12th International Congress on Mathematical Education
Topic Study Group 9
8 July – 15 July, 2012, COEX, Seoul, Korea

REPRODUCTION OF ALGEBRAIC STRUCTURES BY 16-18 YEAR OLD STUDENTS

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In this exploratory research we analyze the structure sense evidenced by 33 secondary students (16-18 years old) in tasks requiring to reproduce the structure of given algebraic expressions. The expressions used were algebraic fractions related to algebraic identities. There were big differences between the students performance which allowed differencing levels in students´ structure sense. Questions and conjectures to be addressed in future research are presented.
Keywords: algebraic identities, algebraic expressions, structure, structural sense.

INTRODUCTION

Novotná and Hoch (2008), among others, have highlighted the inability of secondary students to apply basic algebraic techniques in contexts other than those they have experienced. According to Booth (1982), Wagner, Rachlin and Jensen (1984), Steinberg, Sleeman and Ktorza (1990) and Pirie and Martin (1997), students have difficulty in conceiving a complex expression as a whole and recognizing similarities in the structures of equivalent equations, despite showing the ability to solve these equations following standard procedures. The repeated perception of these and other difficulties shown by the students when working with algebraic expressions of different kinds, have given rise to an increasing interest on researching what and how the algebraic knowledge developed by secondary education students is (Kaput, 1998; Kieran, 2007; Puig, Ainley, Arcavi & Bagni 2007; Vega-Castro, Molina & Castro, 2010). This concern has led to the consideration of the structural sense construct, which aims to clarify the abilities needed to make efficient use of learned algebraic techniques. With this term, Hoch and Dreyfus propose a new way of dealing with the problem of algebra learning and teaching.

STRUCTURAL SENSE

The construct structural sense emerges from the analysis of work on algebraic expressions. It aims to distinguish between the possible actions that make effective use of the particular structure of the expressions in use. By structure we understand the terms that make up the expressions, the signs that connect them, the order of the different elements and the relations between them (Molina, 2010). This idea is what Esty (1992) names *grammatical form* of the expressions, Kieran (1991) refers as *superficial structure* and Kirshner (1989) calls it *syntactic structure*.

The term structural sense was first used by Linchevski and Livneh (1999). Later, Hoch and Dreyfus conducted several studies focusing on this notion and advanced in the definition of this term. Their first tentative definition was presented by Hoch in the CERME of 2003: "*to recognize algebraic structure and to use the appropriate features of that structure in the given context as a guide for choosing which operations to perform*" (p.2). Later, Hoch and Dreyfus (2004, 2005) specified particular abilities that the structural sense encompass in the context of school algebra: to see an algebraic expression or sentence as an entity, recognize an algebraic expression or sentence as a previously seen structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations it is possible to perform, and recognize which manipulations it is useful to perform.

From these abilities, in 2006 they presented an operational definition of structural sense through three descriptors, which enable to identify whether a student is using structural sense in the context of the algebra of secondary education. The authors say that a student shows structural sense in that context if he or she performs the actions detailed in Table 1.

Table 1: Definition and examples of the structure sense descriptors.

Descriptor	Definition of Hoch and Dreyfus (2006)
SS1	Recognize a familiar structure in its simplest form. Example: To factorize $81 - x^2$ recognize the expression as a difference of squares, identify the factors.
SS2	Deal with a compound term as a single entity and through an appropriate substitution recognizes a familiar structure in a more complex form. Example: To factorize $(x-3)^4 - (x+3)^4$ deal with the binomials $(x-3)^2$ y $(x+3)^2$ as a single entity, recognize the expression as a difference of squares, identify the factor.
SS3	Choose appropriate manipulations to make best use of a structure. Example: In previous task apply the notable equal difference of squares $a^2 - b^2 = (a-b)(a+b)$ to factorize these expressions.

We observed that this definition is influenced by Hoch and Dreyfus consideration of tasks that require transforming algebraic expressions. So, abilities such as divide an entity in substructures and perceive mutual connections between structures, pointed by the authors in their previous work, are not emphasized in these descriptors. Having observed this absence, we want to add a descriptor to the previous ones (SS4): "Distinguish substructures within an entity and recognize the relations between them". In this way we aim to bring attention to some abilities implicit in the structural sense construct, which might be use in tasks that do not require transforming expressions (e.g. grouping expressions according to their structure, building expressions with equal structure to another). This four descriptors form a non-exhaustive list of components of structure sense.

EMPIRICAL STUDY

Our research seeks to analyze the structure sense shown by a group of secondary students when constructing expressions with the same algebraic structure as others previously given. The algebraic expressions we considered are algebraic fractions that involve algebraic identities studied in secondary education (see Table 2). We centered our attention on these identities because of the relevance that they have in secondary mathematics curricula and its frequent applications in later topics, both in mathematics and in other areas. Some curricular goals related with the use of these expressions are to "recognize and generate equivalent forms of algebraic expressions" and "understand the meaning of equivalent forms of expressions" (NCTM, 2000, p.226 & p.300).

Study type and sample

This research is exploratory, descriptive and qualitative. The subjects who participated were a group of 33 students of a Spanish secondary school with ages 16 to 18 year old. The sample is intentional. It was selected by educational level and their availability to participate in this research. These students have studied algebra but have not explicitly worked on reproducing the structure of a given algebraic expression previously to the data collection.

Instrument Design

Not having found any instrument used in previous studies that would have enabled us to reach our research objective, we designed one ourselves taking as a guide the structural sense descriptors listed above. It was developed in two phases: a pilot test and a final second version. In its final form, the instrument included four similar tasks and in each one we presented an algebraic fraction. The student was asked to transform the expression into another simpler equivalent one, and to construct a different expression with the same structure than the given expression. In the second part we suggested them to use different numbers and letters in order to make clearer that the expression should be different, not equivalent to the one given. In both cases, we asked students to explain their response to obtain additional information to interpret their productions. In this paper we focus on the analysis of the expressions constructed by the students when being asked to reproduced the structures of given algebraic fractions[1.]

Table 2 presents the algebraic expressions proposed to the students and the different variables considered in their design: the four most common algebraic identities and the inclusion of simple or complex terms. The expressions where designed to allow SS1 to be displayed in task 1 and SS2 in tasks 2, 3 & 4. SS4 could be displayed in all the tasks but SS3 could not as the tasks did not require manipulating the expressions.

[1] For an analysis of the answers to the first part of the tasks see Vega-Castro, Molina & Castro (2011) y Vega-Castro, Molina & Castro (in press).

Table 2: Expressions included in each task and its mean characteristics

Tasks	Algebraic Fraction	Algebraic Identities	Complexity of the terms
1	$\frac{x^2-14x+49}{(x-7)^2(x-7)}$	Square of a difference	Simple
2	$\frac{2m(2m-1)}{4m^5-2m^4}$	Distributive Property/ Common factor	Complex
3	$\frac{(4x^2-1)(4x^2+1)}{(2x+1)(2x-1)}$	Difference of squares	Complex
4	$\frac{(5a^2-1)(5a^2+1)}{25a^4+1+10a^2}$	Square of a sum & Difference of squares	Complex

ANALYSIS OF PRODUCTION

The students' productions were analysed according to whether or not they conserved the structure of the given fractions. Such test enabled us to distinguish three types of production: successful, partly successful and unsuccessful. Table 3 shows the definition of each of these types.

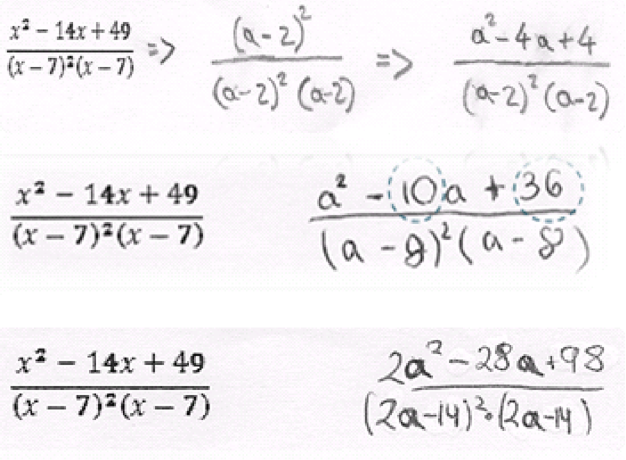
Table 3: Types of production in tasks. P(x) = numerator, Q(x) = denominator

Production	Code	Description of production
Successful	SP	Conserve the structures P(x) and Q(x) relation between them.
Partly successful	PSP	Conserves the structures of P(x) and/or Q(x) without relating them.
Unsuccessful	UP	Not conserve any of the structures.

In the first case, students showed good structural sense because they recognized the (familiar) structure of the numerator and denominator, and perceived connections between both substructures of the fraction. In the second case, students showed some structural sense as they perceive part of the structure of the fraction. In the case of unsuccessful productions, there are no signs of structural sense. Figure 1 shows examples of each type of production with the explanations[2] give by the student. In the first example, we can observe that the student recognized the structure in the numerator and denominator and the equivalence of the numerator and the binomial square at the denominator by making use of an algebraic identity. In the second example the student showed recognition of the structure of the denominator and

[2] Students' explanations have been translated from Spanish to English by the authors.

correctly generated the product of binomials. She also, recognized the second order polynomial structure of the numerator but did not perceive the relation between its coefficients nor those with the independent term of the binomials at the denominator. In the third example, the explanation of the student indicates that he did not recognize the structure of the numerator nor the denominator although an analysis of his production may suggest another interpretation.



Successful Production
Student N°5: "*I have used the algebraic identity square of the difference*".

Partly Successful Production
Student N°9: "*I have done the same fraction but with different numbers and another letter*".

Unsuccessful Production
Student N°1: "*With the same structure, I have changed x by a, and I have changed the numbers multiplying them by 2*".

Figure 1. Examples of every production type in task 1.

Classification of student's productions

We classify the students´ production by using the above codification (see Table 4). For such classification, we took into account the students´ production including their explanations. Each production was classified by the three authors and disagreements were discussed till reaching a consensus.

Table 4: Frequency of Productions. P(x) = numerator, Q(x) = denominator.

Code	Production	Tasks				Total
		1	2	3	4	
SP	Conserve the structures P(x), Q(x) and relation between them.	22 66.7%	9 27.3%	10 30.3%	12 36.4%	53 40.1%
PSP	Preserve only the structure P(x) or Q(x) or the structure of both but not the relations between them.	3 9.1%	8 24.2%	9 27.3%	7 21.2%	27 20.4%

Code	Production	Tasks				Total
		1	2	3	4	
UP	Not conserve any structures	8 24.2%	11 33.3%	9 27.3%	6 18.2%	34 25.8%
	Not performed	0 0.0%	5 15.1%	5 15.1%	8 24.2%	18 13.6%

As shown in Table 4, 40% of the productions were successful, 20.4% were partly successful, 25.8% were unsuccessful productions and 13.6% were not performed. In the first task, the percentage of successful productions was about double the one in other tasks. This was probably due to the fact that the algebraic fraction in task 1 only included simple terms. Having to work with composed terms required from the students to conceive then as entities which imply a higher cognitive load. Task 4 also presented more difficulties than the other tasks according to the higher number of lack of response. This might be a consequence of being the last tasks presented so students may have spent less time on working in it due to tiredness as time was not limited.

Levels of Structural Sense

From the codification of the students´ productions in each of the tasks (see figure 2), we identified several levels of structural sense shown. Here we propose four levels although more or less distinctions could be made. The number of levels is not the matter, rather the variety in the students´ structure sense evidenced.

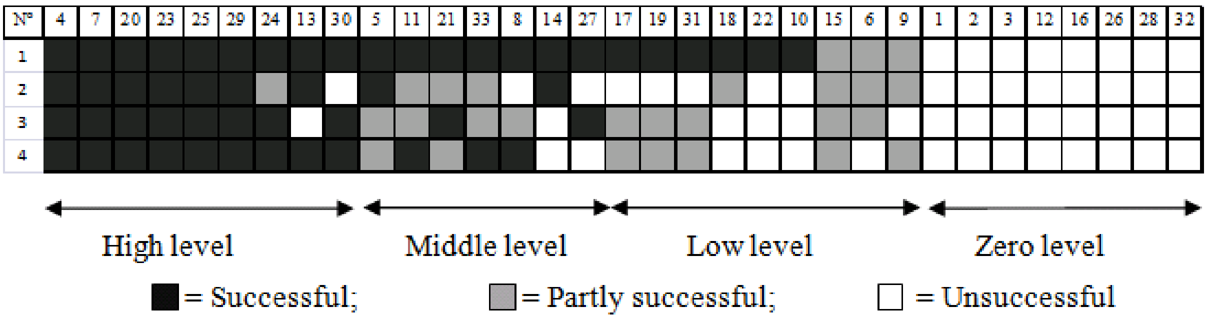


Figure 2. Evidenced students´ levels of structural sense. The left column indicates the number of the task and the upper row refers to the students identified by number.

We associate the high level with the cases in which students have recognized the total structure of the fractions (i.e., their productions are successful) in at least three of the four tasks. The middle level of structural sense corresponds to the cases in which the students exhibited successful productions in half of the proposed tasks. In the other cases the structural sense is considered low or zero depending on whether there is any or no evidence of structural sense.

If we take into account the students´ final qualifications in mathematics in the academic course 2010/2011, we observe that the students (4, 23, 25, 5, 17, 19, 6, 16 and 32) with the highest qualifications, that is between 7 and 10 (over 10), as well as those who did not pass the course (13, 30, 11, 18, 22, 10, 15, 3) are distributed along all the levels (see table 5).

Table 5: Distribution of students across levels according to their qualifications in the mathematics course

Levels of structure sense	Number of students	
	With highest qualifications	Failing the mathematics course
High	3	2
Middle	1	1
Low	3	4
Zero	2	1

DISCUSSION AND CONCLUSIONS

The classification of student productions has allowed us to distinguish degrees to which the structural sense is shown. In that sense the instrument designed in this work is useful in provoking the use of structural sense and differentiating among students. Looking at the tasks as a whole we observe that the structural sense is noticeably variable among the students participating in this study and it is not associated to higher performance in mathematics (in a traditional sense). Students with higher qualifications in the mathematics course can be assumed to have evidenced mastery in the application of learned procedures to typical problems and tasks. The main factor detected limiting the use of structure sense was the appearance of compound terms. The number of students that show structural sense is halved when expressions include compound terms.

Despite not having explicitly work on reproducing the structure of a given number sentence, 66% (22) of the students could do it when the expression only included simple terms and 53% (16) of the students could do it in at least one of the expressions including compound terms. It is remarkable that 24% (8) did not reproduce any of the structure neither of a whole fraction nor of part of it. Further research is needed to identify these cases and analyse what student´s understand as the structure of an algebraic expression and what conditions their ability to use structure sense. Some other questions that we aim to address in next studies are: How does the structural sense develop? What elements determine its use by each student? Another open issue, to continue researching is the identification of more descriptors through the consideration of different contexts or mathematical situations under which number sense can be shown. This would theoretically enrich the structural sense construct. In this paper we have considered necessary to consider a new descriptor for contexts or situations where transformations of expressions are not needed.

Acknowledgements

This work has been developed under the research project EDU2009-11337 "Modelization and representations in mathematics education" from National Plan of Research, Development and Innovation 2010-2012 from the Science and Innovation Ministry of Spain. The first author receives a sponsorship grant from the National Secretary of Science, Technology and Innovation (SENACYT) of the Republic of Panama.

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