Axiomatic districting*

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Summary. *In a framework with two parties, deterministic voter preferences and a type of geographical constraints, we propose a set of simple axioms and show that they jointly characterize the districting rule that maximizes the number of districts one party can win, given the distribution of individual votes (the "optimal gerrymandering rule"). As a corollary, we obtain that no districting rule can satisfy our axioms and treat parties symmetrically.*

Keywords: districting, gerrymandering.

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1 Introduction

The districting problem has received considerable attention recently, both from the political science and the economics viewpoint. Much of the recent work has focused on strategic aspects and the incentives induced by different institutional designs on the political parties, legislators and voters (see, among others, Besley and Preston, 2007, Friedman and Holden, 2008, Gul and Pesendorfer, 2009). Other contributions have looked at the welfare implications of different redistricting policies (e.g. Coate and Knight, 2007). Finally, there is also a sizable literature on the computational aspects of the districting problem (see, e.g. Puppe and Tasnádi, 2008, and the references therein).

In contrast to these contributions, the present paper takes a *normative* point of view. We formulate desirable properties ("axioms") and investigate which districting functions satisfy them. There are several reasons for exploring this approach. First, the axiomatic method allows one to endow the vast space of conceivable districting rules with useful additional structure: each combination of desirable properties characterizes a specific class of districting rules, and thereby helps one to assess their respective merits. Second, one may hope that specific combinations of axioms single out a few, perhaps sometimes even a unique districting rule, thus reducing the space of possibilities. Finally, the axiomatic approach may reveal incompatibility of certain axioms by showing that *no* districting rule can satisfy certain combinations of desirable properties, thereby terminating a futile search.

In a framework with two parties and geographical constraints on the shape of districts, we propose a set of simple axioms and show that they jointly characterize the districting rule that maximizes the number of districts one party can win, given the distribution of individual votes (the "optimal gerrymandering rule"). While some of the axioms have a more pragmatic justification, others have straightforward normative foundations such as the neutrality property which requires that a districting rule should treat parties symmetrically. Evidently, by generating a maximal number of winning districts for one of the parties, the optimal gerrymandering rule violates the neutrality axiom. Therefore, as a straightforward corollary of our main result, we obtain that no districting rule can satisfy a set of reasonable properties and treat parties symmetrically at the same time.

The work closest to ours in the literature is Chambers (2008, 2009) who also takes an axiomatic approach. However, one of his central conditions is the requirement that the election outcome be *independent* of the way districts are formed ("gerrymandering-proofness"), and the main purpose of his analysis is to explore the consequences of this requirement. By contrast, our focus is precisely on the districting process which we try to structure by means of simple governing principles. In particular, geographical constraints which are absent in Chambers' model play an important role in our analysis.

The districting rules that we consider depend among other things on the distribution of votes for each party in the population. One might argue, perhaps on grounds of some "absolute" notion of *ex ante* fairness, that a districting rule must not depend on voters' party preferences since these can change over time. From this perspective, the districting problem is not really an issue and it would seem that any districting which partitions the population in (roughly) equally sized subgroups should be acceptable. By contrast, in the present paper we are interested in a "relative" or *ex post* notion of fair districting, i.e. in the question of what would constitute an acceptable districting rule *given* the distribution of the supporters of each party in the population. This question seems particularly important for practical purposes since a districting policy can be successfully implemented only if it receives sufficient support by the *actual* legislative body.

2 The Framework

We assume that parties A and B compete in an electoral system consisting only of single member districts, where the representatives of each districts are determined by plurality. The parties as well as the independent bodies face the following districting problem.

Definition 1 (Districting problem)**.** A *districting problem* is given by the structure $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, where

- the voters are located within a subset X of the plane $\mathbb{R}^2$,
- $\mathcal{A}$ is the σ-algebra on X consisting of all districts that can be formed without geographical or any other type of constraints,
- the distribution of voters is given by a measure μ on $(X, \mathcal{A})$,
- the distributions of party A and party B supporters are given by measures μ_A and μ_B on $(X, \mathcal{A})$ such that $\mu = \mu_A + \mu_B$,
- t is the given number of seats in parliament,
- $G \subseteq \mathcal{A}$, also called *geography*, is the set of admissible districts satisfying $\mu(g) = \mu(X)/t$ and
$$\mu_A(g) \neq \mu_B(g) \tag{1}$$
for all $g \in G$, and possessing a partitioning of X, i.e there exist mutually disjoint sets $g'_1, \ldots, g'_t \in G$ such that $\cup_{i=1}^{t} g'_i = X$.

Condition (1) excludes ties in the distribution of party supporters in all admissible districts to avoid the necessity of introducing tie-breaking rules. This condition is satisfied, for instance, if the set of voters is finite, μ, μ_A, μ_B are the counting measures and the district sizes are odd.

Definition 2 (Districting)**.** A *districting* for the problem $(X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is a subset $D \subseteq G$ such that D forms a partition of X and $\#D = t$.

We shall denote by $\delta_A(D)$ and $\delta_B(D)$ the number of districts won by party A and party B under D, respectively. We write $\mathcal{D}_\Pi$ for the set of all admissible districtings of problem Π and let $\delta_A(\mathcal{D}) = \{\delta_A(D) : D \in \mathcal{D}\}$ and $\delta_B(\mathcal{D}) = \{\delta_B(D) : D \in \mathcal{D}\}$ for any $\mathcal{D} \subseteq \mathcal{D}_\Pi$.

Definition 3 (Solution)**.** A *solution* F associates to each districting problem Π a non-empty set of chosen districtings $F_\Pi \subseteq \mathcal{D}_\Pi$.

3 Several Solutions

We now present a number of simple solution candidates. The first solution determines the optimal partisan gerrymandering from the viewpoint of party A.

Definition 4 (Optimal solution for A)**.** The optimal solution O^A for party A determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of those districtings that maximize the number of winning districts for party A, i.e.

$$O^A_\Pi = \arg \max_{D \in \mathcal{D}_\Pi} \delta_A(D).$$

Evidently, in the absence of other objectives, O^A is the solution favored by party A supporters. The optimal solution O^B for party B is defined analogously. If we are referring to an optimal solution O, then we have either O^A or O^B in mind.

The next solution minimizes the difference in the number of districts won by the two parties. It has an obvious egalitarian spirit.

Definition 5 (Most equal solution)**.** The solution ME determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of most equal districtings, i.e.

$$ME_\Pi = \arg \min_{D \in \mathcal{D}_\Pi} \left| \delta_A(D) - \delta_B(D) \right| . \tag{2}$$

Since an equal solution does not always exist the most equal solution aims to get as close as possible to equality in terms of the number of winning districts for the two parties.

The third solution maximizes the difference in the number of districts won by the two parties. The objective to maximize the winning margin of the ruling party could be motivated, for instance, by the desire to avoid too much political compromise.

Definition 6 (Most unequal solution)**.** The solution MU determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of most unequal districtings, i.e.

$$MU_\Pi = \arg \max_{D \in \mathcal{D}_\Pi} \left| \delta_A(D) - \delta_B(D) \right| . \tag{3}$$

Fourth, we consider the solution that minimizes partisan bias. It has a clear motivation from the point of view of maximizing representation of the "people's will" in the sense that the share of the districts won by each party is as close as possible to its share of votes in the population.

Definition 7 (Least biased solution)**.** The solution LB determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of those districtings that minimize the absolute difference between shares in winning districts and shares in votes, i.e.

$$LB_\Pi = \arg \min_{D \in \mathcal{D}_\Pi} \left| \frac{\delta_A(D)}{t} - \frac{\mu_A(X)}{\mu(X)} \right| = \arg \min_{D \in \mathcal{D}_\Pi} \left| \frac{\delta_B(D)}{t} - \frac{\mu_B(X)}{\mu(X)} \right| . \tag{4}$$

Finally, we mention the trivial solution that associates to each problem the set of *all* admissible districtings.

Definition 8 (Complete solution)**.** The complete solution C associates with any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of all possible districtings $\mathcal{D}_\Pi$.

4 Axioms

In this section, we formulate five simple axioms each of which has an appeal either from a normative or a pragmatic point of view.

The case of two districts plays a fundamental role in our analysis. Note that by (1) it is not possible that a party can win both districts under one districting and lose both districts under another districting, i.e. if $t = 2$ then $\delta_A(\mathcal{D}_\Pi)$ (respectively, $\delta_B(\mathcal{D}_\Pi)$) cannot contain both 0 and 2. Our first axiom requires that a solution must in fact be "determinate" in the two-district case in the sense that it must not leave open the issue whether there is a draw between the two parties or a victory for one party. In other words, if a solution chooses a districting that results in a draw between the parties for a given problem it cannot choose another districting for the *same* problem that results in a victory for one party.

Axiom 1 (Two-district determinacy)**.** A solution F satisfies *two-district determinacy* if for any districting problem Π with $t = 2$, the sets $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ are singletons.

Evidently, all solutions considered in Section 2 with the exception of the complete solution C satisfy Axiom 1. Also observe that on the family of all two-district problems the most equal solution *ME* and the least biased solution *LB* coincide.[1]

Our next axiom requires that a solution behaves "uniformly" on the set of two-district problems in the sense that the solution must treat different two-district problems in the same way, provided they admit the same set of possible distributions of the number of districts won by each party.

Axiom 2 (Two-district uniformity)**.** A solution F satisfies *two-district uniformity* if for any districting problems Π and Π' with $t = 2$ such that $\delta_A(\mathcal{D}_\Pi) = \delta_A(\mathcal{D}_{\Pi'})$ (and therefore also $\delta_B(\mathcal{D}_\Pi) = \delta_B(\mathcal{D}_{\Pi'})$) we have $\delta_A(F_\Pi) = \delta_A(F_{\Pi'})$ (and therefore also $\delta_B(F_\Pi) = \delta_B(F_{\Pi'})$).

Even though it is imposed only in the two-district case, Axiom 2 is admittedly a strong requirement. It is motivated by the desire to keep the complexity of a solution manageable. Evidently, without Axiom 2, characteristics other than the possible distributions of the number of districts won by each party would have to enter the definition of a solution. Whatever these characteristics may be – whether derived from the underlying distribution of party supporters or from geographical information – their influence would complicate the definition and implementation of a districting rule considerably. In any case, it is easily seen that all solutions presented in Section 2 above satisfy Axiom 2.

Our third axiom, imposed on districting problems of any size, has a motivation similar to that of the previous axiom. It states that if a possible districting induces the same distribution of the number of winning districts for each party than some districting chosen by a solution, it must be chosen by this solution as well.

Axiom 3 (Indifference)**.** A solution F satisfies *indifference* if for any districting problem Π we have that $D \in F_\Pi$, $D' \in \mathcal{D}_\Pi$, $\delta_A(D) = \delta_A(D')$ and $\delta_B(D) = \delta_B(D')$ implies $D' \in F_\Pi$.

[1] To verify this, observe that if there exist admissible districtings $D, D' \in \mathcal{D}_\Pi$ with $\delta_A(D) = 2$ and $\delta_A(D') = 1$, then one must have $0.5 < \mu_A(X)/\mu(X) < 0.75$. Thus, D' must be chosen both by *ME* and *LB*.

Again, it is evident that all solution presented so far satisfy this condition. The following consistency axiom plays a central role. It requires that a solution to a problem should also deliver appropriate solutions to specific subproblems. Its spirit is very similar to the *uniformity principle* in Balinski and Young's (2001) theory of apportionment ("every part of a fair division should be fair").

Axiom 4 (Consistency)**.** A solution F satisfies *consistency* if for any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, any $D \in F_\Pi$ and any $D' \subseteq D$ we have for $Y = \cup_{d \in D'} d$ that

$$D|_Y = D' \in F_{\Pi|_Y} = F_{(Y, \mathcal{A}|_Y, \mu|_Y, \mu_A|_Y, \mu_B|_Y, \#D', G|_Y)},$$

where $\mathcal{A}|_Y = \{A \cap Y : A \in \mathcal{A}\}$, $G|_Y = \{g \in G : g \subseteq Y\}$ and $\mu|_Y, \mu_A|_Y, \mu_B|_Y$ stand for the restrictions of measures μ, μ_A, μ_B to $(Y, \mathcal{A}|_Y)$.

The optimal and complete solutions satisfy consistency. This is evident for the complete solution. To verify it for the optimal solution suppose, by contradiction, that there would exist $D' \subset D \in O^A_\Pi$ such that $D' \notin O^A_{\Pi|_Y}$, where $Y = \cup_{d \in D'} d$. This would imply $\delta_A(D'' \cup (D \setminus D')) > \delta_A(D)$ for any $D'' \in O^A_{\Pi|_Y}$, a contradiction.

By contrast, the other solutions considered in Section 2 violate consistency. This can be verified by considering the districting problem Π with $t = 3$ shown in Figure 1. It consists of 27 voters of which 11 are supporters of party A (indicated by empty circles) and 16 are supporters of party B (indicated by solid circles), and four admissible districtings $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_1, d_4, d_5\}$, $D_3 = \{d_3, d_7, d_8\}$ and $D_4 = \{d_5, d_7, d_9\}$. Note that party A wins two out of the three districts in D_1 and D_2, respectively, and one of the three districts in D_3 and D_4, respectively. Consider the solution *ME* first. Since the difference in the number of winning districts for the two parties is one in all cases, we have $ME_\Pi = \{D_1, D_2, D_3, D_4\}$. Consider the districting $D_1 \in ME_\Pi$ and $Y = d_1 \cup d_2$. Consistency would require that the districting $\{d_1, d_2\}$ is among the chosen districtings if the solution is applied to the restricted problem on Y. But obviously, we have $ME_{\Pi|_Y} = \{\{d_7, d_8\}\}$, because the districting $\{d_7, d_8\}$ induces a draw between the winning districts on Y while the districting $\{d_1, d_2\}$ entails two winning districts for party A (and zero districts won by party B). Similarly, *MU* violates consistency with $D_3 \in MU_\Pi$ and $Y = d_7 \cup d_8$ since $MU_\Pi = \{D_1, D_2, D_3, D_4\}$ and $MU_{\Pi|_Y} = \{\{d_1, d_2\}\}$.

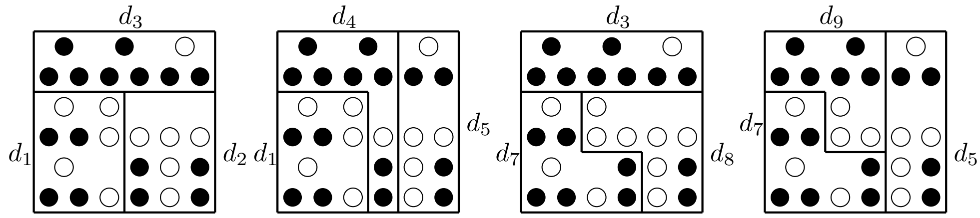

Figure 1: *ME*, *MU* and *LB* violate consistency

To verify, finally, that also *LB* violates consistency observe first that $LB_\Pi = \{D_3, D_4\}$ in Figure 1. Consider $D_4 \in LB_\Pi$ and $Y = d_7 \cup d_9$. Consistency would require that the districting $\{d_7, d_9\}$ is among the districtings chosen by the solution on the restricted problem on Y. But it is easily seen that $LB_{\Pi|_Y} = \{\{d_1, d_4\}\}$, since the districting

$\{d_1, d_4\}$ gives rise to a draw between the parties on Y which is closer to their respective relative shares of votes on Y. Thus the least biased solution also violates consistency.

Our final axiom expresses a fundamental principle of fairness in our context, namely the symmetric treatment of parties *ex ante*.

Axiom 5 (Neutrality)**.** A solution F satisfies *neutrality* if for any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ and any $D \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,t,G)}$ it follows that $D \in F_{(X,\mathcal{A},\mu,\mu_B,\mu_A,t,G)}$.

It is easily seen that all solutions presented so far with exception of the optimal solution(s) satisfy the neutrality axiom.

In the following we will show that for a large class of geographies no solution can satisfy all five axioms simultaneously. While we consider the neutrality condition to be an indispensable fairness requirement, our proof strategy is to show that the first four axioms characterize the optimal partisan gerrymandering solution O. Since this solution evidently violates the neutrality requirement the impossibility result follows.

5 A Characterization Result and an Impossibility

First, we consider districting problems with only two districts.

Lemma 1. *F satisfies two-district determinacy, two-district uniformity and indifference if and only if $F = O$, $F = ME$ or $F = MU$ for $t = 2$.*

Proof. Observe that two-district determinacy and two-district uniformity reduces the number of possible districting rules for $t = 2$ to O, *ME* and *MU* if only the number of winning districts matters (recall that *ME* = *LB* on all two-district problems). Now indifference ensures that either *all* two-to-zero, *all* one-to-one, or *all* zero-to-two districtings admissible for problem Π have to be selected by solution F.

Finally, we have seen that O, *ME* and *MU* satisfy two-district determinacy, two-district uniformity and indifference, which completes the proof. □

Consider districting problems for $t = 3$ with the 9 possible districts and the 3 resulting districtings shown in Figure 2, in which party A voters are indicated by empty circles and party B voters by solid circles, μ equals the counting measure on $(X, \mathcal{A})$ and μ_A, μ_B determine the respective number of party A and party B voters. It can be verified that, considering the districtings from left to right, we obtain 3 to 0, 2 to 1 and 1 to 2 winning districtings for party A, respectively. Thus, e.g. the optimal solution for party A would choose the first districting from the left, while the least biased solution would choose the middle districting. The geography in the depicted problem is "thin" in the sense that all proper subproblems allow only one possible districting. Therefore, the consistency condition has no bite at all in this problem. In order to make use of the consistency property, we will restrict the family of admissible geographies in the following way.

Definition 9. The geography G of a problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is *linked* if for any two possible districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $D' = D_k$ and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$.

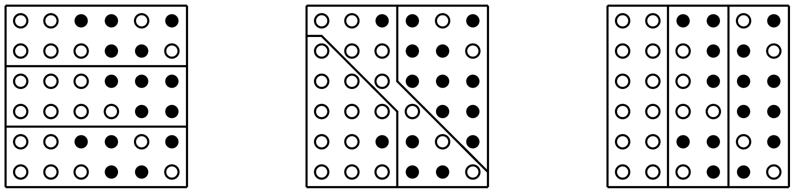

Figure 2: Unlinked districtings

In the appendix, we present a large and natural class of linked geographies. While the linkedness condition clearly limits the scope of our analysis, there is no hope to obtain characterization results of the sort derived here without further assumptions on the family of geographies.

Proposition 1. *If F equals O^A for $t = 2$ and F is consistent and indifferent, then $F = O^A$ for linked geographies.*

Proof. Consider a districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ with $t \geq 3$ and suppose that $F_\Pi \neq O^A_\Pi$ but F is consistent and indifferent. Since F_Π is not O^A_Π, there exist $D' \in O^A_\Pi$ and $D \in F_\Pi$ such that $\delta_A(D') > \delta_A(D)$ by indifference. Since Π has a linked geography there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D' = D_1$, $D = D_k$ and $\# D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$. Let $i' = \max\{i \in \{1, \ldots, k-1\} : \delta_A(D_1) = \delta_A(D_2) = \ldots = \delta_A(D_i) > \delta_A(D_{i+1})\}$ and $j' = \min\{j \in \{2, \ldots, k\} : \delta_A(D_{j-1}) \neq \delta_A(D_j) = \ldots = \delta_A(D_k)\}$. It follows by indifference that $D_{i'} \in O^A_\Pi$ and $D_{j'} \in F_\Pi$.

If $i' = j' - 1$, then $D_{i'}$ and $D_{j'}$ just differ in two districts, which we shall denote by d, d', e and e', where the first two districts belong to $D_{i'}$ while the latter two to $D_{j'}$. Observe that $D_{i'} \setminus \{d, d'\} = D_{j'} \setminus \{e, e'\}$ by linkedness. Let $Y = d \cup d' = e \cup e'$. Since O^A and F are consistent we have $\{d, d'\} \in O^A_{\Pi|_Y}$ and $\{e, e'\} \in F_{\Pi|_Y}$. Our assumption that F equals O^A for $t = 2$ and $D_{i'} \setminus \{d, d'\} = D_{j'} \setminus \{e, e'\}$ implies $\delta_A(D_{i'}) = \delta_A(D_{j'})$; a contradiction.

Assume that $i' < j' - 1$. Employing (1), consistency and linkedness, we have

$$|\delta_A(D_i) - \delta_A(D_{i+1})| \leq 1 \tag{5}$$

for all $i = i', \ldots, j' - 1$ because D_i and D_{i+1} just differ in two districts. Moreover, by the definition of j', by consistency and by our assumption that F equals O^A for $t = 2$ we must have $\delta_A(D_{j'-1}) < \delta_A(D_{j'})$, which in turn implies by (5) and $\delta_A(D_{i'}) > \delta_A(D_{j'-1})$ that there exists a smallest $j^* \in \{i' + 1, \ldots, j'\}$ such that $\delta_A(D_{j^*}) = \delta_A(D_{j'})$. Clearly, $D_{j^*} \in F_\Pi$ by indifference. We cannot have $j^* > i' + 1$ since this would contradict the definition of j^*, $D_{j^*} \in F_\Pi$, $\delta_A(D_{j^*-1}) < \delta_A(D_{j^*})$ and (5). However, if $i' = j^* - 1$, then we can repeat the argument of the previous paragraph by replacing j' with j^* to obtain a contradiction. □

Since neither the most equal or most unequal solutions satisfy consistency we cannot extend *ME* or *MU* for $t = 2$ to arbitrary t in a manner of Proposition 1. However, it might be the case that *ME* or *MU* for $t = 2$ can be extended to another consistent solution. The next proposition demonstrates that such an extension does not exist.

Proposition 2. *There does not exist a consistent and indifferent solution F that equals ME or MU for $t = 2$ even for linked geographies.*

Proof. Suppose that there exists a consistent and indifferent solution F that equals *ME* for $t = 2$. Consider the districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, 3, G)$, where X consists of 27 voters, $\mathcal{A}$ equals the set of all subsets of X, μ is the counting measure, and $G = \{d_1, \ldots, d_9\}$ is as shown in Figure 3 in which party A supporters are indicated by empty circles and party B supporters by solid circles.

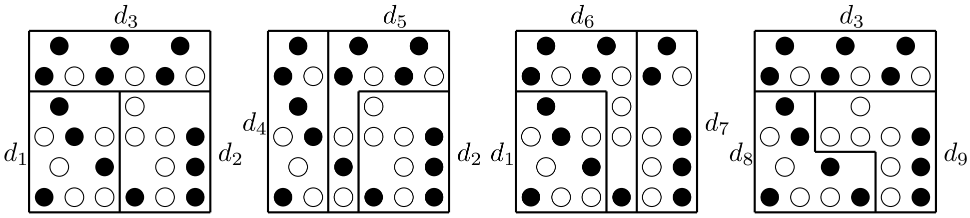

Figure 3: *ME* and *MU* cannot be extended

We can see from Figure 3 that the four possible districtings are $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_2, d_4, d_5\}$, $D_3 = \{d_1, d_6, d_7\}$ and $D_4 = \{d_3, d_8, d_9\}$. It can be checked that the given geography is linked. Since $\delta_A(D_1) = 2$ and $\delta_A(D_2) = \delta_A(D_3) = \delta_A(D_4) = 1$ we must have either $\{D_1\} = F_\Pi$, $\{D_2, D_3, D_4\} = F_\Pi$ or $\{D_1, D_2, D_3, D_4\} = F_\Pi$ by indifference. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By consistency we must have $\{d_1, d_2\} \in F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$, where $X' = d_1 \cup d_2$, $G' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}'$, μ', μ'_A, μ'_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X', respectively. However, $F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$ should equal $\{d_8, d_9\}$ since $F = ME$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of D_3. By consistency we must have $\{d_6, d_7\} \in F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$, where $X'' = d_6 \cup d_7$, $G'' = \{d_2, d_3, d_6, d_7\}$ and $\mathcal{A}''$, μ'', μ''_A, μ''_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X'', respectively. However, $F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$ should equal $\{d_2, d_3\}$ since $F = ME$ for $t = 2$; a contradiction.

Now suppose that there exists a consistent and indifferent solution F that equals *MU* for $t = 2$. Consider once again the problem shown in Figure 3. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By consistency we must have $\{d_1, d_3\} \in F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$, where $X' = d_1 \cup d_3$, $G' = \{d_1, d_3, d_4, d_5\}$ and $\mathcal{A}'$, μ', μ'_A, μ'_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X', respectively. However, $F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$ should equal $\{d_4, d_5\}$ since $F = MU$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of D_4. By consistency we must have $\{d_8, d_9\} \in F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$, where $X'' = d_8 \cup d_9$, $G'' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}''$, μ'', μ''_A, μ''_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X'', respectively. However, $F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$ should equal $\{d_1, d_2\}$ since $F = MU$ for $t = 2$; a contradiction. □

Our main theorem follows from Lemma 1 and Propositions 1 and 2.

Theorem 1. *The optimal solution O is the only solution that satisfies two-district determinacy, two-district uniformity, indifference and consistency on linked geographies.*

We obtain the following result as a simple corollary.

Corollary 1. *There does not exist a two-district determinate, two-district uniform, indifferent, consistent and neutral solution on linked geographies.*

Appendix

We provide an example showing that linkedness is satisfied by a quite natural planar geography. A bounded subset A of $\mathbb{R}^2$ will be called *strictly connected* if its boundary ∂A is a Jordan curve. A subset A of a strictly connected set $B \subseteq \mathbb{R}^2$ *separates* B if $B \setminus A$ is not strictly connected. We call a continuous function $f : X \to \mathbb{R}$ *nowhere constant* if for any $x \in X$ and any neighborhood $N(x)$ of x there exists a $y \in N(x)$ such that $f(x) \neq f(y)$.

Example 1 (Regular). A districting problem $\Pi = (X, \mathcal{B}(X), \mu, \mu_A, \mu_B, t, G)$ is called *regular* if

1. X is a bounded and strictly connected subset of $\mathbb{R}^2$,
2. μ, μ_A and μ_B are finite and absolutely continuous measures on $(X, \mathcal{B}(X))$ with respect to the Lebesgue measure,
3. G consists of all bounded, strictly connected and $\mu(X)/t$ sized subsets of $\mathcal{B}(X)$ and
4. there exists a continuous nowhere constant function $f : X \to \mathbb{R}$ such that $\mu_A(C) = \int_C f(\omega) d\mu(\omega)$ for all $C \in \mathcal{B}(X)$.

The last assumption is a purely technical one providing a sufficient condition to ensure that the districtings emerging in the proof of Lemma 2 can be selected in a way that they satisfy (1).

In what follows we write $D \sim D'$ if for two districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $D' = D_k$ and $\#D_i \cap D_{i+1} = t-2$ for all $i = 1, \ldots, k-1$.

Lemma 2. *Regular districting problems are linked.*

Proof. Linkedness is clearly satisfied if $t = 1$ or $t = 2$. We show that the linkedness of all regular districting problems for $t \leq n$ implies the linkedness of all regular districting problems for $t = n + 1$, which yields by induction the proof of our statement.

Take two arbitrary districtings D and E of a districting problem with $t = n + 1$. We can pick a district $d \in D$ such that d and X have at least a non-degenerate curve as a common boundary and d does not separate X, i.e. there exists a curve C of positive length such that $C \subseteq \partial d \cap \partial X$ and $X \setminus d$ remains strictly connected. Moreover, there exist a district $e \in E$ and a curve $C \subset \mathbb{R}^2$ of positive length such that $\mu(d \cap e) > 0$ and $C \subseteq \partial d \cap \partial e \cap \partial X$.

Case 1: Assume that e does not separate X. Since μ is absolutely continuous there exists a set h such that $\mu(h) = 2\mu(X)/(n + 1)$, $d \cup e \subset h$, $d' = h \setminus d \in G$ and $e' = h \setminus e \in G$ and h does not separate X. Let H be a districting of $Y = X \setminus h$ into $n - 1$ strictly connected districts. Then $\Pi\ |_{Y \cup d'}$ and $\Pi\ |_{Y \cup e'}$ are regular districting problems, and therefore it follows by the induction hypothesis that $D \sim H \cup \{d, d'\}$ and $H \cup \{e, e'\} \sim E$. Clearly $\{d, d'\} \sim \{e, e'\}$, which gives $H \cup \{d, d'\} \sim H \cup \{e, e'\}$.

Case 2: Assume that e does separate X, where the number of strictly disconnected regions of $X \setminus \{e\}$ equals $k \leq n$. Then $d^c \cap \partial e \cap \partial X \neq \emptyset$. We can find a district $e' \in E$ with a unique boundary element $x \in \partial e'$ satisfying $x \in d^c \cap \partial e \cap \partial X$ and that $\partial e \cap \partial e'$ has a common curve of positive length starting from x. Hence, one can exchange territories between e and e' so that for the resulting new districts h and h' we have that $d \cap e \subset h$, h separates X into at most $k-1$ strictly disconnected regions. Clearly, $E' = (E \setminus \{e, e'\}) \cup \{h, h'\} \sim E$ and we can continue with either Case 1 or Case 2, where now E' and h plays the role of E and e, respectively. Observe that we arrive to Case 1 after at most k steps. $\square$

References

[1] Balinski, M. and Young, H.P. (2001), *Fair Representation. Meeting the Ideal of One Man, One Vote*, Second Edition, Brookings Institution Press, Washington D.C.

[2] Besley, T. and Preston, I. (2007), "Electoral Bias and Public Choice: Theory and Evidence," *Quarterly Journal of Economics* 122, 1473-1510.

[3] Chambers, P.C. (2008), "Consistent Representative Democracy," *Games and Economic Behavior* 62, 348-363.

[4] Chambers, P.C. (2009), "An Axiomatic Theory of Political Representation," *Journal of Economic Theory*, 144, 375-389.

[5] Coate, S. and Knight, B. (2007), "Socially Optimal Districting: A Theoretical and Empirical Exploration," *Quarterly Journal of Economics* 122, 1409-1471.

[6] Friedman, J.N. and Holden, R.T. (2008), "Optimal Gerrymandering: Sometimes Pack, But Never Crack," *American Economic Review*, 98, 113-144.

[7] Gul, R. and Pesendorfer, W. (2009), "Strategic Redistricting," *American Economic Review*, forthcoming.

[8] Puppe, C. and Tasnádi, A. (2008), "A Computational Approach to Unbiased Districting," *Mathematical and Computer Modelling* 48, 1455-1460.