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Solving General Second Order Ordinary Differential Equations by a One-Step Hybrid Collocation Method

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ABSTRACT

A one-step hybrid method is developed for the numerical approximation of second order initial value problems of ordinary differential equations by interpolation and collocation at nonstop and step points respectively. The method is zero stable and consistent with very small error term. Numerical experiment of the method on sample problem shows that the method is more efficient and accurate than the results obtained from our earlier methods.

Keywords: *Consistency, hybrid, nonstep, approximation, interpolation, block method*

AMS Subject Classification: 65L05, 65L06, 65L12

1. INTRODUCTION

The direct integration of higher order initial value problem of ordinary differential equations has been widely discussed in the literature [2, 10,11]. There is in fact, a consensus that this method of solution is more convenient and accurate than the reduction order method; which implies that the problem will be reduced to a system of first order problems and solved with suitable first order methods.

Phenomena arising from physical sciences, engineering, economics, etc, are mostly modeled as initial value problems of ordinary differentials. Very often, these models do not have closed form solutions which necessitates the development of numerical methods to approximate their solutions. Indeed, a number of methods have been proposed in the literature, [6,7, 8,12,13].

The method proposed in this paper, is an extension of our previous results; where we used one and two nonstep points respectively, to augment a one-step scheme for the solution of second order initial value problems of ordinary differential equations. Here, three nonstep will be incorporated in the development of the new one-step scheme. The resulting scheme allow function in evaluations at nonstep points like the Runge-kutta methods however, with fewer number of function evaluations per step. The method is implemented as a simultaneous integrator using a modified block method that generates values at a block of points and their derivative values as well. No other method is required to generate staring values for the integration process.

The paper is arranged as follows; in section two we present the derivation and specification of the method followed by the analysis of the method in section three. In section four, the method is experimented on sample problems and the results obtained are compared with our results in terms of the global error and time of computation.(ODE) of the form

2. DERIVATION AND SPECIFICATION OF THE METHOD

In this section, a representation of a continuous one-step method is derived. In the sequel, the main method and other methods required to set up the block method will be generated.

Suppose we approximate the analytical solution of problem

$$\begin{cases} y''=f(x,y,y') & x\in[a,b] \\ y(a)=\zeta_0,\ y'(a)=\zeta_1 \end{cases} \qquad (1)$$

Where f is continuous in $[a,b]$, by a power series polynomial of the form

$$Y(x)=\sum_{j=0}^{m} a_i t^i \qquad (2)$$

on the partition $\Delta_N : a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$ of the integration interval [a, b], with a constant step size h, given by $h = x_{n+1} - x_n;\ n = 0,1,\cdots,N-1$

By carefully choosing and incorporating three nonstep points, $x_{n+u}, x_{n+v}, x_{n+w} \in \{x_n, x_{n+1}\}:\quad u,v,w\in(0,1)$ in such a manner that the zero stability of the main method is guaranteed. We are able to interpolate (2), the Stormer-Cowell way, at a sufficient number of points namely: $x_{n+i},\ i=u,v,w,$ to achieve our purpose and collocate (2) at $x_{n+i},\ i=0,u,v,w,1.$

This way, we obtain a system of seven equations each degree at most six,(i.e m=6), as follows:

$$\sum_{j=0}^{6} a_j x_{n+i}^{j} = y_{n+i}, \qquad i=v,w \qquad (3)$$

$$\sum_{j=0}^{6} j(j-1)a_j x_{n+i}^{j-2} = f_{n+i}, i=0,u,v,w,1 \qquad (4)$$

The system of equations $(3)-(4)$ is solved for the unknown parameters $a_j;\ j=0,1,\ldots,6$. Now substituting these

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values into (2) yields a representation of a continuous implicit one-step hybrid method in the form:

$$Y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} +h^2\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}+\beta_{\frac{1}{2}}f_{n+\frac{1}{2}}\right] \qquad (5)$$

where $i=u,v,w$; $\alpha_v(x),\alpha_w(x)$ and $\beta_j(x)$ are continuous coefficients, $y_{n+j}=y(x_n+jh)$ is the numerical approximation of the analytical solution at x_{n+j} and $f_{n+j}=f\left(x_{n+j},y_{n+j},y'_{n+j}\right)$.

Now if we let $u=\frac{1}{4},v=\frac{1}{2},w=\frac{3}{4}$ and evaluate (5) at $x=x_{n+l},l=0,u,1,$ we obtain three discrete methods as follows:

$$2y_{n+\frac{3}{4}}-3y_{n+\frac{1}{2}}+y_n=\frac{h^2}{3840}[-3f_{n+1} +52f_{n+\frac{3}{4}}+402f_{n+\frac{1}{2}}+252f_{n+\frac{1}{4}}+17f_n \qquad (6)$$

$$y_{n+\frac{3}{4}}-2y_{n+\frac{1}{2}}+y_{n+\frac{1}{4}}=-\frac{h^2}{3840}[f_{n+1} -24f_{n+\frac{3}{4}}-194f_{n+\frac{1}{2}}-24f_{n+\frac{1}{4}}+f_n \qquad (7)$$

$$y_{n+1}-2y_{n+\frac{3}{4}}+y_{n+\frac{1}{2}}=\frac{h^2}{3840}[19f_{n+1} 204f_{n+\frac{3}{4}}-14f_{n+\frac{1}{2}}+4f_{n+\frac{1}{4}}-f_n \qquad (8)$$

Our block method is set up by obtaining additional equations from evaluating the first derivative of (5);

$$hY'(x)=\alpha_0'(x)y_n+\alpha_{\frac{1}{2}}'(x)y_{n+\frac{1}{2}} +h^2\left[\sum_{j=0}^{1}\beta_j'(x)f_{n+j}+\beta_{\frac{1}{2}}'f_{n+\frac{1}{2}}\right] \qquad (9)$$

at $x=x_{n+l},l=0,u,v,w,1$ respectively. This yields five discrete derivative schemes as follows:

$$hy'_n-4y_{n+\frac{3}{4}}+4y_{n+\frac{1}{2}}=\frac{h^2}{5760}\left[33f_{n+1}-284f_{n+\frac{3}{4}}-966f_{n+\frac{1}{2}}-1908f_{n+\frac{1}{4}}-475f_n\right] \quad (10)$$

$$hy'_{n+\frac{1}{4}}-4y_{n+\frac{3}{4}}+4y_{n+\frac{1}{2}}=\frac{h^2}{5760}\left[-5f_{n+1}-72f_{n+\frac{3}{4}}-1494f_{n+\frac{1}{2}}-616f_{n+\frac{1}{4}}+27f_n\right] \quad (11)$$

$$hy'_{n+\frac{1}{2}}-4y_{n+\frac{3}{4}}+4y_{n+\frac{1}{2}}=\frac{h^2}{5760}\left[17f_{n+1}+220f_{n+\frac{3}{4}}-582f_{n+\frac{1}{2}}+76f_{n+\frac{1}{4}}-11f_n\right] \quad (12)$$

$$hy'_{n+\frac{1}{2}}-4y_{n+\frac{3}{4}}+4y_{n+\frac{1}{2}}=\frac{h^2}{5760}\left[-21f_{n+1}+472f_{n+\frac{3}{4}}+330f_{n+\frac{1}{2}}-72f_{n+\frac{1}{4}}+11f_n\right] \quad (13)$$

$$hy'_{n+1}-4y_{n+\frac{3}{4}}+4y_{n+\frac{1}{2}}=\frac{h^2}{5760}\left[481f_{n+1}+1764f_{n+\frac{3}{4}}-198f_{n+\frac{1}{2}}+140f_{n+\frac{1}{4}}-27f_n\right] \quad (14)$$

3. BLOCK METHOD

The definition of the block method, adopted for the implementation of our scheme is a modification of the one in 11. The modified definition vector notation is given, as:

$$h^{\lambda}\overline{a}Y_m=h^{\lambda}\overline{E}y_m+h^{\mu-\lambda}\left[\overline{d}F(y_m)+\overline{b}F(Y_m)\right] \qquad (15)$$

where $\overline{a},\overline{b},\overline{c},\overline{d}$ are constant coefficient matrices;

$$h^{\lambda}Y_m=\left(y_{n+\frac{1}{4}},y_{n+\frac{1}{2}},y_{n+\frac{3}{4}},y_{n+1},hy'_{n+\frac{1}{4}},hy'_{n+\frac{1}{2}},hy'_{n+\frac{3}{4}},hy'_{n+1}\right)^T$$

$$h^{\lambda}y_m=\left(y_{n-1},y_n,hy'_{n-1},hy'_n\right)^T,$$

$$F(Y_m)=\left(f_{n+\frac{1}{4}},f_{n+\frac{1}{2}},f_{n+\frac{3}{4}},f_{n+1}\right)^T, \text{ and}$$

$$F(y_m)=\left(f_{n-\frac{3}{4}},f_{n-\frac{1}{2}},f_{n-\frac{1}{4}},f_n\right),$$

λ is the power of the derivative in (9) and μ is the order of problem.

To set up our block method, (6-8) is combined with equations (10) – (14) with the constant coefficient matrices $\overline{a},\overline{b},\overline{c},\overline{d}$ are obtained as follows

$$\begin{bmatrix} 0 & -11520 & 7680 & 0 & 0 & 0 & 0 & 0 \\ 3840 & -7680 & 3840 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3840 & -7680 & 3840 & 0 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 5760 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 5760 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 5760 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 0 & 5760 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3840 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5760 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 17 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -475 \\ 0 & 0 & 0 & 27 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & -27 \end{bmatrix}$$

$$\begin{bmatrix} 252 & 24 & 4 & 1908 & -616 & 76 & -72 & 140 \\ 402 & 194 & 204 & -966 & -1494 & -582 & 330 & -198 \\ 52 & 24 & 204 & -284 & -72 & -220 & 472 & 1764 \\ -3 & -1 & 19 & 33 & -5 & 17 & -21 & 481 \end{bmatrix}$$

Normalizing (15) yields the block solution:

$$h^{\lambda}\bar{A}Y_m=h^{\lambda}\bar{E}y_m+h^{\mu-\lambda}[\bar{D}f(y_m)+\bar{B}F(Y_m)] \qquad (16)$$

Where $\bar{A}$ is 8 x 8 identity matrix, $\bar{E}$, $\bar{D}$ and $\bar{B}$ are constant coefficient matrices.

A single application of the formula generates simultaneously, the approximate solutions and their derivatives at the points $x_{n+\frac{1}{4}},x_{n+\frac{1}{2}},x_{n+\frac{3}{4}},x_{n+1}$, as the following discrete schemes

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{h^2}{23040}\left[-21f_{n+1} + 116f_{n+\frac{3}{4}} - 282f_{n+\frac{1}{2}} + 540f_{n+\frac{1}{4}} + 365f_n\right] (17)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{h^2}{1440}\left[-3f_{n+1} + 16f_{n+\frac{3}{4}} - 30f_{n+\frac{1}{2}} + 144f_{n+\frac{1}{4}} + 53f_n\right] (18)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + \frac{h^2}{2560}\left[-3f_{n+1} + 20f_{n+\frac{3}{4}} + 18f_{n+\frac{1}{2}} + 156f_{n+\frac{1}{4}} + 49f_n\right] (19)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{90}\left[8f_{n+\frac{3}{4}} + 6f_{n+\frac{1}{2}} + 24f_{n+\frac{1}{4}} + 7f_n\right] (20)$$

$$y'_{n+\frac{1}{4}} = y'_n + \frac{h}{2880}\left[-19f_{n+1} + 106f_{n+\frac{3}{4}} - 264f_{n+\frac{1}{2}} + 646f_{n+\frac{1}{4}} + 251f_n\right] (21)$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{h}{360}\left[-f_{n+1} + 4f_{n+\frac{3}{4}} + 24f_{n+\frac{1}{2}} + 124f_{n+\frac{1}{4}} + 29f_n\right] (22)$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{h}{320}\left[-3f_{n+1} + 42f_{n+\frac{3}{4}} + 72f_{n+\frac{1}{2}} + 102f_{n+\frac{1}{4}} + 27f_n\right] (23)$$

$$y'_{n+1} = y'_n + \frac{h}{90}\left[7f_{n+1} + 32f_{n+\frac{3}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{1}{4}} + 7f_n\right] (24)$$

The one-step block method is implemented as a simultaneous integrator, without requiring other methods to supply starting values nor for the development of predictors, over the subintervals, $[x_0, x_1], \ldots, [x_{N-1}, x_N]$ of the partition $\Delta_N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. This way, the initial conditions are obtained at $x_{n+1},\ n = 0,1,\ldots,N-1$.

4. ANALYSIS OF THE BLOCK METHOD

In this section, fundamental properties of the one-step block method are discussed.

4.1 Order and Error Constant

In what follows, we will define, in the spirit of Awoyemi, et al [19], the linear difference operator associated with the one-step block method with some modifications. We will proceed by first of all, recasting (15) as:

$$\sum_{ij} \bar{\alpha}_{ij}^{\lambda} y_{n+j}^{\lambda} = h^2 \sum_{j} \bar{\beta}_{ij}^{\lambda} f_{n+j} \qquad (25)$$

where $i, j = 0, u, v, w, 1$ and λ is the degree of the derivative in (9).

Definition 1: *The linear difference operator L assotiociated with (15) is defined as:*

$$L\left[y(x);h\right] = \sum_{ij}\left[\bar{\alpha}_{ij}^{\lambda} y(x_n + jh) - h^2 \bar{\beta}_{ij}^{\lambda} y''(x_n + jh)\right] (26)$$

where $i, j = 0, u, v, w, 1;\ y(x)$ *is an arbitrary test function which is continuously differentiable on [a,b].* Expanding $y(x_n + jh)$ and $y''(x_n + jh)$ in Taylor series and collecting like terms in powers of h yields the linear equation:

$$L[y(x);h] = \bar{C}_0 y(x) + \bar{C}_1 h y^{(1)}(x) + \ldots + \bar{C}_p h^p y^{(p)}(x)$$
$$+ \bar{C}_{p+1} h^{p+1} y^{(p+1)}(x) + \bar{C}_{p+2} h^{p+2} y^{(p+2)}(x) + \ldots \quad (27)$$

where they, $\bar{C}_i;\ i = 0,1,\ldots$ are vectors.

Definition 2: *The one-step block method (15) and the associated linear difference operator (15) are said to have order p if* $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_{p+1} = 0$ *and* $\bar{C}_{p+2} \neq 0$.

Definition 3: *The term* $\bar{C}_{p+2}$ *called the error constant implies that the one-step block method (15) has local truncation error given by*

$$t_{n+k} = \bar{C}_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3}) \qquad (28)$$

Hence the block method from our calculation, is of order $p = (5,5,5,5,5,5,5,5)^T$ and has error constant $\bar{C}_{p+2} = \left(\frac{26561}{41287680000}, \frac{1}{645120}, \frac{9}{3670016}, \frac{123}{819200}, \frac{1}{368640}, \frac{3}{655360}, -\frac{7}{1350}\right)^T$.

4.2 Zero Stability, Consistency and Convergence

Definition 4: *The one-step block method (16) is said to be zero stable as* $h \to 0$ *if the first characteristic polynomial* $\bar{\rho}(z)$ *, of (16) satisfies*

$$\bar{\rho}(z) = \det\left[z\bar{A} - \bar{E}\right]$$
$$= z^{r-\mu}(z-1)^{\mu} \qquad (29)$$
$$= 0$$

Where r is the order of the matrices $\bar{A}, \bar{E}$; *and the roots* $z_s,\ s = 1,\ldots,8$ *of (29) satisfy the condition* $|z_s| \le 1$. *Furthermore, those roots with* $|z_s| = 1$ *have multiplicity not exceeding two.*

Applying definition 4 to our one-step block method with $r = 8$ and $\mu = 2$, $\bar{A}$ is a 4×4 identity matrix yields ,

$$\bar{\rho}(z) = z^6 (z-1)^2 = 0$$

Clearly, the conditions of (29) are satisfied hence, the method is zero stable.
The consistency of the method follows from the fact that the order of the block method is greater than one.
Following [9], our method is also convergent.

4.3 Region of Absolute Stability of the Block Method

The stability polynomial of our one-step block method is obtained applying the scalar test problem

$$y'' = -\lambda^2 y \qquad (30)$$

To the block formula (15), in the spirit of (2), such that

$$Y_m = W(\bar{h}) y_m \qquad (31)$$

Where $\bar{h} = \lambda^2 h^2$ and

$$W(\bar{h}) = (\bar{a} - \overline{hb})^{-1}(\bar{c} + \overline{hd})$$

is called the amplification matrix.

Definition 5: *The interval* $(0, \bar{h}_0)$ *of the real line is said to be the interval of absolute stability if in this interval* $\rho(\bar{h}) < 1$, *where* $\rho(\bar{h})$ *is the spectral radius of* $W(\bar{h})$ *[see [5]].*

Our block method is found to satisfy the condition $\rho(\bar{h}) < 1$, if $\bar{h} \in (-806.86, 0)$.

5. NUMERICAL EXAMPLE

In this section, the efficiency and accuracy of our one-step method implemented as a block method is tested on some numerical examples. The absolute errors computed are compared with those obtained in [18], which used a numerical scheme implemented in the predictor corrector mode. Each of the following examples is tested using step size h=1/320. The tables of results of the problems given in Tables 1,2 and 3 respectively, are obtained from a FORTRAN 95 program using a fixed step size; h= 1/320 for computation.

Problem 1:

$$y'' - x(y')^2 = 0;\ y(0) = 1, y'(0) = \frac{1}{2}$$

Theoretical solution:

$$y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

Source: [3]

Problem 2:

$$xy'' - x + 3y' - \frac{3y}{x};\ y(1) = 2,\ y'(1) = 10$$

Theoretical Solution:

$$y = 3x^3 - 2x + x^2(1 + x\ln x)$$

Source: [2]

Problem 3:

$$y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0;\ y(1) = y'(1) = 1$$

Theoretical Solution

$$\frac{5}{3x} - \frac{3}{3x^4}$$

Source [5]

6. DISCUSSION

By extending an earlier result, we have improved on the performance of our one-step method developed by the interpolation and collocation technique with the incorporation of o_step points for the approximation of the solutions of initial value problems of general second order ordinary differential equations of the form (1). Implementing the new scheme by the block method allows the generation of solutions at different grid points simultaneously, in a single application of the method.

Three test problems solved by [2], [3] using a numerical scheme developed in the predictor corrector mode; and also by [5] and [1] implemented in the block mode, have been solved here using the new method. The absolute errors generated by the new method, as reported in Tables 1, 2 and 3, show clearly that by increasing the number of the non-step points from one to three, the accuracy of our new method improved. Furthermore, this new order five method yielded a very small truncation error term and enjoys a wide interval of absolute stability.

We conclude that the new one-step method is accurate, reliable and efficient. Thus, we recommend the new method for numerical approximation of the solutions of, mathematical models describing phenomena in science and engineering in the form of, higher order initial value problems of ordinary differential equations.

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