)(\eta,\xi) - e^{-\lambda(r-s)} V_\lambda(y,s)(\eta,\xi)| \leq h_{\eta\xi}(r) - h_{\eta\xi}(s).$$

Hence, by (5.4.10) we get the inequality

$$\begin{aligned}
&|V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \leq |V_\lambda(y,r)(\eta,\xi) - e^{-\lambda(r-s)} V_\lambda(y,s)(\eta,\xi)| \\
&+ |1 - e^{-\lambda(r-s)}||V_\lambda(y,s)(\eta,\xi)| \\
&\leq h_{\eta\xi}(r) - h_{\eta\xi}(s) + (1 - e^{-\lambda(r-s)})\|y\|_{\eta\xi} \\
&\leq h_{\eta\xi}(r) - h_{\eta\xi}(s) + (1 - e^{-\lambda(r-s)})a
\end{aligned}$$

because $\|y\|_{\eta\xi} \leq a$. In this way we have obtained (5.4.11).

Assume that $s = 0$ and $r > 0$. Then by (5.4.10) and by the definition given in (5.4.5) we get

$$V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi) = V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,0)(\eta,\xi)$$

$$= V_\lambda(y,r)(\eta,\xi) - \|y\|_{\eta\xi} \le 0 \tag{5.4.14}$$

We derive an estimate from below. Assume that $\varphi \in A$. By (5.4.1) in Lemma 5.4.1 and Lemma 1.9.9, we have

$$e_\lambda Var_{[0,r]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\ge e_\lambda Var_{[0,r]}\varphi - e_\lambda Var_{[0,r]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\ge e^{-\lambda r} Var_{[0,r]}\varphi - Var_{[0,r]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\ge e^{-\lambda r}|\varphi(\sigma)(\eta,\xi) - \varphi(0)(\eta,\xi)| - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

$$= e^{-\lambda r}\|y\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

Passing again to the infimum for $\varphi \in A$ on the left hand side of this inequality we get

$$V_\lambda(y,r)(\eta,\xi) \ge e^{-\lambda r}\|y\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

and

$$V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,0)(\eta,\xi) = V_\lambda(y,r)(\eta,\xi)\|y\|_{\eta\xi}$$

$$\ge (e^{-\lambda r} - 1)\|y\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

$$= -(1 - e^{-\lambda r})\|y\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

This together with (5.4.14) yields

$$V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,0)(\eta,\xi)| \le (1 - e^{-\lambda r})\|y\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0)),$$

and this means that the inequality (5.4.11) holds in this case too. The remaining case of $r = s = 0$ is evident because

$$|V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| = 0 = \left(1 - e^{-\lambda|r-s|}\right)a + |h_{\eta\xi}(r) - h_{\eta\xi}(s)|$$

For the case when $r < s$ we obtain

$$|V_\lambda(y,s)(\eta,\xi) - V_\lambda(y,r)(\eta,\xi)| \le \left(1 - e^{-\lambda(s)}\right) |\|y\|_{\eta\xi} - (h_{\eta\xi}(s) - h_{\eta\xi}(r))$$

because the situation is symmetric in s and r. We have thus established results for the case when $s \ge 0, s$ and r

By the previous Lemmas 5.4.3 and 5.4.5, we immediately conclude that the following holds.

5.4.6 Corollary: For $x, y \in \tilde{\mathcal{A}}$, $r, s \in [0,T]$ and $\lambda \ge 0$ the inequality

$$|V_\lambda(x,s)(\eta,\xi) - V_\lambda(y,r)(\eta,\xi)| \le \|x-y\|_{\eta\xi} + (1-e^{-\lambda|r-s|})a + |h_{\eta\xi}(r) - h_{\eta\xi}(s)| \quad (5.4.15)$$

holds.

Next, we shall discuss the behaviour of the function $V_\lambda(x,t)(\eta,\xi)$ defined by (5.4.5) along the solutions of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF(X,t)(\eta,\xi) \quad (5.2.3)$$

We still assume that the assumptions given at the beginning of this chapter are satisfied for the right hand side $F(x,t)(\eta,\xi)$.

The next result will be employed in what follows.

5.4.7 Lemma: Assume that $\psi : [s, s+\beta(s)] \to \tilde{\mathcal{A}}$ is a solution of (5.2.3), $s \ge 0$, $\beta(s) > 0$, then for every λ the inequality

$$\limsup_{\beta \to 0} \frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) - V_\lambda(\psi(s), s)(\eta,\xi)}{\beta} \le -\lambda V_\lambda(\psi(s), s)(\eta,\xi) \quad (5.4.16)$$

holds.

Proof: Let $s \in [0,T]$ and $x \in \tilde{\mathcal{A}}$ be given. Let us choose $a > 0$ such that $a > \|x\|_{\eta\xi} + h_{\eta\xi}(s+1) - h_{\eta\xi}(s)$. Assume that $\varphi \in A$ is given and let $\psi : [s, s+\beta(s)] \to \tilde{\mathcal{A}}$

be a solution of (5.2.3) on $[s, s+\beta(s)]$ with $\psi(s) = x$ where $0 < \beta(s) < 1$. The existence of such a solution is guaranteed by the existence theorem in chapter 4.
For $0 < \beta < \beta(s)$ define

$$\varphi_\beta(\sigma)(\eta, \xi) = \varphi(\sigma)(\eta, \xi) \text{ for } \sigma \in [0, s]$$

and

$$\varphi_\beta(\sigma)(\eta, \xi) = \psi(\sigma)(\eta, \xi) \text{ for } \sigma \in [s, s+\beta].$$

we have $\varphi(s) = \psi(s) = \varphi_\beta(s) = x$. Then $\varphi_\beta \in A$, for $\beta \in [s, s+\beta]$ and since ψ is weakly absolutely continuous and by the definition of a solution we have

$$|\langle \eta, \psi(\sigma)\xi\rangle| = \left| \langle \eta, x(s)\xi\rangle + \int_s^\sigma DF(\psi(\tau), t)(\eta, \xi) \right|$$

$$\leq \|x\|_{\eta\xi} + h_{\eta\xi}(\sigma) - h_{\eta\xi}(s) \leq \|x\|_{\eta\xi} + h_{\eta\xi}(s+1) - h_{\eta\xi}(s) < a$$

for $\sigma \in [s, s+\beta]$ and

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta, \xi) \leq e_\lambda Var_{[0,s+\beta]} \left(\varphi(\sigma)(\eta, \xi) - \int_0^\sigma DF(\varphi_\beta(\tau), t)(\eta, \xi) \right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]} \left(\varphi(\sigma)(\eta, \xi) - \int_0^\sigma DF(\varphi(r), t)(\eta, \xi) \right)$$

$$+ e_\lambda Var_{[s,s+\beta]} \left(\varphi(\sigma)(\eta, \xi) - \int_0^s DF(\varphi(\tau), t)(\eta, \xi) - \int_s^\sigma DF(\psi(\tau), t)(\eta, \xi) \right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]} \left(\varphi(\sigma)(\eta, \xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) \right)$$

$$+ e_\lambda Var_{[s,s+\beta]} \left(\langle \eta, x\xi\rangle - \int_0^s DF(\varphi(\tau), t)(\eta, \xi) \right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]} \left(\varphi(\sigma)(\eta, \xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) \right).$$

Taking the infimum for all $\varphi \in A$ on the right hand side of this inequality we obtain

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta, \xi) \leq e^{-\lambda\beta} V_\lambda(x, s)(\eta, \xi) = e^{-\lambda\beta} V_\lambda(\psi(s), s)(\eta, \xi)$$

This inequality yields

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) - V_\lambda(\psi(s), s)(\eta,\xi) \le (e^{-\lambda\beta}-1)V_\lambda(\psi(s), s)(\eta,\xi)$$

and also

$$\frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) - V_\lambda(\psi(s), s)(\eta,\xi)}{\beta} \le \frac{e^{-\lambda\beta}-1}{\beta}V_\lambda(\psi(s), s)(\eta,\xi)$$

for every $0 < \beta < \beta(s)$.

Since $\lim_{\beta\to 0}\frac{e^{-\lambda\beta}-1}{\beta} = -\lambda$ we immediately obtain (5.4.16).

Now we establish the converse theorems to Theorems 5.3.3 and 5.3.4.

5.4.8 Theorem: Assume that the trivial solution $x \equiv 0$ of equation (5.2.3) is variationally stable then for every $0 < a < c$, there exists a real-valued map $V(x,t)(\eta,\xi)$ satisfying the following conditions:

(i) for every $x \in \tilde{\mathcal{A}}$ the function $t \to V(x,t)(\eta,\xi)$ is of bounded variation in t and continuous in t

(ii) $V(0,t)(\eta,\xi) = 0$ and $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \le \|x-y\|_{\eta\xi}$ for $x, y \in \tilde{\mathcal{A}}, t \in [0,T]$,

(iii) the function $V(x,t)(\eta,\xi)$ is non-increasing along the solutions of the equation (5.2.3),

(iv) the function $V(x,t)(\eta,\xi)$ is positive definite if there is a continuous nondecreasing real-valued function $b : [0,+\infty) \to \mathbb{R}$ such that $b(\rho) = 0$ if and only if $\rho = 0$ and

$$b(\|x\|_{\eta\xi}) \le V(x,t)(\eta,\xi).$$

for every $x \in \tilde{\mathcal{A}}, t \in [0,T]$.

Proof: The candidate for the function $V(x,s)(\eta,\xi)$ is the function $V_0(x,s)(\eta,\xi)$

defined by (5.4.5).

For $\lambda = 0$, i.e. we take $V_\lambda(x,s)(\eta,\xi) = V_0(x,s)(\eta,\xi) = V(x,s)(\eta,\xi)$. Hypothesis (i) is established by Corollary 5.4.9. Hypothesis (ii) follow from (5.4.6) and from Lemma 5.4.3 i.e. The trivial process $\varphi \equiv 0$ evidently belongs to A for $x = 0$ and therefore we have

$$V(0,s)(\eta,\xi) = 0 \tag{5.4.6}$$

for every $s \geq 0$ and $\lambda \geq 0$, because

$$\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi) = 0$$

for $\sigma > 0$. The inequality $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq \|x-y\|_{\eta\xi}$ follows from the proof of Lemma 5.4.3.

By Lemma 5.4.7 for every solution $\psi : [s, s+\sigma] \to \tilde{\mathcal{A}}$ of equation (5.2.3) we have

$$\limsup_{\beta\to 0} \frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) - V_\lambda(\psi(s),s)(\eta,\xi)}{\beta} \leq 0$$

and therefore (iii) is also satisfied.

It remains to show that the function $V(x,t)(\eta,\xi)$ given in this way is positive definite. This is the only point where the variational stability of the solution $x \equiv 0$ of equation (5.2.3) is used.

Assume that there is an ϵ, $0 < \epsilon < a$ and a sequence (x_k, t_k), $k = 1, 2, \ldots,$

$\epsilon \leq \|x_k\|_{\eta\xi} < a$, $t_k \to \infty$ for $k \to \infty$ such that $V(x_k,t_k)(\eta,\xi) \to 0$ for $k \to \infty$. Let $\delta(\epsilon) > 0$ correspond to ϵ by Definition 5.2.4 of stability with respect to perturbations (the variational stability of $x \equiv 0$ is equivalent to the stability with respect to perturbations of this solution by Theorem 5.2.1). Assume that $k \in \mathbb{N}$ is such that for $k > 0$ we have $V(x_k,t_k)(\eta,\xi) < \delta(\epsilon)$. Then there exists $\varphi_k \in A$ such that for every $t_k \in [0,T]$

$$Var_{[0,t_k]}\left(\varphi_k(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi)\right) < \delta(\epsilon)$$

We set

$$\langle \eta, Q(\sigma)\xi\rangle = \langle \eta, \varphi_k(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi) \ \text{ for } \ \sigma \in [0,t_k]$$

$$\langle \eta, Q(\sigma)\xi \rangle = \langle \eta, x_k(\xi) \rangle - \int_0^{t_k} DF(\varphi_k(\tau), t)(\eta, \xi) \text{ for } \sigma \in [t_k, T], t_k > 0$$

We then have

$$Var_{[0,T]} \langle \eta, Q(\sigma)\xi \rangle = Var_{[0,t_k]} \left(\varphi_k(\sigma)(\eta, \xi) - \int_0^{\sigma} DF(\varphi_k(\tau), t)(\eta, \xi) \right) < \delta(\epsilon)$$

and the function $\langle \eta, Q(.)\xi \rangle$ is continuous on $[0, T]$. For $\sigma \in [0, t]$, we have

$$\begin{aligned} \langle \eta, \varphi_k(\sigma)\xi \rangle &= \int_0^{\sigma} DF(\varphi_k(\tau), t)(\eta, \xi) + \langle \eta, \varphi_k(\sigma)\xi \rangle \\ &\quad - \int_0^{\sigma} DF(\varphi_k(\tau), t)(\eta, \xi) \\ &= \int_0^{\sigma} DF(\varphi_k(\tau), t)(\eta, \xi) + \langle \eta, Q(\sigma)\xi \rangle - \langle \eta, Q(0)\xi \rangle \\ &= \langle \eta, \varphi_k(0)\xi \rangle + \int_0^{\sigma} D[F(\varphi_k(\tau), t)(\eta, \xi) + \langle \eta, Q(t)\xi \rangle] \end{aligned}$$

because $\varphi_k(0) = 0$. Hence, φ_k is a solution of the equation

$$\frac{d}{d\tau} \langle \eta, y(\tau)\xi \rangle = D[F(y(\tau), t)(\eta, \xi) + Q(t)(\eta, \xi)]$$

and therefore, by the variational stability we have $\|\varphi_k(s)\|_{\eta\xi} < \epsilon$ for every $s \in [0, t_k]$. Hence we also have $\|\varphi_k(t_k)\|_{\eta\xi} = \|x_k\|_{\eta\xi} < \epsilon$ and this contradicts our assumption. In this way we obtain that the function $V(x, t)(\eta, \xi)$ is
positive definite and (iv) is also satisfied.

The next statement is the converse for Theorem 5.3.4 on variational asymptotic stability.

5.4.9 Theorem: Assume that the trivial solution $x \equiv 0$ of equation (5.2.3) is variationally asymptotically stable then for every $0 < a < c$ there exists a real-valued map $U(x, t)(\eta, \xi) : \tilde{\mathcal{A}} \times [0, T] \to \mathbb{R}$ satisfying the following conditions:

(i) For every $x \in \tilde{\mathcal{A}}$ the map $t \to U(x, t)(\eta, \xi)$ is continuous on $[0, T]$ and is locally of bounded variation on $[0, T]$,

(ii) $U(0,t)(\eta,\xi) = 0$ and

$$|U(x,t)(\eta,\xi) - U(y,t)(\eta,\xi)| \le \|x-y\|_{\eta\xi} \text{ for } x,y \in \tilde{\mathcal{A}},\ t \in [0,T],$$

(iii) For every solution $\psi(\sigma)$ of the equation (5.2.3) defined for $\sigma \ge t$, where $\psi(t) = x \in \tilde{\mathcal{A}}$ the relation

$$\limsup_{\beta\to 0} \frac{U(\psi(t+\beta), t+\beta)(\eta,\xi) - U(x,t)(\eta,\xi)}{\beta} \le -U(x,t)(\eta,\xi)$$

holds,

(iv) the function $U_{\eta\xi}(x,t)$ is positive definite.

Proof: For $x \in \tilde{\mathcal{A}},\ s \ge 0$ we set

$$U(x,s)(\eta,\xi) = V(x,s)(\eta,\xi)$$

where $V_0(x,s)(\eta,\xi)$ is the function defined by (5.3.5) for $\lambda = 1$. In the same way as in the proof of Theorem 5.4.8 the map $U(x,s)(\eta,\xi)$ satisfies (i), (ii) and (iii). (The item (iii) is exactly the statement given in Lemma 5.4.7).

It remains to show that (iv) is satisfied for this choice of the function $U(x,s)(\eta,\xi)$. Since the solution $x \equiv 0$ of equation (5.2.3) is assumed to be variationally attracting and by Theorem 5.2.1 it is also attracting with respect to perturbations and therefore there exists $\delta_0 > 0$ and for every $\epsilon > 0$ there is a $A = A(\epsilon) \ge 0$ and $B = B(\epsilon) > 0$ such that if $\|y_0\|_{\eta\xi} < \delta_0$, $y_0 \in \tilde{\mathcal{A}}$ and $Q \in BV(\tilde{\mathcal{A}}) \bigcap (\tilde{\mathcal{A}})_{wac}$ on $[t_0,t_1] \subset [0,T]$, and

$$Var_{[t_0,t_1]} p = Var_{[t_0,t_1]} Q < B(\epsilon)$$

then

$$\|y(t)\|_{\eta\xi} < \epsilon$$

for all $t \in [t_0,t_1] \cap [t_0 + A(\epsilon), T]$ and $t_0 \ge 0$ where $y(t)$ is a solution of

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = D[F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)] \tag{5.2.7}$$

with $y(t_0) = y_0$.

Assume that the map U is not positive definite then there exists ϵ, $0 < \epsilon < a = \delta_0$, $a > 0$ and a sequence $(x_k, t_k),\ k = 1, 2, \dots$, assume also that $\epsilon \leq \|x_k\|_{\eta\xi} < a$, $t_k \to \infty$ for $k \to \infty$ such that $U(x_k, t_k) \to 0$ for $k \to \infty$. Choose $k_0 \in \mathbb{N}$ such that for $k \in \mathbb{N}$, $k > k_0$ we have $t_k > A(\epsilon) + 1$ and

$$U(x_k, t_k)(\eta, \xi) < B(\epsilon)e^{-(A(\epsilon)+1)}, x_k \in \tilde{\mathcal{A}}$$

According to the definition of the map U we choose $\varphi \subset A$ such that

$$e_1 Var_{[0,t_k]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right) < B(\epsilon)e^{-(A(\epsilon)+1)}.$$

Define $t_0 = t_k - (A(\epsilon) + 1)$. Then $t_0 > 0$ because $t_k > A(\epsilon) + 1$ and also $t_k = t_0 + A(\epsilon) + 1 > t_0 + A(\epsilon)$.

Therefore,

$$e_1 Var_{[t_0,t_k]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right) < B(\epsilon)e^{-(A(\epsilon)+1)},$$

by inequality (5.4.1) in Lemma 5.4.1 also

$$e^{-(A(\epsilon)+1)} Var_{[t_0,t_k]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right)$$

$$= e^{-(t_k-t_0)} Var_{[t_0,t_k]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right) < B(\epsilon)e^{-(A(\epsilon)+1)}.$$

and therefore, we get

$$Var_{[t_0,t_k]}\left(\varphi(\sigma)(\eta,\xi) - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right) < B(\epsilon). \tag{5.4.17}$$

For $\sigma \in [t_0, t_k]$ define

$$\langle \eta, Q(\sigma)\xi\rangle = \langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)$$

The function $Q : [t_0, t_k] \to \tilde{\mathcal{A}}$ evidently lie in $BV(\tilde{\mathcal{A}}) \bigcap (\tilde{\mathcal{A}})_{wac}$ and by the inequality (5.4.17) we have

$$Var_{[t_0,t_k]}Q = Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi)\right) < B(\epsilon)$$

$$Var_{[t_0,t_k]}Q < B(\epsilon)$$

Moreover,

$$\begin{aligned}\langle \eta, \varphi(\sigma)\xi \rangle &= \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) + \langle \eta, \varphi(\sigma)\xi \rangle - \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) = \\ &= \int_0^\sigma DF(\varphi(\tau), t)(\eta, \xi) + \langle \eta, Q(\sigma)\xi \rangle\end{aligned}$$

and also

$$\langle \eta, \varphi(s)\xi \rangle - \langle \eta, \varphi(t_0)\xi \rangle = \int_{t_0}^s DF(\varphi(\tau), t)(\eta, \xi) + \langle \eta, Q(s)\xi \rangle - \langle \eta, Q(t_0)\xi \rangle$$

$$= \int_{t_0}^s D[F(\varphi(\tau), t)(\eta, \xi) + \langle \eta, Q(t)\xi \rangle],$$

and this means that the function $\varphi : [t_0, t_k] \to \tilde{\mathcal{A}}$ is a solution of the equations (5.2.7) and (5.2.8) with

$$\|\varphi(t_0)\|_{\eta\xi} \leq a = \delta_0$$

because $\varphi \in A$ for each $t_k \in [0, T]$. By the definition of variational attracting the inequality $\|\varphi(t_0)\|_{\eta\xi} < \epsilon$ holds for every $t > t_0 + A(\epsilon)$. This is of course valid also for the value $t = t_k > t_0 + A(\epsilon)$, i.e.

$$\|\varphi(t_k)\|_{\eta\xi} = \|x_k\|_{\eta\xi} < \epsilon$$

and this contradicts the assumption $\|x_k\|_{\eta\xi} \geq \epsilon$. This yields the positive definiteness of the real-valued map U. And the result is established.

Chapter 6

Continuous Dependence on Parameters of Kurzweil Equations associated with Quantum Stochastic Differential Equations

6.1 Introduction

This chapter is devoted to the investigation of continuous dependence on parameters of solutions of Kurzweil equations associated with the quantum stochastic differential equations. Continuous dependence of solution on parameters has been used by some authors to establish general results on existence of solution especially for the Kurzweil equations associated with the classical differential equations and to derive other special results such as averaging for generalized ordinary differential equations [47, 51, 87].

The motivation for studying continuous dependence of solutions on parameters for this class of noncommutative quantum stochastic differential equation (1.5) is to include in the theory of non classical ordinary differential equation and the associated Kurzweil equation the convergence effect of equations (1.5) when it depends on a parameter.

The next section will consist of two sections: section 6.2 and section 6.3. In section 6.2, we shall first establish some preliminary results. The main results will be established in section 6.3. The main results will be establised under two conditions: When the stochastic processes are simple processes of bounded variation and when they are just stochastic processes of bounded variation.

6.2 Preliminary Results

Through out this chapter, $\eta, \xi \in I\!D \underline{\otimes} I\!E$ is an arbitrary pair of elements. In what follows, we consider the sequence of coefficients $E_k, F_k, G_k, H_k : \tilde{\mathcal{A}} \times [t_0, T] \to \tilde{\mathcal{A}}$ belonging to the appropriate space as in equation (1.3) giving rise to a sequence of QSDEs of the form (1.3) given by

$$\begin{aligned} dX(t) &= E_k(X(t),t)dA_f(t) + F_k(X(t),t)dA_g^+(t) \\ &\quad + G_k(X(t),t)d\Lambda_\Pi(t) + H_k(X(t),t)dt \\ X(t_0) &= X_0, t \in [t_0,T], k = 0,1,2,.. \end{aligned} \tag{6.2.1}$$

The equivalent form of equation (6.2.1) is the following sequence of nonclassical ordinary differential equation

$$\begin{aligned} \frac{d}{dt}\langle \eta, x(t)\xi\rangle &= P_k(x,t)(\eta,\xi) \\ X(t_0) &= X_0,\ t \in [t_0,T] \end{aligned} \tag{6.2.2}$$

Where the sequence of sesquilinear forms $(X,t) \to P_k(X,t)(\eta,\xi),\ k = 0,1,2,...$
is assumed to be of class $C(\tilde{\mathcal{A}} \times [t_0,T], W)$.
By equation (1.6) in chapter one, the map P_k appearing in equation (6.2.2) has the form

$$\begin{aligned} P_k(x,t)(\eta,\xi) &= (\mu E_k)(x,t)(\eta,\xi) + (\gamma F_k)(x,t)(\eta,\xi) + (\sigma G_k)(x,t)(\eta,\xi) \\ &\quad + H_k(x,t)(\eta,\xi), (x,t) \in \tilde{\mathcal{A}} \times [t_0,T] \end{aligned} \tag{6.2.3}$$

where $H_k(x,t)(\eta,\xi) := \langle \eta, H_k(x,t)\xi\rangle$.
The map P_k may some times be written as $P_k(x,t)(\eta,\xi) = \langle \eta, P_{k,\alpha\beta}(x,t)\xi\rangle$ where

$P_{k,\alpha\beta} : \tilde{\mathcal{A}} \times I \longrightarrow \tilde{\mathcal{A}}, I = [t_0, T]$ is given by

$$P_{k,\alpha\beta}(x,t) = \mu_{\alpha\beta}(t)E_k(x,t) + \gamma_\beta(t)F_k(x,t) + \sigma_\alpha(t)G_k(x,t) + H_k(x,t)$$

for $(x,t) \in \tilde{\mathcal{A}} \times I$.

6.2.1 Definition

(i) Let the sequence of maps $P_k : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow \text{sesq}(I\!\!D \underline{\otimes} I\!\!E)$ be given by equation (6.2.3), define

$$F_k(X,t)(\eta,\xi) = \int_{t_0}^{t} P_k(X(s),s)(\eta,\xi)ds.$$

Then we refer to the equation

$$\frac{d}{dt}\langle \eta, X(\tau)\xi \rangle = DF_k(X(\tau),t)(\eta,\xi) \tag{6.2.4}$$

as the Kurzweil equation associated with equation (6.2.2).

let the space $\tilde{\mathcal{A}}$ and the functions $h_{\eta\xi}, W$ be given as in chapter one.

6.2.1 Lemma: Assume the following hold:

$F_k : G \to sesq(I\!\!D \underline{\otimes} I\!\!E)$ is of class $\mathbb{F}(G, h_{\eta\xi}, W)$ for $k = 0, 1, \ldots$ and that

$$\lim_{k\to\infty} F_k(x,t)(\eta,\xi) = F_0(x,t)(\eta,\xi) \tag{6.2.5}$$

for $(x,t) \in G, G = \tilde{\mathcal{A}} \times [t_0, T]$.

If $x : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subset [t_0, T]$, $x \in BV(\tilde{\mathcal{A}})$ then

$$\lim_{k\to\infty} \int_a^b DF_k(x(\tau),t)(\eta,\xi) = \int_a^b DF_0(x(\tau),t)(\eta,\xi) \tag{6.2.6}$$

We shall proof this Lemma when X is a simple process of bounded variation and also the case when X is only of bounded variation.

Proof: Let $\epsilon > 0$ be given. Assume that $\beta_{\eta\xi} := \beta > 0$ and

$$W(\beta) \leq \frac{\epsilon}{2(h_{\eta\xi}(b) - h_{\eta\xi}(a) + 1)}$$

Since $x : [a,b] \to \tilde{\mathcal{A}}$ lie in $BV(\tilde{\mathcal{A}})$ for every $\beta > 0$, there is a stochastic process $\varphi : [a,b] \to \tilde{\mathcal{A}}$ such that

$$\|x(\tau) - \varphi(\tau)\|_{\eta\xi} \leq \beta_{\eta\xi} \text{ for } \tau \in [a,b] \tag{6.2.7}$$

Therefore,

$$|F_k(x(\tau),t_2)(\eta,\xi) - F_k(x(\tau),t_1)(\eta,\xi) - F_k(\varphi(\tau,t_2)(\eta,\xi) + F_k(\varphi(\tau),t_1)(\eta,\xi)|$$
$$\leq W(\|x(\tau) - \varphi(\tau)\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$
$$\leq W(\beta)|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$

for $\tau \in [a,b], t_1, t_2 \in [a,b]$ and $k = 0,1,\ldots$ because F_k is of class $\mathbb{F}(G, h_{\eta\xi}, W)$.

By Theorem 1.9.8 the integrals

$$\int_a^b DF_k(x(\tau),t)(\eta,\xi), \quad \int_a^b DF_k(\varphi(\tau),t)(\eta,\xi)$$

exist and yields the estimate

$$\left|\int_a^b D[F_k(x(\tau),t)(\eta,\xi) - F_k(\varphi(\tau),t)(\eta,\xi)]\right|$$
$$\leq \int_a^b W(\beta) dh_{\eta\xi}(s) = W(\beta)(h_{\eta\xi}(b) - h_{\eta\xi}(a)) \tag{6.2.8}$$

for every $k = 0,1,\ldots$

Again since $F_k \in \mathbb{F}(G, h_{\eta\xi}, W)$ we have,

$$|F_k(x,t_2)(\eta,\xi) - F_k(x,t_1)(\eta,\xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|,$$

for every $x \in \tilde{\mathcal{A}}$ and $t_1, t_2 \in [t_0, T]$ and this leads to the conclusion that

$$\lim_{\rho\to 0} F_k(x, t+\rho)(\eta,\xi) = F_k(x,t)(\eta,\xi)$$

for every $(x,t) \in G, k = 0,1,\ldots$. Hence by (6.2.5), we obtain

$$\begin{aligned}
\lim_{k\to\infty} F_k(x,t)(\eta,\xi) &= \lim_{k\to\infty}\lim_{\rho\to 0} F_k(x,t+\rho)(\eta,\xi) \\
&= \lim_{\rho\to 0}\lim_{k\to\infty} F_k(x,t+\rho)(\eta,\xi) \\
&= \lim_{\rho\to 0} F_0(x,t+\rho)(\eta,\xi) \\
&= F_0(x,t)(\eta,\xi).
\end{aligned}$$

Using this equality and assuming that $\varphi(s) : [a, b] \to \tilde{\mathcal{A}}$ is a simple process then there is a partition $a = s_1 < s_2 < ... < s_k = b$ of $[a, b]$ such that $\varphi(s) = c_j \in \tilde{\mathcal{A}}$ for $s \in (s_{j-1}, s_j), j = 1, 2, ..., k$ where $c_j, j = 1, 2, ..., k$ are finite number of elements of $\tilde{\mathcal{A}}$, $s_{j-1} < s_{j+1} < \sigma_0 < \sigma_1 < \sigma_2 < \sigma_{j+1} < \sigma_j \in (s_{j-1}, s_j)$, we obtain by Theorem 1.9.7

$$\begin{aligned}\int_{s_{j-1}}^{s_j} DF_k(\varphi(\tau), t)(\eta, \xi) &= F_k(c_j, s_{j-1})(\eta, \xi) - F_k(c_j, \sigma_0)(\eta, \xi) - F_k(\varphi(s_j), s_{j-1})(\eta, \xi) \\ &\quad + F_k(\varphi(s_j), s_j)(\eta, \xi) + F_k(c_j, \sigma_0)(\eta, \xi) - F_k(c_j, s_{j+1})(\eta, \xi) \\ &\quad + F_k(\varphi(s_{j-1}), s_{j+1})(\eta, \xi) - F_k(\varphi(s_{j-1}), s_{j-1})(\eta, \xi) \\ &= F_k(c_j, s_{j-1})(\eta, \xi) - F_k(c_j, s_{j+1})(\eta, \xi) + \\ &\quad + F_k(\varphi(s_{j-1}), s_{j+1})(\eta, \xi) - F_k(\varphi(s_{j-1}), s_{j-1})(\eta, \xi) - \\ &\quad - F_k(\varphi(s_j), s_{j-1})(\eta, \xi) + F_k(\varphi(s_j), s_j)(\eta, \xi).\end{aligned}$$

Repeating the above for the case when k is replaced with 0 and taking the limit of the above as $k \to \infty$, we therefore,have

$$\lim_{k\to\infty} \int_{s_{j-1}}^{s_j} D[F_k(\varphi(\tau), t)(\eta, \xi) - F_0(\varphi(\tau), t)(\eta, \xi)] = 0 \tag{6.2.9}$$

Since φ is a simple process, we obtain from (6.2.5) using the additivity of the integral the relation

$$\lim_{k\to\infty} \int_{a}^{b} D[F_k(\varphi(\tau), t)(\eta, \xi) - F_0(\varphi(\tau), t)(\eta, \xi)] = 0 \tag{6.2.10}$$

Next, we consider the stochastic process φ satisfying (6.2.7) which is not necessarily a simple process but of bounded variation.

We get the following estimate by using (6.2.8) as follows:

$$\left|\int_a^b DF_k(x(\tau),t)(\eta,\xi) - \int_a^b DF_0(x(\tau),t)(\eta,\xi)\right|$$

$$\leq \left|\int_a^b DF_k(x(\tau)t)(\eta,\xi) - \int_a^b DF_k(\varphi(\tau),t)(\eta,\xi)\right|$$

$$+\left|\int_a^b DF_0(x(\tau),t)(\eta,\xi) - \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)\right|$$

$$+\left|\int_a^b DF_k(\varphi(\tau),t)(\eta,\xi) - \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)\right|$$

$$\leq 2W(\beta)(h_{\eta\xi}(b) - b_{\eta\xi}(a)) + \left|\int_a^b DF_k(\varphi(\tau),t)(\eta,\xi) - \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)\right|$$

$$\leq \epsilon + \left|\int_a^b DF_k(\varphi(\tau),t)(\eta,\xi) - \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)\right|$$

by the choice of β. By taking the limit as $k \to \infty$ on both sides of this inequality we obtain

$$\lim_{k\to\infty}\left|\int_a^b DF_k(x(\tau),t)(\eta,\xi) - \int_a^b DF_0(x(\tau),t)(\eta,\xi)\right|$$

$$\leq \epsilon + \lim_{k\to\infty}\left|\int_a^b DF_k(\varphi(\tau),t)(\eta,\xi) - \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)\right|$$

and since ϵ can be taken arbitrarily small we obtain the result

$$\lim_{k\to\infty}\int_a^b DF_k(\varphi(\tau),t)(\eta,\xi) = \int_a^b DF_0(\varphi(\tau),t)(\eta,\xi)$$

6.3 Major Results

6.3.1 Theorem: Assume that the following hold:

(i) $F_k : G \to \mathrm{Sesq}(I\!\!D \underline{\otimes} I\!\!E)$ is of class $\mathbb{F}(G, h_{\eta\xi}, W)$ for $k = 0, 1, ...$

(ii) $\lim_{k\to\infty} F_k(x,t)(\eta,\xi) = F_0(x,t)(\eta,\xi), \quad (x,t) \in G$ (6.3.1)

(iii) $x_k : [a,b] \to \tilde{\mathcal{A}}, \quad k = 1, 2, \ldots$ is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi \rangle = DF_k(x,t)(\eta,\xi), \quad \text{on} \quad [a,b] \subset [t_0, T] \tag{6.3.2}$$

(iv) $\lim_{k\to\infty} x_k(s) = x(s), \quad s \in [a,b]$ (6.3.3)

Then

$x : [a,b] \to \tilde{\mathcal{A}}$ is of bounded variation on $[a,b]$ and it is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi \rangle = DF_0(x,t)(\eta,\xi) \quad \text{on} \quad [a,b] \tag{6.3.4}$$

Proof. By (1.9.12) of Lemma 1.9.10, we have

$$\|x_k(s_2) - x_k(s_1)\|_{\eta\xi} \leq |h_{\eta\xi}(s_2) - h_{\eta\xi}(s_1)|$$

for every $k = 1, 2, \ldots$ and $s_1, s_2 \in [a,b]$.

Hence,

$$\begin{aligned} \|x_k(s)\|_{\eta\xi} &\leq \|x_k(a)\|_{\eta\xi} + h_{\eta\xi}(s) - h_{\eta\xi}(a) \\ &\leq \|x_k(a)\|_{\eta\xi} + h_{\eta\xi}(b) - h_{\eta\xi}(a) \end{aligned}$$

The last inequality is a consequence of the following statement.

Remark. Since x lie in $BV(\tilde{\mathcal{A}})$, we have from definition that

$$Var_{[a,b]}X_k(\eta,\xi) = \sup_{\tau}\left(\sum_{j=1}^{n} \|X(t_j) - X(t_j - 1)\|_{\eta\xi}\right)$$

then the above inequality holds.

and

$$\text{var}_{[a,b]} x_k(\eta,\xi) \leq h_{\eta\xi}(b) - h_{\eta\xi}(a) \tag{6.3.5}$$

By (6.3.3) we have $x_k(a) \to x(a)$ for $k \to \infty$ and therefore the sequence (x_k) of simple processes on $[a,b]$ is bounded and by (6.3.5) of bounded variation on $[a,b]$. By Helly's choice Theorem [8'] there exists a subsequence of (x_k) which converges uniformly to a function

$x \in Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$.

Hence we conclude by (6.4.3) that $x : [a, b] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}}_{wac} \cap BV(\tilde{\mathcal{A}})$, and Theorems 1.9.7 and 1.9.8 lead to the conclusion that the integral

$$\int_a^b DF_0(x(\tau), t)(\eta, \xi) \text{ exists.}$$

By definition of a solution of the Kurzweil equation (6.3.2) we have

$$\langle \eta, x_k(s_2)\xi \rangle - \langle \eta, x_k(s_1)\xi \rangle = \int_{s_1}^{s_2} DF_k(x_k(\tau), t)(\eta, \xi) \qquad (6.3.6)$$

for every $s_1, s_2 \in [a, b]$ and $k = 1, 2, \ldots$.

The aim here is to show that

$$\lim_{k\to\infty} \int_{s_1}^{s_2} DF_k(x_k(\tau), t)(\eta, \xi) = \int_{s_1}^{s_2} DF_0(x(\tau), t)(\eta, \xi) \qquad (6.3.7)$$

for any $s_1, s_2 \in [a, b]$ because passing to the limit $k \to \infty$ in (6.3.6) we obtain

$$\langle \eta, x(s_2)\xi \rangle - \langle \eta, x(s_1)\xi \rangle = \int_{s_1}^{s_2} DF_0(x(\tau), t)(\eta, \xi)$$

for every $s_1, s_2 \in [a, b]$ provided (6.3.7) is true, and this means

that $x : [a, b] \to \tilde{\mathcal{A}}$ is a solution of (6.3.4) on the interval $[a, b]$.

To prove (6.3.7), consider the difference

$$\int_{s_1}^{s_2} DF_k(x_k(\tau), t)(\eta, \xi) - \int_{s_1}^{s_2} DF_0(x(\tau), t)(\eta, \xi)$$

$$= \int_{s_1}^{s_2} D[F_k(x_k(\tau), t)(\eta, \xi) - F_k(x(\tau), t)(\eta, \xi)]$$

$$+ \int_{s_1}^{s_2} D[F_k(x(\tau), t)(\eta, \xi) - F_0(x(\tau), t)(\eta, \xi)]$$

for $a \le s_1 \le s_2 \le b$.

By lemma 6.2.1 we have

$$\lim_{k\to\infty} \int_{s_1}^{s_2} D[F_k(x(\tau), t)(\eta, \xi) - F_0(x(\tau), t)(\eta, \xi)] = 0 \qquad (6.3.8)$$

Since the map $F_k : G \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is of class $\mathbb{F}(G, h_{\eta\xi}, W)$ for $k = 0, 1, \ldots$ we have

$$|F_k(x_k(\tau), t_2)(\eta, \xi) - F_k(x_k(\tau), t_1)(\eta, \xi) - F_k(x(\tau), t_2)(\eta, \xi) + F_k(x(\tau), t_1)(\eta, \xi)|$$

$$\leq W(\|x_k(\tau) - x(\tau)\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \qquad (6.3.9)$$

for $\tau, t_1, t_2 \in [a, b]$.

The stochastic processes $x_k - x, k = 1, 2, \ldots$ lie in $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{\mathcal{A}})$ and therefore the functions $W(\|x_k(\tau) - x(\tau)\|_{\eta\xi})$ also lie in $Ad(\tilde{\mathcal{A}})_{wac} \cap BV(\tilde{A})$. Hence by Theorem 1.9.8 and Lemma 1.9.9 the integrals

$$\int_{s_1}^{s_2} W(\|x_k(s) - x(s)\|_{\eta\xi}) dh_{\eta\xi}|(s)$$

exist for every $k = 1, 2, \ldots$.

Because every process $X : [t_0, T] \to \tilde{\mathcal{A}}$ in $L^2_{loc}(\tilde{\mathcal{A}})$ of bounded variation is the uniform limit of finite simple processes, by (6.3.9) and Theorem 1.9.7, we obtain the inequality

$$\left| \int_{s_1}^{s_2} D[F_k(x_k(\tau), t)(\eta, \xi) - F_k(x(\tau), t)(\eta, \xi)] \right|$$

$$\leq \int_{s_1}^{s_2} W(\|x_k(s) - x(s)\|_{\eta\xi}) dh_{\eta\xi}(s) \qquad (6.3.10)$$

for every $s_1, s_2 \in [a, b]$ and $k = 1, 2, \ldots$

Moreover, (6.3.3) implies

$$\lim_{k\to\infty} W(\|x_k(s) - x(s)\|_{\eta\xi}) = 0, \quad s \in [a, b]$$

and we also have $0 \leq W(\|x_k(s) - x(s)\|_{\eta\xi}) \leq C_{\eta\xi,s}$ a constant for every $s \in [a, b]$. Hence by Theorem 1.9.2 of convergence theorem we obtain

$$\lim_{k\to\infty} \int_{s_1}^{s_2} W(\|x_k(s) - x(s)\|_{\eta\xi}) dh_{\eta\xi}(s) = 0.$$

and by (6.3.10) also

$$\lim_{k\to\infty} \int_{s_1}^{s_2} D[F_k(x_k(\tau), t)(\eta, \xi) - F_k(x(\tau), t)(\eta, \xi)] = 0.$$

This relation together with (6.3.8) yields (6.3.7) and this concludes the proof.

Remark. Theorem 6.3.1 is in a certain sense a weak form of continuous dependence results for the Kurzweil equation associated with the quantum stochastic differential equation introduced in chapter one. The most important assumption is the relation (6.3.1) which ensures that if a sequence of simple processes $x_k : [a,b] \to \tilde{\mathcal{A}}$ of solutions of (6.3.2), $k = 1, 2, \ldots$. Converges absolutely to a certain function $x : [a,b] \to \tilde{\mathcal{A}}$ then the limit is a solution of the equation (6.3.4). There are different additional conditions on the right hand sides F_k of (6.3.2) and F_0 of (6.3.4) in Theorem 6.3.1. Now we present a result with an additional uniqueness condition for the "limit" equation (6.3.4).

6.3.2 Theorem. Assume that the following hold:

(i) $F_k : G \to \text{Sesq}(I\!\!D \underline{\otimes} I\!\!E)$ is of class $\mathbb{F}(G, h_{\eta\xi}, W)$ for $k = 0, 1, \ldots.$

(ii) $\lim_{k\to\infty} F_k(x,t)(\eta,\xi) = F_0(x,t)(\eta,\xi), \quad (x,t) \in G$ (6.3.1)

(iii) $x : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subset [t_0, T]$ is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF_0(x,t)(\eta,\xi), \quad \text{on} \quad [a,b] \subset [t_0,T] \tag{6.3.4}$$

which has the following uniqueness property:

(a) If $x : [a,c] \to \tilde{\mathcal{A}}$, $[a,c] \subset [a,b]$ is a solution of (6.3.4) such that $y(a) = x(a)$ then $y(t) = x(t)$ for every $t \in [a,c]$.

(b) $e > 0$ such that if $s \in [a,b]$ and $|||y - x(s)\|_{\eta\xi} < e$ then

$(y,s) \in G = \tilde{\mathcal{A}} \times [a,b]$

(c) $y_k \in \tilde{\mathcal{A}}, k = 1, 2, \ldots$ satisfy

$$\lim_{k\to\infty} y_k = x(a).$$

Then for sufficiently large $k \in \mathbb{N}$ there exists a solution x_k of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF_k(x,t)(\eta,\xi) \tag{6.3.2}$$

on $[a,b]$ with $x_k(a) = y_k$ and

$$\lim_{k\to\infty} x_k(s) = x(s), \quad s \in [a,b].$$

Proof. By assumption we have $(y,a) \in G$ provided

$$\|y - x(a)\|_{\eta\xi} < \frac{e}{2^*}$$

or

$$\|y - x(a)\|_{\eta\xi} \;=\; |\langle \eta, y\xi\rangle - \langle \eta, x(a)\xi\rangle - F_0(x(a), a_1)(\eta,\xi) + F_0(x(a), a)(\eta,\xi)| < \frac{e}{2^*}$$

where $a_1 > a$.

Since $\langle \eta, y_k\xi\rangle \to \langle \eta, x(a)\xi\rangle$ for $k \to \infty$, we have by (6.3.1) also

$$\langle \eta, y_k\xi\rangle + F_k(y_k, a_1)(\eta,\xi) - F_k(y_k, a)(\eta,\xi) \to$$

$$\to \langle \eta, x(a)\xi\rangle + F_0(x(a), a_1)(\eta,\xi) - F_0(x(a), a)(\eta,\xi)$$

for $k \to \infty$ because

$$|F_k(y_k, a_1)(\eta,\xi) - F_k(x(a), a_1)(\eta,\xi) - F_k(y_k, a)(\eta,\xi) + F_k(x(a), a)(\eta,\xi)|$$

$$\leq W(\|y_k - x(a)\|_{\eta\xi})(h_{\eta\xi}(a_1) - h_{\eta\xi}(a))$$

and

$$F_k(x(a), a_1)(\eta,\xi) - F_k(x(a), a)(\eta,\xi) - F_0(x(a), a)(\eta,\xi) + F_0(x(a), a)(\eta,\xi) \to 0$$

for $k \to \infty$. Hence we can conclude that there is a $k_1 \in \mathbb{N}$ such that for $k > k_1$ we have $(y_k, a) \in G$ as well as

$$(y_k + F_k(y_k, a_1) - F_k(y_k, a)) \in G.$$

Similarly

$$(\langle \eta, y_k\xi\rangle, \langle \eta, a\xi\rangle) \in \mathbb{C}$$

as well as

$$(\langle \eta, y_k\xi\rangle + F_k(y_k, a_1)(\eta, \xi) - F_k(y_k, a)(\eta, \xi)) \in \mathbb{C}.$$

Let $d > a$, such that $t \in [a, b]$, and

$$|\langle \eta, x\xi\rangle - (\langle \eta, yk\xi\rangle + F_k(y_k, a_1)(\eta, \xi)) - F_k(y_k, a)(\eta, \xi)| \leq h_{\eta\xi}(t) - h_{\eta\xi}(a_1)$$

then $(x, t) \in G$ for $k > k_1$.

Using the result on existence of solution in chapter 4, we obtain that for $k > k_1$ there exists a solluiton $x_k : [a, d] \to \tilde{\mathcal{A}}$ of the Kurzweil equation (6.3.2) on $[a, d]$ such that $x_k(a) = y_k$, $k > k_1$. We claim that

$$\lim_{k\to\infty} x_k(t) = x(t) \text{ for } t \in [a, d].$$

Note that the solution x_k of (6.3.2) exist on the interval $[a, d]$ and that this interval is the same for all $k > k_1$.

By Theorem 6.3.1, if the sequence $\{x_k\}$ of simple processes contain absolutely convergent subsequence on $[a, d]$ then the limit of this subsequence is necessarily $x(t)$ for $t \in [a, d]$ by the uniqueness assumption on the solution x of (6.3.4). By lemma 1.9.10 the sequence $\{x_k\}, k > k_1$ of simple processes on $[a, d]$ belongs to $BV(\tilde{\mathcal{A}})$. Therefore, by Helly's choice Theorem the sequence contains a convergent subsequence and $x(t)$ is therefore the only accumulation point of the sequence $x_k(t)$ for every $t \in [a, d]$, i.e.

$$\lim_{k\to\infty} x_k(t) = x(t) \text{ for } t \in [a, d].$$

In this way we have shown that the theorem holds on $[a, d]$, $d > a$. Assume that the convergence result does not hold on the whole interval $[a, b]$. Then there exists $d^* \in (a, b)$ such that for every $d < d^*$ there is a solution x_k of equation (6.3.4) with $x_k(a) = y_k$ on $[a, d]$ provided $k \in \mathbb{N}$ is sufficiently large and $\lim_{k\to\infty} x_k(t) = x(t)$ for $t \in [a, d]$ but this does not hold on $[a, d]$ for $d > d^*$.

By Lemma 1.9.10 we have

$$\|x_k(t_2) - x_k(t_1)\|_{\eta\xi} \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|, \quad t_2 t_1 \in [a, d^*]$$

for $k \in \mathbb{N}$ sufficiently large. Therefore the limits $x_k(d^*)$ exist and since the solution x is continuous on $[a, b], [a, d] \subset [a, b]$, we obtain $\lim_{k\to\infty} x_k(d^*) = x(d^*)$ and this means that Theorem 6.3.2 holds on the closed interval $[a, d^*]$ too. Using now $d^* < b$ as the starting point we can show in the same way as above that the theorem holds also on the interval $[d^*, d^* + \triangle]$ with some $\triangle > 0$ and this contradicts our assumption. Therefore the theorem holds on the whole interval $[a, b]$.

Remark: Theorem 6.3.2 is derived from the result given in Theorem 6.3.1

Next, we use theorem 6.3.1 above to establish continuous dependence of solution for the quantum stochastic differential equation (QSDE) introduced in chapter one. The relationship between equation (6.2.2) and equation (6.2.4) is summarized as follows. If $F_k(x, t)(\eta, \xi)$ is a sequence of sesquilinear form and

$$F_k(x, t)(\eta, \xi) = \int_{t_0}^{t} P_k(x, s)(\eta, \xi)ds \tag{6.3.11}$$

since the integrals $\int_{t_0}^{t} P_k(x, s)(\eta, \xi)ds$ and the maps $(x, t) \to F_k(x, t)(\eta, \xi)$ are almost identical for every $x \in \tilde{\mathcal{A}}, s, t \in [t_0, T]$ we have from Theorem 1.10.3 that every solution of

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle = P_k(x, t)(\eta, \xi)$$

is at the same time a solution of

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF_k(x, t)(\eta, \xi)$$

and conversely.

Assume that $[t_0, T] \times Q$ is a compact neighbourhood of $[t_0, T] \times \tilde{\mathcal{A}}$, where

$$[t_0, T] \times Q \subseteq [t_0, T] \times \tilde{\mathcal{A}},$$

and z_0 is an accumulation point of Q. Assume further that the map $(x, t) \longrightarrow P(x, t)(\eta, \xi)$ is of class $C([t_0, T] \times \tilde{\mathcal{A}}, W)$, for $(x, t) \in [t_0, T] \times \tilde{\mathcal{A}}$ and $\eta, \xi \in I\!D \underline{\otimes} I\!E$

is arbitrary.

6.3.1 Definition Assume that the

map $P : \tilde{\mathcal{A}} \times [t_0, T] \times Q \to sesq[\mathbb{D}\underline{\otimes}\mathbb{E}]$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ satisfies the following conditions:

(i) $P(x, ., z)(\eta, \xi)$ is measurable for $(x, z) \in \tilde{\mathcal{A}} \times Q$,

(ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ such that $\int_{t_0}^{t} M_{\eta\xi}(s)ds < \infty$ and $|P(x, ., z)(\eta, \xi)| \leqslant M_{\eta\xi}(s), \quad (x, s, z) \in \tilde{\mathcal{A}} \times [t_0, T] \times Q$

(iii) There exists measurable functions $K_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ such that for each $t \in [t_0, T], \quad \int_{t_0}^{t} K_{\eta\xi}(s)ds < \infty$, and

$$|P(x, s, z)(\eta, \xi) - P(y, s, z)(\eta, \xi)| \leq K^{p}_{\eta\xi}(s)W(\|x - y\|_{\eta\xi})$$

For $(x, s, z), (y, s, z) \in \tilde{\mathcal{A}} \times [t_0, T] \times Q$ and where from (i) - (iii) $W(t) \neq t$ and

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K_{\eta\xi}(s)ds$$

6.3.2 Definition We define

$$F_k(x, t, z)(\eta, \xi) = \int_{t_0}^{t} P_k(x, s, z)(\eta, \xi)ds \tag{6.3.12}$$

for $(x, t, z) \in \tilde{\mathcal{A}} \times [t_0, T] \times Q$

6.3.3 Theorem. Assume that for some $c \in [a, b]$ we have

$$\text{(i)} \ \lim_{z \to z_0} \int_{c}^{t} P(x, s, z)(\eta, \xi)ds = \int_{c}^{t} P(x, s, z_0)(\eta, \xi)ds \tag{6.3.13}$$

for $(x, s, z) \in \tilde{\mathcal{A}} \times [a, b] \times Q$

(ii) Let $x(t, z) : [a, b] \times Q \longrightarrow \tilde{\mathcal{A}}, z \neq z_0$ be a solution of

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(x, t, z)(\eta, \xi) \tag{6.3.14}$$

on $[a,b] \subset [t_0, T]$, such that

$$\text{(iii)} \quad \lim_{z \to z_0} x(t,z) = y(t), \quad t \in [a,b], \quad y \in \tilde{\mathcal{A}} \tag{6.3.15}$$

Then $y : [a,b] \longrightarrow \tilde{\mathcal{A}}$ is a solution of

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(x,t,z_0)(\eta,\xi) \tag{6.3.16}$$

on $[a,b]$.

Proof.

By the hypothesis above, Theorem 1.10.1 yields that the map $(x,t) \to F(x,t,z)(\eta,\xi)$ given by (6.3.12) is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0,T], h_{\eta\xi}, W)$ for all $z \in Q$ where

$$h_{\eta\xi}(t) = \int_c^t K^p_{\eta\xi}(s)ds + \int_c^t M_{\eta\xi}(s)ds$$

$s, t \in [a,b]$.

The relation (6.3.13) can be written in the form

$$\lim_{z \to z_0} F(x,t,z)(\eta,\xi) = F(x,t,z_0)(\eta,\xi)$$

when (6.3.12) is taken into account. By Theorem 1.10.3, equation (6.3.14) has the same set of solutions as the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi \rangle = DF(x,t,z)(\eta,\xi) \tag{6.3.17}$$

for all $z \in Q, t, \tau \in [a,b] \subset [t_0,T]$.

Consequently, using (6.3.15) and Theorem 6.3.1 we obtain that the stochastic process $y : [a,b] \longrightarrow \tilde{\mathcal{A}}$ is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi \rangle = DF(x,t,z_0)(\eta,\xi).$$

Therefore by theorem 1.10.3 again y is a solution of (6.3.16) on $[a,b]$ and this proves the theorem.

Remark. Theorem 6.3.3 is a corollary of continuous dependence results for the Kurzweil equation associated with QSDE. It represents, continuous dependence theorem for the non classical ordinary differential equation introduced in chapter one under the relatively weak "$integralcontinuity$" assumption represented by equation (6.3.13).

Chapter 7

Summary, Conclusion and Recommendations

7.1 Introduction

In this chapter, a summary on the findings of the rearch work is presented. The outstanding contributions to knowlegde are also discussed. Recommendations on the proposed application of the technique of topological dynamics to the study of QSDEs and further research are suggested.

7.2 Summary and Conclusion

The main objective of this research work is to establish existence and uniqueness of solution of Kurzweil equation associated with the quantum stochastic differential equations (QSDEs) that satisfy a more general Lipschitz condition and to establish a basis for the application of the technique of topological dynamics. Hence, we also studied several kinds of stability viz; variational stability, relationship between variational attracting and variational asymptotic stability, converse Lyapunov type results and continuous dependence of solution on parameters. The motivation for studying this class of equations is to create a frame work for the technique of topological dynamics to be applicable in quantum stochastic differential equations as in the classical setting [4, 51-53, 87].

For existence of solution, we reviewed several results on QSDEs within the frame work of the Hudson and Parthasarathy formulation of QSDEs [3, 11, 12, 14, 15-18, 30, 44]. This is very important since we are extending the results in [6, 30] to a class of equation that satisfy a general Lipschitz condition. For variational stability and continuous dependence of solution on parameters, we reviewed several other results within the context of classical Kurzweil equations associated with ordinary differential equations (ODEs). This allows for extension of these methods to the present non-commutative quantum setting and therefore the background knowledge is necessary for investigating other qualitative properties of solution of QSDEs.

We have established the existence of a unique solution for a class of equation that satisfy a general Lipschitz condition. The technique of investigation involves the application of the method of successive approximations used in Ayoola and Ekhaguere [7, 30]. This method guaranteed the study of existence and uniqueness of solution for the map P that satisfy the conditions of the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, with $W(t) \neq t$ instead of $W(t) = t$.

The existence of solution for the Kurzweil equation associated with the QSDE was established using the equivalent form of the Hudson and Patharsarathy's formulation. This is possible since the equivalence of the Kurzweil equation and the associated QSDE has been established in [6] independent of the Lipschitz condition $W(t) = t$. The result on existences of solution generalizes the result in [6], so that the result in [6] becomes a special case of this result.

We also established results on all kinds of variational stability of solution of the perturbed and unperturbed differential equation. Although, unlike variational stability of ordinary differential equations which have been investigated by various authors, not much has been done in quantum stochastic differential equations concerning variational stability atleast within the consulted literatures. Hence this is quite a new approach for stability of solution of non commutative QSDE (1.5) introduced in chapter one. The study of variational stability is very effective when studying dynamical systems. The Lyapunov method was employed to investigate the stability of the solution without knowing the exact solution of the given differential equation. The Lyapunov's method makes use of a real- valued function to establish stability results. This is guaranteed here, since the complex field $\mathbb{C} \cong \mathbb{R}^2$. Every other conditions such as continuity of the Lyapunov function is also guaranteed because the stochastic processes are adapted, weakly and absolutely continuous.

Variational stability guaranteed that any solution of the Kurzweil equation that starts near $x = 0$ remains close to it in the future, while variational asymptotic stability implies that the solution converges to zero in the future. The converse variational stability guaranteed the existence of a Lyapunov function when the solution is variationally stable. Lastly we established results on continuous dependence of solution on parameters. These results show the convergence effect of QSDE (1.5) when it depends on a parameter. With This method, we were able to obtain results on existence of solution independent of any Lipschitz condition. This we were able to achieve because the stochastic processes are simple processes and also of bounded variation.

The conclusion is that existence of solution will not only depend on the Lipschitz condition $W(t) = t$ but on a more general condition $W(t) \neq t$. Also the results on variational stability and continuous dependence of solution on parameters have provided a basis for the application of the technique of topological in quantum stochastic differential equation and the associated Kurzweil equations.

7.3 Outstanding contribution to knowledge

(1) Existence of solution of Kurzweil equation associated with quantum stochastic differential equations has been extended to a class of equation that is not restricted to the Lipschitz condition. This will subsequently widen the solution space of quan-

tum stochastic differential equation especially for the class of equations that will fail to satisfy the Lipschitz condition.

(2) The results on variational stability, asymptotic variational stability and continuous dependence of solution on parameters of equation (1.5) have not been considered before now. This is the first time such results will be establihed.

(3) The theory of Kurzweil equations associated with quantum stochastic differential equation provides a basis for future application of the technique of topological dynamics to the study of quantum stochastic differential equation as in classical cases.

7.4 Practical applications of QSDEs

The nature of occurrence of events in the world is chaotic, unpredictable hence the need to use stochastic differential equations to model real life problems becomes imperative [60, 64, 89]. Stochastic differential equations have found many real life applications viz: medicine, engineering, psychology, economics, stock markets, conflict management, etc. See the references [11,34] for more on practical applications.

7.5 Recommendations

The following are possible areas for further investigation of quantum stochastic differential equations and the associated Kurzweil equation.

1 The possibility of investigating the existence of solution of the Kurzweil equation independent of the associated QSDE as in the case of classical differential equations.

2 The possibility of investigating other analytical properties of solution of the Kurzweil equation associated with QSDE, such as averaging for generalized

QSDEs, Maximal solutions, etc.

3 Investigating the convergence effect that occurs when the right hand side of the equations converge to other functions that are not necessarily QSDEs like in the case of classical ODE where it converges to a Dirac function.

4 Extending the concept of measure differential equations to the present non commutative quantum setting for systems that exhibit discontinuous solutions caused by the impulsive behaviour of the differential system.

5 Establishing convergence schemes for QSDEs that will depend on the general Lipschitz condition established here.

7.6 Appendix A

Statement of Helly's selection theorem

Let X be a separable Hilbert space and let $\mathrm{Reg}([0,T];X)$ denote the space of regulated functions $f : [0,T] \to X$, equipped with the supremum norm. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Reg}([0,T];X)$ satisfying the following condition: for every $\epsilon > 0$, there exists some $L_\epsilon > 0$ so that each f_n may be approximated by a $u_n \in \mathrm{BV}([0,T];X)$ satisfying

$$\|f_n - u_n\|_\infty < \epsilon$$

and

$$|u_n(0)| + Var(u_n) \leq L_\epsilon,$$

where $|.|$ is defined to be the norm in X and $\mathrm{Var}(u)$ denotes the variation of u, which is defined to be the supremum

$$\sup_\Pi \Sigma_{j=1}^m |u(t_j) - u(t_{j-1})|$$

over all partitions

$$\Pi = 0 = t_0 < t_1 < ... < t_m = T, m \in \mathbb{N}$$

of $[0,T]$. Then there exists a subsequence

$$(f_{n(k)}) \subseteq (f_n) \subset \mathrm{Reg}([0,T];X)$$

and a limit function $f \in \mathrm{Reg}([0,T];X)$ such that $f_{n(k)}(t)$ converges weakly in X to $f(t)$ for every $t \in [0,T]$. That is, for every continuous linear functional $\lambda \in X^*$,

$$\lambda(f_{n(k)}) \to \lambda(f(t))$$

in $\mathbb{R}$ as $k \to \infty$.

7.7 Appendix B

Statement of Gronwall's inequality

Let I denote an interval of the real line of any of the form $[a,\infty), [a,b], [a,b)$ with $a < b$. Let α and u be measurable functions defined on I and let μ be a local finite measure on the Borel σ-algebra of I(we need $\mu([a,t]) < \infty$, for all t in I). Assume that u is integrable with respect to μ in the sense that

$$\int_a^t |u(s)\mu(ds) < \infty, \quad t \in I,$$

and that u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_{[a,t)} u(s)\mu(ds), \quad t \in I.$$

If, in addition,

(i) the function α is non-negative or

(ii) the function $t \to \alpha([a,t])$ is continuous for t in I and the function α is integrable with respect to μ in the sense that

$$\int_a^t |u(s)\mu(ds) < \infty, \quad t \in I,$$

then u satisfies the Gronwall's inequality

$$u(t) \leq \alpha(t) + \int_{[a,t)} \alpha(s)\exp(\mu(I_{s,t}))\mu(ds)$$

for all t in I, where $I_{s,t}$ denotes the open interval (s,t).

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