# THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT ONE POINT AS EXPLAINED BY STUDENTS OF NON-COMPULSORY SECONDARY EDUCATION

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*Abstract*

*We review various educational studies of the mathematical concept of limit of a function at a point that indicate how colloquial uses of the terms "to approach," "to tend toward," "to reach," "to exceed" and "limit" influence students' conceptions of these terms. We then present the results of an exploratory study of this question performed with Spanish students in non-compulsory secondary education and analyze the responses they provide to justify the truth or falsity of statements related to the different characteristics of the concept of finite limit of a function at a point when they use these terms. Finally, we organize their answers according to the kinds of arguments made. Using the response profiles detected, we discuss the influence of everyday usage on the students' arguments.*

## PROBLEM

The language used to describe the properties of the concept of limit includes terms that have diverse colloquial uses in everyday life. Monaghan (1991), Tall (1980), Tall and Vinner (1981), and Cornu (1991), among others, analyze students' use of the terms "to approach," "to converge at," "to tend toward," "to reach," "to exceed," and "limit" in a mathematical context and the conflicts that arise from using these terms due to the diversity of their meanings. Their analysis suggests a focus for investigating how students' mathematical use of these terms is conditioned by their everyday meanings.

We propose to review and update these studies. Our report characterizes the students' conceptions, as demonstrated in their explanations of certain statements on the concept of the finite limit of a function at a point, when they use the key terms identified in the literature.

## ANTECEDENTS

Tall (1980) finds that most students conceive of the limit as a dynamic process and not as a numerical quantity, due to the no less problematic characteristics of the intuitive definition of limit. Tall and Vinner (1981) confirm that the expression "f(x) tends toward L, when x tends toward $x_0$" is a source of conflict in the formal definition of limit. Among studies that focus on specific terminology, Monaghan (1991) analyzes in greater depth the influence of language on students' ideas about the terms "to tend toward," "to approach," "to converge at" and "limit" when used to refer to different graphs of functions and the examples the students use to explain these terms. Monaghan concludes that many students do not distinguish between "tend toward"

and "approach" in a mathematical context. Cornu (1991) studies the influence that the terms "to reach" and "to exceed" associated with the term "limit" in a colloquial context have on students' conceptions when they classify the value of the limit as not exceedable and not reachable in a dynamic or static environment. This influence is also reported in subsequent studies published in the Proceedings of the PME. Juter and Grevholm (2004) identify subjects who argue that the limit of a function at infinity is not reached because x cannot reach infinity. In analyzing the evolution of different conceptualizations of the limit of a series, Roh (2007) also finds that the unreachability of the limit is an obstacle.

## METHOD

### Instrument

We worked with a survey of six open-response questions presented in two different questionnaires, A and B. The items are adapted from Lauten, Graham and Ferrini-Mundy (1994); the statements are given in the Appendix.

The questions are open answer. They require assigning, and justifying, the values of true (T) or false (F) to a statement about a property related to the concept of the limit of a function at a point.

### Sample

The sample was composed of 36 Spanish students in the first year of non-compulsory secondary education, 16-17 years of age, who were taking Mathematics for the Science and Technology track. The students were chosen deliberately based on their availability.

The survey was administered in the middle of the academic year 2010/2011; the subjects had received prior instruction on the concept of limit. Of the total number of subjects, 18 answered questionnaire A and the other 18 answered questionnaire B. The survey was administered during a regular session of the math class.

## RESULTS

### Arguments on object-process interrelation in the concept of limit

Item 1.A requires justifying the truth or falsity of the statement: "A limit describes how a function moves as x moves towards a certain point." This item revises a question posed by Tall (1980).

The key idea of the item is that of the movement of the function (dynamic process). Table 1 summarized the three profiles provided by the students' arguments.

| Profiles | Profile descriptions |
|---|---|
| **Profile I (Process)** | Responds true. The limit object describes the dynamic process that gives rise to it. |

| | |
|---|---|
| **Profile II (Object/Process)** | Responds true. Stresses the limit as the result of the dynamic process that, implicitly, is also part of the concept; expresses a dual conception of object and process. |
| **Profile III (Object)** | Responds false. Differentiates the limit object from the dynamic process that gives rise to it. |

Table 1: Profile descriptions related to Item 1.A

Sample answer for Profile I: "True: Yes, it does, because the limit provides all the possible points that a function can take."

Sample answer for Profile III: "False. A limit is that toward which the function f(x) tends."

**Arguments on the exceedable character of the value of the limit**

Item 2.A requires justifying the truth or falsity of the statement: "A limit is a number or point past which the function cannot go." This question was analyzed by Monaghan (1991) and Cornu (1991).

The key idea of this item is not to exceed (a limit property). Table 2 summarized the two profiles provided by the students arguments.

| Profiles | Profile descriptions |
|---|---|
| **Profile I (Not exceedable)** | The value of the limit is recognized as not exceedable in any case. Most of the arguments stress the value of the limit as unreachable **(Subprofile I.1) (Not reachable)**. This property is due to the infinite nature of the numerical process and the exclusion of the image of the point. |
| **Profile II (Exceedable)** | The value of the limit is exceedable in certain cases; in fact, examples are given in which the limit is exceedable; in some cases it is considered reachable **(Subprofile II.1) (Reachable).** |

Table 2: Profile descriptions related to Item 2.A

Sample answer for Subprofile I.1: "True: Because a limit is a point that a function approaches infinitely but never reaches"

**Arguments on the relation between the finite character of the practical process vs. the potentially infinite character of the formal iterative process**

Item 3.A proposes justifying the truth or falsity of the statement: "A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached." Key ideas for this item are, following Tall (1980): *to try values* and *to reach*. We find four profiles, or arguments, as shown in Table 3.

| Profiles | Profile descriptions |
|---|---|
| **Profile I (Potential infinity)** | Responds true. The process is potentially infinite, without further consideration. There are references to both-sides conditions of the process **(Subprofile I.1) (bilaterality)**. |
| **Profile II (Practical finiteness)** | Responds false. Stresses the finiteness of the process in practice vs. the potentially infinite formal process. |
| **Profile III (Bilateralness)** | Responds false. Does not assume the existence of the limit and requires the both-sides condition not included in the wording of the item. |
| **Profile IV (Not reachable)** | Responds false. The limit is not reachable, possibly due to the potentially infinite character of the process. |

Table 3: Profile descriptions related to Item 3.A

**Arguments on the reachable character of the limit**

Item 1.B asks students to argue the truth or falsity of the statement: "A limit is a number or point the function gets close to but never reaches." The item is related to the studies done by Tall (1980), Tall and Vinner (1980), Monaghan (1991), and Cornu (1991). Key terms in this item are *to get close* and *to reach*. Three profiles of arguments emerge, as summarized in Table 4:

| Profiles | Profile descriptions |
|---|---|
| **Profile I (Not reachable)** | Responds true. Affirms that the limit is unreachable in any case, due to the infinite nature of the process of approaching in some cases. |
| **Profile II (Reachable)** | Responds false. The limit can be reachable in some cases. A transition profile. |
| **Profile III (Always reachable)** | Responds false. Affirms that the value of the limit is always reachable. Only two cases affirm that the value of the limit is reachable but never exceedable **(Subprofile III.1)(Not exceedable).** |

Table 4: Profile descriptions related to Item 1.B

Sample answer for Profile II: "False: It is not necessary to reach the limit exactly but only to approach it enough to know what the limit is"

Sample answer for Subprofile III.1: "False: Function attains the limit but does not go past it."

## Arguments on the interrelation of precision and the process of approximating the limit

Item 2.B asks for arguments for the truth or falsehood of the statement: "A limit is an approximation that can be made as accurate as you wish." The main idea of this item is to consider a limit as a process of approximation that can be as accurate as desired. The arguments collected demonstrate the profiles presented in Table 5.

| Profiles | Profile descriptions |
| --- | --- |
| **Profile I (Exactitude)** | Responds false. Considers the limit as an exact value, not a process of approximation giving increasingly accurate values. Limit is an object, not a process. |
| **Profile II (Arbitrary precision)** | Responds true. Shows the infinite nature of the process followed to calculate the value of the limit. Contrary what we would logically expect, some responses hold that it is not the precision that matters but the values that approximate the limit. |
| **Profile III (Restricted precision)** | Responds true. Precision is restricted, given the finite nature of the method of calculating in practice. Only one subject fits **Subprofile III.1 (Conditional precision)**, which establishes that the precision depends on the function and the values of x. |

Table 5: Profile descriptions related to Item 2.B

Sample answer for Profile II: "False: Because accuracy is not important—finding the numbers that get closer to the limit is"

## Arguments on the arbitrariness of the process of approximation

Item 3.B asks for arguments for the truth or falsity of the statement: "A limit is a number that the y-values of a function can be made arbitrarily close to by restricting the x-values." The key terms in this item are to be made arbitrarily close to and x-value restrictions. The answers show two profiles of response, presented in Table 6.

| Profiles | Profile descriptions |
| --- | --- |
| **Profile I (Non-arbitrariness)** | Responds false. Differentiates between approximate and arbitrary approach. The values of f(x) do not approach the limit arbitrarily. Rather, depending on the values of x chosen, their respective images approach or recede from the limit. |
| **Profile II (Arbitrariness)** | Responds true. Some subjects consider the monotony of convergence at the limit. |

Table 6: Profile descriptions related to Item 3.B

Sample answer for Profile I: "The values of the function become close to the limit in an approximate but not an arbitrary way."

## DISCUSSION AND CONCLUSIONS

The subjects themselves provide the criterion for discriminating between profiles based on the arguments used. When we find several different arguments simultaneously, we consider the true/false dichotomy as a secondary criterion for discriminating between the profiles.

### Conceptions about the concept of limit

The profiles of response for Item 1.A show subjects who conceive the limit exclusively as a process. There are also subjects who establish the interdependence of the process of obtaining the limit of a function at a point and the object that they call limit. That is, the object determines and describes the process followed to obtain the limit, even when using a dynamic interpretation, conclusions that agree with those of Tall (1980). Finally, we identify subjects who do conceive the limit not as a process but as an object, point or number; the concept of limit establishes not a "how" but a "where."

Second, we can infer from Items 3.A and 2.B that most of the subjects recognize that the practical method for calculating a limit is finite, although some believe that assigning values close to it can be done as many times as one wishes, using the intuitive notion of potential infinity. However, others stress that precision is restricted, or that it is enough to intuit the limit. It is possible that the use of the expression "as one wishes" in Item 2.B introduced some subjectivity and hindered students' understanding of the arbitrary precision of the approximation of a limit.

Third, we do not find indications that the term "restriction" in the wording of Item 3.B led subjects to an intuitive use of the continuum (intervals or graphic representation), since they persist in discrete reasoning. These questions develop those studied by Monaghan (1991).

### Exceedable or reachable character of the value of the limit

It is significant that these properties are considered primarily in general and not in particular. The unexceedable character of the limit is attributed to a great extent to its unreachability. However, unreachability is not due to unexceedability; even if the limit is reachable, it is not exceedable for some subjects. It is significant that only 2 of the 18 subjects are aware of the particular character of these and that only one admits different possibilities. Misuse of the expression "f(x) tends toward L" and of strictly monotonous functions may contribute to these obstacles, detected previously by Cornu (1991). Furthermore, we suggest that these properties are considered locally and not globally.

**The notion of arbitrariness**

The students interpreted the term "arbitrariness" in different ways. One case differentiates between "approaching approximately" and "approaching arbitrarily." Another case interprets this as: "The approximation to the limit is arbitrary if any value of the variable x is verified," an understanding based on its everyday meaning.

**Balance**

No previous study has analyzed the response profiles presented here. The richness and variety of the responses obtained for the items proposed is organized according to the profiles, which demonstrate the complexity of the notions implied in the concept of the finite limit of a function at a point. It remains to establish connections between the profiles described for the different questions.

**Acknowledgements**

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## APPENDIX: Items included in the questionnaire

**Questionnaire A.**

(1.A.) A limit describes how a function moves as x moves towards a certain point.

(2.A.) A limit is a number or point past which the function cannot go.

(3.A.) A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

**Questionnaire B.**

(1.B.) A limit is a number or point the function gets close to but never reaches

(2.B.) A limit is an approximation that can be made as accurate as you wish.

(3.B.) A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.