Invariance under type morphisms: the Bayesian Nash Equilibrium*

Miklós Pintér
Corvinus University of Budapest[†]

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Abstract

Ely and Peski (2006) and Friedenberg and Meier (2010) provide examples when changing the type space behind a game, taking a "bigger" type space, induces changes of Bayesian Nash Equilibria, in other words, the Bayesian Nash Equilibrium is not invariant under type morphisms.

In this paper we introduce the notion of strong type morphism. Strong type morphisms are stronger than ordinary and conditional type morphisms (Ely and Peski, 2006), and we show that Bayesian Nash Equilibria are not invariant under strong type morphisms either. We present our results in a very simple, finite setting, and conclude that there is no chance to get reasonable assumptions for Bayesian Nash Equilibria to be invariant under any kind of reasonable type morphisms.

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1 Introduction

In incomplete information situations the uncertainties about the situation and about the players' information about the situation etc. are modeled

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[†]Department of Mathematics, Corvinus University of Budapest, 1093 Hungary, Budapest, Fővám tér 13-15., miklos.pinter@uni-corvinus.hu

by types (Harsányi, 1967-68). A type gives the complete description of the players' epistemological and non-epistemological characteristics, that is, of what the players believe about the situation, what the players believe about the other players' beliefs about the situation, and so on to infinity, and the players' preferences, tastes among others. The formal definition of type spaces is introduced by Heifetz and Samet (1998).

Incomplete information situations can be modeled by Bayesian games. A Bayesian game, more precisely its description, encapsulates the descriptions of the players' epistemic and non-epistemic properties, that is, it contains a type space (see Definition 3). Concrete problems, situations, "call for" concrete type spaces. However, it can happen that the modeler misses a type, so the applied model (Bayesian game and the corresponding type space) is inadequate. Then it is necessary to incorporate the missed type into the Bayesian game, that is, to consider a type space which contains the previously missed type too.

In general, a modeler never can be sure that she does not miss any type, so it is recommended to work with a universal type space (Heifetz and Samet, 1998), with a type space which encapsulates all types. It is well known that whether a universal type space exists or not depends on the model setting (Heifetz and Samet, 1998; Pintér, 2010). In this paper we ignore this problem, since in the simple setting we apply, this problem is not relevant[1].

A type space is bigger than an other type space if the former contains all types of the latter. Formally, type space $\mathcal{T}_1$ is broader (bigger) than type space $\mathcal{T}_2$ if there is a type morphism from type space $\mathcal{T}_2$ to type space $\mathcal{T}_1$[2]; where the type morphism is a function, which maps the types of a type space into an other type space (see Definition 2).

In other words, if a model misses a type, then we have to take a bigger type space (a type space into which the original type space can be mapped by a type morphism) which encapsulates the missed type.

The Bayesian Nash Equilibrium is introduced by Harsányi (1967-68), and it is a commonly used solution concept for Bayesian (incomplete information) games. By an example Ely and Peski (2006) demonstrate that even if two type spaces induce the same beliefs hierarchies for the players, the Bayesian Nash Equilibria can be different in the two games with the different type spaces Friedenberg and Meier (2010) also notice that if we change the type space in a Bayesian game to a bigger type space, then the Bayesian Nash

[1]We mean the purely measurable and the topological settings coincide in our simple setting.

[2]Notice that it is common in mathematics to use mappings for comparing sets (objects). E.g. set A is "bigger" than set B, i.e. A has more elements than that B has, if there is an injective mapping from set B to set A.

Equilibria may change. In other words, examples show that the Bayesian Nash Equilibrium is not invariant under type morphisms.

Furthermore, we can see an interesting duality. While in simple models, for Bayesian games with "simple" type spaces, we can give the Bayesian Nash Equilibria, in these models we never can be sure that all relevant types are considered. However, if we consider the universal (or a "big") type space, when we can be sure that we do not miss any relevant type, then we get a Bayesian game (with the universal or a "big" type space) for which we can hardly give the Bayesian Nash Equilibria.

An attempt to get invariance is made by Ely and Peski (2006) who introduce conditional type morphisms (the authors call it type morphism which preserves conditional beliefs). By the conditional type morphisms less type spaces are related to each other, so there is smaller chance for violating the invariance. We must mention that Ely and Peski (2006)'s goal is not to ensure the invariance of Bayesian Nash Equilibrium, so our results are not directly related to theirs.

We also remark that as the above mention papers (except Harsányi (1967-68)), the literature of type spaces use advanced mathematical tools (see Mertens and Zamir (1985); Brandenburger and Dekel (1993); Heifetz (1993); Mertens et al. (1994); Heifetz and Samet (1999); Battigalli and Siniscalchi (1999); Pintér (2005); Dekel et al. (2006, 2007) among others)

In this paper we introduce the notion of strong type morphism (see Definition 6). The strong type morphism is stronger than the (ordinary) type morphism and the conditional type morphism (Ely and Peski, 2006), it preserves not only the hierarchies of beliefs and the conditional hierarchies of beliefs of the players, but it does the "aggregated" beliefs of the players as well. By "aggregated" beliefs we mean that if players pool their information, then they can refine their beliefs, they can use an "aggregated" belief which reflects all information the considered players have. The strong type morphism is a very strong concept, it relates two states of the world in two type spaces only if at the two considered states of the world the players pooled (aggregated) information is the same.

By an example we demonstrate that the Bayesian Nash Equilibria are not invariant under strong type morphisms either. Therefore, since in our opinion the strong type morphism is the strongest reasonable type morphism notion, we conclude that there is no relevant concept of type morphisms under which the Bayesian Nash Equilibrium is invariant.

Therefore, contrary to the notions of interim correlated and interim independent rationalizable actions which are invariant under ordinary and conditional type morphisms (in the two player case) respectively – see Dekel et al. (2007) for interim correlated rationalizability and Ely and Peski (2006) for interim independent rationalizability – the Bayesian Nash Equilibrium is a problematic solution concept for incomplete information games.

Furthermore, we provide all results and concepts in a simple, finite game and finite type space setting. By doing so (1) we make our results accessible for readers who are not familiar with advanced mathematics (e.g. with measure theory), (2) we demonstrate that the discussed problem (the invariance of Bayesian Nash Equilibrium) does not call for tools of advanced mathematics, and last but not least, (3) we can avoid certain technical difficulties.

In point (3) above, we refer to the following problem. When we introduce a type morphism which work with not only beliefs, but conditional beliefs, or aggregated beliefs, then we should introduce these concepts for the type spaces too. In the abstract settings (see e.g. Heifetz and Samet (1998); Pintér (2010)) it could be very difficult to introduce conditional and aggregated beliefs. In the finite setting we consider, we can introduce these notions (conditional beliefs and aggregated beliefs) in very simply ways and forms.

The setup of the paper is as follows: in Section 2 we introduce the main notions of type spaces, Bayesian games and type morphisms. Section 3 presents the main results. The last section briefly concludes.

2 Type spaces and other notions

Harsányi (1967-68) introduces the notion of types so as to avoid that the players' hierarchies of beliefs[3] come in models explicitly. A type of a player gives a complete description of the player, it encompasses her all relevant beliefs, knowledges, features and properties. Heifetz and Samet (1998) introduce the formal concept of type spaces. The type space is a tuple, a mathematical object, which collects certain types of the players, and explicitly gives the players' beliefs at their types.

In the following definition we present the notion of type spaces we use in this paper.

Definition 1. *A type space based on parameter space P is a tuple $(P, \{T_i\}_{i \in N}, \Pi)$ such that*

- *P is the non-empty, finite parameter set,*
- *T_i is the non-empty, finite type set of player i, $i \in N$,*
- *Π is a probability distribution on $P \times T$, it is called common prior,*

[3] Harsányi (1967-68) calls them "infinite regress in reciprocal expectations".

where $N = \{1, 2, \ldots, n\}$ *is the player set, and* $T = \times_{i \in N} T_i$.

Parameter set P is a set containing all states of nature, that is, it consists of parameters which are relevant for the modeled situation, but cannot be assigned to any player. In this paper we assume that the parameter set is finite, and every subset of it is an event, put it differently, the players are capable of recognizing (naming) any subset of P.

T_i is the type set of player i, it consists of parameters which are relevant for the modeled situation and can be assigned to player i. Typically, the players' taste and beliefs are represented by types. We can also say that the parameter set P is the type set of the nature, in this sense the nature is a player. As in the case of the parameter set, we assume that the type sets are finite sets, and every subset of a type set is an event.

The space $P \times T$ is the state space, its points are the states of the world. Each state of the world is a complete description of a possible state: of the state of nature (the component belonging to P) and of the players' types (the components belonging T). Therefore, at state of the world $(p, t) \in P \times T$, where $t = (t_1, t_2, \ldots, t_n)$, p is the state of nature, and t_i is the type of player i.

Harsányi (1967-68) assumes a common prior from which the players' beliefs can be deduced. The assumption of common prior is also called Harsányi Doctrine. The Harsányi Doctrine's philosophical basis is well discussed in the literature (see e.g. the original paper Harsányi (1967-68)), so we do not discuss it here. The common prior assumption heavily restricts the class of type spaces we can consider. In this paper, however, it is enough to consider this narrow class of type spaces, because any counterexample in this setting is valid in the general setting (Heifetz and Samet, 1998) as well.

In this paper we assume that each type has positive Π-probability, that is, for each player i and type $t_i \in T_i$

$$\Pi(P \times T_1 \times \cdots \times T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots \times T_n) > 0 \ . \tag{1}$$

By Assumption (1) we avoid technical difficulties and can provide our message in a simple setting. If player i's type is $t_i \in T_i$, then player i' belief at type t_i is given by the probability distribution $\Pi(\cdot \mid P \times T_1 \times \cdots T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots T_n)$, which is a conditional distribution. Put it differently, for any event $A \subseteq P \times T$, player i believes at type t_i that event A occurs (by Bayes' rule) with probability

$$\begin{aligned} &\Pi(A \mid P \times T_1 \times \cdots \times T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots \times T_n) \\ = &\frac{\Pi(A \cap P \times T_1 \times \cdots \times T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots \times T_n)}{\Pi(P \times T_1 \times \cdots \times T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots \times T_n)} \ . \end{aligned}$$

Therefore, for each player at each type the type space tells us what the player believes. For each player we can define a function $\tau_i : T_i \to \Delta(P \times T_{-i})$, where Δ is the set of probability distributions (measures) and $T_{-i} = \times_{j \in N \setminus \{i\}} T_j$, which gives what player i believes at her types. We call function τ_i player i's type function.

In the general setting (see e.g. Heifetz and Samet (1998); Pintér (2010)) type spaces contain type functions instead of priors. It is because, in general, even if the players' have priors we cannot apply Bayes' rule to get the type functions since usually the probability of a type is zero (the denominator is zero in Bayes' rule). Therefore, our definition of type space (see Definition 1) is purely a simplified version of the general definition of type space (see Heifetz and Samet (1998)), and it reflects the same intuitions as that the general notion does.

When we would like to compare two type spaces, intuitively, we can say that a type space is bigger than an other, if it contains every type the other does. Moreover, as in the case of sets, where the cardinality of sets are related to each other by functions, in this case we also use functions to establish relations between type spaces. These functions are called type morphisms.

Definition 2. *Mapping* (id_P, φ) *– which is the product mapping of the identity mapping on* P *and* $\varphi = (\varphi_i)_{i \in N}$, $\varphi_i : T_i \to T'_i$, $i \in N$ *– is a type morphism between type spaces* $\mathcal{T} = (P, \{T_i\}_{i \in N}, \Pi)$ *and* $\mathcal{T}' = (P, \{T'_i\}_{i \in N}, \Pi')$, *if for each player* $i \in N$, *type* $t_i \in T_i$ *and set* $A \subseteq P \times T'$

$$\begin{aligned} &\Pi'(A \mid P \times T'_1 \times \cdots \times T'_{i-1} \times \{\varphi_i(t_i)\} \times T'_{i+1} \times \cdots \times T'_n) \\ &= \Pi((\mathrm{id}_P, \varphi)^{-1}(A) \mid P \times T_1 \times \cdots \times T_{i-1} \times \{t_i\} \times T_{i+1} \times \cdots \times T_n) \ . \end{aligned} \tag{T1}$$

The type morphism formalizes the intuition that player i has the same information at types $t_i \in T_i$ and $\varphi_i(t_i) \in T'_i$. However, when one considers players' information, we can mean different things by the word information. Usually (Heifetz and Samet, 1998), we say that player i has the same beliefs at types $t_i \in T_i$ and $\varphi_i(t_i) \in T'_i$. More precisely, player i's belief hierarchies at types $t_i \in T_i$ and $\varphi_i(t_i) \in T'_i$ coincide.

Therefore, a type morphism maps states of the world of a type space to an other type space's states of the world in the way that it preserves the players' hierarchies of beliefs. It is worth noticing, however, that type morphisms preserve not only the belief hierarchies but more. Ely and Peski (2006)'s example (in the Introduction) clearly shows this fact. In that example Ely and Peski (2006) give two different type spaces in which every type induces the same belief hierarchy, that is, all types in the two type space represent the same belief hierarchy. Then they put the type space in a Bayesian game and show that the Bayesian Nash Equilibria are different in the two games. However, even if the two considered type space represent the same belief hierarchy, those are not equal by type morphisms, that is, one of the two is strictly bigger than the other, we can type-map the smaller into the bigger, but cannot the bigger into the smaller. Therefore, even if the two type spaces represent the same belief hierarchy those are not equal by type morphism, that is, type morphism preserve not only the belief hierarchies, but something more. For further discussion on this topic see Friedenberg and Meier (2011).

As a consequence of that type morphisms do not only preserve belief hierarchies, we can say that our results are different from Ely and Peski (2006)'s and Sadzik (2010)'s results who focus on the problem that the belief hierarchies are not sufficient to determine interim correlated rationalizable actions and Bayesian Nash Equilibria respectively. In this paper, however, we focus on type morphisms and not on belief hierarchies, and consider type spaces which may reflect different (hierarchies of) beliefs for the players.

Back to the discussion of the intuitions lying behind the notion of type morphisms, by type morphisms we can compare type spaces. We say that type space $\mathcal{T}'$ is bigger than type space $\mathcal{T}$, if there is a type morphism from type space $\mathcal{T}$ to type space $\mathcal{T}'$. As we mentioned it in the Introduction, missing relevant types is an important issue in modeling incomplete information situations. If we want to put a missing type in the model, then we have to give a type space, which is bigger than the previously used and encompasses the missing type. In this finite setting, it is easy to incorporate missing types in a model.

We emphasize that, even if the notion of type morphisms used in this paper is different from the general concept introduced by Heifetz and Samet (1998), both formulations capture the same idea, so we do not depart from the main stream of discussing type spaces and type morphisms.

Next, we introduce notion of Bayesian game, the model of incomplete information situations we consider in this paper.

Definition 3. *A Bayesian game is a tuple* $(N, \{A_i\}_{i \in N}, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}, \mathcal{T})$ *such that*

- N *is the non-empty, finite set of players,*
- A_i *is the non-empty, finite action set of player* i, $i \in N$,
- $s_i : T_i \to A_i$ *is a strategy of player* i, S_i *is player* i*'s strategy set,* $i \in N$,
- $u_i : P \times A \to \mathbb{R}$ *is the payoff function of player* i, $i \in N$,
- $\mathcal{T}$ *is a type space,*

where $A = \times_{i\in N} A_i$.

The definition above is a simple textbook definition of Bayesian games (see e.g. Mas-Colell et al. (1995)) except that instead of giving the type sets and the common prior explicitly, we refer to the type space encompassing those.

Harsányi (1967-68) introduces also the notion of Bayesian Nash Equilibrium.

Definition 4. *Strategy profile* $s^* \in S = \times_{i\in N} S_i$ *is a Bayesian Nash Equilibrium in Bayesian game* $(N, \{A_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}, \mathcal{T})$*, if for each player* $i \in N$ *and strategy* $s_i \in S_i$

$$\int\limits_{P\times T} u_i(s^*)\ \mathrm{d}\Pi \geq \int\limits_{P\times T} u_i(s_i, s^*_{-i})\ \mathrm{d}\Pi\ ,$$

where $(s_i, s^*_{-i}) = (s^*_1, s^*_2, \ldots, s^*_{i-1}, s_i, s^*_{i+1}, \ldots, s^*_n)$[4].

It is well-known that in this finite setting the two notions of Bayesian Nash Equilibrium – first when it is defined as a Nash Equilibrium in the ex-ante strategic form, and second when it is defined as a Nash Equilibrium in the interim strategic form – coincide. Therefore, our results are meaningful for both approaches.

Furthermore, it follows from the above definitions that in this paper we do not allow the players to play mixed actions, so we do not consider mixed, but pure Bayesian Nash Equilibria. This assumption also helps us to keep the setting simple, to avoid (unnecessary) technical difficulties. As we discuss it in Subsection 3.1 that we use pure Bayesian Nash Equilibria instead of the more general mixed Bayesian Nash Equilibria has no impact on the message of this paper. Our results are valid in any reasonable general setting, including the setting when one considers mixed Bayesian Nash Equilibria.

3 Results

In this section we present an example for Bayesian Nash Equilibria being not invariant under type morphisms. Even if the example is very simple, it grabs the very essence of the problem, and clearly shows that there is no chance for assumptions ensuring the invariance of Bayesian Nash Equilibria under type morphisms.

[4]It is worth noticing that in the above definition $u_i(s^*)$ is a composite function.

3.1 Invariance under ordinary type morphisms

First, we take (ordinary) type morphisms (see Definition 2).

Example 5. Consider a 2×2 game, and let the parameter space (which contains the basic uncertainties of the model, the states of nature) be $P = \{\theta_1, \theta_2\}$, the payoffs be given by Tables 1 and 2.

		Player 2 L	R
Player 1	U	(0,0)	(0,0)
	D	(0,0)	(0,0)

Table 1: The state of nature is θ_1

		Player 2 L	R
Player 1	U	(0,1)	(1,0)
	D	(1,0)	(0,1)

Table 2: The state of nature is θ_2

Type space 1: Consider the following type space $\mathcal{T}_1 = (P, \{\{t_i\}\}_{i=1,2}, \Pi)$, where

$$\Pi(\{x\}) = \begin{cases} 1, & \text{if } x = (\theta_1, t_1, t_2) \\ 0 & \text{otherwise} \end{cases},$$

that is, it is commonly believed that the actual state of nature is θ_1[5]. Therefore, it is enough to consider the game in Table 1, so every strategy profile $((s_1 = U, s_2 = L)$, $(s_1 = U, s_2 = R)$, $(s_1 = D, s_2 = L)$, and $(s_1 = D, s_2 = R))$ is a Bayesian Nash Equilibrium.

Type space 2: Consider the following type space $\mathcal{T}_2 = (P, \{\{t_i^1, t_i^2\}\}_{i=1,2}, \Pi')$, where

$$\Pi'(\{x\}) = \begin{cases} \frac{1}{2}, & \text{if } x = (\theta_1, t_1^1, t_2^1) \text{ or } x = (\theta_2, t_1^2, t_2^2) \\ 0 & \text{otherwise} \end{cases},$$

[5] We mean both players believe with probability one that the state of nature is θ_1, both players believe with probability one that the other player believes with probability one that the state of nature is θ_1, and so on to infinity.

that is, at type profile (t_1^1, t_2^1) it is commonly believed that the state of the world is (θ_1, t_1^1, t_2^1), and at type profile (t_1^2, t_2^2) it is commonly believed that the state of the world is (θ_2, t_1^2, t_2^2).

Since at type profile (t_1^2, t_2^2) it is commonly believed that the players play a game having no Nash Equilibrium, there is no Bayesian Nash Equilibrium in this game.

Let mapping $\varphi = (\varphi_i)_{i=1,2}$ be defined as $\varphi_i(t_i) = t_i^1$, $i \in N$. Then, it is a slight calculation to see that (id_P, φ) is a type morphism from type space $\mathcal{T}_1$ to type space $\mathcal{T}_2$.

Since at type space $\mathcal{T}_1$ there are four Bayesian Nash Equilibria and at type space $\mathcal{T}_2$ there is no Bayesian Nash Equilibrium, we can conclude that the Bayesian Nash Equilibrium is not invariant under (ordinary) type morphisms.

As we have mentioned earlier, in this paper we do not allow the players to play mixed actions, that is, we consider pure Bayesian Nash Equilibria (see Definition 4). It is obvious, however, that all results and anomalies we present in this paper remain valid in a more complex setting, where e.g. mixed Bayesian Nash Equilibria are considered. The more complex setting could only make the discussion more technical, it could not add anything to the message of this paper.

In order to demonstrate this argument, consider the following example. Take two complete information game, one where at each action profile all players get 0 payoff, and an other in which there is no mixed Nash Equilibrium. Then in the first game there are many Nash Equilibria, more precisely, every strategy profile is a Nash Equilibrium. Moreover, w.l.o.g. we can assume that in the two games the players' sets coincide and so do the strategy sets[6]. Let the parameter set (P) consist of states: θ_1 represents the case when the first (the trivial) game is played, and θ_2 stands for the case when the second game (having no mixed Nash Equilibrium) is played.

Consider the two following type spaces: first when each player has only one type, and it is commonly believed that the players play the first game, that is, it is commonly believed that the state of nature is θ_1. In this game there are many Bayesian Nash Equilibria[7].

The second, when each player has two types, we refer to the types as the first type and as the second type of a given player. Moreover, at the type profile, where each player's type is the first type, let commonly believed that the state of the world is that where the state of nature is θ_1, and the type

[6]Notice that, we can define the first game as it is played by the players of the second game, and the players' strategies are also coincide with the strategies of the second game. The two games differ only in the payoffs.

[7]The analogy with type space $\mathcal{T}_1$ of Example 5 is clear.

profile is as it is. Therefore, at the "first type" type profile it is commonly believed that the players play the first (trivial) game. At the type profile, where each player's type is the second type, let commonly believed that the state of the world is that where the state of nature is θ_2, and the type profile is as it is. Therefore, at the "second type" type profile it is commonly believed that the players play the second game, the game having no mixed Nash Equilibrium[8].

Therefore, at the first type space there are many Bayesian Nash Equilibria, while at the second type space there is no Bayesian Nash Equilibrium. Finally, consider the function, which maps the types of the first type space into the first types of the second type space. Then, it is easy to see that the product of the identity mapping on the parameter set and this function is a type morphism[9], so we can conclude again that the Bayesian Nash Equilibria are not invariant under (ordinary) type morphisms.

If we liked to present the above example precisely not only "verbally", however, then we would have to define all notions in an abstract setting, therefore we would care about many technical details, which as the above discussion clearly shows, is unnecessary for the results of this paper.

3.2 Invariance under strong type morphisms

Ely and Peski (2006) introduce the notion of conditional type morphism. Conditional type morphisms are type morphisms that preserve not only the players' beliefs, but their conditional beliefs too[10]. Ely and Peski (2006) consider only the two player setting, where the conditional beliefs of player i give at each type of player i that what she believes about the parameter set contingent on the types of the other player. Practically, the conditional beliefs give what the given player believes about that what information she and the other player would have together if they pooled their information. Therefore, in a certain sense, the conditional beliefs are aggregated beliefs of two given players.

In the following we "extend" the definition of conditional type morphism by Ely and Peski (2006), and introduce a notion where not only two, but all players can pool (aggregate) their information. We call this notion strong type morphism. In our simple, finite setting, however, it is possible to formulate aggregated beliefs without sophisticated mathematical tools, and we can ignore certain conceptual aspects either. Since we use type spaces with

[8]The analogy with type space $\mathcal{T}_2$ of Example 5 is obvious.

[9]The analogy with mapping (id_P, φ) of Example 5 is apparent.

[10]We are not precise here, since the conditional type morphisms preserve not only the players' conditional beliefs, but their hierarchies of conditional beliefs too.

common prior, we implicitly assume that all players believe the same about their aggregated beliefs, that is, e.g. both players i and j believe the same about what information they would have if they pooled their information.

In order to consider the aggregated beliefs of two or more players we must have introduced these beliefs in the type spaces too. Doing so, however, may cause many technical difficulties, but does not add anything new for the message of the paper. E.g. in type space $\mathcal{T}_2$ (see Example 5) the aggregated beliefs of the players at type profiles (t_1^1, t_2^1) and (t_1^2, t_2^2) are well defined by the common prior and the Bayes' Rule. However, at type profiles (t_1^1, t_2^2) and (t_1^2, t_2^1) this type space says nothing, since the probabilities of these type profiles are zero, so we cannot apply the Bayes' Rule. In this case we can introduce an "aggregated type" function, $f_a : T_1 \times T_2 \to \Delta(P)$, into the model, which tells us the two players' beliefs about the states of nature at their type profiles.

At type profiles (t_1^1, t_2^1) and (t_1^2, t_2^2) the common prior tells us the aggregated beliefs of the players, $f_a(t_1^1, t_2^1)(\{x\}) = \begin{cases} 1, & \text{if } x = \theta_1 \\ 0 & \text{otherwise} \end{cases}$, and $f_a(t_1^2, t_2^2)(\{x\}) = \begin{cases} 1, & \text{if } x = \theta_2 \\ 0 & \text{otherwise} \end{cases}$. In the cases of the two remaining type profiles, however, we must explicitly give the aggregated beliefs, e.g. let $f_a(t_1^1, t_2^2)(\{\theta_1\}) = f_a(t_1^1, t_2^2)(\{\theta_2\}) = f_a(t_1^2, t_2^1)(\{\theta_1\}) = f_a(t_1^2, t_2^1)(\{\theta_2\}) = \frac{1}{2}$, that is, the players believe at both considered type profiles that both states of nature occur with equal probability.

In order to avoid giving the technically non trivial definition of “extended“ type spaces (which encapsulate aggregated beliefs), we define the notion of strong type morphism as follows:

Definition 6. *Mapping* (id_P, φ) *– which is the product mapping of the identity mapping on* P *and* $\varphi = (\varphi_i)_{i \in N}$, $\varphi_i : T_i \to T_i'$, $i \in N$ *– is a strong type morphism between type spaces* $\mathcal{T} = (P, \{T_i\}_{i \in N}, \Pi)$ *and* $\mathcal{T}' = (P, \{T_i'\}_{i \in N}, \Pi')$, *if for each set of players* $S \subseteq N$, $S \neq \emptyset$, *types* $t_i \in T_i$*:*

$$\Pi(P \times \{t_S\} \times T_{-i}) > 0 \ , \tag{S1}$$

and for any set $A \subseteq P \times T'$*:*

$$\Pi'(A \mid P \times \{\varphi_S(t_S)\} \times T'_{-S}) = \Pi((\mathrm{id}_P, \varphi)^{-1}(A) \mid P \times \{t_S\} \times T_{-S}) \ , \tag{2}$$

where $t_S = \times_{i \in S}\{t_i\}$, $T_{-S} = \times_{i \in N \setminus S} T_i$ *and* $\varphi_S = \times_{i \in S} \varphi_i$.

Intuitively, strong type morphisms preserve not only the players' beliefs and conditional beliefs, but their aggregated beliefs too. In other words, a strong type morphism maps a type profile to such an other type profile at which the players have the same aggregated information (as that at the original type profile). In our setting with two players Ely and Peski (2006)'s conditional type morphism and the strong type morphism coincide. On the other hand, in our setting we deviate from Ely and Peski (2006)'s more technical definition in two points:

(1) In point (S1) we require that a strong type morphism can take only type spaces as domain where the common prior determines the aggregated beliefs. This is because we do not want to introduce the notion of an "extended" type space. Point (2) also imposes a requirement on the involved type spaces, it says (among others) that at the type profiles in the range of a strong type morphism the aggregate beliefs must be deducible from the common prior. It is clear that in a general setting, e.g. in Ely and Peski (2006)'s setting, much more type space can be related than that in our setting. Therefore, our results are valid in any reasonable general setting as well.

(2) An other, but minor point is that in Ely and Peski (2006)'s approach different players can have different aggregated beliefs at the same type profile. In our example, we give only one aggregated beliefs for the two players in type space $\mathcal{T}_1$ (of Example 5). First, as we said in Point (1) we do consider aggregated beliefs deducible from the common prior. Second, since we assume a common prior, it is very natural to suppose that the aggregated beliefs of the two players are the same. We mean, this implicit assumption fits the setting we consider.

It is easy to see that strong type morphisms are stronger than (ordinary) type morphisms. That is, if there is a strong type morphism from type space $\mathcal{T}$ to type space $\mathcal{T}'$, then that is also an ordinary type morphism between the two type spaces.

It is also straightforward that type morphism φ in Example 5 is a strong type morphism from type space $\mathcal{T}_1$ to type space $\mathcal{T}_2$. To sum up, Bayesian Nash Equilibria are not invariant under strong type morphisms either. Moreover, since in the two player setting the notions of strong morphism and of Ely and Peski (2006)'s conditional type morphism coincide, we can also conclude that Bayesian Nash Equilibria are not invariant under conditional type morphisms either.

Finally, we close our discussion with some remarks on a related paper, on Friedenberg and Meier (2010). Friedenberg and Meier (2010) give two results on invariance of Bayesian Nash Equilibria under type morphisms (see Proposition 7.1 on p. 23 and Section 8). They assume that either (i) the two involved type spaces can differ in only countably many types or (ii) each type has positive probability under the common prior[11]. Since Friedenberg and Meier (2010) work in general setting, we compare the general setting consequences of our results to their results.

As we have already mentioned in the Introduction, there is an interesting duality in the relation of Bayesian Nash Equilibria and type spaces. For Bayesian games with simple type spaces we can determine the set of Bayesian Equilibria, but in this case we cannot be sure that we do not miss (important) types. If we consider a "big" (the universal if it exists) type space (Heifetz and Samet, 1998), then we can be sure that we do not miss any type, however for Bayesian games with a "big" type space we can hardly give the set of Bayesian Nash Equilibria.

Considering Friedenberg and Meier (2010)'s results we can see the followings. If we take a "big" (the universal if it exists) type space, then any type space differing from it in only countable many types is so big (practically as big as the "big" is) that we can hardly determine the Bayesian Nash Equilibria. Put it differently, Friedenberg and Meier (2010)'s Result (i) does not answer the problem of how we can make conclusions about Bayesian Nash Equilibria for big type spaces from simple, easy to handle type spaces.

Furthermore, the interesting ("big") type spaces are uncountably infinite type spaces, so it is impossible that each type has positive probability by the common prior. Therefore, Friedenberg and Meier (2010)'s positive Result (ii) is not a solution for the problem we consider in this paper either.

However, it is also worth mentioning that Friedenberg and Meier (2010)'s goal is different from ours, they show that even if an analyst works with the universal type space (or a "big" type space, since the universal type space might not exist), her predictions will not coincide with the players' own predictions (assuming they know more and can preclude certain types). This is an alternative way to consider the problem discussed in this paper.

4 Conclusion

In this paper we have showed that Bayesian Nash Equilibria are not invariant under type morphisms. We have presented our results in a very simple, finite setting, by doing so we have demonstrated that this problem can be examined without any heavy mathematical tools. The example we have provided clearly shows that there is no chance for assumptions ensuring the invariance of Bayesian Nash Equilibria under either ordinary or conditional or strong type morphisms.

[11]Friedenberg and Meier (2010) use mixed Bayesian Nash Equilibria, so our results do not contradict with their results.

Our conclusion is as follows: contrary to the notions of interim correlated and interim independent rationalizable actions which are invariant under ordinary and conditional type morphisms respectively (see (Dekel et al., 2007; Ely and Peski, 2006) respectively)[12], the Bayesian Nash Equilibrium is a problematic solution concept for incomplete information games.

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