DYNAMIC GEOMETRY, IMPLICATION AND ABDUCTION: A CASE STUDY

Carmen Samper, Leonor Camargo, Patricia Perry, & Óscar Molina
Universidad Pedagógica Nacional, Bogotá, COLOMBIA

In this paper we illustrate the role of dynamic geometry as an environment that propitiates the use of empirical explorations to favor learning to prove. This is possible thanks to abductive processes, related to the establishment of implications that university students of a plane geometry course carry out when, supported by a dynamic geometry program, they solve a problem in which they must discover a geometric fact, formulate a conjecture and prove it.

INTRODUCTION

The potential of dynamic geometry programs to favor the connection between empirical exploration of geometric figures, the formulation of conjectures and the production of deductive chains is widely recognized (Laborde, 2000; Olivero, 2002; Cerulli & Mariotti, 2003; Mariotti, 2007; Arzarello, Olivero, Paola & Robutti 2007, Fujita, Jones & Kunimune, 2010). In such a connection, as our study reveals, both the establishment of implications and abductive argumentation play a primary role. From our point of view, the frequent use of the program permits students to recognize that any result obtained with it is possibly valid in the theory that the program models, and to take advantage of this circumstance to look for its justification in that theory.

In the problem solving process analysis, the above mentioned authors have signaled out the role of abductive argumentation as the contact point between conjecture production and proof construction. Yet, from our point of view, there is a need for greater research evidence of the role dynamic geometry plays to orient the search of a specific thematic core within a theory and to identify those properties that can be used to justify a conjecture. This deficiency leads us to pay close attention to the student's arguments in the different moments of a problem solving process, to analyze the effect of the use of dynamic geometry in those arguments and link between the establishment of implications and the abductive processes.

THEORETICAL REMARKS

We consider *exploration*, in general, as a heuristic type of activity that can be carried out in the world of phenomena and in the theoretical world. In the world of phenomena, exploration is realized on geometric figure representations and it has an empirical character. We therefore refer to it as *empirical exploration*. When it is carried out in a dynamic geometry environment, the objective is to detect invariants and formulate them as regularities as properties, through inductive arguments. We name this activity *dynamic exploration*. In the theoretical world, the exploration is realized on the

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 43-49. Taipei, Taiwan: PME.

statements that make up individual knowledge. We refer to it as *theoretical exploration*. It is carried out with the purpose of recognizing or finding statements that permit justifying an affirmation or making decisions about where to direct empirical exploration.

From a mathematical point of view, an implication is a narrative which expresses that a statement is a logical consequence of a theory (Arzarello, 2007) and therefore, if such theory, or the part of it that is of interest, is admitted to be valid, then the statement is also, once a proof of its validity is produced. In other words, a conditional statement, $p \rightarrow q$, is a logical consequence of a theory if q can be obtained, from using the theory. In the educational realm, we are interested in identifying possible *p*, implication manifestations linked to the recognition of a work space in which efforts in finding a path to justify the conditional and being able to affirm that it is a logical consequence of the theory are concentrated, even if there is no clear way to construct the justification.

Once placed in a work space, we consider as an *abductive process* the act of evoking specific conditional statements with the same consequent as the formulated conjecture that is going to be proven, to obtain a possible antecedent which leads to the consequent. This notion is compatible with Peirce's abduction (Arzarello, Olivero, Paola & Robutti 2007); evidencing this lets us assure that the evocation of that theoretical work space actually gave place to the establishment of an implication. We differentiate the process of establishing an implication from that of formulating an abduction because in the first case a theory is referred to, or part of one, and in the second case, the reference is to one or more rules or specific statements that can be later connected with the consequent found.

RESEARCH CONTEXT

In our research study we adopted a qualitative methodology situated within the descriptive-interpretive tradition. We gather information in the natural classroom context that is interpreted through analysis categories that arise from the data study and from the conceptualization we develop. Guided by the framework, we search for evidence of the connections students make between their experimental activity and the recognition that a property is logical consequence of a theory, giving way to possible implications throughout the problem solving process.

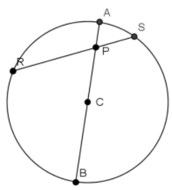
The students, who in the first semester of 2008 were enrolled in a Euclidian geometry course that is part of the mathematics requirements of the pre-service teacher program of the Universidad Pedagógica Nacional (Bogotá, Colombia), were invited to solve the following problem: Using dynamic geometry, construct $\odot C$ and a fixed point P in its interior. For which chord AB of the circle, containing point P, is the product $AP \times BP$ maximum?

The problem requires that students recognize that to theoretically be able to establish that there is no maximum value it is necessary to use the relationships between the lengths of the segments determined by point *P* in two chords that contain it, and to discover that angles subtending the same arc are congruent, a fact they did not know. We consider that the problem has optimal characteristics for our study: (i) it is similar to other problems proposed to the students in previous courses, in which they are asked to find the conditions so that a certain property is satisfied; (ii) the students had a theoretical knowledge in geometry that permits interpreting the representation phenomena, due to the construction of geometric figures and their exploration; (iii) the students had sufficient experience in the dynamic geometry software management, reason why we supposed they would use it ideally as an exploration environment to establish a regularity that would become a conjecture; (iv) a first approximation to the problem generally favors an anticipating a result that is later discarded or ratified with exploration, and therefore, the interest to justify their findings is impulse.

CASE STUDY

Due to space limitations, to illustrate the analysis we made we will present only some moments of the work carried out by the group conformed of Susana (S), Juan (J) and Felipe (F) who established a correct implication, developed work rich in abductions, and were able to progress in the proof of their conjecture.

S initially represented their anticipation using dynamic geometry, constructing a chord that contained both *P* and *C*, the center of the circle, and calculating the product of the measurement of the two segments determined by *P* on the chord. As soon as she obtained the product, S expressed that they should have constructed "any" chord containing *P* so that they could compare results. She then constructed another chord without erasing the one she already had, mechanism that she used to carry out the dynamic exploration of the situation. Immediately, J realizes the invariant: "Constant!" and asks himself why.



143. J: I don't know. Maybe it is because each time one moves, this one diminishes and this one increases. [Shows $\overline{PR}$ and $\overline{PS}$.]

148. J: But it diminishes proportionally.

153. S: […] [Speaking to J] But why does it diminish proportionally? That is the doubt. Because then it always is, that is they diminish and increase the same amount. [She refers to the lengths of the two subsegments.]

154. J: Let's look at the ratio to see what happens. Ahh, which is which?

[…] [Since they can't find the adequate combination to obtain equal ratios, they decide to analyze algebraically the factors of the constant product.]

270. J: No, because no, uhmmm, up to now, what do we have? Of what S has there, we found that *AP* times *BP* is a constant, right?

271. S: Aha.

272. J: And that *RP* and *SP* gave us the same constant, therefore they are the same.

273. S: *PA* over *PS* …*PR*? [Writes in the notebook: $PA \cdot PB = PR \cdot PS;$

PA/PS = *PR/PB*]

317. J: We are assuming that this ratio is always the same.

318. S: Well, in theory it must be so.

J observes that as he drags one of the ends of chord $\overline{RS}$, the length of $\overline{RP}$ increases when that of $\overline{PS}$ diminishes. This observation is bizarre for S reason why J adventures a first explanation: "it diminishes proportionally" [143]. Initially, J is referring to the length reduction and increase of the two segments in one chord and it seems that mentioning proportion is not, at that moment, because he is thinking of the theory of proportions. However, the action that they carry out with dynamic geometry leads J to connect the idea of constant product with constant ratio. Without explicitly mentioning a definition, a theorem or a postulate from which the fact can be derived, J invites his partners to find reasons that lead to a proportion from which the constant product is derived [154].

His proposal to examine proportions is for us an implication especially because, in what follows, the students devote their time to form ratios between the segment measurements they have found. Recurrently, they mention that their conclusion must be a logical consequence of the theory, something we consider as another factor to assure that this is a manifestation of implication. Later, dynamic geometry becomes an instrument to determine which ratios permit establishing the evoked theory as the theoretic foundation of the result they have established. The manifestation of implication can be schematized as follows:

Theory A:	Proportions
Concluded fact:	The product of the measures of the lengths of the segments in which the chord is divided is constant (*q*)

When the students are able to correctly establish the ratios, a process of geometric implication begins, because they evoke triangle similarity.

324. S. Do we have it? In theory, we should have similar triangles, right?

325. J: Triangles?

326. S: Similar triangles.

327. J: Yes, yes, yes.

328. S: That is, what we have drawn are similar triangles.

329. J: In theory, yes.

330. S: Well, we haven't drawn the triangles as such, right? But implicitly, there are similar triangles.

[…] [In what follows, they discuss about how to express in their conjecture that the product is the same regardless of which chord containing *P* is chosen.]

529. S: Products and ratios… Well construct the triangles because …

Without having represented triangles in the constructed figure, S alludes to the possibility of having similar triangles. So again, there is a manifestation of an implication. It is not an abduction because she does not evoke a specific rule but a theory and they do not know yet what to use from it. Such theory becomes the space for their future work. Once they have finished the process of writing their conjecture so that it clearly expresses the generality found, S again evokes the theory of similar triangles, stressing that it is where they can find an explanation. In intervention [529], S explicitly says why she establishes the implication with similar triangles. The scheme represents the above:

Theory B:	Similar triangles
Fact that wants to be concluded:	Ratios between the measures of the lengths of the segments in which the chord is divided are equal (*q*)

The students look for arguments that in the Theory of similar triangles guarantee that this relation exists for a pair of triangles. They begin an eminently theoretic search (in an abductive way) to guarantee the similarity and to be able make a deduction. They allude to more specific elements to assure the similarity, as the Angle-angle Criteria, and they propose an auxiliary construction for that effect: the construction of a line parallel to one of the sides of a triangle, since they must establish the congruence of another pair of angles that are not vertical angles.

514. J: For similarity, what must we do?

515. S: For similarity we have the criterion…

516. J: Yes.

517. S: The theorem…

518. J: But, what do we have here? That is …

519. S: Here we have only two congruent angles, and that is all. [They have marked, in a paper representation, the congruence of the vertical angles.]

543. S: We need another angle at least.

[…] [In this interval, the students theoretically explore possible auxiliary constructions to obtain the congruency of another pair of angles, all of them unsuccessful.]

Even though the students do not mention the Angle-angle criteria explicitly, they are referring to it, since S alludes to the pair of angles they can already assure as congruent [519] and she mentions the need to establish that another pair of angles are congruent [543]. This is an abduction process that can be represented as:

Fact they want to justify:	Similar triangles (*q*)
Theoretic backing:	Angle-angle similarity criteria

Abduction product:	The necessity of two congruent pairs of corresponding angles (*p*)

544. J: But I do not see anything.

545. S: No, there isn't … [A few seconds of silence go by.]

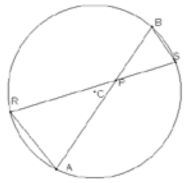
546. J: The angle…

547. F: Construct a parallel line…

548. J: But, parallel to whom? To this? [the short side of the triangle that does not contain *P*.]

549. S: We would need a parallel line to this one [the same side already mentioned] through this point [point *P*].

551. J: If we construct a parallel to this through here? [It can't be seen what he is referring to.] There, what would we get? We would have only this angle with this one [angles *ARP* and *BSP*], right? [He is referring to the graph on the paper.]



The students clearly know what the consequent of the conditional statement they want to establish is but they have not identified the specific angles. F suggests constructing a parallel line [547]; S and J accept the idea, and specify the point the line must contain and to which line it must be parallel. It is an abductive process because they mention a possible antecedent, referring to alternate interior angles [651], to be able to conclude the consequent they have established. This process is schematized:

Fact they want to justify:	The existence of another pair of congruent angles (alternate interior) (*q*)
Possible theoretic backing:	Paralelism
Abduction product:	Existence of a line parallel to a side of the triangle (*p*)

CONCLUSIONS

Our research interest is centered mainly in the student's search for the nexus between the theory they count on, the information dynamic geometry provides, the explicit establishment of implications and the abductive processes that ultimately lead to a proof. We presuppose that the evocation of a work space directs the exploration in search of an explanation of why a statement is true, and that the proof construction can include a mixture of deductions and abductions through which they advance towards finding the correct path.

When one is learning to prove, generally the argumentation on which lies the construction of a justification is of an abductive character. In a dynamic geometry environment, the students carry our empirical explorations, not only to establish a conjecture, but also to work within the frame of a theory evoked through their implication processes and to determine the viability of the ideas that emerge from their abductive processes. It is worth noting that this type of environment plays an important role in the evocation of theories; the first explorations carried out by the students induced them to frame themselves within the theory of ratios, fact that lead them to the theory of similar triangles.

This study well portrays the usually hidden nature of genuine and creative mathematical activity in which a mathematician is involved when he wants to justify a statement he believes to be true. Even though the expected product is a deductive chain that shows the statement's validity, the path to attain that involves other type of processes related to the search of generalizations and of ideas or rules that can be warrants in the justification of the induced property.

References

Arzarello, F. (2007). The proof in the 20th century. From Hilbert to automatic theorem proving introduction. In P. Boero (Ed.), *Theorems in school. From history, epistemology and cognition to classroom practice* (pp. 43-59). The Netherlands: Sense Publishers.

Arzarello, F., Olivero, F., Paola, D. & Robutti, O. (2007). The transition to formal proof in geometry. In P. Boero (Ed.), *Theorems in schools: From history, epistemology and cognition to classroom practice* (pp. 305-323). The Netherlands: Sense Publishers.

Cerulli, M. & Mariotti, A. (2003). Building theories: Working in a microworld and writing the mathematical notebook. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 181-188). Honolulu: University of Hawaii.

Fujita, T., Jones, K. & Kunimune, S. (2010). Students'geometrical constructions and proving activities: A case of cognitive unitiy? In M.F. Pinto & T.F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 9 - 16). Belo Horizonte, Brasil: PME.

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics, 44*(1-3), 151-161.

Mariotti, M.A. (2007). Geometrical proof. The mediation of a microworld. In P. Boero (Ed.), Theorems in school. From history, epistemology and cognition to classroom practice (pp. 285-304). The Netherlands: Sense Publishers.

Olivero, F. (2002). The proving process within a dynamic geometry environment. Doctoral thesis not published. University of Bristol, Graduate School of Education.