IMPROVING AWARENESS ABOUT THE MEANING OF THE PRINCIPLE OF MATHEMATICAL INDUCTION

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This work is based on our conviction that it is possible to minimize difficulties students face in learning the principle of mathematical induction by means of clarifying its logical aspects. Based on previous research and theory, we designed a method of fostering students' understanding of the principle. We present results that support the effectiveness of our method with teachers in training who are not specializing in mathematics.

Keywords: Learning and teaching with understanding; Methods of teaching; Principle of mathematical induction; Proof in mathematics; Teachers' training

Fomentar la Conciencia sobre el Significado del Principio de Inducción Matemática

Este trabajo está basado en nuestra convicción de que es posible minimizar las dificultades de los alumnos cuando se enfrentan al aprendizaje del principio de inducción matemática mediante la clarificación de sus aspectos lógicos. Basándonos en la investigación y teoría previas, diseñamos un método para fomentar la comprensión del principio por los alumnos. Presentamos resultados que respaldan la efectividad de nuestro método con profesores en formación no especializados en matemáticas.

Términos clave: Aprendizaje y enseñanza con comprensión; Demostración en matemáticas; Formación de profesores; Métodos de enseñanza; Principio de inducción matemática

The principle of mathematical induction (PMI) represents a key topic in the education of teachers in Italy. The approach traditionally used in Italian schools devotes little time to the teaching of a solid understanding of the principle. Most textbooks do not cover the PMI in depth and only require students to "blindly"

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apply it in proving equalities. Students learn to mechanically reproduce the exercises but do not develop a true understanding of the PMI. We propose that it is important and also possible to promote understanding of the PMI, rather than just its application, using non traditional methods. In this paper we present some findings from a study that used a non-traditional approach to teaching the PMI with 44 pre- and in-service middle school —grades 6-8— teachers who were completing a teacher training course. Most of these trainees were not mathematics graduates, but had had some exposure to the PMI during their studies and therefore are a good sample for both examining the traces of their education history and assessing the usefulness of a non-traditional approach to teaching the PMI. In particular, we were interested in promoting comprehension and correcting previously learned misconceptions.

THEORETICAL FRAMEWORK

Previous research has highlighted difficulties that students encounter learning the PMI due to certain misconceptions about it. For example, Ron and Dreyfus (2004) argue that three aspects of knowledge are required to foster a meaningful understanding of a proof by mathematical induction (MI): (a) understanding the structure of proofs by MI, (b) understanding the induction basis, and (c) understanding the induction step. Based on our experience teaching the PMI, we believe that the third aspect, the induction step, is the most important in fostering an understanding of it. Ernest (1984) observes that a typical misconception among students is the idea that in MI "you assume what you have to prove and then prove it" (p. 181). Fishbein and Engel (1989) also stress that many students are "inclined to consider the absolute truth value of the inductive hypothesis in the realm of the induction step" (p. 276). Both Fishbein and Engel (1989) and Ernest (2004) argue that the source of this misconception is in students' lack of understanding of the meaning of proofs of implication statements. They suggest that a proper approach to teaching the PMI must include logical implication and its methods of proofs. In Malara (2002), we agree with Avital and Libeskind (1978) who suggest that a way to overcome students' bewilderment in front of the jump from induction basis to induction step is to approach MI by means of *naïve induction*, which consists of showing the passage from k to $k+1$ for particular values of k "not by simple computation but by finding a structure of transition which is the same for the passage from each value of k to the next" (p. 431).

Another conceptual difficulty experienced by students that is highlighted by research is that many students look at the PMI as something which is neither self evident nor a generalization of previous experience. Ernest (1984) suggests that a way to overcome this problem is to refer to the well ordering of natural numbers. That is, if a number has a property and "if it is passed along the ordered sequence from any natural number to its successors, then the property will hold for all numbers, since they all occur in the sequence" (p. 183). Harel (2001) also refers to this way of introducing the PMI, calling it *quasi-induction*, but he observes that there is a conceptual gap between the PMI and quasi-induction which students are not always able to grasp. The quasi-induction has to do with steps of local inference, while PMI has to do with steps of global inference.

In addition, Ron and Dreyfus (2004) highlight the usefulness of using analogies with students when teaching the PMI for two reasons: (a) Analogies illustrate the relationship between the method of induction and the ordering of natural numbers, and (b) They are tools for fostering understanding of the use of MI in proofs.

RESEARCH HYPOTHESIS AND PURPOSES

We propose that an effective approach to teaching the PMI requires a combination of the different points described above. In particular, we propose that the essential steps in a constructive path toward PMI should include:

1. A thorough analysis of the concept of logical implication.
2. An introduction of PMI through the naïve approach, drawing parallels between PMI and the ordering of natural numbers, and the use of reference metaphors.
3. A presentation of examples of fallacious induction to stress the importance of the inductive basis.

Our hypothesis is that a path in which all of these aspects are considered leads to a real understanding of the meaning of the principle and therefore to a more conscientious use of it in proofs. Furthermore, a real understanding of the principle does not necessarily mean being able to apply it, since many proofs through MI require being able to use and interpret algebraic language.

The purpose of our research is to test the usefulness of this proposed path in instilling a deeper understanding of the PMI. We do this by monitoring trainees during a range of activities and ending with a final exam designed to assess students' true understanding of the PMI. In this paper we present the experience of one trainee, which supports the effectiveness of this approach.

METHOD

The path we propose can be divided into six main phases:

1. An initial diagnostic test.
2. Activities which lead students from conditional propositions in ordinary language to logical implications.

3. Numerical explorations of situations aimed at producing conjectures to be proved in a subsequent phase.
4. An introduction to the method of proofs by MI and to the statement of the principle.
5. Analysis of the statement of PMI and production of proofs.
6. A final test —given 3 weeks after the last lesson—.

Because of space limitations, we focus on one central phase in the path, because it contains the aspects we propose as essential to a meaningful approach to teaching PMI. The following proof, which was a starting point in the construction of a lesson, was proposed by a trainee —the teacher R—, during the numerical exploration phase[1]. R intended to prove the conjecture she produced on the sum of the powers of 2: $2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}-1$. After having observed that proving this equality is the same as proving $2^0+2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}$, R proceeded in this way:

$$2^0+2^0+2^1+2^2+2^3+\ldots+2^n=$$
$$2\cdot 2^0+2^1+2^2+2^3+\ldots+2^n=$$
$$2^1+2^1+2^2+2^3+\ldots+2^n=$$
$$2\cdot 2^1+2^2+2^3+\ldots+2^n=$$
$$2^2+2^2+2^3+\ldots+2^n=$$
$$2\cdot 2^2+2^3++\ldots+2^n=$$
$$\ldots=$$
$$2^n+2^n=$$
$$2\cdot 2^n=$$
$$2^{n+1}$$

We showed to trainees R's proof of $P(n): 2^0+2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}$ and we observed with them that the individual steps of her proof constitute "micro-proofs" of the individual implications: $P(0)\to P(1)$, $P(1)\to P(2)$,…; the dots testify that she made a generalization. The formal aspects we used in this discussion were:

[1] R's proof represents what Harel (2001) defines as quasi-induction.

$$P(0): 2^0 + 2^0 = 2 \cdot 2^0 = 2^{0+1} \Rightarrow$$
$$P(0) \rightarrow P(1) \Rightarrow$$
$$P(1): (2^0 + 2^0) + 2^1 = 2^1 + 2^1 = 2 \cdot 2^1 = 2^{1+1} \Rightarrow$$
$$P(1) \rightarrow P(2) \Rightarrow$$
$$P(2): (2^0 + 2^0 + 2^1) + 2^2 = 2^2 + 2^2 = 2 \cdot 2^2 = 2^{2+1} \Rightarrow$$
$$\ldots \Rightarrow$$
$$P(k): 2^0 + 2^0 + 2^1 + 2^2 + \ldots + 2^k = 2^{k+1} \Rightarrow$$
$$P(k) \rightarrow P(k+1) \Rightarrow$$
$$(2^0 + 2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} = 2^{k+1} + 2^{k+1} = 2 \cdot 2^{k+1} = 2^{k+2} \Rightarrow \ldots$$

We discussed the following points with the trainees:

- The structure of natural numbers is such that every number *n* could be obtained from the previous *n-1* adding 1;
- Every sum S_n is obtained by the previous sum adding 2^n—the nth power of 2—; and
- The terms of the successions have in common the property of strictly depending on the terms which precede them.

These observations allowed the trainees to agree on the fact that every proposition could be derived recursively from its prior. Starting with this intuition, we highlighted the common structure of R's proofs of the "particular implications" and guided trainees to observe that this structure can be followed every time it is necessary to prove a proposition *P(k+1)* starting from the previous proposition *P(k)*. Trainees became aware that the complete proof of the statement is based on a chain of implications, such as the ones highlighted in R's proof, that can be summarized as $P(k) \rightarrow P(k+1)$, ∀k∈N. Together we constructed the proof of this general implication, as a generalization of the step-by-step micro-proofs. Because of the previous activities on logical implication, trainees were aware that an implication could also be valid when the two components are not valid. It was easy for them therefore gradually to become aware that proving $P(k) \rightarrow P(k+1)$ ∀k∈N means proving that *P(n)* is valid ∀n∈N, only if the first proposition of the chain, *P(0)*, is valid.

Analysis of Trainees' Work During the Path: The Case of L

During the activities, trainees also worked individually. We collected their protocols in order to analyze the evolution of their acquisition of meaning of the PMI. In particular, we compared the answers they gave in the initial and final tests in order to highlight their effective acquisition of awareness of the meaning and use of PMI. The final test consisted of four questions, two following the questionnaire included in Fishbein and Engel (1989), the other two concerning the proof of two statements. The purpose was to verify

- whether trainees really understood the meaning of the inductive step and the importance of the inductive basis as an integral part of the proofs by MI; and
- whether trainees were able to single out the key passages which are necessary to perform proofs by MI concerning new conjectures.

The results of the questionnaires were really satisfactory because almost all trainees produced correct proofs and, more importantly, many of them demonstrated having acquired an effective comprehension of the sense of the principle. As an example, we focus on the analysis of the evolution of another trainee —the teacher L—, because we observed a remarkable difference between the problematical nature of her initial situation and the level of awareness and the abilities she displayed in her answers on the final test. We present two excerpts from her protocols. The first one is taken from the initial test and the second concerns an answer she gave in the final test.

Excerpt from the Initial Test

The excerpt refers to the proof of the inequality $2n > 3n + 1$, where $n \geq 4$. L wrote:

1) $2^4 > 3 \cdot 4 + 1$

$16 > 13$ *ok*

2) $2^k > 3k + 1$

$k > 4$ *It is true.*

Proof: $2^{k+1} > 3(k+1) + 1$

$2 \cdot 2^k > 3k + 3 + 1$

$2 \cdot 2^k > 3k + 1 + 3 \cdot$

$2 \cdot P(k) > P(k) + 3$, *which is always true because the hypothesis is true (∀k≥4)… but it something I can see at a glance!*

First of all, let us notice L's erroneous used of the specific symbology. Instead of referring to *P(k)* as the proposition which represents the statement to be proved, she dealt with it as representing each of the expressions at the two sides of the inequality. The logical aspects involved in the use of the principle should also be considered. For example, L directly took into account the inequality to be proved and tried to justify it on the basis of the hypothesis, but her arguments relied only on "evidence." L's difficulties have to be ascribed to a lack of knowledge about logical implication, which is also documented in other answers.

Excerpt from the Final Test

The second excerpt we present refers to a part of the answer L gave to the following question:

> *During a class activity on PMI, Luigi speaks to his mathematics teacher in order to remove a doubt: "We have just proved a theorem, represented by the proposition P(n), by MI, but this method is not clear… I am not sure that the theorem is really true because, in order to prove P(n+1), we had to hypothesize that P(n) is true, but we do not know if P(n) is really true until we prove it!" If you were his teacher, how would you answer to Luigi?*

After correctly enunciating the principle, L commented:

> *It is necessary for Luigi to understand that in the inductive step we do not prove either P(n) or P(n+1), we only prove that the validity of P(n) implies the validity of P(n+1), that is, we prove the implication P(n)→P(n+1).*

Because of space limitations, we do not report the correct proofs L produced. This excerpt, however, demonstrates the level of comprehension she attained during the laboratory activities.

CONCLUSIONS

Our observations of the laboratory activities and analysis of trainees' protocols allow us to draw some conclusions on the validity of our research hypothesis. L represents a prototype of an individual for whom a traditional way of teaching left only few confused ideas on the proving method by MI. The different approach L adopted and her ability both to understand the problem pointed out by Luigi and to respond in a synthetic and precise way to his doubts, represent evidence of the effectiveness of the choices we made in our approach to teaching the PMI. L is just one example from a large group of trainees who developed a deeper understanding of the PMI in a similar way. The positive outcomes on the final tests testify to the validity of our research hypothesis regarding the aspects fundamental to a productive introduction to the use of PMI as a "proving tool." As a future development of our research, in order to test further the effects of this approach, we plan to test the same method in secondary school, with students learning the PMI for the first time. In particular, our aim is to highlight the role played by the teacher in the management of the lessons.

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