ADAPTING THE HYPOTHETICAL LEARNING TRAJECTORY NOTION TO SECONDARY PRESERVICE TEACHER TRAINING

Pedro Gómez[a], María José González[b] and Jose Luis Lupiáñez[a]

[a]Universidad de Granada, [b]Universidad de Cantabria

We adapt the idea of hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) to preservice teacher training. Our approach is based on the notion of capacity, which is used to characterize a concrete teacher's learning goal. Using links between capacities, we propose some tools that can be used by a teacher to analyze and select tasks and to produce hypotheses about students' learning processes. We describe possible uses of these tools by future teachers, considering that they will use standard available resources in preservice teacher training: meanings of a concept in school mathematics and students difficulties when facing the tasks. We exemplify this process considering a particular learning goal in a lesson on the quadratic function.

INTRODUCTION

When a teacher plans mathematical tasks he carries out some kind of anticipation about his students' learning processes. This can be considered one basic assumption for any of the teachers' planning responsibilities, from the annual subject design to the planning of every daily class period. The problem of anticipating students' learning processes with the purpose of designing or selecting mathematical tasks, when stated into a constructivist framework, highlights the so-called *planning paradox* (Ainley & Pratt, 2002, p. 18). This paradox points out the tension between impoverished mathematical tasks, focused on learning objectives, in opposition to students' engaging tasks that, however, imply difficult learning assessment.

Trying to resolve this paradox, Simon (1995) proposed a particular view of the teacher planning process in a constructivist framework. He introduced the *hypothetical learning trajectory* (HLT) construct in order to structure the relationships between the teacher goal for the students learning, the mathematical tasks proposed and the hypotheses about the process of students' learning. This notion has gone through different interpretations and elaborations (see, for example, Clements & Sarama, 2004 for an overview).

Acknowledging that the HLT notion does not provide a framework to think about the role of mathematical tasks in the learning process, Simon and Tzur (2004) have proposed the *reflection on activity-effect relationships* mechanism for explaining mathematics concept development. This reflection mechanism elaborates the constructivist assimilation construct by taking into account learners' reflection on the relationship between their activities and their effects. It is established that a learner develops a new concept by recording units of experience consisting of iterations of the activity linked to its effect, sorting and comparing those records, and identifying relationships and patterns among the activities and their effect.

This is a cognitive description that can be used with a practical purpose: to determine processes to generate HLT's, that is, to make operative the cyclic relation between the design and selection of mathematical tasks and the students' development of conceptual learning. Simon and Tzur exemplify one possible process on a lesson involving equivalent fractions. But some open questions arise in this approach. Considering the researchers' viewpoint, the literature shows different HLT's developments, which have been produced by focusing on particular aspects of the notion. These analyses have been framed on a variety of educational domains (curriculum, learning, instruction…). But the fact of considering a teacher —and not a researcher— producing his own planning under the previous foundations deserves an adaptation to this particular professional context. However, the teacher does not need fixed instructional sequences, but some framework of reference, together with a set of exemplary activities that serve as source of inspiration for his own designs (Gravemeijer, 2004). Teacher knowledge and experience and the literature available to him are the basic resources for the teacher in order to generate HLT's that support his own daily planning task. This information needs to be organized in some systematic process and require to be supported by some specific tools to serve to a concrete teacher planning purpose.

In this paper we address this question by focusing in the initial phase of the teacher professional development. Thus, we propose a process that adapts Simon's proposal to preservice teacher training programs. This functional view of the HLT is supported on the notion of capacity. We use this term to refer to the successful performance of an individual with respect to a given task. Using this notion we give a concrete meaning to the notion of a teacher's learning goal. The first section of this paper is devoted to present these two ideas. Then, in the second section, we describe the notion of learning path of a task, which can be used with a practical purpose by a teacher to reflect on and to be able to justify his decisions concerning the analysis and selection of mathematical tasks. In this section, we present some other associated tools to serve for this purpose. In the third section, we describe how the future teacher can use these tools, based on previously identified school mathematics meanings of a topic and students' difficulties when solving types of tasks corresponding to the learning goal. Thus, we give an account of the utility of this instrument for the teacher's prevision about the students' learning processes. All these ideas are exemplified considering a preservice teacher planning a lesson on the quadratic function. We finish the paper presenting some remarks on the previous process, addressing its role in a more general framework of a teachers training course and suggesting some future work.

TEACHER EXPECTATIONS: LEARNING GOALS AND CAPACITIES

We adapt Simon's construct to preservice teacher training and propose that foreseeing students' learning process implies the identification, description and relation of five elements (Gómez & Lupiáñez, 2005): (a) the knowledge that students have before instruction, (b) the teacher's goal for students' learning, (c) the tasks involved in instruction, (d) the difficulties that students might face when solving the tasks, and (e) the hypothesis about the paths through which learning can develop.

All these elements should be suitably characterized in order to produce an operative version of the process. With this purpose, we introduce the notions of capacity and learning goal. In the context of school mathematics we use the term *capacity* to refer to the successful performance of an individual with respect to a given task. Therefore, we will say that an individual has developed a capacity when he is able to solve the tasks requiring it. For instance, we will say that an individual has developed his capacity for completing squares when there is evidence that he can solve the tasks that involve this mathematical topic. Capacities, in this sense, are specific to singular mathematical topics and are bound to types of tasks. We will use the notion of capacity as our basic unit of analysis with respect to the teacher's learning expectations. A capacity is characterized by a type of tasks and depends on the current knowledge of the individuals it refers to. In a planning context for upper secondary mathematics, a teacher can formulate "completing the square" as a capacity if he considers that students of this level should know the procedures for solving tasks requiring it. For lower secondary students, the teacher might set "completing the square" as a learning goal. The core of this paper deals with the relationship between these two notions.

We assume that the teacher has chosen a specific learning goal for which he is planning a lesson [1]. This learning goal is the framework of reference that delimit and conditions the procedures that the teacher is expected to perform in order to formulate his hypotheses about the process of the students' learning. A learning goal is a complex notion. If the teacher wants to design tasks for promoting his students' achievement of that goal, then it is necessary to characterize it in such a way that he can conjecture how and to which extent a task (or a sequence of tasks) can contribute to it's attainment.

For the sake of the argument, let us assume that the teacher is planning a lesson on the quadratic function for upper secondary mathematics students and that he has chosen as learning goal the following:

LG: To recognize and use the graphical meaning of the parameters of the symbolic forms of the quadratic function and communicate and justify the results of its use.

A concrete learning goal, as the above, delimits a specific part of the subject matter. The learning goal does not refer to the quadratic function as a whole. Figure 1 shows a partial result of the quadratic function subject matter analysis involving only two symbolic forms and some of the graphical elements of the parabola.

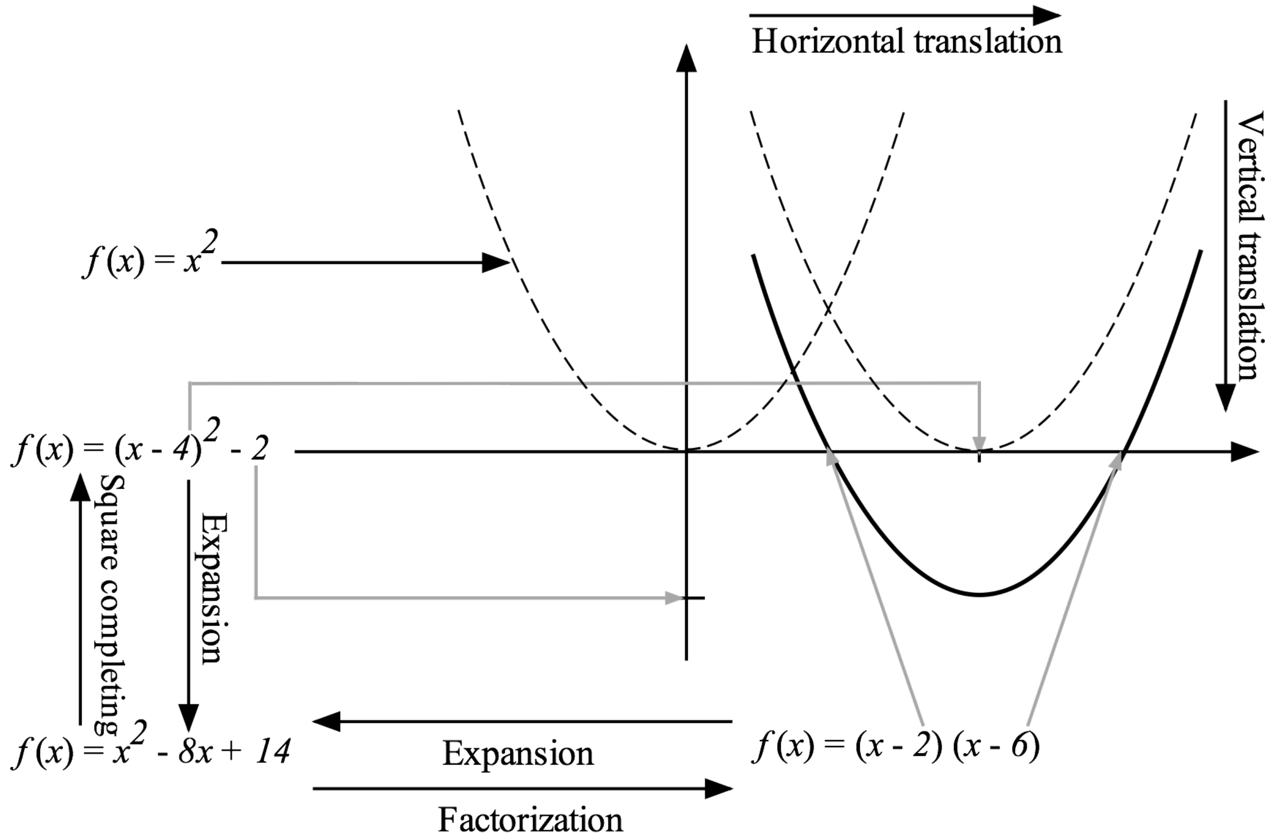

Figure 1. Partial result of a subject matter analysis

On the basis of the information in Figure 1, the teacher can identify some of the capacities involved in this topic [2]. Table 1 shows such a list.

Perform, communicate and justify symbolic transformation procedures	*Identify, show and justify graphical elements*
C1 Square completion	C8 Vertex coordinates
C2 Expansion	C9 Y-axis intersections
C3 Factorization	C10 X-axis intersections
	C11 Focus coordinates
	C12 Directrix equation
	C13 Symmetry axis equation
Identify, show and justify symbolic elements	*Perform, communicate and justify graphical transformation procedures*
C4 Canonical form (a, h, k)	C14 Horizontal translation
C5 Focus form (p, h, k)	C15 Vertical translation
C6 Standard form (a, b, c)	
C7 Multiplicative form (a, r_1, r_2)	C16 Vertical scaling

Table 1: Capacities for graphical meaning of the parameters of the symbolic forms of the quadratic function

LEARNING PATHS

We introduce the idea of *learning path of a task* as a sequence of capacities that students might put in place in order to solve it. For instance, for the task T_1: “given that 2 and 6 are the X-intercepts of a parabola with $a = 1$, find the coordinates of its vertex”, a learning path is the sequence C10 → C7 → C2 → C6 → C1 → C4 → C8. Learning paths can be displayed graphically. If we group capacities according to Table 1, then the above sequence can be displayed as shown in Figure 2: recognize the X-axis intersections as a graphical element (C10), recognize that those intersections correspond the values of r_1 and r_2 in the multiplicative form of the quadratic function (C7), use the expansion procedure (C2) in order to obtain the standard form and recognize it (C6), use the square completion procedure (C1) in order to obtain the canonical form and identify and recognize its parameters *h* and *k* (C4), and recognize the values of those parameters as the coordinates of the vertex in the graphical representation (C8).

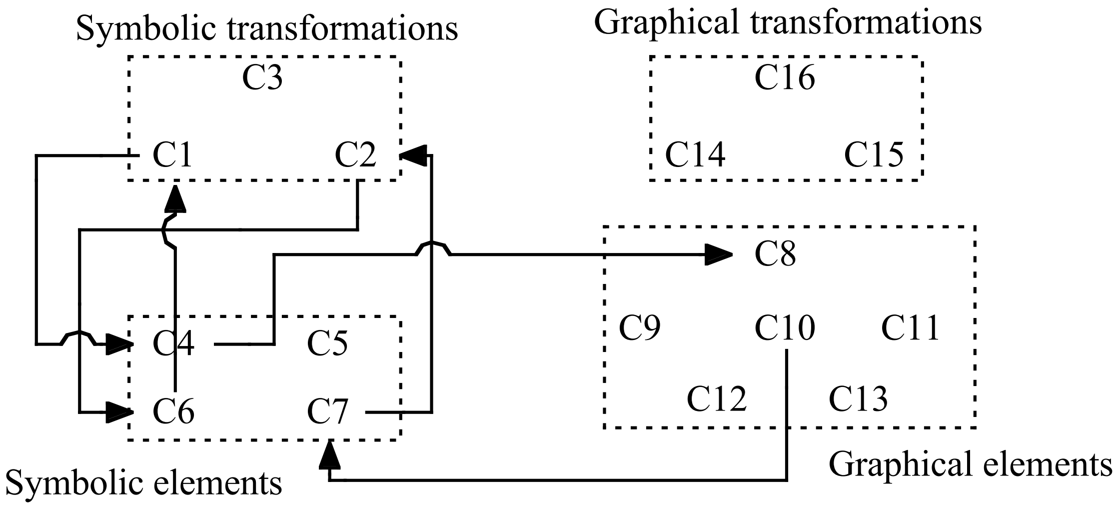

Figure 2: A learning path for a task T_1

A learning path for a task, such as the one depicted in Figure 2, informs the teacher about an *ideal* sequence of capacities that the students might execute when facing the task. It is ideal in the sense that it emerges form the task and the school mathematics meanings of the subject matter related to the learning goal, under the assumption of the students’ previous knowledge (each C_i in the learning path is a capacity, as defined previously). It does not take into account, for the time being, the difficulties that the students might have when trying to solve the task or alternative sequences of capacities that they might execute and that do not correspond to the learning goal’s subject matter.

From this ideal perspective, we can talk of the learning paths of a learning goal. For that purpose, the teacher can identify and characterize the set of tasks, *T*, whose successful solution distinguishes, in his opinion, an individual that has achieved the learning goal. The *learning paths of a learning goal* are those that correspond to the tasks in *T*. It is confusing to represent the graph of the learning paths of a learning goal. In Table 2 we depict the links between capacities that can be established on the basis of the school mathematics meanings of the learning goal's topic. A "1" in a cell indicates that it is possible to link the capacity in that row with the capacity in the corresponding column. The learning paths for a goal, from this ideal perspective, are the sequences of capacities that can be constructed from the information in the table. For instance, C8 → C4 → C2 → C6 is a learning path for *LG*. Learning paths inform the teacher about the tasks that he can consider when planning a lesson for a given learning goal. A task is established by the information required for executing the fist capacity in the learning path (the information given by the task) and the information required for the last capacity in the path (the information asked for by the task). A *type of task* can be characterized by the learning paths required for the solution of the tasks composing it.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15	C16
C1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
C2	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
C3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
C4	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	1
C5	0	1	0	0	0	0	0	0	0	0	1	1	1	0	0	0
C6	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1
C7	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
C8	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0
C9	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
C10	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
C11	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
C12	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
C13	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
C14	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
C15	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0
C16	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0

Table 2: Possible links among capacities for the learning goal *LG*

A learning goal can be made explicit with the help of its learning paths. A learning path is more than the capacities that constitute it: it is the sequence of capacities with which it is possible to solve a certain type of tasks. Therefore, an individual has achieved a goal if, for the type of tasks that characterize it, he is able to recognize the learning path that corresponds to each type of task and to execute them successfully. Simon and Tzur's activity-effect relationship can be adapted to this setting: students will learn (achieve a learning goal) while reflecting on the effects of executing a suitable combination of the learning paths that characterize that goal.

The teacher can use the information in Table 2 as a reference for identifying, analyzing and comparing the learning paths associated to different tasks. For instance, he can take into account the number of capacities involved and whether he considers the sequences challenging to his students. He can also dynamically use this tool in order to adapt learning sequences to students or to select and design tasks.

The teacher has to take into account the difficulties his students might have when facing the tasks. Difficulties refer to sequences of capacities that the students do not recognize or are not able to execute. For instance, he might know, by experience or from the literature, that students tend to recognize only two symbolic forms of the quadratic function. He can include this information in his analysis of the goal's learning paths by, for instance, marking capacities C4 and C5 and their links in Table 2. He should then favour those tasks whose learning paths involve those capacities and those links. On the other hand, the teacher might know that, for certain tasks, his students use sequences of capacities that do not correspond to the subject matter that he has delimited for the learning goal. For instance, the students might use numerical procedures for producing the graph of the function and, then, estimate the value of some of its elements. In this case, the teacher has to enlarge the subject matter analysis and reformulate the goal's capacities and learning paths in order to take into account these difficulties.

A learning goal's learning paths can be used for analyzing and selecting tasks (and sequences of tasks). For instance, the teacher might decide that the learning path for task T_1 involves too few capacities and he might find a task that he considers can be more challenging for the students. He might use the task T_2: "given a parabola with focus at $(0,\frac{9}{4})$, directrix $y=\frac{7}{4}$, and that can be obtained with a vertical translation of two units from $y=x^2$, find the coordinates of its vertex, its intersections with the X and Y axes, its symmetry axis and its symbolic forms. Justify your results". Figure 3 shows the learning paths for this task.

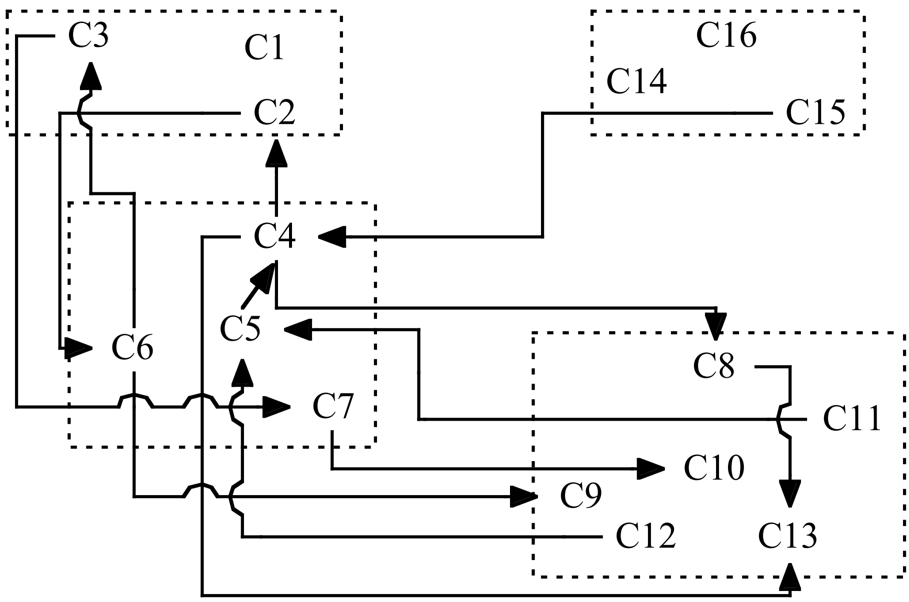

Figure 3: Learning paths for the second task

On the basis of the information in Figure 3, the teacher might consider that task T_2 might better contribute to the goal achievement. He might even consider that he can combine T_2 with other similar tasks with different initial data (Gómez, Mesa, Carulla, Gómez & Valero, 1996, pp. 77-79). In this sense, tasks' analysis on the basis of the goal's learning paths provides the teacher with information that he can use for comparing and selecting tasks, and for designing sequences of tasks that can contribute to the learning goal achievement.

LEARNING PATHS IN PRESERVICE TEACHER TRAINING

The design of preservice teacher training courses should be based on a conceptualization of the activities that the teacher has to do in order to promote students' learning and of the knowledge that is necessary to perform those activities. We call the structuring of a cycle of these activities a didactical analysis (Gómez & Rico, 2002). It is organized around four analyses: subject matter, cognitive, instruction, and performance. Didactical analysis allows the teacher to examine and describe the complexity and multiple meanings of the subject matter, and to design, select, implement, and assess teaching/learning activities.

Any cycle of the didactical analysis begins with the identification of students' knowledge for the subject matter at hand on the basis of the information provided by the last phase of the previous cycle. With this information, and taking into account the global planning of the course, we expect the preservice teacher to make a proposal for the goals he wants to achieve and the mathematics content he wants to work on. The next step of the cycle involves the description of the mathematical content from the viewpoint of its teaching and learning in school. The subject matter analysis is a procedure that allows the preservice teacher to identify and organize the multiple meanings of a mathematical topic. It is based on three aspects of any given topic: its representations, conceptual structure and phenomenology (Gómez, 2006). The preservice teacher can use this information in the cognitive analysis, in which he establishes his hypothesis about how students construct their knowledge when they face the learning activities that are proposed to them. The information from the subject matter and cognitive analysis allows the teacher to carry out an instruction analysis: the analysis, comparison and selection of the tasks that can be used in the design of the teaching and learning activities that will compose the instruction in class. In the performance analysis the teacher observes, describes, and analyzes students' performance in order to produce better descriptions of their current knowledge and review the planning in order to start a new cycle.

The learning path of a task can be used as a pivotal notion in the cognitive and instruction analysis. Preservice teachers in a methods course can use the information from the subject matter analysis of the topic in order to identify and formulate the capacities related to a given learning goal and to establish an adjacency matrix for it, as the one shown in Table 2. The information in this matrix enables the preservice teacher to characterize the learning paths of a sequence of tasks, to locate students' difficulties and, therefore, to analyze, compare and select those tasks that, in his opinion, can better promote students' learning goal achievement. With this procedure for the cognitive and instruction analysis, task design is not left to intuition or trial and error. Teachers can make hypothesis about students' learning process when facing the tasks with the help of a systematic analysis of the mathematical topic and of its cognitive implications. Our conjecture is that this type of detailed analysis of a concrete mathematical topic enables preservice teachers to acknowledge the complexity of school mathematics and provides them with tools and procedures that they can use in their future practice.

FINAL REMARKS

We have followed the principles of HLT to propose a procedure that can be used by teachers for planning their instruction —in a systematic and reflective manner— and promoting the achievement, to the extent of their knowledge, of students' learning goals. We have shown that, when analyzed in detail, a concrete learning goal is a complex object: it can be characterized by a numerous set of learning paths. The question remains of whether there are criteria for selecting the combination of learning paths (and the corresponding sequence of tasks) that can promote students' goal achievement in the most efficient manner.

The notion of learning path and its associated tools and procedures tackles the planning paradox because it allows the teacher to establish an explicit link between learning goals and tasks. Planning is produced on the basis of the relationship between the two. This notion does not give an automatic answer to the issue of which tasks are challenging or relevant. This is a question that teacher has to answer for himself on the basis of his knowledge of his students. However, learning paths allow the teacher to distinguish the universe of tasks he can choose from and to analyze those tasks in terms of the activity that his students might get involved in when solving them.

Future teachers performing cognitive and instruction analysis are involved in other issues, besides the analysis of a learning goal and a task learning paths, for the purpose of selecting tasks and foreseeing students' learning (Gómez, 2006). These analysis deal as well with procedures for identifying and formulating learning goals for a topic, with how learning goals should be combined when planning the teaching and learning of that topic, and with how tasks should be organized in sequences. Lupiáñez & Rico (2006) have also used the notion of capacity to develop an instrument for assessing the relevance of a learning goal or a task: the extent with which the goal or the task contributes to the development of a given list of competencies. All these processes represent instruments that the future teacher can use for articulating mathematical and learner-centered perspectives when planning tasks in the context of a learning goal.

NOTES

1. A lesson can span over more than one class period.

2. Capacities can be formulated in many ways, depending on many factors. We do not have space here to consider this problem.

REFERENCES

Ainley, J., & Pratt, D.: 2002, *Purpose and utility in pedagogical task design*, Norwich, Reino Unido*,* Paper presented at the Twenty-sixth annual Annual Conference of the International Group for the Psychology of Mathematics Education.

Clements, D. H., & Sarama, J.: 2004, 'Learning trajectories in mathematics education', *Mathematical Thinking and Learning* 6 (2), 81-89.

Gómez, P.: 2006, 'Análisis didáctico en la formación inicial de profesores de matemáticas de secundaria' [In P. Bolea, M. J. González & M. Moreno (Eds.), *X Simposio de la Sociedad Española de Investigación en Educación Matemática* (pp. 15-35), Instituto de Estudios Aragoneses, Huesca.

Gómez, P., & Lupiáñez, J. L.: 2005, *Trayectorias hipotéticas de aprendizaje en la formación inicial de profesores de matemáticas de secundaria*, Oporto, Portugal*,* Paper presented at the V Congresso Ibero-Americano de Educaçao Matemática.

Gómez, P., Mesa, V. M., Carulla, C., Gómez, C., & Valero, P. (Eds.): 1996, *Situaciones problemáticas de precálculo. El estudio de funciones a través de la exploración con calculadoras gráficas*, una empresa docente y Grupo Editorial Iberoamérica, México.

Gómez, P., & Rico, L.: 2002, 'Didactical analysis and planning of school tasks in preservice mathematics teacher training' [In D. S. Mewborn, White, D. Y. White, H. G. Wiegel, R. L. Bryant & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 3)* (pp. 1214-1217), ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, OH.

Gravemeijer, K.: 2004, 'Local instruction theories as means of support for teachers in reform mathematics education', *Mathematical Thinking and Learning* 6 (2), 105-128.

Lupiáñez, J. L., & Rico, L.: 2006, *Análisis didáctico y formación inicial de profesores: competencias y capacidades en el aprendizaje de los escolares*, Huesca, Paper presented at the X Simposio de la SEIEM.

Simon, M.: 1995, 'Reconstructing mathematics pedagogy from a constructivist perspective', *Journal for Research in Mathematics Education* 26 (2), 114-145.

Simon, M. A., & Tzur, R.: 2004, 'Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory', *Mathematical Thinking and Learning* 6 (2), 91-104.