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A new proportional procedure for the n−person cake−cutting problem

Attila Tasnádi
Budapest University of Economic Sciences and Public Administration, Department of Mathematics

Abstract

We present a new cake−cutting procedure which guarantees everybody a proportional share according to his own valuation.

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1. Introduction

Divide-and-choose is a well-known procedure to divide a divisible object, henceforth a cake, between two persons in such a way that each of them can guarantee himself at least half of the cake with respect to his own evaluation. According to this procedure the divider cuts the cake into two pieces and the chooser takes one of the two pieces. Though the two persons may have different evaluations of certain pieces of the cake this procedure ensures that both persons receive at least their proportional shares according to their own evaluations of the two pieces. Both persons can receive less than their proportional share only by they own fault, which might happen if the divider does not cut the cake into two identical pieces with respect to his own evaluation, or if the chooser selects the piece which he evaluates as less than a half of the cake by mistake.

Steinhaus (1948) was the first to consider and solve an extension of the cake division problem described above, namely, to the case of three persons. By applying his solution everybody could assure himself a proportional share with respect to his own evaluation. However, he could not extend his method for the n-person cake division problem. Using a completely different approach a first solution to the n-person cake division problem was given by Steinhaus's students Banach and Knaster (Steinhaus, 1948). Later on, many other solutions to the n-person cake division problem were given. For an overview of these procedures we refer to Brams and Taylor (1996) and Robertson and Webb (1998).

In this note we present a new and simple recursive procedure for the n-person cake division problem. The present procedure shows similarities to Fink's (1964) procedure and for the three person case there are also some similarities to Steinhaus's (1948) procedure.

2. The procedure

Suppose that we have n persons who want to divide a cake between themselves in a way that everybody can assure himself a proportional share according to his own evaluation of possible pieces of the cake. We have to assume that everybody evaluates the cake in a 'non-atomic' way, i.e., for any piece of the cake with a corresponding evaluation $\alpha \in \mathbb{R}_+$ and for any value $\beta \in [0, \alpha]$ there exists a subpiece with evaluation β.

For the two person case we will apply the celebrated divide-and-choose procedure. First, for expositional reasons, we consider the three person case. The procedure consists of the following steps.

1. Ask one person, say person 1, to divide the cake into three pieces.

2. Let persons 2 and 3 mark two pieces.

3. The three pieces will be divided by applying the divide-and-choose procedure in the following way.

   (a) If persons 2 and 3 mark the same two pieces, then they will share each of these two pieces by applying the divide-and-choose procedure, while person 1 receives the piece, which was not selected by either of the other two persons.

   (b) If persons 2 and 3 make different marks, then they share the piece marked by both of them again by applying the divide-and-choose procedure and they share the piece marked by only one of them with person 1 also by applying the divide-and-choose procedure.

It is easy to see that this procedure guarantees each player a proportional share. Clearly, if person 1 does not cut the cake into exact thirds according to his evaluation, he risks that the other players do not mark a piece which he values less than a third and he will end up with less than his proportional share. Hence, player 1 has to cut the cake into three parts which he values to be identical. Turning to persons 2 and 3 we can see that by applying the divide-and-choose procedure two or three times, they both will obtain at least a half of each marked piece. Since persons 2 and 3 value the two pieces they marked to be at least two thirds, they will receive at least a third of the cake each.

Now we turn two the n-person case. Our recursive procedure goes as follows:

1. Suppose that we have already solved the $n-1$-person case.

2. Let one person, say person 1, divide the cake into n pieces.

3. Let persons 2 through n mark $n-1$ pieces.

4. Since persons 2 through n made $(n-1)^2$ marks and since we want to apply the $n-1$-person procedure from step 1 to divide each of the n pieces, person 1 has to fill out the empty places at each piece by an appropriate number of copies. In particular, if the mth piece was selected by $k_m \in \{0, 1, \ldots, n-1\}$ persons $\left(\sum_{m=1}^{n} k_m = (n-1)^2\right)$, then person 1 takes part by $n-1-k_m$ copies in the division of the mth piece. Hence, we can apply to each piece an $n-1$-person procedure, which guarantees person 1 at least $(n-1-k_m)/(n-1)$ of the mth piece and persons 2 through n at least $1/(n-1)$ of a marked mth piece.

Finally, we have to check that our procedure ensures everybody a proportional share. Obviously, person 1 has to cut the cake into n equal pieces according to his own evaluation, since otherwise it could happen that none of the remaining persons marked a piece, which person 1 values as less than $1/n$. Now, if person 1 cuts the cake into n identically valued pieces he receives at least a fraction of $\sum_{m=1}^{n} \frac{n-1-k_m}{n-1}\frac{1}{n} = \left(n - \frac{(n-1)^2}{n-1}\right)\frac{1}{n} = \frac{1}{n}$ of the entire cake. Furthermore, any other person receives at least $\frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n}$ of the cake. Thus, by induction we have established that our procedure guarantees a proportional share to everybody.

References

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