EXTREMAL PROBLEMS CONCERNING CYCLES IN GRAPHS AND THEIR COMPLEMENTS

TA Kusmayadi [1] and L Caccetta [2]

[1] Department of Mathematics Faculty of Mathematics and Natural Sciences
Sebelas Maret University Surakarta 57126
Indonesia
e-mail : trikusma@uns.ac.id

[2] Department of Mathematics Curtin University of Technology
GPO Box U1987 Perth 6845
Western Australia

Abstract

Let $\mathcal{G}_t(n)$ be the class of connected graphs on n vertices having the longest cycle of length t and let $G \in \mathcal{G}_t(n)$. Woodall (1976) determined the maximum number of edges of G, $\mathcal{E}(G) \leq w(n,t)$, where $w(n, t) = (n - 1)\, t/2 - r(t - r - 1)/2$ and $r = (n - 1) - (t - 1) \lfloor (n - 1)/(t - 1) \rfloor$. An alternative proof and characterization of the extremal (edge-maximal) graphs given by Caccetta and Vijayan (1991). The edge-maximal graphs have the property that their complements are either disconnected or have a cycle going through each vertex (i.e. they are hamiltonian). This motivates us to investigate connected graphs with prescribed circumference (length of the longest cycle) having connected complements with cycles . More specifically, we focus our investigations on :

> *Let $\mathcal{G}(n, c, \bar{c})$ denote the class of connected graphs on n vertices having circumference c and whose connected complements have circumference $\bar{c}$. The problem of interest is that of determining the bounds of the number of edges of a graph $G \in \mathcal{G}(n, c, \bar{c})$ and characterize the extremal graphs of $\mathcal{G}(n, c, \bar{c})$.*

We discuss the class $\mathcal{G}(n, c, \bar{c})$ and present some results for small c. In particular for $c = 4$ and $\bar{c} = n - 2$, we provide a complete solution.

***Key words** : extremal graph, circumference*

1. Introduction

The properties of the graphs usually involve certain graph parameters. A great deal of graph theory is concerned with establishing the best bounds for graph parameters and characterizing the graphs for which the bounds are achieved. This important area of graph theory, called extremal graph theory, forms the main focus of this paper. The property we consider is expressed in terms of the length of the largest cycle in the graph and the length of the largest cycle in the graphs complement. In particular, we focus our attention on the problem of determining the bounds of the number of edges of graph G if given $c(G)$ and $c(\overline{G})$.

2. Notation and Terminology

We use standard set theoretic notation and terminology. As there is considerable variation in the graph theoretic notation and terminology used in the literature, we present, in this section, the basic notation and terminology that we use in this paper. For the most part, our notation and terminology follows that of Bondy and Murty (1976). We denote the vertex set of a graph G by V(G) and the edge set of G by E(G); the cardinalities of these sets are denoted by $\nu(G)$ and $\varepsilon(G)$, respectively. We use the standart notation denoting the complete graph on n vertices by K_n and the complete bipartite graph with bipartitioning sets of order m and n by $K_{m,n}$. The path and cycle on n vertices are denoted by P_n and C_n, respectively. The join between two graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H.

3. Preliminary Lemmas

Let $G \in \mathcal{G}_t(n)$ be the class of connected graphs on n vertices having the longest cycle of length t. The following three lemmas formed an important component of the method of proof used by Caccetta and Vijayan (1991).

Lemma 1:
Let $G \in \mathcal{G}_t(n)$ and let $x \in H$ be joined to the vertices $i_1, i_2, \ldots, i_k$ of C. Then, for $1 \le \alpha \ne \beta \le k$, we have :

(a) $|i_\alpha - i_\beta| \ge 2$;

(b) $(i_\alpha - 1,\ i_\beta - 1), (i_\alpha + 1,\ i_\beta + 1) \notin E(G)$. □

This lemma tells us that any vertex of G – V(C) cannot be joined to two consecutives vertices of a cycle C in G. So the graph G can be depicted as in Figure 1 below (broken lines indicate edges in $\overline{G}$) :

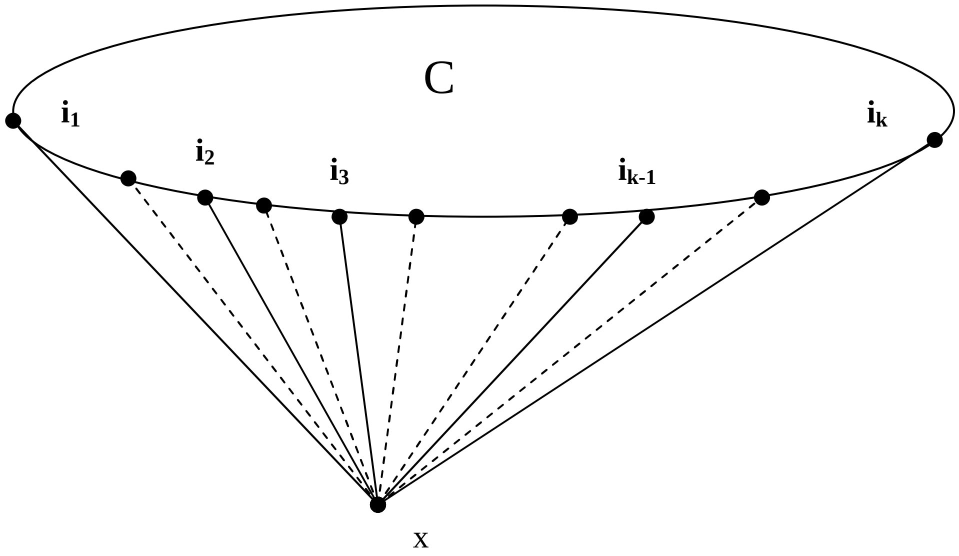

Figure 1

Lemma 2 :

Let $G \in \mathcal{G}_t(n)$ and let $P = i, x_1, x_2, \ldots, x_{d-1}, j$ be an (i, j)-path, $i \neq j$, of length d whose internal vertices are not in C. Then

(a) $d \leq |i - j| \leq t - d$, so $t \geq d$,

(b) for positive integers a, b with $a + b \leq d$: $(i + a, j + b), (i - a, j - b) \notin E(G)$. □

Instead of a single vertex of $G - V(C)$, this lemma considers a path of length d whose internal vertices are not in C. So, this result generalizes Lemma 1. Then graph G can be depicted as in Figure 2 below (broken lines indicate edges in $\overline{G}$) :

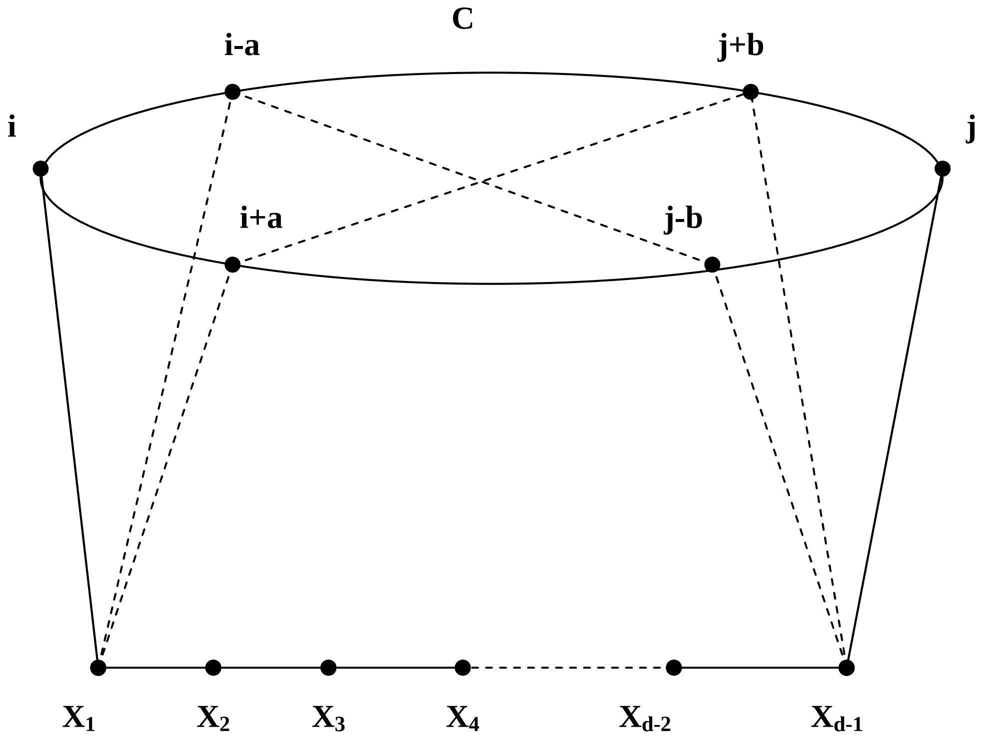

Figure 2

Lemma 3 :

Let $G \in \mathcal{G}_t(n)$ and let $P = i, x_1, x_2, \ldots, x_{d-1}, j$ be an (i, j)-path, $i \neq j$, of length $d \geq 2$ whose internal vertices are not in C. Suppose x_1 is joined to k vertices of C. Let $A = G[V(C)]$. Then

$$\varepsilon(A) \leq \tfrac{1}{2}\, t(t - 1) - \tfrac{1}{2}\,(k + d - 2)(k + d - 3),$$

with equality holding only if $d = 2$ and $t = 2k$. □

We often make use of the above three lemmas since our focus in this paper is to determine the maximum number of edges of a graph G having certain properties. Our properties specify that G and its complement have a longest cycle of specified length and both must be connected.

Lemma 4 (Xu 1987) : Let G be a graph of order $n \geq 6$, and both G and $\overline{G}$ have cycles, then

$$n + 2 \leq c(G) + c(\overline{G}) \leq 2n$$

and

$$3(n - 1) \leq c(G) \,.\, c(\overline{G}) \leq n^2.$$ □

The following lemma is due to Kusmayadi (1995).

Lemma 5 :
Let $G \in \mathcal{G}_{2k}(n)$, $k \geq 2$, be a k-connected graph. Then $\overline{G}$ is not connected. □

We now consider the class $\mathcal{G}(n, 4, n - 2)$ of connected graphs having a cycle of length 4 and a connected complement with a cycle of length $n - 2$. Let $G \in \mathcal{G}(n, 4, n - 2)$. Kusmayadi (1995) found the bounds of $\varepsilon(G)$ as stated in the following theorem.

Theorem 6 : Let $G \in \mathcal{G}(n, 4, n - 2)$, $n \geq 9$. Then

$$\varepsilon(G) = 2n - 4 \text{ or } 2n - 5.$$

Moreover, these bounds are sharp.

4. Characterization of $\mathcal{G}(n, 4, n - 2)$

The main goal of this section is to give the characterization of the extremal graphs of $\mathcal{G}(n, 4, n - 2)$. Let $G \in \mathcal{G}(n, 4, n - 2)$. The following few results deal with the diameter d(G) of $G \in \mathcal{G}(n, 4, n - 2)$.

Lemma 7 :
Let $G \in \mathcal{G}(n, 4, n - 2)$, $n \geq 11$. Then $d(G) \geq 3$.

Proof :
Let $G \in \mathcal{G}(n, 4, n - 2)$. Then , by Lemma 5, G has a cut vertex, v say. Suppose that $d(G) \leq 2$. Then every vertex of G is adjacent to v and hence $d_G(v) = n - 1$. But then $\overline{G}$ cannot be connected. Hence $d(G) \geq 3$. □

Lemma 8 :
Let $G \in \mathcal{G}(n, 4, n - 2)$, $n \geq 11$. Then $d(G) \leq 3$.

Proof :
Suppose that $d(G) \geq 4$ and let G be the smallest graph on n vertices satisfying the hypothesis in the lemma. We will prove that $\delta(G) \geq 2$.

Suppose $\delta(G) = 1$ and let $d_G(x) = 1$. Consider $G - x$. Clearly, $G - x$ is connected, $c(G - x) = c(G) = 4$, $\varepsilon(G - x) = \varepsilon(G) - 1$ and $d(G - x) \le d(G)$. By Lemma 4, in $\overline{G - x}$, we have :

$$c(G- x) + c(\overline{G - x}) \ge (n - 1) + 2$$
$$= n + 1.$$

Therefore

$$c(\overline{G - x}) \ge (n + 1) - c(G - x)$$
$$= n - 3 \text{ (since } c(G - x) = 4).$$

By the choice of G, we know that

$$c(\overline{G - x}) \ne (n - 1) - 2.$$

Since $c(\overline{G - x}) \le c(\overline{G}) = n - 2$ and $c(\overline{G - x}) \ne n - 3$, then the only possibility is that

$$c(\overline{G - x}) = n - 2.$$

But then we have $c(\overline{G}) > n - 2$ (since $d_G(x) = 1$), a contradiction. So we must have $\delta(G) \ge 2$.

From Theorem 6, we know that $\varepsilon(G) = 2n - 4$ or $2n - 5$. So, the average degree $\bar{d}$ of G is

$$\bar{d} = 2\varepsilon(G)/n \le (4n - 8)/n < 4.$$

This implies that $\delta(G) \le 3$ and hence $\delta(G) = 2$ or 3.

Suppose $d_G(x) = \delta$. Consider $G - x$. Obviously, $c(G - x) \le c(G)$. We will show that $c(G - x) = c(G)$.

In $G - x$, we have

$$\varepsilon(G - x) = \varepsilon(G) - \delta \ge 2n - 5 - \delta = 3(n - 2)/2 + n/2 - 2 - \delta$$
$$\ge 3(n - 2)/2 + (n - 10)/2$$
$$> 3(n - 2)/2 = w(n - 1, 3) \text{ \{Woodall's number\}}.$$

Therefore, $c(G - x) \ge 4$ and so $c(G - x) = c(G) = 4$, as required.

We claim that $c(\overline{G - x}) \ne c(\overline{G}) - 1$. Suppose $c(\overline{G - x}) = c(\overline{G}) - 1$. Then $c(\overline{G - x}) = (n - 2) - 1 = n - 3$. Since $d_G(x) = \delta \le 3$, then $d_{\overline{G}}(x) \ge n - 4$. Consider $\overline{G}$.

The situation is as depicted in the following figure :

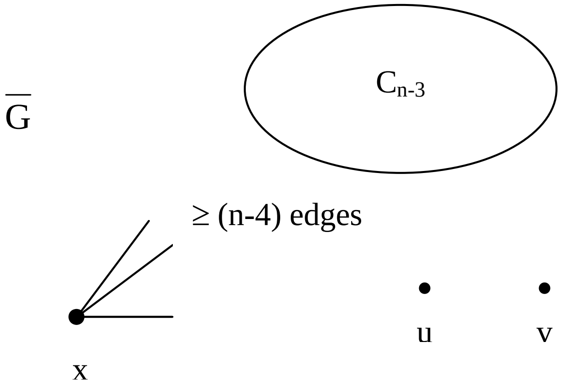

Figure 3

We consider four cases according to the values of $e(x, C_{n-3})$.

Case 1 : $e(x, C_{n-3}) = n - 3$.

Since the maximum degree of x in $\overline{G}$ is at most $n - 3$, then x cannot be joined to vertices u and v of $\overline{G}$. In addition, vertices u and v are joined to at most one vertex of C_{n-3}, as otherwise $c(\overline{G}) > n - 2$. Therefore

$$\begin{aligned} \varepsilon(\overline{G}) &\le \tfrac{1}{2}(n-3)(n-4) + (n-3) + 3 \\ &\le \tfrac{1}{2}(n^2 - 5n + 12), \end{aligned}$$

with equality achieved when $\overline{G}$ is as shown below :

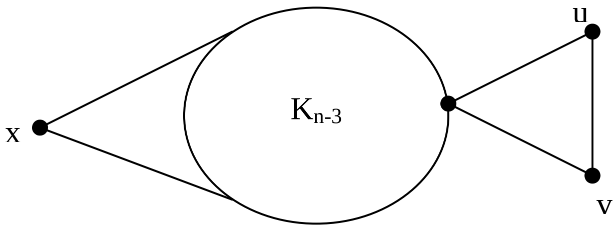

Figure 4

But then G is disconnected. Therefore

$$\varepsilon(\overline{G}) \le \tfrac{1}{2}(n^2 - 5n + 10).$$

Now, we claim that there exists vertices c_i and c_j of $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$. Suppose not. Then $\overline{G}[V(C_{n-3}) \cup \{x\}] \cong K_{n-2}$ and hence $d(G) < 4$, a contradiction. Therefore, there exists c_i and c_j in $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$.

Suppose $c_i, c_j \in N_{C_{n-3}}(u)$. Then, in $\overline{G}$, we have a cycle C : x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length $n - 1$, a contradiction. This implies that u and v are each joined to at most one of c_i and c_j.

Now, suppose $c_i \in N_{C_{n-3}}(u) \cap N_{C_{n-3}}(v)$. It is easy to check that, in G, we have $d(G) = 2$, a contradiction. So, without no loss of generality, we can assume that $vc_i \notin E(\overline{G})$ and $uc_j \notin E(\overline{G})$. But then, in G, we have a cycle C : u, c_k, v, c_i, c_j, u of length 5, a contradiction.

Case 2 : $e(x, C_{n-3}) = n - 4$.

Then $d_{\overline{G}}(x) = n - 4$ or $n - 3$. We consider these two possibilities separately.

Suppose that $d_{\overline{G}}(x) = n - 4$. Then x cannot be joined to vertices u and v of $\overline{G}$. In addition, u and v can only be joined to at most one vertex of C_{n-3}. The reason for this is as follows :

Suppose, without no loss of generality, uc_i and $uc_j \in E(\overline{G})$. Then, in $\overline{G}$, we have a cycle C: x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length n – 1, a contradiction. Hence,

$$\begin{aligned}\varepsilon(\overline{G}) &\le \tfrac{1}{2}(n-3)(n-4) + (n-4) + 3 \\ &\le \tfrac{1}{2}(n^2 - 5n + 10),\end{aligned}$$

with equality achieved when $\overline{G}$ is as shown below :

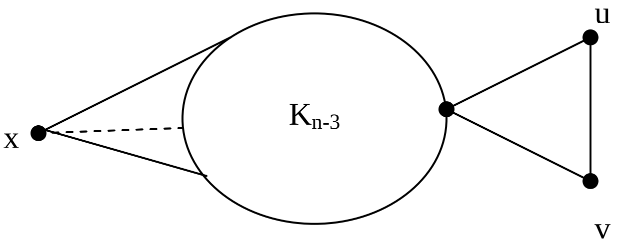

Figure 5

But then G is disconnected or d(G) = 2, a contradiction.
Therefore,

$$\varepsilon(\overline{G}) \le \tfrac{1}{2}(n^2 - 5n + 8).$$

Again, we claim that there exists c_i and c_j of $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$. Suppose not. Let $e = xc_r$, where $c_r \in V(C_{n-3})$. Then $\overline{G}[V(C_{n-3}) \cup \{x\}] \cong K_{n-2}\backslash e$ and hence d(G) < 4, a contradiction. Therefore, there exists c_i , $c_j \in V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$.

Suppose c_i, $c_j \in N_{C_{n-3}}(u)$. Then, in $\overline{G}$, we have a cycle C : x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length n – 1, a contradiction. This implies that u and v are each joined to at most one of c_i and c_j.

Now, suppose $c_i \in N_{C_{n-3}}(u) \cap N_{C_{n-3}}(v)$. It is easy to check that, in G, we have d(G) = 2 or there is a cycle C : x, c_r, u, c_j, v, x (note that c_r could be the same as c_i) of length 5, a contradiction. So, with no loss of generality, we can assume that $vc_i \notin E(\overline{G})$ and $uc_j \notin E(\overline{G})$. But then, in G, we have a cycle C : u, c_k, v, c_i, c_j, u of length 5, a contradiction.

Suppose now that $d_{\overline{G}}(x) = n - 3$. Then, without loss of generality, we may assume that $ux \in E(\overline{G})$. Clearly, u cannot be joined to any vertices of C_{n-3} and vertex v can only be joined to at most one vertex of C_{n-3}, as otherwise, suppose vc_i and $vc_j \in E(\overline{G})$. Then, in $\overline{G}$, we have a cycle C : x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length n – 1, a contradiction.

But then, in G, we can find a cycle C : u, c_1,v, x, c_2, u (c_1, $c_2 \in V(C_{n-3})$) of length 5, a contradiction.

Case 3 : $e(x, C_{n-3}) = n - 5$.

If $d_{\overline{G}}(x) = n - 3$, then x must also be joined to u and v. Clearly, u and v cannot be joined to any vertex of C_{n-3}, as otherwise $c(\overline{G}) > n - 2$. But then, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$, i = 1, 2) of length 6, a contradiction.

If $d_{\overline{G}}(x) = n - 4$, then, without loss of generality, we may assume that $ux \in E(\overline{G})$. Clearly, u cannot be joined to any vertices of C_{n-3} and vertex v can only be joined to at most one vertex of C_{n-3}, as otherwise $c(\overline{G}) > n - 2$. Again, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$, i = 1, 2, 3) of length 6, a contradiction.

Case 4 : $e(x, C_{n-3}) = n - 6$.

Clearly $d_{\overline{G}}(x) = n - 4$ and hence x must also be joined to u and v. In addition, u and v cannot be joined to any vertex of C_{n-3}, as otherwise $c(\overline{G}) > n - 2$. But then, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$) of length 6, a contradiction.

Therefore,

$$c(\overline{G - x}) = c(\overline{G}) \text{ or } c(\overline{G - x}) \leq c(\overline{G}) - 2.$$

If $c(\overline{G - x}) = n - 2$ and since $d_{\overline{G}}(x) \geq n - 4$, then $c(\overline{G}) > n - 2$, a contradiction.
Now, if $c(\overline{G - x}) \leq (n - 2) - 2 = (n - 1) - 3$ and since $d_{\overline{G}}(x) = n - 3$ or $n - 4$, then clearly that $c(\overline{G}) < n - 2$, a contradiction. This completes the proof of the lemma. $\square$

Lemmas 7 and 8 together give :

Theorem 9 :
Let $G \in \mathcal{G}(n, 4, n - 2)$, $n \geq 11$. Then $d(G) = 3$.

The following result deals with the minimum degree $\delta(G)$ of a graph $G \in \mathcal{G}(n, 4, n - 2)$.

Lemma 10 :
Let $G \in \mathcal{G}(n, 4, n - 2)$, $n \geq 11$. Then $\delta(G) = 1$.

Proof :
Let G be the smallest graph on $n \geq 11$ vertices satisfying the hypothesis in the lemma. By Theorem 6, the average degree $\overline{d}$ of G is

$$\overline{d} = 2\varepsilon(G)/n \leq (4n - 8)/n < 4,$$

and so $1 \leq \delta(G) \leq 3$.

Suppose $\delta(G) \geq 2$ and $d_G(x) = \delta$. Consider $G - x$. Clearly, $c(G- x) \leq c(G) = 4$.
By Lemma 4 we have

$$\begin{aligned} c(G- x) + c(\overline{G - x}) &\geq (n - 1) + 2 \\ &= n + 1, \end{aligned}$$

and hence

$$\begin{aligned} c(\overline{G - x}) &\geq (n + 1) - c(G - x) \\ &= (n + 1) - 4 \text{ (Since } c(G- x) \leq 4) \\ &= n - 3. \end{aligned}$$

Obviously, $c(\overline{G - x}) \leq c(\overline{G}) = n - 2$.

By the choice of G, we know that

$$c(\overline{G - x}) \neq (n - 1) - 2.$$

So the only possibility is $c(\overline{G - x}) = n - 2$. Since $d_G(x) = 2$ or 3 then we have $c(G) > n - 2$, a contradiction. This completes the proof of the lemma. □

Let $G \in \mathcal{G}(n, 4, n - 2)$ and let C be a cycle of length $n - 2 = c(\overline{G})$ in the connected complement $\overline{G}$. Suppose $\overline{A} = \overline{G}[C]$. The following lemma gives the lower bound of the number of edges of $\overline{A}$.

Lemma 11 :

Let $G \in \mathcal{G}(n, 4, n - 2)$ and let C be a cycle of length $n - 2 = c(\overline{G})$ in the connected complement $\overline{G}$. Suppose $\overline{A} = \overline{G}[C]$. Then

$$\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1 .$$

Proof :

Let $G \in \mathcal{G}(n, 4, n - 2)$ and let $C = \{x_1, x_2, \ldots, x_{n-2}\}$ be a cycle of length $n - 2$ in $\overline{G}$. Let $\overline{H} = \overline{G} - V(C) = \{u_1, u_2\}$ and $\overline{A} = \overline{G}[V(C)]$. Consider $\overline{G}$. We, first, show that $d_{\overline{G}}(u) \leq 2$ for any $u \in V(\overline{H})$. Suppose $d_{\overline{G}}(u) \geq 3$ for some $u \in V(\overline{H})$. Then, at least one of the vertices of $\overline{H}$ must be joined to at least two vertices of C. Suppose $u_1 x_i$ and $u_1 x_j \in E(\overline{G})$.
By Lemma 1, we get

$$|i - j| \geq 2 \text{ and } x_{i+1} x_{j+1} \notin E(\overline{G}).$$

Now, suppose

$$N_{\overline{G}}(u_1) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

By lemmas 1 and 2, we get

$$x_{i_\ell+1} x_{i_m+1} \notin E(\overline{G}),\ 1 \leq \ell \neq m \leq k.$$

This implies that, in G, we can find a path P_k and vertex u_1 is joined to every vertex of this path P_k. If $k \geq 4$, we can get a cycle of length at least 5 in G, a contradiction.

Therefore

$$\left| N_{\overline{G}}(u_1) \cap C \right| \leq 3.$$

If $\left| N_{\overline{G}}(u_1) \cap C \right| = 2$ and $d_{\overline{G}}(u_2) \leq 2$, then by lemmas 1 and 2, we can find, in G, an edge e and vertex x_k such that e is incident to u_1 and u_2 and both u_1 and u_2 are joined to vertex x_k. Consequently, there exists a cycle u_1 x_{i+1} x_{j+1} u_1 x_k u_1 of length 5 in G, a contradiction.

Now, suppose $\left| N_{\overline{G}}(u_1) \cap C \right| = 3$ and $d_{\overline{G}}(u_2) \leq 2$. Again, by lemmas 1 and 2, we can find a path P_3 in G such that vertex u_1 is joined to every vertex of P_3 and vertex u_2 is joined to at least two vertices of P_3. Hence, we get a cycle of length 5 in G, a contradiction.

So, the only possibility is $d_{\overline{G}}(u_1) = d_{\overline{G}}(u_2) = 3$. Without any loss of generality, the situation can be depicted as follows :

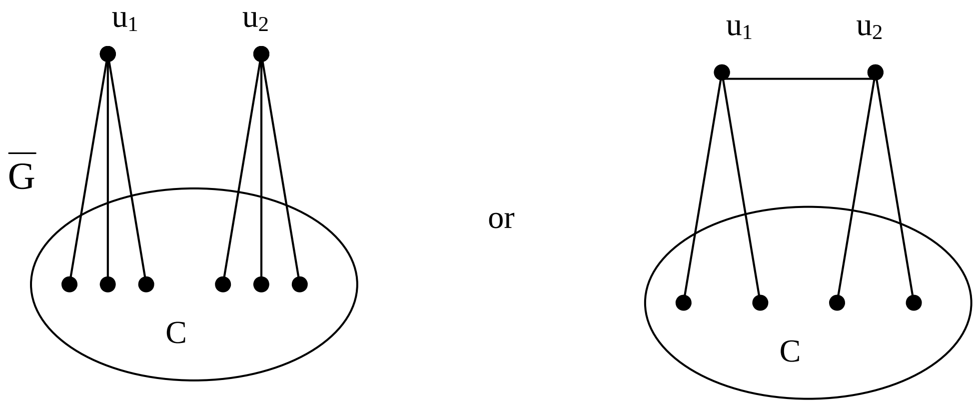

Figure 6

If $u_1u_2 \notin E(\overline{H})$, then clearly $\left| N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2) \right| \leq 3$. Hence, we can find a path P_3 in G such that vertices u_1 and u_2 are joined to every vertex of this path P_3. Therefore, we get a cycle of length at least 5 in G, a contradiction.

Now, if $u_1u_2 \in E(\overline{H})$, then $\left| N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2) \right| \leq 2$. If $\left| N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2) \right| = 0$, then, in G, we can find two K_2's such that vertices u_1 and u_2 are joined to every vertex of these K_2's. Hence, we get a cycle of length at least 5 in G, a contradiction.

If $\left| N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2) \right| \geq 1$, then we can find K_2 and a vertex x_k in G such that u_1 and u_2 are joined to x_k and every vertex of K_2. This implies that G has a cycle of length 5 : u_1 x_{i+1} x_{j+1} u_2 x_k u_1, a contradiction. Therefore, we have $d_{\overline{G}}(u) \leq 2$ for any vertex $u \in V(\overline{H})$.

If vertices u_1 and u_2 of $\overline{H}$ are joined to the same vertex of C, then by Lemma 10, we have $\Delta(\overline{G}) = n - 2$, and hence

$$\varepsilon(\overline{A}) = \binom{n-2}{2} - 1. \tag{1}$$

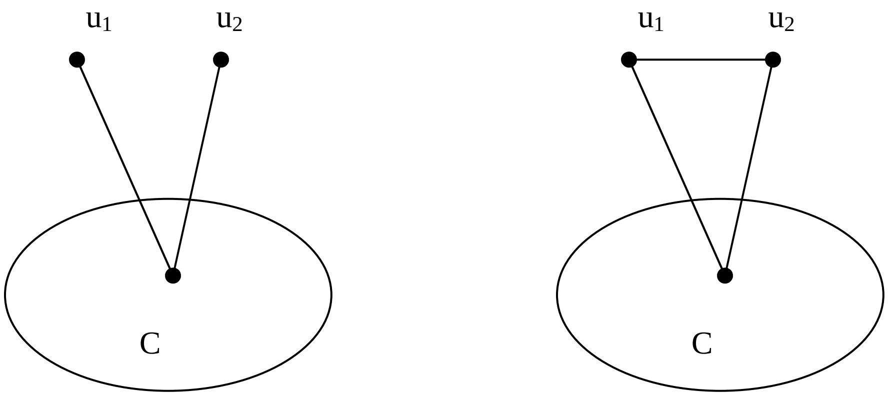

Figure 7

If vertices u_1 and u_2 of $\overline{H}$ are joined to the different vertex of C, then again, by Lemma 10, we have $\Delta(\overline{G}) = n - 2$, and then

$$\varepsilon(\overline{A}) = \binom{n-2}{2}. \tag{2}$$

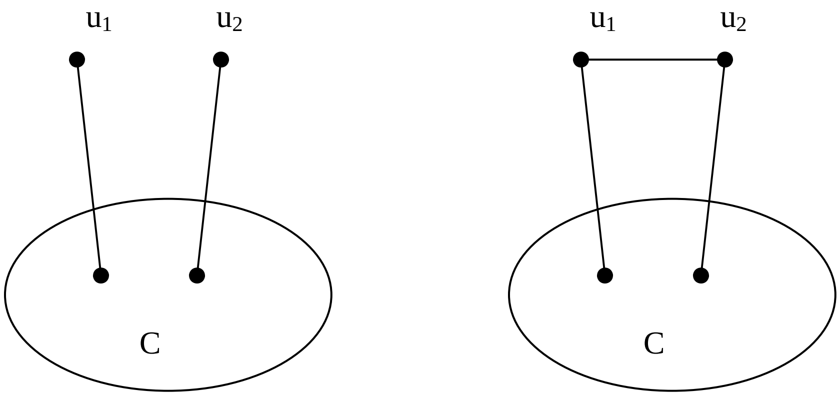

Figure 8

From (1) and (2) we get

$$\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1,$$

as required. □

REMARK 1:

Let $G \in \mathcal{G}(n, 4, n-2)$ and let $\overline{G}$ be the connected complement of G. Then $d(\overline{G}) = 3$. This follows from Theorem 9 and Lemma 11.

We are now ready to characterize the extremal graphs of $\mathcal{G}(n, 4, n-2)$ as stated in the following theorem.

Theorem 12 :

Let $G \in \mathcal{G}(n, 4, n-2)$. Then $G \cong G_i$, i = 1, 2, 3, 4,

where $$\varepsilon(G_i) = \begin{cases} 2n-5, & i = 1, 2, 3 \\ 2n-4, & i = 4 \end{cases}$$

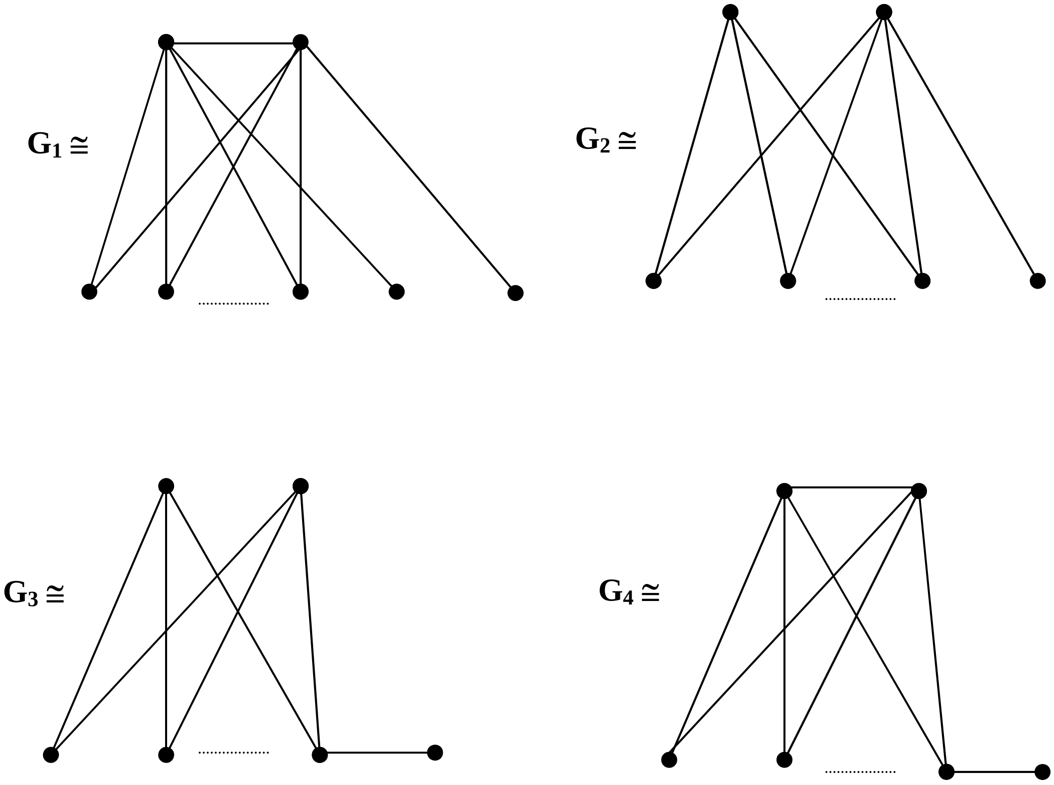

Figure 9

Proof :

Let $G \in \mathcal{G}(n, 4, n-2)$ and let $\overline{G}$ be the connected complement of G having a cycle C of length $n - 2 = c(\overline{G})$ in $\overline{G}$. Let $C = \{ x_1, x_2, \ldots, x_{n-2}\}$, $\overline{A} = \overline{G}[V(C)]$ and $\overline{H} = \overline{G} - V(C) = \{u_1, u_2\}$.

By Lemma 11,

$$\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1.$$

So, we have two cases to consider concerning the number of edges in $\overline{A}$.

Case 1 : $\varepsilon(\overline{A}) = \binom{n-2}{2}$.

Then $\overline{A} \cong K_{n-2}$. Therefore, any vertex of $\overline{H}$ can be joined to at most one vertex of $\overline{A}$, as otherwise $c(\overline{G}) > n - 2$.

If $u_1u_2 \in E(\overline{G})$, then without loss of generality, we can take $u_1x_i \in E(\overline{G})$ and $u_2x_j \notin E(\overline{G})$, $1 \leq j \leq n - 2$. So, we get $G \cong G_2$. If $u_1u_2 \notin E(\overline{G})$, then u_1x_i and $u_2x_j \in E(\overline{G})$, $i \neq j$. Hence, we get $G \cong G_1$.

Case 2 : $\varepsilon(\overline{A}) = \binom{n-2}{2} - 1$.

Then,

$$\overline{A} \cong K_{n-2}\backslash e$$

Suppose $e = x_\ell x_m$, with x_ℓ and $x_m \in V(\overline{A})$.
By Remark 1, we get $d(\overline{G}) = 3$ and hence any vertex of $\overline{H}$ must be joined to the same vertex x_ℓ or x_m , as otherwise $c(\overline{G}) > n - 2$ or $d(\overline{G}) > 3$.

If $u_1u_2 \in E(\overline{H})$, then without loss of generality, we can take $u_1x_\ell \in E(\overline{G})$ and $u_2x_\ell \notin E(\overline{G})$, and so we get $G \cong G_3$.

If $u_1u_2 \notin E(\overline{H})$, again, without loss of generality, we can take u_1x_ℓ and $u_2x_\ell \in E(\overline{G})$ and hence we get $G \cong G_4$.

This completes the proof of the theorem. □

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