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ABSTRACT

Student responses to arithmetical questions that can be solved by using arithmetical structure can serve to reveal the extent and nature of relational, as opposed to computational thinking. Here, student responses to probes which require them to justify-on-demand are analysed using a conceptual framework which highlights distinctions between different forms of attention. We analyse a number of actions observed in students in terms of forms of attention and shifts between them: in the short-term (in the moment), medium-term (over several tasks), and long-term (over a year). The main factors conditioning students´ attention and its movement are identified and some didactical consequences are proposed.

KEYWORDS:

Forms of attention; Number sentences; Relational Thinking

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In the context of elementary arithmetic, students´ thinking usually focuses on performing operations and getting a result. It is perfectly reasonable for students to display a computational mindset, since this is what is promoted by traditional ways of teaching arithmetic and may be favoured by informal pre-school arithmetic experience. Where little teaching time or attention is devoted to appreciating the structure of expressions, looking for and expressing patterns, or using mathematical properties to justify calculation choices, computation is likely to dominate. This continues despite the wide recognition that understanding is just as important as facility (Kilpatrick, Swafford, & Findell, 2001; Pirie & Kieren, 1989; Sierpinska, 1994) which can be traced back at least to Plato (Republic II 488ff) (see Hamilton & Cairns, 1961, pp. 353-384).

In this paper we consider a particular arithmetic activity which aims to promote students´ development of awareness of arithmetic structure: to decide and provide justifications-on-demand about the truth or falsity of addition and subtraction statements which depend upon arithmetic properties (e.g., $10+4=4+10$; $13+11=12+12$; $100+94-94=100$). While it is the case that solving missing-number sentences (e.g., $12+\square=13+4$) seems to elicit computational approaches, justifying-on-demand the truth or falsity of number sentences can prompt students to look instead at the sentence as a whole and to recognize and make use of some relations between numbers (Molina & Ambrose, 2008).

Our aim in this paper is to use the construct of `shifts of attention´ (Mason, 1998) to make sense of the approaches over time (short, middle and long term) to tasks involving number sentences based on arithmetic relations, displayed by a group of Spanish elementary students. We are not concerned here with whether the instructional intervention developed was helpful for students in acquiring and developing relational approaches (however see Molina, 2006). Rather our focus is on describing and analysing the extent and nature of relational thinking detectable in students´ productions overtime, by means of the analysis of the movement and structure of students´ attention (Mason, 1998).

Conceptual Framework

Our approach is eclectic, combining analysis of previously collected and analysed empirical data, with a phenomenologically-based experiential stance. The data comes from a study on the use or presence of relational thinking (Molina, 2006). Here we re-analyse that data by making use of an analytical tool derived from phenomenological enquiries into the nature and role of attention, using the discipline of noticing (Mason, 2002). We share Marton and Booth´s (1997) view of learning as a change in the person's way of experiencing a phenomenon/situation/object. Through the idea of the structure of attention, we try to capture how students experience number sentences in order to understand how they act in this context. We first present a theoretical description of relational thinking and of the notion of structure of attention. Afterwards, we invite the reader to work on an activity to let him/her experience how attention may shift when considering numerical statements.

Relational Thinking

Promoting the integration of arithmetic and algebra in the elementary curriculum is an issue of intense current interest (Becker & Rivera, 2008; Kaput, Carraher, & Blanton, 2007). These authors among others make the case for algebraic thinking to be promoted and supported in the earliest grades, and Hewitt (1998) makes the case that in order to understand arithmetic, it is necessary to engage in algebraic-type thinking by thinking in generalities. No-one expects students to memorise the results of all possible two and three digit additions and subtractions. Rather, the methods which students use for these are themselves generalities involving properties which can be perceived as being instantiated in each and every particular calculation.

Several researchers (Carpenter, Franke, & Levi, 2003; Molina, 2006; Stephens, 2007) have drawn attention particularly to the use of relational thinking. This type of thinking has mainly been considered in the context of number sentences. It is deemed to be taking place when students use arithmetical relations between terms contained in a number sentence in order to expedite the calculation or to judge directly the validity of an arithmetical statement. For example, when asked to decide on the truth or falsity of sentences such as $257 - 34 = 257 - 30 - 4$ or $27 + 48 - 48 = 26$ [1], instead of performing the calculations and then checking for equality, students may recognize and make use of arithmetical relations so as to avoid computation. The sentence can be seen as a whole, its components and structure can be appreciated, and relations between its elements (e.g., some numbers and/or some operational signs are repeated in the sentence) together with knowledge of the structure of arithmetic can be used to conclude about its true value. Some students may even be able to justify their use of these relations by referring to general properties (such as $a - a = 0$).

These two different approaches —one totally computational, the other making use of the structure of the expressions, with or without explicit awareness of instantiation of properties—, but in a broader context, have been referred to by Hejny, Jirotkova and Kratochvilova (2006) as *procedural* and *conceptual meta-strategies*, respectively.

In the sentence $257 - 34 = 257 - 30 - 4$, for example, the use of relational thinking would mean appreciating that two expressions are being related by the equal sign, considering those expressions as wholes and recognizing that in both of them the same quantity is being subtracted from 257. Thus both expressions are equal. Similarly, using this type of thinking in the sentence $27+48-48=26$ requires treating the expression on the left side as a whole and discerning the presence of 48 – 48. Being aware (implicitly or explicitly) that $a-a=0$, would allow obtaining the numeric value of the left side of the sentence and so, concluding the falsity of the sentence by comparing it to 26.

In order to think relationally, students need to consider expressions from a structural perspective rather than simply from a procedural one. Sentences and parts of sentences need to be considered as wholes (sub-expressions) instead of as processes to carry out step by step. Substructures within the whole expression need to be identified and compared. In this way, relationships between them are recognized. All these are components of structural-sense as defined by Hoch and Dreyfus (2004). The use of relational thinking also implies drawing upon number-sense and operation-sense (Slavit, 1999) as relations between numbers, operations and expressions involved in the sentence are established and knowledge about the structure of the number system, properties of operations, and relations between operations, among other elements, are acted upon, implicitly or explicitly.

Considering number sentences as a context to reveal and promote students´ use of relational thinking requires taking into account their understanding of the equal sign. According to previous studies (Behr, Erlwanger, & Nichols, 1980; Carpenter et al., 2003; Kieran, 1981; Molina & Ambrose, 2008; Warren, 2003), students tend to see and use the equal sign as a "do something" signal. When confronting arithmetic expressions, they tend to focus on performing the operations expressed, usually by reading form left to right. However, both Carpenter et al. (2003) and Molina, Castro & Castro (2009) have observed that although elementary students initially tend to interpret the equal sign as an operational symbol, if teaching is designed to promote a relational understanding of this symbol in the context of number sentences, they are able to develop this understanding.

Structure of Attention

We analyse students´ responses about the truth value of number sentences used in Molina (2006) by trying to detect evidence of subtly different ways of attending. Whatever the purposes and utility (Ainley & Pratt, 2002) perceived by students, once they engage with a task, what matters is both what they are attending to and how they are attending to it.

Consider the statement $50+52=51+51$, as an example. When asked to determine the truth value of this statement, some students might detect that 51 is both 1 more than 50 and 1 less than 52, in this particular situation without being explicitly aware of the general property that adding and subtracting the same amount leaves the sum invariant. Their attention may be concentrated on the particularities of 51 and 1. Some may be more or less explicitly aware, in the sense that they make confident and consistent use of the fact in multiple instances, perhaps telling someone else to do it, and some may offer an explicit articulation or formulation when asked to justify their actions. Some students may be aware that this is an instantiation of a general property. Of course when asked to justify why the sum is invariant, they may be led to make use of properties such as associativity of addition, either explicitly and articulately, or implicitly as a 'theorem-in-action' (Vergnaud, 1981). In other words, they can be 'reasoning on the basis of properties' while at the same time unaware of a relationship as an instance of a property.

In attending to something such as an arithmetic statement, it is possible to be predominantly gazing at or holding a whole (this may be literally 'the whole sentence' or some component of it). It is also possible to be discerning details (which themselves may become 'wholes' which are not at the time further dissected into sub-details). It is possible to be (re)cognizing relationships between specific discerned details (such as noticing that 14 is 2 less than 16), and it is possible to be aware of the relationships between specifics as instantiations of properties that hold in many different situations or contexts. It is important for learners to develop the flexibility to shift between these various forms of what can be attended to, and in what ways.

Finally, it is possible to reason on the basis of perceived and agreed properties, that is, to engage in formal mathematical reasoning. These are different forms or states of attention identified by Mason (2004). He conjectured that one of the reasons that mathematical reasoning proves to be so difficult to teach is that students may not have accumulated necessary experience of the different forms of attention to have reached a point at which reasoning on the basis of specified agreed properties has sufficient foundation. Furthermore, they may not have developed sufficient flexibility of shifts between forms of attention. In other words, it is difficult to display formal reasoning if your attention does not readily perceive relationships as instances of properties. This in turn is difficult if the attention is on discerning details rather than on recognizing relationships, which depends on more than gazing at wholes.

The states of attention discerned in this framework are neither levelled nor ordered. They are often transitory states visited for micro-seconds, but they can also become stable and robust against alteration for varying periods of time. Indeed they can become ingrained habits which block further development. It is a matter of self-observation to discover that these states or structures of attention can be fleeting as well as stable, and that in different situations they occur in different ways. In other words, different states can be triggered more prominently than others by different cues.

Experiencing what it Might be Like for Students

From a phenomenological perspective, and before looking at data from students, it is important as a researcher to try to enter for oneself something of the experience that students might have, in order to be sensitised to the sorts of things that students say and do and how to interpret them. In particular, it is important to become aware of how our own attention is differently and variously structured at different times when working on a mathematical problem or task. Here for example are two tasks taken from a Hungarian secondary problems book (Tankønyviado Budapest 1988):

Calculate $\frac{10000 \cdot 10004 - 10002 \cdot 9998}{10000 \cdot 10000 - 10001 \cdot 9999}$

and $\frac{1234321234321 \times 2468642468641 - 1234321234320}{1234321234320 \times 2468642468641 + 1234321234321}$

These questions provide some initial phenomenological data by offering an opportunity to find yourself resisting calculation, and perhaps gazing, or ‘holding wholes’ while waiting for something to emerge, discerning details in the expressions, recognizing relationships amongst the discerned elements and perceiving those relationships as particular instances or instantiations of more general properties. Both tasks are constructed in such a way as to be quite “unfriendly” to calculation (even using calculator is likely to present difficulties in storing the numbers). The second one particularly absorbs a good deal of attention as you seek something invariant amongst discerned details. This gives a taste of the movements of attention which a young student might experience when facing simpler probes such as

$257-34=257-30-4$ and $27+48-48=26$. Of course the relationships are somewhat more sophisticated here, but not significantly so.

Some people who are familiar with algebra may choose to replace long strings of symbols with letters, displaying their awareness that complexity in number names often obscures relationships. Others, seeking a way to deal with the numerator of the first one, may think of 10004 as $10002+2$ and expand before factoring, or by replacing the first product by $(10002-2)\cdot(10002+2)$, may then decide whether to do the same thing to the second product or to use the repetition of 10002 to achieve a simplification. These transformations are developed in the anticipation of reaching a simplified equivalent expression. The importance of anticipation in guiding algebraic thinking, and here, relational thinking, has been stressed by Boero (2001).

Similar remarks apply to the second calculation. The size of the numbers provokes resistance to calculation and a search for some other relationships to use, an observation exploited by Zazkis (2001). Even checking how close the second number in the numerator is to being double the first number (having discerned the two and recognized a potential relationship), requires careful attention, successively discerning 'the next few digits' in each and checking the doubling relationship.

In the context of simply being asked for an answer, many of us might accept a first ‘intuitive sense’ of relationships and be content with our first conjecture about the true value of the sentence. Perhaps this is the experience of many students in classrooms. In the context of expecting to be asked to justify a response, it is likely that greater care would be taken in the discerning of relevant details through recognizing relationships and using these as instantiations of familiar properties to achieve simplification.

Research Setting and Probes

The data analyzed in this paper come from a teaching experiment which shares the features of research design identified by Cobb and his colleagues (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) as it consists of iterative attempts developed in a (complex and real) teaching/learning context, aiming to understand and improve educative processes (Steffe & Thompson, 2000). This methodology is characterized by recurrent cycles formed by the formulation of hypothesis and conjectures, the design of an in-class intervention, the experimentation in the classroom, the analysis of the data collected and the reconstruction of the hypotheses and conjectures to start a new cycle (Steffe & Thompson, 2000). Therefore, each session has its own aims depending on the particular hypothesis and conjectures to be tested in it in relation to the broader objective of the teaching experiment.

In the teaching experiment we refer here, the guiding broad objective was the study of the use and development of relational thinking that third grade students display when being asked to determine the validity of addition and subtraction true/false number sentences (Molina, 2006). The first author acted as teacher for a group of twenty-six eight-nine years old Spanish students during six one-hour in-class sessions, over a period of one year. The regular classroom teacher was present in all the sessions but did not participate because he had not become involved in the research process. In these conditions, it is frequent in teaching experiments that one of the researchers takes the role of the teacher in order to experience at first hand students´ learning and reasoning (Kelly & Lesh, 2000; Steffe & Thompson, 2000).

The chronology of the sessions was as follows. The second session was two months after the first one. The rest of the sessions were from one to two weeks apart except for the final session which took place at the beginning of the following academic year, eight months later. This timeline was chosen intentionally (except for vacation periods) so that (a) the intervention took place over a significant period of time so as to have some effect, (b) to reduce the probability of assessing a memory-based learning and, (c) to have sufficient time to analyze the data from each session and to make decisions about the next in-class intervention.

The teacher-researcher proposed to the students various missing-number and true/false number sentences in individual written activities, whole group discussions and individual interviews. This variety of intervention formats was chosen to (a) provide time for individual work and reflection, (b) promote students´ exchange and comparison of ideas, and (c) have opportunities to more closely access some students´ thinking.

The proposed sentences included numbers using one, two or three digits and the operations of addition and subtraction. Some were sentences with all the operations on one side of the equation (e.g., $10+9-4=15$) where the equal sign is mostly interpreted by students as indicating the answer to the computation on its left side (Behr, Erlwanger & Nichols, 1980). Others were sentences with operational symbols on both sides (e.g., $14+6=10+10$) or no operational symbols (e.g., $27=27$). These sentences were based on the arithmetic properties indicated in Table 1. Therefore, they could be solved by using relational thinking as well as by calculation.

Arithmetic property	Example of sentences considered
Commutative Property of Addition	$10+4=4+10$
Non-Commutability of Subtraction	$15-6=6-15$
Inverse Relation of Addition and Subtraction	$100+94-94=100$
Compensation Relation	$13+11=12+12$, $78-45=77-44$
Zero as Unity Element	$0+325=326$, $125-0=125$
Inverse Element	$100-100=1$
Composition/Decomposition Relationships[2]	$78-16=78-10-6$
	$7+7+9=14+9$
Relative Size Comparisons	$37+22=300$, $72=56-14$

Table 1. Arithmetic properties used in the design of the number sentences

Description of the Sessions

Due to the different objectives of each session (described below) missing-number number sentences were used in session 1 and part of session 2, and true/false sentences in the other sessions (see the particular sentences considered in Table 2). Missing-number sentences have proven to be useful for revealing different conceptions and challenging children to reconsider their interpretations of the equal sign, while true/false sentences help challenge students' computational mindset (Molina & Ambrose, 2008).

Session 1 sentences	Session 2 sentences	Session 3 sentences	
8 + 4 = □ + 5	12 – 4 = 13 – □	72 = 56 – 14	7 + 7 + 9 = 14 + 9
□ = 25 – 12	9 – 4 = □ – 3	78 – 16 = 78 – 10 – 6	10 – 7 = 10 – 4
14 + □= 13 + 4	□ – 6 = 15 – 7	24 – 15 = 24 – 10 – 5	7 + 3 = 10 + 3
13 – 7 = □ – 6	14 – 9 = □ – 10	78 – 45 = 77 – 44	62 – 13 + 13 = 65
□ + 4 = 5 + 7	17 – □ = 18 – 8	100 + 94 – 94 = 100	19 – 3 = 18 – 2
12 + 7 = 7 + □		27 – 14 + 14 = 26	13 + 11 = 12 + 12
		231 + 48 = 231 + 40 + 8	10 + 4 = 4 + 10
		13 – 5 + 5 = 13	0 + 325 = 326
		51 + 51 = 50 + 52	37 + 22 = 300
		15 – 6 = 6 – 15	125 – 0 = 125
		27 – 14 + 14 = 26	7 = 12
		93 = 93	100 – 100 = 1
		24 – 24 = 0	

Table 2. Sentences used in the tasks of the first three sessions

Students were asked to complete the missing-number sentences and to explain how they solved it. In the true/false sentences, they were asked to determine their truth value and to be prepared to provide justifications-on-demand for their answers. In the discussions students were encouraged to articulate their strategies, to look for strategies different to those already proposed by the other students for the same sentence, and to provide justifications when asked. In this way everyone was exposed to a range of approaches, from the computational to the relational.

Justifications for responses were sought as a way to access student thinking, their ways of "seeing" the sentences, and to elicit their appreciation and verbalization of relations as well as the recognition of properties. Seen in terms of attention, being asked for justifications can shift attention from the details of particular tasks to the actions that can be short-cut by making use of arithmetical properties.

In session 1, students were asked to solve a written task individually concerning the sentences shown in Table 2. After this task, there was a plenary discussion about their answers and the way they got them. In this way students´ conceptions about the equal sign were assessed and the approaches they used started to become explicit.

In session 2, students solved individually a written task with another set of missing-number sentences (see Table 2) which were designed to explore some of the difficulties evidenced by the students in the previous session and to examine the stability of students´ understanding of the equal sign. After having a whole group discussion about these sentences which involved justifications-on-demand, students were asked to construct their own addition and subtraction sentences with operations on both sides. The aim of this last task was to (a) test our assessment of each student's understanding of the equal sign by asking them to make active use of this sign; and (b) to check whether they had noticed the richness in relations of the proposed sentences, even though only two students had so far evidenced relational thinking. Finally, in this session there was a short guided discussion of some students´ sentences which were considered to have the potential to lead to verbalizations of relational thinking: $15-15=0-0$, $10+120=100+20$, $11+11=11+11$,

$10+4=10+4$, $1000+100=0$. This discussion made explicit that some sentences could be solved without performing any operation.

From the third session on, the use and display of relational thinking was promoted by encouraging the use of multiple ways of determining the truth value of a number sentence, asking students for ways of doing so without doing all the computations, and by showing a special appreciation of explanations based on relations. The learning of specific relational strategies was not promoted. Emphasis was placed on the development of a habit of looking for relations, trying to help students make explicit and apply the knowledge of structural properties which they had from their previous arithmetic experience. Sessions 3, 4, 5 and 6 aimed to identify students´ approaches when working on the sentences and to detect and analyze obstacles arising from shifts or absence of shifts of attention. Below we briefly describe the design of each of these sessions (see more in Molina, 2006).

In session 3 a plenary discussion was developed in which justifications were demanded in response to students´ assertions as to whether various statements were true or false (see the statements used in Table 2). In the design of these sentences the following elements were balanced: (a) true and false sentences, (b) numbers lower than 30 or numbers from 50 to 326, (c) sentences based on each of the arithmetic properties before mentioned. Students were also asked to propose a correction for the sentences that they considered false.

In session 4 students individually solved a written task consisting of true/false sentences similar to those used in session 3 (see the particular sentences considered in Table 3). Students had to decide if the sentences were true or false, justify their decision and, if they considered the sentence false, to propose a correction.

In session 5 semi-structured interviews were conducted with half of the students. The students were chosen depending on the use of relational thinking that they had evidenced, with the requirement of having attended all the previous sessions. At least two students of each of the following categories were selected: (a) No use of relational thinking evidenced; (b) Use of relational thinking evidenced just occasionally; (c) Half use and half non-use of relational thinking evidenced; (d) Use of relational thinking evidenced in most sentences, (e) unclear behaviour. The aim of the interviews was to deepen the study of students´ use of relational thinking. These students were probed with sentences which were based on different arithmetic properties (see Table 1) than those sentences in which they had evidenced use of relational thinking in previous sessions. The sentences considered were similar to those of session 3. As before, they were asked to be prepared to provide justifications-on-demand.

Session 6 was an assessment session in which students were probed with the same individual written task used in session 4.

Classroom Atmosphere

During the six sessions of the teaching experiment, students actively participated in the discussions and written activities right from the start. They knew how to solve the open sentences through computation and enjoyed participating in discussions and getting opportunities to explain their thinking. The teacher-researcher asked them for different ways in which they could justify their answers, and they responded by making efforts to provide explanations different to the ones of the other students. They sometimes shifted the order in which the operations were performed or looked for different strategies to compute the same computation (e.g. by an addition instead of by a subtraction, through counting, etc.). By asking students for different ways of solving the same sentence, the teacher-researcher tried to force them to listen to the other students´ explanation. They sometimes referred to other students´ explanations to try to show that theirs was different, but never asked for clarification of other students' responses.

Students evidenced more difficulties in explaining their thinking when it was based on relations. In many cases their explanations were less mathematically precise, so the teacher-researcher often translated them to the whole class in order to ease understanding. As an example we present below an extract[3] from the discussion of the sentence $51+51=50+52$ in session 3.

> **Fran:** Like fifty-one plus fifty-one are one hundred and two, but fifty-one, if you subtract fifty, you can add to fifty-one, one from the other, one more, and you get fifty-two.
>
> **Teacher-Researcher:** Ah, that is interesting. You said that you can take one from here [pointing to the first fifty] and add it to this one [pointing to the second fifty one]. Isn't it? Is that what you said?
>
> **F:** And you get there, one hundred… fifty plus fifty-two.
>
> **T-R:** Ah, Did you understand what Fran said? He said that if we take one from the first fifty-one (pointing to the first fifty-one), we get fifty (pointing to the fifty), and if we give it to the other fifty-one (pointing to the second fifty-one), we get fifty-two (pointing to the fifty-two).

Students with weak arithmetic skills rarely participated in the discussions. They seemed frightened to state their thinking to the whole group and displayed little trust in their mathematical competence. They were not forced to participate orally but their answers were noted whenever they were willing to provide them to the whole group. On several occasions they only stated the validity of the sentence and claimed not to be able to explain their thinking. The written activities and the interviews were good opportunities to access these students´ thinking as they seemed to feel less intimidated.

Students´ Previous Experience Related to These Tasks

Unlike reports form various researchers (Behr et al., 1980; Carpenter et al., 2003; Kieran, 1981; Molina & Ambrose, 2008; Warren, 2003) previously commented on, the group of elementary students that participated in the teaching experiment did not tend to perceive the equal sign as an operational symbol. In the first session assessment most students displayed evidence that they were looking for a number that would make the expressions in both sides of the equal sign to have the same numeric value. During the teaching experiment some students displayed an occasional instability in their understanding when the sentences involved computations which required a higher cognitive demand or which caused them some difficulties. It was then that they altered the structure of the sentence or ignored some of the terms. But, in general, students evidenced a relational understanding of the equal sign[4]. This is probably a consequence of the fact that their regular classroom teacher was especially concerned about learning the meaning of mathematical symbols and used to emphasize it in his daily teaching.

Students were not, however, used to working on number sentences with operational signs on both sides, even though occasionally they would encounter this type of sentence in their textbook in activities or explanations regarding the use of brackets or the commutative property. Relational thinking was not promoted in their regular instruction. Only some mental computation strategies were addressed, presented as ‘tricks’, and they were not promoted in the regular practice. According to the regular classroom teacher, students´ previous experience about structural properties was reduced to direct instruction about commutativity of addition and multiplication, and associativity of addition, as well as their own awareness about structural properties resulting of their personal arithmetic experience.

The regular mathematics instruction in the classroom was very traditional and did include neither discussions nor group work. The communication in the classroom was based on the teacher explaining the lesson, asking for answers and giving feedback about their correctness. Students used to work individually at their desks and sometimes went to the blackboard to solve some computation or problem in front of all the students. The classroom teacher faithfully followed a textbook which was mainly centered on promoting computational practice and occasionally included some word problems.

Data Collection and Analysis

In all the sessions the students´ individual written work was collected. All the whole-group discussions were video-recorded and the interviews were audio-recorded. These recordings were complemented with the researchers´ field-notes about the in-class interventions as well as about the researchers´ meetings.

The analysis combined data from the different sources: students´ written work and students´ oral explanations in discussions and interviews. Each students´ answer in each sentence was individually analyzed and also compared to his/her responses to other sentences in the same session and in other sessions, as well as to other students´ responses.

Because the analysis reported here is a re-analysis of the data after it had all been collected, in order to identify where students were placing the attention when working on each sentence we focused on what the students said and wrote. The data collected does not allow us to locate or observe all the ways in which students focused their attention, but only for those that the students displayed through their explanations or written work.

We were especially careful about not over-reading or over-hearing in students´ productions; a characteristic challenge of the complex process of interpreting students´ talk and actions (Wallach & Even 2005). There are some instances in which the data do not reveal where students were placing their attention. For example, César's work on the sentence $122+35-35=122$ does not reveal whether he recognized that the same number was added and subtracted to 122 (although it suggests so). In the rest of the sentences of the session 4 assessment, he computed and compared the numeric values of the expressions in both sides of the equal sign; however, no written computation appeared in his submitted paperwork for this sentence. In addition, his explanation for this sentence was fairly incomplete: "True because $122+35$ but if you 35 to 22, you get 122".

As a consequence of this limitation in identifying the students´ focus of attention from the data collected, when we comment on the result of this study in the next sections of the paper, we provide ranges of percentages and of numbers of students, instead of exact quantities.

Movements of Attention Looking Across Students´ Thinking Over Time

As might be expected, at least 50% of the time learners computed both sides of the equations and then declared whether the equality was true or false. In about 5% to 6 % of the time, they started to calculate but then something in what they said or did, led them to recognize or at least to act upon a relationship through which they could detect the truth or falsity of the equality. Between 25% to 30% of the time students made overt use of relationships without doing any computations at all.

The notion of shifts of attention or alterations in the structure of attention provides an explanation for this observed behaviour. When analyzing students´ structure of attention, we detect shifts over three different scales of time. Short-term shifts occurred while deciding the validity of a single sentence. Medium-term shifts took place while a student was working on a set of number sentences. Long-term shifts are detected when comparing a student's behaviour in different sessions of the teaching experiment.

Working on a Sentence. Short term shifts.

Students´ responses show that, faced with an equation involving numbers, they discerned at least the numbers, the equal sign, and some operations. Some did not display recognition of any relationship. For example, in the sentence $257-34=257-30-4$ Clara's work consisted of computing the numeric value of both sides using the addition standard algorithm for the computations $257-30=227$, $227-4=223$ and $257-34=223$. She explained: "True because $257-34$ is 223 and it is equal to $257-30-4$ [that] is 223". Noelia did similarly in the sentence $125-125=13$. She computed the numeric value of the left side using the subtraction standard algorithm and then concluded the falsity of the sentence by explaining that the result of the computation was zero and not thirteen. We conjecture that the equal sign or the presence of operation signs may have triggered them into a computational mindset to obtain the numeric value in order to test validity. Their attention seemed to be focused on, even absorbed by the calculations. As we later discuss, it may be that the more challenging the computation, the more attention is required to carry out the calculation, holding temporary results and so on, so that other features of the statement fade into the background.

Others, however, while performing a calculation For example, this was observed in Fabian´s work on the sentence $51+51=50+52$. He explained: "Like fifty-one plus fifty-one are one hundred and two, but fifty-one, if you subtract fifty, you can add to fifty-one, one from the other, one more, and you get fifty-two". Maite´s work on the sentence $75 + 23 = 23 + 75$ also evidenced this shift in attention. She first wrote $75 + 23$75 + 23 in a vertical format to add them by columns but suddenly stopped and explained "

In other cases such as in the sentence $7+7+9=14+9$, initiating a calculation and getting a partial result suddenly resonated with what else was visible in the statement, prompting recognition of a relationship. For example in this sentence Clara explained: "adding seven plus seven…. adding seven plus seven you get fourteen, the same than there, nine the same than there too". Having calculated or just by knowing the fact $7+7=14$, the presence of the 14 was strong enough to lead Clara to direct attention to the right hand side, so she saw or became aware of two identical statements $14+9$ and $14+9$, which can be seen to be equal without further computation. Not only is it a repetition of a specific number, but a particular instance of a general property.

What we can't find out very easily is whether these students were aware of or perceived the general property explicitly, with the particular as an instantiation, or whether their action was more like a theorem-in-action (Vergnaud, 1981), in which they acted as if they knew the property, while only being explicitly aware of the relationship in the particular case.

Between 25% to 30% of the time, students went directly and immediately to relationships without any evident attempt at computation and their thinking can be described as being relational. This is for example the case of David in the $122+35-35=122$ who did not perform any written work on this sentence and explained "True because it is as if you give the number and then you get [take] it back". Another example of this use of relational thinking is provided by Carmen explanation to the sentence $13+11=12+12$: "I have thought that I can take one from thirteen and you get twelve, and I added that one to the eleven, and I get twelve plus twelve equals twelve plus twelve". To do this, they would have held the sentence as a whole, directly discerning and attending to the numbers and operations involved and recognizing relations between them. Their attention would appear to have been dominated by recognizing relationships.

Sameness was one of the relationships used and in sentences like $18-7=7-18$ (mis)led some students to declare the result true. This can be result of having recognized a relationship but continued to hold the elements as wholes rather than checking details of the sameness. Similarly lack of sameness was used occasionally to conclude that a sentence was false (e.g., $10-7=10-4$; $53+41=54+40$).

In order to make use of relational thinking, a learner needs to have sufficient free attention so as not have it all taken up with any calculations that are initiated, or else to be centered in recognizing relationships rather than being drawn into calculating as a first response. Details need to be discerned on both sides of the equals sign, and relationships amongst these need to be recognized. The essence of relational thinking is the recognition of a relationship between some features of the statement, usually items from both sides but not necessarily. This in turn requires awareness of the equal sign not as a trigger to calculate, but rather as statement of relationship. Justification, when demanded, can either be in terms of the particulars, or in terms of generality, indicating a perception of properties being instantiated.

Medium-Term Shifts from one Sentence to another

Sometimes students displayed evidence of a shift in the structure of their attention during the movement from one sentence to another. In Table 3 we have an example in the working of a student, Jose, in the written assessment of session 4. Although his thinking is not clear in some sentences, other responses suggest movement in his attention. Jose proceeded relationally in the sentences $75-14=340$ and $6+4+18=10+18$. In the former he compared the relative size of the numbers in both terms and concluded that the equality was impossible; and in the latter he recognized the equivalence of both sides of the sentence by computing $6+4=10$ and recognizing sameness between both terms (which he expressed by writing $10+18=10+18$). However, in the sentences $122+35-35=122$ and $16+14-14=36$ he computed the left side, from left to right, and compared the numeric value obtained with the number on the right side. Jose's attention at any given moment may have been influenced by the type of sentence: with operations on both sides of the equal sign or just on one of them. His approach tended to be computational when addressing the later type of sentences and relational in the former type, with some exceptions.

Sentence	Explanation
18 – 7 = 7 – 18	True because 18 – 7 is = [the same] as 7 – 18, it is the same.
75 – 14 = 340	False because 75–14 cannot give 340
17 – 12 = 16 – 11	True because 17 – 12 is = [the same] as 16 – 11
122 + 35 – 35 = 122	False, because 122 + 35 = 175 and 175 – 35 = 140, so it doesn't give 122 (He computes 175 – 35 = 140 using columns)
6 + 4 + 18 = 10 + 18	True because 6 + 4 + 10 + 18 = 10 + 18, it is the same in both computations
75 + 23 = 23 + 75	True because 75 + 23 is the same as 23 + 75
7 + 15 = 8 + 15	False because 7 + 15 is 22 and 8 + 15 is 23
53 + 41 = 54 + 40	True because 53 + 41 is 91 and 54 + 40 is 94
16 + 14 – 14 = 36	False because 16 + 14 is and 30 – 14 is 20
257 – 34 = 257 – 30 – 4	True because 257 –34 is 223 and 257 – 30 – 4 is 223

Table 3. Jose's responses and justifications in session 4 written assessment

In other students' responses to the same assessment we observe different patterns in the movement of their attention. David computed the numeric value of both sides in all except the sentences $18-7=7-18$ and $122+35-35=122$. David's thinking seems mainly computational but in $122+35-35=122$, without doing any computation, he appreciated that "it is as if you give it a number and later you get (take) it back". In this case we conjecture that the bigger numbers in the sentence may have led him to attend to relations between the terms which he did not recognize again later in the sentence $16+14-14=36$ where his thinking might be more concrete and calculational, due to his familiarity with these smaller numbers. Although some students treated large numbers as obstacles, they can help students focus on form and structure rather than on computation. Zazkis (2001) uses this insight as a pedagogical tool to help students see the general in the particular and focus on noticing structure, reasoning with it and expressing it.

In the case of Elena, we detect a different behaviour as she thought relationally when working on almost all the sentences in the assessment. She computed the values of each side to answer the sentence $17-12=16-11$ but, after doing the same in the sentence $53+41=54+40$, she explained that it was true because "we put the 1 from the 54 to the 40 and you get the same". In the sentence $7+15=8+15$ she also did the computation before appreciating that it could not be true due to a difference of size between the expressions on both sides. She seemed to recognize relationships as an after-thought rather than before she embarked on computation. In her case the size of the numbers in the sentence did not appear to influence her approach but the relations involved in the design of the sentences did. The sentences based on the compensation relation were for her the ones where relations were harder to recognize.

Maite, in the same assessment, only showed evidence of use of relational thinking in the following sentences which include operations on both sides: $75+23=23+75$, $7+15=8+15$ and $18-7=7-18$. In the other sentences she computed and compared the numeric value of each side, while in these three sentences recognized sameness or "almost sameness" between the expressions on both sides as she expressed (see Table 4). In these sentences she initially proceeded to compute one of the sides, or just to write it vertically, before recognizing any relation. With her behaviour Maite displayed a tendency to calculating when approaching the sentences but in some sentences her attention was not completely taken by the computations, allowing her to recognize some basic relations between both sides of the sentences, mostly when there was an operation on both sides of the equal sign.

Sentence	Explanation
18 – 7 = 7 – 18	True because both are equal (She computed 18 – 7 = 01 by using the addition standard algorithm)
75 + 23 = 23 + 75	True because is equal (She wrote 75 + 23 in a vertical format but then stop)
7 + 15 = 8 + 15	False because it is almost the same but it is not the same (She computed 15 + 7 = 22 using the addition standard algorithm)

Table 4. Maite's responses and justifications in session 4 written assessment that displayed use of relational thinking

These examples illustrate a mid-term shift detectable in students´ attention as they worked by themselves in a set of sentences. The movement of their attention seems to be due to a variety of possible influences. As we further discuss below focusing on the arithmetic relations used in the design of the sentences, in some cases sentences with operations on both sides of the equal sign promoted more relational approaches than sentences with operations on just one side. In others, big numbers dissuaded students from initiating calculations and led their attention to the structure of the sentence.

Some relations (e.g., sameness) were more easily recognized by the students than others (e.g., compensation). Even those students who were most likely to compute tended not to do so on number sentences which include zero relations ($a+0=a$; $a-0=a$; $a-a=0$). Sentences involving the commutative property also seemed to promote relational approaches. In the discussion of session 3, none of the students that participated in the discussion of the sentence $10+4=4+10$ solved it by computing. In sessions 4 and 6, only 5 and 8 students, respectively, solved the sentence $75+23=23+75$ by computing the numeric value of each side.

In the sentences based on composition/decomposition relation as well as on the inverse relation of addition and subtraction, half of the students preceded computationally while the other half used relational thinking. However, in the latter we detected more use of computational approaches when the sentences included small numbers. In the sentences based on "relative size comparisons" initially, during the whole group discussion of session 3, students evidenced both approaches but computational approached became more frequent in the sessions 4 and 6. This tendency was specially appreciated in the action sentences considered. We conjecture that this may be result of the fact that they did not include equal numbers in both sides, while the others did. The sentences based on the compensation relation were the ones least frequently approached relationally, especially those involving subtraction.

Long-Term Shifts from one Session to another

Initially, in sessions 1 and 2, computing and comparing the numeric values of each side was the most common strategy. When asked to solve the sentences in different ways, students tended to propose a different order in which to perform the computations. Only three students spontaneously evidenced some use of relational thinking in the sentences $12 + 7 = 7 + \square$, $9 - 4 = \square - 3$, $12 - 4 = 13 - \square$ and $15 - 15 = 0 - 0$. However, from session 3 on, when paying attention to recognizing relations was promoted and various strategies based on the use of relational thinking were made explicit, more students, and more frequently, recognized relations and perceived properties which they used to solve the sentences. In all, about 14 out of 18, 19 out of 24, 11 out of 13 and 17 out of 24 students respectively, evidenced used of relational thinking in the third, forth, fifth and sixth sessions (See Table 5).

Session	Students attending the session	Students participating	Students evidencing relational thinking	
			Minimum	Maximum
1	26	26	1	1
2	21	21	2	3
3	22	18	12	14
4	25	24	17	19
5	13	13	10	11
6	24	24	13	17

Table 5. Number of students evidencing relational thinking in each session

By the time of the sixth session, all but two or three students solved the sentences at least once using this type of thinking instead of computing and comparing the numeric values of both sides. Interestingly, three students gave evidence of relational thinking only during the interview. For example Roberto used it in several sentences by providing the following explanations: "[$13+5=5+13$] It is true because it is the same; just the other way around", "[$26-8=100$] It is false because if it were… it is twenty-six minus eight and it is equals to one hundred, and then as it is minus, it is to take away, and it has to be still one hundred…One hundred is higher than that one, than the subtraction". This student did not provide any evidence of having attended to relations between terms in the sentences in previous sessions but, on the other hand, never displayed any special difficulty in the proposed tasks. From these results, we conjecture that relational thinking may not become evident unless students are immersed in a culture which explicitly values recognizing relationships and perceiving properties of which they are instantiations. It is not that students can't, or even don't, think relationally, but rather what is encouraged and promoted through classroom practices.

In the interviews other students (5 of the 13 interviewed) displayed a more frequent use of relational thinking than in previous sessions. For example, David had provided just an isolated use of relational thinking in previous sessions (e.g., "True because it is as if you give it the number and then you get (take) it back" in the sentence $122+35-35=122$) but in the interview he used relational thinking in all the sentences he encountered:

$13+5=5+13$, $26-8=100$, $8+6=4+4+6$, $11-6=10-5$, $11+7=10+8$, $19-13=9-3$.

We conjecture that how frequently students used relational thinking or in which type of sentences they did, also depended on their arithmetic knowledge and on their awareness of this knowledge. Being asked to justify a conclusion, result, or conjecture can be responded to in many ways. At first there is the "it just is" response (Freudenthal, 1978) associated with students who take everything they are told at school as factual and requiring learning. It requires a change in their perception of the implicit *contrat didactique* (Brousseau, 1984) to recognize the socio-cultural practice of mathematics to provide reasons based on structural properties. But to engage in such a practice also requires an awareness of the fact of these properties as properties, with particular instantiations as relationships. A student whose attention is fully taken up by numbers, operations and calculations is not in a position to recognize relationships, much less to perceive them as instantiations of perceived properties.

Conclusions

The teaching intervention developed in this teaching experiment aimed to promote the display of relational thinking and tried to alter how students attended to calculations and expressions through justifications-on-demand in a classroom atmosphere where the focus was not on numeric results nor on calculations but on recognizing and expressing relationships, and using them to get an answer. In this context we identified changes in students´ attention by looking across their thinking over time.

It is clear from the data that students´ focus of attention varied from moment to moment and from time to time. We showed that these movements were influenced by the characteristics of the sentence: structure, size of the numbers and relations used in its design. The results presented here as well as others of previous studies (Molina & Ambrose, 2008) suggest that they also depended on the student's previous arithmetic experience (which conditions the cognitive demand of a task, their conception of numbers and of the equal sign meaning, their conscious and unconscious awareness of arithmetic relations, and the strength of their disposition to compute) and on the classroom culture. Regarding this last element, we observed that our emphasis in getting justifications about the validity of a sentence which did not require computations encouraged students to look for relations between terms or parts of the sentences and try to reason using them. When the teacher-researcher worked individually with students in the interviews, this influence was even more evident. Our special appreciation of this type of explanation together with the particular design of the sentence considered (where the numbers are used as quasivariables to express properties, see Fujii & Stephens, 2001) naturally led to relational thinking being made explicit since it is well known that elementary students are capable of this type of reasoning (Carpenter, Franke, & Levi, 2003; Molina & Ambrose, 2008).

By looking closely at students' responses to a specific range of arithmetic statements, it seems reasonable that it would be of benefit to teachers to become attuned to subtle variations in how students are attending, in order to be in a better position to prompt students both to recognize relations, and to perceive properties as being instantiated. Thus one possible didactical consequence is that by working on their own awareness so as to sensitise themselves to different ways students might attend, teachers can extend the didactic choices available to them for directing students´ attention appropriately.

Martino and Maher (1999) found that, in general, students do not naturally seek to build a justification or proof of the validity of a solution as they tend to think that proposing a solution is sufficient evidence for justification. So, the teachers´ role in demanding justifications would appear to be essential in order to take students´ thinking further. Even if it is not addressed in teaching, many students (although not all) eventually tend to follow a developmental progression in the use of relational thinking as result of their arithmetic experience (Knuth, Alibali, McNeil, Weinberg & Stephens, 2005; Stephens, 2007). By incorporating justifications-on-demand into classroom practices, it may be that this vital shift in thinking can be accelerated and exploited to enrich and connect the learning of arithmetic and algebra. Rather than being the focus of a few specific lessons, attending, using and expressing relations and properties as well as providing justifications could become part of regular mathematical practice. Even if some students are disposed to think relationally, this may not be evident unless students are immersed in a cultural practice which calls upon the expression of relationships and properties.

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FOOTNOTES

[1] Along this paper, we will refer to this type of sentences as “true/false number sentences”.

[2] Although we acknowledge that the compensation relation is not strictly the same property for both operations, addition and for subtraction, we present them together because both refer to how to compensate for an increment or decrease in one of the terms in a sum or subtraction.

[3] All extract from discussions as well as students´ explanations presented in this paper were translated from Spanish.

[4] See Molina, Castro, & Castro (2009) for a deeper and more detailed analysis of the students´ understanding of the equal sign evidenced along the teaching experiment.