Optimal redistricting under geographical constraints: Why "pack and crack" does not work.*

Clemens Puppe[1] and **Attila Tasnádi**[2]

[1] Department of Economics, University of Karlsruhe, D – 76128 Karlsruhe, Germany, puppe@wior.uni-karlsruhe.de

[2] Department of Mathematics, Corvinus University of Budapest, H – 1093 Budapest, Fővám tér 8, Hungary, attila.tasnadi@uni-corvinus.hu *(corresponding author)*†

Revised: May 2009

Appeared in *Economics Letters* 105(2009), 93-96. ©Elsevier
The original article is available at www.sciencedirect.com.
doi:10.1016/j.econlet.2009.06.008

Abstract. We show that optimal partisan redistricting with geographical constraints is a computationally intractable (NP-complete) problem. In particular, even when voter's preferences are deterministic, a solution is generally not obtained by concentrating opponent's supporters in "unwinnable" districts ("packing") and spreading one's own supporters evenly among the other districts in order to produce many slight marginal wins ("cracking").

Keywords: districting, gerrymandering, NP-complete problems.

JEL Classification Number: D72

*We would like to thank an anonymous referee for his/her helpful comments. The second author gratefully acknowledges financial support from the Hungarian Academy of Sciences (MTA) through the Bolyai János research fellowship.

†Telephone/fax: +36 1 3870834

1 Introduction

The problem of determining the shape of electoral districts in democratic elections has received a significant amount of attention recently.[1] From the viewpoint of an involved political party the optimal policy consists in maximizing the number of districts won by that party (by simple majority, say). This is known as "optimal partisan gerrymandering." According to common wisdom the solution to this problem in a two-party system is obtained by the so-called "pack and crack" procedure: concentrate the supporters of the opponent party in "unwinnable" districts ("pack") and spread one's own supporter evenly over the other districts which are then won by the smallest possible margin ("crack"). It has been noted that, while this intuition is valid in the deterministic case with no uncertainty about voters' preferences, it does not generally carry over to models with incomplete information (see Friedman and Holden, 2008).

Here, we point out that the "pack and crack" procedure fails to produce an optimal solution even in the deterministic case once geographical constraints are taken into account. From a practical perspective this is an important observation since geographical constraints, such as contiguity of districts, are part of the legal requirements (in the U.S.) on partisan gerrymandering. Furthermore, we demonstrate that the failure of the "pack and crack" procedure is related to a more general structural feature of the underlying problem: in the case of geographical constraints there exists no simple (i.e. polynomial time) algorithm for determining an optimal districting. Specifically, we prove that in this case deciding whether there exists a districting such that a party wins at least a given number of districts is an NP-complete problem.[2]

2 The Framework

We assume that voters have to be divided into a given number of equal districts in which candidates of two parties, parties A and B, compete for a seat. A district is won by a candidate if she receives the majority of votes. We shall denote the number of voters by n, the set of voters by N, the number of districts by d, and the set of districts by $\mathcal{D}$. We assume that the voters have deterministic and known party preferences given by the mapping $v : N \to \{A, B\}$, which thus determines the number of supporters of parties A and B, denoted by n_A and n_B, respectively. For simplicity, we also assume that d divides n and that each district must consist of $2k + 1$ voters with $k \geq 2$. Therefore, assuming full participation, each district is won by either party A or party B. We introduce the following simple, but quite general framework, to incorporate geographical constraints.

Definition 1 (Geography)**.** A set system $\mathcal{S} \subset 2^N$ of $2k + 1$ sized subsets of N such that there exist appropriately chosen sets $S_1, \ldots, S_d \in \mathcal{S}$ partitioning N is called a *geography.* A *districting problem* for geography S is a pair $(N, \mathcal{S})$.

Definition 2 (Districting)**.** For a given geography $\mathcal{S} \subset 2^N$ a mapping $f : N \to \mathcal{D}$ is called a *districting* if $f^{-1}(i) \in \mathcal{S}$ for all $i \in \mathcal{D}$ and $\cup_{i \in \mathcal{D}} f^{-1}(i) = N$.

[1] See, among many others, Owen and Grofman (1988), Sherstyuk (1998), Gilligan and Matsusaka (1999), and for more recent contributions, Friedman and Holden (2008), Gul and Pesendorfer (2007), Puppe and Tasnádi (2008).

[2] Altman (1997) also points out that several problems related to achieving an ex ante unbiased districting are NP-hard. For details on computer aided districting we refer to Altman, MacDonald and McDonald (2005).

Observe that if $\mathcal{S}$ consists of all $2k+1$ sized subsets of N, then we obtain as a special case *districting without geographical constraints.* A districting f and voters' preferences v determine the number of districts won by parties A and B, which we denote by $F(f,v,A)$ and $F(f,v,B)$, respectively.

Definition 3 (Optimal districting)**.** For a given problem $(N,\mathcal{S})$ and given voters' preferences $v: N \to \{A,B\}$ a districting $f: N \to \mathcal{D}$ is *optimal* for party $I \in \{A,B\}$ if $F(f,v,I) \geq F(g,v,I)$ for any districting $g: N \to \mathcal{D}$.

Note that since there are finitely many districtings, there exists at least one optimal districting for each party.

3 Pack and crack

The "pack and crack" principle, which informally requires the construction of losing districts consisting entirely of the opponent's supporters and forming winning districts with slight marginal wins, is usually believed to produce an optimal districting in the deterministic case and serves as a benchmark in a number of recent papers (e.g. Gul and Pesendorfer, 2007 and Friedman and Holden, 2008). We show by example that an algorithm respecting the pack and crack principle does not necessarily produce an optimal districting.

In our subsequent analysis, we will employ the following notion of *waste function.* Observe that if a party, say party A, constructs a winning district D with $j \geq k+1$ own supporters and $2k+1-j$ supporters of party B, then party A *wastes* $j-(k+1)$ voters. Similarly, if party A constructs a loosing district D with $j \leq k$ own supporters and $2k+1-j$ supporters of party B, then party A *wastes* j voters. Therefore, we define the waste function of party A associated with a given district $D \in \mathcal{S}$ by $w_A(D) = j-(k+1)$ if D is a winning district for A and $w_A(D) = j$ if D is a loosing district for A, where j stands for the number of supporters of party A in D. The waste function $w_B: \mathcal{S} \to \{0,1,\ldots,k\}$ of party B is defined analogously.

Definition 4 (Pack and crack)**.** A procedure that produces a districting $(D_1, D_2, \ldots, D_d)$ for $(N,\mathcal{S})$ in the given order is a *pack and crack procedure* for party $I \in \{A,B\}$ if for any $i = 1,\ldots,d-1$ we have $w_I(D_i) \leq w_I(D_{i+1})$, and there does not exist an $i = 0,\ldots,d-1$ and a district $D' \in \mathcal{S}$ such that $w_I(D') < w_I(D_{i+1})$ and $(D_1, D_2, \ldots, D_i, D')$ is extendable to an admissible districting.

Without geographical constraints we have the following well-known result.

Proposition 1. *There exists a pack and crack procedure that determines an optimal districting in polynomial time in case of no geographical constraints.*

Proof. The following procedure, already described in Gilligan and Matsusaka (1999), finds an optimal districting for party $I \in \{A,B\}$. In case of $n_I \leq d(k+1)$ fill the first $m = \lfloor \frac{n_I}{k+1} \rfloor$ districts with $k+1$ supporters of party I and k supporters of its opponent, party J.[3] Secondly, fill the next $m' = \lfloor \frac{n_J - km}{2k+1} \rfloor$ districts entirely with party J supporters. Finally, if $m+m' < d$, fill the last district with the remaining voters.

[3]In what follows $\lfloor x \rfloor$ stands for the largest integer not greater than x.

In case of $n_I > d(k+1)$ fill the first $m = \lfloor \frac{n_J}{k} \rfloor$ districts with $k+1$ supporters of party I and k supporters of its opponent, party J. Secondly, fill the last $m' = \lfloor \frac{n_I-(k+1)m}{2k+1} \rfloor$ districts entirely with party J supporters. Finally, if $m + m' < d$, fill the omitted district with the remaining voters. □

The algorithm given in the proof of Proposition 1 starts with "cracking" and as a result after the first m steps there are (almost) no other possibilities for the remaining districts than "packing." Assuming $2(k+1) \leq n_I < d(k+1)$, note that a procedure starting with packing $m^* = \lfloor \frac{n_J}{2k+1} \rfloor$ districts entirely with party J supporters results in at least m^* winning districts for party I's opponent, which is larger than $d - m$. Therefore, not all pack and crack procedures result in an optimal districting even in case of no geographical constraints. This latter example highlights that the order in which cracking and packing is carried out matters.

To be more precise, let us call procedures that determine solutions in the manner of the proof of Proposition 1 simply "crack" procedures.

Definition 5 (Crack)**.** A procedure that produces a districting $(D_1, D_2, \ldots, D_d)$ for $(N, \mathcal{S})$ in the given order, is a *crack procedure* for party $I \in \{A, B\}$ if the sequence starts with $m \in \{0, 1, \ldots, d\}$ winning districts for party I and terminates with $d - m$ loosing districts for party I such that $w_I(D_i) \leq w_I(D_{i+1})$ for all $i = 1, \ldots, m-1$, and there does not exist an $i = 0, \ldots, m-1$ and a party I winning district $D' \in \mathcal{S}$ such that $w_I(D') < w_I(D_{i+1})$ and that $(D_1, D_2, \ldots, D_i, D')$ is extendable to an admissible districting.

Clearly, crack procedures constitute a subclass of pack and crack procedures. Proposition 1 shows that crack procedures determine optimal districtings in case without geographical constraints. However, this is no longer true if geographical constraints are present. We verify this based on the simple "rectangular country" shown in Figure 1 with $n_A = 24$, $n_B = 26$, $k = 2$, $d = 10$, and in which party A supporters are indicated by solid circles, while party B supporters are indicated by empty circles. Two voters are *adjacent* if they have a common boundary (edge) and a district is *connected* if two voters living in the same district are "reachable" through a sequence of adjacent voters. We impose the simple restriction on the districting that only connected districts may be formed, which defines a geography $\mathcal{S}$ for the given country.

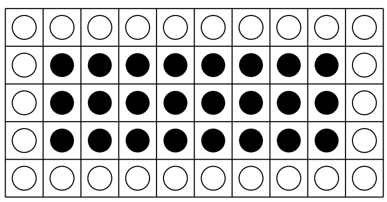

Figure 1: Rectangular country

Figure 2 shows two possible districtings obtained by different crack procedures. Numbers refer to the order in which districts are formed, omitting for simplicity party B supporters. Evidently, the districting shown on the left hand side of Figure 2 gives 8 winning districts to party A, while the districting on the right hand side gives only 6 winning districts to party A. In particular, crack procedures may fail to result in an optimal solution already in the case of the single additional "contiguity" constraint.

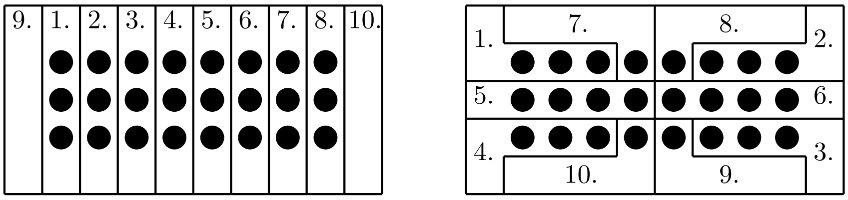

Figure 2: Two crack solutions

But the situation becomes even worse if one allows for more general geographies. As an example, consider Figure 3, and let the geography be given by the districts corresponding to the five rows and the five columns, respectively. Party A supporters are again indicated by solid circles. Any crack procedure has to choose the top row district as the first district. This necessarily leads to the horizontal districting and thus results in just one winning district for party A, whereas the vertical districting would give two winning districts.

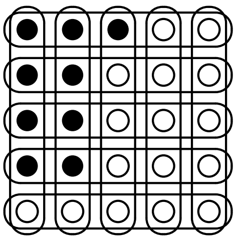

Figure 3: Crack is not optimal

4 Determining an optimal districting is NP-complete

The above negative examples suggest to consider the problem of optimal redistricting for a given geography $\mathcal{S}$ on N from a computational perspective. We establish that even the associated decision problem, i.e. deciding whether there exists a districting with at least m winning districts for a party, say party A, is a computationally intractable NP-complete problem; we call this problem WD.[4] In order to prove this, we shall reduce a well-known variant of SET PACKING (henceforth, SP), a proven NP-complete problem (see Garey and Johnson; 1979, pp. 221), to WD. SP asks whether a given set system $\mathcal{C}$ of subsets of X such that $|C| \leq k+1$ for all $C \in \mathcal{C}$ (with $k \geq 2$) possesses at least m mutually disjoint sets.

Theorem 2. *WD is NP-complete.*

Proof. Whether a districting f possesses at least m winning districts for party A can be verified easily in polynomial time, and therefore WD $\in$ NP.

We take an instance of SP for which we can assume without loss of generality that $X = \cup_{C \in \mathcal{C}} C$. The elements of the set X will all be taken to be party A supporters.

[4]Observe that if there would exist a polynomial time algorithm for optimal districting one would obtain a polynomial time algorithm for WD as well.

First, we associate with an arbitrarily chosen set $C \in \mathcal{C}$ a district D_C as follows: If $|C| = j \leq k+1$, then we add k new voters for party A to X, $k+1-j$ new voters for party B to the set of voters, and define district D_C a consisting of the party A supporters from C and the newly added $k+(k+1-j)$ voters. Clearly, D_C is a winning district for party A. By carrying out the above procedure, we obtain $|\mathcal{C}|$ districts and a set of voters Y. We illustrate the types of districts that can occur in this manner so far for $k = 2$ in Figure 4. As above, party A supporters are indicated by solid circles.

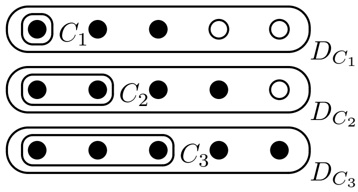

Figure 4: First step in case $k = 2$

Secondly, Y and the $2k|Y|$ newly added party B supporters complete the set of voters N. We partition N into $|Y|$ equally sized sets such that each partition element contains exactly one voter from the set Y, and we include these sets in the geography $\mathcal{S}$. We shall denote the district containing $y \in Y$ by D_y. Clearly, the above defined partition of N gives an admissible districting of N in which party B wins all districts.

Thirdly, we complete the geography $\mathcal{S}$. Take an arbitrarily chosen set $C \in \mathcal{C}$ and its associated district D_C as described in the first step. We partition $N_C = \cup_{y \in D_C} D_y$ into $2k+1$ equally sized sets such that one set equals D_C and the remaining $2k$ sets all contain exactly one element from each set $D_y \setminus \{y\}$ (where $y \in D_C$), which gives us districts D_C^i for $i = 2, \ldots, 2k+1$. We illustrate in Figure 5 the districts obtained in this way through our second and third steps for the case $k = 2$ and for two given sets $C, C' \in \mathcal{C}$ with two common elements. The "vertical sets" were derived in our second step, while the "horizontal sets" in our third step. To illustrate the interplay between the vertical and horizontal sets, for example, if C is contained in a set packing, then C' cannot be contained in the same set packing; and therefore, turning to winning districts, the derived districting containing D_C would contain the 5 horizontal districts on the left hand side and the 3 vertical districts on the right hand side.

Formally, to obtain the districting problem $(N, \mathcal{S})$, let

$$\mathcal{S} = \{D_C\}_{C \in \mathcal{C}} \bigcup \{D_y\}_{y \in Y} \bigcup \{D_C^i\}_{i=2, C \in \mathcal{C}}^{2k+1}.$$

A districting for the geography $\mathcal{S}$ contains at least m winning districts for party A if and only if it does contain at least m sets from $(D_C)_{C \in \mathcal{C}}$, since these sets are exactly the winning sets for party A by construction of $\mathcal{S}$. Observe that if we can take $p \geq m$ winning sets for party A, then the districting contains for any winning district $D = D_C$ of party A the districts $\left(D_C^i\right)_{i=2}^{2k+1}$ by our third step ("horizontal sets") and for any $y \in Y$ not contained in a winning district for party A the associated set D_y defined by our second step ("vertical sets"). Therefore, the necessary and sufficient condition for the existence of a districting with at least m winning districts for party A is the existence of m mutually disjoint sets from $\mathcal{C}$. Thus, we have given (since k is fixed) a polynomial time reduction of SP to WD, which completes the proof. □

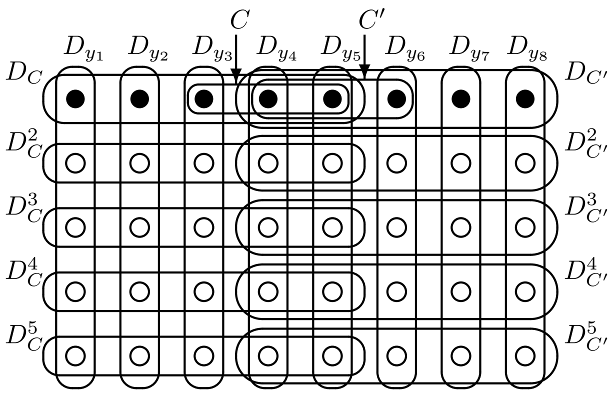

Figure 5: $C = \{y_3, y_4, y_5\}$ and $C' = \{y_4, y_5, y_6\}$

References

[1] Altman, M., 1997, Is automation the answer? The computational complexity of automated redistricting, Rutgers Computer and Law Technology Journal 23, 81–142.

[2] Altman, M., MacDonald, K. and M.P. McDonald, 2005, From crayons to computers: The evolution of computer use in redistricting, Social Science Computer Review 23, 334–346.

[3] Friedman, J.N. and R.T. Holden, 2008, Optimal gerrymandering: Sometimes pack, but never crack, American Economic Review 98, 113–144.

[4] Garey, M.R. and D.S. Johnson, 1979, Computers and Intractability: A Guide to the Theory of NP-Completeness. (W.H Freeman and Company, San Francisco).

[5] Gilligan, T.W. and J.G. Matsusaka, 1999, Structural constraints on partisan bias under the efficient gerrymander, Public Choice 100, 65–84.

[6] Gul, R. and W. Pesendorfer, 2007, Strategic Redistricting, mimeographed.

[7] Owen, G. and B. Grofman, 1988, Optimal partisan gerrymandering, Political Geography Quarterly 7, 5–22.

[8] Puppe, C. and A. Tasnádi, 2008, A computational approach to unbiased districting, Mathematical and Computer Modelling 48, 1455–1460.

[9] Sherstyuk, K., 1998, How to gerrymander: A formal analysis, Public Choice 95, 27–49.