COMPLEXITY OF FAMILIES OF COMPACT SETS IN $\mathbb{R}^N$

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ABSTRACT

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The space of all compact subsets of $\mathbb{R}^N$ with the Vietoris topology, denoted $\mathcal{K}(\mathbb{R}^N)$, is a Polish space, i.e. separable and completely metrizable. It is naturally stratified by dimension. In this work, we study the zero and one dimensional compact subsets of $\mathbb{R}^N$, and two equivalence relations on $\mathcal{K}(\mathbb{R}^N)$: the homeomorphism relation and the embedding relation induced by the action of autohomeomorphisms of $\mathbb{R}^N$.

Among the zero dimensional compact subsets, Cantor sets are generic and form a Polish subspace. We study the topological properties of this space as well as the structure with respect to the embedding relation. Moreover, we show that the classification of Cantor sets up to embedding relation is at least as complex as the classification of countable structures.

Next, we look into one dimensional compact subsets, particularly those that are connected, i.e. curves. The curves also form a Polish subspace. We introduce a new connectedness property, namely strong arcwise connectedness. We study the complexity of curves with this property using descriptive set theory tools, and show that the space of all curves which are strong arcwise connected, is not Borel, and is exactly at the second level of the projective hierarchy. In addition, we examine the classification of curves up to either equivalence relation and show that the curves are not classifiable by countable structures.

Keywords: Cantor sets, embedding, classification, turbulence, curves, Borel hierarchy, difference hierarchy, Projective hierarchy, strong arcwise connectedness, rational continuum, dendrites, dendroids.

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PREFACE

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0.0 INTRODUCTION

Embeddings of a space into another space is a major research topic in topology. Circles embedded in $\mathbb{R}^3$ are knots. While Antoine's famous *necklace* is an example of a wild embedding of the Cantor set in $\mathbb{R}^3$. Most results on this topic has been achieved by studying examples one–by–one. My research has been on a method of taking a big–picture view of all embeddings simultaneously. We use tools from topology and descriptive set theory, and actions of Polish groups to understand how hard it is to characterize certain important classes of embeddings.

A topological space X is *Polish* if it is separable and completely metrizable, and it is a *Polish group* if it is a topological group and the topology is Polish. Lie groups, separable Banach spaces, automorphism groups of certain countable objects (trees, groups, and so on) are examples of Polish spaces or groups. In a Polish space the *Borel sets* are those that can be defined from the open sets in countably many steps. These countable steps form a hierarchy of Borel classes; at the first level are the classes of open and closed sets, at the second level the classes of F_σ and G_δ sets, at the third level the classes of $F_{\sigma\delta}$ and $G_{\delta\sigma}$ and so on. Intuitively, Borel sets are those that are 'computable', and the sets in higher levels of this hierarchy are more complex ('harder to compute') than the sets in lower levels. Between levels of the Borel hierarchy lies the *difference hierarchy*, which gives a construction of the intersection of Borel classes of the same level.

Throughout this work, we study the hyperspace of $\mathbb{R}^N$ consisting of all compact subsets of $\mathbb{R}^N$, denoted $\mathcal{K}(\mathbb{R}^N)$, and its various subspaces. The topology of this hyperspace is the Vietoris topology, which is the same as the topology induced by the Hausdorff metric. This space is separable, and Hausdorff metric complete, hence $\mathcal{K}(\mathbb{R}^N)$ is Polish.

On the hyperspace $\mathcal{K}(\mathbb{R}^N)$, there are two kinds of natural equivalence relations. One is the homeomorphism relation, i.e. two compact sets are equivalent if and only if there is a homeomorphism that takes one set to the other. The other is induced by the action of the group of autohomeomorphisms of $\mathbb{R}^N$, denoted by $Aut(\mathbb{R}^N)$, on $\mathcal{K}(\mathbb{R}^N)$. Under the compact–open topology, $Aut(\mathbb{R}^N)$ is a Polish group. The standard action of this group on $\mathbb{R}^N$, defined by taking a homeomorphism and an element in $\mathbb{R}^N$ to the image of this element under the homeomorphism, is continuous. Moreover, this action extends in the obvious way to the space of all compact subsets, $\mathcal{K}(\mathbb{R}^N)$, by taking a compact set to its image under the homeomorphism.

Under this action there are two kinds of natural invariant subsets:

- $\mathcal{K}_{\mathcal{P}}(\mathbb{R}^N)$: the set of all compact subsets of $\mathbb{R}^N$ with the topological property $\mathcal{P}$, for example connectedness, perfectness, being m-dimensional, etc.
- $\mathcal{K}_K(\mathbb{R}^N)$: all homeomorphic copies of compact set K in $\mathbb{R}^N$, where K is a specific compact set like Cantor space $\mathcal{C}$, circle S^1, the unit interval I, etc.

Using the Borel hierarchy, we can give upper and lower bounds on the complexity of these invariant sets. Thus we can *prove* that the problem, for example, of deciding if a given compact subset of $\mathbb{R}^N$ is homeomorphic to the circle is of a certain complexity (not higher, and – provably – not lower).

It is clear that the space of all compact subsets of $\mathbb{R}^N$ stratifies under this action by dimension. So our strategy towards understanding the action of $Aut(\mathbb{R}^N)$ on $\mathcal{K}(\mathbb{R}^N)$ is to work our way up through the dimensions. We start with the zero–dimensional compact subsets of $\mathbb{R}^N$, which we verify is a complete G_δ subset in $\mathcal{K}(\mathbb{R}^N)$. On the other hand, we have shown that the class of one–dimensional compact sets is a little more complicated than that, namely it is in the difference hierarchy of $D_2(\mathbf{\Sigma}^0_2)$ (for $N \geq 3$). While the set of N-dimensional compact subsets of $\mathbb{R}^N$, $\mathcal{K}_N(\mathbb{R}^N)$, is a complete F_σ set. We also show that for any m with $1 \leq m < N$, the space of m-dimensional compact subsets, $\mathcal{K}_m(\mathbb{R}^N)$ is in the same difference class.

Among the zero–dimensional compact subsets, the class of Cantor subsets, denoted $\mathcal{C}(\mathbb{R}^N)$, form a G_δ subset of $\mathcal{K}(\mathbb{R}^N)$. We have unraveled topological properties as well as

Borel structure of this space. For example, we know that it is separable, completely metrizable, path connected and locally path connected but not locally compact. On the space of Cantor subsets, the homeomorphism relation is not interesting since all Cantor sets are homeomorphic. However, the action of $Aut(\mathbb{R}^N)$ induces a non–trivial equivalence relation on Cantor sets, which is the same as the equivalence relation defined in earlier studies of Cantor sets, [2, 20, 21]. Two Cantor sets are equivalent if there is a homeomorphism of $\mathbb{R}^N$ which takes one Cantor set to the other. A Cantor set equivalent to the standard Cantor set is called *tame*, and otherwise it is called *wild*. We have shown that tame Cantor sets are generic in the space $\mathcal{C}(\mathbb{R}^N)$ in Theorem 3.2.3.

A very important class in $\mathcal{K}(\mathbb{R}^N)$ is the set of all compact connected subsets of $\mathbb{R}^N$, namely the space of all *subcontinua*, denoted $\mathbf{C}(\mathbb{R}^N)$. This is a closed subset of $\mathcal{K}(\mathbb{R}^N)$ and hence it is Polish as well. Moreover, being an invariant subspace, the induced action also stratifies $\mathbf{C}(\mathbb{R}^N)$ by dimension. In this case, the zero–dimensional compact connected sets are exactly the one point sets, and this class is a closed set. One dimensional subcontinua of $\mathbb{R}^N$ are called *curves*. We show that the curves form a G_δ subset of $\mathbf{C}(\mathbb{R}^N)$, see Theorem 2.2.1.

For many subclasses of curves, including some homeomorphism classes, the Borel complexity of the class has been determined, see [3]. These examples use descriptive set theory methods to classify previously studied topological objects. In this work, we will examine new classes of curves, namely the class of n–sac and ω–sac curves, and use both topological and descriptive set theoretic methods to characterize them.

A continuum X is called *n-sac (n strongly arc-connected)* if for any n points $x_1, \ldots, x_n$ X there is an arc in X that visits the points in the given order. It is called *ω-sac* if for any n, X is n-sac. In the case of finite graphs, which is a subclass of curves, being 3-sac has been characterized in section 4.1 and we know that no finite graph is 4-sac. Additionally, we know that regular curves are never n-sac for all n. The next natural class is the rational curves, and we prove that it is essentially impossible to characterize rational ω-sac curves by showing that the subspace of rational ω-sac curves in $\mathcal{K}(\mathbb{R}^N)$ is very complex, indeed not even Borel. This relies on being able to construct many widely different rational ω-sac curves.

Throughout this work, we not only study the Borel complexity of classes of sets arising from the two equivalence relations defined above but also the complexity of the classification up to these equivalences. For example, the classification of curves up to either equivalence relation is strictly more complex than that of classifying countable groups. While classification of Cantor sets is at least as complex as the classification of countable groups.

Chapter 1 provides fundamental definitions and facts on the Polish spaces, Borel and projective hierarchy as well as on Polish groups and actions, which will be needed in the following chapters.

In Chapter 2 we develop the general setting for the complexity questions we will answer later on. We prove that the class of m-dimensional compact subsets of $\mathbb{R}^N$ is in the difference hierarchy $D_2(\mathbf{\Sigma}^0_2)$ for $1 < m < N$, and it is simpler for $m = 0$ or $m = N$. Additionally, in Section 2.2 we introduce the space of all subcontinua of $\mathbb{R}^N$ and some important examples of continua.

Chapter 3 is on zero dimensional compact subsets of $\mathbb{R}^N$. It turns out most of these are Cantor sets. We will examine the space of Cantor sets in $\mathbb{R}^3$ in detail, in Section 3.1. Section 3.2 is devoted to complexity of classes of Cantor sets, whereas Section 3.3 is on the complexity of classification of Cantor sets.

The natural successor to the zero dimensional compact subsets are the one dimensional compact subsets in $\mathbb{R}^N$, which is discussed in Chapter 4. The organization of this chapter is similar to the previous chapter. First we discuss the classes of n-strongly arc connected and ω-strongly arc connected curves in Section 4.1 and their complexity in Section 4.2. Then in Section 4.3, we discover complexity of classification problems of curves and some of its subclasses.

1.0 BOREL HIERARCHY AND BOREL EQUIVALENCE RELATIONS

This chapter gives some background in classical descriptive set theory and equivalence relations. This includes basic notions and facts about Polish spaces and groups. Most of these can be found in [15] and [12], which are our main references on descriptive set theory and equivalence relations.

1.1 POLISH SPACES

A topological space is *Polish* if it is separable and completely metrizable. $\mathbb{R}, \mathbb{C}, \mathbb{R}^N, 2^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}$ are some examples of Polish spaces.

Given a topological space X, *Borel sets* is the smallest collection containing the open sets that is closed under complements, countable unions and intersections, and is denoted by $\mathcal{B}(X)$. For metrizable space X, this class can be analyzed in a transfinite hierarchy of length ω_1, called the *Borel hierarchy.* It consists of classes of subsets of the space and these classes are defined inductively from the open sets with the following rules:

- $\mathbf{\Sigma}^0_1$ is the class of open subsets
- $\mathbf{\Pi}^0_\alpha$ consists of complements of $\mathbf{\Sigma}^0_\alpha$
- A set A is in the class $\mathbf{\Sigma}^0_\alpha$, $\alpha > 1$ if there is a sequence of sets $(A_i)_{i\in\mathbb{N}}$ such that A_i is $\mathbf{\Pi}^0_{\alpha_i}$ for some $\alpha_i < \alpha$ and $A = \bigcup_i A_i$

In the first level are the open sets ($\mathbf{\Sigma}^0_1$) and closed sets ($\mathbf{\Pi}^0_1$), in the second level F_σ's ($\mathbf{\Sigma}^0_2$) and G_δ's ($\mathbf{\Pi}^0_2$), in the third level we have $G_{\delta\sigma}$'s ($\mathbf{\Sigma}^0_3$) and $F_{\sigma\delta}$'s ($\mathbf{\Pi}^0_3$), etc. We can also consider the *ambiguous classes*, denoted Δ^0_α, where $\Delta^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. So the Δ^0_1 is the class of both open and closed subsets, Δ^0_2 is the set of both G_δ and F_σ sets, etc. Hence $\mathcal{B}(X)$ ramifies as follows:

$$\begin{array}{ccccccccccc} & \boldsymbol{\Sigma}^0_1 & & \boldsymbol{\Sigma}^0_2 & \dots & & \boldsymbol{\Sigma}^0_\alpha & \dots & & \boldsymbol{\Sigma}^0_\gamma & \dots \\ \Delta^0_1 & & \Delta^0_2 & & \dots & \Delta^0_\alpha & & \dots & \Delta^0_\gamma & & \dots \\ & \boldsymbol{\Pi}^0_1 & & \boldsymbol{\Pi}^0_2 & \dots & & \boldsymbol{\Pi}^0_\alpha & \dots & & \boldsymbol{\Pi}^0_\gamma & \dots \end{array}$$

where every class is contained in any class to the right of it. We think of sets in higher levels as more 'complex', as their definitions in terms of open sets are more complex.

Beyond the Borel sets we have projective sets, these are obtained by continuous images and complementation from Borel sets. A subset A of Polish space X is *analytic* (denoted $\boldsymbol{\Sigma}^1_1$) if it is the continuous image of a Borel subset of a Polish space. Borel subsets are analytic, but the converse is not true. A set is *co-analytic* ($\boldsymbol{\Pi}^1_1$) if its complement is analytic. The class of projective sets ramifies in an infinite hierarchy of length ω:

$$\begin{array}{cccccccc} & \boldsymbol{\Sigma}^1_1 & \boldsymbol{\Sigma}^1_2 & & \boldsymbol{\Sigma}^1_n & \boldsymbol{\Sigma}^1_{n+1} & \\ \mathcal{B}(X) & & & \dots & & & \dots \\ & \boldsymbol{\Pi}^1_1 & \boldsymbol{\Pi}^1_2 & & \boldsymbol{\Pi}^1_n & \boldsymbol{\Pi}^1_{n+1} & \end{array}$$

where every class is contained in any class to the right of it. Moreover, a set A is in the class $\boldsymbol{\Sigma}^1_n$ if it is continuous image of a set from the class $\boldsymbol{\Pi}^1_{n-1}$, and $\boldsymbol{\Pi}^1_n$ consists of complements of the sets in $\boldsymbol{\Sigma}^1_n$.

Another hierarchy that is interesting is the difference hierarchy, which gives a construction of the ambiguous class $\Delta^0_{\alpha+1}$ from the class $\boldsymbol{\Sigma}^0_\alpha$. We will only use the first levels of this hierarchy, mainly $D_2(\boldsymbol{\Sigma}^0_\alpha)$ and $\hat{D}_2(\boldsymbol{\Sigma}^0_\alpha)$. A set is in $D_2(\boldsymbol{\Sigma}^0_\alpha)$ if it is the intersection of a $\boldsymbol{\Sigma}^0_\alpha$ set and a $\boldsymbol{\Pi}^0_\alpha$ set. And a set is in $\hat{D}_2(\boldsymbol{\Sigma}^0_\alpha)$ if the complement is in $D_2(\boldsymbol{\Sigma}^0_\alpha)$. Also note that $D_1(\boldsymbol{\Sigma}^0_\alpha) = \boldsymbol{\Sigma}^0_\alpha$ and $\hat{D}_1(\boldsymbol{\Sigma}^0_\alpha) = \boldsymbol{\Pi}^0_\alpha$.

Example 1. Let $Q_2 = \big\{(x_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}} \mid \exists m, \forall n \geq m(x_n = 0)\big\}$. It is a known $\boldsymbol{\Sigma}^0_2$ (F_σ) set, hence the complement Q^c_2 is $\boldsymbol{\Pi}^0_2$ (G_δ) in $2^{\mathbb{N}}$. And thus the product $Q_2 \times Q^c_2$ is in the difference hierarchy -$D_2(\boldsymbol{\Sigma}^0_2)$- of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Note that a basic neighborhood of an element $x = (x_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$ is denoted by $N(x \restriction m)$ for some $m \in \mathbb{N}$, where it is the following set $\big\{y = (y_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}} \mid y_i = x_i, \forall i \leq m\big\}$.

It is important to know the hierarchy of a natural set, not only because it is a beautiful result of descriptive set theory, but also because it can lead us to learn more about the topological properties. So it is important to place a given set in the lowest possible level of Borel or projective hierarchies.

1.2 WADGE HIERARCHY

Wadge hierarchy is another hierarchy of sets which extends the Borel hierarchy and gives us a way to decide which sets belong to which level of the hierarchy. Knowing the exact level for a certain set may lead to nicer characterizations.

Let X, Y be topological spaces and $A \subset X$, $B \subset Y$. A is *Wadge reducible* to B, denoted $A \leq_W B$, if there is a continuous reduction of A to B, i.e. there is a continuous map $f : X \to Y$ with $f^{-1}(B) = A$. We denote this continuous reduction by $f : (X, A) \to (Y, B)$. Intuitively, if $A \leq_W B$, then A is 'simpler' than B. This reduction leads to an equivalence relation: $A \equiv_W B \iff A \leq_W B$ and $B \leq_W A$

It turns out the Borel hierarchy forms the initial segment of the Wadge hierarchy given by $\leq_W$. If B is $\mathbf{\Sigma}^0_\alpha$ set and $A \leq_W B$, then A is $\mathbf{\Sigma}^0_\alpha$ set. The same holds for $\mathbf{\Pi}^0_\alpha$.

Let Γ be a class of sets like $\mathbf{\Sigma}^i_\alpha$ and Y be a Polish space. Then a set $B \subset Y$ is called Γ-*hard* if $A \leq_W B$ for any $A \in \Gamma(X)$, where X is a zero dimensional Polish space. If additionally, $B \in \Gamma(Y)$, then B is called Γ-complete.

Let $\hat{\Gamma}$ denote the class of sets which are complements of those in Γ.

Facts. *1. If $\hat{\Gamma} \neq \Gamma$ on zero dimensional Polish spaces and Γ is closed under continuous preimages, then no Γ-hard set is in $\hat{\Gamma}$.*

2. If A is Γ-hard then A^c is $\hat{\Gamma}$-hard.

3. If B is Γ-hard and $A \leq_W B$, then A is Γ-hard.

This gives us a very common method to show that a given set B is Γ-hard: choose a known Γ-hard set A and show $A \leq_W B$. Though for the Borel sets there is another good way:

Theorem 1.2.1. *Let X be a Polish space, and $A \subset X$. If $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$, then A is $\mathbf{\Sigma}^0_\alpha$-complete. (Similarly, interchanging $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$).*

This theorem also holds for the difference hierarchy $D_2(\mathbf{\Sigma}^0_\alpha)$ and its complement $\hat{D}_2(\mathbf{\Sigma}^0_\alpha)$, [3].

Example 2. $Q_2 \times Q_2^c$ in Example 1 is actually $D_2(\mathbf{\Sigma}^0_2)$-complete since Q_2 is a $\mathbf{\Sigma}^0_2$-complete.

Example 3. Let $S_3^* = \{\alpha \in 2^{\mathbb{N}\times\mathbb{N}} : \exists J \, \forall j > J \, \exists k \; \alpha(j,k) = 0\}$, it is a known $\mathbf{\Sigma}^0_3$-complete set. Also let $P_3 = \{\beta \in 2^{\mathbb{N}\times\mathbb{N}} : \forall j \, \exists K \, \forall k \geq K \; \beta(j,k) = 0\}$, which is known to be $\mathbf{\Pi}^0_3$-complete in $2^{\mathbb{N}\times\mathbb{N}}$. Thus $S_3^* \times P_3$ is a $D_2(\Sigma_3)$ subset of $\left(2^{\mathbb{N}\times\mathbb{N}}\right)^2$.

1.2.1 TREES

A tree is a basic tool in descriptive set theory. However, it is not the same notion as the one used in graph theory.

Let A be a nonempty set, by $A^{<\mathbb{N}}$ we denote the set of all finite sequences on A. Let $s \in A^{<\mathbb{N}}$, so $s = (a_0, \ldots, a_{n-1})$ for some $n \in \mathbb{N}$. Then we denote the length of s by *length*(s) which is n, and for $m \leq n$, the restriction is $s \restriction m = (a_0, \ldots, a_{m-1})$. For finite sequences s and t on A, we say s is initial segment of t if $s = t \restriction m$ for some $m \leq$ *length*(t). The *concatenation* of the finite sequences $s = (a_i)_{i<n}$ and $t = (b_j)_{j<m}$ is the sequence $s^\frown t = (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1})$.

A *tree* on a set A is a subset τ of $A^{<\mathbb{N}}$, which is closed under initial segments, i.e. if $t \in \tau$ and s is an initial segment of t, then $s \in T$. An *infinite branch* of τ is a sequence $x \in A^{\mathbb{N}}$ such that $x \restriction n \in \tau$ for all $n \in \mathbb{N}$.

A tree is *well-founded* if it has no infinite branches. If, on the other hand, T has an infinite branch then it is called an *ill-founded* tree. We will denote the space of all trees on $\mathbb{N}$ by **Tr**, the space of all well-founded trees on $\mathbb{N}$ by **WF** and the space of all ill-founded trees on $\mathbb{N}$ by **IF**. It turns out **Tr** is a Polish space, being a closed subset of the Polish space $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$, see [15].

For a tree T on a product of sets, for example in the form $A = B \times C$, we will identify elements of T with pairs of finite sequences. So if $s \in T$, then $s = (s_i)_{i<n}$ for some n, and $s_i = (b_i, c_i) \in B \times C$, and we will write $s = (t, r)$ where $t = (b_i)_{i<n}$ is a finite sequence on

B and $r = (c_i)_{i<n}$ is a finite sequence on C. For such a tree T, and an infinite sequence $x \in B^{\mathbb{N}}$, $T(x) = \{s \in C^{<\mathbb{N}} : (x \upharpoonright length(s), s) \in T\}$ is called the section tree on C.

Fact. **IF** *is* $\mathbf{\Sigma}^1_1$*-complete, hence* **WF** *is* $\mathbf{\Pi}^1_1$*-complete. (See [15])*

1.3 POLISH GROUPS AND BOREL EQUIVALENCE RELATIONS

A topological group is *Polish group* if the topology is Polish. Countable groups with discrete topology, separable Banach spaces and locally compact second countable groups are examples of Polish groups.

Let G be a Polish group and X be a Polish space, X is called a *Polish G-space* if G acts continuously on X, i.e. there is a continuous map $a : G \times X \to X$ defined by $a(g, x) = g \cdot x$ with the following properties:

- $g \cdot (h \cdot x) = (gh) \cdot x$
- $e \cdot x = x$, where e is identity of G

This action induces an equivalence relation on X - *orbit equivalence relation*, denoted E^X_G, given by:

$$xE^X_G y \iff \exists g \in G(g \cdot x = y)$$

The *orbit* of x, denoted $[x]_G$, is the set $\{g \cdot x \mid g \in G\}$. The set $G_x = \{g \in G \mid g \cdot x = x\}$ is the *stabilizer* of x.

For a subset $A \subset X$, $g \cdot A = \{g \cdot x \mid x \in A\}$, and A is *invariant* if for each $g \in G$, $g \cdot A = A$.

For a Polish space X, an equivalence relation E on X is *Borel* if it is Borel subset of the product space $X \times X$. For example, the equality relation on $\mathbb{R}$, denoted $\mathrm{id}(\mathbb{R})$, is the set $\{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and it is a closed subset of $\mathbb{R} \times \mathbb{R}$, thus it is a Borel equivalence relation.

Equivalence relations give us a way to classify elements of the set we work on. Classification of objects up to some notion of equivalence is an important problem in many areas of mathematics. However, it is also important to be able to understand how complicated this classification is. For equivalence relations we have a way to compare them, namely the Borel reduction.

A function $f : X \to Y$ is *Borel* if the inverse image of any open set in Y is Borel in X. f is called a *Borel isomorphism* if it is a bijection and both f and f^{-1} are Borel maps.

Given equivalence relations E and F on X and Y, we say that E is *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel function $f : X \to Y$ so that for all $x_1, x_2 \in X$, $x_1 E x_2 \iff f(x_1) F f(x_2)$. We write $E \leq_c F$ if there is a continuous reduction and $E <_B F$ if $E \leq_B F$ holds and $F \leq_B E$ fails. Intuitively, if $E \leq_B F$, then E has a classification problem which is at most as difficult as that of F.

An equivalence relation E is called *smooth* if $E \leq_B \mathrm{id}(\mathbb{R})$. If E is a smooth equivalence relation, and F is any other equivalence relation which is Borel reducible to E, then F is also smooth.

Let LO be the set of all linear orderings, i.e. for any element $\alpha \in LO$, there is a relation $<_\alpha$ on $\mathbb{N}$ which is a linear order. Two elements $\alpha, \gamma \in LO$ are equivalent if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ with $n <_\alpha m \iff f(n) <_\gamma f(m)$ for each $m, n \in \mathbb{N}$, such an f is called an *order isomorphism*. Let $\sim_{LO}$ denote the equivalence relation on linear orders. An equivalence relation E is said to *admit classification by countable structures* if $E \leq_B \sim_{LO}$. If additionally, $\sim_{LO} \leq_B E$, then E is called S_∞-universal.

Theorem 1.3.1 (Becker, Kechris)**.** *S_∞ is the group of all permutations of $\mathbb{N}$. Let G be a closed subgroup of S_∞ and X be a Polish G-space. Then E_G^X admits classification by countable structures.*

It turns out there are classification problems that are beyond the level of countable structures. A tool to show that a given classification problem is beyond this level or not comparable is called turbulence.

Let G be Polish group and X be a Polish G-space. The action of G on X is called *turbulent* if:

1. every orbit is dense
2. every orbit is meager

3. for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there is $y_0 \in [y]_G$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with $x_0 = x$, $x_{i+1} = g_i \cdot x_i$ and for some subsequence $(x_{n_i})_{i\in\mathbb{N}}$, $x_{n_i} \to y_0$

A turbulent orbit equivalence relation refuses classification by countable structures. Actually, turbulence is necessary for non-classification.

Theorem 1.3.2 (Hjorth, [12])**.** *Let G be a Polish group and X a Polish G-space. Then exactly one of the following holds:*

1. *the orbit equivalence relation is reducible to isomorphism on countable models*
2. *there is a turbulent Polish G-space Y and a continuous G-embedding from Y to X.*

A common method for showing a given equivalence relation E is turbulent is to find a known turbulent equivalence relation F and find a Borel reduction from F to E. One such example is:

Example 4. The space $c_0 = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid x_n \to 0\right\}$ with the sup norm acts on the ambient space $\mathbb{R}^{\mathbb{N}}$ by translation. $\mathbb{R}^{\mathbb{N}}$ is a Polish c_0-space as the inclusion map from c_0 to $\mathbb{R}^{\mathbb{N}}$ is continuous. This action is one of the simplest examples of a turbulent action. (see [12], p. 52)

2.0 SPACE OF COMPACT SUBSETS

In this chapter, we will define the space of all compact subsets of a metric space X and its topology, and then we will study the case when $X = \mathbb{R}^N$.

Let X be a separable metric space, and $\mathcal{K}(X)$ be the space of all non-empty compact subsets of the space X with the Vietoris topology. A basic open set in $\mathcal{K}(X)$ has the form:

$$B(U_1, U_2, \ldots, U_n) = \left\{ K \in \mathcal{K}(X) \mid K \subseteq \bigcup_i U_i \text{ and } K \cap U_i \neq \emptyset, \forall i \right\},$$

where $U_1, \ldots, U_n$ are open sets in X.

The topology arising from the Hausdorff metric gives us the same topology [7], which is defined as:

$$d(K, L) = \max\{\sup_{a \in K} d(a, L), \sup_{b \in L} d(b, K)\}$$

where d is the metric on X. If the metric d is complete then so is the induced Hausdorff metric on $\mathcal{K}(X)$. Moreover, $\mathcal{K}(X)$ is second countable metric space hence separable since X is a separable metric space. Hence for a Polish space X, the space of all compact subsets, $\mathcal{K}(X)$, is also Polish.

On the space of compact subsets of X we can talk about two kind of equivalence relations:

($\sim_H$) The equivalence relation induced by homeomorphism defined by:

$K \sim_H L \iff$ there is a homeomorphism $h : K \to L$

($\sim$) The equivalence relation induced by the action of $Aut(X)$ on $\mathcal{K}(X)$ defined by:

$K \sim L \iff$ there is a homeomorphism $h \in Aut(X)$ such that $h(K) = L$

We will look into Borel complexity of certain invariant subspaces as well as some classification problems arising from these equivalence relations on some of the invariant subspaces of $\mathcal{K}(\mathbb{R}^N)$.

2.1 COMPACT SUBSETS OF $\mathbb{R}^N$

In this section, we will examine certain subspaces of $\mathcal{K}(\mathbb{R}^N)$ for a fixed natural number N. We will see that the space of at most m-dimensional compact subsets form a dense G_δ, whereas the space of all m-dimensional subsets form a provably more complex class, and they are in the difference hierarchy $D_2(\mathbf{\Sigma}_2^0)$.

The *topological dimension* of a space X is the minimal number n such that any finite open cover of X has a finite open refinement in which no point of X is included in more than $n+1$ elements.

Lemma 2.1.1 (Lebesgue Covering Lemma)**.** *Let (X,d) be a compact metric space. For any open cover of X, there is a $\delta > 0$ such that any subset of X with diameter $< \delta$ is contained in some member of the cover. δ is called the Lebesgue number.*

The space $\mathcal{K}(\mathbb{R}^N)$ stratifies with dimension. Let

$$\mathcal{K}_{\leq m}(\mathbb{R}^N) = \big\{A \in \mathcal{K}(\mathbb{R}^N) \mid \dim A \leq m\big\}$$

Theorem 2.1.2. *$\mathcal{K}_{\leq m}(\mathbb{R}^N)$ is a G_δ subset of $\mathcal{K}(\mathbb{R}^N)$.*

Proof. Consider the sets:

$$\mathcal{U}_n = \bigcup \big\{B(U_1, \ldots, U_r) \mid r \in \mathbb{N}, \operatorname{diam} U_i < 1/n, U_{i_1} \cap U_{i_2} \ldots \cap U_{i_{m+2}} = \emptyset, \forall i_1 \neq \ldots \neq i_{m+2}\big\}$$

where $U_i \subset \mathbb{R}^N$ are open sets. Obviously, each $\mathcal{U}_n$ is an open set in $\mathcal{K}(\mathbb{R}^N)$. We will show that the intersection of these $\mathcal{U}_n$'s is the set $\mathcal{K}_{\leq m}(\mathbb{R}^N)$.

Let $A \subset \mathbb{R}^N$ be compact and $\dim A \leq m$. Fix $s \in \mathbb{N}$ arbitrary. Then $\{B(x, 1/(3s))\}_{x\in A}$ covers A, so it has a finite subcover $\{B(x_i, 1/(3s))\}_{i=1}^k$. Then since $\dim A \leq m$ there is a finite open refinement $\{V_i\}_{i=1}^t$ of the finite cover $\{B(x_i, 1/(3s))\}_{i=1}^k$ such that $V_{i_1} \cap V_{i_2} \ldots \cap V_{i_{m+2}} = \emptyset$ for distinct i_j's and $\operatorname{diam} V_i < 1/s$ for each $i \leq t$. Thus $A \in \mathcal{U}_s$.

For the converse, suppose $A \in \bigcap_s \mathcal{U}_s$. Let $\{V_i\}_{i=1}^l$ be a finite open cover for A. By Lebesgue Covering Lemma, there is $r > 0$ such that for any $E \subset A$ with $\operatorname{diam} E < r$, there is i such that $E \subset V_i$. Take $s \in \mathbb{N}$ so that $1/s < r$. For this s, there is $B(U_1, \ldots, U_k)$ in $\mathcal{U}_s$ with $A \in B(U_1, \ldots, U_k)$. Let $E_i = A \cap U_i$, so $A = \bigcup_i E_i$, and $\operatorname{diam} E_i < 1/s < r$, hence there exists j_i such that $E_i \subset V_{j_i}$. Set $W_i = U_i \cap V_{j_i}$ for $i = 1, \ldots, k$. $\{W_i\}_{i=1}^k$ is a finite open refinement of $\{V_i\}_{i=1}^l$ because $A = \bigcup_i E_i \subset \bigcup_i (U_i \cap V_{j_i}) = \bigcup_i W_i$. Moreover, $\{W_i\}_{i=1}^k$ is of order $\leq m$ as:

$$W_{i_1} \cap \ldots \cap W_{i_{m+2}} = (U_{i_1} \cap V_{j_{i_1}}) \cap \ldots \cap (U_{i_{m+2}} \cap V_{j_{i_{m+2}}}) \subset U_{i_1} \cap \ldots \cap U_{i_{m+2}} = \emptyset$$

for $i_1 \neq i_2 \ldots \neq i_{m+2}$. Hence $\dim A \leq m$.

Thus $\mathcal{K}_{\leq m}(\mathbb{R}^N) = \bigcap_m \mathcal{U}_m$. $\square$

Consider the m-dimensional subsets: $\mathcal{K}_m(\mathbb{R}^N) = \mathcal{K}_{\leq m}(\mathbb{R}^N) \setminus \mathcal{K}_{\leq m-1}(\mathbb{R}^N)$.

Corollary. *$\mathcal{K}_m(\mathbb{R}^N)$ is dense, but not a G_δ.*

Proof. Let $B(U_1, \ldots, U_k)$ be a basic open set. Since each $U_i \neq \emptyset$, we can find open balls $B_i \subset U_i$ such that $B_i \cap B_j = \emptyset$, $i \neq j$. Then take F_i a homeomorphic copy of I^m in B_i. So $F = \bigcup_i F_i$ meets each U_i, $\dim F = m$ and it is compact. Hence any basic open set meets $\mathcal{K}_m(\mathbb{R}^N)$.

If $\mathcal{K}_m(\mathbb{R}^N)$ were a G_δ subset, then by Baire Category theorem, $\mathcal{K}_m(\mathbb{R}^N) \cap \mathcal{K}_{\leq m-1}(\mathbb{R}^N)$ is dense, but this intersection is empty. $\square$

Now since for each $m \leq n$, $\mathcal{K}_m(\mathbb{R}^N)$ is the difference of two G_δ sets, it is in the difference hierarchy: $D_2(\mathbf{\Sigma}_2^0)$. However, some of these sets are simpler.

Corollary.

- *$\mathcal{K}_n(\mathbb{R}^N)$ is F_σ-complete.*
- *$\mathcal{K}_0(\mathbb{R}^N)$ is G_δ.*

It turns out the set of all m dimensional compact subsets $\mathcal{K}_m(\mathbb{R}^N)$, for $1 \leq m < n$, is not simpler.

Theorem 2.1.3. *$\mathcal{K}_m(\mathbb{R}^N)$ is $D_2(\mathbf{\Sigma}^0_2)$-complete, for $n > 1$ and $1 \leq m < n$.*

Proof. We will use a known $D_2(\mathbf{\Sigma}^0_2)$-complete set, $Q_2 \times Q_2^c$ which was described in Chapter 1 Example 1. We need a continuous reduction $F : (2^{\mathbb{N}} \times 2^{\mathbb{N}}, Q_2 \times Q_2^c) \to (\mathcal{K}(\mathbb{R}^N), \mathcal{K}_m(\mathbb{R}^N))$ so that:

$$(\alpha, \beta) \in Q_2 \times Q_2^c \iff F(\alpha, \beta) \text{ is } m \text{ dimensional.}$$

Recall that such a continuous reduction is a continuous map $F : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathcal{K}(\mathbb{R}^N)$ with $F^{-1}(\mathcal{K}_m(\mathbb{R}^N)) = Q_2 \times Q_2^c$.

For any $n > m$, $\mathbb{R}^{m+1}$ embeds in $\mathbb{R}^N$ so it is enough to show this for $n = m + 1$. F will still be continuous reduction from $(2^{\mathbb{N}} \times 2^{\mathbb{N}}, Q_2 \times Q_2^c)$ to $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}_m(\mathbb{R}^N))$ when considered for other n.

Fix a sequence $(S_n)_{n\in\mathbb{N}}$ of disjoint cubes (isometric copies of I^n) in $\mathbb{R}^N$ that converge to the origin. And let $(K_n)_{n\in\mathbb{N}}$ be a sequence of curves that converge to a space filling curve in the unit square $I \times I$. And let $(C_n)_{n\in\mathbb{N}}$ be a sequence of Cantor sets such that for each n, $C_n \subset K_n$ and $d_H(C_n, K_n) < 1/n$. We can find such a sequence of Cantor sets, since the Cantor sets are dense in $\mathcal{K}(\mathbb{R}^N)$.

Let L_{pq} be the isometric copy of $C_p \times I^{m-1}$ in the cube S_q and M_{pq} be the isometric copy of $K_p \times I^{m-1}$ in the cube S_q. Since $(K_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ both converge to the square $I \times I$, the sequences $(C_n \times I^{m-1})_{n\in\mathbb{N}}$ and $(K_n \times I^{m-1})_{n\in\mathbb{N}}$ converge to the unit cube I^{m+1}.

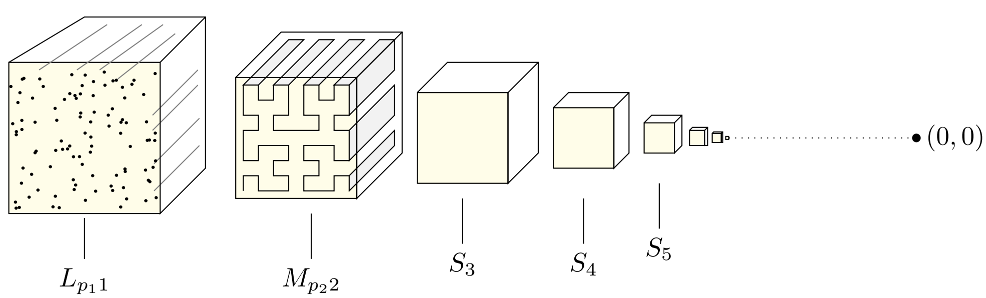


Figure 2.1: $F(\alpha, \beta)$ for some α and β

We define F as follows:

$$F(\alpha,\beta)=\bigcup_q \begin{cases} S_q & \text{if } \beta(n)=0, \forall n\geq q, \\ L_{p_q q} & \text{if } \exists m\geq q \ni \ \alpha(m)=1 \text{ where } p_q=\min\{m\geq q \mid \alpha(m)=1\} \\ M_{s_q q} & \text{if } \forall n\geq q, \alpha(n)=0 \text{ and} \\ & \exists m\geq q \ni \ \beta(m)=1 \text{ where } s_q=\min\{m\geq q \mid \beta(m)=1\} \end{cases}$$

Claim 1. F is a reduction of $Q_2\times Q_2^c$ to $\mathcal{K}_m(\mathbb{R}^N)$.

If $\alpha\in Q_2$ and $\beta\in Q_2^c$, then there is $q\in\mathbb{N}$ with $\forall n\geq q$, $\alpha(n)=0$ and there is $m\geq q$ with $\beta(m)=1$. Thus $F(\alpha,\beta)$ is a disjoint union of finitely many L_{pq}'s, the origin and M_{pq}'s in cubes after the q^{th} cube. Hence it is m dimensional.

If $\alpha\notin Q_2$, then for each q, there is $m\geq q$ with $\alpha(m)=1$, hence $F(\alpha,\beta)$ is a union of disjoint sets L_{pq} and the origin. Hence it is $m-1$ dimensional.

If $\beta\in Q_2$, then there is q such that $\forall n\geq q$, $\beta(n)=0$, hence $F(\alpha,\beta)$ includes the cubes after S_q, hence it is $m+1$ dimensional.

Claim 2. F is continuous.

Fix $\varepsilon>0$ and $(\alpha,\beta)\in 2^{\mathbb{N}}\times 2^{\mathbb{N}}$. Take a neighborhood $B=B(U_1,\ldots,U_r)$ of $F(\alpha,\beta)$, where $U_i=B(x_i,\varepsilon)$ for some $x_i\in F(\alpha,\beta)$. Without loss of generality suppose x_r is the origin. Since $S_q\to x_r$, there is m such that $\forall q\geq m$, $S_q\subset U_r$. Also, for arbitrary q, there is k such that $\forall p\geq k$, $d_H(L_{pq},S_q)<\varepsilon/2$ and $d_H(M_{pq},S_q)<\varepsilon/2$.

We need to show that there is a neighborhood $\mathcal{N}$ of (α,β) such that $F(\mathcal{N})\subset B$.

1. $\alpha,\beta\in Q_2$ implies that there are m_1,m_2 with $\forall n>m_1$, $\alpha(n)=0$ and $\forall n>m_2$, $\beta(n)=0$. Let m_1,m_2 be smallest such integers so that $\alpha(m_1)=1$ and $\beta(m_2)=1$. Also let $M=\max\{m,k,m_1,m_2\}$ and $\mathcal{N}=\{\alpha\}\times\mathcal{N}(\beta\restriction M)$, recall that the set $\mathcal{N}(\beta\restriction M)$ is a basic neighborhood of β which includes all the sequences that start with the same first M values in the sequence β. Let $(\alpha,\theta)\in\mathcal{N}$. Then $F(\alpha,\beta)=\bigcup_{q>m_2}S_q\cup\bigcup_{q=m_1+1}^{m_2}M_{s_q q}\cup\bigcup_{q=1}^{m_1}L_{p_q q}$. If $q\leq\max\{m_1,m_2\}$, then $F(\alpha,\theta)\cap S_q=F(\alpha,\beta)\cap S_q$; if $q>\max\{m_1,m_2\}$, then $F(\alpha,\beta)\cap S_q=S_q\supset F(\alpha,\theta)\cap S_q$. Hence $F(\alpha,\theta)\subset\bigcup_i U_i$.

   We also need, $F(\alpha,\theta)\cap U_i\neq\emptyset$ for each i. For $x_i\in F(\alpha,\beta)\cap S_q$, we have the following cases:

- if $q \leq \max\{m_1, m_2\}$, then $x_i \in F(\alpha, \theta) \cap S_q$, hence $U_i \cap F(\alpha, \theta) \neq \emptyset$.

- if $q > \max\{m_1, m_2\}$, then $F(\alpha, \theta) \cap S_q$ is either M_{pq} for some $p > M$, or it is the whole cube S_q. In either case, $d_H(F(\alpha, \theta) \cap S_q, S_q) < \varepsilon/2$. Hence $F(\alpha, \theta) \cap U_i \neq \emptyset$.

Thus $F(\alpha, \theta) \in B$ and $F(\mathcal{N}) \subset B$.

2. $\alpha \in Q_2$ and $\beta \notin Q_2$ implies there is m_1 with $\forall n > m_1$, $\alpha(n) = 0$.

   Then $F(\alpha, \beta) = \bigcup_{q > m_1} M_{s_q q} \cup \bigcup_{q=1}^{m_1} L_{p_q q}$.

   Let $M = \max\{m, k, m_1\}$ and $\mathcal{N} = \{\alpha\} \times \mathcal{N}(\beta \restriction M)$, and take $(\alpha, \theta) \in N$.

   If $q \leq M$, then $F(\alpha, \theta) \cap S_q = F(\alpha, \beta) \cap S_q$; if $q > M$, then $F(\alpha, \theta) \cap S_q = M_{r_q q}$ or S_q, in either case $F(\alpha, \theta) \subset U_r$. Hence $F(\alpha, \theta) \subset \bigcup_i U_i$.

   For $x_i \in F(\alpha, \beta) \cap S_q$, we have the following cases:

   - if $q \leq M$, then $x_i \in F(\alpha, \theta) \cap S_q$, hence $U_i \cap F(\alpha, \theta) \neq \emptyset$.

   - if $q > M$, then $F(\alpha, \theta) \cap S_q$ is either M_{pq} for some $p > M$, or it is the whole cube S_q. In either case, $d_H(F(\alpha, \theta) \cap S_q, S_q) < \varepsilon/2$. Also, $d_H(M_{s_q q}, S_q) < \varepsilon/2$, so $d_H(F(\alpha, \beta) \cap S_q, F(\alpha, \theta) \cap S_q) < \varepsilon$. Hence $F(\alpha, \theta) \cap U_i \neq \emptyset$.

   Thus $F(\alpha, \theta) \in B$ and $F(\mathcal{N}) \subset B$.

3. $\alpha \notin Q_2$ and $\beta \in Q_2$ implies that there is m_2 with $\forall n > m_2$, $\beta(n) = 0$.

   Then $F(\alpha, \beta) = \bigcup_q L_{p_q q}$.

   Let $M = \max\{m, k, m_2\}$ and $\mathcal{N} = \{\alpha\} \times \mathcal{N}(\beta \restriction M)$, and take $(\alpha, \theta) \in \mathcal{N}$. Then $F(\alpha, \theta) = F(\alpha, \beta)$. Hence $F(\mathcal{N}) \subset B$.

4. $\alpha, \beta \notin Q_2$. Then $F(\alpha, \beta) = \bigcup_q L_{p_q q}$. Let $M = \max\{m, k\}$ and $\mathcal{N} = \{\alpha\} \times \mathcal{N}(\beta \restriction M)$. For $(\alpha, \theta) \in \mathcal{N}$, $F(\alpha, \theta) = F(\alpha, \beta)$. Hence $F(\mathcal{N}) \subset B$.

$\square$

2.2 SUBCONTINUA OF $\mathbb{R}^N$

In this section, we will introduce the space of all subcontinua of $\mathbb{R}^N$ and some important classes of continua, like Peano continuum, dendrites and dendroids. Later in Chapter 4 we will see some complexity results on these classes.

The space of subcontinua of $\mathbb{R}^N$ consists of all connected compact subsets of $\mathbb{R}^N$, denoted $\mathbf{C}(\mathbb{R}^N)$. Being a closed subset of $\mathcal{K}(\mathbb{R}^N)$ it is also a Polish space with Vietoris topology.

Similar to $\mathcal{K}(\mathbb{R}^N)$, $\mathbf{C}(\mathbb{R}^N)$ also stratifies with dimension under the action of $Aut(\mathbb{R}^N)$. Let $\mathbf{C}_{\leq m}(\mathbb{R}^N)$ be the set of all subcontinua of dimension $\leq m$ and $\mathbf{C}_m(\mathbb{R}^N)$ be the set of all subcontinua of dimension m. Then $\mathbf{C}_0(\mathbb{R}^N)$ consists of all one point subsets of $\mathbb{R}^N$, which is known to be a closed subset. And for $m \geq 1$;

Corollary.

- $\mathbf{C}_{\leq m}(\mathbb{R}^N)$ *is a* G_δ *set.*
- $\mathbf{C}_n(\mathbb{R}^N)$ *is an* F_σ *set.*

Proof. Now $\mathbf{C}_{\leq m} = \mathcal{K}_{\leq m} \cap \mathbf{C}(\mathbb{R}^N)$, so intersection of a G_δ (by Theorem 2.1.2) and a closed set, hence is a G_δ.

Being the complement of the G_δ set $\mathbf{C}_{\leq n-1}(\mathbb{R}^N)$, $\mathbf{C}_n(\mathbb{R}^N)$ is F_σ. □

Theorem 2.2.1. *The class of curves in* $\mathbb{R}^N$ *is a* G_δ *in* $\mathbf{C}(\mathbb{R}^N)$.

Proof. The set of all curves is the difference of the sets $\mathbf{C}_{\leq 1}(\mathbb{R}^N)$ and $\mathbf{C}_0(\mathbb{R}^N)$, where the first one is a G_δ and the second is a closed set. Hence the difference is G_δ. □

Let $K \subset \mathbb{R}^N$ be a non-degenerate continuum, i.e. it has more than one element, and let $\mathcal{H}(K) = \{X \in \mathbf{C}(\mathbb{R}^N) \mid X \text{ is homeomorphic to } K\}$. It is well known that $\mathcal{H}(K)$ is dense in $\mathbf{C}(\mathbb{R}^N)$. In particular, $\mathbf{C}_m(\mathbb{R}^N)$ is dense in $\mathbf{C}(\mathbb{R}^N)$ for $1 \leq m \leq n$.

A *Peano continuum* is a continuum which is locally connected. An *arc* is a continuum homeomorhic to the closed interval.

A *graph* is a continuum which can be written as the union of finitely many arcs which pairwise intersect only in their end points. A *dendrite* is a locally connected compact connected set (continuum) which does not include a subcontinuum homeomorphic to the circle.

A continuum is X a *dendroid* if it is arcwise connected and hereditarily unicoherent, where X is *arcwise connected* if for all $x \neq y \in X$ there is an arc contained in X with the end points x, y and it is *unicoherent* if whenever it is written as the union of two subcontinua then their intersection is a continuum.

It is known that dendrites form a $\mathbf{\Pi}^0_3$-complete set, whereas dendroids are more complex and the class of dendroids is $\mathbf{\Pi}^1_1$-complete, see [3].

A continuum is *regular* if for every point there is a local base where each element of the base has a finite boundary. A continuum is *rational* if for every point there is a local base where each element of the base has a countable boundary.

Any regular or rational continuum is a curve. Dendrites are regular, and dendroids are rational curves. In [6], the authors show that the set of regular continua is a $\mathbf{\Pi}^0_4$-complete set. Also, the set of rational continua is $\mathbf{\Pi}^1_1$-hard, however the complete classification is not known.

3.0 ZERO DIMENSIONAL COMPACT SUBSETS

For a given zero dimensional compact subset K of $\mathbb{R}^N$, by Cantor-Bendixson theorem, there are unique subsets P and C of K where P is a perfect and C is countable such that $K = P \cup C$ (see [15]). Furthermore, if K is uncountable then P is a Cantor set, since it is compact perfect and zero dimensional. So the two natural classes of zero dimensional compact subsets are: countable compact subsets and Cantor subsets.

The set of all countable compact subsets of $\mathbb{R}^N$ is a $\mathbf{\Pi}^1_1$-complete set in $\mathcal{K}(\mathbb{R}^N)$ (see [15], p.210). Moreover, it is an invariant subset of $\mathcal{K}(\mathbb{R}^N)$ under both equivalence relations $\sim_H$ and $\sim$.

In the rest of this chapter, we will examine the class of Cantor subsets of $\mathbb{R}^3$, which is denoted by $\mathcal{C}(R^3)$. We will show that $\mathcal{C}(R^3)$ is a G_δ subset of $\mathcal{K}(\mathbb{R}^3)$, hence it is a Polish space itself. Since all Cantor sets are homeomorphic, we will only consider the equivalence relation $\sim$ on $\mathcal{C}(R^3)$.

3.1 CANTOR SETS IN $\mathbb{R}^3$

This section will introduce the space of Cantor subsets of $\mathbb{R}^3$ and its topology, which is induced from $\mathcal{K}(\mathbb{R}^3)$.

A set $C \subset \mathbb{R}^3$ is a *Cantor set* if and only if it is totally disconnected, perfect, compact metric space. For a Cantor set C in $\mathbb{R}^3$ a sequence $(C_n)_{n\in\mathbb{N}}$ of compact 3-manifolds with boundary is a *defining sequence* for C if and only if:

1. for each $i \in \mathbb{N}$, C_i is the union of a finite number of mutually exclusive polyhedral cubes

with handles,

2. for each $i \in \mathbb{N}$, $C_{i+1} \subseteq \text{Int}(C_i)$, and
3. $C = \bigcap_n C_n$

Two Cantor sets $K, K' \subset \mathbb{R}^3$ are *equivalent* if there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K'$. This is the equivalence relation induced by the action of $Aut(\mathbb{R}^3)$. A Cantor set K is *tame* if it is equivalent to standard Cantor set, $E_{1/3}$. Otherwise, it is called *wild*. A well known example for wild Cantor sets is the Antoine's necklace, [1].

It is known that, every Cantor set has a defining sequence. Moreover, two Cantor sets K and L are equivalent if and only if there are equivalent defining sequences $\{M_i\}$ and $\{N_i\}$ for K and L respectively, where $\{M_i\}$ and $\{N_i\}$ are equivalent means that for each i, there is a homeomorphism $h_i \in Aut(\mathbb{R}^3)$ such that $h_i(M_i) = N_i$ and $h_{i+1} \restriction (\mathbb{R}^3 - M_i) = h_i \restriction (\mathbb{R}^3 - M_i)$. [Sher, [20]]

We will show below that the space of Cantor subsets of $\mathbb{R}^3$ is Polish itself, and then we will discover that some of the natural subclasses are in the first few levels of the Borel hierarchy of this space. The following lemma is a well-known result, we will include the proof here for completeness.

Lemma 3.1.1. *1. $\mathcal{K}_P(\mathbb{R}^3) = \{K \in \mathcal{K}(\mathbb{R}^3) \mid K$ is perfect$\}$ is a dense G_δ set in $\mathcal{K}(\mathbb{R}^3)$.*

2. $\mathcal{K}_C(\mathbb{R}^3) = \{K \in \mathcal{K}(\mathbb{R}^3) \mid K$ is a Cantor set$\}$ is a (dense) G_δ, and thus Polish.

Proof. 1. Let

$$\mathcal{U}_n = \bigcup \{B(U_1, \ldots, U_r) \mid r \in \mathbb{N},\, diam(U_i) < 1/n,\, \forall i,\, \exists j \neq i,\, s.t.\, U_i \cap U_j \neq \emptyset\}$$

where U_i's are open disks (ie homeomorphic to open ball in $\mathbb{R}^3$).

Claim 1. $\bigcap \mathcal{U}_n = \mathcal{K}_P(\mathbb{R}^3)$

Suppose K is not perfect, we will show it is not in the intersection. There is an isolated point $x \in K$, so $d(x, K\backslash\{x\}) > 0$. So there is $N \in \mathbb{N}$ with $d(x, K\backslash\{x\}) > 1/N > 0$. Let $n = 2N$, then $K \notin \mathcal{U}_n$ because, for each open disk U with $x \in U$, $diam(U) < 1/n$, and for any open disk U' with $diam(U') < 1/n$ & $(K\backslash\{x\}) \cap U' \neq \emptyset$, the intersection

$U \cap U'$ is empty. (Since otherwise, there is $y \in U \cap U'$ and $z \in K \cap U'$, and $d(x,z) \leq d(x,y) + d(y,z) < 1/N$ contradicting the choice of N).

For the converse, suppose K is perfect compact set. Fix n. We need to show $K \in \mathcal{U}_n$. Take the open cover $\{B(x, 1/(2n))\}_{x \in K}$, then since K is compact there are finitely many $x_i \in K$ such that $\{U_i = B(x_i, 1/(2n))\}_{i=1}^k$ covers K. Now if for some i, $U_i \cap U_j = \emptyset$, for all $j \neq i$, then there exist $x \in U_i$ such that $x \neq x_i$, as otherwise $K \cap U_i = \{x_i\}$ and so x_i is an isolated point, which is not possible. Then take $U_{k+1} = B(x, 1/(2n))$. Now for the new finite open cover $\{U_i\}_{i=1}^{k+1}$ if there is U_j which does not meet other U_i's, then we can repeat as above and add finitely many more open sets of diameter $< 1/n$. Hence after finitely many steps we have $\{U_i\}_{i=1}^r$, such that $K \in B(U_1, \ldots, U_r)$, $\operatorname{diam} U_i < 1/n$ and for any $1 \leq i \leq r$, there is $j \neq i$ with $U_i \cap U_j \neq \emptyset$. Thus $K \in \mathcal{U}_n$.

For denseness, take a basic open set in $\mathcal{K}(\mathbb{R}^3)$, $B(U_1, \ldots, U_r)$. Then for any i, take $x_i \in U_i$. Then there is $\varepsilon > 0$ with $B(x_i, \varepsilon) \subset U_i$ for each i. Take $K = \bigcup_{i=1}^r \operatorname{Cl} B(x_i, \varepsilon/2)$. Then $K \in B(U_1, \ldots, U_r)$. Moreover, K is perfect, since otherwise an isolated point of K would be isolated in one of the closed balls $B(x_i, \varepsilon)$, which can not happen. Thus $\mathcal{K}_P(\mathbb{R}^N)$ is dense in $\mathcal{K}(\mathbb{R}^N)$.

2. By 2.1.2, when $m = 0$, we get all zero dimensional subsets of $\mathbb{R}^3$, which is a G_δ and $\mathcal{C}(\mathbb{R}^3) = \mathcal{K}_P(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, hence it is a G_δ subset as well.

□

Lemma 3.1.2. *If $K \in \mathcal{C}(\mathbb{R}^3)$ has defining sequence $\{M_n\}_{n \in \mathbb{N}}$, then K has the following as a local base in $\mathcal{K}(\mathbb{R}^3)$:*

$$\left\{ B(V_1^n, \ldots, V_{m_n}^n) \mid V_1^n, \ldots, V_{m_n}^n \text{ are interiors of the components of } M_n \right\}$$

Proof. Let $B(U_1, \ldots, U_k)$ be a basic open set containing K.

$K = \bigcap_n M_n \subset \bigcup_{i=1}^k U_i$, so for some $l_1 \in \mathbb{N}$, $M_{l_1} \subset \bigcup_{i=1}^k U_i$, since otherwise we get a sequence of compact non-empty sets $C_n = M_n - \bigcup_{i=1}^k U_i$, whose intersection is non-empty subset of K but not a subset of $\bigcup_{i=1}^k U_i$, hence gives us a contradiction.

Let $S = \{(i,j) : V_j^{\ell_1} \cap U_i \cap K \neq \emptyset, 1 \leq i \leq k, 1 \leq j \leq m_{\ell_1}\}$. For each (i,j) in S fix $x_{ij} \in V_j^{\ell_1} \cap U_i \cap K$.

Then, for each $(i, j) \in S$, since x_{ij} is in open U_i, and using the definition of defining sequence, there is an ℓ_{ij} such that the component of $M_{\ell_{ij}}$ containing x_{ij} in its interior is a subset of U_i.

Let $\ell = \max(\ell_1, \ell_{ij} : (i, j) \in S)$.

We will show that $K \in B(V_1^\ell, \dots, V_{m_\ell}^\ell) \subseteq B(U_1, \dots, U_k)$ – as required.

As the V_j^ℓ are the interiors of the components of M_ℓ, by definition of defining sequence, K is in $B(V_1^\ell, \dots, V_{m_\ell}^\ell)$.

Now take any K' in $B(V_1^\ell, \dots, V_{m_\ell}^\ell)$. Since $\ell \geq \ell_1$, $K' \subseteq \bigcup_{j=1}^{m_l} V_j^\ell \subseteq M_\ell \subseteq M_{\ell_1} \subseteq \bigcup_{i=1}^k U_i$.

It remains to show that $K' \cap U_i \neq \emptyset$ for all i. So fix i.

Since $U_i \cap K \neq \emptyset$, and $K \subseteq \bigcup_j V_j^{\ell_1}$, for some j_i we have $U_i \cap V_{j_i}^{\ell_1} \cap K \neq \emptyset$. Thus (i, j_i) is in S, and some component of $M_{\ell_{ij_i}}$ containing x_{ij_i} in its interior is a subset of U_i. Since $\ell \geq \ell_{ij_i}$, it follows that some $V_{j'}^\ell$ is contained in U_i. As K' meets this $V_{j'}^\ell$ it meets U_i. $\square$

From now on a basic open set for a Cantor set K will refer to a set $B(U_1, \dots, U_m)$ where each U_i is an open set and $\mathrm{Cl}\, U_i \cap \mathrm{Cl}\, U_j = \emptyset$ when $i \neq j$.

3.1.1 PROPERTIES OF $\mathcal{C}(\mathbb{R}^3)$

A Cantor set K in $\mathbb{R}^3$ is *n-decomposable* if there are n topological open balls $U_1, \dots, U_n$ in $\mathbb{R}^3$ with pairwise disjoint closures such that $K \subset \bigcup_{i=1}^n U_i$ and $K \cap U_i \neq \emptyset$ for all i. A Cantor set which is n-decomposable for each n is ω-decomposable. Clearly, all tame Cantor sets are ω-decomposable. A Cantor set which is not 2-decomposable is *indecomposable*.

Lemma 3.1.3. *The orbit of a Cantor set K –under the equivalence relation $\sim$– is dense if and only if K is ω-decomposable.*

Proof. Suppose the orbit of K is dense in $\mathcal{C}(\mathbb{R}^3)$. Fix $n \in \mathbb{N}$. Let K' be an n-decomposable Cantor set, so there are n topological open balls $U_1, \dots, U_n \subset \mathbb{R}^3$ with $B(U_1, \dots, U_n)$ a basic open neigborhood of K'. And there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) \in B(U_1, \dots, U_n)$. Then $K \subset \bigcup_{i=1}^n h^{-1}(U_i)$, and $h^{-1}(U_i) \cap K \neq \emptyset$, $\forall i$. Moreover $K \in B(h^{-1}(U_1), \dots, h^{-1}(U_n))$, and $h^{-1}(U_i)$'s are topological open balls with pairwise disjoint closures (as h is homeomorphism). Hence K is n-decomposable. Since n is arbitrary, K is ω-decomposable.

On the other hand, let K be ω-decomposable. Let L be arbitrary Cantor set, and let $B(V_1, \ldots, V_m)$ be a basic open set containing L. Since K is ω-decomposable, it is m-decomposable and there are topological open balls $U_1, \ldots, U_m$ such that $B(U_1, \ldots, U_m)$ is a basic open neighborhood of K. Now there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(\mathrm{Cl}\, U_i) \subset V_i$. But then $h(K)$ is in the orbit of K and in $B(V_1, \ldots, V_m)$. Hence the orbit is dense. $\square$

So the space of Cantor subsets of $\mathbb{R}^3$ is connected, as it contains a dense connected subspace - the equivalence class of tame Cantor sets. Actually more is true:

Theorem 3.1.4. *$\mathcal{C}(\mathbb{R}^3)$ is path connected.*

Proof. We will prove a more general statement below:

Lemma 3.1.5. *Let $U \subset \mathbb{R}^3$ be an open subset. And let K be a Cantor set in U. And let C be a tame Cantor set in U. Then there is a path from C to K that lies in U.*

Proof: K has a defining sequence $(K_n)_{n\in\mathbb{N}}$, where each K_n is finite disjoint union of handlebodies. Without loss of generality, $K_n \subset U$ for each n. Let K_n have q_n components.

We can cover C with q_0 disjoint open balls, say $B^0_1, \ldots, B^0_{q_0}$, which are subsets of U. Let $K^0_1, \ldots, K^0_{q_0}$ be a listing of components of K_0. There is an isotopy $H^0 : \mathbb{R}^3 \times [0, 1/2] \to \mathbb{R}^3$, such that H^0_t is an autohomeomorphism of $\mathbb{R}^3$ for each $t \in [0, 1/2]$, and $H^0_0 = \mathrm{id}$ and $H^0_{1/2}(B^0_i) \subset \mathrm{Int}(K^0_i)$. Let $C^0 = H^0_{1/2}(C)$, so $C^0 \subset \mathrm{Int}(K_0)$ and is a tame Cantor set.

Now in each component K^0_i of K_0, there is a piece of C^0, which is in the topological ball $H^0_{1/2}(B^0_i)$. Since it is tame, we can cover the piece in $H^0_{1/2}(B^0_i)$ with q_1-many disjoint open balls, say $B^{1,i}_1, \ldots, B^{1,i}_{q_1}$, which are subsets of the interior of corresponding component K^0_i (hence of U). Let $K^{1,i}_1, \ldots, K^{1,i}_{q_1}$ be a listing of components of K_1 in K^0_i. There is an isotopy $H^1 : \mathbb{R}^3 \times [1/2, 3/4] \to \mathbb{R}^3$ such that H^1_t is an autohomeomorphism of $\mathbb{R}^3$, $H^1 \restriction (\mathbb{R}^3 - K_0) = \mathrm{id}_{\mathbb{R}^3 - K_0}$, $H^1_{1/2} = \mathrm{id}$ and $H^1_{3/4}(B^{1,i}_j) \subset \mathrm{Int}(K^{1,i}_j)$. Let $C^1 = H^1_{3/4}(C^0)$, so $C^1 \subset \mathrm{Int}(K_1)$ and is a tame Cantor set.

Continuing this way we get an isotopy $H^n : \mathbb{R}^3 \times [1-1/2^n, 1-1/2^{n+1}] \to \mathbb{R}^3$, for each n, where at each n, only points in K_n are moved.

Now let define a path as follows: $p : [0,1] \to \mathcal{C}(\mathbb{R}^3)$,

$$p(t) = \begin{cases} H^n(C^{n-1}, t), & 1-1/2^n \leq t \leq 1-1/2^{n+1} \\ K, & t = 1 \end{cases}$$

This is a continuous path from C to K in U. p is obviously continuous for $t < 1$. We only need to check continuity at $t = 1$. Take basic open neighborhood of K, $B(U_1, \ldots, U_{k_n})$, where U_i's are interiors of the components of some K_n. So $C^n = H^n_{1-1/2^{n-1}}(C^{n-1}) \subset \bigcup_{i=1}^{k_n} U_i$, and by definition $C^n \cap U_i \neq \emptyset$, for each i. And for each $t > 1-\frac{1}{2^{n-1}}$, $(t < 1)$ $1-1/2^m \leq t \leq 1-1/2^{m+1}$ for some $m \geq n$, $p(t) = H^m(C^{m-1}, t) \subset K_m \subset \bigcup_{i=1}^{k_n} U_i$, and by definition $C^n \cap U_i \neq \emptyset$, for each i. Hence for each $t > 1 - \frac{1}{2^{n-1}}$, $p(t) \in B(U_1, \ldots, U_{k_n})$. ▪

Now suppose K and L are Cantor sets, and let C be a tame Cantor set. Then by taking $U = \mathbb{R}^3$, we get a path from $K \subset \mathbb{R}^3$ to C and a path from L to C. Now following the path from K to C and then the second one backwards from C to L we get a path from K to L. □

Theorem 3.1.6. *$\mathcal{C}(\mathbb{R}^3)$ is locally path connected.*

Proof. Fix Cantor set K and basic open neighborhood $B = B(U_1, ..., U_n)$. For each i select a tame Cantor set $C_i \subset U_i$. Let $C = \bigcup_{i=1}^n C_i$. Then C is a tame Cantor set in the basic neighborhood B.

Take any other L in B, we want to define a path inside B which takes K to L. Let $L_i = U_i \cap L = \operatorname{Cl} U_i \cap L$, so it is a Cantor set. Then by Lemma 3.1.5 there are paths p_i from C_i to L_i inside U_i. Now let $p : I \to \mathcal{C}(\mathbb{R}^3)$ be defined as $p(t) = \bigcup_{i=1}^n p_i(t)$, then $p(0) = \bigcup_{i=1}^n p_i(0) = \bigcup_{i=1}^n C_i = C$, $p(1) = \bigcup_{i=1}^n p_i(1) = \bigcup_{i=1}^n L_i = L$ and each $p(t)$ is a Cantor set in $B = B(U_1, ..., U_n)$, as it is a finite union of Cantor sets. So we only need to show p is continuous: To this end, fix $t \in [0,1]$, let $B(V_1, \ldots, V_m)$ be a basic open set including $p(t) = \bigcup_{i=1}^n p_i(t)$. Now for each i, let $V_{i_1}, \ldots, V_{i_{k(i)}}$ be the ones that intersect $p_i(t)$, so $p_i(t) \in B(V_{i_1}, \ldots, V_{i_{k(i)}})$. Now let $U = \bigcap_{i=1}^n p_i^{-1}(B(V_{i_1}, \ldots, V_{i_{k(i)}}))$, then U is open neighborhood of t. And $p(U) \subset B(V_1, \ldots, V_m)$, as for any $s \in U$, $p_i(s) \in B(V_{i_1}, \ldots, V_{i_{k(i)}})$, and the union $p(s) = \bigcup_i p_i(s)$ is a Cantor set which is a subset of $\bigcup_{i=1}^m V_i$ and meets each one of V_i's. $\square$

3.2 STRUCTURE OF CANTOR SETS

Recall that a Cantor set is called n-decomposable if there are n disjoint topological open balls covering the Cantor set, where each of them intersects the Cantor set. We know that such open balls will form a basic open neighborhood for a Cantor set in $\mathcal{C}(\mathbb{R}^3)$. Also, a Cantor set is called ω-decomposable if it is n-decomposable for each n. Thus we have:

Proposition 3.2.1.

- *The set of n-decomposable Cantor sets form a dense open subset of $\mathcal{C}(\mathbb{R}^3)$.*
- *The set of ω-decomposable Cantor sets form a dense G_δ subset of $\mathcal{C}(\mathbb{R}^3)$.*

Proof. Let $D_n = \bigcup\{B(U_1, \ldots, U_n) \text{ basic open} | U_i\text{'s are topological open balls}\}$. Then $K \in \mathcal{C}(\mathbb{R}^3)$ is n-decomposable if and only if K is in D_n, and clearly D_n is a dense open subset of $\mathcal{C}(\mathbb{R}^3)$. Also let $D = \bigcap_n D_n$, then D is a dense G_δ. Moreover, K is in D if and only if K is ω-decomposable. $\square$

Now consider the Cantor sets which can be decomposed with open sets homeomorphic to the interior of solid n-tori. We call a Cantor set K to be *$(g, \geq n)$-decomposable* if there are n open sets $U_1, \ldots, U_n$ each homeomorphic to the interior of g-tori with disjoint closures such that $K \subset \bigcup_i^n U_i$ and $K \cap U_i \neq \emptyset$ for each i. K is *(g, n)-decomposable* if it is $(g, \geq n)$-decomposable but not $(g, \geq n+1)$-decomposable. And K is *(g, ω)-decomposable* if for each n, it is $(g, \geq n)$-decomposable. A Cantor set which is not $(g, 2)$-decomposable is *g-indecomposable.*

A Cantor set has *genus less than or equal g* if and only if it has a defining sequence so that the genus of each component at each level is less than or equal to g. It has *genus g* if and only if it has genus $\leq g$ but not $\leq (g-1)$. Note that a genus zero Cantor set is tame, [23].

Let $B(U_1, \ldots, U_n)$ and $B(V_1, \ldots, V_m)$ be basic open sets in $\mathcal{C}(\mathbb{R}^3)$. Write

$$B(U_1, \ldots, U_n) \prec B(V_1, \ldots, V_m)$$

if and only if for each U_j there is a V_i so that $\operatorname{Cl} U_j \subseteq V_i$.

Lemma 3.2.2. *The following statements are equivalent:*

(a) A Cantor set K in $\mathbb{R}^3$ has genus g,

(b) it has a defining sequence where all the components in the sequence are genus-g handlebodies whose diameters $\to 0$,

(c) K has a sequence of open neighborhoods $(B(U_1,^n, \ldots, U^n_{m_n}))_n$ such that

(i) for each n, j, $\operatorname{Cl} U^n_j$ is a genus g handlebody whose diameter $\to 0$ with n,

(ii) the closures of distinct U^n_j and U^n_k are disjoint,

(iii) $B(U^n_1, \ldots U^n_{m_n}) \prec B(U^{n'}_1, \ldots U^{n'}_{m_{n'}})$ if $n > n'$, and

(iv) $(B(U^n_1, \ldots, U^n_{m_n}))_n$ is a local base at K.

Proof. K has genus-g if and only if it has a defining sequence so that the genus of each component at each level is g if and only if (a).

(c) $\implies$ (b): Let $K_n = \bigcup_{i=1}^{m_n} \operatorname{Cl} U^n_i$, then $(K_n)_{n\in\mathbb{N}}$ is a defining sequence for K where each K_n is a disjoint union of genus-g handlebodies whose diameter $\to 0$ (as $n \to \infty$).

(b) $\implies$ (c): Conversely let $(K_n)_{n\in\mathbb{N}}$ as in (a). Without loss of generality, we can assume that for each n, diam(components of K_n) $< \frac{1}{n}$. (We can make sure this happens by removing some levels from the original defining sequence)

Now let $\{U^n_1, \ldots, U^n_{m_n}\}$ be a listing of interiors of the handlebodies at level n of the defining sequence. So $(B(U^n_1, \ldots, U^n_{m_n}))$ is a sequence of basic open neighborhoods of K which satisfies (i), (ii) and (iii) by definition of the defining sequence. Also by Lemma 3.1.2, this forms a local base. □

Theorem 3.2.3. *For any genus g;*

1. *The set of genus-g Cantor sets are dense in $(g-1)$-indecomposables.*
2. *The set of genus-g Cantor sets are a G_δ.*
3. *The set of $(g, \geq n)$-decomposable Cantor sets is an open (invariant) set.*

4. *The set of (g,ω)-decomposable Cantor sets is a G_δ set, containing genus-g ones, hence it is dense.*

5. *The set of all g-indecomposable Cantor sets is closed and nowhere dense in $(g-1)$-indecomposables.*

Proof. We set all Cantor subsets to be -1-indecomposable.

1. Take any $(g-1)$-indecomposable Cantor set K, and a basic neighborhood $B(V_1,\dots,V_m)$, without loss of generality V_i's are interior of handlebodies.

   Since K is $(g-1)$-indecomposable, V_i's must be linked.

   Now shrink the handlebody V_i to a 'skeleton' S_i, so S_i's are linked as V_i's. On each S_i, add skeletons of tori, so that they are linked and they cover S_i. Now replacing each torus with a genus g, $(g-1)$-indecomposable Cantor set, will give us a genus g Cantor set in the basic neighborhood $B(V_1,\dots,V_m)$.

2. Let $T_n = \bigcup\{B(U_1,\dots,U_m)$ basic : the closures of the U_i's are handlebodies of genus g with diameters $< 1/n\}$. Let $T = \bigcap_n T_n$. Then every genus g Cantor set is in T. Moreover, we will show that every $K \in T$ has genus g.

   Since K is in T, for each n there is a $B(U_1^n,\dots,U_{m_n}^n)$ from T_n which contains K. We will prove: $(*)$ given any basic neighborhood $B(V_1,\dots,V_m)$ of K, there is an n such that $B(U_1^n,\dots,U_{m_n}^n) \prec B(V_1,\dots,V_m)$.

   Applying this claim recursively we can easily find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $B(U_1^{n_k},\dots,U_{m_{n_k}}^{n_k}) \prec B(U_1^{n'_k},\dots,U_{m_{n'_k}}^{n'_k})$ if and only if $k > k'$. Then Lemma 3.2.2 completes the proof.

   To prove $(*)$ fix a basic neighborhood $B(V_1,\dots,V_m)$ of K. Let ϵ to be the minimum of $d(K,\mathbb{R}^3 \setminus \bigcup_i V_i)$ and $d(\mathrm{Cl}\,V_i,\mathrm{Cl}\,V_j)$ for all $i \neq j$. Pick n so that $1/n < \epsilon$ and, by the Lebesgue Covering Lemma, so that if x, x' are in K and $d(x,x') < 1/n$ then x and x' are in some V_i.

   We check that each $\mathrm{Cl}\,U_j^n$ is contained in some V_i. To this end take any U_j^n. Let $K' = K \cap U_j^n = K \cap \mathrm{Cl}\,U_j^n$. Then (by the second condition on n) K' is contained in

some V_i, and further $\mathrm{Cl}\, U_j^n$ meets only this V_i (otherwise U_j^n would meet V_i and $V_{i'}$, contradicting $diam(U_j^n) \leq 1/n < \epsilon$ and $d(\mathrm{Cl}\, V_i, \mathrm{Cl}\, V_{i'}) \geq \epsilon$).
Suppose, for a contradiction, that $\mathrm{Cl}\, U_j^n \not\subseteq V_i$. Then there is a y in $\mathrm{Cl}\, U_j^n \backslash V_i \subseteq \mathbb{R}^3 \backslash \bigcup_{i'} V_{i'}$. So $d(K', y) \geq \epsilon$. Pick x in K' so that $d(x, y) \geq \epsilon$. Now we see that x and y are in $\mathrm{Cl}\, U_j^n$, and the diameter of $\mathrm{Cl}\, U_j^n \leq 1/n < \epsilon$, contradiction.

3. Let $D_m = \bigcup \{B(T_1, \ldots, T_m)$ basic $|T_i$'s are open sets homeomorphic to the interior of solid g-torus$\}$. Then K is in D_m if and only if K is $(g, \geq m)$-decomposable. Clearly each D_m is an open set.

4. Let $D = \bigcap_n D_n$, then D is G_δ and K is in D if and only if K is (g, ω)-decomposable. A genus-g, $(g-1)$-indecomposable Cantor set is necessarily (g, ω)-decomposable. Hence (g, ω)-decomposable Cantor sets are dense in $(g-1)$-indecomposables.

5. Let $\mathcal{I} = \bigcap_n D_n^C$ (complements of D_n's), hence $\mathcal{I}$ is a closed nowhere dense subset of $\mathcal{C}(\mathbb{R}^3)$. Also $K \in \mathcal{I}$ if and only if K is g-indecomposable.

□

3.3 COMPLEXITY OF CLASSIFICATION

In this section, we consider the classification of Cantor sets up to the equivalence $\sim$ as defined in Chapter 2. In the previous section, we have seen that there are many different Cantor sets, but some of those may be equivalent. In this section we will show that there are many inequivalent classes of Cantor sets by showing that there are at least as many as the countable linear orders have.

Lemma 3.3.1. *For a given linear order α, we can construct a sequence of pairwise disjoint open intervals $(p_n, q_n)_{n \in \mathbb{N}}$ with end points in $(0, 1)$ and with the following properties:*

1. *The order of $\{p_n \mid n \in \mathbb{N}\}$ is isomorphic to the order coded by α,*

2. $\inf_n p_n = 0$ *if and only if the order has no smallest element,*

3. $\sup_n q_n = 1$ *if and only if the order has no largest element,*

4. For any $x \notin \bigcup_n (p_n, q_n)$, $\sup \{q_n \mid q_n \leq x\} = \inf \{p_n \mid p_n \geq x\}$ *if and only if there is no biggest* q_n *below* x *and no smallest* p_n *above* x*,*

5. $|p_n - q_n| < \frac{1}{n}$

Moreover, we can assign a collection of intervals $\{(p_n^\alpha, q_n^\alpha) \mid n \in \mathbb{N}\}$ *to each linear order* $\alpha \in LO$ *such that if* α *and* β *agree on the order of* $1, \ldots, N$ *then* $(p_n^\alpha, q_n^\alpha) = (p_n^\beta, q_n^\beta)$ *for all* $n = 1, \ldots, N$*.*

See the proof in [11] by Gartside and Pejic, the only difference is that we use the interval $(0, 1)$ rather than $(0, 1/2)$.

Note. Any Antoine's necklace can be embedded into the closed standard double cone, which is the set $\{(x, y, z) \in \mathbf{R}^3 | \sqrt{y^2 + z^2} + |x| \leq 1\}$, such that the intersection of the Antoine set and the boundary of the cone is precisely the set $\{(\pm 1, 0, 0)\}$, which are called the *end points*, (see Figure 3.1).

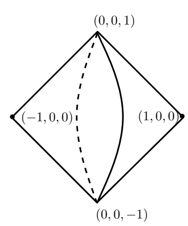


Figure 3.1: Standard double cone

Theorem 3.3.2. *The classification problem of Cantor sets up to equivalence is at least as complicated as classification of countable linear orders.*

Proof. We will define a Borel map $f : LO \to \mathcal{C}(\mathbb{R}^3)$ such that two linear orders $\alpha, \gamma \in LO$ are equivalent if and only if $f(\alpha), f(\gamma)$ are equivalent Cantor sets.

Let A be a rigid Antoine necklace. Now given a linear order α, we will define a corresponding Cantor set C_α as follows:

Let $\{(p_n^\alpha, q_n^\alpha)\}_{n \in \mathbb{N}}$ be the intervals from the Lemma 3.3.1. For each n, let us put an isometric copy of the closed standard double cone in (p_n^α, q_n^α) with end points being the end

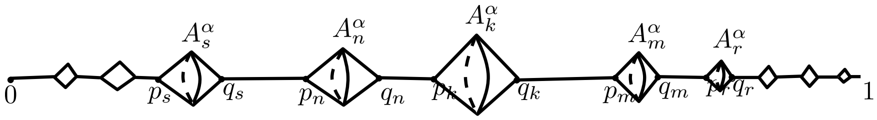


Figure 3.2: Cantor cones in the intervals

points of the interval, then we embed the Cantor set A into this cone, and will call it A_n^α. (see Figure 3.2)

Also for every consecutive $m, k \in \mathbb{N}$ with respect α, put an isometric copy of the closed standard double cone in (q_m, p_k) with end points being the end points of the interval, then we embed the Cantor set A into this cone, and will call it $A_{m,k}^\alpha$

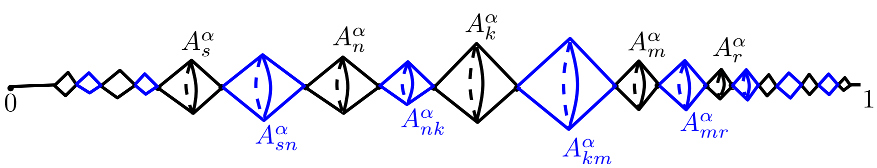


Figure 3.3: Construction of Cantor Set for given α

Then let $C_\alpha = f(\alpha) = \bigcup_n A_n^\alpha \cup \bigcup_{m<_\alpha k} A_{m,k}^\alpha$, where m, k in the second union are consecutive with respect to α. (see Figure 3.3)

Claim 1. C_α is a Cantor set.

C_α is obviously perfect, as none of the points will be isolated. It is bounded as it is a subset of $[0,1]^3$.

Let $x \in \mathbb{R}^3 \setminus C_\alpha$. If $x \notin [0,1]^3$, then we can find an open ball including x and not intersecting C_α. For $x = (x_1, x_2, x_3) \in [0,1]^3$, $(x_1, 0, 0)$ is in at least one of the cones along $[0,1]$, say the one including A_n^α. Let $A_{m,n}^\alpha$ be the necklace preceding A_n^α, and $A_{n,k}^\alpha$ be the succeeding necklace. Since $x \notin C_\alpha$, $x \notin A_n^\alpha \cup A_{m,n}^\alpha \cup A_{n,k}^\alpha$. Since these are compact sets, $\varepsilon = \min(d(x, A_{m,n}^\alpha), d(x, A_n^\alpha), d(x, A_{n,k}^\alpha)) > 0$. Hence $B_\varepsilon(x) \cap C_\alpha = \emptyset$. Hence it is closed (thus compact).

Suppose $C \subset C_\alpha$ is a connected component, let $x \in C$. Then without loss of generality $x \in A_n^\alpha$ for some n. If $C \subset A_n^\alpha$, then C must be a one point set. Suppose for a contradiction that $y \neq x$ in C and in some other Antoine necklace. Then:

- either y is in Antoine necklace right next to A_n^α, so in one of $A_{m,n}^\alpha$, $A_{n,k}^\alpha$, say the second. Then there are open sets $U', V' \subset \mathbb{R}^3$ with $A^\alpha_{n,k} \subset U' \cup V'$, $A^\alpha_{n.k} \cap U' \cap V' = \emptyset$ and $(q^\alpha_n, 0, 0) \in U'$, $y \in V'$.

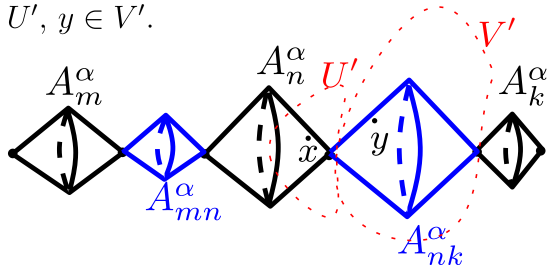


Figure 3.4: Disjoint open sets containing x and y, case I

- or there is at least one Antoine necklace between A^α_n and the Antoine necklace including y, say A^α_k. Without loss of generality, suppose $n <_\alpha k$ are consecutive. Then there are open sets $U', V' \subset \mathbb{R}^3$ with $A^\alpha_{n,k} \subset U' \cup V'$, $A^\alpha_{n,k} \cap U' \cap V' = \emptyset$ and $(q^\alpha_n, 0, 0) \in U'$, $(p^\alpha_k, 0, 0) \in V'$.

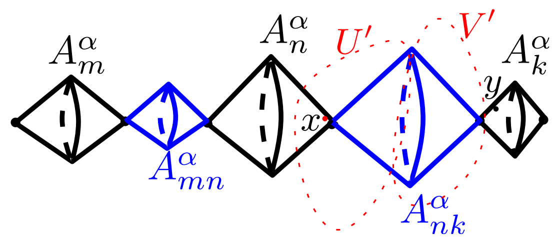


Figure 3.5: Disjoint open sets containing x and y, case II

Also in either case, U' does not intersect any Antoine necklace after $A^\alpha_{n,k}$ and V' does not intersect any Antoine necklace before $A^\alpha_{n,k}$. Take $U = U' \cup \{\tilde{x} \in \mathbb{R}^3 \mid \tilde{x}_1 < q^\alpha_n\}$ and $V = V' \cup \{\tilde{x} \in \mathbb{R}^3 \mid \tilde{x}_1 > p^\alpha_k\}$, then $x \in U$, $y \in V$, $C_\alpha \subset U \cup V$ and $U \cap V \cap C = \emptyset$. Hence C is not connected. Thus C must be a one-point set.

Claim 2. f is Borel.

To show f is Borel, it is enough to work with subbasic open sets in $\mathcal{C}(\mathbb{R}^3)$, which are: $B_1(U) = \{K \mid K \subset U\}$ and $B_2(U) = \{K \mid K \cap U \neq \emptyset\}$, where $U \subset \mathbb{R}^3$ is open.

We will show that $S_1 = f^{-1}(B_1(U))$ is closed and $S_2 = f^{-1}(B_2(U))$ open.

Consider $S_1 = \{\alpha \in LO \mid f(\alpha) \subset U\}$, so $S_1^c = \{\alpha \in LO \mid f(\alpha) \cap U^c \neq \emptyset\}$. For $\alpha \in S_1^c$, $f(\alpha) \cap U^c \neq \emptyset$, hence for some n, $U^c \cap A^\alpha_n \neq \emptyset$.

Now take the basic open set $M = \{\gamma \in LO \mid \alpha, \gamma \text{ agree on the order of } 1, \ldots, n\}$ in LO. Then $(p^\alpha_n, q^\alpha_n) = (p^\gamma_n, q^\gamma_n)$ for all $\gamma \in M$. In particular, $A^\gamma_n = A^\alpha_n$ for all $\gamma \in M$, hence $f(\gamma)$

meets U^c, and thus $M \subset S_1^c$ is a neighborhood of α. Thus S_1 is closed. (If $U^c \cap A^\alpha_{n,m} \neq \emptyset$, then take M to be linear orders that agree on the order of $1, \ldots, \max(n, m)$).

Consider $S_2 = f^{-1}(B_2(U)) = \{\alpha \in LO \mid f(\alpha) \cap U \neq \emptyset\}$. For $\alpha \in S_2$, $f(\alpha) \cap U \neq \emptyset$, hence for some n, $U \cap A^\alpha_n \neq \emptyset$.

Now take the basic open set $N = \{\gamma \in LO : \alpha, \gamma \text{ agree on the order of } 1, \ldots, n\}$ in LO. Then $(p^\alpha_n, q^\alpha_n) = (p^\gamma_n, q^\gamma_n)$ for all $\gamma \in N$. In particular, $A^\gamma_n = A^\alpha_n$ for all $\gamma \in N$, hence $f(\gamma)$ meets U, and thus $N \subset S_2$ is a neighborhood of α. Thus S_2 is open.

Claim 3. α, γ are equivalent if and only if C_α, C_γ are equivalent Cantor sets.

($\Rightarrow$) Suppose α, γ are equivalent linear orders. Then there is an order preserving map $g : \{p^\alpha_n \mid n \in \mathbb{N}\} \to \{p^\gamma_n \mid n \in \mathbb{N}\}$ (i.e. $n <_\alpha m \iff p^\alpha_n < p^\alpha_m \iff g(p^\alpha_n) < g(p^\alpha_m)$ and g is bijection). Now define $h : C_\alpha \to C_\gamma$ as follows:

$h|A^\alpha_n$ is homeomorphism of A^α_n and A^γ_k where $h(p^\alpha_n) = g(p^\alpha_n) = p^\gamma_k$, thus $h(q^\alpha_n) = q^\gamma_k$, and

$h|A^\alpha_{n,m}$ is homeomorphism of $A^\alpha_{n,m}$ and $A^\gamma_{k,l}$ where $h(p^\alpha_m) = g(p^\alpha_m) = p^\gamma_l$

Now h is a homeomorphism of the Cantor sets C_α and C_γ. The only problem can occur at the intersection points, but for a sequence $(x_n)_{n\in\mathbb{N}} \in C_\alpha$ converging to an intersection end point p^α_n, without loss of generality for infinitely many n, $x_n \in A^\alpha_n$, so $h(x_n) \in h(A^\alpha_n)$. h is a homeomorphism on A^α_n, hence $h(x_n)$ converges to $h(p^\alpha_n)$.

We can easily extend this homeomorphism to the cones including the Antoine necklaces, which in turn can be extended to the whole $\mathbb{R}^3$.

($\Leftarrow$) Suppose there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(C_\alpha) = C_\gamma$. Consider $g = h|\, \{p^\alpha_n \mid n \in \mathbb{N}\}$. Since h is a homeomorphism and the points p^α_n, q^α_n are different from the other points on the Antoine necklaces $A^\alpha_n, A^\alpha_{n,m}, A^\alpha_{n,k}$ (as well as the corresponding ones for γ), $g(\{p^\alpha_n \mid n \in \mathbb{N}\}) \subset \{p^\gamma_n, q^\gamma_n \mid n \in \mathbb{N}\}$. Moreover, it is a bijection.

Enough to show that: if $n <_\alpha m$ (so $p^\alpha_n < p^\alpha_m$) then $g(p^\alpha_n) < g(p^\alpha_m)$, hence γ is an equivalent order. Suppose not, then $g(p^\alpha_m) < g(p^\alpha_n)$ (they are not equal since bijection). Then we have two cases:

(i) For each $k \neq m, n$ either $k <_\alpha n$ or $m <_\alpha k$ (i.e. m, n are consecutive wrt α)

Then $h(A^\alpha_{n,m})$ is in the cone with end points $h(q^\alpha_n)$ and $g(p^\alpha_m)$ and should be right after $h(A^\alpha_n)$ but also right before $h(A^\alpha_m)$, which is not possible since $g(p^\alpha_m) < g(p^\alpha_n)$.

(*ii*) There is $k \in \mathbb{N}$ with $n <_\alpha k <_\alpha m$, then by (*i*), the image of Antoine necklace consecutive to A_n^α should be consecutive to $h(A_n^\alpha)$. Following each Antoine necklace in the middle of A_n^α and A_m^α one by one, we get $h(A_m^\alpha)$ should come after $h(A_n^\alpha)$, but $g(p_m^\alpha) < g(p_n^\alpha)$ means $h(A_m^\alpha)$ comes before.

Thus either case leads to a contradiction. $\square$

Now this tells us that the classification of all Cantor subsets is complicated, however the following question remains:

Question 1. Is Cantor sets classifiable by countable structures?

We know, by the work of Curtis and van Mill in [5], that the problem of classifying Cantor sets in $\mathbb{R}^3$ is the same as the problem of classifying open submanifolds of $\mathbb{R}^3$. Thus by the above theorem, we have:

Corollary. *Classifying open 3-manifolds is at least as complex as classifying countable groups.*

In comparison, the recently completed classification of compact 3-manifolds is a simpler classification.

4.0 ONE DIMENSIONAL COMPACT SUBSETS

The most interesting class in the space of all one dimensional compact subsets is the space of curves. This includes many important classes of continua, like dendrites, dendroids, regular and rational continua.

Let $\mathscr{C}(\mathbb{R}^N)$ denote the class of all curves in $\mathcal{K}(\mathbb{R}^N)$, $N \geq 2$. Then $\mathscr{C}(\mathbb{R}^N) \subset \mathbf{C}(\mathbb{R}^N)$ and it is a G_δ set (see Theorem 2.2.1).

In this chapter we will explore curves with additional connectedness properties, namely strongly arc connected (sac) curves. We will characterize 3-sac graphs, find the Borel complexity and show that the ω-sac curves cannot be characterized simpler than the definition. In the last section, we will look into the classification problem of dendroids and dendrites, and show that the classification of dendroids up to equivalence is provably more complex than classifying countable structures.

4.1 STRONGLY ARC-CONNECTED CURVES

A space X is *n–strongly arc connected (n–sac)* if for every distinct $x_1, \dots, x_n$ in X there is an arc $\alpha : [0,1] \to X$ such that $\alpha((i-1)/(n-1)) = x_i$ for $i = 1, \dots, n$ — in other words, the arc α 'visits' the points in order. Further, call a space ω–sac if it is n–sac for every n. Note that being 2-sac is the same as being arc-connected.

Lemma 4.1.1. *Let X be a space.*

If there is a finite F such that $X \setminus F$ is disconnected, then X is not $(|F|+2)$–sac.

Proof. If F is empty then X is disconnected and hence not 2–sac. So suppose F has n

elments, say $x_1, \ldots, x_n$, for $n \geq 1$. Let U and V be an open partition of $X \setminus F$. Pick x_{n+1} in U and x_{n+2} in V. Consider an arc α in X visiting $x_1, \ldots, x_n$, and then x_{n+1}. Then α ends in U and cannot enter V without passing through F. Thus no arc extending α can end at x_{n+2} — and X is not $n+2$–sac, as claimed. □

Corollary. *Let X be a space.*

(1) If there is an open set U with non–empty but finite boundary, then X is not $(|\partial U|+2)$–sac.

(2) No regular continuum is ω–sac.

(3) A continuum containing a free arc is not 4–sac.

(4) No compact continuous image of an interval is 4–sac.

Proof. (1) is simply a restatement of Lemma 4.1.1. Then (2) is immediate from (1). For (3), apply (1) to an open interval inside the free arc. While for (4) note that, by Baire Category, a compact continuous image of an interval contains a free arc, so apply (3). □

Call an arc α in a space X a *'no exit' arc* if every arc β containing the endpoints of α, and meeting α's interior must contain all of α.

Lemma 4.1.2. *If a space contains a no exit arc then it is not 4–sac.*

Proof. Let x_1 and x_2 be the endpoints of α. Pick x_3 and x_4 so that x_1, x_3, x_4, x_2 are in order along α. Suppose, for a contradiction, β is an arc visiting the x_i in order. Since x_3 and x_4 are in the interior of α, by hypothesis, β contains α. Now we see that if β enters the interior of α from x_1 then it visits x_3 before x_2. While if β enters the interior of α from x_2 it visits x_4 before x_3. Either case leads to a contradiction. □

Proposition 4.1.3. *No planar continuum is 4–sac.*

Proof. Let K be a plane continuum. If it is not arc connected then it is not 2–sac, so suppose K is arc connected. Pick $\mathbf{x}_-$ (respectively, $\mathbf{x}_+$) in K to have minimal x–coordinate (resp., maximal x–coordinate). If $\mathbf{x}_-$ and $\mathbf{x}_+$ have the same x–coordinate, then X is an arc, and so not 3–sac.

Otherwise, translating the mid point between $\mathbf{x}_-$ and $\mathbf{x}_+$ to the origin, shearing in the y–coordinate only to move $\mathbf{x}_-$ and $\mathbf{x}_+$ onto the x–axis, and then scaling, we can assume without loss of generality that $\mathbf{x}_- = (-1, 0)$, $\mathbf{x}_+ = (+1, 0)$ and $K \subseteq [-1, 1] \times \mathbb{R}$.

There is an arc α in K from $\mathbf{x}_-$ to $\mathbf{x}_+$. Some sub–arc, α', of α meets $\{-1\} \times \mathbb{R}$ and $\{+1\} \times \mathbb{R}$ in just one point (each). If for every x in $(-1, 1)$ the vertical line $\{x\} \times \mathbb{R}$ meets α' in just one point, then α' is a free arc, and K is not 4–sac, as claimed.

Otherwise there is an $x_0 \in (-1, 1)$ such that there are two distinct points $\mathbf{x}_3$ and $\mathbf{x}_4$ in $K \cap (\{x_0\} \times \mathbb{R})$. We can suppose $\mathbf{x}_3$ has minimal y–coordinate, y_3, while $\mathbf{x}_4$ has maximal y–coordinate, y_4. Assume, for a contradiction, that there is an arc β from $\mathbf{x}_1 = \mathbf{x}_-$ to $\mathbf{x}_4$ visiting $\mathbf{x}_2 = \mathbf{x}_+$ and $\mathbf{x}_3$ in order. Let β_1 be the sub–arc from $\mathbf{x}_1$ to $\mathbf{x}_2$ and β_3 be the sub–arc from $\mathbf{x}_3$ to $\mathbf{x}_4$. Note that $\beta_1 \cap \beta_3 = \emptyset$, and so β_1 meets $\{x_0\} \times \mathbb{R}$ only inside $\{x_0\} \times (y_3, y_4)$. Hence the line $L = (-\infty, -1) \times \{0\} \cup \beta_1 \cup (+1, +\infty) \times \{0\}$ splits the plane into two disjoint open sets, U_3 containing $\mathbf{x}_3$, and U_4 containing $\mathbf{x}_4$. However β_3 is supposed, on the one hand, to be an arc from $\mathbf{x}_3$ to $\mathbf{x}_4$, and so must cross L, and on the other hand, is forced to be disjoint from each part of L: β_1 (by choice of β) and both $(-\infty, -1) \times \{0\}$ and $(+1, +\infty) \times \{0\}$ (since $K \subseteq [-1, 1] \times \mathbb{R}$) — contradiction. □

So many continua are not even 4–sac. But some curves are ω–sac.

Example 5. The Menger curve is ω–sac.

Since graphs are never 4-sac, regular continua are not ω-sac and the Menger curve is not rational, the following questions are natural.

Question 2. 1. Which graphs are 3-sac? Can we characterize them?

2. Is there a rational ω-sac curve? Can we characterize ω-sac continua?

The circle and theta curve are 3-sacs. Three points, p, q, r, on the theta curve below in figure 4.1 do not lie on any simple closed curve.

It turns out, these characterize the 3-sac graphs:

Proposition 4.1.4. *Suppose G is a finite graph. Then the following are equivalent:*

1. *G is 3-sac.*
2. *G has no cut point.*

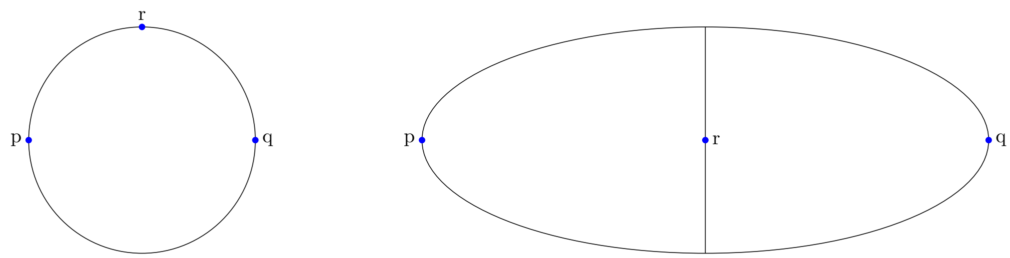


Figure 4.1: Circle and Theta curve with points p, q, r

3. Any three points in G are contained in either a simple closed curve or a theta curve.

Notation: If $\alpha : [0,1] \to G$ is an arc, then for $t < l, t, l \in [0,1]$, $\alpha_{t,l}$ will denote a sub-arc of α from $\alpha(t)$ to $\alpha(l)$.

Lemma 4.1.5. *Suppose G is a finite graph with possibly a vertex taken out. Suppose K and L are non-empty, disjoint (connected) subgraphs of G. Then there is an arc $\alpha : [0,1] \to G$ such that $\alpha(0) \in K$, $\alpha(1) \in L$, $\alpha((0,1)) \cap (K \cup L) = \emptyset$.*

Proof. Pick $a \in K, b \in L$. Since G is arc-connected there is an arc $\beta : [0,1] \to G$ such that $\beta(0) = a$, $\beta(1) = b$. Since K, L, G are finite graphs, there are vertices $c = \beta(t_1) \in L$ and $d = \beta(t_0) \in K$, such that $\beta([0,t_1)) \cap L = \emptyset$ and $\beta((t_0,t_1)) \cap K = \emptyset$. Since K, L are disjoint $t_0 \neq t_1$. If s is a homeomorphism from $[0,1]$ to $[t_0, t_1]$ mapping 0 to t_0, then $\alpha = \beta_{t_0,t_1} \circ s$ is desired arc. □

Lemma 4.1.6. *Suppose G is a finite graph without any cut points. Suppose K and L are non-empty, disjoint (connected) subgraphs of G. Then there exist disjoint arcs α, β in $cl(G-(K \cup L))$ with endpoints a_1, a_2 and b_1, b_2 respectively such that $a_1, b_1 \in K$ and $a_2, b_2 \in L$. If K (L) contains only one point then $a_1 = b_1$ ($a_2 = b_2$).*

Proof. Induction on n, number of edges in $cl(G - (K \cup L))$.

If $n = 1$ then the only edge in $cl(G - (K \cup L))$ will be connecting K and L and any point on it will be a cut point. Hence $n > 1$.

If $n = 2$, suppose α, β are the two edges in $cl(G - (K \cup L))$ with endpoints a_1, a_2 and b_1, b_2 respectively. Then without loss of generality we have the following cases:

- $a_1 \in K, a_2 = b_1, b_1 \notin K \cup L, b_2 \in L$ and all points on α and β are cut points.
- $a_1 \in K, a_2, b_1, b_2 \in L$ and any point on α is a cut point.
- $a_1, b_1 \in K, a_2 \in L, b_2 \notin L$, then there is a cut point on α.
- $a_1, b_1 \in K, a_2, b_2 \in L$ and without loss of generality $a_1 = b_1$. If K is a point then Lemma 2 holds otherwise a_1 is a cut point.
- $a_1, b_1 \in K, a_2, b_2 \in L$ and a_1, b_1, a_2, b_2 are distinct points. Then we have two disjoint arcs starting at K and ending at L.

So Lemma 4.1.6 holds for $n = 2$.

Now suppose the lemma holds for n and $cl(G - (K \cup L))$ has $n + 1$ edges. Then there are the following cases:

1. Each edge starts in K and ends in L: if there are two edges that start at distinct points and end at distinct points then lemma holds. If without loss of generality all edges end at the same point of L, say c, then either L is a point and Lemma 4.1.6 holds or c is a cut point.
2. There is an edge, say γ that starts at $c \in K$ and ends at $d \notin L$ (so it does not intersect L). Then let $K \cup \gamma = K'$ and $\gamma \cup \alpha' = \gamma'$, which is an arc starting at c and ending at a_2. Then K', L satisfy hypothesis of Lemma 4.1.6 and $cl(G - (K' \cup L))$ has n edges. Hence by induction hypothesis there exists a pair of disjoint arcs α', β' with endpoints a'_1, a_2 and b'_1, b_2 such that $a'_1, b'_1 \in K'$ and $a_2, b_2 \in L$ and if L contains only one point then $a_2 = b_2$. Subcases:

   2.1. If $a'_1, b'_1 \in K$ then let $\alpha = \alpha', \beta = \beta'$ and Lemma 4.1.6 holds.

   2.2. If without loss of generality $a'_1 \in K' - K, b'_1 \in K - \{c\}$ then only possibility is $a'_1 = d$. If we let $\alpha = \gamma', \beta = \beta'$, Lemma 4.1.6 holds.

   2.3. Say $a'_1 = d, b'_1 = c$. Since c is not a cut point $G - \{c\}$ is arc-connected (G is a finite graph). Apply Lemma 4.1.5 to $K - \{c\}$ and $(L \cup \gamma' \cup \beta') - \{c\}$ to obtain an arc $\delta : [0, 1] \to G - \{c\}$ such that $\delta(0) \in K - \{c\}, \delta(1) \in (L \cup \gamma' \cup \beta') - \{c\}, \delta((0, 1)) \cap (K \cup L \cup \gamma' \cup \beta') = \emptyset$.

Now if $\delta(1) \in L$, let $\alpha = \delta$ and $\beta = \beta'$ or α' whichever does not have $\delta(1)$ as an endpoint, Lemma 4.1.6 holds.

If $\delta(1) \in \beta'$, hence $\delta(1) = \beta'(t)$ for some $t \in (0, 1]$. Let $\alpha = \gamma', \beta = \delta \cup \beta'_{t,1}$ and Lemma 4.1.6 holds.

If $\delta(1) \in \gamma'$, hence $\delta(1) = \gamma'(t)$ for some $t \in (0, 1]$. Let $\alpha = \delta \cup \gamma'_{t,1}, \beta = \beta'$ and Lemma 4.1.6 holds.

□

Proof. (Proposition 4.1.4)

"(1) ⇒ (2)" follows immediately from Lemma 4.1.1.

"(2) ⇒ (3)" let $p, q, r \in G$ be any three points. Apply Lemma 4.1.6 to $\{p\} = K, \{q\} = L$ and resulting $\alpha \cup \beta$ gives a simple closed curve containing p, q. If $r \in \alpha \cup \beta$ then all three lie on a simple closed curve. If $r \notin \alpha \cup \beta$ then apply Lemma 4.1.6 to $K = \alpha \cup \beta, L = \{r\}$ to obtain two disjoint arcs connecting $\alpha \cup \beta$ and r, say α', β' then $\alpha \cup \beta \cup \alpha' \cup \beta'$ is a theta curve that contains all three of p, q, r.

"(3) ⇒ (1)" obvious. □

Lemma 4.1.7 (Finite Gluing)**.** *If X and Y are $2n-1$-sac, and Z is obtained from X and Y by identifying pairwise $n-1$ points of X and Y, then Z is n-sac (but not $n+1$–sac by Lemma 4.1.1).*

Proof. Pick any $z_1, z_2, \ldots, z_n$ in Z. For each i, if $z_i \in X - Y$ and $z_{i+1} \in Y - X$ or $z_i \in Y - X$ and $z_{i+1} \in X - Y$, pick $z_{(i,i+1)} \in (X \cap Y) - \{z_1, z_2, \ldots, z_n, z_{(1,2)}, z_{(2,3)}, \ldots, z_{(i-1,i)}\}$ (if these $z_{(1,2)}, z_{(2,3)}, \ldots, z_{(i-1,i)}$ were picked). This is possible since $|X \cap Y| = n - 1$. Let $\mathcal{Z}$ be a sequence of z_j's with $z_{(i,i+1)}$'s inserted between z_i and z_{i+1} whenever they exist. And let $\mathcal{Z}_{\mathcal{X}}$ be a sequence derived from $\mathcal{Z}$ by deleting terms that do not belong to X. Define $\mathcal{Z}_{\mathcal{Y}}$ similarly. Since elements of $\mathcal{Z}_{\mathcal{X}}$ come either from $\{z_1, z_2, \ldots, z_n\}$ or from $X \cap Y$, $|\mathcal{Z}_{\mathcal{X}}| \leq 2n-1$. Similarly, $|\mathcal{Z}_{\mathcal{Y}}| \leq 2n - 1$. Let β be an arc in X going through elements of $\mathcal{Z}_{\mathcal{X}}$ in order and γ be an arc in Y going through elements of $\mathcal{Z}_{\mathcal{Y}}$ in order. Let $a_1, a_2, \ldots, a_k$ be $z_{(1,2)}, z_{(2,3)}, \ldots, z_{(n-1,n)}$ whenever they exist, respectively. Without loss of generality, suppose $z_1 \in X$. Define α to be the arc consisting of the following parts:

- part of β from z_1 to a_1;
- part of γ from a_1 to a_2;
- part of β from a_2 to a_3 ...
- part of β or γ (depending on whether k is even or odd) from a_k to z_n.

□

Now we turn to our second question, and give an example of a rational ω-sac curve.

Theorem 4.1.8. *There is a rational continuum which is ω–sac.*

Proof. Write $B(\mathbf{x}, r)$ for the open disk in the plane of radius r centered at $\mathbf{x}$. Write $S(\mathbf{x}, r)$ for the boundary circle of $B(\mathbf{x}, r)$. Pick any sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ in $(0, 1)$ increasing to 1. Let $c_0 = 0$, $r_0 = x_0$ and $c_n = (x_n + x_{n-1})/2$, $r_n = (x_n - x_{n-1})/2$ for $n \geq 1$. Let θ be rotation of the plane by 90^o clockwise.

Let $U = \bigcup_{i=0}^{3} \theta^i \left(\bigcup_{n=0}^{\infty} B((c_n, 0), r_n)\right)$, and $T = [-1, +1]^2 \setminus U$.

Let S be the geometric boundary of T, so $S = \bigcup_{i=0}^{3} \theta^i \left(\bigcup_{n=0}^{\infty} S((c_n, 0), r_n)\right)$. Let S^- and S^+ be the two circles in S immediately to the left and right of the center circle, i.e. $S^- = S((-x_1, 0), r_1)$ and $S^+ = S((x_1, 0), r_1)$. For i=0 (respectively, i=1) pick a two sided sequence of points, $(p^-_{m,i})_{m\in\mathbb{Z}}$, on the top (respectively, bottom) edge of S^- converging on the left to $(-x_1 - r_1, 0)$ (the leftmost point of S^-) and on the right to $(-x_1 + r_1, 0)$ (the rightmost point of S^-). Find a corresponding pair, $(p^+_{m,i})_{m\in F}$ for $i = 0$, and 1 of double sequences on the top and bottom edges of S^+ converging to the leftmost and rightmost points of S^+. Let $T_i = \theta^i ([0, 1]^2 \cap T)$ for $i = 0, 1, 2, 3$. Note that each T_i is a topological rectangle with natural 'corners' and 'midpoints' of the sides.

Let $X_1 = T$, $R_{(i)} = T_i$ and $S_1 = S$. Let $h_{(0)}$ be a homeomorphism of $[-1, +1]^2$ with T_0 carrying top–right corner to top–right corner etc, and midpoints to midpoints. Let $h_{(i)} = \theta^i \circ h_{(0)}$ for $i = 1, 2, 3$.

Inductively, suppose we have continuum X_n, geometric boundary S_n, and for each $\sigma \in \Sigma_n = \{0, 1, 2, 3\}^n$ a rectangle R_σ and a homeomorphism h_σ of $[-1, +1]^2$ with R_σ. Fix a σ for a moment. Then R_σ has four subrectangles $h_\sigma(T_i)$. For $i = 0, 1, 2, 3$ let $h_{\sigma\frown i}$ be a

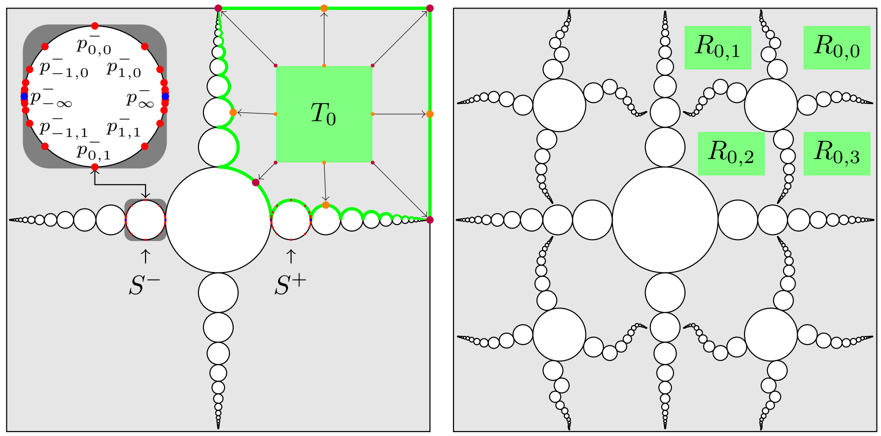


Figure 4.2: T and X_2

homeomorphism of $[-1,+1]^2$ with $h_\sigma(T_i)$ taking corners to corners etcetera. Let $R_{\sigma\frown i} = h_\sigma(T_i) = R_\sigma \setminus h_{\sigma\frown i}(U)$. Let $X_{n+1} = \bigcup_{\sigma\in\Sigma_n}\bigcup_{i=0}^{3} R_{\sigma\frown i} = \bigcup_{\sigma\in\Sigma_{n+1}} R_\sigma$. Let S_{n+1} be the natural geometric boundary.

Let $X = \bigcap_n X_n$. Then X is a variant of Charatonik's description of Urysohn's locally connected, rational continuum in which every point has countably infinite order, see [4]. Thus X is rational (and locally connected). Since it is planar it is not 4–sac. Note that each $R_\sigma \cap X$ has a countable boundary contained in the sides of R_σ. Call a side of R_σ 'finite' if it contains only finitely many boundary points. A side containing infinitely many boundary points, is said to be 'infinite'.

For each n and σ in Σ_n, there are two circles, $h_\sigma(S^-)$ and $h_\sigma(S^+)$. Identify, for all $m \in \mathbb{Z}$ and $i \in \{0,1\}$, the points $h_\sigma(p^-_{m,i})$ and $h_\sigma(p^+_{m,i})$ (creating a 'rational bridge' between the circles). Note that the diameters of the circles shrink to zero with n. It follows that the resulting quotient space, Y, is a locally connected, rational continuum. We show that Y is ω–sac.

Fix distinct points $x_1,\ldots,x_n$ in Y. The diameters of the rectangles, R_σ for $\sigma\in\Sigma_m$, shrink to zero with m, so we can find an N such that if $i\neq j$, $x_i\in R_\sigma$, and $x_j\in R_\tau$ where $\sigma,\tau\in\Sigma_N$ then R_σ and R_τ are disjoint. For each i, let R_i be the unique R_σ containing x_i.

Subdivide the square $r=[-1,+1]^2$ into four subsquares $r_{(i)} = \theta^i\left([0,1]^2\right)$. And continue subdividing to get a final subdivision of $[-1,+1]^2$ into subsquares r_σ for $\sigma \in \Sigma_N$. Note that two squares r_σ and r_τ are adjacent if and only if the corresponding rectangles R_σ and R_τ are adjacent. For each σ in Σ_N, consider R_σ. It has four sides, at most two are 'finite' sides. For each finite side remove the line segment in r which is the corresponding side in r_σ. The result r' is an open, connected subset of the plane. It follows that r' is ω–sac. Hence there is an arc α' which visits the interior of the squares r_i in order: $r_1, r_2, \ldots, r_n$ (indeed we can suppose α' visits the centers of the r_i in turn). Further, we can suppose that α' consists of a finite union of horizontal or vertical line segments of the form $\{p/q\} \times J$ or $J \times \{p/q\}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and J is a closed interval. Let M be a common denominator of all the denominators (q's) used. Then α' is an arc on the grid $r' \cap \left(\left(\bigcup_{p\in\mathbb{Z}} \{p/M\} \times \mathbb{R}\right) \cup \left(\bigcup_{p\in\mathbb{Z}} \mathbb{R} \times \{p/M\}\right)\right)$.

Consider X_1. There is a connected chain of circles, V_0, in X_1 from the bottom edge to the top, and a connected chain of circles, H_0, from the left side to the right. Note that V_0 and H_0 are in X. Now consider X_2. There is a connected chain of circles to the right of V_0 from the top edge to a circle in H_0, and another to the right of V_0 from the bottom edge to a circle in H_0. By construction, both chains end at the *same circle* of H_0. Call the union of these two chains, along with the circle they connect to, V_1. It is a vertical connected chain of circles from the top edge to the bottom. Similarly, there is a vertical connected chain of circles, V_{-1} from the top edge to the bottom, lying to the left of V_0. Further there are two horizontal connected chains of circles, H_1 and H_{-1}, above and, respectively, below, H_0. Observe that V_0 and $V_{\pm 1}$ are disjoint, as are H_0 and $H_{\pm 1}$. Together these six chains form a three–by–three 'grid' in X. Repeating, we find a 'grid' of horizontal, $H_{\pm n}$ and vertical, $V_{\pm n}$, chains, all in X, where the $H_{\pm n}$ converges to the left and right sides, $\{\pm 1\} \times [-1,1]$ and $V_{\pm n}$ converges to the top and bottom edges $[-1,1] \times \{\pm 1\}$.

Now consider a rectangle R_σ for some σ in Σ_N. It has at least two 'infinite' sides. For concreteness let us suppose that the bottom and right sides of R_σ are infinite, with the limit point on the bottom edge being to the right, and the limit along the right edge being at the top (all other cases are very similar). The vertical chains, V_n in X, for $n \in \mathbb{N}$, have analogues in R_σ. By construction, each V_n meets the bottom edge of R_σ in an arc of a circle whose ends are points in the R_τ 'below' R_σ. Extend V_n to include this arc. Repeat at the top edge, if it is infinite. Apply the same procedure to the horizontal chains, H_n for $n \in \mathbb{N}$.

The horizontal chains, and respectively the vertical chains, remain disjoint. Note that, by construction, if R_τ is the rectangle 'below' R_σ, then the nth vertical chain in R_σ connects to the nth vertical chain in R_τ (and similarly for the rectangle to the right of R_σ). If x_i is in R_σ but not on the geometric boundary of R_σ, then let P_i be sufficiently large that x_i is to the left of V_{P_i} and below H_{P_i}.

Now let P be the maximum of the P_i. Return to an individual rectangle, R_σ, as in the previous paragraph. Take the union of the vertical chains, $V_P, \ldots, V_{P+M}$, and the horizontal chains, $H_P, \ldots, H_{P+M}$. Take the union now over all σ in Σ_N. This gives a 'grid', G, naturally containing an isomorphic copy of the grid G' in r'. Think of the grid, G', as a graph, and α' as an edge arc in this graph. Then we can realize the arc α' in G' as a connected chain of circles in X. Evidently (by traveling along the 'top' or 'bottom' edges of the circles in the union) we can extract an arc, α_*, contained in this union. The arc α_* visits the R_i in order. Note that α_* is not (necessarily) an arc in Y, but it can easily be modified to be so, call this arc, α_0.

To complete the proof, we modify α_0, to another arc α in Y, which visits the points x_i in order. As α_0 visits the R_i in turn, there are sub–arcs β_i of α_0, where β_i comes before β_j if $i < j$, such that β_i crosses from one infinite edge of R_i to another (along the 'grid' G inside R_i). We will replace β_i in α_0 by another sub–arc, visiting x_i, contained inside R_i, with the same start and end as β_i, but otherwise disjoint from the 'grid' G. Doing this for all i, gives the arc α in Y.

Fix i. Again for concreteness, orient $R_i = R_\sigma$ as above. Suppose that β_i enters R_i at y_i, a point on the bottom edge, and exits at z_i, a point on the right edge. Pick Q in $\mathbb{N}$ sufficiently large that, V_Q is to the right of the rightmost vertical chain in $G \cap R_i$, above the highest horizontal chain in $G \cap R_i$ (i.e. $Q > P + M$) and if x_i is not on the geometric boundary of R_i, the vertical chain V_{-Q} is to the left of x_i and the horizontal chain H_{-Q} is below x_i. The union of $V_{\pm Q}$ and $H_{\pm Q}$ contains an obvious 'ring', a connected cycle of circles, just interior to the geometric boundary of R_i. Observe that this ring meets each arc component of $\alpha_0 \cap R_i$ in two circles, which are bridges. Select a simple closed curve (in Y), S, contained in this ring, which connects with β_i at two points (one, call it y_i', near y_i, and another, call it z_i', near z_i), but which uses the bridges to prevent intersection with any other (arc component of) α_0.

We will modify S so that it visits x_i. If this is possible then either the arc 'travel along β_i from y_i to y_i', then *clockwise* along S until we reach z_i', followed by traveling along the arc β_i to z_i'; or the arc obtained by following S *anti–clockwise*, is the required modification of β_i.

Two cases arise. If x_i is on the geometric boundary of R_i, we can find two arcs starting at x_i, and otherwise disjoint, both meeting S (but disjoint from the grid G). The required modification of S is now obvious (follow S, then the first arc met, to x_i, back to S along the second arc, and finish following S). If x_i is not on the geometric boundary, then it is to the left and below the grid. It is also in some rectangle, R_τ, τ from some $\Sigma_{N'}$ where $N' > N$, where R_τ is disjoint from the geometric boundary of R_i. Following S anticlockwise we can get to a point, a_i, below the lowest horizontal line of the grid, but above and to the left of the top–left corner of R_τ. Following S clockwise we can get to a point, b_i, left of the leftmost vertical line of the grid $G \cap R_i$, but below and to the right of the bottom–right corner of R_τ. We can now find disjoint arcs from a_i to the top–left corner of R_τ, and from b_i to the bottom–right corner of R_τ. And these can be extended to disjoint (except at x_i) arcs a_i to x_i and b_i to x_i. Again, using these arcs, we can modify S to detour through x_i. $\square$

4.2 COMPLEXITY OF 3-SAC GRAPHS AND ω-SAC CONTINUA

In this section we will examine the complexity of 3-sac graphs and ω-sac rational curves.

4.2.1 3-SAC GRAPHS

Lemma 4.2.1. *Let $\mathcal{G}$ be any collection of graphs and $N \in \mathbb{N}$ a fixed number. Then $H(\mathcal{G})$, the set of all subcontinua of I^N homeomorphic to some member of $\mathcal{G}$, is $\mathbf{\Pi}^0_3$–hard and in the difference hierarchy $D_2(\mathbf{\Sigma}^0_3)$.*

Proof. That $H(\mathcal{G})$ is $\mathbf{\Pi}^0_3$–hard is immediate from Theorem 7.3 of [3]. It remains to show it is in $D_2(\mathbf{\Sigma}^0_3)$.

For spaces X and Y, write $X \leq Y$ if X is Y*–like*, which means that for each $\varepsilon > 0$ there is a continuous map $f : X \to Y$ such that f is onto and $\{x \in X \mid f(x) = y\}$ has diameter less than ε for each $y \in Y$. We also write $X < Y$ if $X \leq Y$ but $Y \not\leq X$ and $X \sim Y$ if $X \leq Y$ and $Y \leq X$. Further write $\mathcal{L}_X = \{Y : Y \leq X\}$ and $Q(X) = \{Y : X \sim Y\}$.

Up to homeomorphism there are only countably many graphs. So enumerate $\mathcal{G} = \{G_m : m \in \mathbb{N}\}$. According to Theorem 1.7 of [3], for a graph G and Peano continuum, P, we have that P is G–like if and only if P is a graph obtained from G by identifying to points disjoint (connected) subgraphs. For a fixed graph G, then, there are, up to homeomorphism, only finitely many G–like graphs. For each G_m in $\mathcal{G}$ pick graphs $G_{m,i}$ for $i = 1, \ldots, k_m$ such that each $G_{m,i}$ is G–like but G is not $G_{m,i}$–like, i.e. $G_{m,i} < G$, and if G' is a graph such that $G' < G$ then for some i we have $H(G') = H(G_{m,i})$.

For a graph G, $H(G) = Q(G)$ ([14]). Hence, writing $\mathcal{P}$, for the class of Peano continua, we have that $H(\mathcal{G}) = \bigcup_m Q(G_m) = \mathcal{P} \cap (\bigcup_m R_m)$, where $R_m = \mathcal{L}_{G_m} \setminus \bigcup_{i=1}^{k_m} \mathcal{L}_{G_{m,i}} = \mathcal{L}_{G_m} \cap \left(C(I^N) \setminus \bigcup_{i=1}^{k_m} \mathcal{L}_{G_{m,i}}\right)$.

By Corollary 5.4 of [3], for a graph G, the set $\mathcal{L}_G$ is $\mathbf{\Pi}^0_2$. Hence each R_m, as the intersection of a $\mathbf{\Pi}^0_2$ and a $\mathbf{\Sigma}^0_2$, is $\mathbf{\Sigma}^0_3$, and so is their countable union. Since $\mathcal{P}$ is $\mathbf{\Pi}^0_3$, we see that $H(\mathcal{G})$ is indeed the intersection of a $\mathbf{\Pi}^0_3$ set and a $\mathbf{\Sigma}^0_3$ set. □

Proposition 4.2.2. *Let SG_3 be the set of subcontinua of I^N which are 3–sac graphs. Then SG_3 is $D_2(\mathbf{\Sigma}^0_3)$–complete.*

Proof. According to Lemma 4.2.1 SG_3 is $D_2(\mathbf{\Sigma}^0_3)$, so it suffices to show that it is $D_2(\mathbf{\Sigma}^0_3)$–hard. To show that SG_3 is $D_2(\mathbf{\Sigma}^0_3)$–hard it suffices to show that there is a continuous map $F : \left(2^{\mathbb{N}\times\mathbb{N}}\right)^2 \to C(I^N)$ such that $F^{-1}(SG_3) = S^*_3 \times P_3$. We do the construction for $N = 2$. Since $\mathbb{R}^2$ embeds naturally in general $\mathbb{R}^N$, the proof obviously extends to all $N \geq 2$.

For x, y in $\mathbb{R}^2$, let $\mathrm{Cl}\, xy$ be the straight line segment from x to y. Set $O = (0, 0)$, $T = (3, 1)$, $B_1 = (1, 0)$, $B_2 = (4/3, 0)$, $B_3 = (5/3, 0)$, $B_4 = (2, 0)$ and $T_1 = (1, 1)$, $T_2 = (4/3, 1)$, $T_3 = (5/3, 1)$, $T_4 = (2, 1)$. Let $K_0 = \mathrm{Cl}\, OB_4 \cup \mathrm{Cl}\, B_4T \cup \mathrm{Cl}\, TT_1 \cup \mathrm{Cl}\, T_1O \cup \cup \mathrm{Cl}\, B_2T_2 \cup \mathrm{Cl}\, B_3T_3$. Then K_0 is a 3–sac graph. Define $b_j = (1/j, 0)$, $t_j = (1/j, 1/j)$, $t^k_j = (1/j, 1/j - 1/(kj))$ and $s^k_j = (1/j - 1/(kj(j+1)), 0)$. Then $K_J = K_0 \cup \bigcup_{j=1}^{J} \mathrm{Cl}\, b_jt_j$ — for each J — is also a 3–sac graph.

Let K'_0 be K_0 with the interior of the line from O to B_1, and the interior of the line from T_4 to T, deleted.

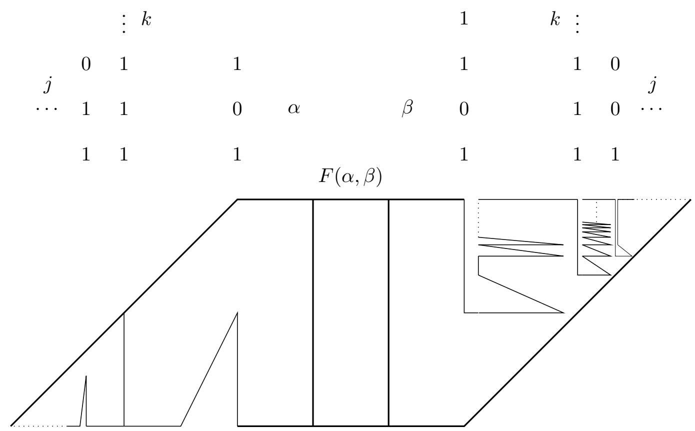


Figure 4.3: Graph of $F(\alpha, \beta)$

We now define F at some α and β in $2^{\mathbb{N}\times\mathbb{N}}$. Fix j. If $\alpha(j,k) = 1$ for all k, then let $R_j = \mathrm{Cl}\, b_j t_j \cup \mathrm{Cl}\, b_j b_{j+1}$. Otherwise, let $k_0 = \min\{k : \alpha(j,k) = 0\}$, and let $R_j = \mathrm{Cl}\, b_j t_j^{k_0} \cup \mathrm{Cl}\, t_j^{k_0} s_j^{k_0} \cup \mathrm{Cl}\, s_j^{k_0} b_{j+1}$.

For any j, k set $p_j = 3 - 1/j$, $q_j^k = 1 - 1/(j+k)$, $\ell_j = p_j + (1/8)(p_{j+1} - p_j)$ and $r_j = p_j + (7/8)(p_{j+1} - p_j)$. Fix j. Define

$$
\begin{aligned}
S_j &= \mathrm{Cl}\,(p_j,1)(p_j,q_j^1) \cup \mathrm{Cl}\,(p_j,q_j^1)(\ell_j,q_j^1) \cup \mathrm{Cl}\,(\ell_j,1)(p_{j+1},1) \\
&\cup \bigcup\{\mathrm{Cl}\,(\ell_j,q_j^k)(\ell_j,q_j^{k+1}) : \beta(j,k) = 0\} \\
&\cup \bigcup\{\mathrm{Cl}\,(\ell_j,q_j^k)(r_j,q_j^k) \cup \mathrm{Cl}\,(r_j,q_j^k)(\ell_j,q_j^{k+1}) : \beta(j,k) = 1\}.
\end{aligned}
$$

Let $F(\alpha,\beta) = K_0' \cup \bigcup_j (R_j \cup S_j)$. Then it is straightforward to check F maps $\left(2^{\mathbb{N}\times\mathbb{N}}\right)^2$ continuously into $C([0,4]^2)$.

Take any α. For any j, the set R_j connects the bottom edge $\mathrm{Cl}\, OB_1$ with the diagonal edge $\mathrm{Cl}\, OT_1$ if $\alpha(j,k) = 1$ for all k, and otherwise is an arc from b_j to b_{j+1}. Hence $\bigcup_j R_j$ is a free arc from B_1 to O if α is in S_3^*, and otherwise can't be a subspace of a graph (because it contains infinitely many points of order 3).

Take any β. For any j, S_j is an arc from $(p_j,1)$ to $(p_{j+1},1)$ if $\beta(j,k) = 0$ for all but finitely many k, but contains a 'topologists sine curve' if $\beta(j,k) = 1$ for infinitely many k. Thus $\bigcup_j S_j$ is a free arc from T_4 to T if β is in P_3, and otherwise can't be a subspace of a graph (because it contains a 'topologists sine curve').

Hence if (α, β) is in $S_3^* \times P_3$, $F(\alpha, \beta)$ is homeomorphic to some K_J, which in turn means it is a graph which is 3–sac. On the other hand, if either α is not in S_3^* or β is not in P_3, then $F(\alpha, \beta)$ contains subspaces which can't be subspaces of a graph — and so is not a graph. Thus $F^{-1}(SG_3) = S_3^* \times P_3$ as required. □

4.2.2 ω-SAC CONTINUA

From now on we will look into ω-sac curves. First we will define examples of rational ω-sac curves, then we will show that the set of such curves is very complex - not Borel.

We will build examples of spaces by laying out 'tiles'. A *'tile'* is simply any space T which is (i) a subspace of the solid square pyramid in $\mathbb{R}^3$ with base $S = [-1, +1]^2 \times \{0\}$ and vertex at $(0, 0, 1)$ (so it has height 1) and (ii) contains the four corner points of the base, (i, j) for $i, j = \pm 1$. Call the intersection of a tile T with S, the base of T. Call the intersection of T with the boundary $B = ([-1, 1] \times \{-1, 1\} \times \{0\}) \cup (\{-1, 1\} \times [-1, 1] \times \{0\})$ of the base S, the boundary of T. Call the point $(-1, 1, 0)$ the top–left corner of the base.

Lemma 4.2.3. *There are (homeomorphic) subspaces T_0 and T_1 of $[-1, +1]^2 \times \mathbb{R}$ such that: (i) T_0 and T_1 are ω-sac rational curves, (ii) T_0 and T_1 are contained in the pyramid with base $[-1, +1]^2 \times \{0\}$ and with height 1, (iii) T_0 contains the boundary of the square $[-1, +1]^2 \times \{0\}$, and (iv) the intersection of T_1 and the boundary of the square $[-1, +1]^2 \times \{0\}$ is $(A \times \{-1, 1\} \times \{0\}) \cup (\{-1, 1\} \times A \times \{0\})$ where A is a sequence on $[-1, 0]$ converging to 0 along with -1 and 0.*

Proof. The example, Y, of an ω-sac rational curve given in Theorem 4.1.8 is derived from a space X. This space X is a subspace of $[-1, +1]^2$. We may suppose that X is in fact a subspace of the square $S = [-1, +1]^2 \times \{0\} \subseteq \mathbb{R}^3$. The space Y is obtained from X by identifying a sequence of pairs of double sequences. These double sequences all are disjoint from the boundary, B, of the square S, and the diameters and distance between pairs of sequences converges to zero. This identification process can be repeated in $(-1, +1)^2 \times \mathbb{R}$, keeping the boundary, B, of the square, S, fixed, to get a space T_0' homeomorphic to Y. Applying a homeomorphism of $[-1, +1]^2 \times \mathbb{R}$ fixing B, the boundary of the square, and changing only the z–coordinates, to T_0', we get a space T_0, also homeomorphic to Y and containing B, and which is contained in the pyramid with base $[-1,+1]^2 \times \{0\}$ and height 1.

Scaling $\mathbb{R}^3$ around the center point of the the base square, S, we can shrink T_0 away from the boundary B of S and still have it inside the required pyramid. Instead of doing this transformation, shrink T_0 while keeping fixed the set $(A \times \{-1,1\} \times \{0\}) \cup (\{-1,1\} \times A \times \{0\})$. This gives T_1. $\square$

Let X be a space and A an infinite subset. We say that X is *ω–sac$^+$ (with respect to A)* if for any points $x_1, \dots, x_n$ in X there is an arc α in X visiting the x_i in order, such that α meets A only in a finite set. Observe that if X is ω–sac$^+$ with respect to A, and A' is an infinite subset of A, then X is ω–sac$^+$ with respect to A'.

Lemma 4.2.4 (ω–Gluing)**.** *Let $Z = X \cup Y$, where X, Y and $A = X \cap Y$ are infinite. If X is ω–sac$^+$ with respect to A, and Y is ω–sac, then Z is ω–sac.*

Proof. Take any finite sequence of points $z_1, \dots, z_N$ in Z. By adding points to the start and end of the sequence, if necessary, we can suppose that z_0 and z_N are in X. Group the sequence, $z_1, \dots, z_{n_1}, z_{n_1+1}, \dots, z_{n_2}, \dots, z_{n_{k-1}}, z_{n_{k-1}+1}, \dots, z_{n_k}$, where $z_1, \dots, z_{n_1}$ are in X, $z_{n_1+1}, \dots, z_{n_2}$ are in $Y \setminus X$, and so on, until $z_{n_{k-1}+1}, \dots, z_{n_k} = z_N$ are in X. Pick $t_1^\pm, \dots, t_k^\pm$ in $A \setminus \{z_i\}_{i \leq N}$.

Using the fact that X is ω–sac$^+$, pick arc α^- in X visiting in order, $z_1, \dots, z_{n_1}, t_1^-, t_1^+$, $z_{n_2+1}, \dots, z_{n_2+1}$, $\dots, z_{n_3}, t_2^-, t_2^+$ and so on, ending with z_{n_k}, such that α^- meets A only in a finite set F.

Using the fact that Y is ω–sac, pick an arc α^+ in Y visiting in order the points, t_1^-, z_{n_1+1}, $\dots, z_{n_2}, t_1^+, t_2^-$ and so on, avoiding $F \setminus \{t_1^\pm, \dots, t_k^\pm\}$.

Now we can interleave α^- and α^+ to get an arc, α, visiting all the specified points in order. So we start α by following α^- to visit $z_1, \dots, t_1^-$, then pick up α^+ at t_1^- to visit $z_{n_1+1}, \dots, z_{n_2}, t_1^+$, and back to α^- from t_1^+, and so on. $\square$

Lemma 4.2.5.

(i) The tile T_0 is ω-sac$^+$ with respect to any infinite discrete subset of its boundary.

(ii) The tile T_1 is ω-sac$^+$ with respect to its boundary.

Proof. Recall that T_0 and T_1 are both homeomorphic. In turn, T_0 is a homeomorph of Y from Theorem 4.1.8 with the boundary square for both not just homeomorphic but identical (when we identify the plane, $\mathbb{R}^2$, with $\mathbb{R}^2 \times \{0\}$). So we argue this for Y only. Looking at the proof that Y is ω–sac it is clear that the arc, α_0, visiting some specified points, $x_1, \ldots, x_n$, in order, need only touch the boundary in an arbitrarily small neighborhood of any x_i which happens to be on the boundary. This immediately gives the first claim — Y (and so T_0) is ω-sac$^+$ with respect to infinite discrete subsets of the boundary square.

Further, the point $(0, -1)$ can be reached from the interior of Y (away from the boundary square) by two disjoint arcs which meet the set $(A \times \{-1, 1\}) \cup (\{-1, 1\} \times A)$ only at $(0, -1)$ — for one arc, α^-, follow one side of the sequence of circles converging to $(0, -1)$ and for the other, α^+, start at $(0, -1)$ go right along the boundary edge a short way, and then go into the interior. The same is true for the points $(0, 1), (-1, 0)$, and $(1, 0)$.

Now to get the desired arc, if every x_i is not one of $(0, -1), (0, 1), (-1, 0)$, or $(1, 0)$, then just use α_0. While if x_i, is say, $(0, -1)$, then pick α_0 to visit $x_1, \ldots, x_{i-1}, t^-, t^+, x_{i+1}, \ldots$, where t^-, t^+ are points close to $(0, -1)$ on α^- and α^+ respectively. Now let α be the arc that follows α_0 to t^-, then follows α^- to $x_i = (0, -1)$, then α^+ to t^+, and then resumes along α_0. □

For any tile T, $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$ and $a, b > 0$, denote by $T(\mathbf{x}, a, b)$ the space T scaled in the x and y coordinates so its base has length a and width b, then scaled in the z coordinate so that the pyramid containing it has height no more than the smaller of a and b, and then translated in the x, y-plane so that the top–left corner is at $(x, y, 0)$.

From Lemma 4.2.4, part (ii) of Lemma 4.2.5, and an easy induction argument, the following is clear.

Lemma 4.2.6. *Any space obtained by gluing along matching edges a finite family of translated and scaled copies of T_1 is a rational ω-sac curve.*

Proof. Let S be a space obtained by gluing along matching edges a finite family of translated and scaled copies of T_1. Fix a point $x \in S$, then x is in one of the tiles say t_1. Then there are two cases, either there is another tile t_2 which meets t_1 at one side of its base, and x is in this intersection. Then since each t_i is rational, there is a neighborhood base $\mathcal{B}_i$ for x in

t_i, for each $i = 1, 2$. Let $\mathcal{B} = \{B_1 \cup B_2 \mid B_i \in \mathcal{B}_i\}$. Then $\mathcal{B}$ is a neighborhood base at x in $t_1 \cup t_2$, since for any open set U including x, $U \cap t_i$ is open in t_i and includes x, so there is $B_i \in \mathcal{B}_i$ with $B_i \subset U \cap t_i$, and thus $x \in B_1 \cup B_2 \subset U$. Also each $B \in \mathcal{B}$ has countably many boundary points, because when we combine two sets we add at most countably many more points to the boundary as the intersection of two tiles is a sequence of points in the boundary. Otherwise, there is a neighborhood base of x in t_1 such that all elements of this base has countably infinite boundary.

Let now, $\{x_1, \ldots, x_n\} \subset S$. We want to find an arc α through these points in order. If all of these points are in one tile, we have such an arc since each tile is ω-sac. Otherwise, there are at least two different tiles in which the points lie, and we need a finite number of tiles from S to have a connected subset of it including the points. Say we need at least m tiles, then we will proceed by induction on m:

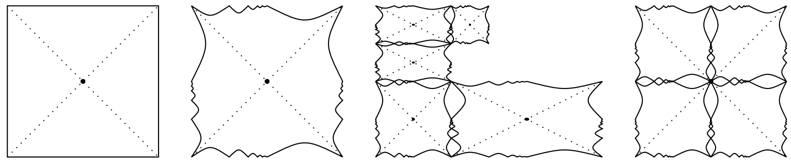
$m = 1$ is the first case. Suppose $m = 2$, so the points lie in two tiles t_1 and t_2, and the tiles intersect along one edge of their bases. Then by using ω-gluing lemma, $t_1 \cup t_2$ is also ω-sac, hence we can find an arc through the points in the given order. Suppose now, for any $m - 1$ tiles that form a connected subset of S, this subset is ω-sac. Then for m tiles that form a connected subset, there is at least one tile t such that when we remove this tile the rest of them are still connected, call the union of the rest Y. Then again using ω-gluing lemma for $X = t$ and Y, we get that the union is ω-sac. $\square$

We define recursively a sequence of tiles. The first in the sequence is T_1 from above. Given tile T_n, where $n \geq 1$, define T_{n+1} to be $T_n((-1,1),1,1) \cup T_n((-1,0),1,1) \cup T_n((0,1),1,1) \cup T_n((0,0),1,1)$ scaled in the z–coordinate only so as to fit inside the pyramid with base S and height 1. Then all the tiles T_n are rational ω-sac continua.

Theorem 4.2.7. *Fix $N \geq 3$. For $n \geq 2$ or $n = \omega$, let R_n be the set of rational n-sac continua, and let $R_{n,\neg(n+1)}$ be the set of rational continua which are n-sac but not $n+1$-sac.*

Then all the sets R_n and $R_{n,\neg(n+1)}$ are $\mathbf{\Sigma}^1_1$-hard subsets of the space $\mathcal{K}(\mathbb{R}^N)$.

Proof. We prove that there is a continuous map K of the space $\mathcal{T}$ of all trees on $\mathbb{N}$ into the space $\mathcal{K}(\mathbb{R}^3)$ such that: if the tree τ has no infinite branch then K_τ is a rational continuum which is not arc-connected (in other words, 2-sac), while if τ has an infinite branch, then K_τ



Generic tile   Tile T_1 Gluing T_1's   Tile T_2

Figure 4.4: Examples of Tiles

is an ω-sac rational continuum. The claim that R_n is a $\mathbf{\Sigma}^1_1$-hard subset of the space $\mathcal{K}(\mathbb{R}^N)$, follows simultaneously for all n and N. We then give the minor modifications necessary to have that K_τ is n-sac but not $n+1$-sac when τ has an infinite branch. The remaining claims follows immediately.

A basic building block for K_τ is $S(T)$ a variant of the topologist's sine-curve based on a tile T. This sine-curve lies in the rectangular box

$$\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 13/3, 0 \leq y \leq 5/2, 0 \leq z \leq 1\}.$$

We call the point $(0, 5/2, 0)$ the top left corner of $S(T)$.

Explicitly $S(T)$ is $(D_0 \cup AB_0 \cup U_0 \cup AT_0) \cup \bigcup_{n\geq 1}(D_n \cup C_n \cup AB_n \cup U_n \cup AT_n)$ where

$$\begin{aligned}
D_n &= \bigcup_{i=1}^{2\cdot 4^n} T(10/(3\cdot 4^n), 1/2 + i/4^n, 1/4^n),\\
C_n &= T(10/(3\cdot 4^n), 1/2 - 1/4^n, 1/4^n) \cup T(10/(3\cdot 4^n), 1/2, 1/4^n) \quad (n \geq 1),\\
AB_n &= T((7/(3\cdot 4^n), 1/2 + 1/4^n), 1/4^n),\\
U_n &= \bigcup_{i=1}^{2\cdot 4^n - 1} T((4/(3\cdot 4^n), 1/2 + i/4^n), 1/4^n)\\
&\quad \cup T((4/(3\cdot 4^n), 5/2 - 1/4^{n+1}, 1/4^n, 3/4^{n+1})\\
&\quad \cup T((4/(3\cdot 4^n), 5/2, 1/4^n, 1/4^{n+1}), \quad \text{and}\\
AT_n &= T((4/(3\cdot 4^n) - 1/4^{n+1}, 5/2, 1/4^{n+1}, 1/4^{n+1}).
\end{aligned}$$

For each $m \geq 1$, let $c_m = T(10/(3\cdot 4^n), 1/2 - 1/4^n, 1/4^n)$ be the tile in $S(T)$ at the bottom of the mth connector, C_m.

For any point y in $\mathbb{R}$, and any $a > 0$, let $S(y, a, T)$ be the sine curve $S(T)$ scaled (in all directions) by a, and translated so that its top left corner is at $(0, y, 0)$.

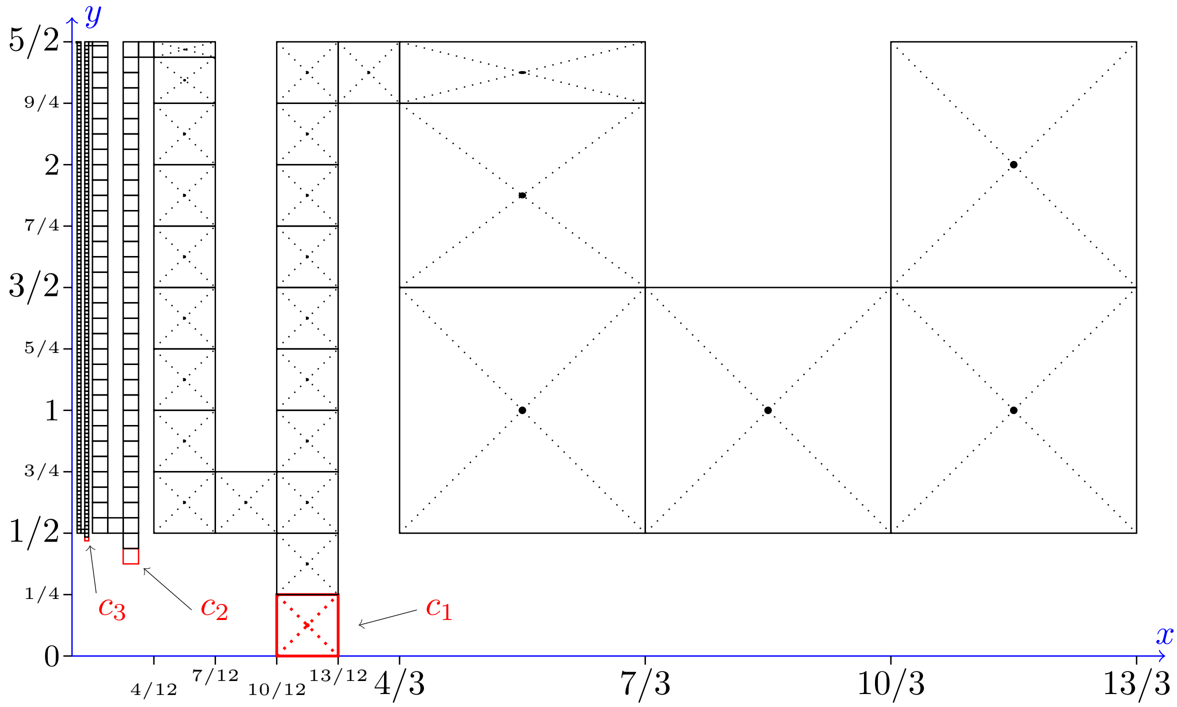


Figure 4.5: Sine-curve made of tiles

Next, given a tree τ and a tile T, we define a 'branch space', $B(T,\tau)$, lying in the rectangular box,

$$\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 13/3, 0 \le y \le 10/3, 0 \le z \le 1\},$$

which is $\bigcup\{S_s : s \in \tau\}$, where each S_s is defined with the aid of some connecting tiles, c_s, and numbers, y_s, by induction on the length of s, as follows:

Step 1: Let $y_{()} = 5/2 + 5/6 = 10/3$, and let $S_{()} = S(y_{()}, 1, T)$ (i.e. the sine-curve defined above based on T, translated along the y-axis by $5/6$).

Step 2: The sine curve, $S_{()}$ has a family of connecting tiles c_m. Set $c_{(m)} = c_m$. Let $y_{(m)} = y_{()} - 2/4^0 - 2/4^m$, and let $S_{(m)} = S(y_{(m)}, \frac{1}{4^m}, T)$. Note, critically, that the top–right tile of this sine-curve, $S_{(m)}$, is such that its top edge coincides with the bottom edge of $c_{(m)}$.

Step $n+1$: Fix an $s \in \tau$ with length n. We again will have connecting tiles, c_m, from the sine-curve S_s. Set $c_{s^\frown m} = c_m$. Let $y_{s^\frown m} = y_s - 2/4^L - 2/4^{L+m}$ where $L = \sum_{i=1}^n s_i$, and let $S_{s^\frown m} = S(y_{s^\frown m}, 1/4^{L+m})$. Again note that the top–right tile of $S_{s^\frown m}$ has its top edge coinciding with the bottom edge of $c_{s^\frown m}$.

Assume, for this paragraph only, that $\tau = \tau_c$ is the complete tree, and T is the solid tile. For any s in τ, let $\tau_s = \{s' \in \tau : s' \text{ extends } s\}$, and let $B_s = \bigcup\{S_s : s \in \tau_s\}$. By construction, $B_{(1)}$ is a 1/4th copy of $B(T,\tau) = B_{()}$, and $B_{(2)}$ is a 1/16th copy. It is easy to check that the height (in the y-coordinate) of $B_{()}$ is exactly $10/3$. So the height of $B_{(2)}$ is 1/16th of this, which is $5/24$. The gap between the top edge of $B_{(1)}$ and the top edge of $B_{(2)}$ is $9/24$. Thus $B_{(2)}$ is disjoint from $B_{(1)}$. By self–similarity it follows that B_s and B_t meet if and only if one of s and t is an immediate successor of the other. This all shows that, for any tree and any tile, $B(T,\tau)$ is well defined, and is the edge connected union of tiles meeting along matching edges.

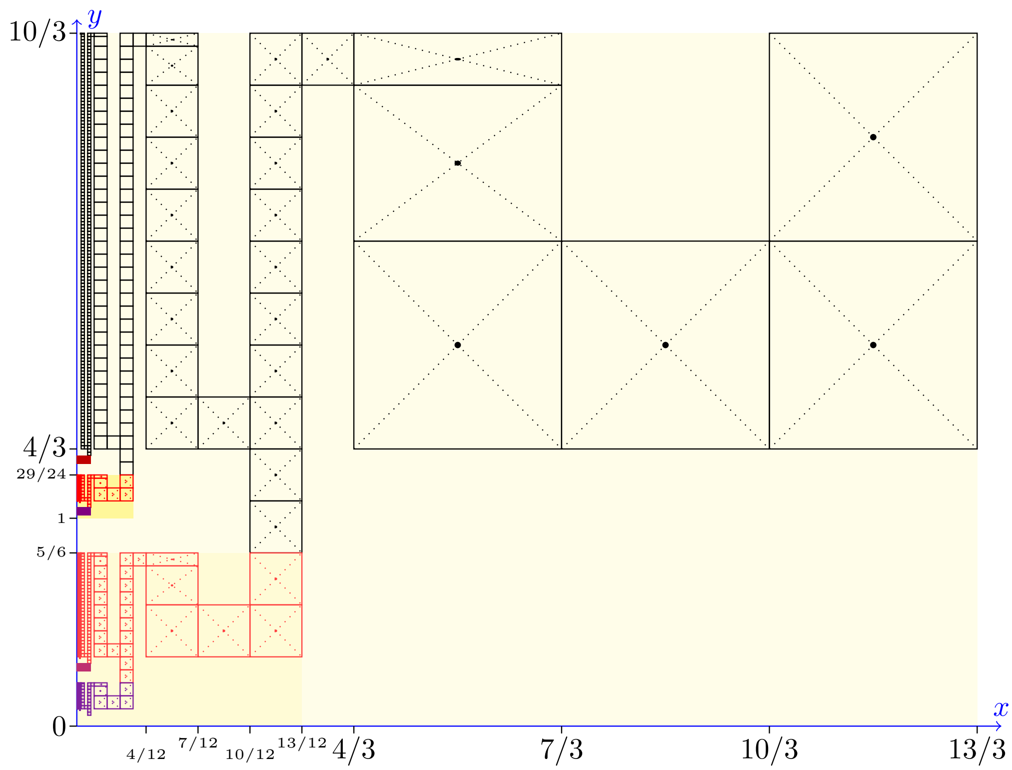


Figure 4.6: Branch space $B(T,\tau)$

We call the point $(0, 10/3, 0)$ the top left corner of $B(T,\tau)$. For y in $\mathbb{R}$ and $a > 0$, let $B(y, a, T, \tau)$ be $B(T,\tau)$ scaled in the y-coordinate only by a, and translated so its top left corner is at $(0, y, 0)$.

Now our K_τ will consist of $\bigcup_{n\geq 0} B_n \cup L \cup S$, where $B_n = B(y_n, 1/2^n, T_{n+1}, \tau)$, for $y_n = 7/2^n$, and the two pieces L and S are defined as follows.

The set L is a homeomorphic copy of the tile T_0, bent in the middle so that its base is contained in the L-shaped area

$$\{(x,y,0) \in \mathbb{R}^3 : -2/3 \leq x \leq 0, -1 \leq y \leq 7 \text{ or } 0 \leq x \leq \tfrac{22}{3}, -1 \leq y \leq 0\}$$

and the boundary of the base of the tile is the boundary of this area.

The set S is a sine curve variant based on the tile T_1, which connects the branch spaces B_n, and converges down to the x-axis. Concretely, $S = \bigcup_{n\geq 0}(AR_n \cup D_n \cup AL_n \cup C_n)$ where

$$\begin{aligned}
AR_n &= \bigcup_{i=0}^{3\cdot 4^n-1} T_1((13/3 + i/4^n, 7/2^n), 1/4^n),\\
D_n &= \bigcup_{i=1}^{3\cdot 2^n-1} T_1((13/3 + 3 - 1/4^n, 7/2^n - i/4^n), 1/4^n),\\
AL_n &= \bigcup_{i=1}^{2\cdot 4^n-2} T_1((13/3 + 1 + i/4^n, 7/2^n + 1/4^n - 3/2^n), 1/4^n, 1/4^n)\\
&\quad \cup T_1((13/3 + 1, 7/2^n + 1/4^n - 3/2^n), 1/4^{n+1}, 1/4^n)\\
&\quad \cup T_1((13/3 + 1 + 1/4^n, 7/2^n + 1/4^n - 3/2^n), 3/4^{n+1}, 1/4^n), \quad \text{and}\\
C_n &= \bigcup_{i=1}^{2^{n+1}} T_1(13/3 + 1, 7/2^{n+1} + i/4^{n+1}, 1/4^{n+1}).
\end{aligned}$$

Claim 1. K_τ is a rational continuum.

Proof: Let $R = \bigcup_n B_n \cup S$.

Let $L' = \{(x,y,0) \in \mathbb{R}^3 : x = 0, -1 \leq y \leq 7 \text{ or } -2/3 \leq x \leq 22/3, y = 0\}$, be the inner boundary of the base of L.

Since $cl(R) \subseteq R \cup L'$, K_τ is clearly compact. Since L and R are connected, and S is a variant topologists sine curve, clearly K_τ is connected.

For all the points of K_τ except those on L', we have a natural neighborhood base at the point for which each element has a countable boundary (which comes from the tile(s) the point is in).

Take any point $\mathbf{x}$ in L'. We suppose now, $\mathbf{x} = (x_0, 0, 0)$ (the other case is similar). Because B_n is based on the tile T_n, combined with the fact that the T_1's in the connecting sine curve, S, have size shrinking to zero, the set M of all x-components of the left and right edges of the base of tiles in R is dense in $[0, 22/3]$.

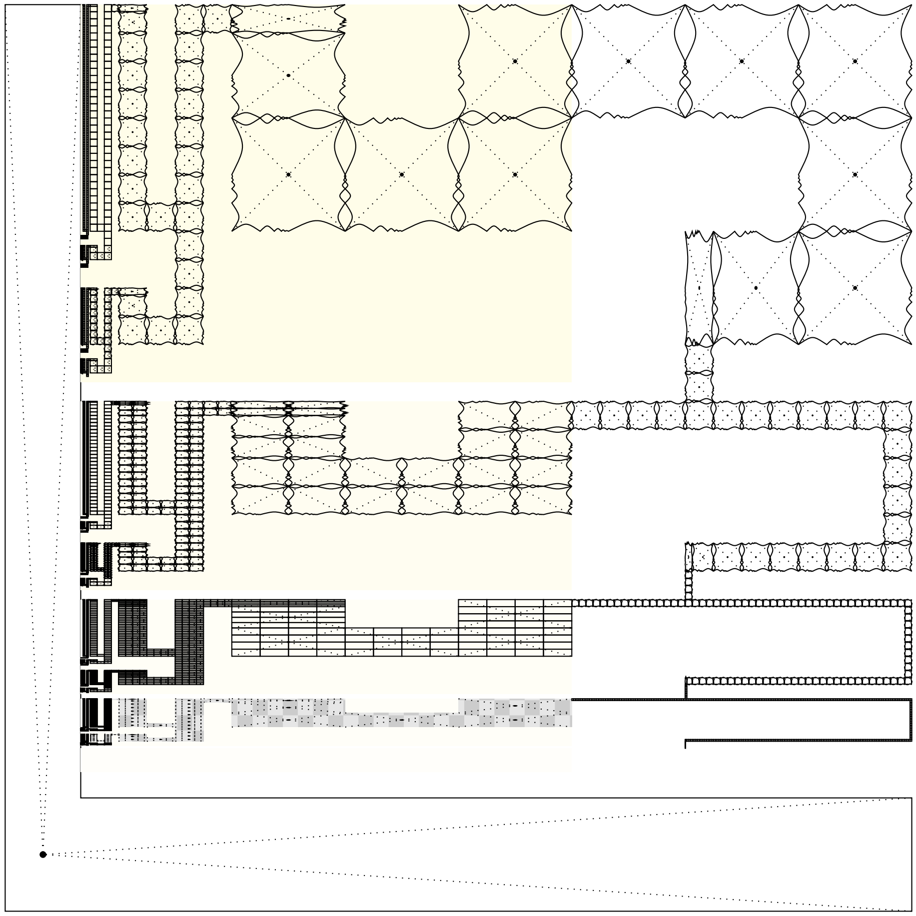


Figure 4.7: Construction of rational continuum K_τ

Let U be a rectangular neighborhood of $\mathbf{x}$ in $\mathbb{R}^3$, and $r_{min} = \min\{x : (x,0,0) \in U\}$ and $r_{max} = \max\{x : (x,0,0) \in U\}$. Without loss of generality, if y_{max} is the value of the maximum y-component in U then $\{(x,y,z) \in U \mid y = y_{max}\}$ do not intersect with any of the B_n, i.e. the top of U is in between B_n and B_{n+1} for some n.

The set $U \cap L$ includes a neighborhood N of x which has countable boundary. Let $a = \min\{x : (x,0,0) \in N\}$ and $b = \max\{x : (x,0,0) \in N\}$. Since M is dense there are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in M such that a_n increases to a, b_n decreases to b, and for each n, $r_{min} \le a_n \le a < b \le b_n \le r_{max}$. Let $m_1 \ge n$ be such that both of the lines $x = a_1$ and $x = b_1$ intersect the xy-projection of $B_{m_1} \cup \{(x,y,z) \in S : y \le 7/2^{m_1}\}$ along edges of tiles only. Let $S_n = \{(x,y,z) \in S : y \le 7/2^n\}$. And inductively, let $m_i \ge m_{i-1}$ such that the lines $x = a_i$ and $x = b_i$ intersect the xy-projection of $B_{m_i} \cup S_{m_i}$ along edges of tiles only. Now take $N' = \bigcup_i ((S_{m_i} \setminus S_{m_{i+1}} \cup \bigcup_{k+m_i<m_{i+1}} B_{m_i+k}) \cap \{(x,y,z) : a_i \le x \le b_i\})$.

Here for each i, we cut B_{m_i+k} along edges of finitely many tiles, hence the boundary is

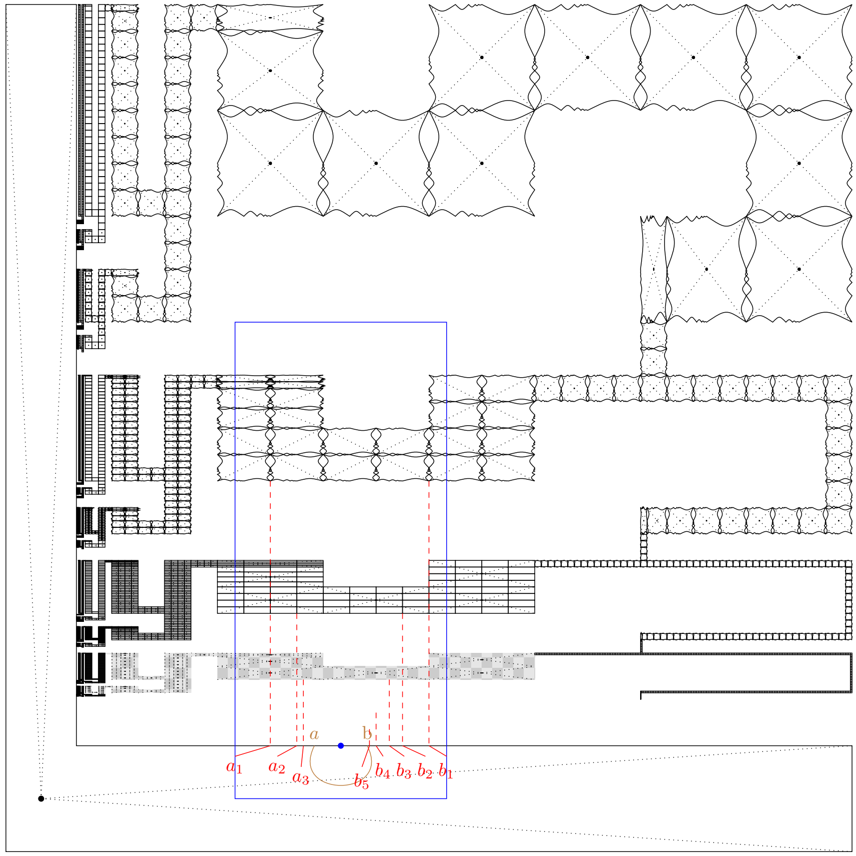


Figure 4.8: A neighborhood with countable boundary in K_τ

at most countable. And similarly for $S_{m_i} \setminus S_{m_{i+1}}$, we cut along the edges of finitely many tiles. Thus N' has countable boundary. Moreover, $N \cup N' \subset R$ is a neighborhood of $\mathbf{x}$ with countable boundary. ∎

Claim 2. If τ has an infinite branch (i.e. $\tau \in \mathbf{IF}$) then K_τ is ω-sac.

Proof: Suppose τ has an infinite branch. Note that if T is any tile, then there is a branch of edge connected tiles in $B(T, \tau)$ which converges to a point $\mathbf{y}_\tau$ on the y-axis.

We first show that for any $m \geq 1$, the branch space $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$ is ω-sac. To do so we only need to check that if $\mathbf{y}_\tau$ is one of the n-points $x_1, \ldots, x_n$ in $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$, then we can find an arc joining them in that order. Suppose $x_k = \mathbf{y}_\tau$. Then the points $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ are in some finite family of edge connected tiles of $B(T_m, \tau)$. Let t be a tile in the branch space $B(T_m, \tau)$ such that none of the x_i's is in the tiles to the left and bottom of this tile except for $x_k = \mathbf{y}_\tau$. Let $\mathbf{y}_1$ be the bottom left corner of t, and $\mathbf{y}_2$ be the top left corner of t. Then by Lemma 4.2.6 there is an arc α_0 in $B(T_m, \tau)$ through the points $x_1, \dots, x_{k-1}, \mathbf{y}_1, \mathbf{y}_2, x_{k+1}, \dots, x_n$ in the given order. Let α_1 be the part of α_0 through $x_1, \dots, x_{k-1}, \mathbf{y}_1$. Let β_1 be the arc starting at $\mathbf{y}_1$ and ending at $\mathbf{y}_\tau$ obtained by traveling along the right and bottom edges of tiles of the branch converging to $\mathbf{y}_\tau$. Similarly, let α_2 be the part of α_0 through $y_2, x_{k+1}, \dots, x_n$. And let β_2 be the arc starting at $\mathbf{y}_\tau$, and following the left and top edges of tiles of the branch converging to $\mathbf{y}_\tau$, back to $\mathbf{y}_2$. Then the arc α obtained by following α_1, β_1, β_2 and then α_2, is the desired arc through the points $x_1, \dots, x_n$ in the given order.

Back now to K_τ, when τ has an infinite branch. For each n, B_n a has a corresponding branch converging to a point $\mathbf{y}_n$ on the y-axis. An easy modification of the argument for $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$ shows that the space $R \cup \{\mathbf{y}_n : n \in \mathbb{N}\}$ is ω-sac.

Since $R \cup \{\mathbf{y}_n\}_n$ is ω-sac, $(R \cup \{\mathbf{y}_n\}_n) \cap L = \{\mathbf{y}_n\}_n$, and L is ω-sac$^+$ with respect to discrete sets (Lemma 4.2.5, part (i)), it follows from the ω–Gluing Lemma that $K_\tau = R \cup L$ is indeed ω-sac. ∎

Claim 3. If τ has no infinite branch (i.e. $\tau \in \mathbf{WF}$) then K_τ is not 2-sac.

Proof: If τ does not have any infinite branches, then there are no arcs connecting L to R. This is clear because, without infinite branches, any path starting in R and attempting to reach L is forced to travel along a topologist's sine curve variant — which is impossible. ∎

Claim 4. The map $\tau \mapsto K_\tau$ is continuous.

Proof: Let $K : \mathbf{Tr} \to \mathcal{K}(\mathbb{R}^3)$ given by $K(\tau) = K_\tau$. Let s be in $\mathbb{N}^{<\mathbb{N}}$, and write $[s]$ for the set of all trees containing s. Then $[s]$ is a closed and open subset of $\mathbf{Tr}$. Subbasic open sets in $\mathcal{K}(\mathbb{R}^3)$ are of one of two forms: (i) $\langle U \rangle = \{C : C \subseteq U\}$ and (ii) $\langle X; V \rangle = \{C : C \cap V \neq \emptyset\}$, where U and V are open subsets of $\mathbb{R}^3$. We show inverse images under K of both types of subbasic open set are open in $\mathbf{Tr}$, thus confirming continuity of the map $\tau \mapsto K_\tau$.

For subbasic sets of type (ii), the sets V may be taken to come from any basis for $\mathbb{R}^3$; we will take for V open balls in $\mathbb{R}^3$ which either meet, or have closure disjoint from, $L \cup S$. Fix such a V. If V meets $L \cup S$, then $K^{-1}\langle X; V \rangle = \mathbf{Tr}$. If the closure of V is disjoint from L, then for any tree τ, V meets only finitely many $B_n(\tau)$, and in each of these branch spaces, meets only finitely many sine curves. Suppose V meets sine curves labelled by $s_1, \ldots, s_k$. Then $K^{-1}\langle X; V\rangle = \bigcup\{[s_i] : 1 \leq i \leq k\}$, which is open (each $[s_i]$ is open).

For subbasic sets of type (i), if $L \cup S$ is not contained in U, then $K^{-1}\langle U\rangle = \emptyset$. So suppose, $L \cup S \subseteq U$. Let τ_c be the complete tree. Then all but finitely many of the sine curves making up the $B_n(\tau_c)$'s are contained in U. Let them be labelled by $s_1, \ldots, s_k$. Then $K^{-1}\langle U\rangle = \mathcal{K}(\mathbb{R}^3) \setminus \bigcup\{[s_i] : 1 \leq i \leq k\}$, which is open (each $[s_i]$ is closed). ■

Claims 1–4 show that $\tau \mapsto K_\tau$ is a continuous reduction $(\mathcal{T}, \mathbf{IF}) \to (\mathcal{K}(\mathbb{R}^3), R_\omega)$ where K_τ is a rational continuum which is ω-sac if τ has an infinite branch, but is not even 2-sac when τ has no infinite branches.

We now turn to the case for $R_{n,\neg(n+1)}$. To start fix $n \geq 2$. Select $n-2$ points $a_1, \ldots, a_{n-2}$ from the interior of the right hand edge of the base of T_0. Similarly to the definition of T_1, shrink T_0 while keeping fixed the set $\{a_1, \ldots, a_{n-2}\}$ and the top edge of the base. This gives a tile $\widehat{T}_n$. Now consider the map $\tau \mapsto K^n_\tau$ where K^n_τ is K_τ along with the tile $\widehat{T}_n(22/3, 0, 1)$. Then it is easy to see (given our previous work) that K^n_τ is a rational continuum and the map $\tau \mapsto K^n_\tau$ is continuous. Because the extra tile, $\widehat{T}_n(22/3, 0, 1)$, meets the rest of K^n_τ in exactly $n-1$ points (namely $a_1, \ldots, a_{n-2}$ and the topleft corner of the base of the tile), K^n_τ is never $n+1$-sac. When τ has no infinite branch, then K^n_τ is not 2-sac, so definitely not in $R_{n,\neg(n+1)}$. But when τ has an infinite branch, both $\widehat{T}_n(22/3, 0, 1)$ and the rest of K^n_τ are ω-sac, and (again) meet in $n-1$ points — so by Lemma 4.1.7, K^n_τ is n-sac. □

Theorem 4.2.8. *The sets S_n of n-sac continua, for a natural number $n \geq 2$ or $n = \omega$, are $\mathbf{\Pi}^1_2$-complete subsets of the space $\mathcal{K}(\mathbb{R}^N)$, where $N \geq 4$.*

Proof. First note that the definition of n-sac is a $\mathbf{\Pi}^1_2$ statement. Thus each S_n is a $\mathbf{\Pi}^1_2$ set. Also, note that in the case of $n = 2$, S_n is the set of all arc connected continua, and this was proved to be $\mathbf{\Pi}^1_2$-complete by Ajtai and Becker, see [15] for details.

We prove the claim, for all n and N simultaneously, by proving that there is a continuous map Φ from the space $\mathbb{N}^\mathbb{N}$ into the space $\mathcal{K}(\mathbb{R}^4)$ such that: given a $\mathbf{\Pi}^1_2$ set $A \subset \mathbb{N}^\mathbb{N}$, if $x \in A$ then $\Phi(x) = P_x$ is a continuum which is not arc–connected (i.e. 2–sac), while if $x \in A$, then P_x is an ω–sac continuum. (See [15] 37.11 for a similar argument).

Let A be a $\mathbf{\Pi}^1_2$ subset of $\mathbb{N}^{\mathbb{N}}$ and B be a $\mathbf{\Sigma}^1_1$ subset of $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $x \in A$ if and only if for each $y \in 2^{\mathbb{N}}$, $(x, y) \in B$. Now let τ be a tree on $\mathbb{N} \times 2 \times \mathbb{N}$ with

$B = \{(x, y) \mid \exists z \in \mathbb{N}^{\mathbb{N}}(x, y, z) \text{ is a branch of } \tau\} = \{(x, y) \mid \tau(x, y) \notin \mathbf{WF}\}$. Recall that $\tau(x, y) = \{s : (x \restriction \mathit{length}(s), y \restriction \mathit{length}(s), s) \in \tau\}$ is a tree on $\mathbb{N}$.

Now for each $x \in \mathbb{N}^{\mathbb{N}}$, we will construct a continuum $P_x \subset \mathbb{R}^4$ as follows:

First, we identify the Cantor space $2^{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}}$ with the standard Cantor set in $[0, 1]$. Then for each $y \in 2^{\mathbb{N}}$, let $L_{x,y} = K_{\tau(x,y)}$, as described in Theorem 4.2.7, placed in the cube $\{(a, b, c, d) \mid -\frac{2}{3} \leq a \leq \frac{22}{3}, -1 \leq b \leq 7, c \geq 0, d = y\}$. Thus the outside edges of the tile L in $K_{\tau(x,y)}$ is on $a = -\frac{2}{3}$ or $b = -1$. Now we will connect the continua $L_{x,y}$ along the edges on $a = -\frac{2}{3}$ with Menger cube M which is placed in the cube $\{(a, b, c, d) | -2 \leq a \leq -\frac{2}{3}, -1 \leq b \leq 7, c = 0, 0 \leq d \leq 1\}$.

Now let $P_x = \bigcup_{y \in 2^{\mathbb{N}}} L_{x,y} \cup M$.

Then, P_x is a continuum and the map $x \mapsto P_x$ from $\mathbb{N}^{\mathbb{N}} \to \mathcal{K}(\mathbb{R}^4)$ is continuous. Moreover, $x \in A$ if and only if for each $y \in 2^{\mathbb{N}}$, $\tau(x, y) \in \mathbf{IF}$. Thus if $x \notin A$, then there is $y \in 2^{\mathbb{N}}$ with $\tau(x, y) \in \mathbf{WF}$, so the corresponding rational ω-sac continua $L_{x,y} = K_{\tau(x,y)}$ is not 2-sac, hence the union $P_x = M \cup \bigcup_{y \in 2^{\mathbb{N}}} L_{x,y}$ is not 2-sac.

On the other hand, if $x \in A$, then for each $y \in 2^{\mathbb{N}}$, $L_{x,y}$ is ω-sac by Theorem 4.2.7. We also know that the cube M is ω-sac. To show P_x is ω-sac, we will only go through one example, all other cases will be similar:

Let $x_1, \ldots, x_n$ be points in P_x. Suppose that for all odd indices $k \leq n$, $x_k \in L_{x,y_0}$ for some fixed $y_0 \in 2^{\mathbb{N}}$. Also suppose that for each even index $m \leq n$, $x_m \in L_{x,y_m}$ for some $y_m \in 2^{\mathbb{N}}$ so that $y_i \neq y_j$ for $i \neq j$. Then choose distinct $z_k \in L_{x,y_0} \cap M$ for odd numbers $k \leq n$ and choose distinct $z^1_m, z^2_m \in L_{x,y_m} \cap M$ for even numbers $m \leq n$. Since L_{x,y_0} is ω-sac, there is an arc α through the points $x_1, z_1, x_3, z_3, \ldots, x_K, z_K$ (where K is the largest odd integer less than or equal to n). Similarly, for each even m less than or equal to n, there is an arc α_m through the points z^1_m, x_m, z^2_m. Additionally, as M is ω-sac there is an arc β through the points $z_1, z^1_2, z^2_2, z_3, z^1_4, z^2_4, z_5, \ldots, z_K, z^1_n, z^2_n$ (without loss of generality n is even). Now we define an arc in P_x as follows:

Starting at x_1 follow α until z_1, then we switch to β and follow until z^1_2, then switch to α_2 and follow until z^2_2, then switch back to β until z_3, and switch to α to follow until z^1_4, etc. In this way, we will go through all x_i's in the given order, which gives us an arc inside P_x. $\square$

4.3 COMPLEXITY OF CLASSIFICATION

In [3], Camerlo, Darji and Marcone have shown that the classification problem for homeomorphism on dendrites is S_∞-universal, hence classifiable by countable structures.

Question 3. Are dendrites up to equivalent embedding classifiable by countable structures?

In this section we show that the dendroids have very complicated classification problem, hence the classification of all curves is very complicated.

4.3.1 DENDROIDS

Since the dendroids include all dendrites, the classification problem of dendroids up to homeomorphism is at least S_∞-universal. However, it is not classifiable by countable linear orders, in fact it is strictly more complex than the classification of any countable structures. Let $\mathscr{D}$ denote the set of all dendroids.

Theorem 4.3.1. *Homeomorphism on dendroids is a turbulent equivalence relation.*

Proof. It's known that the equivalence on $\mathbb{Z}^{\mathbb{N}}$ defined as $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are equivalent if $\dfrac{x_n - y_n}{n} \to 0$, is a turbulent one. We will reduce this equivalence relation to the homeomorphism relation on dendroids.

We will define a Borel map $f : \mathbb{Z}^{\mathbb{N}} \to \mathscr{D}$ so that $\dfrac{x_n - y_n}{n} \to 0 \iff f(x)$ is homeomorphic to $f(y)$.

Let L be the following set: $\{0\}\times\{0\}\times[-1,1]\cup[0,1]\times\{0\}\times\{0\}\cup\bigcup_n\{1/n\}\times\{0\}\times[-1,1]$. It is a dendroid.

Fix two sequences of distinct prime numbers $(p_n)_{n\in\mathbb{Z}}$ and $(q_n)_{n\in\mathbb{N}}$. Also fix a countable dense subset $(d_n)_{n\in\mathbb{N}}$ of $[-1,1]$, say $d_0 = 0$. Let $g : [-\pi/2, \pi/2] \to [-1,1]$ be the map $g(x) = \dfrac{2x}{\pi}$.

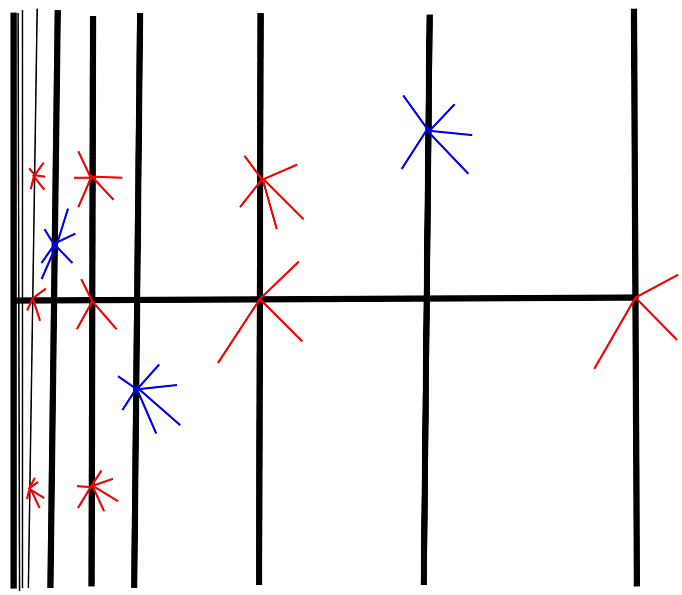


Figure 4.9: Construction of $L(x_n)$

Fix a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ and consider the following set:

On L put branches at the point $\mathbf{x}_n = (1/2n, 0, g(\arctan(x_n)))$ so it is a branching point of order p_n, where the new branches meet the set L only at branching point $\mathbf{x}_n$.

Also put branches on the line $x = 1/n$, for $n = 2k+1$, at the points $\mathbf{x}_{n,i} = (1/n, 0, d_i)$ for $i = 1, \ldots, k$ so that $\mathbf{x}_{n,i}$ is of order q_i. And the new branches meet the modified compact set only at the branching point. Call this set $L(x_n)$. (See Figure 4.9)

Claim 1. $L(x_n)$ is a dendroid.

$L(x_n)$ is closed and bounded subset of $\mathbb{R}^3$, and it is a countable union of connected sets with non-empty intersection. So it is a continuum.

It is arcwise connected as there are only finitely many branches at each branching point.

To show it is also hereditarily unicoherent, it is enough to show that $L(x_n)$ is unicoherent, since the possible subcontinua are: points, arcs, finite trees or copies of $L(x_n)$. If A, B are two subcontinua with $A \cup B = L(x_n)$, then possible intersections are: a point, an arc, a finite graph or a dendroid, which are continua.

Claim 2. $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(y_n)_{n\in\mathbb{N}}$ if and only if $L(x_n) \sim_H L(y_n)$

Suppose there is a homeomorphism $h : L(x_n) \to L(y_n)$. Since a homeomorphism should map a branching point of order p to a branching point of the same order, $h(\mathbf{x}_n) = \mathbf{y}_n$ and $h(\mathbf{x}_{n,i}) = \mathbf{y}_{n,i}$. Hence the line $l = \{0\} \times \{0\} \times [-1, 1]$ is fixed.

For a contradiction, suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are not equivalent, then $\dfrac{x_n - y_n}{n} \not\to 0$, so $g(\arctan(\frac{x_n}{n})) - g(\arctan(\frac{y_n}{n})) \not\to 0$. So that $\mathbf{x}_n - \mathbf{y}_n \not\to (0, 0, 0)$. So for some $\varepsilon > 0$ there is a subsequence with $|\mathbf{x}_{n_k} - h(\mathbf{x}_{n_k})| \geq \varepsilon$, $\forall k$. But there is a convergent subsequence $(\mathbf{x}_{n_{k_i}})_{i\in\mathbb{N}}$ of $(\mathbf{x}_{n_k})_{k\in\mathbb{N}}$, say to $\mathbf{x}$. Then $h(\mathbf{x}_{n_{k_i}}) \to h(\mathbf{x}) = \mathbf{x}$, as the line l is fixed by any homeomorphism. Thus we get a contradiction, as $|h(\mathbf{x}_{n_{k_i}}) - \mathbf{x}_{n_{k_i}}| \leq |h(\mathbf{x}_{n_{k_i}}) - \mathbf{x}| + |\mathbf{x}_{n_{k_i}} - \mathbf{x}| < \varepsilon$ will be true for sufficiently large i.

For the other direction, suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are equivalent sequences in $\mathbb{Z}^{\mathbb{N}}$, i.e. $\dfrac{x_n - y_n}{n} \to 0 \iff g(\arctan(\frac{x_n}{n})) - g(\arctan(\frac{y_n}{n})) \to 0$. Let $\mathbf{x}_n$ and $\mathbf{y}_n$ be as above. Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$h(\mathbf{x}_n) = \mathbf{y}_n$, and $h(\mathbf{x}_{n,i}) = \mathbf{y}_{n,i}$. And h fixes the line I. To have $h(L(x_n)) = L(y_n)$, we can make sure h is defined continuously between branching points, as in both sets they are copies of unit interval. We need to check continuity on the line l. Let $\mathbf{z} = (0, 0, z) \in l$ be fixed, we know $h(\mathbf{z}) = \mathbf{z}$.

If $\mathbf{z}$ is a cluster point of $(\mathbf{x}_n)_{n\in\mathbb{N}}$, then there exist a subsequence $(\mathbf{x}_{n_i})_{i\in\mathbb{N}}$ that converges to $\mathbf{z}$. Then we have $g(\arctan(\frac{x_{n_i}}{n_i})) \to z$ and $g(\arctan(\frac{x_{n_i}}{n_i})) - g(\arctan(\frac{y_{n_i}}{n_i})) \to 0$, hence $g(\arctan(\frac{y_{n_i}}{n_i})) \to z$. Thus $h(\mathbf{x}_{n_i}) = \mathbf{y}_{n_i} \to h(\mathbf{z}) = \mathbf{z}$.

If $\mathbf{z}$ is not a cluster point of $(\mathbf{x}_n)_{n\in\mathbb{N}}$, then there is some $\varepsilon > 0$ so that $B(\varepsilon, \mathbf{z})$ does not include any of the branching points $\mathbf{x}_n$ $(n > 0)$ in the definition of $L(x_n)$. Suppose for a contradiction $\mathbf{z}$ is a cluster point for $\mathbf{y}_n$, say $\mathbf{y}_{n_i} \to \mathbf{z}$. Then $g(\arctan(\frac{y_{n_i}}{n_i})) \to z$ and $g(\arctan(\frac{x_{n_i}}{n_i})) - g(\arctan(\frac{y_{n_i}}{n_i})) \to 0$, hence $g(\arctan(\frac{x_{n_i}}{n_i})) \to z$. So for large enough i, $\mathbf{x}_{n_i} \in B(\varepsilon, \tilde{z})$. Thus $\mathbf{z}$ can not be a cluster point of $\mathbf{y}_n$. $\square$

This proves that curves up to homeomorphism are not classifiable by countable structures as well, since curves include all the dendroids.

Theorem 4.3.2. *Equivalence of dendroids is a turbulent equivalence relation.*

Proof. The construction in Theorem 4.3.1 works for equivalence as well. Also the proof with minor modifications will work here. For a given sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$, we use $L(x_n)$ as defined in previous proof. (See Figure 4.9).

If there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(L(x_n)) = L(y_n)$, then proving that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are equivalent is exactly the same.

If on the other hand, we have equivalent sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, then we modify the definition of the homeomorphism by extending continuously to all $\mathbb{R}^3$ (since the homeomorphism is defined inside the cube $[0,1]\times[-1,1]\times[-1,1]$, we can define h as identity outside this cube and extend continuously in between). And the only set we have to check continuity is still the line l, which follows from previous proof. □

4.3.2 EQUIVALENCE ON AN ORBIT OF HOMEOMORPHISM

Let $\mathcal{E}$ be a set of curves all of which are homeomorphic. We know that this is a Borel subset of $\mathcal{K}(I^n)$. Although they are all homeomorphic, they are not necessarily equivalent.

Question 4. How complex is the classification problem of $\mathcal{E}$ under equivalence?

One class we can look into is the class of all Warsaw circles, denoted $\mathcal{W}$, i.e. all circles homeomorphic to the standard Warsaw circle. It turns out this classification is very complex as well. To prove that we will use a sequence of inequivalent prime knots. A *prime knot* is a knot which can not be decomposed into non-trivial knots as a connected sum of knots. For example, trefoil knot, figure-eight knot are prime knots. It is known that there are infinitely many prime knots which are not equivalent under the equivalence relation $\sim$.

Theorem 4.3.3. *The equivalence on $\mathcal{W}$ is a turbulent equivalence relation.*

Proof. Will define a Borel map $f : \mathbb{Z}^{\mathbb{N}} \to \mathcal{W}$ so that $\dfrac{x_n - y_n}{n} \to 0 \iff f(x_n)$ is equivalent to $f(y_n)$.

For $n \in \mathbb{Z}$, fix a prime knot K_n, which is not equivalent to any of the previous ones. Let $0 < \varepsilon < 1/2$. Also fix a countable dense subset (d_n) of $[-1,1]$, say $d_0 = 0$. Let $g : [-\pi/2, \pi/2] \to [-1,1]$ be the map $g(x) = \dfrac{2x}{\pi}$.

Now fix a sequence $z = (z_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$. Consider the set $S = \{(x, 0, \sin(1/x)) \mid x \in (0,1]\}$ and $L = \{(0,0,t) \mid t \in [-1,1]\}$. And let W be the usual Warsaw circle. Let

$$S_n = \left\{(x, 0, \sin(1/x)) \in S \mid \tfrac{2}{(2n+1)\pi} \le x \le \tfrac{2}{(2n-1)\pi}\right\} \text{ and}$$

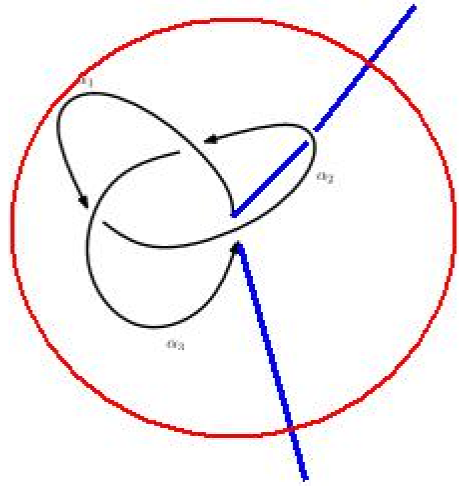


Figure 4.10: Tying a knot inside given ball

$$z'_n \in [\tfrac{(4n-1)\pi}{2}, \tfrac{(4n+1)\pi}{2}] \text{ be such that } \sin(z'_n) = g(\arctan(z_n/n)), \text{ and}$$
$$d'_{n,i} \in [\tfrac{(4n-3)\pi}{2}, \tfrac{(4n-1)\pi}{2}] \text{ for } 1 \leq i \leq n \text{ be such that } \sin(d'_{n,i}) = d_{i-1}$$

And then put a ball B_n with center at $\mathbf{z}_n = (1/z'_n, 0, g(\arctan(z_n/n))) \in S_{2n}$ so that $B_n \cap S_m = \emptyset$ for $m \neq n$ and $\operatorname{diam} B_n < \frac{1}{2(n+1)}$. Also put balls $B_1^k, \ldots, B_k^k$ on S_{2k-1} as follows: B_i^k has center $\mathbf{d}_{k,i} = (1/d'_{k,i}, 0, d_{i-1})$ so that $B_i^k \cap S_m = \emptyset$ for $m \neq k$ and $\operatorname{diam} B_i^k < \frac{1}{2(i+k+1)}$.

Now we will tie copies of knots in the following way (see Figure 4.10): The copy of the knot will be in the interior of the ball specified. Cut the knot K_n at some point so we have two end points and remove the piece $S_{2n} \cap B(\mathbf{z}_n, \varepsilon^{n+1})$ from S_{2n}, which is in the interior of the ball B_n, then identify the end points of the knot K_n with the end points in the line S_{2n}. We will call S_{2n} with the attached knot K_n to be S'_{2n}

Also tie a copy of K_i to S_{2k-1} where $i < 0$ and $0 < |i| < k$ as explained above, denote it as K_i^k, so that K_i^k lies inside the ball $B_{|i|}^k$. Let S'_{2k-1} denote the set S_{2k-1} together with the attached knots on it.

Now let $C(z) = W \cup \bigcup_n S'_n$ (See Figure 4.11).

Claim 1. $C(z)$ is a Warsaw circle.

Two knots are always homeomorphic, so adding knots to a standard Warsaw circle we still get a Warsaw circle.

Claim 2. z and y are equivalent sequences if and only if $C(z) \sim C(y)$

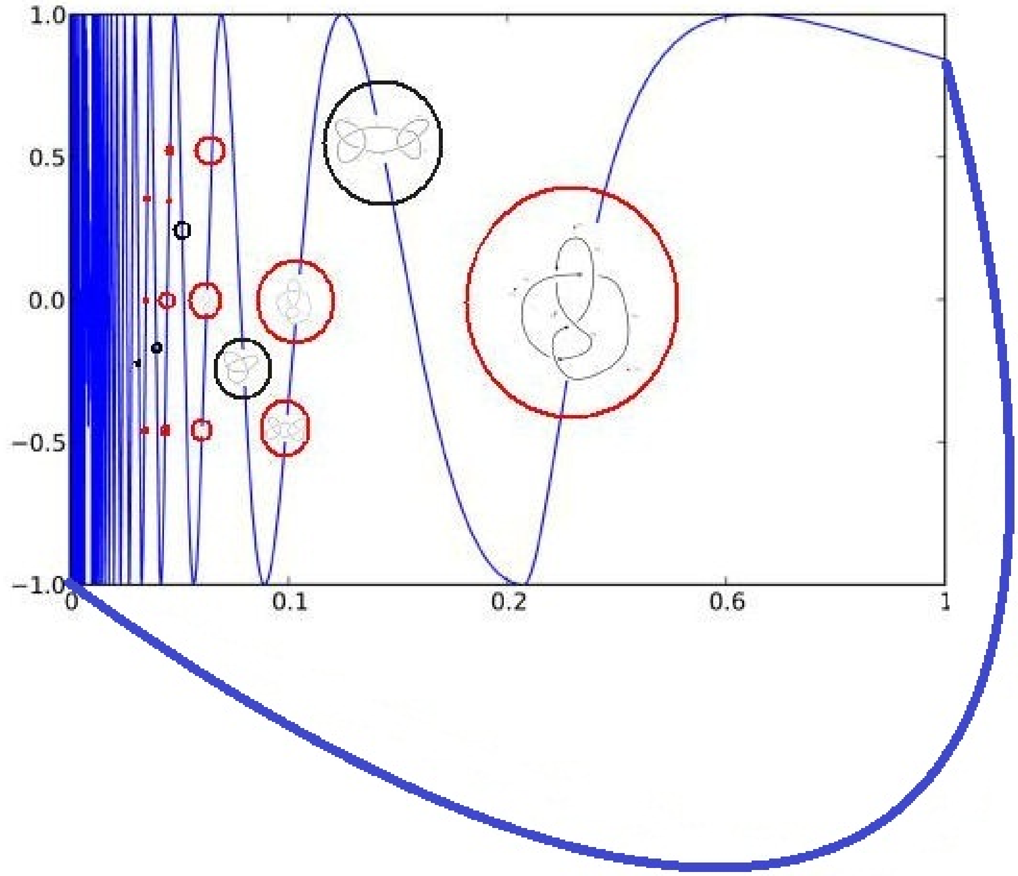


Figure 4.11: Construction of curve $C(z)$

Suppose there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(C(z)) = C(y)$. Since each knot K_n in the definition of these curves are inequivalent knots, h should map each to itself. So in particular, h fixes L and $|h(\mathbf{z}_n) - \mathbf{y}_n| \to 0$, where $\mathbf{y}_n$ is defined accordingly.

For a contradiction, suppose z and y are non-equivalent sequences, so $\dfrac{z_n - y_n}{n} \not\to 0$, and $g(\arctan(\frac{z_n}{n})) - g(\arctan(\frac{y_n}{n})) \not\to 0$. Then $\mathbf{z}_n - \mathbf{y}_n \not\to (0,0,0)$. So for some $\varepsilon > 0$ there is a subsequence with $|\mathbf{z}_{n_k} - \mathbf{y}_{n_k}| \geq \varepsilon$, $\forall k$ (*). But there is a convergent subsequence $(\mathbf{z}_{n_{k_i}})_{i\in\mathbb{N}}$ of $(\mathbf{z}_{n_k})_{k\in\mathbb{N}}$, say to $\mathbf{z}$. Then $h(\mathbf{z}_{n_{k_i}}) \to h(\mathbf{z}) = \mathbf{z}$, as the line L is fixed by any homeomorphism. Thus we get a contradiction, as $|\mathbf{y}_{n_{k_i}} - \mathbf{z}_{n_{k_i}}| \leq |h(\mathbf{z}_{n_{k_i}}) - \mathbf{y}_{n_{k_i}}| + |h(\mathbf{z}_{n_{k_i}}) - \mathbf{z}_{n_{k_i}}| < \varepsilon$ will be true for sufficiently large i, contradicting (*).

For the other direction, suppose z and y are equivalent sequences in $\mathbb{Z}^{\mathbb{N}}$, i.e. $\dfrac{z_n - y_n}{n} \to 0$, and thus $g(\arctan(\frac{z_n}{n})) - g(\arctan(\frac{y_n}{n})) \to 0$. Let $\tilde{z_n}$ and $\tilde{y_n}$ be as above. Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

h maps corresponding knots as expected, and h fixes the line L, and the curves from the first knot to the point $(0,0,-1)$ is extended homeomorphically. Thus $h(C(z)) = C(y)$. Moreover, h is identity outside the box $[-1,1]\times[-1,1]\times[-1,1]$.

As the balls are disjoint we can make sure h is defined continuously outside $C(z)$ and inside the box (start with first and second balls of $C(z)$ and $C(y)$ we can extend definition of h between the planes $x = 1/z'_1$ and $x = 1/z'_2$ using the definition of h given above, and then proceed in order...). Thus we need to only check continuity on the line L. Let $\mathbf{z} = (0,0,z) \in L$ be fixed. We know $h(\mathbf{z}) = \mathbf{z}$. Moreover, since the corresponding knots are mapped to each other, $|h(\mathbf{z}_n) - \mathbf{y}_n| \to 0$ (**).

If $\mathbf{z}$ is a cluster point of $(\mathbf{z}_n)_{n\in\mathbb{N}}$, then there exist a subsequence $(\mathbf{z}_{n_i})_{i\in\mathbb{N}}$ that converges to $\mathbf{z}$. Then we have $g(\arctan(\frac{z_{n_i}}{n_i})) \to z$ and $g(\arctan(\frac{z_{n_i}}{n_i})) - g(\arctan(\frac{y_{n_i}}{n_i})) \to 0$, hence $g(\arctan(\frac{y_{n_i}}{n_i})) \to z$. Thus $\mathbf{y}_{n_i} \to h(\mathbf{z}) = \mathbf{z}$, hence by (**) $h(\mathbf{z}_{n_i}) \to \mathbf{z}$.

If $\mathbf{z}$ is not a cluster point of $(\mathbf{z}_n)_{n\in\mathbb{N}}$, then there is some $\varepsilon > 0$ so that $B(\varepsilon, \mathbf{z})$ does not intersect any of the knots K_n $(n > 0)$ in the definition of $C(z)$. Suppose for a contradiction $\mathbf{z}$ is a cluster point for $h(\mathbf{z}_n)$, say $h(\mathbf{z}_{n_i}) \to \mathbf{z}$. Then by (**), $\mathbf{y}_{n_i} \to \mathbf{z}$. Then $g(\arctan(\frac{y_{n_i}}{n_i})) \to z$ and $g(\arctan(\frac{z_{n_i}}{n_i})) - g(\arctan(\frac{y_{n_i}}{n_i})) \to 0$, hence $g(\arctan(\frac{z_{n_i}}{n_i})) \to z$. So for large enough i, $K_{n_i} \cap B(\varepsilon, \tilde{z}) \neq \emptyset$. Thus $\mathbf{z}$ can not be a cluster point of $(h(\mathbf{z}_n))_{n\in\mathbb{N}}$. $\square$

This theorem tells us that Warsaw circles are not classifiable by countable structures but it might be true that they are not comparable.

APPENDIX

EMBEDDINGS OF A COMPACT METRIC SPACE

Let K be a compact metric space, consider the following space of functions:

$\mathrm{Emb}(K, \mathbb{R}^N) = \{e : K \to \mathbb{R}^N | e \text{ is an embedding}\}$.

It is known that the space of continuous functions from K to $\mathbb{R}^N$, $C(K, \mathbb{R}^N)$ is a Polish space, (see [15], p. 24). Moreover, $\mathrm{Emb}(K, \mathbb{R}^N)$ is a G_δ subset of $C(K, \mathbb{R}^N)$ (see [13], p.56), hence $\mathrm{Emb}(K, \mathbb{R}^N)$ is a Polish space.

A.1 RELATION OF $\mathrm{Emb}(K, \mathbb{R}^N)$ AND $\mathcal{K}_K(\mathbb{R}^N)$

Let $\Phi : \mathrm{Emb}(K, \mathbb{R}^N) \to \mathcal{K}_K(\mathbb{R}^N)$ - all copies of K in $\mathbb{R}^N$- be defined as $\Phi(h) = h(K)$. Obviously, Φ is onto but not one to one.

Proposition A.1.1. *Φ is continuous.*

Proof. Fix $e \in \mathrm{Emb}(K, \mathbb{R}^N)$ and $\varepsilon > 0$. There are elements $x_1, \dots, x_n \in 2^{\mathbb{N}}$ such that $\{B(\varepsilon, e(x_i))\}_{i=1}^n$ cover $e(K)$. Let $V = B(B(2\varepsilon, e(x_1)), \dots, B(2\varepsilon, e(x_n)))$ is an open neighborhood of $\Phi(e) = e(K)$. Any open neighborhood of $\Phi(e)$ contains an open set in the form of V.

Let $\gamma < \varepsilon$, and $W = B(\gamma, e) = \left\{g \in \mathrm{Emb}(K, \mathbb{R}^N) \mid |g(x) - e(x)| < \gamma, \forall x \in K\right\}$, then

- for $x \in K$, there is i such that $e(x) \in B(\varepsilon, e(x_i))$, for this i and any $g \in W$,

$$|g(x) - e(x_i)| \leq |g(x) - e(x)| + |e(x) - e(x_i)| < 2\varepsilon$$

- Since $|g(x_i) - e(x_i)| < \varepsilon$, $\Phi(g) \cap B(2\varepsilon, e(x_i)) \neq \emptyset$

Hence $\Phi(g) \in V$. And since g is arbitrary, $\Phi(W) \subset V$. $\square$

Now, fix a compact set $K \in \mathcal{K}(\mathbb{R}^N)$. Consider the group $G = Aut(K)$ acting on $\mathrm{Emb}(K, \mathbb{R}^N)$ by $(h, e) \mapsto (e \circ h)$. This is a continuous Polish group action.

Moreover, we have the continuous map $\Phi : \mathrm{Emb}(K, \mathbb{R}^N) \to \mathcal{K}_K(\mathbb{R}^N)$, which satisfies all the requirements of a theorem by Ryll-Nardzewski in [19] as a level set of Φ is $\Phi^{-1}(L) = \big\{e \in \mathrm{Emb}(K, \mathbb{R}^N) \mid e(K) = L\big\}$, where $L = e_0(K)$ and the orbit of e_0 is $\{e_0 \circ h \mid h \in Aut(K)\}$ are equal sets. The reverse inclusion is clear by definition of the action. Suppose $e \in \Phi^{-1}(L)$, then $e(K) = L$. Define $h : L \to L$ by $h(x) = e_0^{-1}(e(x))$, then $h \in Aut(L)$ and $e = e_0 \circ h$. Thus we have,

Theorem A.1.2. *For any $K \in \mathcal{K}(\mathbb{R}^N)$, the set of all instances of K in $\mathbb{R}^N$, $\mathcal{K}_K(\mathbb{R}^N)$ is an absolutely Borel set.*

Theorem A.1.3. *Φ is an open map, for $K = \mathcal{C}$ and $n = 3$.*

Proof. Fix $e \in \mathrm{Emb}(\mathcal{C}, \mathbb{R}^3)$ and $\varepsilon > 0$. We will show that $\Phi(B(e, \varepsilon))$ includes a neighborhood of $K = \Phi(e)$.

Since K is a Cantor set, it has a defining sequence $\{M_n\}_n$. And there is $n \in \mathbb{N}$ such that the diameters of components of $M_n < \varepsilon$. Let $F_1, \ldots, F_m$ be the components of M_n.

Now take $L \in B(F_1, \ldots, F_m)$. Let $K_i = K \cap F_i$ and $L_i = L \cap F_i$, they are also Cantor sets. Also let $C_i = e^{-1}(K_i)$, which is Cantor and clopen subset of $\mathcal{C}$. Define f_i on C_i as embedding of C_i onto L_i, and let $f = f_i$ on C_i. Then $f(\mathcal{C}) = L$, C_i's are disjoint, so f is a well defined embedding. Morevoer, for $x \in \mathcal{C}$, $x \in C_i$ for some i, so $|e(x) - f(x)| < \operatorname{diam} F_i < \varepsilon$, as both K_i and L_i are subsets of F_i.

Now $K \in B(\mathrm{Int}(F_1), \ldots, \mathrm{Int}(F_m)) \subset B(F_1, \ldots, F_m) \subset \Phi(B(e, \varepsilon))$. $\square$

Thus all the structure we have on Cantor sets can be moved to the embedding space $\mathrm{Emb}(\mathcal{C}, \mathbb{R}^3)$. In particular, most embeddings of Cantor set in $\mathbb{R}^3$ are tame. On the other hand, it is known that most embeddings of the circle in $\mathbb{R}^N$ ($N \geq 4$) are unknotted (tame), while most embeddings of circle in $\mathbb{R}^3$ are wildly knotted by Milnor [17].

Remark. Φ is not open when $K = I$- unit interval.

Consider the embedding $e(x) = (x, 0, 0)$ and fix $0 < \gamma < 1/4$. Let $W = B(\gamma, e)$. If $\Phi(W)$ were open, then in particular it includes a basic open neighborhood of $e(I)$. So there exist $\varepsilon > 0$ so that $x_1 = (0, 0, 0)$ and $x_k = (1, 0, 0)$ are connected by a simple chain consisting of open sets of the form $B(\varepsilon, x)$, say $U_1, \dots, U_m$. So $(U_i)_{i=1}^m$ is a finite open cover for $e(I)$ and without loss of generality $\varepsilon < \gamma$. Let $V = B(U_1, \dots, U_m)$ be that basic open neighborhood in $\Phi(W)$. Now consider the compact set C defined as follows: $C = h(I)$, where

$$h(x) = \begin{cases} (2x, \frac{\varepsilon}{2}\sqrt{1-2x}, 0) & 0 \le x \le 1/2 \\ (2-2x, -\frac{\varepsilon}{2}\sqrt{2x-1}, 0) & 1/2 \le x \le 1 \end{cases}$$

$C \in V$, but $C \notin \Phi(W)$: Note that the non-cut points $0, 1$ of I should map to the non-cut points of C and $e(I)$. Suppose for a contradiction $g \in W$ with $g(I) = C$, then without loss of generality $g(0) = (0, \frac{\varepsilon}{2}, 0)$ and $g(1) = (0, -\frac{\varepsilon}{2}, 0)$. But then $|g(1) - e(1)| > 1 > \gamma$, contradicting $g \in W$.

Question 5. What about when $K = S^1$? $K = I \times I$?

The Polish group $G = Aut(\mathbb{R}^3)$ acts on $X = \mathrm{Emb}(\mathcal{C}, \mathbb{R}^3)$ by $h \cdot e = h \circ e$. This is a continuous action. The image of an orbit of this equivalence relation under the map Φ is an orbit of the action of G on $\mathcal{K}_C(\mathbb{R}^N)$.

Let $K \subset \mathbb{R}^3$ be a Cantor set, consider the fiber of K in $\mathrm{Emb}(\mathcal{C}, \mathbb{R}^3)$,

$$F_K = \left\{ e \in \mathrm{Emb}(\mathcal{C}, \mathbb{R}^3) \mid e(\mathcal{C}) = K \right\}$$

Proposition A.1.4. *F_K is closed nowhere dense.*

Proof. $F_K = \Phi^{-1}(K)$, so closed.

Suppose for a contradiction $e \in \mathrm{Int}(F_K)$, so there is $\varepsilon > 0$ such that $B(\varepsilon, e) \subset F_K$. If K is wild, then since tame Cantor sets is dense there is $g \in B(\varepsilon, e)$ with $g(\mathcal{C})$ is tame, so $g \notin F_K$. If K is tame, then there is an ω-decomposable non-tame Cantor set in this open set, hence in either case there is $g \in B(\varepsilon, e)$ with $g(\mathcal{C}) \neq K$. □

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