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COMPACT FINITE DIFFERENCE SCHEMES FOR POISSON EQUATION USING DIRECT SOLVER

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ABSTRACT

The Compact Finite Difference Schemes for the solution of one, two and three dimensional Poisson equation is considered in this paper. The discretization using the truncation errors of the Taylor's series method are of $o\left(h^4\right)$ and $o\left(h^6\right)$. In the one and two dimension cases, the stencils are of 9-point. In the three dimension case, the $o\left(h^4\right)$ scheme is 19-point stencil and the $o\left(h^6\right)$ scheme is 27-point. Two numerical experiments were conducted and the results confirm that Compact Finite Difference Schemes are accurate and efficient methods.

Key words: Poisson equation, compact difference scheme, Taylor's series method, LU decomposition.

1. INTRODUCTION

The efforts to compute more accurate solution using limited grid sizes have directed researchers' attention to developing high-order compact finite difference schemes. Compact difference schemes are high-order implicit methods which feature higher-order accuracy and spectral like resolution with smaller stencils. In the past two decades several strategies have been devised for the construction of compact difference schemes. The schemes include: the Taylor series method, Pade approximation methods and Birkhoff interpolation methods. The original Pade methods go back to the 1950s. The Taylor's series methods were popularised in the 1990s and the interpolation methods were recently presented in a paper on the derivation of high-order compact schemes for non-uniform grids [1]. The methods of compact difference have been used widely in the large area of computational problems, for example, the convergence and solution for the compact difference method on parabolic equations were discussed in [1, 2, 3, 4, and 5]. There were also some works on applying the compact difference scheme for steady convection-diffusion problem [6, 7], the Helmholtz equations [8, 9, and 10] and the hyperbolic equation [11, 12].

In this paper, we present the 9-point compact schemes of $o\left(h^4\right)$ and $o\left(h^6\right)$ for 1D and 2D Poisson equations. A high-order compact scheme for 3 dimensional Poisson equation of $o\left(h^4\right)$ and $o\left(h^6\right)$ is also presented. They are of 19-point and 27-point respectively [13].The schemes lead to a large system of linear equations $Ax = b$, where A is sparse. A MATLAB direct solver using LU decomposition is implemented for the computation of the numerical examples.

2. FORMULATION OF HIGH-ORDER COMPACT SCHEMES

The archetypal elliptic equation in spatial dimensions is represented by the Poisson equation. Here, we develop schemes for Poisson equation for one, two and three dimensional uniform grids on a structured grid of uniform mesh size $\Delta x = \Delta y = \Delta z = h$. First, let us introduce the following notations: [see 14].

$$\delta_+ u_i = \frac{u_{i+1} - u_i}{h} = \delta_{x+} \quad and \quad \delta_- u_i = \frac{u_i - u_{i-1}}{h} = \delta_{x-}$$

denote the standard forward finite difference and backward finite difference schemes for first derivative. Also,

$$\delta_0 u_i = \frac{1}{2}\left(\delta_+ u_i + \delta_- u_i\right) = \frac{u_{i+1} - u_{i-1}}{2h} \qquad (1)$$

is the first-order central finite difference with respect to x where $u_i = u\left(x_i\right)$. The standard second –order central finite difference is denoted as $\delta_X^2 u_i$ and is defined as

$$\delta_+\delta_- u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \frac{\delta_+ - \delta_-}{h}$$

(2)

Difference operators $\delta_o y, \delta_o z, \delta^2 y\, and\, \delta^2 z$ are defined similarly. By using the Taylor's series expansion, a fourth and sixth order accurate finite difference for the first and second derivatives can be approximated as follows:

$$\delta_o u = \frac{du}{dx} + \frac{h^2}{3!}\frac{d^3 y}{dx^3} = \left(1 + \frac{h^2}{6}\frac{d^2}{dx^2}\right)\frac{du}{dx} = \left(1 + \frac{h^2}{6}\delta^2\right)\frac{du}{dx} + o\left(h^4\right)$$

$$= \frac{du}{dx} + \frac{h^2}{3!}\frac{d^3 u}{dx^3} + \frac{h^4}{5!}\frac{d^5 u}{dx^5} = \left(1 + \frac{h^2}{6}\delta^2 + \frac{h^4}{120}\delta^4\right)\frac{du}{dx} + o\left(h^6\right) \quad (3)$$

We may rewrite equation (3) for $\delta_0 u$ as

$$\frac{du}{dx} = \left(1 + \frac{h^2}{6}\delta^2\right)^{-1}\delta_0 u + 0\left(h^4\right)$$

$$= \left(1 + \frac{h^2}{6}\delta^2 + \frac{h^4}{120}\delta^4\right)^{-1}\delta_0 u + o\left(h^6\right). \quad (4)$$

Also,

$$\delta_x^2 u = \frac{d^2 u}{dx^2} + \frac{h^2}{12}\frac{d^4 u}{dx^4} = \left(1 + \frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{d^2 u}{dx^2} = \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + o\left(h^4\right) \quad (5a)$$

$$= \frac{d^2 u}{dx^2} + \frac{h^2}{12}\frac{d^4 u}{dx^4} + \frac{h^4}{360}\frac{d^6 u}{dx^6} = \left(1 + \frac{h^2}{12}\delta^2 + \frac{h^4}{360}\delta^4\right)\delta^2 u + o\left(h^6\right) \quad (5b)$$

2.1 One – dimensional case

For an illustration purpose, we first consider the one – dimensional problem which can be represented as

$$u''(x) = f(x).\ x \in I, \qquad I = [0,1] \quad (6)$$

From equation (5a), the fourth order accurate finite difference estimate for $u''(x)$ is

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12}u_i^{(iv)} = \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + o\left(h^4\right) \quad (7)$$

The idea behind the higher order compact scheme is to approximate $u_i^{(iv)}$ in equation (7) to second order accuracy to achieve an overall truncation accuracy of fourth order. To this end, we simply double differentiate equation (6) to get

$$u_i^{(iv)}(x) = f''(x). \quad (8)$$

Also, applying the central difference scheme to $f''(x)$, we have

$$f''(x) = \delta_x^2 f_i + o\left(h^2\right) \quad (9)$$

Hence, from equation (7), we get

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12}\left(\delta_x^2 f_i + o\left(h^2\right)\right) + o\left(h^4\right)$$

or

$$u_i'' = \delta_x^2 u_i - \frac{h^2}{12}\delta_x^2 f_i + o\left(h^4\right) \quad (10)$$

Using this estimate and considering the discrete solution of equation (6) which satisfies the approximation, we get

$$\delta_x^2 u_i = f_i + \frac{h^2}{12} f_i'' + o\left(h^4\right) \tag{11}$$

where U_i is the discrete approximation to u_i satisfying the discrete formulation of equation (6), which implies $u_i = U_i + o\left(h^4\right)$. Using equation (2), equation (11) can be expressed in the form

$$U_{i+1} - 2U_i + U_{i-1} = \frac{h^2}{12}\left(f_{i+1} + 10 f_i + f_{i-1}\right) \tag{12}$$

For the sixth – order accurate finite difference estimate of equation (6), we have from equation (5b):

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12} u_i^{(iv)} + \frac{h^4}{360} u_i^{(vi)} + o\left(h^6\right) \tag{13}$$

Both $o\left(h^2\right) and\ o\left(h^4\right)$ are included in equation (13). We approximate both of them to construct an $o\left(h^6\right)$ scheme. Applying $\delta_x^2\ to\ u_i^{(iv)}$, we get

$$u_i^{(vi)} = \delta_x^2 u_i^{(iv)} + o(h^2). \tag{14}$$

Substituting equation (14) into equation (13) yields

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12} u_i^{(iv)} + \frac{h^4}{360}\left(\delta_x^2 u_i^{(iv)} + o\left(h^2\right)\right) + o\left(h^6\right). \tag{15}$$

To get the compact $o\left(h^6\right)$ approximation, we again apply equation (9) that is

$$u_i^{(iv)}(x) = f''(x), where\ \ f_i = f\left(x_i\right) and\ \ f_i'' = f''\left(x_i\right).$$

Inserting this equation into equation (15) results in

$$\delta_x^2 u_i = u_i'' + \left(\frac{h^2}{12} + \frac{h^4}{360}\delta_x^2\right) f_i''(x_i) + o\left(h^6\right)$$

Again,

$$u_i'' = \delta_x^2 u_i - \left(\frac{h^2}{12} + \frac{h^4}{360}\delta_x^2\right) f''(x_i) \tag{16}$$

Or from equation (6),

$$\delta_x^2 U_i = f_i + \left(\frac{h^2}{12} + \frac{h^4}{360}\delta_x^2\right) f''(x_i) \tag{17}$$

where U_i is the discrete approximation to u_i satisfying the discrete formulation of equation (6) which implies $u_i = U_i + o\left(h^6\right)$. The $o(h^6)$ approximation of equation (6) can be given as

$$U_{i+1} - 2U_i + U_{i-1} = \frac{h^2}{12}\left(f_{i+1} + 10 f_i + f_{i-1}\right) + \frac{h^4}{360}\left(f_{i+1}'' + f_{i-1}''\right) - \frac{h^4}{180} f_i''. \tag{18}$$

2.2 Two – Dimensional Case

Consider the two – dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f\left(x, y\right) \qquad for\, x, y \in \Omega.,\ \ \Omega = [0,1]\times[0,1]. \tag{19}$$

The central difference scheme for equation (19) in two – dimensions can be written as

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + \tau_{ij} = f_{ij} \tag{20}$$

where $u_{ij} = u\left(x_i, y_j\right)\ and\ \ f_{ij} = f\left(x_i, y_j\right)$

and

$$\tau_{ij} = -\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right]_{ij} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{ij} + o\left(h^6\right) \tag{21}$$

We need a compact $o\left(h^2\right)$ approximation of the first square bracket in equation (21). This can be done by taking the following approximate derivative of equation (19), to get

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2}, \quad \frac{\partial^4 u}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 u}{\partial x^2 \partial y^2}. \tag{22}$$

Substituting equation (22) into equation (21), we obtain the alternative form for the exact truncation error at node ij :

$$\tau_{ij} = -\frac{h^2}{12}\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{ij} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{ij} + o\left(h^6\right)$$

$$= \frac{h^2}{12}\left(-\nabla^2 f_{ij}\right) + \frac{h^2}{6}\left[\frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{ij} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{ij} + o\left(h^6\right). \tag{23}$$

In our derivation of the $o\left(h^4\right)$ scheme, we use equation (22) and the expressions of the first square bracket in equation (23) that is:

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} - \frac{h^2}{12}\nabla^2 f_{ij} + \frac{h^2}{6}\left[\frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{ij} = f_{ij} + o\left(h^4\right)$$

or

$$\nabla^2 u_{ij} + \frac{h^2}{6}\left[\delta_x^2 \delta_y^2\right] u_{ij} = f_{ij} + \frac{h^2}{12}\nabla^2 f_{ij} + o\left(h^4\right) \tag{24}$$

The scheme may be written explicitly as

$$\frac{1}{6}\left(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}\right) + \frac{2}{3}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) - \frac{10}{3}u_{ij}$$
$$= \frac{h^2}{12}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 8 f_{ij}\right) \tag{25}$$

This is the well known $o\left(h^4\right)$ accurate nine-point compact scheme.

For the $o\left(h^6\right)$ scheme, we need a fourth-order approximation of $\frac{\partial^4 u}{\partial x^2 \partial y^2}$ in equation (23). This can be written as

$$\left[\frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{ij} = \partial^2 x \partial^2 y u_{ij} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right]_{ij} + o\left(h^4\right) \tag{26}$$

Substituting equation (26) into equation (23), gives

$$\tau_{ij} = \frac{h^2}{12}\left(-\nabla^2 f_{ij}\right) + \frac{h^2}{6}\left[\partial^2 x \partial_y^2 u_{ij} - \frac{h^2}{12}\left(\frac{\partial^6 u}{\partial x^4 \partial y^4}\right)\right]_{ij} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{ij} + o\left(h^6\right).$$

or

$$\tau_{ij} = \frac{h^2}{12}\left(-\nabla^2 f_{ij} + 2\partial^2 x \partial^2 y u_{ij}\right) - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + 5\frac{\partial^6 u}{\partial x^4 \partial y^2} + 5\frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial y^6}\right]_{ij} + o\left(h^6\right) \tag{27}$$

A compact expressions of the $o\left(h^6\right)$ approximation is required and this can be done by further differentiating equation (19), that is

$$\frac{\partial^6 u}{\partial x^6} = \frac{\partial^4 f}{\partial x^4} - \frac{\partial^6 u}{\partial x^4 \partial y^2}, \quad \frac{\partial^6 u}{\partial y^6} = \frac{\partial^4 f}{\partial y^4} - \frac{\partial^6 u}{\partial x^2 \partial y^4} \tag{28}$$

and

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^4 f}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} \tag{29}$$

Substituting equations (28) and (29) into equation (27) gives

$$\tau_{ij} = \frac{h^2}{12}\left(-\nabla^2 f_{ij} + 2\partial_x^2\partial_y^2 u_{ij}\right) - \frac{h^4}{360}\left[\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} - \left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right) + 5\left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right)\right]_{ij}$$

$$= \frac{h^2}{12}\left(-\nabla^2 f_{ij} + 2\partial_x^2\partial_y^2 u_{ij}\right) - \frac{h^4}{360}\left[\nabla^4 f_{ij} + 4\partial_x^2\partial_y^2 f_{ij}\right] + o\left(h^6\right) \tag{30}$$

The compact sixth order approximation of the two dimensional Poisson equation can the be obtained as

$$\nabla^2 u_{ij} + \frac{h^2}{6}\partial_x^2\partial_y^2 u_{ij} = f_{ij} + \frac{h^2}{12}\nabla^2 f_{ij} + \frac{h^4}{360}\nabla^4 f_{ij} + \frac{h^4}{90}\partial_x^2\partial_y^2 f_{ij} + o\left(h^6\right) \tag{31}$$

where ∇^2 is the Laplacian operator and ∇^4 is the biharmonic operator. In equation (33), we assume the derivative of f can be determined analytically. In the case where f is not known analytically, we need only a fourth – order accurate approximation of $\nabla^2 f_{ij}$ and a second-order accurate approximation of

$\nabla^4 f_{ij}$ *and* $\left[\partial^4 f \big/ \partial x^2 \partial y^2\right]_{ij}$ [9]. A fourth-order accurate approximation of $\nabla^2 f_{ij}$ can be obtained using

$\left(1 - \frac{h^2}{12}\partial^2\right)\partial^2 f_{ij}$ to get

$$\frac{1}{12h^2}\left[-f_{i+1,j} - f_{i-1,j} + 16 f_{i+\frac{1}{2},j} + f_{i-\frac{1}{2},j} - 60 f_{ij} + 16 f_{i,j+\frac{1}{2}} + 16 f_{i,j-\frac{1}{2}} - f_{i,j+1} - f_{i,j-i}\right]$$

A second-order accurate approximation of $\nabla^4 f_{ij}$ can also be obtained as

$$\nabla^4 f_{ij} = \left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right]_{ij} = \partial_x^2\partial_y^2 f_{ij} = \frac{1}{h^4}\left[\begin{array}{l} f''_{i+1,j+1} + f''_{i+1,j-1} + f''_{i-1,j+1} + f''_{i-1,j-1} \\ -2\left(f''_{i+1,j} + f''_{i-1,j} + f''_{i,j+1} + f''_{i,j-1}\right) + 4 f''_{ij} \end{array}\right] \tag{32}$$

A nine-point $o\left(h^6\right)$ scheme for 2D Poisson may be expressed as

$$\frac{1}{6}C + \frac{2}{3}D - \frac{10}{3}U_{ij} = h^2\left[\frac{1}{9}E - \frac{1}{144}G + \frac{7}{12}f_{ij} + \frac{1}{90}L - \frac{1}{45}M + \frac{2}{45}f''_{ij}\right] + h^4\left[\frac{1}{360}N - \frac{1}{90}f^{iv}_{ij}\right]. \tag{33}$$

$C = U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}$, $\qquad D = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1.}$

$E = f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}$, $\qquad G = f_{i+2,j} + f_{i-2,j} + f_{i,j+2} + f_{i,j-2}$.

$M = f''_{i+1,j} + f''_{i-1,j} + f''_{i,j+1} + f''_{i,j-1}$ $\qquad N = f^{iv}_{i+1,j} + f^{iv}_{i-1,j} + f^{iv}_{i,j+1} + f^{iv}_{i,j-1}$.

2.3 Three-Dimension Case

In this section, we perform a similar derivation of the high-order difference scheme for Poisson equation in 3spatial dimensions which is given as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f\left(x, y, z\right) \ for \ \ x, y, z \in \Omega. \tag{34}$$

Here, Ω is taken as a cubic solid, $[0,1]\times[0,1]\times[0,1]$. The central difference scheme for equation (34) can be written as:

$$\partial_x^2 u_{ijk} + \partial_y^2 u_{ijk} + \partial_z^2 u_{ijk} + \tau_{ijk} = f_{ijk} \tag{35}$$

where

$$u_{ijk} = u(x_i, y_j, z_k), \; f_{ijk} = f(x_i, y_j, z_k)$$

and

$$\tau_{ijk} = \frac{-h^2}{12}\left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4}\right]_{ijk} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} + \frac{\partial^6 u}{\partial z^6}\right]_{ijk} + o(h^6) \tag{36}$$

We take $o(h^2)$ approximation of the first square bracket in equation (36). This is done by taking appropriate derivative of equation (34) which is,

$$\begin{aligned}
\frac{\partial^4 u}{\partial x^4} &= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 u}{\partial y^2 \partial x^2} - \frac{\partial^4 u}{\partial z^2 \partial x^2} \\
\frac{\partial^4 u}{\partial y^4} &= \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial z^2 \partial y^2} \\
\frac{\partial^4 u}{\partial z^4} &= \frac{\partial^2 f}{\partial z^2} - \frac{\partial^4 u}{\partial x^2 \partial z^2} - \frac{\partial^4 u}{\partial y^2 \partial z^2}
\end{aligned} \tag{37}$$

Equation (37), when substituted into equation (36), gives the alternative form for the exact truncation error at mode ijk. That is

$$\tau_{ijk} = \frac{-h^2}{12}\left(\nabla^2 f_{ijk} - 2\left[\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2}\right]_{ijk}\right) - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} + \frac{\partial^6 u}{\partial z^6}\right]_{ijk} + o(h^6) \tag{38}$$

$$\nabla^2 u_{ijk} + \frac{h^2}{6}\left[\partial_x^2 \partial_y^2 + \partial_x^2 \partial_z^2 + \partial_y^2 \partial_z^2\right] u_{ijk} = f_{ijk} + \frac{h^2}{12}\nabla^2 f_{ijk} + o(h^4) \tag{39}$$

The order four 3D scheme of the Poisson equation may be written explicitly as

$$\begin{aligned}
&\frac{1}{6}\begin{bmatrix} u_{i+1,j+1,k} + u_{i+1,j-1,k} + u_{i-1,j+1,k} + u_{i-1,j-1,k} \\ u_{i+1,j,k+1} + u_{i+1,j,k-1} + u_{i-1,j,k+1} + u_{i-1,j,k-1} \\ u_{i,j+1,k+1} + u_{i,j+1,k-1} + u_{i,j-1,k+1} + u_{i,j-1,k-1} \end{bmatrix} \\
&+ \frac{1}{3}\left[u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1}\right] - 4u_{ijk} \\
&= \frac{h^2}{12}\left[f_{i+1,j,k} + f_{i-1,j,k} + f_{i,j+1,k} + f_{i,j-1,k} + f_{i,j,k+1} + f_{i,j,k-1} + 6f_{ijk}\right]
\end{aligned} \tag{40}$$

For the $o(h^6)$ scheme of the 3D problem we need a fourth order approximation of the following:

$$\begin{aligned}
\left[\frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{ijk} &= \partial_x^2 \partial_y^2 u_{ijk} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right]_{ijk} + o(h^4) \\
\left[\frac{\partial^4 u}{\partial x^2 \partial z^2}\right]_{ijk} &= \partial_x^2 \partial_z^2 u_{ijk} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4 \partial z^2} + \frac{\partial^6 u}{\partial x^2 \partial z^4}\right]_{ijk} + o(h^4) \\
\left[\frac{\partial^4 u}{\partial y^2 \partial z^2}\right]_{ijk} &= \partial_y^2 \partial_z^2 u_{ijk} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial y^4 \partial z^2} + \frac{\partial^6 u}{\partial y^2 \partial z^4}\right]_{ijk}
\end{aligned} \tag{41}$$

We substitute equation (41) into equation (38), to get:

$$\tau_{ijk} = +\frac{h^2}{12}\left\{\begin{array}{l} -\nabla^2 f_{ijk} + 2\left[\partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2\right]u_{ijk} \\ -\frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial x^4 \partial z^2} + \frac{\partial^6 u}{\partial y^4 \partial z^2} + \frac{\partial^6 u}{\partial y^2 \partial z^4}\right]_{ijk} \end{array}\right\}$$

$$-\frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} + \frac{\partial^6 u}{\partial z^6}\right]_{ijk} + o\left(h^6\right)$$

$$= \frac{h^2}{12}\left\{-\nabla^2 f_{ijk} + 2\left[\partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2\right]u_{ijk}\right\}$$

$$-\frac{h^4}{360}\left[\begin{array}{l} \frac{\partial^6 u}{\partial x^6} + 5\frac{\partial^6 u}{\partial x^4 \partial y^2} + 5\frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial y^6} + 5\frac{\partial^6 u}{\partial x^2 \partial z^4} + \frac{\partial^6 u}{\partial z^6} \\ +5\frac{\partial^6 u}{\partial y^4 \partial z^2} + 5\frac{\partial^6 u}{\partial y^2 \partial z^4} \end{array}\right]_{ijk} + o\left(h^6\right) \quad (42)$$

Getting a compact sixth – order approximation requires compact expressions for the nine derivatives of order six in equation (42), which can be done by further differentiating equation (34), that is

$$\begin{aligned} \frac{\partial^4 f}{\partial x^2 \partial y^2} &= \frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial x^2 \partial y^2 \partial z^2} \\ \frac{\partial^4 f}{\partial x^2 \partial z^2} &= \frac{\partial^6 u}{\partial z^4 \partial x^2} + \frac{\partial^6 u}{\partial z^2 \partial x^4} + \frac{\partial^6 u}{\partial x^2 \partial y^2 \partial z^2} \\ \frac{\partial^4 f}{\partial y^2 \partial z^2} &= \frac{\partial^6 u}{\partial y^4 \partial z^2} + \frac{\partial^6 u}{\partial y^2 \partial z^4} + \frac{\partial^6 u}{\partial x^2 \partial y^2 \partial z^2} \end{aligned} \quad (43)$$

Also,

$$\begin{aligned} \frac{\partial^6 u}{\partial x^6} &= \frac{\partial^4 f}{\partial x^4} - \frac{\partial^6 u}{\partial y^2 \partial x^4} - \frac{\partial^6 u}{\partial z^2 \partial x^4} \\ \frac{\partial^6 u}{\partial y^6} &= \frac{\partial^4 f}{\partial y^4} - \frac{\partial^6 u}{\partial x^2 \partial y^4} - \frac{\partial^6 u}{\partial z^2 \partial y^4} \\ \frac{\partial^6 u}{\partial z^6} &= \frac{\partial^4 f}{\partial z^4} - \frac{\partial^6 u}{\partial x^2 \partial z^4} - \frac{\partial^6 u}{\partial y^2 \partial z^4} \end{aligned} \quad (44)$$

Substitute equations (43) and (44) into equation (42), gives

$$\tau_{ijk} = \frac{h^2}{12}\left\{-\nabla^2 f_{ijk} + \left[\partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2\right]u_{ijk}\right\}$$

$$-\frac{h^4}{360}\left\{\nabla^4 f_{ijk} + 4\left[\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial x^2 \partial z^2} + \frac{\partial^4 f}{\partial y^2 \partial z^2}\right]_{ijk} - 12\left[\frac{\partial^6 u}{\partial x^2 \partial y^2 \partial z^2}\right]_{ijk} + o\left(h^6\right)\right\} \quad (45)$$

after simplification of the expressions in the second square bracket, using equation(34) and equation (45), the compact sixth-order approximation of the three-dimensional Poisson equation can thus be obtained as

$$\nabla^2 u_{ijk} + \frac{h^2}{6}\left[\partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2\right]u_{ijk} + \frac{h^4}{30}\partial_x^2\partial_y^2\partial_z^2$$

$$= f_{ijk} + \frac{h^2}{12}\nabla^2 f_{ijk} + \frac{h^4}{360}\nabla^4 f_{ijk} + \frac{h^4}{90}\left[\partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2\right]f_{ijk} \quad (46)$$

3. NUMERICAL EXAMPLES

In this section, we performed two numerical experiments to solve a 2 dimensional Poisson equation (19) on the unit square domain $[0,1]\times[0,1]$. In both examples, pure Dirichlet boundary conditions are prescribed on all sides of the unit square.

In order to compare the numerical solution U_{ij} to the exact solution u_{ij}, we used $L_2 - norm$ of the error vector e, define as

$$\|e\|_2 = \frac{1}{N}\sqrt{\sum_{ij=0}^{N} e_{ij}^2},$$

where $e_{ij} = u_{ij} - U_{ij}$ *and* N is the number of nodes.

Problem 1.

$$F(x,y) = -2\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2}x \sin\frac{\pi}{2}y, \qquad 0 \le x, \quad y \le 1.$$

The exact solution is $u(x,y) = \sin\frac{\pi}{2}x\sin\frac{\pi}{2}y.$

Problem 2.

$$F(x,y) = -2\pi^2\cos(\pi x)\sin(\pi y) \qquad 0 \le x, \ y \le 1.$$

The exact solution is $u(x,y) = \cos(\pi x)\sin(\pi y)$

4. CONCLUSIONS

In this paper, we present a compact finite difference schemes for one, two and three dimensional Poisson equation. The discretization are of $o(h^4)$ *and* $o(h^6)$. The one and two dimensional equations are of 9-point. The three dimensional equation are 17-point stencil for $o(h^4)$ and 27-point for $o(h^6)$.

Our numerical experiments confirm that compact finite difference schemes are accurate and efficient methods. This work is extendable to 9-point scheme for three dimensional Poisson equations.

Table 1. Computation result for test problem 1

Location(x , y)	Exact Solution	Errors	
		$o(h^4)$ Scheme	$o(h^6)$ Scheme
(0.25,0.25)	0.1464	1.7931E-06	4.1833E-08
(0.50,0.25)	0.2706	6.6047E-06	1.3697E-08
(0.75,0.25)	0.3536	6.5863E-06	3.4281E-08
(0.25,0.50)	0.2706	2.0554E-06	7.0399E-08
(0.50,0.50)	0.5000	3.4787E-06	2.6034E-07
(0.75,0.50)	0.6533	1.0964E-05	3.6875E-07
(0.25,0.75)	0.3536	2.7446E-05	4.3768E-07
(0.50,0.75)	0.6533	3.5855E-05	8.1433E-08
(0.75,0.75)	0.8536	1.0773E-06	2.9210E-08

Table 2. Computational results for test problem 2

Location(x , y)	Exact Solution	Errors	
		$o(h^4)$ Scheme	$o(h^6)$ Scheme
(0.20, 0.20)	0.4755	7.585E-07	3.321E-09
(0.40, 0.20)	0.1816	1.635E-07	8.763E-09
(0.60, 0.20)	-0.1816	3.417E-07	4.954E-09
(0.80, 0.20)	-0.4755	2.926E-07	4.346E-09
(0.20, 0.40)	0.7694	5.359E-07	6.017E-09
(0.40, 0.40)	0.2939	3.843E-07	1.739E-09
(0.60, 0.40)	-0.2939	4.967E-07	8.334E-09
(0.80, 0.40)	-0.7694	3.052E-07	5.678E-09
(0.20, 0.60)	0.7694	1.874E-07	2.346E-09
(0.40, 0.60)	0.2939	4.967E-07	3.864E-09
(0.60, 0.60)	-0.2939	2.397E-07	8.763E-09
(0.80, 0.60)	-0.7694	3.052E-07	6.497E-09
(0.20, 0.80)	0.4755	1.335E-06	2.936E-09
(0.40, 0.80)	0.1816	6.763E-06	7.384E-08
(0.60, 0.80)	-0.1816	5.243E-07	3.741E-08
(0.80, 0.80)	-0.4755	3.438E-07	8.307E-09

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