Algebraic duality theorems for infinite LP problems☆

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Abstract

In this paper we consider a primal-dual infinite linear programming problem-pair, i.e. LPs on infinite dimensional spaces with infinitely many constraints. We present two duality theorems for the problem-pair: a weak and a strong duality theorem. We do not assume any topology on the vector spaces, therefore our results are algebraic duality theorems. As an application, we consider transferable utility cooperative games with arbitrarily many players.

Keywords: Infinite dimensional duality theorems, TU games with infinitely many players, Core, Bondareva–Shapley theorem, Exact games

1. Introduction

In this paper we consider LP problems on infinite dimensional spaces with infinitely many constraints. We take pure vector spaces over the real field, i.e. we do not assume any topological structure on them. We present a weak and a strong duality theorem in this setting. From the proofs of these results it can be deduced that not only the finite dimensional statement can be generalized, but the concept of the proof of the finite case as well. In this sense our results elucidate that the weak and the strong duality theorem are for infinite problems, so it is natural to discuss them in such a setting.

[6] contains the first complete proof of the Strong Duality Theorem for finite LPs. It is worth to mention that [6] used the Strong Duality Theorem

☆I thank the anonymous referee for the valuable comments to the manuscript and the constructive suggestions. Financial supports by the Hungarian Scientific Research Fund (OTKA) and the Jnos Bolyai Research Scholarship of the Hungarian Academy of Sciences are gratefully acknowledged.

to prove the minimax theorem of zero-sum, two-person games. About duality theorems and results for infinite LPs [2] provides a thorough overview. Moreover, some of the papers, we consider when we discuss applications, refer to [5].

As an application we present a result for transferable utility cooperative games with arbitrarily many players. In cooperative game theory perhaps the most important solution concept is the core ([7]). In the finitely many player case the Bondareva–Shapley theorem, see [3] and [13], provides a necessary and sufficient condition for the non-emptiness of the core, it states that the core of a given game is non-empty if and only if the given game is balanced. The textbook proof of the Bondareva–Shapley theorem goes by the Strong Duality Theorem, see e.g. [10].

The most important previous results on this field are as follows: [11] proves a Bondareva–Shapley theorem type result for arbitrarily many player games and so does [8] for the countably many player case (the latter result is based on [5]). [9] gives an overview on the Bondareva–Shapley theorem type results for both the finitely and the infinitely many player settings.

In this paper we generalize the Bondareva–Shapley theorem for games with arbitrarily many players differently from that the above papers do. Our result is line with the finitely many player case result, i.e. we apply a purely algebraic argument, the purely algebraic Strong Duality Theorem.

We also discuss [12]'s result on exact games in the same vein as we do [11]'s. We show that the finitely many player characterization that [4] presents can be generalized into the setting described in the previous paragraph. Again, this result is also an application of the purely algebraic duality theorems.

The setup of the paper is as follows: in the next section we provide the algebraic duality theorems and in the last section we apply those to transferable utility cooperative games with arbitrarily many players.

2. Algebraic duality theorems

Notation: for any vector space X, X^* denotes its algebraic dual, i.e. X^* is for the set of the linear functionals on X. For any linear mapping $A : X \to Y$, A^* is for its adjoint mapping, i.e. $A^* : Y^* \to X^*$ is such that $\forall x \in X$, $\forall y \in Y^*$: $y(A(x)) = A^*(y)\ (x)$. Moreover, for any (real) vector space X, $C \subseteq X$ is a convex cone, if $\forall x, y \in C$, $\forall \alpha, \beta \in \mathbb{R}_+$: $\alpha x + \beta y \in C$.

Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be arbitrary functions, and $A \subseteq X$ be an arbitrary set. Then $f \leq_A g$ if $\forall x \in A$: $f(x) \leq g(x)$. If f and g are linear functionals and X is a coordinate space, then $f \leq g$ means $\forall x \in X_+$: $f(x) \leq g(x)$.

Let X be a vector space and $A \subseteq X$ be an arbitrary set. Then Lin (A) is for the vector space spanned by A, i.e. it is the smallest vector space which contains A.

First we provide the problem-pair we consider in this section.

Definition 1. *Let X, Y be vector spaces, $A : X \to Y$ be a linear mapping, $C \subseteq X$ be a convex cone, $b \in Y$ and c be a linear functional on X, i.e. $c \in X^*$. Consider the following problems:*

$$\begin{array}{cccc} (P) & c(x) & \to & \sup \\ & A(x) & = & b \\ & x & \in & C \end{array} \qquad \begin{array}{cccc} (D) & y(b) & \to & \inf \\ & A^*(y) & \geq_C & c \\ & y & \in & Y^* \end{array} \tag{1}$$

We say (P) / (D) has an optimal solution if $\sup\limits_{A(x)=b,\ x\in C} c(x)$ / $\inf\limits_{A^*(y)\geq_C c} y(b)$ is finite, i.e. $\sup\limits_{A(x)=b,\ x\in C} c(x) \in \mathbb{R}$ / $\inf\limits_{A^*(y)\geq_C c} y(b) \in \mathbb{R}$.

The above (P)-(D) problem-pair is a straightforward infinite reformulation of the well-known finite (dimensional) primal-dual problem-pair. It is worth noticing that since we do not impose any order on the vector space Y, without loss of generality we cannot give the canonical form (containing only inequalities) of the primal problem.

Next we consider a weak duality theorem.

Theorem 2 (Weak Duality Theorem)**.** *Consider the problem-pair of* (1)*. Then $\forall x \in X$, $\forall y \in Y^*$ such that x and y are feasible solutions of (P) and (D) respectively:*

$$c(x) \leq y(b) \ .$$

Proof. From the definition of an adjoint mapping: $\forall x \in X$, $\forall y \in Y^*$: $A^*(y)\ (x) = y(A(x))$. From the constraints $\forall x \in X$, $\forall y \in Y^*$ such that x and y are feasible solutions of (P) and (D) respectively:

$$c(x) \leq A^*(y)\ (x)$$

and

$$A^*(y)\ (x) = y(A(x)) = y(b)\ .$$

Summing up, $\forall x \in X$, $\forall y \in Y^*$ such that x and y are feasible solutions of (P) and (D) respectively:

$$c(x) \leq y(b)\ .$$

□

The following theorem is well-known, its proof can be found e.g. in [1] (Theorem 5.46 pp. 188-189).

Theorem 3 (Basic Separating Hyperplane Theorem)**.** *Let X be a vector space and A, B be disjoint convex sets of X such that A has an internal point*[1]*. Then there is a non-zero linear functional properly separating A and B.*

The following theorem is the main result of this section.

Theorem 4 (Strong Duality Theorem)**.** *Consider the problem-pair of* (1)*, moreover, assume that C has an internal point. Then one and only one of the following alternatives is always true:*

1. *(P) and (D) have optimal solutions and* $\sup\limits_{A(x)=b,\ x\in C} c(x) = \inf\limits_{A^*(y)\geq_C c} y(b)$.
2. *(P) has no feasible solution, and (D)'s feasibility set is not empty and its objective function is unbounded on it.*
3. *(P)'s feasibility set is not empty and its objective function is unbounded on it, and (D) has no feasible solution.*
4. *Both (P) and (D) have no feasible solution.*

Proof. Point *1.*: Assume that (P) has an optimal solution, and let $z > \sup\limits_{A(x)=b,\ x\in C} c(x)$ be arbitrarily fixed. Furthermore, let $d \doteq (b, z)$ and $B : X \to Y \times \mathbb{R}$ be a mapping such that $\forall x \in X$: $B(x) \doteq (A(x), c(x))$.

Then $d \notin B(C)$ and $B(C)$ is a convex set. Two cases can happen: (1) $d \in \text{Lin}\ (B(C))$. In this case, since C has an internal point, $B(C)$ also has

[1]For the definition of an internal point see e.g. [1] 5.45 Definition p. 188.

an internal point in Lin $(B(C))$, therefore we can apply Theorem 3 and get g a non-zero linear functional on Lin $(B(C))$ such that $\forall x \in B(C)$: $g(x) \geq 0$ and $g(d) \leq 0$. It is clear that g can be extended onto $Y \times \mathbb{R}$, and let f be such an extension of g.

(2) $d \notin$ Lin $(B(C))$. In this case let f be a linear functional on $Y \times \mathbb{R}$ such that $\forall x \in$ Lin $(B(C))$: $f(x) = 0$ and $f(d) = -1$ (it is clear that there is such a linear functional).

Then $\exists \beta \in \mathbb{R} \setminus \{0\}$ such that $\forall x \in C$:

$$\begin{aligned} f(B(x)) = f((A(x), c(x))) &= f((A(x), 0)) + f((0, c(x))) \\ &= f|_Y(A(x)) + \beta c(x) \geq 0 \ , \end{aligned} \tag{2}$$

and

$$f(d) = f((b, z)) = f((b, 0)) + f((0, z)) = f|_Y(b) + \beta z \leq 0 \ , \tag{3}$$

where $f|_Y$ is the restriction of f on $Y \times \{0\}$, so $f|_Y$ is a linear functional on Y.

Since the primal problem (P) has a feasible solution, $\exists x' \in C$ such that $A(x') = b$ and $c(x') < z$. By putting x' into the inequalities (2) and (3) we get that $\beta < 0$.

Let $y_0 \mathrel{\hat{=}} -\dfrac{f|_Y}{\beta}$. Then y_0 is a linear functional on Y, $y_0(A(x)) = A^*(y_0)\ (x)$ implies that $A^*(y_0) \geq_C c$, i.e. the feasibility set of (D) is not empty, and $y_0(b) \leq z$.

Summing up the discussion above, from the Weak Duality Theorem (Theorem 2) we get

$$\sup_{x \in C,\ A(x)=b} c(x) \leq \inf_{y \in Y^*,\ A^*(y) \geq_C c} y(b) \leq y_0(b) \leq z \ .$$

z was arbitrary fixed, therefore

$$\sup_{x \in C,\ A(x)=b} c(x) = \inf_{y \in Y^*,\ A^*(y) \geq_C c} y(b) \ .$$

Finally, assume that (D) has an optimal solution. If (P) has a feasible solution, then its objective function is bounded above (see the Weak Duality Theorem (Theorem 2)), therefore (P) also has an optimal solution and we can apply the above reasoning.

It is also clear from the above discussion that the following two cases cannot happen: (1) (P) has an optimal solution and (D) has no feasible solution, (2) (D) has an optimal solution and (P) has no feasible solution (in this case we can choose z as an arbitrary negative number).

Points *2* and *3*: From the Weak Duality Theorem (Theorem 2) if (P)'s / (D)'s objective function is not bounded, then (D) / (P) cannot have any feasible solution, and if (P) / (D) has a feasible solution, then the objective function of (D) / (P) cannot be unbounded.

Point *4*: It is left for the reader. □

Remark 5. In the above theorem we have assumed that C has an internal point. From the proof above we can conclude that it is possible to weaken this assumption. It is enough to assume that for any $z > \sup\limits_{A(x)=b,\ x \in C} c(x)$: $B(C)$ and d can be properly separated by a non-zero linear functional.

3. An application

Notation: Let N, the player set be an arbitrary non-empty set, $\mathcal{A} \subseteq \mathcal{P}(N)$ be a field and $v : \mathcal{A} \to \mathbb{R}$ be a mapping such that $v(\emptyset) = 0$. Then v is called transferable utility (TU) cooperative game (henceforth game) with player set $(N, \mathcal{A})$. Furthermore, let $\mathcal{G}^{(N,\mathcal{A})}$ denote the class of games with player set $(N, \mathcal{A})$, and $a(\mathcal{A})$ be for the set of additive set functions on field $\mathcal{A}$. Finally, for any $S \subseteq N$: χ_S denotes the characteristic function of set S.

Definition 6. *The core ([7]) of game $v \in \mathcal{G}^{(N,\mathcal{A})}$ is defined as follows:*

$$\text{Core}\ (v) \doteq \{\mu \in a(\mathcal{A}) \mid \forall A \in \mathcal{A} :\ \mu(A) \geq v(A)\ \textit{and}\ \mu(N) = v(N)\}\ .$$

In other words, the core of a game is the set of allocations such that (1) the total value of the grand coalition is allocated, and (2) no coalition has an incentive to deviate.

Definition 7. *Game $v \in \mathcal{G}^{(N,\mathcal{A})}$ is balanced, if*

$$\sup_{\lambda \in \Lambda_{\mathcal{A}},\ \sum\limits_{A \in \mathcal{A}_\lambda} \lambda_A \chi_A = 1} \sum_{A \in \mathcal{A}_\lambda} \lambda_A v(A) \leq v(N)\ ,$$

where $\Lambda_{\mathcal{A}} \doteq \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid |\{A \in \mathcal{A} \mid \lambda_A > 0\}| < \infty\}$ *and* $\mathcal{A}_\lambda \doteq \{A \in \mathcal{A} \mid \lambda_A \neq 0\}$.

In cooperative game theory, perhaps the most important solution concept is the core. In the finitely many player case the Bondareva–Shapley theorem, see [3] and [13], provides a necessary and sufficient condition for the non-emptiness of the core, it states that the core of a given game v is non-empty if and only if v is balanced. The textbook proof of the Bondareva–Shapley theorem goes by the Strong Duality Theorem, see e.g. [10]. The primal problem belongs to the concept of balancedness and the dual problem belongs to the non-emptiness of the core.

In the case of arbitrarily many players [11] proves a Bondareva–Shapley theorem type result and so does [8] for the countably many player case. The proofs of these results, however, are based on topological concepts.

The next theorem is a generalized and non-topological Bondareva-Shapley theorem type result.

Theorem 8. *For any* $v \in \mathcal{G}^{(N,\mathcal{A})}$*:* $\text{Core}\,(v) \neq \emptyset$ *if and only if* v *is balanced.*

Proof. Consider the (P)-(D) problem-pair of (1), where

(i) $X \doteq \text{Lin}\,(\Lambda_{\mathcal{A}} \cup \{\mathbf{1}\})$, where $\mathbf{1}$ is the constant 1 function on $\mathcal{A}$,

(ii) v is a linear functional defined as follows: for any $x \in \Lambda_{\mathcal{A}}$: $v(x) \doteq \sum\limits_{A \in \mathcal{A}_x} x_A v(A)$, and $v(\mathbf{1}) = 0$,

(iii) $Y \doteq \text{Lin}\left(\left\{\sum\limits_{A \in \mathcal{A}_x} x_A \chi_A \mid x \in \Lambda_{\mathcal{A}}\right\}\right)$, i.e. the elements of Y are linear combinations of finitely many characteristic functions on sets of $\mathcal{A}$. Therefore the elements of Y are $\mathcal{A}$-measurable functions and Y is a vector space,

(iv) $C \doteq X_+$, $\mathbf{1}$ is an internal point of set C,

(v) $A : X \to Y$ is a linear mapping defined as follows: $\forall x \in \Lambda_{\mathcal{A}}$: $A(x) \doteq \sum\limits_{A \in \mathcal{A}_x} \chi_A x_A$, and $A(\mathbf{1}) = 0_Y$. Notice that, for any $x \in X$: $x = x' + \alpha \mathbf{1}$, where $x' \in \text{Lin}\,(\Lambda_{\mathcal{A}})$ and $\alpha \in \mathbb{R}$,

(vi) It is easy to verify that $Y^* = a(\mathcal{A})$,

(vii) A^* is defined as follows: $\forall y \in Y^*$: let $A^*(y)$ be a linear functional such that for any $x \in \Lambda_{\mathcal{A}}$: $A^*(y)\ (x) \circeq \int x\ dy$, and $A^*(\mathbf{1}) = 0$. Then for any $x \in X$, where $x = x' + \alpha\mathbf{1}$ (see point (v)), for any $y \in Y^*$:

$$
\begin{aligned}
y(A(x)) = \int \sum_{A \in \mathcal{A}_{x'}} \chi_A x'_A\ dy = \sum_{A \in \mathcal{A}_{x'}} \int \chi_A x'_A\ dy \\
= \sum_{A \in \mathcal{A}_{x'}} y(A) x'_A = A^*(y)\ (x)\ .
\end{aligned}
$$

Therefore A^* is the adjoint of A,

(viii) $b \circeq \chi_N$, i.e. $b \circeq 1$,

(ix) $c \circeq v \in \mathcal{G}^{(N,\mathcal{A})}$.

Core $(v) \neq \emptyset$ if and only if (D) has an optimal solution and that is not greater than $v(N)$.

v is balanced if and only if (P) has an optimal solution and that is not greater than $v(N)$ (see points (ii) an (v)).

We can apply point *1* of the Strong Duality Theorem (Theorem 4). □

In Definition 6 we have defined the core as a set of certain additive set functions. It is possible, however, to generalize the above concept so that the core, call it σ-core consists of σ-additive set functions. Then the non-emptiness of the σ-core would mean that the (ordinary) core contains a σ-additive element. Since in our opinion the purely algebraic duality theorems do not imply a Bondareva-Shapley theorem type result for this generalized core concept, in this paper we do not discuss this problem.

Notice that in the proof of Theorem 8 the vector spaces are coordinate spaces. Therefore, we can deploy this fact and get to the following problem-pair, where $A \in \mathcal{A}$ is an arbitrary non-empty set:

$$
\begin{array}{lrcll}
(P') & c(x) & \rightarrow & \sup & \\
& A(x) & = & b & \\
& x_S & \geq & 0 & S \in \mathcal{A} \setminus \{A\} \\
& x_A & \in & \mathbb{R} & \\
\\
(D') & y(b) & \rightarrow & \inf & \\
& A^*(y)\ (\chi_S) & \geq & c(\chi_S) & S \in \mathcal{A} \setminus \{A\} \\
& A^*(y)\ (\chi_A) & = & c(\chi_A) & \\
& y & \in & Y^* &
\end{array}
\tag{4}
$$

Since the above problem-pair is a special case of the problem-pair (1), both duality theorems, the Weak and the Strong Duality Theorem (Theorems 2 and 4) hold for it.

[14] and [12] introduce the concept of exact games for the finitely many and the infinitely many player case respectively.

Definition 9. *Game* $v \in \mathcal{G}^{(N,\mathcal{A})}$ *is exact, if* $\forall A \in \mathcal{A}$*:* $\exists \mu \in \mathrm{Core}\ (v)$ *such that* $\mu(A) = v(A)$*.*

The problem pair (P')-(D') (see (4)) implies a natural generalization of the concept of balancedness (see Definition 7) introduced by [4]:

Definition 10. *Game* $v \in \mathcal{G}^{(N,\mathcal{A})}$ *is exactly balanced, if*

$$\sup_{\substack{\lambda \in \Lambda^e_{\mathcal{A}},\ \sum\limits_{A \in \mathcal{A}_\lambda} \lambda_A \chi_A = 1}} \sum_{A \in \mathcal{A}_\lambda} \lambda_A v(A) \leq v(N) \ ,$$

where $\Lambda^e_{\mathcal{A}} \doteq \{\lambda \in \mathbb{R}^{\mathcal{A}} \mid |\{A \in \mathcal{A} \mid \lambda_A > 0\}| < \infty$ *and* $|\{A \in \mathcal{A} \mid \lambda_A < 0\}| \leq 1\}$ *and* $\mathcal{A}_\lambda \doteq \{A \in \mathcal{A} \mid \lambda_A \neq 0\}$*.*

Then we can generalize [12] in the direction of that we do not use any topology in the proof, and [4] in the direction that there can be arbitrarily many players in the game, and we get the following result:

Theorem 11. *For any* $v \in \mathcal{G}^{(N,\mathcal{A})}$*:* v *is exact if and only if* v *is exactly balanced.*

Proof. Apply the proof of Theorem 8 to the problem-pair (4). □

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