THREE ESSAYS ON AUCTIONS

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This dissertation studies new bidding behaviors in richer environments where bidders can either communicate or intertemporally interact. We focus on such three perspectives as collusion, strategic information disclosure and intertemporal inference. In the collusion chapter, we propose a framework to investigate the structure of endogenous collusion and show that an endogenously formed ring shall in general be a partial ring. In the information disclosure chapter, we study the auctioneer's optimal choice of interperiod information release and show the standard sequential Dutch auction or the sequential first-price auction with the announcement of each stage's winning bid can generate the highest revenue among all considered sequential auction formats. In the intertemporal inference chapter, we suggest a resale explanation for the price path in sequential auctions with multi-unit demand.

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1.0 INTRODUCTION

Traditional auction models generally treat auctions in a static environment where there exists neither communication nor intertemporal interaction among bidders. But in many realistic situations, auctions are run in multiple rounds. Bidders can not only communicate, but also obtain rivals' private information though interperiod inference. In such richer situations, whether bidders will display certain new behaviors is the question that this dissertation is going to focus on.

The following three chapters will study auctions from the perspectives of collusion, strategic information disclosure and intertemporal inference respectively. In Chapter 2, we endogenize collusion by allowing a collusion initiator to select a particular scheme and then propose it to a chosen number of other bidders. The main finding is that, when there are at least three bidders, an endogenously formed ring includes at least two members and is in general not all-inclusive. Since a partial ring creates bidder asymmetry, it makes a first-price auction inefficient. This finding provides a basis for laws that outlaw collusion in auctions. Chapter 3 characterizes equilibria in various two-stage sequential auction formats under all possible forms of interperiod information disclosure in an IPV model. We study the role of interperiod information disclosure in affecting bidders' intertemporal learning, bidding, and auction revenue. Unlike Milgrom and Weber (1982), who show in their model that it is always good for the auctioneer to commit to complete information revelation, we find that this is not necessarily the case in sequential auctions due to bidders' intertemporal substitution. In our model, only selective information release can be revenue enhancing. We show that the standard sequential Dutch auction or the sequential first-price auction with the announcement of each stage's winning bid generates the highest revenue among all considered auction formats. Chapter 4 studies price trends in a sequential first-price common-value auction with resale.

It differs from the previous research in that it considers sequential auctions with *multi-unit* demand. In the two-stage case, we propose a condition that guarantees the existence of a symmetric monotonic equilibrium which exhibits a declining trend. This is because bidders have the incentive to overbid in the first round to lower their rivals' intertemporal inference on the object value so that they can obtain a second-stage advantage. We also characterize the necessary properties of symmetric monotonic equilibria in the finite N-stage and the infinite-stage cases. In the former case, the price trend remains constant and drops only at the last stage; in the latter case, we have a constant price trend throughout. In the final chapter, we summarize the new results that this dissertation obtains and lay out the future research agenda.

2.0 ENDOGENOUS FORMATION OF A BIDDING RING

Auctions are a prevalent mechanism to allocate resources. It is natural for bidders to collude in order to capture the surplus that should have been transferred to the auctioneer. The format of collusion in auctions has been widely studied and the current literature follows two major trends. The first trend applies a mechanism design approach to study different forms of collusion in a one-shot auction game. Recent papers include McAfee and McMillan (1992), Marshall, Meurer, Richard and Stromquist (1994), and Marshall and Marx (2004). The other trend explores the implicit collusion of bidders in a repeated game framework. Recent works include Blume and Heidhues (2002), Aoyagi (2003) and Skrzypacz and Hopenhayn (2004). The auction literature on collusion is also related to the industrial organization literature on cartel and merger, such as Salant, Switzer and Reynolds (1983), Deneckere and Davidson (1985), Cave and Salant (1995); and to the game theory literature on coalition formation such as Bloch (1996) and Ray and Vohra (1999). In this paper, we will adopt the mechanism design approach to study the endogenous formation of a bidding ring in a one-shot first-price auction. Related works are briefly discussed as follows.

McAfee and McMillan (1992) analyze collusion in the first-price auction within a homogeneous Independent Private Value (IPV) framework (we will refer to their paper as MM (1992) hereafter). Their emphasis is on a surplus division game for an all-inclusive cartel. They show that collusion is inefficient when the cartel is weak (no internal transfer) and efficient when the cartel is strong (side payments allowed). An incomplete cartel is also studied for a discrete special case where bidders' valuations follow a two-point distribution. In this special case, they analyze a cartel formation game, which suggests a way to model the ring formation when bidders can make endogenous decisions. The major problem with this approach, however, is that in general it may not yield a unique equilibrium outcome, i.e., there may be more than one coalition structure that satisfies the stipulated equilibrium conditions. This point can be supported by a similar numerical example in Deneckere and Davidson (1985) where merger with Bertrand competition is studied and we see that almost all coalition sizes can satisfy the MM (1992) equilibrium conditions. Another problem of this approach is concerned with one equilibrium condition, which requires that being an outsider captures at least as big surplus as joining the ring. This condition aims to prevent the deviation of the outsiders. But a successful deviation of an outsider not only hinges on her own payoff comparison, but also on whether the ring will want to accept her. Even if the outsider finds it profitable to join the ring, once her joining decreases the ring members' surplus, the ring may not accept her. Therefore, the above equilibrium condition is too strong for the purpose of preventing an outsider's deviation. Even if the condition is not satisfied, once a ring finds it unprofitable to accept an outsider, an equilibrium will emerge.

The incomplete cartel problem in MM (1992) is not fully tackled because the bidders become asymmetric under collusion in the first-price auction, which makes the equilibrium bid functions analytically unsolvable. Marshall, Meurer, Richard and Stromquist (1994) approach this analytical difficulty with numerical methods. They provide numerical results for a K-member ring of N bidders ($K \leq N$) in a homogeneous IPV setting where bidders' valuations are drawn from a uniform distribution over the support of [0, 1]. The results reported in their paper give us a good understanding as to how the surplus of ring members and outsiders evolves when the ring size increases.

Bidder collusion in a heterogeneous IPV environment is studied in Marshall and Marx (2004). They divide collusive mechanisms into two categories, the bid coordination mechanism and the bid submission mechanism and carry out a characterization of collusive behavior in the first and second-price auctions under each. One important feature of their collusive scheme is that once a ring member deviates, the ring does not operate and all bidders bid non-cooperatively. As they mention in their paper, this is a common, but not innocent assumption. In fact, the collapse of the ring as one member defects is not a credible threat if it is in the remaining members' own interests to continue the collusive operation. A more common equilibrium condition adopted in the coalition formation literature is that the coalition will still operate but with one member fewer, like in MM (1992), Bloch (1996) and Ray and

Vohra (1999). Also, the ring setup is exogenously given and the endogenous ring formation problem is not dealt with.

In the IO literature, Cave and Salant (1995) propose a majority-rule voting game to endogenize the decision of cartel structure. Notice that an important variable, the cartel size is exogenously given as all-inclusive. It is not obvious how a cartel size can be endogenized under a voting setup. Bloch (1996) and Ray and Vohra (1999) study endogenous coalition formation via a sequential bargaining game. Their methods however, are hardly tractable if applied to the standard auction setup.

Our objective in this paper is thus to propose a tractable approach to analyze bidders' collusive behavior in an environment where they can endogenously choose and implement a mechanism for themselves. A sketch of the approach is that nature selects a *collusion initiator* from N symmetric bidders. This initiator then proposes a collusive scheme that maximizes her own ex ante surplus to a chosen number of other bidders. If these invited bidders agree to join the ring, then collusion works according to the proposed scheme. If an invited bidder declines the proposal, she will bid non-cooperatively along with the uninvited bidders. Detailed description of the whole game and related discussions will be left for the next section.

The main contribution of this paper is to propose a tractable framework to model endogenous ring formation in auctions. Within this framework, we find that when there are at least three bidders, an endogenously formed ring includes at least two members and is in general not all-inclusive. As the first-price auction becomes asymmetric after the collusion, the auction outcome will be inefficient. An implication is that outlawing the collusion will be socially beneficial. The approach in this paper can be readily extended to the other auction formats.

The remainder of the paper proceeds as follows. In Section 2, we set up the auction environment and define the collusive scheme. Section 3 formulates the endogenous collusive problem that bidders need to solve. Section 4 gives the solution and Section 5 concludes.

2.1 AUCTION SETTING AND COLLUSIVE SCHEME

This is an N-bidder single object first-price auction model ($N > 1$) with no reserve price.[1] All bidders are risk-neutral and their private valuations, $v_1, v_2, ... v_N$, are independent random variables with the same density $f(v_i)$ on the common support $[\underline{v}, \overline{v}]$. $f(v_i)$ is continuously differentiable and bounded away from zero on $[\underline{v}, \overline{v}]$.

The endogenous ring formation game consists of the following four stages.

Stage 1. Nature selects a bidder to be the collusion initiator with probability $\dfrac{1}{N}$ from N symmetric bidders.

Stage 2. Before all bidders (including the collusion initiator) observe their private valuations, the collusion initiator selects a collusive scheme Γ, which contains a bid assignment rule and a transfer rule, from the feasible set of schemes defined below. To a chosen number of other bidders, she announces the collusive rules.

Stage 3. After considering the above proposal, each invited bidder decides whether to join the ring or not. If an invited bidder agrees to join, she will commit to the scheme proposed by the collusion initiator. If an invited bidder refuses to join, she will then bid non-cooperatively along with the uninvited bidders in the main auction.

Stage 4. All bidders observe their private valuations and bid according to their plans decided in stage 3.

We will address four points concerning the above collusion game. First, in the first stage, since bidders are ex ante identical, each bidder will have an equal chance to be chosen by nature as the collusion initiator, hence the selection probability is $\dfrac{1}{N}$. Second, in stage 3, we assume that once an invited bidder turns down the proposal, she will bid non-cooperatively. This means we assume *only* the collusion initiator has the ability to organize a ring. This is the major simplifying assumption that makes our future analysis tractable. Finally, we assume both the collusive proposal and the decisions to accept or decline the proposal must be made before all bidders observe their valuations. This assumption fits into the circumstances where bidders will have to collude before they know exactly the object value. These

[1] A reserve price complicates our derivation, but does not affect our basic conclusions. We do not analyze its effect in this paper.

circumstances arise when bidders find that as the precise information on the object value has been collected, it may be too late to organize a ring.

Now we will define the feasible set of schemes, from which the collusion initiator will select one. A collusive scheme is a mechanism denoted by $\Gamma = (K,\ \beta_K,\ T_K)$. K is the number of bidders that are included in a ring. β_K: $[\underline{v}, \overline{v}]^K \to R^K$, is the function that maps from all members' reports to their respectively assigned bids given the ring size is K. Let $V = (v_1, v_2, ...v_K)$, which denotes the vector of K ring members' valuations. We use $\hat{\beta}_K(V)$ to denote the *largest* bid that $\beta_K(V)$ assigns given the report vector V. So $\hat{\beta}_K(V)$ is a mapping from $[\underline{v}, \overline{v}]^K$ to R. T_K: $[\underline{v}, \overline{v}]^K \to R^K$, is the payment function based on all ring members' reports. We impose only one restriction on the mechanism. The payment rule $T_K(V)$ can be decomposed into two parts: $P_K(V)$ and X_K, where $P_K(V)$ is a function from $[\underline{v}, \overline{v}]^K$ to R and X_K is a scalar, for any given K. Only the member whose assigned bid is the highest according to $\beta_K(V)$ pays the amount $P_K(V)$ to the Center. All members obtain an equal non-contingent transfer X_K from the Center. So we are employing an equal surplus sharing rule.

2.2 FORMULATION OF THE PROBLEM

2.2.1 Optimality Criterion

Without loss of generality, we make bidder 1 the collusion initiator. So Bidder 1 will choose the appropriate K, $\beta_K(V)$, $P_K(V)$ and X_K to maximize her own ex ante payoff defined as G_K^1, where the subscript K denotes the number of ring members. Then G_K^1 equals the following expression:

$$\frac{1}{K}\{X_K + E_{v_1 \ldots v_K}[(v_1 - \hat{\beta}_K(V))\Pr(\text{bidder 1 wins the object}|\text{bidder 1 is assigned the largest bid by } \beta_K(V)) - E_{v_1 \ldots v_K}[\ P_K(V)|\text{bidder 1 is assigned the largest bid by } \beta_K(V)]\} + \frac{K-1}{K} \times X_K$$

$\frac{1}{K}$ is the probability that bidder 1 is assigned the highest bid by the bid assignment rule $\beta_K(V)$, given that ring members' signals are iid. $\frac{K-1}{K}$ is the probability that her assigned bid is not the highest. X_K is the non-contingent transfer. The term $E_{v_1 \ldots v_K}[\ P_K(V)|$bidder

1 is assigned the largest bid by $\beta_K(V)$] is the expected payment conditional on that the bidder's assigned bid is the highest among all members. The expression:

$E_{v_1 \ldots v_K}[(v_1 - \hat{\beta}_K(V)) \Pr(\text{bidder 1 wins the object}|\text{bidder 1 is assigned the largest bid by } \beta_K(V))]$ is the expected surplus captured from the main auction conditional on bidder 1 has the highest assigned bid.

Notice that G_K^1 can also be rearranged as:

$\frac{1}{K} E_{v_1 \ldots v_K}[(v_1 - \hat{\beta}_K(V)) \Pr(\text{bidder 1 wins the object}|\text{bidder 1 is assigned the largest bid by } \beta_K(V))] - \frac{1}{K} E_{v_1 \ldots v_K}[\ P_K\ (V)|\text{bidder 1 is assigned the largest bid by } \hat{\beta}_K(V)] + X_K$

It is easy to see that the ex ante expected surplus for the other ring members has the same expression as G_K^1 because a) they have iid signals and b) the collusive proposal adopts an equal surplus sharing rule. Both of these factors help to preserve the symmetry of ring members.

2.2.2 Constraints

In order to make the mechanism implementable, the collusion initiator's optimization problem will have to be subject to the following four regular constraints: budget balance, the incentive compatibility for reports, incentive compatibility for bidding and the participation constraint.

2.2.2.1 Budget Balance It is a standard requirement that the mechanism should be ex ante budget-balanced. Based on the mechanism's equal surplus sharing rule, we need that $X_K = \frac{1}{K} E_{v_1 \ldots v_K}[\ P_K\ (V)]$. An immediate consequence of this budget balance constraint is that G_K^1 collapses to:

$\frac{1}{K} E_{v_1 \ldots v_K}[(v_1 - \hat{\beta}_K(V)) \Pr(\text{bidder 1 wins the object}|\text{bidder 1 is assigned the largest bid by } \beta_K(V))]$. This is because $E_{v_1 \ldots v_K}[\ P_K\ (V)] = E_{v_1 \ldots v_K}[\ P_K\ (V)|\text{bidder 1 is assigned the largest bid by } \beta_K(V)]$.[2]

[2] $E_{v_1 \ldots v_K}[P_K\ (V)] = \sum_{i=1}^{K} \Pr(\text{bidder } i \text{ is assigned the largest bid by } \beta_K\ (V)) \times E_{v_1 \ldots v_K}[P_K\ (V)\ |\text{bidder } i \text{ is assigned the largest bid by } \beta_K\ (V)]$. As both $\Pr(\text{bidder } i \text{ is assigned the largest bid by } \beta_K\ (V)))$ and $E_{v_1 \ldots v_K}[P_K\ (V)\ |\text{bidder } i \text{ is assigned the largest bid by } \beta_K\ (V)]$ are the same for all i because of the bidder symmetry, we then have $E_{v_1 \ldots v_K}[P_K\ (V)] = E_{v_1 \ldots v_K}[P_K\ (V)\ |\ \text{bidder } i \text{ is assigned the largest bid by } \beta_K\ (V)]$ for all i.

Notice that the term:

$E_{v_1 \ldots v_K}[(v_1 - \hat{\beta}_K(V)) \Pr(\text{bidder 1 wins the object}|\text{bidder 1 is assigned the largest bid by } \beta_K(V))]$ is the total surplus captured by the ring, so G_K^1 also becomes the *per member* surplus of the ring. Because of the bidder symmetry, maximizing the collusion initiator's own surplus now is equivalent to maximizing the per member surplus of the ring. Define the per member surplus as G_K, so $G_K = \dfrac{1}{K} E_{v_1 \ldots v_K}[(v - \hat{\beta}_K(V)) \Pr(\hat{\beta}_K(V) \text{wins the object})]$, where v is defined as the report in V that is assigned the largest bid by $\beta_K(V)$.

The following lemma shows an important property of the optimal collusive mechanism.

Lemma 1 *Under the defined collusive scheme, the optimal $\beta_K(V)$ always assigns the largest bid $\hat{\beta}_K(V)$ to the member with the highest report and $\hat{\beta}_K(V)$ only needs to condition on the highest report v.*

Proof. See the Appendix.

Lemma 1 tells us that there is no contradiction between optimality and efficiency inside the ring. For any given ring size K, each ring member's surplus will be maximized if the member with the highest object value is assigned the largest bid to compete in the main auction. Also, the optimal bid assigned to the highest-report member does not need to condition on other members' reports. This lemma largely simplifies our future analysis. The intuition is that the highest-report member could capture more surplus for her own hence have more surplus to share with all the other members. Since now the whole ring would equally share the surplus captured *only* by the highest-report member, other members' reports are then extraneous information in term of the ring's surplus maximization. With this lemma, the per member surplus G_K becomes:

$$\frac{1}{K} E_{v_1 \ldots v_K}[(v - \hat{\beta}_K(v)) \Pr(\hat{\beta}_K(v) \text{wins the object})|v \text{ is the highest report in } V].$$

Another consequence from the budget balance constraint is that the payment rule $P_K(V)$ and X_K disappear from G_K explicitly. It means that the payment rule will not be an element that can affect ring members' surplus. So the ring's surplus will now only depend on $\{K, \hat{\beta}_K(v)\}$. An implication is that various payment rules, as long as they satisfy the equal surplus sharing and budget balance assumptions, can all be used to implement the efficient mechanism here. For example, the first-price pre auction knockout used in MM (1992) can be readily generalized to implement efficient collusion for any given ring size K. The second-price pre auction knockout in Marshall and Marx (2004) is also an effective implementation method. It does not matter which knockout the ring chooses: it will not affect any member's ex ante expected surplus. This fact is summarized as the following lemma.

Lemma 2 *Under the defined collusive schemes, different constructions of payment rule will not affect the optimal collusive surplus.*

Actually, the irrelevance of the payment rule to the collusive surplus can also be obtained from the incentive comparability for reports constraint. It is easy to check that the commonly known property in mechanism design that only the allocation rule affects the equilibrium payoff can be applied here to produce Lemma 2. But the incentive compatibility for reports constraint alone can not lead to the result stated in Lemma 1.

2.2.2.2 Incentive Compatibility for Bidding In order to maximize the per capita surplus G_K, we necessarily require that for any ring size K, the corresponding total ring surplus must be maximized. As the ring can not endogenously choose the main auction rule, it can only design an appropriate $\hat{\beta}_K(v)$ to bring maximum total surplus to itself given the outside bidders' strategies. This leads to the following incentive compatibility for bidding condition:

(I) $\hat{\beta}_K(v) \in \arg\max_b \, (v-b) \times \Pr[b \geq \max_{j\in\{K+1,...N\}} \beta_K^{out}(v_j)]$

(II) $\beta_K^{out}(v_j) \in \arg\max_b (v_j - b) \times \Pr\{b \geq \max[\max_{j'\in\{K+1,...N\}\backslash\{j\}} \beta_K^{out}(v_{j'}), \, \hat{\beta}_K(v)]\}$

$\beta_K^{out}(v_j)$ denotes the bid function for any outsider j, who competes against a ring of size K. Since the outsiders are ex ante identical, they will follow the same equilibrium bid function $\beta_K^{out}(.)$. Therefore, (I) and (II) simultaneously determine the collusive scheme's optimal bid assignment rule $\hat{\beta}_K(v)$ and the outsiders' bid function $\beta_K^{out}(.)$ as a pair of mutual best responses.

2.2.2.3 Participation Constraint Finally, we come to the participation constraint, which aims to ensure the stability of the proposed scheme. In our setup, the stability condition only needs to make sure that joining the ring is better than bidding non-cooperatively. Marshall and Marx (2004) adopt a strong punishment to the deviant by collapsing the whole ring. In this paper, we assume that when a single member deviates, the ring can still operate but with one member fewer. The punishment here is much weaker, and due to a free-rider effect, an outsider may capture more surplus than a ring member. A similar effect is reported in IO literature such as Salant et al (1983) and Deneckere and Davidson (1985), who study merger under Cournot and Bertrand competition respectively. Both papers find that outside firms can capture more surplus than coalition members by free riding on the overall suppressed competition. The free-rider effect from collusion in auctions can also be seen in the numerical results in Marshall et al (1994). In a similar collusive scheme with iid uniform valuations, they show that as the ring size increases, the outsiders' surplus also increases. So the collusive benefit of the ring leaks out to the outsiders too. Can the free-rider effect be so big as to disable a ring? They do not provide a numerical example in which this is the case. But later in this paper, we will show analytically that this is a possibility.

Before we write down the specific participation constraint, let us introduce some notation. We use D_K to denote the ex ante expected deviation profit for a single member i in a ring of size K. Since all members are ex ante identical, their deviation profits will be the same. We will use v' to denote the valuation of the deviating member and $v_1^{(K-1)}$ to denote the highest order statistic among the valuations of the $K-1$ remaining members. Then $D_K = E_{v'}\{(v' - \beta^{out}_{K-1}(v')) \times \Pr[\beta^{out}_{K-1}(v') \geq \max[\max_{j\in\{K+1,\dots N\}} \beta^{out}_{K-1}(v_j), \hat{\beta}_{K-1}(v_1^{(K-1)})]]\}$. So the participation constraint will be: $G_K \geq D_K$, i.e., a ring member should obtain equal or more surplus as she stays in a ring than when she becomes a outsider. Once this constraint is satisfied, in equilibrium all invited bidders will accept the collusion initiator's proposal since it is not profitable for them to reject.

2.2.3 Optimization Problem

The collusion initiator's problem has now been largely simplified. The budget balance constraint narrows our search for the optimal mechanism to those efficient ones. The incentive compatibility for bidding constraint in fact pins down the bid assignment rule. Therefore, the number of ring members K remains as the only choice variable to maximize the ex ante per member surplus G_K subject to the participation constraint. Hence, an initially complicated endogenous ring formation problem now boils down to an optimal ring size problem.

Let us use $\beta_K^{in}(v)$ to denote the ring's bid assignment rule $\hat{\beta}_K(v)$ so that the notation $\beta_K^{in}(v)$ can more clearly represent the ring's bid in counter to the outsider's bid $\beta_K^{out}(v)$ in our subsequent analysis. Formally, this optimization problem can be formulated as follows:

$$\max_{K\in\{1,2,..N\}} G_K$$

subject to:

$$\beta_K^{in}(v) \in \arg\max_b \ (v-b) \times \Pr[b \geq \max_{j\in\{K+1,...N\}} \beta_K^{out}(v_j)]$$

$$\beta_K^{out}(v_j) \in \arg\max_b (v_j - b) \times \Pr\{b \geq \max[\max_{j'\in\{K+1,...N\}\setminus\{j\}} \beta_K^{out}(v_{j'}),\ \beta_K^{in}(v)]\}$$

$$G_K \geq D_K$$

where v is the highest order statistic in the K-dimensional vector of reports V and v_j is the valuation of the outsider j.

2.3 SOLUTION

Although the endogenous ring formation problem has been much simplified, a complete solution to it is still analytically impossible. The major difficulty is that the above incentive compatibility for bidding constraint can not yield a pair of closed form equilibrium bid functions. Nevertheless, our assumption that $f(v_i)$ is continuously differentiable and bounded away from zero on the common support $[\underline{v}, \overline{v}]$ will guarantee the existence and uniqueness of a monotonically increasing pure strategy equilibrium $\beta_K^{in}(v)$ and $\beta_K^{out}(v_j)$ for any given ring size K.[3] Therefore, for a given density, the above formulation can lead to a numerical solution to the optimal ring size, hence an endogenous collusion format. This paper will only deal with the analytical solutions, which will also bring us several important results. We will proceed by first characterizing some properties of the equilibrium bid functions.

2.3.1 Characterization of the Bid Functions

As before, $\beta_K^{in}(v)$ denotes the equilibrium bid function of the ring and $\beta_K^{out}(v_j)$ denotes the equilibrium bid function for the outside bidder j. Let $\lambda_K^{in}(\beta)$ and $\lambda_K^{out}(\beta)$ denote the inverse

[3] See the existence and uniqueness results in Maskin and Riley (2000b), (2003), Athey (2001) and Lebrun (2002) etc.

bid functions for $\beta_K^{in}(v)$ and $\beta_K^{out}(v)$ respectively. $\tilde{F}$ denotes the c.d.f of the highest order statistic v of $v_1, ...v_K$. Then using the monotonicity of $\lambda_K^{in}(\beta)$ and $\lambda_K^{out}(\beta)$, $\beta_K^{in}(v)$ and $\beta_K^{out}(v_j)$ must satisfy:

(I') $\beta_K^{in}(v) = \arg\max_b \{(v-b)\times[F(\lambda_K^{out}(b))]^{N-K}\}$

(II') $\beta^{out}(v_j) = \arg\max_b \{(v_j - b) \times \tilde{F}[\lambda_K^{in}(b)] \times [F(\lambda_K^{out}(b))]^{N-K-1}\}$

The first-order conditions derived from (I') and (II') will establish a system of differential equations, from which the following properties of bid functions can be easily obtained.

Lemma 3 *Properties of the equilibrium bid functions:*

(a) $\beta_K^{in}(\underline{v}) = \beta_K^{out}(\underline{v}) = \underline{v}$, $\beta_K^{in}(\overline{v}) = \beta_K^{out}(\overline{v}) = \beta_K^*$.

(b) $\beta_K^{in}(v) < \beta_K^{out}(v)$ *for all* $v \in (\underline{v}, \overline{v})$.

(c) $\frac{d\beta_K^{in}(v)}{dv}|_{v=\underline{v}} = 1 - \frac{1}{N-K+1}$, $\frac{d\beta_K^{out}(v)}{dv}|_{v=\underline{v}} = 1 - \frac{1}{N}$.

Proof. See the Appendix.

Property (a) is standard, which says that both bid functions have the same starting and ending points for any given ring size K. Property (b) shows that the outsiders always bid more aggressively than the ring for all $v \in (\underline{v}, \overline{v})$. Again, it is the standard weakness leading to aggression result. Property (c) deals with the motion of bids at the lower support. Marshall et al (1994) derive the same derivative results for the uniform distribution when they pursue a numerical solution. Here, we find that these derivative results actually hold for any density. All these properties will be useful for our future characterization of the optimal ring size.

2.3.2 Unconstrained Optimal Ring Size

The following proposition is the only analytical result on the optimal ring size that we can obtain for the general density of bidders' valuations. That is, if we ignore the participation constraint, the all-inclusive ring will maximize each member's expected surplus. We follow the common assumption that an all-inclusive ring wins the object with zero cost, which enables the ring to capture the whole surplus in the auction. But as the surplus has to be spread to all members, we can not immediately tell whether the per member surplus is also maximized. The following proposition gives us an affirmative answer.[4]

Proposition 1 *Under the defined collusive schemes, the all-inclusive ring maximizes the ex ante per member surplus once the participation constraint is ignored.*

Proof. See the Appendix.

Does the above proposition still hold once some participation constraint is imposed? Under the participation constraint assumption in Marshall and Marx (2004), i.e., a ring collapses when a single member deviates, an all-inclusive ring can definitely prevent any deviation, hence will be the optimal ring size.[5] However, we rarely if ever to observe an all-inclusive ring in practice especially when the number of bidders is large. As we argued before, the participation constraint adopted in this paper gives rise to a free-rider effect for outsiders. If the suppressed competition resulting from the collusion of large number of bidders creates a big free-rider effect, the all-inclusive ring may be very hard to sustain. Then a non all-inclusive ring will be justified as the endogenous collusion format in equilibrium. Because of the analytical difficulties, this result will be shown in the following section only for the uniform distribution.

2.3.3 Constrained Optimal Ring Size

The task in this section is to show that in order to guarantee the stability of the ring, the collusion initiator may only approach a fraction of all the bidders before she announces the collusion idea. So a non all-inclusive ring can be rationalized as bidders' endogenous optimal choice.

We assume now that bidders' valuations are iid uniform over the support $[0, \bar{v}]$. So the density function is $f(v) = \frac{1}{\bar{v}}$ and the c.d.f is $F(v) = \frac{v}{\bar{v}}$. Besides those properties of the bid functions in Lemma 3, we will derive some additional ones for this particular density. Let $\beta^{in}_{N-1}(v)$ denote the equilibrium bid functions for the ring of size $N-1$, and $\beta^{out}_{N-1}(v)$ the bid function for the only outsider. $\lambda^{in}_{N-1}(\beta)$ and $\lambda^{out}_{N-1}(\beta)$ denote their inverse functions

[4]Notice that the all-inclusive ring maximizes the per member surplus even with a posted reserve price.

[5]The per member surplus of the all-inclusive ring is $\frac{1}{N}\int_{\underline{v}}^{\overline{v}}[v-\underline{v}]d[F(v)]^N$, which is bigger than the surplus from noncooperative bidding $\frac{1}{N}\int_{\underline{v}}^{\overline{v}}[v-\beta(v)]d[F(v)]^N$, where $\beta(v)$ is the equilibrium bid function.

respectively. β^* denotes the common terminal bid at the valuation $\overline{v}$. Then we have the following lemma.

Lemma 4 *When bidders' valuations are iid uniform over the support* $[0, \overline{v}]$*, the equilibrium bid functions* $\beta^{in}_{N-1}(v)$ *and* $\beta^{out}_{N-1}(v)$ *have the following properties:*

(a) $\beta^{out}_{N-1}(v)$ *and* $\beta^{in}_{N-1}(v)$ *are concave and convex respectively.*

(b) $\lambda^{in}_{N-1}(\beta) > \dfrac{\overline{v}}{\beta^*}\,\beta$ *for all* $\beta \in [0, \beta^*]$.

(c) $\beta^* = \overline{v}\left(1 - C^{\frac{1}{N-2}}\right)$ *where* $C = \dfrac{N^N}{2^{2N-2}(N-1)^{N-1}}$.

Proof. See the Appendix.

Property (a) in the above lemma shows that both bid functions behave regularly. Property (b) is an immediate consequence of (a), which says that $\lambda^{in}_{N-1}(\beta)$, the inverse bid function of the ring, must lie above the line joining the origin and the common terminal bid β^* at the valuation $\overline{v}$. Property (c) gives an analytical expression for the terminal bid β^*, which can be shown to increase with the number of bidders. With these properties, Proposition 6 gives us a sufficient condition under which a non all-inclusive ring will be the endogenous collusive scheme.

Proposition 2 *If bidders' valuations are iid uniform over the support* $[0, \overline{v}]$*, a non all-inclusive ring will be the endogenous collusive scheme when the total number of bidders is bigger or equal to 10.*

Proof. See the Appendix.

Unfortunately, we can not solve analytically for the constrained optimal ring size for an arbitrarily given total number of bidders N. But we can be sure that it is definitely not the degenerate single-bidder ring. This assertion is stated in the following proposition.

Proposition 3 *If bidders' valuations are iid uniform over the support* $[0, \overline{v}]$*, any ring with at least two members can always capture more per member surplus than the noncooperative bidding.*

Proof. See the Appendix.

Therefore in the first-price auction a ring is very likely to exist since we show that endogenous collusion should include at least two members. At the same time, the collusion is generally not all-inclusive, especially when the number of bidders participating in the auction is large. So our conclusion is that bidders tend to form a nontrivial ring (ring size belongs to $\{2, .., N-1\}$ when $N > 2$) endogenously. This gives rise to two important implications. First, in MM (1992) the overall auction outcome is efficient under a strong cartel, which consists of all bidders. In contrast, we show here that once the collusion is endogenized, a nontrivial ring will be formed and it will create bidder asymmetry and consequently lead to inefficient outcome in the main auction. Therefore, it will be *socially beneficial* to outlaw collusion in auctions. The associated policy implication is that more resources need to be devoted to the enforcement of anti-collusion laws. Second, the nontrivial ring result heavily hinges on the free-rider effect in the first-price auction. Such an effect disappears in the second-price auction, which means that an all-inclusive ring might be more common there.

2.4 CONCLUSION

This paper studies how collusion emerges in a first-price auction. We endogenize collusion by allowing a collusion initiator to select a particular scheme and then propose it to a chosen number of other bidders. The main finding is that, when there are at least three bidders, an endogenously formed ring includes at least two members and is in general not all-inclusive. Since a partial ring creates bidder asymmetry, it makes a first-price auction inefficient. This finding provides a basis for laws that outlaw collusion in auctions.

This paper is only a first step to delve into the endogenous collusion problem in auctions. There are still many open questions left for future research. For example, we could consider cases where bidders are ex ante heterogeneous; where there is more than one collusion initiator among bidders leading to multiple rings in the auction; and other possible relaxations of those assumptions defining the bidders' collusive technology. With these extensions, we might discover more intriguing or profitable forms of collusion.

3.0 INFORMATION DISCLOSURE IN SEQUENTIAL AUCTIONS

3.1 INTRODUCTION

Many nondurable goods auctions are carried out repeatedly across periods, such as flower auctions and fish auctions, etc. It is interesting to observe that these goods are often sold via the Dutch auction format or its variants. The sale of flowers is a well known example for its use of the Dutch auction (about 85% of Netherlands cut flowers are handled by the Dutch auctions annually[1]). The fish auction is a natural variant of the Dutch format. Here then comes the puzzle. In the auction literature, we know that if bidder are risk-neutral and their valuations are independent and identically distributed, the first-price, second-price, Dutch and English auctions are revenue equivalent; and once their valuations are affiliated, the English auction generates the highest revenue followed by the second-price auction and then the Dutch and first-price auctions. In both cases, the Dutch auction never beats the other auction procedures in term of revenue. But why do people stick to it in various nondurable goods auctions? In this paper, we provide one explanation, that is, the Dutch auction can be revenue superior in a sequential environment that captures the essential features of the nondurable goods sale.[2]

There are four important features for most nondurable goods auctions. First, those goods are sold period by period because they are nondurable and the goods sold each period are approximately the same. Second, bidders' identities are the same across periods, and these bidders tend to be large buyers in the same line of business aiming for the retail

[1] The figure is quoted from the International Labor Organization working paper *The World Cut Flower Industry: Trends and Prospects*.

[2] Notice that in practice flowers are sold repeatedly via a kind of multi-unit Dutch auction. Because of the analytical difficulties, this paper only models the sequential single-unit auction.

resale of the auctioning goods. Third, because of the similarity of each period's goods, the valuations of the same bidder in two consecutive rounds tend to be correlated. Finally, due to this valuation correlation and constant bidder identity, each bidder can always infer some information on her rivals' current valuations from the released bidding results of previous rounds. This interperiod learning makes bidding behavior very different from that in a one-shot auction.

In this paper, we analyze the sequential sale via a highly stylized independent private value (IPV) model, where two bidders compete for two identical nondurable objects, each per period. In connection with the flower sale, the private value can be interpreted as the private gross profit (before deducting the bid) of each buyer in the industry. Then the goal of the auctioneer is to select an auction format with appropriate interperiod information release to maximize her overall revenue. We assume that the auctioneer will commit to one auction format for both periods. The commonly adopted formats include: the first-price and second-price auctions, the English and Dutch auctions. The choices of information disclosure at the end of each stage auction include: announcing winning or losing status, winning or losing bids or both.[3] This in turn defines 16 sequential auction formats, most of which are outcome equivalent as we will discuss later. The objective of this paper is to characterize the Perfect Baysian Nash Equilibrium (PBNE) in various sequential auction formats and then carry out the revenue comparison to find out the optimal one.

The distinction of this paper from other works on sequential auctions is briefly summarized as follows. It differs from McAfee and Vincent (1997) in the following respect. McAfee and Vincent (1997) deal with the sale of a single object, which will be resold if it can not be auctioned when all bids in a given period are below the reserve price. This paper, however, studies the sequential sale of two objects, each of which can always be auctioned at each period since we will assume no reserve price. The difference between this paper and Weber (1983) is that Weber considers a sequential auction with unit-demand and a bidder quits the auction once she obtains one unit, while this paper studies a sequential sale where a bidder pursues a unit every period.

[3] Notice that the *minimum* information released in each period is the winning or losing status because bidders need to know their entitlement to the object, which is nondurable hence must be consumed in the current period.

There are three contributions in this paper. First, we characterize the equilibria in various two-stage sequential auctions with multi-unit demand. Second, we analyze the effect of interperiod information release on bidding behavior and the revenue. Finally, from the revenue comparison of all sequential formats, we obtain the result that the standard Dutch auction is tied with the first-price auction with the winning-bid announcement, generating the highest revenue. Considering the implementation simplicity of the standard Dutch format relative to its first-price counterpart, this result may explain the pervasive employment of Dutch auctions in flowers and fish sale.

The remainder of the paper proceeds as follows. Section 2 describes the model. Equilibrium bidding behavior will be characterized in Section 3. Section 4 carries out revenue comparison. Section 5 concludes.

3.2 THE MODEL

This is a two-period sequential auction model with two risk-neutral bidders. The auctioneer has two identical nondurable goods for sale, one per period. The auction formats we consider include: sequential first-price, second-price, English and Dutch auctions with the announcement of winning and losing status, of winning bid, of losing bid and of both bids, at the end of the first stage. Bidder i's valuation is v_i, where $i = 1, 2$. Both valuations are iid over the unit interval [0, 1] according to the density $f(v)$, which is continuously differentiable and bounded away from zero. After each bidder observes her valuation at the first round, her valuation remains constant since then, i.e., her second-stage valuation will be the same as her first-stage one. So here we assume a perfect cross-period correlation of valuations. The discount factor is δ, where $\delta \in [0, 1]$. Each bidder will maximize her discounted sum of surplus and the auctioneer her discounted sum of revenue.

3.3 CHARACTERIZATION OF EQUILIBRIUM

Since we assume bidders are ex ante identical, it is natural for us to focus on symmetric monotonic PBNE. As we will see later, in some formats, bidders will have to adopt a mixed strategy. In this paper, we are going to restrict ourselves to those equilibria with a monotonicity requirement defined as follows. Let $I(v)$ and $S(v)$ denote the inf and sup of a bidder's randomized bids with valuation v at a given stage.[4] Whenever $v_2 > v_1$, we should have $I(v_2) > I(v_1)$ and $S(v_2) > S(v_1)$. Notice that the cost of this monotonicity requirement is that it may rule out pooling equilibria and may give rise to a non-existence problem, which is indeed the case as we see later. The benefit of it, however, is to ensure the revenues are compared within the same category of equilibria (monotonic equilibria only) so that we can maintain the maximum uniformity when discussing the ranking here relative to the other rankings in literature that typically use monotonic equilibria.

3.3.1 Outcome Equivalence Simplification

In the sequential IPV environment, the standard Dutch auction has the same equilibrium outcome as the first-price auction with the announcement of winning bid at each stage. This is because when a standard Dutch auction, e.g., the flower auction, ends, all bidders can publicly observe the winning bid. Of course, the standard Dutch auction technology can also be altered to accommodate all other information disclosure requirements. These variants of the standard Dutch auction are outcome equivalent to their sequential first-price counterparts too. So there is no loss of generality for us to focus our subsequent analysis on the sequential first-price auctions only.

Similarly, in the sequential environment, the standard English auction has the same equilibrium outcome as the second-price auction with the announcement of losing bid at each stage. That is because when a standard English auction ends, all bidders can publicly observe all the losing bids. Also, the standard English auction technology can be modified to meet all other information release requirements. Again, because of the outcome equivalence,

[4]A pure strategy is considered a degenerate mixed strategy.

our analysis can only be concentrated on the sequential second-price auctions.

Therefore, we will study the sequential first-price and second-price auctions with the announcement of winning or losing status, winning bid, losing bid and both bids in the rest of the paper.

3.3.2 Sequential Second-Price Auctions

The bidding behavior in the sequential second-price auctions is easy to characterize because bidders bid the same way regardless of interperiod information release structures. The following proposition states the equilibrium strategy.

Proposition 1 *In the two-stage sequential second-price auctions with perfect cross-period correlation of valuations, bidding one's own valuation at both stages constitutes a monotonic equilibrium.*

Proof. See the Appendix.

However here bidding one's own valuation will not constitute a dominant strategy equilibrium any more as in the one-shot second-price auction. Also, it is interesting to observe that different interperiod information disclosures do not affect bidders' bidding behavior at all. This is because bidding one's own valuation is still a dominant strategy for the last stage, which gives bidders no incentive to deviate from the equilibrium at the first stage. It is not obvious whether there exist other symmetric monotonic equilibria or not. The current equilibrium will be the only one if we assume bidders simply want to play a stage dominant strategy at each period.

3.3.3 Sequential First-Price Auctions

The bidding behavior in the sequential first-price auctions is much more complicated. Under different interperiod information release structures, bidders bid differently. Consequently the interperiod information disclosure plays a crucial role for the auctioneer's revenue. We begin with the analysis of information release of the first-stage winning/losing status, where the identity of winner or loser will be announced once the first-stage auction ends.[5]

3.3.3.1 Announcement of the First-Stage Winning/Losing Status

We are looking for a monotonic pure strategy symmetric equilibrium in this game and it has the following structure.

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.

2. At the second stage, if the bidder wins the first stage at the valuation v, she believes that her rival's valuation $\hat{v}$ falls in the interval of $[0,\, v)$ with the conditional density $\frac{f(\hat{v})}{F(v)}$.[6] Then she bids according to $\beta_1(v)$. While if she loses the first stage, she believes that her rival's valuation $\hat{v}$ will be in $[v,\, 1]$ with the conditional density $\frac{f(\hat{v})}{1-F(v)}$, and bids according to $\beta_2(v)$.

Notice that at the second stage each bidder' belief on her rival's valuation distribution is parameterized by her own private signal. However, in a standard one-shot auction environment, bidders' beliefs on valuation distribution are common knowledge. This difference makes it impossible to directly apply the standard equilibrium existence results in the auction literature to here. But similarity between the structure of our setting and the standard auction environment enables us to extend established approaches to the current case. The following proposition gives us a confirmation of the existence of the above equilibrium in the sequential first-price auction.

Proposition 2 *In the two-stage sequential first-price auction with perfect cross-period correlation of valuations, there exists a monotonic pure-strategy symmetric equilibrium under the first-stage winning/losing status announcement.*

Proof. See the Appendix.

The proof makes use of the results in Landsberger, Rubinstein, Wolfstetter and Zamir (2001), who study an auction environment with commonly known ranking of valuations. It is exactly our second-stage problem. Theorem 1 in Landsberger et al (2001) gives us

[5] Notice that announcing the winner's or the loser's identity discloses the same information in the two-bidder model.

[6] As we are looking for a symmetric monotonic equilibrium, a bidder's winning implies her rival's valuation is smaller than hers.

an equilibrium existence and uniqueness result among all differentiable bid functions for the second-stage problem. Using this result, the existence of the first-stage equilibrium bid function is immediate. In fact, we can further show that both stages' equilibrium bid functions must be differentiable drawing on the method in Maskin and Riley (2003), which combined with Theorem 1 in Landsberger et al (2001) yields the uniqueness of pure-strategy equilibrium for this two-stage problem. As we will not focus on the uniqueness issue in this paper, its proof is forgone.

The symmetric monotonic pure-strategy equilibrium can not be analytically solved. This makes a final revenue comparison impossible. But a revenue ranking is important to illuminate the role of information disclosure in sequential auctions. So we assume the valuation density $f(v) = 1$, i.e., we assume bidders' valuations are iid uniform over $[0, 1]$. Our subsequent revenue ranking will be under this simplifying assumption. In the current case the closed-form solutions to the bid functions still can not be obtained under the iid uniform assumption. However, we can find some qualitative features for the equilibrium bid functions, which are collected in the following proposition and sufficient for our final revenue comparison.

Proposition 3 *Under the assumption that bidders' valuations are iid uniform, the equilibrium bid functions have the following properties:*

a) $\beta_1(0) = \beta_2(0) = 0$ *and* $\beta_1(1) = \beta_2(1) = t^*$, *where* t^* *is both bidders' common terminal bid.*

b) $\beta_1(v)$ *and* $\beta_2(v)$ *are strictly concave and convex respectively.*

c) $\frac{3}{4}v > \beta_2(v) > t^*v > \beta_1(v) > \frac{1}{2}v$ *for all* $v \in (0, 1)$.

d) $\frac{2}{3} \geq t^* \geq \frac{5}{8}$

e) $\beta(v) = \frac{v}{2}(1-\delta) + \frac{\delta}{v}\int_0^v \beta_2(t)\,dt < \frac{v}{2}$ *for all* $v \in (0, 1)$.

Proof. See the Appendix.

Property a) says that two bid functions at the second stage has the same starting and ending points. This result is standard. Property b) shows that both bid functions behave regularly, which is a result due to the uniform distribution assumption. Property c) offers us some bounds to approximate the two bid functions. $\frac{v}{2}$ and $\frac{3v}{4}$ can be shown to be tangent to $\beta_1(v)$ and $\beta_2(v)$ at the point $v = 0$ respectively. So the two bid functions are enclosed by their respective tangent lines at the origin and separated by the line t^*v. Property d) gives us rather narrow bounds for the common end bid t^*. Property e) gives us an expression for the first-stage bid function, which is smaller than $\frac{v}{2}$ for all $v \in (0, 1)$, where $\frac{v}{2}$ is just the equilibrium bid function in a one-shot auction of two bidders with iid uniform signal. Also from Property e), we can see that the first-stage bid function is an increasing function of δ.

These properties will be useful for our revenue comparison. Also they give us a very intuitive understanding of bidders' behavior in this sequential first-price auction. First, because of information disclosure, bidders have less private information at the second stage than in a standard one-shot auction. So they can obtain less informational rent, which explains why both of them bid more aggressively in the second round than in a one-shot auction. Second, after the first-stage auction, the first-period loser will believe that her rival is stronger than her previous expectation while the winner will believe her rival is weaker than her previous expectation. This induces the loser to bid more aggressively than the winner at the second stage. Third, both bidders will bid less aggressively than in a one-shot auction in the first round, which we call *bid reduction* in this paper. Bid reduction is a direct result of bidders' optimal decision of intertemporal substitution. In the second stage, the first stage winner and loser bid quite differently, which provides an intertemporal arbitrage opportunity for bidders. In the first stage, it is profitable for a bidder to bid less than in a one-shot auction only. This is because by doing so at the second stage this bidder will have a higher chance to meet a first-stage winner, who is easier to defeat than a first-stage loser. This explains why in equilibrium a bid reduction can occur in the first stage. Finally, as to the effect of the discount factor δ, once it becomes bigger, the second-stage payoff has higher value for the bidder, which naturally promotes her intertemporal substitution, i.e., leading to larger first-stage bid reduction.

3.3.3.2 Announcement of the First-Stage Winning Bid Now, we come to the analysis of bidding behavior under more interperiod information disclosure—announcing the winning bid at the end of the first stage. So not only the winning and losing status, but also the winning bid becomes common knowledge at the second stage. Again, we are looking for a symmetric monotonic equilibrium. It is easy to show that there is no pure-strategy monotonic equilibrium in this game. So we focus our attention to the following equilibrium, where both bidders follow the same monotonic pure-strategy bid function in the first stage and the first-stage winner will adopt a mixed strategy at the second stage.

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.

2. At the second stage, if the bidder wins the first stage with her valuation v, then she believes that her rival's valuation falls in the interval of $[0,\ v)$ with the conditional density $\frac{f(\hat{v})}{F(v)}$. She will randomly choose a bid b on the support $(t_*, t^*]$ with density $g^v(b)$.[7] While if a bidder loses the first stage at the valuation v and infers that the winning valuation is $\tilde{v}$ from the winning bid announcement, she will bid $\beta_2^{\tilde{v}}(v)$.[8]

There are two important features of the above equilibrium. One is that the first-stage winner adopts a randomized strategy at the second stage. The other is that both bidders have to condition their second-stage bids on the announcement of the first-stage winning bid. The equilibrium existence result for general density $f(.)$ can be shown in the similar backward induction manner as in the Proof of Proposition 2. Notice that the second-stage equilibrium is a generalization of the asymmetric auction example in Vickrey (1961). So by applying the refinement argument in Vickery (1961), we will have a unique equilibrium outcome here with an additional assumption that bidders will always choose strategies involving least mixing. For the purpose of revenue comparison, we will derive the specific equilibrium strategies only for the uniformly distributed valuations.

Proposition 4 *Under the assumption that bidders' valuations are iid uniform, in the two-stage sequential first-price auction with perfect cross-period correlation of valuations, bidders will exhibit the following equilibrium behavior under the first-stage winning bid announcement: both bidders bid* $\frac{v}{2}$ *in the first stage; the first-stage winner randomizes over* $(\frac{v}{2}, \frac{3v}{4}]$ *according to the c.d.f.* $G^v(b) = \frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}$*; the first-stage loser will not bid if her valuation* $v \in [0, \frac{\tilde{v}}{4})$*, where* $\tilde{v}$ *is the inferred valuation of the first-stage winner, and bid* $\tilde{v} - \frac{\tilde{v}^2}{4v}$ *if* $v \in [\frac{\tilde{v}}{4}, \tilde{v})$.

[7] Notice that for different v, the randomization density g is also different.

[8] Also the functional form of $\beta_2^{\tilde{v}}(.)$ is parameterized by the announcement.

Proof. See the Appendix.

The above is a symmetric monotonic equilibrium as we define it at the beginning of Section 3. Notice that if the first-stage winner's valuation is $\tilde{v}$ and the first-stage loser's valuation is v, the winner will bid above $\frac{\tilde{v}}{2}$ and the loser will bid $\tilde{v} - \frac{\tilde{v}^2}{4v}$. $\tilde{v} - \frac{\tilde{v}^2}{4v}$ is smaller than $\frac{\tilde{v}}{2}$ for all $v < \frac{\tilde{v}}{2}$. This means that in equilibrium when the first-stage loser's valuation is smaller than a half of the winner's, the loser always loses the second stage. Under this contingency, other bid functions for the first-stage loser can also constitute a monotonic equilibrium as long as the submitted bid is smaller than a half of the winner's valuation and at the same time prevents the winner's deviation. Of course, these equilibria will all yield the same outcome. But our equilibrium in Proposition 4 is the only one that can describe the first-stage loser's strategy in just one function, hence making the derivation of the first-stage bid function tractable.[9]

The intuition for the above equilibrium bidding behavior is straight forward. First, since the first-stage winner's valuation is always commonly known at the beginning of the second stage, it is not surprising that the winner will randomize in order to offset this information asymmetry. Second, the first-stage loser obtains some informational advantage at the second stage, i.e., knowing the winner's valuation. As the loser is weaker than the winner in the first place, this extra information enables both bidders to compete on a relatively *level* ground. So in equilibrium no bidder's second-stage bid function can dominate the other. In contrast, in the previous case the loser's bid function always lies above that of the winner. Third, there is no bid reduction in the first stage because there exists no intertemporal arbitrage opportunity. At the second stage, it is as hard to defeat a first stage loser as to defeat a winner since now two bidders' strength is brought in line with each other by the loser's informational advantage. So no bidder has the incentive to overbid or underbid in the first stage. Finally, the discount factor δ does not enter the first-stage bid function. This is natural since bidders do not need to consider the intertemporal substitution at all.

[9] Kaplan and Zamir (2000) derive almost the same second-stage equilibrium in their Proposition 5.2. The only difference is that they assume that the first-stage loser bids her own valuation when her valuation is less than a half of the winner's inferred valuation. The equilibrium in Kaplan and Zamir (2000) truncates the bid function $\beta_2^{\tilde{v}}(v)$, which makes the analytical derivation of the first-stage bid function rather difficult.

3.3.3.3 Announcement of the First-Stage Losing Bid Now instead of announcing the winning bid, the auctioneer can also choose to only release the first-stage losing bid in a sequential first-price auction. Before we start to solve for the equilibrium, our intuition from the previous analysis will lead to the following conjecture. The first-stage loser is already on the weak side at the second stage. Announcing her bid gives further informational advantage to the winner, which only exacerbates the loser's position and will induce quite aggressive bidding for the loser at the second stage. Consequently there will be a large bid reduction in the first stage. This is because by following the same intertemporal arbitrage reasoning, the first-stage bidder will have a high incentive to underbid so that her chance to meet an aggressive first-stage loser can be decreased. Although the above conjecture is in the right direction, the following proposition shows that the strength of the first-stage bid reduction can be so big that no symmetric monotonic equilibrium can be supported regardless of the value of discount factor δ.

Proposition 5 *In the two-stage sequential first-price auction with perfect cross-period correlation of valuations, there is no symmetric monotonic equilibrium under the first-stage losing bid announcement.*

Proof. See the Appendix.

It is not clear if there exist other asymmetric or non-monotonic equilibria in this case. But the non-existence of an important class of equilibria here may shed light on the phenomenon that it is very rare to observe any real world sequential first-price or Dutch auction arrangements where the auctioneer discloses each stage's losing bid only.

3.3.3.4 Announcement of both the First-Stage Winning and Losing Bids Under this format, the auctioneer releases all the information available to her, i.e., both the winning and losing bids, by the end of the first stage. If we assume a symmetric monotonic pure-strategy bid function for the first stage, then both bidders will exactly infer each other's valuation from the interperiod information release. At the second stage, if a bidder loses the first stage with a valuation v, then she will face a rival whose valuation is $\hat{v}$ where $\hat{v} > v$. The type of the second-stage equilibria we are looking for is similar to those discussed in Blume (2003). The loser will randomize according to the density $h(v)$ over the support $[v' - \eta, v')$

where $v' \in [v, \hat{v})$, $\eta > 0$ and the winner with high valuation $\hat{v}$ bids v'. So the sup of the loser's randomized bids should be between her and the winner's valuations and the whole randomization builds a wall that prevents the winner from bidding less than this sup. We assume that bidders choose strategies with least mixing. This assumption will lead to a large class of equilibria parameterized by v', η, and the randomization density $h(v)$. It is analytically intractable to derive those equilibria when the parameters remain general. So we will only focus on a subset of this class of equilibria. As we will see below, this subset of equilibria turn out to generate the same auction revenue.

Since $v' \in [v,\ \hat{v})$, it is natural to set v' as a weight average of v and $\hat{v}$, i.e., $(1-k)v + k\hat{v}$, where $k \in (0,\ 1)$. For analytical convenience, we set η as $v' - v$, i.e., we simply let the first-stage loser randomize over the support of $[v,\ (1-k)\,v + k\hat{v})$. We also assume $h(v)$ to be a uniform density. So the equilibrium will be as follows:

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.

2. At the second stage, both the first-stage winner's and loser's valuations $\hat{v}$ and v become common knowledge. The loser randomizes uniformly over the support $[v,\ (1-k)\,v + k\hat{v})$, while the winner bids $(1-k)\,v + k\hat{v}$.

The above equilibrium is a symmetric and monotonic one according to our definition. But in order to support it, we need an extra assumption specified in the following lemma.

Lemma 1 *The above equilibrium is supportable only when* $\frac{1}{2} \geq k \geq \frac{\delta}{1+\delta}$.

Proof. See the Appendix.

When $k > \frac{1}{2}$, the first-stage loser's randomization density wall is not high enough to prevent the winner's penetration (deviation) at the second stage. While if $k < \frac{\delta}{1+\delta}$, a bidder can always profitably mimic the zero valuation at the first stage. This deviation can only be prevented by asking the first-stage loser to bid sufficiently above her valuation so as to stop the winner from bidding too leniently at the second stage, which in turn will eliminate bidders' incentive to underbid in the first stage. The final equilibrium is summarized in the following proposition.

Proposition 6 *Under the assumption that bidders' valuations are iid uniform, in the two-stage sequential first-price auctions with perfect cross-period correlation of valuations, bid-*

ders will exhibit the following behavior under both the first-stage winning and losing bids announcement: both bidders bid $\frac{(1-\delta k)v}{2}$ *in the first stage, where* $k \in [\frac{\delta}{1+\delta}, \frac{1}{2}]$. *The first-stage loser randomizes uniformly over* $[v_l,\ (1-k)\, v_l + kv_h)$ *and the first-stage winner bids* $(1-k)\, v_l + kv_h$, *where* v_l *and* v_h *are the realized valuations of the loser and the winner respectively.*

Proof. See the Appendix.

The above result has almost the same interpretation as in the case of sequential first-price auctions with the announcement of winning/losing status. The first-stage loser bids aggressively at the second stage. Bid reduction appears in the first stage, which becomes more serious when discount factor δ gets bigger. We can also check that the total revenue in this two-stage auction is $\frac{1}{3} + \frac{1}{3}\delta$ (see Lemma 4 in next section), which does not contain the weight k. It means that bidders' intertemporal substitution exactly cancels out the effect of k. So the revenue remain the same within this subset of equilibria. We conjecture that this property may be extended to the original large class of equilibria.

3.4 REVENUE COMPARISON

By now, we have either derived equilibria or their properties in all considered sequential auction formats under the iid uniform assumption. We are ready to compare the revenue generated from each of them. Due to the simplification at the beginning of Section 3, we only need to consider the following four revenues. First, the revenue from the sequential second-price auctions, which is denoted as R_1. Second, the revenues from the sequential first-price auctions with the announcement of the first-stage winning/losing status, winning bid, and both winning and losing bids. We denote them as R_2, R_3, R_4 respectively. Notice that there is no symmetric monotonic equilibrium in the sequential first-price auction with the announcement of the first-stage losing bid, so we leave it out of our revenue comparison.

From Proposition 1, we know that $R_1 = \frac{1}{3} + \frac{\delta}{3}$. The calculations for the other revenues are much more involved. Lemma 2 gives us an upper bound for R_2 although no bid functions can be analytically obtained for the case.

Lemma 2 *The revenue* $R_2 < \frac{1}{3} + \frac{16}{49}\delta$.

Proof. See the Appendix.

The bound of R_3 is stated in Lemma 3.

Lemma 3 *The revenue* $R_3 > \frac{1}{3} + \frac{1}{3}\delta$.

Proof. See the Appendix.

Finally, we need to calculate R_4. Lemma 4 states the result.

Lemma 4 *The revenue* $R_4 = \frac{1}{3} + \frac{1}{3}\delta$.

Proof. See the Appendix.

Proposition 7 *Under the assumption that bidders valuations are iid uniform, the revenue ranking is:* $R_3 > R_1 = R_4 > R_2$.

The following table summarizes all the revenue results we have obtained. We use I, II and III to denote R_3, R_1 and R_2 respectively. So I, II and III represent revenues in descending order. The first row of the table represents 4 different information release structures, where from left to right more and more information is disclosed. The first column represents four basic stage auction rules, where the first-price and second-price are outcome equivalent to the Dutch and English auctions respectively.

Table I. Revenue from Sequential Auctions with iid Uniform Valuations

Auction Formats	W/L Status	W Bid	L Bid	W & L Bids
First/Dutch	III	I	—	II
Second/English	II	II	II	II

From the above table, we can easily see that the interperiod information disclosure is immaterial in the sequential second-price and English auctions because the intertemporal learning does not affect bidders' equilibrium bidding. Notice that in sequential environment, the first-price and Dutch formats can revenue dominate the second-price and English formats. This is because under the first two the auctioneer has an extra device, i.e., the intertemporal information disclosure, to affect bidders' bidding and increase the total revenue.

It is commonly known that more information revelation from the auctioneer can further facilitate bidders' competition, hence increasing the revenue. A series of theorems in Milgrom and Weber (1982), which we abbreviate as MW hereafter, show that the public reporting policy never decreases the revenue in all auction formats. But the above table tells us that more information does not necessarily increase the revenue. From the announcement of the winning/losing status to the announcement of the winning bid only, revenue from the sequential first-price auctions increases. But as more information is released, i.e., both the winning and losing bids are announced, the revenue drops again.

Then we might ask why our results are so different from those in MW and how to explain the relationship between the intertemporal information disclosure and the auction revenue in our environment. There are two critical differences between our environment and that in MW. First, MW considers one-shot auction, while this paper studies sequential auctions. Second, in MW, the auctioneer's information affects both bidders' own valuations and their inference of their rivals' valuations. So the information disclosure creates both the valuation-increasing effect because bidders' valuations are assumed to be monotonically increasing with the announced signal, and the inference effect. But in this paper, the auctioneer's information does not change bidders' own valuations and we only have the inference effect here. In our setting, it is still true that more information disclosure never decreases the *stage* auction revenue. Drawing on the results in the proofs of Lemma 2 to 4, it is easy to check that the *second-stage* revenues in the first-price and Dutch formats are consistently improved by more and more intertemporal information release. However, as auctions are conducted sequentially, increased second-stage revenue does not guarantee a higher overall two-stage revenue because of bidders' intertemporal substitution. If the interperiod information structure is such that an intertemporal arbitrage opportunity exists for bidders to profitably shade their bids in the first stage, then in equilibrium a first-stage bid reduction will occur which in turn will lead to a decreased first-stage revenue. So overall, the two-stage revenue may not be enhanced. Therefore, the best information release structure should be the one that not only intensifies the second-stage bidding competition but also eliminates the intertemporal arbitrage opportunity, hence the first-stage bid reduction. The announcement of the first-stage winning bid in the sequential first-price and Dutch auctions just satisfies this informational requirement hence yielding the highest revenue among all considered formats. Then we conclude that it is not the information volume but the information structure that actually matters in term of revenue maximization in sequential auctions. The general rule is to give the auction loser some informational advantage. If we push the above argument further, it is natural to ask whether there exists such an information structure that provides an opposite intertemporal arbitrage incentive to induce overbidding in the first stage. This is possible only when the first-stage loser bids less aggressively than the winner at the second stage, which may happen under other valuation evolution assumptions but not in our setting.

3.5 CONCLUSION

This paper characterizes equilibria in various two-stage sequential auction formats under all possible forms of interperiod information release in an IPV model. We study the role of interperiod information disclosure in affecting bidders' intertemporal learning, bidding, and auction revenue. Unlike Milgrom and Weber (1982), who show in their model that it is always good for the auctioneer to commit to complete information revelation, we find that this is not necessarily the case in sequential auctions. Information disclosure does not affect the revenue in the sequential second-price and English auctions. In the sequential first-price and Dutch auctions, more information release can even decrease the revenue. This is because in sequential environment bidders' intertemporal substitution may lead to bid reduction in the first stage, which outweighs the second-stage revenue gain from the interperiod information release, hence decreasing the overall revenue. We show that the standard sequential Dutch auction or the first-price auction with the first-stage winning bid announcement generates the highest revenue among all considered formats just because their information release structure facilitates the second-stage competition and at the same time avoids the first-stage bid reduction. As a first step study of sequential auctions with multi-unit demand, our results are derived in a two-stage two-bidder model. Extensions to arbitrary number of bidders and stages; more general distribution and interperiod relation of bidders' valuations, etc., are left for future work.

4.0 A RESALE EXPLANATION FOR DECLINING PRICES IN SEQUENTIAL AUCTIONS

4.1 INTRODUCTION

The declining price trend in sequential auctions has long been a puzzle in auction research. Empirical findings of declining prices have been reported in such papers as Milgrom and Weber (1982*b*) in transponder-leases auctions; Ashenfelter (1989) , McAfee and Vincent (1993) and Ginsburgh (1998) in wine auctions; Ashenfelter and Genesove (1992) in real-estate auctions; Beggs and Graddy (1997) in art auctions and Gerard J. van den Berg, Jan C. van Ours and Menno P. Pradhan (2001) in flower auctions, etc. In a sequential auction of identical objects, we normally expect a similar sale price for each object. This is analytically shown in Weber (1983). So the phenomenon that prices decline in a repeated sale of identical objects poses an anomaly, which is often termed as *declining price anomaly* in the auction literature. A number of theoretical studies explain this declining price anomaly from various perspectives. From the perspective of bidder preferences, McAfee and Vincent (1993) attribute the declining price trend to the non-decreasing absolute risk aversion of bidders; and Branco (1997) to synergies. From the perspective of auction structures, Milgrom and Weber (1982*b*) suggest that the use of agents in auctions may explain the declining prices; Black and De Meza (1993) explain this price trend with a buyer's option, which is that the winner of the first auction has the opportunity to buy the remaining objects at the winning price; Von der Fehr (1994) and Menezes and Monteiro (1997) relate the declining price trend to the auction participation costs. From the perspective of the nature of the objects, Engelbrecht-Wiggans (1994), Bernhardt and Scoones (1994), and Gale and Hausch (1994) explain the declining prices with heterogeneity of the objects.

A common characteristic of the above theoretical literature is that almost all of them assume bidders have single-unit demand, i.e., once a bidder obtains one unit of the object at a given stage auction, she will not participate in following stages. However, in many cases where a declining price path is detected, this assumption does not seem to be appropriate. For example, in the condominium auctions (Ashenfelter and Genesove, 1992) and flower auctions (Gerard J. van den Berg, Jan C. van Ours and Menno P. Pradhan, 2001), bidders tend to be the investors, who purchase the objects not for their own consumption but for the resale values. Therefore, these bidders will participate in the auction every period as long as they remain in business. Then it is natural to ask if there are some new theoretical explanations for the declining price anomaly that rest upon the assumption that bidders have multi-unit demand.

The difficulty with the sequential auction with multi-unit demand is that bidders become asymmetric after the intertemporal inference following the first-round auction. This in general makes an analytical solution impossible. But if we dispose of the intertemporal inference by allowing each bidder to have a random draw of signal in every round, the sequential auction will then be reduced to the repeated auction. This paper proposes a sequential common-value auction framework that can both maintain certain degree of intertemporal inference and at the same time accommodate bidder symmetry. The crucial assumption of our model is that the common market resale price at a given period is an aggregation of both a common fundamental and all bidders' idiosyncratic tastes.[1] The fundamental is assumed to be a martingale and bidders' idiosyncratic tastes are assumed to be cross-period iid. Then for a given object, its current-stage resale price will be parameterized by the realized resale price of the previous stage. This cross-period correlation of resale prices aims to capture the persistence of consumer preferences in the final consumption market.[2]

In this paper, we obtain the following two findings. First, bidding prices decline in expectation in the two-stage sequential auction under an sufficient equilibrium condition we propose. Second, we characterize the necessary properties of symmetric monotonic equilibria in the finite N-stage and the infinite-stage cases. In the former case, the price trend remains

[1] Notice that the declining price anomaly refers to the declining *bidding* prices in a sequential auction, not the resale prices we mention here.

[2] Notice that here bidders are the investors who aim for the resale of the objects to the final consumers.

constant and drops only at the last stage; in the latter case, we have a constant price trend throughout.[3]

The remainder of the paper proceeds as follows. In Section 2, we set up the model. Section 3 gives the equilibrium solutions. Section 4 concludes.

4.2 THE MODEL

We first set up the sequential first-price common-value auction model in a two-stage environment. There is one nondurable object for sale at each period t, where $t = 1, 2$. There are N risk-neutral bidders. There is no reserve price. At the end of each period t, all bidders' bids submitted in this period will be announced. Bidders pursue the object not for their own consumption, but aim to resell it at the current-period resale price P_t. The value of P_t is realized immediately *after* the object is auctioned at period t and P_t becomes publicly observable since then. An object can always be auctioned at a given period because there is no reserve. We assume that it can always be resold at that period due to its perishability. Bidder $1, 2, ...N$ observe their private signals $X_t^1, X_t^2, ...X_t^N$ respectively at period t. All bidders' signals are iid according to the density $f(.)$ over the support $[0, \omega]$. These signals represent bidders' idiosyncratic tastes for the object at a given period. We assume that P_t is the aggregation of all bidders' individual

tastes and some fundamental value.[4] The fundamental value is an unobservable random variable Θ_t, which is drawn from a density with mean $\bar{\theta}$ at period 1 and follows a martingale process at period 2, i.e., $E\Theta_2 = \theta_1$. So the fundamental value itself does not exhibit any trend in expectation. We assume $P_t = \alpha\Theta_t + (1-\alpha) U(X_t^1, X_t^2, ...X_t^N)$, where $U(.)$ is both

[3]Weber (1983) produces a constant price trend too but in a finite-stage, single-unit demand setting.

[4]Ashenfelter and Genesove (1992) produce a time-series plot of the condominium auction bids, which reflects bidders' private signals, and the associated resale prices. They discover that there exists a strong correlation between the resale prices and the bids submitted in auctions. It seems that these two variables are linked and we assume here that the linkage is through an unobservable fundamental.

symmetric and increasing in each of its arguments. The resale price P_t is a weighted average of the fundamental and an aggregation of the collective signals represented by $U(.)$.[5] We assume the fundamental value and bidders' tastes are statistically independent. We will use the lower case notations p_t, θ_t and x_t^i to denote the realized values of P_t, Θ_t and X_t^i respectively hereafter. The discount factor is δ.

The generalization of the setup to arbitrary stages is straightforward—just change $E\Theta_2 = \theta_1$ to $E\Theta_{t+1} = \theta_t$ where t is any natural number smaller than the total number of stages.

4.3 EQUILIBRIUM

4.3.1 The Two-stage Problem

In the two-stage sequential auction, we are looking for a monotonic symmetric equilibrium. The equilibrium will take the following form:

Time 1. At the first stage, bidder i observes a signal x_1^i and bids according to $\beta_1(x_1^i)$.

Time 2. The object is awarded to the bidder with the highest bid and all bids are announced. Then all bidders' private signals become common knowledge from the bid function $\beta_1(x_1^i)$. Whenever a bid not belonging to the support $[\beta_1(0), \beta_2(\omega)]$ is observed, its associated signal will always be inferred as zero.[6]

Time 3. The unobservable fundamental value Θ_1 is realized as θ_1. The resale price p_1 is formed according to $p_1 = \alpha\theta_1 + (1-\alpha) U(x_1^1, x_1^2, ...x_1^N)$ and then becomes publicly observable.

Time 4. The first-stage winner resells the object at the price p_1.

Time 5. At the second stage, bidder i observes a signal x_2^i and bids according to $\beta_2\left(p_1, x_1^1, x_1^2, ...x_1^N, x_2^i\right)$, where the second-stage bid function conditions on all available information.

[5] We can make more general assumption for P_t like $P_t = \hat{U}(\Theta_t, X_t^1, X_t^2, ...X_t^N)$. While in such situations, once $\hat{U}(.)$ is not linear in $\Theta_t, \hat{U}(\Theta_t, .)$ tends to exhit certain trend even when Θ_t is a martingale. For example, if $\hat{U}(.)$ is concave or convex in $\Theta_t, \hat{U}(\Theta_t, .)$ turns out to be a submartingale or supermartingale respectively. To avoid a price trend brought by the evolution of the fundmental vaule itself rather than bidders' strategic interactions, we choose to focus on $\hat{U}(.)$ being linear in Θ_t in this paper.

[6] Since we can freely specify the off-equilibrium path belief, this assumption helps to restrict the equilibrium bids to a closed interval in the simplest way.

We will start to solve for the above equilibrium from the second stage.

4.3.1.1 The Second-stage Problem Let us assume for the moment that all bidders follow a symmetric monotonic pure-strategy bid function truthfully at the first stage. With the disclosure of all bidders' bids, the first-stage private signals $x_1^1, x_1^2, ..., x_1^N$ become common knowledge. Since p_1 is publicly observable at the second stage, the unobservable θ_1 can be inferred from the equation of resale price formation described above. Bidders' signals are cross-period independent, while Θ_1 is correlated with Θ_2, so the *only valuable* information bidders will condition on at the second stage is the inferred true value of θ_1. The first-stage information $p_1, x_1^1, x_1^2, ...x_1^N$ will affect the bid only through the aggregated variable θ_1. Then the second-stage problem becomes a standard one-shot auction with public information. Now we introduce some new notation: $X_t^{-i} = \{X_t^1, X_t^2, ...X_t^N\} \backslash \{X_t^i\}$; $x_t^{-i} = \{x_t^1, x_t^2, ...x_t^N\} \backslash \{x_t^i\}$; $Y_t^i = \max_{s \neq i} X_t^s$. Then the following proposition is immediate.

Proposition 1 *Assuming that a symmetric monotonic equilibrium exists, the second-stage equilibrium bid function* $\beta_2 \left(p_1, x_1^i, x_1^{-i}, x_2^i\right)$ *equals:*

$p_1 - (1-\alpha)\, U(x_1^i, x_1^{-i}) + (1-\alpha)\, E_{Y_2^i}\{E_{X_2^{-i}}[U(Y_2^i, X_2^{-i})|Y_2^i]|Y_2^i < x_2^i\}$.

Proof. See the Appendix.

It is easy to see that the above second-stage bid function is monotonically increasing in a bidder's second-stage signal but decreasing in all bidders' first-stage signals. In our model, this feature is the key that leads to a declining price path as we will see soon.

4.3.1.2 The First-stage Problem Now we start to solve for the first stage equilibrium bid function. According to the second-stage bid function in Proposition 1, we observe that if bidder i mimics a higher type than her true one in the first stage, her opponents will bid lower in the second stage. This fact can be understood through bidder i's opponents' intertemporal inference of the unobservable fundamental. Once the opponents observe a high bid at the first stage, i.e., a high signal from bidder i, their inference of the first-stage fundamental value will be low for a given resale price p_1 according to its functional form. Since the fundamental is a martingale, the opponents will be induced to believe that the mean of the second-stage fundamental is low hence bid low.

So bidder i's logic for mimicking a higher type at the first stage can be described as follows: "I know that I may bid a little bit higher than the expected resale value of the object at the first stage. But I just want to create a false image of narrow margin between my bid and the realized resale price. Then my opponents will believe that this object's resale value does not quite live up to my high taste, hence may have a low fundamental. They then bid low at the second stage under their pessimistic expectations and I can easily win the object with a low bid at that time. So my loss from overbidding in the first stage can be compensated by the gain from underbidding in the second stage." In equilibrium of course, each bidder will have to bid truthfully. The above argument on bidders' intertemporal incentives explains why in equilibrium the truthful bidding in the first stage can turn out to be more aggressive than that in the second stage.

The formal equilibrium derivation is outlined as follows. At the first stage, bidder i with a private signal x_1^i will maximize the expected overall two-stage payoff $\Pi = \Pi_1 + \delta E_{X_1^2,\ldots,X_1^N,\Theta_1,X_2^1}\Pi_2$, where Π_1 and Π_2 are the first and second-stage payoffs respectively. We assume bidder i mimics a type $z \neq x_1^i$ at the first stage. Then following any pair of (z, x_1^i), there is an expected continuation optimal second-stage payoff $E_{X_1^2,\ldots,X_1^N,\Theta_1,X_2^1}\Pi_2(z, x_1^i)$ for bidder i. So the first-stage bid function will be derived in such a way that setting $z = x_1^i$ will maximize the overall payoff Π. Let $[E_{X_1^{-i}}U(.,.)]'$ represent the derivative with respect to the first argument of the function $U(.,.)$. Then the following proposition gives the first-stage equilibrium bid function.

Proposition 2 *Assuming that a symmetric monotonic equilibrium exists, the first-stage equilibrium bid function $\beta_1(x_1^i)$ equals:*

$$\alpha\bar{\theta} + (1-\alpha)\, E_{Y_1^i}\{E_{X_1^{-i}}[U(Y_1^i, X_1^{-i})|Y_1^i] + \frac{\delta[E_{X_1^{-i}}U(Y_1^i, X_1^{-i})]'}{N[F^{N-1}(Y_1^i)]'}|Y_1^i < x_1^i\}.$$

Proof. See the Appendix.

We have derived both stages' bid functions by assuming a symmetric monotonic equilibrium exists, the key point that remains to be checked is whether this pair of bid functions indeed constitute a symmetric monotonic equilibrium. The following proposition gives the answer.

4.3.2 A Sufficient Equilibrium Condition

This section shows that a symmetric monotonic equilibrium exists for certain types of distribution $F(.)$ and aggregation function $U(.)$. The following proposition states a sufficient equilibrium condition and confirms the declining equilibrium price trend.

Proposition 3 *When the expression* $\dfrac{[E_{X_1^{-i}}U(X_1^i, X_1^{-i})]'}{[F^{N-1}(X_1^i)]'}$ *is increasing in* X_1^i*, the bid functions derived in the above two propositions constitute a symmetric monotonic equilibrium. Also,* $E_{X_1^i}\beta_1(X_1^i) > E_{P_1,X_1^i,X_1^{-i},X_2^i}\beta_2(P_1, X_1^i, X_1^{-i}, X_2^i)$ *.*

Proof. See the Appendix.

Mathematically, the monotonicity of $\dfrac{[E_{X_1^{-i}}U(X_1^i, X_1^{-i})]'}{[F^{N-1}(X_1^i)]'}$ guarantees the monotonicity of the first-stage bid function, which in turn ensures the monotonicity of the second-stage bid function. Under the above equilibrium condition, the bidding prices will exhibit declining trend. Intuitively, this sufficient equilibrium condition can be understood through the analysis of bidders' incentives. On the one hand, as we argued in Section 3.1.2., bidders have the incentive to "overbid" in the first stage in order to obtain the second-stage advantage. The term $[E_{X_1^{-i}}U(X_1^i, X_1^{-i})]'$ is just reduced from the expression that measures the gain from the first-stage over-bidding. On the other hand, it is commonly known that bidders should shade their bids sufficiently below their signals in a common-value auction to avoid the winner's curse. The term $[F^{N-1}(X_1^i)]'$ represents the winning probability and can be considered as a measure for the loss from the first-stage over-bidding due to an exacerbated winner's curse. The first-stage bidders then face these two conflicting incentives and they need to evaluate their aggregated effect, which is measured by the quotient of these two terms. To preserve the pure-strategy solution to the monotonic decision problem at the first stage, it is natural to require that this overall effect to be monotonic in each bidder's type. At the second stage, there is no such conflicting incentives, hence the equilibrium solution is standard.

It is easy to check that the difference between $E_{X_1^i}\beta_1(X_1^i)$ and $E_{P_1,X_1^i,X_1^{-i},X_2^i}\beta_2(P_1, X_1^i, X_1^{-i}, X_2^i)$ is $\dfrac{\delta(1-\alpha)}{N}E_{X_1^i,Y_1^i}\{\dfrac{[E_{X_1^{-i}}U(Y_1^i, X_1^{-i})]'}{[F^{N-1}(Y_1^i)]'}|Y_1^i < X_1^i\}$. From the expression of this difference, we can see that the price drop tends to be more severe when we have a larger δ or a smaller α. Bigger δ means that gaining the second-stage advantage is more important for the overall payoff. In order to obtain bigger second-stage advantage, bidders should bid more aggressively in the first stage, leading to a larger price decline. Smaller α implies that the individual tastes are more important in the formation of the resale price. So the strategic revelation of bidders' types can affect the total payoff in a higher degree, which in turn generates a larger price drop. The above result of declining prices will still remain valid in a sequential second-price common-value auction and other auction variants as long as the incentive structure described in Section 3.1.2 is preserved.

4.3.3 Generalization

We then ask whether the above framework can be generalized to finite N (N>2) or infinite stages and whether the declining price trend is still preserved there. In principle, our setup can accommodate an analytical solution for the finite-stage problem since bidders remain symmetric at any given round after processing the information from all previous rounds. Then a backward induction will derive each stage's bid function. However, this backward induction is tractable only when we know the specific forms of the aggregation function $U(.)$ and the density $f(.)$. In a two-bidder three-stage example, where $f(.)$ is a uniform density over $[0,1]$ and $P_t = \frac{1}{3}(X_t^1 + X_t^2) + \frac{2}{3}\Theta_t$, we can find that the equilibrium bid functions are $\beta_1(x_1^1) = \frac{1}{3}x_1^1 + \frac{2}{3}\bar{\theta} + \frac{\delta}{6}$, $\beta_2(x_2^1, \theta_1) = \frac{1}{3}x_2^1 + \frac{2}{3}\theta_1 + \frac{\delta}{6}$ and $\beta_3(x_3^1, \theta_2) = \frac{1}{3}x_3^1 + \frac{2}{3}\theta_2$. Replacing $\frac{2}{3}\theta_t$ with $p_t - \frac{1}{3}(x_t^1 + x_t^2)$ for $t = 1, 2$, we can obtain the final expressions for the three bid functions.

When $U(.)$ and $f(.)$ remain general, the backward induction becomes intractable as the number of stages is more than 2. It is not clear whether a symmetric monotonic equilibrium exists for the more-than-two-stage problem. But we can still obtain some equilibrium properties if an equilibrium exits, which are summarized in the following proposition.

Proposition 4 *When a symmetric monotonic equilibrium exists in the finite N-stage* ($N > 2$) *sequential auction, the equilibrium has the property:* $E\beta_1(.) = E\beta_2(.) ... = E\beta_{N-1}(.) > E\beta_N(.)$; *when a stationary symmetric monotonic equilibrium exists in the infinite-stage sequential auction, the equilibrium has the property:*$E\beta_t(.) = E\beta_{t+1}(.)\ \forall\ t \in \{1, 2, ...\infty\}$.

Proof. See the Appendix.

The above proposition shows that for a finite-stage problem, the bidding price drops in expectation only at the last stage. The reason for this phenomenon hinges on the Markovian property of the fundamental and the cross-period independence of bidders' signals. These two factors make a bidder able to affect her rivals' intertemporal inference only *one* period ahead. Then in equilibrium, the gain from affecting future payoff will be the same for all periods (since only the next period matters) except the last one (since there is no next period), hence generating a constant price trend with a price drop only at the last stage. In the infinite-stage case, there is no last stage throughout, so the price trend remains constant.

The intuition derived from the analysis of Proposition 4 gives rise to the following conjecture, which will be left for future study. We conjecture that the *length* of the cross-period persistence of the object fundamental plays an important role in determining the declining price trend. In a two-stage sequential auction, the fundamental is persistent across both stages and the intertemporal inference generates a definite declining price trend. Now let us consider a more general model where $P_t = U\left(\Theta_t.\theta_{t-1}, ...\theta_{t-s},\ X_t^i, X_t^{-i}\right)$, i.e., the resale price not only depends on the current period fundamental but also the fundamentals in previous periods. Then if the fundamental is more persistent than a martingale, the current fundamental will be affected by the realized fundamentals from more-than-one previous rounds. This general model seems to be hard to solve analytically at the moment. But its solution may be conjectured from the learning interpretation obtained from the martingale case. For example, if the fundamental is persistent across all stages, then at stage t, bidders can always affect more future stages hence bigger future payoff than at stage $t+1$. Therefore, bidders will have higher incentives to raise their bids at stage t than stage $t+1$, which may lead to a continuously declining price path. In another example, if the fundamental is persistent only for 3 stages while the sequential auction has 5 rounds, then we would expect $E\beta_1(.) = E\beta_2(.) = E\beta_3(.) > E\beta_4(.) > E\beta_5(.)$. This is because bidders can affect future 2 rounds' payoff (high future payoff) at stage 1, 2 and 3, while at stage 4 they can only affect 1 future round (medium future payoff) and at stage 5 zero round (low future payoff), then bidders' intertemporal decisions similar as those discussed in Section 3.1.2 tend to yield the stated price trend. In a third example, if the auction rounds are far more than the length of the persistence of the fundamental, like 50 rounds with a martingale fundamental, we should then expect an relatively constant price trend. Therefore, price trends with varying degrees of declining can all be generated from adjusting the length of the persistence of the fundamental.

4.4 CONCLUSION

This paper studies price trends in a sequential first-price common-value auction with resale. It differs from the previous research in that it considers sequential auctions with *multi-unit* demand. In the two-stage case, we propose a condition that guarantees the existence of a symmetric monotonic equilibrium which exhibits a declining trend. This is because bidders have the incentive to overbid in the first round to lower their rivals' intertemporal inference on the object value so that they can obtain a second-stage advantage. We also characterize the necessary properties of symmetric monotonic equilibria in the finite N-stage and the infinite-stage cases. In the former case, the price trend remains constant and drops only at the last stage; in the latter case, we have a constant price trend throughout.

Future work will be devoted to showing the existence of equilibrium in general N-stage (N>2) sequential auction and to investigating price trends when the fundamental is more persistent than a martingale.

5.0 CONCLUSION

This dissertation studies new bidding behaviors in such richer environments as with collusion, strategic information disclosure and intertemporal inference. We first find that an endogenously formed ring will include at least two members and is in general not all-inclusive. We then show that in various two-stage sequential auctions, the standard sequential Dutch auction or the first-price auction with the first-stage winning bid announcement shall generate the highest revenue. Finally, we propose a resale explanation for the declining price path in sequential auctions with multi-unit demand.

Future research agenda aims to extend the current frameworks to more general settings. For example, in the collusion analysis, we could consider cases where bidders are ex ante heterogeneous; where there is more than one collusion initiator among bidders leading to multiple rings in the auction; and other possible relaxations of those assumptions defining the bidders' collusive technology. In the information disclosure investigation, we could extend the two-bidder two-stage setting to arbitrary number of bidders and stages; more general distribution and interperiod relation of bidders' valuations. In the sequential auctions' intertemporal inference study, future work will be devoted to showing the existence of equilibrium in general N-stage ($N>2$) sequential auction and to investigating price trends when the fundamental is more persistent than a martingale.

APPENDIX

Proofs for Chapter 2

Proof of Lemma 1.

Consider a contingency, where V is realized such that v_1 is not the highest element and v_2 is, but $\beta_K(V)$ assigns the largest bid $\hat{\beta}_K(V)$ to member 1. Now look at another bid assignment rule $\beta^*_K(V)$, which is the same as $\beta_K(V)$ except that under the previously stated contingency, $\beta^*_K(V)$ assigns the largest bid $\hat{\beta}_K(V)$ to member 2. Notice that the largest bid assigned under these two rules is still the same, the only difference is who obtains this bid. The outsiders will bid the same under these two rules since they always compete against the same bid function $\hat{\beta}_K(V)$ of the ring. Given the outsiders bid the same, Pr($\hat{\beta}_K(V)$wins the object) is the same under both rules. Since $\beta^*_K(V)$ assigns the largest bid to member 2, conditional on winning member 2 can capture more surplus $v_2 - \hat{\beta}_K(V)$ for the ring than what member 1 can, i.e., $v_1 - \hat{\beta}_K(V)$. Therefore, the rule $\beta^*_K(V)$ outperforms $\beta_K(V)$ under this particular contingency. It follows that the bid assignment rule that maximizes the per member surplus G_K must always assigns the largest bid to the highest report.

So G_K becomes $\dfrac{1}{K}E_{v_1 \ldots v_K}[(v - \hat{\beta}_K(V)) \Pr(\hat{\beta}_K(V) \text{wins the object})|v \text{ is the highest report in } V]$. Consider two contingencies where the valuations $V = (v_1, \ldots v_K)$ and $V' = (v'_1, \ldots v'_K)$ are realized such that $\max\{v_1, \ldots v_K\} = \max\{v'_1, \ldots v'_K\} = v$ and $V \backslash \{v\} \neq V' \backslash \{v\}$, i.e., the highest report in V and V' is the same but the other reports are not all the same. Assume $\hat{\beta}_K(V)$ assigns different bids to the highest-report member under these two contingencies. If the per member surplus G_K is different under these two bids, then there exists a profitable deviation from $\hat{\beta}_K(V)$ in at least one of the contingencies. If G_K is the same under these two bids, assigning either bid to both contingencies does not change the surplus. Therefore, to constitute an equilibrium, an optimal $\hat{\beta}_K(V)$ (if there exists one) shall have such a property that for the same highest report v, the assigned bid shall be the same regardless of other reports. It implies that $\hat{\beta}_K(V)$ is solely determined by the element v, where v is the highest report in V. Hence Then there is no loss of generality for us to focus on the bid assignment rule that conditions on v only. Q.E.D.

Proof of Lemma 3.

From the results of Maskin and Riley (2000b), (2003), Athey (2001) and Lebrun (2002) etc, we know that there exists a unique pair of equilibrium bid functions $\beta_K^{in}(v)$ and $\beta_K^{out}(v)$ that solves equations (I') and (II') in Section 4.1. Furthermore, $\beta_K^{in}(v)$ and $\beta_K^{out}(v)$ are monotonic and $\beta_K^{in}(\underline{v}) = \beta_K^{out}(\underline{v})=\underline{\mathrm{v}}$, $\beta_K^{in}(\overline{v}) = \beta_K^{out}(\overline{v}) = \beta_K^*$. So Property (a) is the standard boundary result.

Since all the properties that we are going to prove should hold for any given ring size K, we can drop the subscript K to simplify the notation without causing confusion. We let $\lambda_1(\beta)$ and $\lambda_2(\beta)$ denote $\lambda_K^{in}(\beta)$ and $\lambda_K^{out}(\beta)$ respectively. The first-order conditions of (I') and (II') lead to:

$$(1) \quad \frac{(N-K)\, f(\lambda_2(\beta))}{F(\lambda_2(\beta))} \times \lambda_2'(\beta) = \frac{1}{\lambda_1(\beta)-\beta}$$

$$(2) \quad \frac{(N-K-1)\, f(\lambda_2(\beta))}{F(\lambda_2(\beta))} \times \lambda_2'(\beta) + \frac{Kf(\lambda_1(\beta))}{F(\lambda_1(\beta))} \times \lambda_1'(\beta) = \frac{1}{\lambda_2(\beta)-\beta}$$

We will first show $\lambda_1(\beta) > \lambda_2(\beta)$ holds in a neighborhood of β^*. Substituting $\beta = \beta^*$ into equation (1) and (2), we obtain:

$$(3) \quad \frac{(N-K)\, f(\lambda_2(\beta^*))}{F(\lambda_2(\beta^*))} \times \lambda_2'(\beta^*) = \frac{1}{\lambda_1(\beta^*)-\beta^*}$$

$$(4) \quad \frac{(N-K-1)\, f(\lambda_2(\beta^*))}{F(\lambda_2(\beta^*))} \times \lambda_2'(\beta^*) + \frac{Kf(\lambda_1(\beta^*))}{F(\lambda_1(\beta^*))} \times \lambda_1'(\beta^*) = \frac{1}{\lambda_2(\beta^*)-\beta^*}$$

Since $\lambda_1(\beta^*) = \lambda_2(\beta^*) = \overline{v}$, by combining (3) and (4) we have:

$$(5) \quad \frac{(N-K)\, f(\lambda_1(\beta^*))}{F(\lambda_1(\beta^*))} \times \lambda_2'(\beta^*) = \frac{(N-K-1)\, f(\lambda_2(\beta^*))}{F(\lambda_2(\beta^*))} \times \lambda_2'(\beta^*) + \frac{Kf(\lambda_1(\beta^*))}{F(\lambda_1(\beta^*))} \times \lambda_1'(\beta^*)$$

Equation (5) can be reduced to:

(6) $K\lambda_1'(\beta^*) = \lambda_2'(\beta^*)$

So for any $K > 1$, we must have $\lambda_1'(\beta^*) < \lambda_2'(\beta^*)$, hence there exists a neighborhood of β^* where $\lambda_1(\beta) > \lambda_2(\beta)$ for all β belong to this neighborhood. Suppose $\lambda_1(\beta)$ and $\lambda_2(\beta)$ first cross at $\beta' \in (\underline{\mathrm{v}}, \beta^*)$ such that $\lambda_1(\beta') = \lambda_2(\beta')$. Then with the continuity of the inverse bid functions, we must have $\lambda_1'(\beta') > \lambda_2'(\beta')$. However, through the same derivation as the above, we can obtain $\lambda_1'(\beta') < \lambda_2'(\beta')$, which is a contradiction. Therefore, $\lambda_1(\beta)$ and $\lambda_2(\beta)$ will never cross over the support $(\underline{v}, \beta^*)$. Consequently, we must have $\lambda_1(\beta) > \lambda_2(\beta)$ hold for all $\beta \in (\underline{v}, \beta^*)$, i.e., $\beta^{in}(v) < \beta^{out}(v)$ for all $v \in (\underline{v}, \overline{v})$.

For Property (c), we can take the limit from both sides of (1) and (2), which yields:

(7) $\lim_{\beta \to \underline{\mathrm{v}}} \dfrac{(N-K) f(\lambda_2(\beta))}{F(\lambda_2(\beta))} \times \lambda_2'(\beta) = \lim_{\beta \to \underline{\mathrm{v}}} \dfrac{1}{\lambda_1(\beta) - \beta}$

(8) $\lim_{\beta \to \underline{\mathrm{v}}} \dfrac{(N-K-1) f(\lambda_2(\beta))}{F(\lambda_2(\beta))} \times \lambda_2'(\beta) + \dfrac{K f(\lambda_1(\beta))}{F(\lambda_1(\beta))} \times \lambda_1'(\beta) = \lim_{\beta \to \underline{\mathrm{v}}} \dfrac{1}{\lambda_2(\beta) - \beta}$

Rearranging (7) and (8) and using L'Hospital's Rule and the fact that $\beta_K^{in}(\underline{\mathrm{v}}) = \beta_K^{out}(\underline{\mathrm{v}}) = \underline{\mathrm{v}}$, we can easily obtain $\lambda_1'(\underline{\mathrm{v}}) = 1 + \dfrac{1}{N-K}$ and $\lambda_2'(\underline{\mathrm{v}}) = 1 + \dfrac{1}{N-1}$. Hence, $\dfrac{d\beta_K^{in}(v)}{dv}|_{v=\underline{v}} = 1 - \dfrac{1}{N-K+1}$, $\dfrac{d\beta_K^{out}(v)}{dv}|_{v=\underline{v}} = 1 - \dfrac{1}{N}$. Q.E.D.

Proof of Proposition 1.

Again, $\lambda_2(\beta)$ denotes the inverse function of $\beta_K^{out}(v)$ in this proof. Then the ex ante per member surplus of a ring with size $K < N$ is:

$$G_K = \frac{1}{K} E_{v,\, v_{K+1},\dots,v_N}[(v - \beta_K^{in}(v))\, 1_{\beta_K^{in}(v) \geq \max_{j \in \{K+1,\dots N\}} \beta_j^{out}(v_j)}]$$
$$= \int_{\underline{v}}^{\overline{v}} \{ \frac{[v - \beta_K^{in}(v)] \times [F(\lambda_2(\beta_K^{in}(v)))]^{N-K}}{K} \times K f(v)[F(v)]^{K-1} \} dv$$
$$= \int_{\underline{v}}^{\overline{v}} \{ [v - \beta_K^{in}(v)] \times [F(\lambda_2(\beta_K^{in}(v)))]^{N-K} \times f(v)[F(v)]^{K-1} \} dv$$

By Property (b) in Lemma 3, we have $\beta_K^{in}(v) < \beta_K^{out}(v)$, which leads to the fact that:

$G_K < \int_{\underline{v}}^{\overline{v}} \{ [v - \beta_K^{in}(v)] \times [F(\lambda_2(\beta_K^{out}(v)))]^{N-K} \times f(v)[F(v)]^{K-1} \} dv.$

The right hand side of the above inequality equals $\frac{1}{N} \int_{\underline{v}}^{\overline{v}} [v - \beta_K^{in}(v)] d[F(v)]^N$, which is less than $\frac{1}{N} \int_{\underline{v}}^{\overline{v}} [v - \underline{v}] d[F(v)]^N$. Since $\frac{1}{N} \int_{\underline{v}}^{\overline{v}} [v - \underline{v}] d[F(v)]^N$ is the per member surplus for the all-inclusive ring assuming that the all-inclusive ring can obtain the object by only submitting $\underline{v}$, we can conclude that the all-inclusive ring maximizes the unconstrained ex ante per member surplus. Q.E.D.

Proof of Lemma 4.

For a deviating member from an all-inclusive ring, she bids $\beta^{out}_{N-1}(v)$ to compete a ring of $N-1$ members following the bid function $\beta^{in}_{N-1}(v)$. The density is $f(v) = \frac{1}{\overline{v}}$ and c.d.f is $F(v) = \frac{v}{\overline{v}}$. We use $\lambda_1(\beta)$ and $\lambda_2(\beta)$ to denote the inverse function of $\beta^{in}_{N-1}(v)$ and $\beta^{out}_{N-1}(v)$ respectively. Then the first-order conditions for the incentive compatibility of bidding lead to:

(9) $\dfrac{1}{\lambda_2(\beta)} \times \lambda_2'(\beta) = \dfrac{1}{\lambda_1(\beta)-\beta}$

(10) $\dfrac{(N-1)}{\lambda_1(\beta)} \times \lambda_1'(\beta) = \dfrac{1}{\lambda_2(\beta)-\beta}$

From Lemma 3, we know that $\lambda_1(0){=}\lambda_2(0){=}0$ and $\lambda_1(\beta^*){=}\lambda_2(\beta^*){=}\bar{v}$, where β^* is the common terminal point for two bid functions. We now start to show the monotonicity of $\lambda_1'(\beta)$ and $\lambda_2'(\beta)$. Rearrange (9) and (10) as (11) and (12):

(11) $\lambda_2'(\beta)\lambda_1(\beta) - \lambda_2'(\beta)\beta = \lambda_2(\beta)$

(12) $\lambda_1'(\beta)\lambda_2(\beta) - \lambda_1'(\beta)\beta = \dfrac{\lambda_1(\beta)}{N-1}$

Differentiating both sides of (11) and (12), we have:

(13) $\lambda_2''(\beta) = \dfrac{\lambda_2'(\beta)\left(2-\lambda_1'(\beta)\right)}{\lambda_1(\beta)-\beta}$

(14) $\lambda_1''(\beta) = \dfrac{\lambda_1'(\beta)\left(1+\dfrac{1}{N-1}-\lambda_2'(\beta)\right)}{\lambda_2(\beta)-\beta}$

From Lemma 3, we know that $\lambda_1'(0) = 2$, $\lambda_2'(0) = 1 + \dfrac{1}{N-1}$ and $\lambda_1(\beta) > \lambda_2(\beta)$. Also $\lambda_1(\beta) > \beta$ and $\lambda_2(\beta) > \beta$. Suppose $\lambda_1'(\varepsilon) \geq 2$ for a small increment ε from 0. We have $\lambda_2''(\varepsilon) \leq 0$ from (13), so $\lambda_2'(\varepsilon) \leq 1 + \dfrac{1}{N-1}$, which implies that $\lambda_1''(\varepsilon) \geq 0$ from (14). Iterating this law of motion for each small increment starting from 0 makes it impossible for $\lambda_1(\beta)$ and $\lambda_2(\beta)$ to meet at a common terminal point. So we must have $\lambda_1'(\varepsilon) < 2$, which implies $\lambda_1''(\varepsilon) < 0$ and $\lambda_2''(\varepsilon) > 0$. This initial condition combined with the law of motion defined in equation (13) and (14) gives us the fact that $\lambda_1''(\beta) < 0$ and $\lambda_2''(\beta) > 0$ for all $\beta \in (0, \beta^*]$. Consequently, $\lambda_1(\beta)$ always lies above the line joining the origin and the common terminal bid at the valuation $\overline{v}$. Then $\lambda_1(\beta) > \dfrac{\overline{v}}{\beta^*}\beta$ is immediate. We next look for β^*. Notice that (9) and (10) are the same system as (2) and (3) in Marshall et al (1994) except the terminal condition. Follow the same calculation in Appendix A in Marshall et al (1994), we can obtain β^* in our setting, which equals $\overline{v}\left(1 - C^{\frac{1}{N-2}}\right)$ where $C = \dfrac{N^N}{2^{2N-2}(N-1)^{N-1}}$. Q.E.D.

Proof of Proposition 2.

Again, we use $\lambda_1(\beta)$ to denote the inverse function of $\beta^{in}_{N-1}(v)$. The per member surplus for an all-inclusive ring when the total number of bidders is N is $\int_0^{\overline{v}} v[F(v)]^{N-1} f(v)\,dv = \dfrac{\overline{v}}{N+1}$. The corresponding deviation profit is D_N, which equals:
$\int_0^{\overline{v}} [v - \beta^{out}_{N-1}(v)] \times [F(\lambda_1(\beta^{out}_{N-1}(v)))]^{N-1} f(v)dv$. Since $\beta^{out}_{N-1}(v)$ is the best response to $\beta^{in}_{N-1}(v)$, we must have $[v - \beta^{out}_{N-1}(v)] \times [F(\lambda_1(\beta^{out}_{N-1}(v)))]^{N-1} > [v - (1 - \frac{1}{N})v] \times [F(\lambda_1((1 - \frac{1}{N})v))]^{N-1}$ when we replace $\beta^{out}_{N-1}(v)$ with its upper bound $(1 - \frac{1}{N})v$ in the bidder's payoff function. Also, we have $\lambda_1(\beta) > \dfrac{\overline{v}}{\beta^*}\beta$ from Lemma 4, then we can obtain a lower bound for the deviation profit by replacing $\lambda_1(\beta)$ with $\dfrac{\overline{v}}{\beta^*}\beta$. Notice that
$[v - (1 - \frac{1}{N})v] \times [F(\lambda_1((1 - \frac{1}{N})v))]^{N-1} > [v - (1 - \frac{1}{N})v] \times [F(\frac{\overline{v}}{\beta^*}((1 - \frac{1}{N})v))]^{N-1}$. Therefore, we have $D_N > \int_0^{\overline{v}} [v - (1 - \frac{1}{N})v] \times [F(\frac{\overline{v}}{\beta^*}(1 - \frac{1}{N})v))]^{N-1} f(v)dv = \dfrac{(N-1)^{N-1}\,\overline{v}^N}{N^N(N+1)\,\beta^{*N-1}}$.

A sufficient condition for $D_N > G_N$ is that $\dfrac{(N-1)^{N-1}\,\overline{v}^N}{N^N(N+1)\,\beta^{*N-1}} > \dfrac{\overline{v}}{N+1}$. Substituting the expression for β^* into the left hand side of the inequality and rearrange, we need $\dfrac{(N-1)^{N-1}}{N^N\left(1 - C^{\frac{1}{N-2}}\right)^{N-1}} > 1$. As $1 - C^{\frac{1}{N-2}} < 1$, the left hand side will increase dramatically as N gets large. Substituting the expression for C into the left hand side of the inequality, we need $\dfrac{(N-1)^{N-1}}{N^N[1 - (\frac{N^N}{2^{2N-2}(N-1)^{N-1}})^{\frac{1}{N-2}}]^{N-1}} > 1$. Let $\alpha = \dfrac{N^N}{2^{2N-2}(N-1)^{N-1}}$. It is easy to show that $\dfrac{d\alpha}{dN} < 0$ when $N > 1$. So $\alpha^{\frac{1}{N-2}}$ decreases with the increase of N. Also, $\lim\limits_{N\to+\infty} \alpha^{\frac{1}{N-2}} = \dfrac{1}{4}$. Then $1 - \alpha^{\frac{1}{N-2}}$ monotonically converges to $\dfrac{3}{4}$ from below. Therefore, $\dfrac{(N-1)^{N-1}}{N^N\left(1 - \alpha^{\frac{1}{N-2}}\right)^{N-1}} > \dfrac{(N-1)^{N-1}}{N^N(\frac{3}{4})^{N-1}}$. The term $\dfrac{(N-1)^{N-1}}{N^N(\frac{3}{4})^{N-1}}$ is monotonically increasing when

$N > 4$ and bigger than 1 when $N > 13$. That means $\dfrac{(N-1)^{N-1}}{N^N\left(1-\alpha^{\frac{1}{N-2}}\right)^{N-1}}$ is bigger than 1 at $N \leq 13$. We can evaluate $\dfrac{(N-1)^{N-1}}{N^N\left(1-\alpha^{\frac{1}{N-2}}\right)^{N-1}}$ up to $N = 13$ and easily see that when $N \geq 10$, the all-inclusive ring will not be stable. Q.E.D.

Proof of Proposition 3.

We use $\lambda_2(\beta)$ to denote the inverse function of $\beta_K^{out}(v)$. From the proof of Proposition 1, we know that the per member surplus of a ring with K members is: $G_K = \int_{\underline{v}}^{\overline{v}}\{[v-\beta_K^{in}(v)]\times[F(\lambda_2(\beta_K^{in}(v)))]^{N-K}\times f(v)[F(v)]^{K-1}\}dv$, which equals: $\frac{1}{\overline{v}^N}\int_0^{\overline{v}}\{[v-\beta_K^{in}(v)]\times[\lambda_2(\beta_K^{in}(v))]^{N-K}\times v^{K-1}\}dv$ for the uniform distribution on the support $[0,\overline{v}]$. Use G_1 to denote the surplus for each bidder under the noncooperative bidding, then $G_1 = \frac{1}{\overline{v}^N}\int_0^{\overline{v}}\{[v-(\frac{N-1}{N}v)]\times v^{N-1}\}dv$. As $\beta_K^{in}(v)$ is the best response to $\beta_K^{out}(v)$, we must have $[v-\beta_K^{in}(v)]\times[\lambda_2(\beta_K^{in}(v))]^{N-K} > [v-(\frac{N-1}{N}v)]\times[\lambda_2(\frac{N-1}{N}v)]^{N-K}$ when replacing $\beta_K^{in}(v)$ with $\frac{N}{N-1}v$ in the bidder's payoff function. So if we can show that $\lambda_2(\beta) > \frac{N}{N-1}\beta$, we can immediately obtain $G_K > G_1$ for $K \geq 2$. Again, we have the system of differential equations for the uniform distribution with the ring size $K \geq 2$:

(15) $$\frac{N-K}{\lambda_2(\beta)}\times\lambda_2'(\beta) = \frac{1}{\lambda_1(\beta)-\beta}$$

(16) $$\frac{(N-K-1)}{\lambda_2(\beta)}\times\lambda_2'(\beta) + \frac{K}{\lambda_1(\beta)}\times\lambda_1'(\beta) = \frac{1}{\lambda_2(\beta)-\beta}$$

From Lemma 3, we know that $\lambda_1'(0) = 1+\frac{1}{N-K}$ and $\lambda_2'(0) = 1+\frac{1}{N-1}$. Now we are ready to show $\lambda_2(\beta) > \frac{N}{N-1}\beta$, which consists of the following two steps.

Step 1. We will show that for a small increment ε from 0, $\lambda_1'(\varepsilon) \leq 1+\frac{1}{N-K}$. Let us prove by contradiction and suppose that $\lambda_1'(\varepsilon) > 1+\frac{1}{N-K}$. Similar as in the proof of Lemma 4, we can obtain $\lambda_2''(\beta) = \dfrac{\lambda_2'(\beta)\left(1+\frac{1}{N-K}-\lambda_1'(\beta)\right)}{\lambda_1(\beta)-\beta}$ from (15). Then $\lambda_2''(\varepsilon) < 0$, which implies that $\lambda_2(\varepsilon) < (1+\frac{1}{N-1})\varepsilon$. So $\frac{1}{\lambda_2(\varepsilon)-\varepsilon} > \frac{N-1}{\varepsilon}$. Combine (15) and (16), we can derive that: $\frac{(N-K-1)}{(N-K)} \times \frac{1}{\lambda_1(\varepsilon)-\varepsilon} + \frac{K}{\lambda_1(\varepsilon)} \times \lambda_1'(\varepsilon) > \frac{N-1}{\varepsilon}$. This inequality together with the fact that $\frac{\lambda_1(\varepsilon)}{\varepsilon} = \lambda_1'(\varepsilon)$ when ε is sufficiently small, gives us $\lambda_1'(\varepsilon) < 1 + \frac{1}{N-K}$, which is a contradiction to our hypothesis. So we must have $\lambda_1'(\varepsilon) \leq 1 + \frac{1}{N-K}$ and $\lambda_2''(\varepsilon) \geq 0$.

Step 2. We will show that for all $\beta \in (0, \beta_K^*)$, if $\frac{\lambda_1(\beta)}{\beta} \leq 1 + \frac{1}{N-K}$ and $\frac{\lambda_2(\beta)}{\beta} \geq 1 + \frac{1}{N-1}$, then $\lambda_1'(\beta) \leq 1 + \frac{1}{N-K}$. This assertion is proved as follows. Merge (15) with (16), we obtain

$$(17)\ \lambda_1'(\beta) = \frac{1}{K}\left[\frac{\frac{\lambda_1(\beta)}{\beta}}{\frac{\lambda_2(\beta)}{\beta}-1} - \frac{\frac{\lambda_1(\beta)}{\beta}}{\frac{\lambda_1(\beta)}{\beta}-1} \times \frac{N-K-1}{N-K}\right]$$

If $\frac{\lambda_1(\beta)}{\beta} \leq 1 + \frac{1}{N-K}$ and $\frac{\lambda_2(\beta)}{\beta} \geq 1 + \frac{1}{N-1}$, then it is immediate that:

$\lambda_1'(\beta) \leq \frac{1}{K}\left[\frac{1+\frac{1}{N-K}}{1+\frac{1}{N-1}-1} - \frac{1+\frac{1}{N-K}}{1+\frac{1}{N-K}-1} \times \frac{N-K-1}{N-K}\right] = 1 + \frac{1}{N-K}$ by replacing $\frac{\lambda_1(\beta)}{\beta}$ and $\frac{\lambda_2(\beta)}{\beta}$ with their upper and lower bound respectively in equation (17).

The result in step 1 gives the initial condition and the one in step 2 stipulates the law of motion of bids. Combining these two steps and the fact that both bid functions meet at the same ending point, we must have $\lambda_2''(\beta) > 0$ for all $\beta \in (0, \beta_K^*)$. So we obtain the desired result $\lambda_2(\beta) > \frac{N}{N-1}\beta$. Q.E.D.

Proofs for Chapter 3

Proof of Proposition 1.

Given bidder 1 follows and believes her rival also follows the equilibrium strategy, if bidder 2 mimics the valuation other than her own, she does not gain in the first round. In the second round, bidder 1 still bids her valuation, which is optimal even though the inferred valuation from bidder 2 is wrong because bidding one's own valuation in a single second-price auction is an ex post equilibrium. So the optimal response for bidder 2 at the second stage is to still bid her own valuation. Then there is nothing to gain for bidder 2 to mimic the other valuation at the first stage, which leads to the conclusion that in equilibrium both bidders will bid their own valuations. Q.E.D.

Proof of Proposition 2.

The existence proof consists of two standard steps. The first step produces the equilibrium candidate and the second step verifies that the candidate is indeed an equilibrium.

Step 1. Producing the equilibrium candidate. We start from the second stage. Let us assume that a bidder observes a valuation v and submit a bid $\beta(z)$ at the first stage, i.e., she mimics z type, while the other bidder follows the specified strategy truthfully. Let $\lambda(\beta)$, $\lambda_1(\beta)$ and $\lambda_2(\beta)$ denote the inverse functions of $\beta(v)$, $\beta_1(v)$ and $\beta_2(v)$ respectively.

If the bidder loses at the bid $\beta(z)$, then she bids $\beta_2(m)$, i.e., she mimics type m at the second stage, and her rival will bid $\beta_1(\hat{v})$. The losing bidder believes that $\hat{v}$ is in $(z, 1]$ with density $\dfrac{f(\hat{v})}{1 - F(z)}$. So the second-stage expected payoff for the first-stage losing bidder who mimics type m is $\pi_2 = (v - \beta_2(m)) \Pr(\beta_2(m) > \beta_1(\hat{v}))$. $\Pr(\beta_2(m) > \beta_1(\hat{v}))$ is the probability of winning at the second stage for the first-stage losing bidder. In our case, $\Pr(\beta_2(m) > \beta_1(\hat{v})) = \dfrac{F[\beta_1^{-1}(\beta_2(m))] - F(z)}{1 - F(z)}$. In a standard auction environment, this probability is only a function of the current stage bid, while here the probability is also parameterized by her previous stage bid.

If the bidder wins at the bid $\beta(z)$, then she bids $\beta_1(n)$, i.e., she mimics type n, and her rival will bid $\beta_2(\hat{v})$, where the winning bidder believes that $\hat{v}$ is in $[0, z)$ with density $\dfrac{f(\hat{v})}{F(z)}$. So the second-stage expected payoff for the first-stage winning bidder who mimics type n is $\pi_1 = (v - \beta_1(n)) \Pr(\beta_1(n) > \beta_2(\hat{v}))$. $\Pr(\beta_1(n) > \beta_2(\hat{v}))$ is the probability of winning at the second stage for the first-stage winning bidder. Here $\Pr(\beta_1(n) > \beta_2(\hat{v})) = \dfrac{F[\beta_2^{-1}(\beta_1(n))]}{F(z)}$.

We then start to consider the first-period bid function. The first-period bid function has to balance the second-period payoff. Since we have assumed that the bidder mimics type z at the first stage, her overall expected payoff for two stages is:

$$
\begin{aligned}
\pi(v) &= F(z)[(v - \beta(z)) + \delta\pi_1] + (1 - F(z))[\delta\pi_2] \\
&= F(z)\{(v - \beta(z)) + \delta[(v - \beta_1(n))\frac{F[\beta_2^{-1}(\beta_1(n))]}{F(z)}]\} \\
&+ (1 - F(z))[\delta(v - \beta_2(m))\frac{F[\beta_1^{-1}(\beta_2(m))] - F(z)}{1 - F(z)}] \\
&= F(z)(v - \beta(z)) + \delta(v - \beta_1(n))F[\beta_2^{-1}(\beta_1(n))] + \delta(v - \beta_2(m))[F[\beta_1^{-1}(\beta_2(m))] - F(z)]
\end{aligned}
$$

The optimality of the symmetric bid functions requires that truthful bidding is optimal for all three bid functions simultaneously. Then we can use the following conventional method: The first-order condition of z is:

(1) $f(z)\,v - f(z)\,\beta(z) - F(z)\,\beta'(z) - \delta f(z)\,(v - \beta_2(m)) = 0$

The first-order condition of m is:

(2) $(v - \beta_2(m))\beta_1^{-1'}(\beta_2(m))\,f[\beta_1^{-1}(\beta_2(m))] - \{F[\beta_1^{-1}(\beta_2(m))] - F(z)\} = 0$

The first-order condition of n is:

(3) $(v - \beta_1(n))\beta_2^{-1'}(\beta_1(n))\,f[\beta_2^{-1}(\beta_1(n))] - F[\beta_2^{-1}(\beta_1(n))] = 0$

Since the truthful bidding is the equilibrium solution to the above three equations, we replace m, n and z with v. Then (2) and (3) become:

(4) $(v - \beta_2(v))\beta_1^{-1'}(\beta_2(v))\,f[\beta_1^{-1}(\beta_2(v))] - \{F[\beta_1^{-1}(\beta_2(v))] - F(v)\} = 0$

(5) $(v - \beta_1(v))\beta_2^{-1'}(\beta_1(v))\,f[\beta_2^{-1}(\beta_1(v))] - F[\beta_2^{-1}(\beta_1(v))] = 0$

Let $\lambda(\beta)$, $\lambda_1(\beta)$ and $\lambda_2(\beta)$ denote the inverse functions of $\beta(v)$, $\beta_1(v)$ and $\beta_2(v)$ respectively. The equations (4) and (5) can be transformed into:

(6) $(\lambda_2(t) - t)\lambda_1'(t)\,f(\lambda_1(t)) = F[\lambda_1(t)] - F[\lambda_2(t)]$

(7) $(\lambda_1(t) - t)\lambda_2'(t)\,f(\lambda_2(t)) = F[\lambda_2(t)]$

The equilibrium boundary conditions must be $\lambda_1(0) = \lambda_2(0) = 0$ and $\lambda_1(1) = \lambda_2(1) = t^*$ as usual, where t^* is the common terminal bid when a bidder observes the valuation 1. Theorem 1 in Landsberger et al (2001) gives the existence result of a *monotonic* solution to the system (6) and (7). The first-stage bid function can be directly solved from the equation transformed from equation (1) by imposing the equilibrium condition, i.e., replacing z and m with v. Let y and y' denote $\beta(v)$ and $\beta'(v)$ respectively. Then the transformed equation can be rewritten as: $y' = -\frac{f(v)}{F(v)}y + \frac{f(v)\,v}{F(v)} - \frac{\delta f(v)\,v}{F(v)} + \frac{\delta f(v)}{F(v)}\beta_2(v)$. This is a nonhomogeneous first-order linear differential equation. Given the boundary condition $\beta(0) = 0$, the unique solution is: $\frac{1}{F(v)}\int_0^v [f(t)\,t - \delta f(t)\,t + \delta f(t)\,\beta_2(t)]dt$. It is easy to check that $\beta'(v) = \frac{f(v)\int_0^v F(t)\,(1 - \delta + \beta_2'(t))dt}{F^2(v)}$. Given the fact that $\beta_2'(t) > 0$ as we have obtained above, $\beta'(v) > 0$ is immediate.

Step 2. Verifying the equilibrium. Since the three bid functions produced above are all monotonic, the application of the standard verification approach, i.e., to show that all other mimicking types will lead to less payoff, is straight forward. So we forgo its detailed derivation here. Q.E.D.

Proof of Proposition 3.

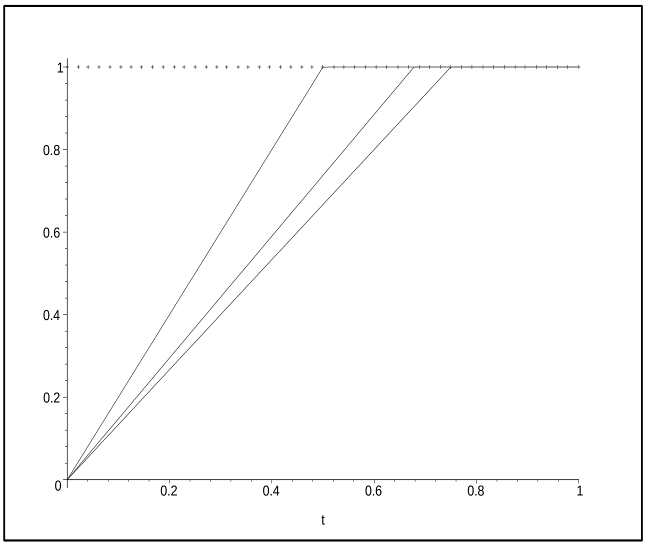
Under the assumption that $f(v) = 1$, the differential equation system (6) and (7) can be transformed into:

(8) $(\lambda_2(t) - t)\lambda_1'(t) = \lambda_1(t) - \lambda_2(t)$

(9) $(\lambda_1(t) - t)\lambda_2'(t) = \lambda_2(t)$

Property a) is just the standard boundary conditions. The following graph helps us to see the proof for the rest of the properties.

Figure 1. Bounds of Inverse Bid Functions



In the above figure, from the left to the right, the three lines are $2t$, $\frac{t}{t^*}$ and $\frac{4t}{3}$ respectively. We need to show that $\lambda_1(t)$ and $\lambda_2(t)$ behave regularly within the regions between $2t$ and $\frac{t}{t^*}$, $\frac{t}{t^*}$ and $\frac{4t}{3}$ respectively. Property b) and c) are proved by ruling out all other possibilities by contradiction, which is a tedious process. First, we can easily show that $\lambda_1'(0) = 2$ and $\lambda_2'(0) = \frac{4}{3}$. Second, we can show that $\lambda_1'(t^*) = 0$ and $\lambda_2'(t^*) = \frac{1}{1-t^*}$. Third, differentiate both sides of equation (9), we have $\lambda_1'(t)\lambda_2'(t) + (\lambda_1(t) - t)\lambda_2''(t) = 2\lambda_2'(t)$, i.e., $\lambda_2''(t) = \frac{2\lambda_2'(t) - \lambda_1'(t)\lambda_2'(t)}{\lambda_1(t) - t}$. As $\lambda_1(t) - t > 0$, we can obtain the following relations: (i) If $\lambda_1'(t) < 2$, then $\lambda_2''(t) > 0$. (ii) If $\lambda_1'(t) > 2$, then $\lambda_2''(t) < 0$. The above results prepare us to show the bounds for $\lambda_1(t)$ and $\lambda_2(t)$ with the following steps.

A. Suppose $\lambda_1(t)$ lies entirely above the line $2t$. Notice that $\lambda_1'(0) = 2$ and $\lambda_1'(t^*) = 0$, which combined with the fact that $\lambda_1(t)$ lies entirely above $2v$ implies that $\lambda_1'(t)$ must have first increased above two and then decreased to zero. Given the smoothness of $\lambda_1(t)$, this means that there exists at least a $t > 0$ such that $\lambda_1'(t) = 2$. Let $\bar{t}$ be the inf of the set of such t . Then $\lambda_1'(\bar{t}) > 2$ for all $t < \bar{t}$. By relation (ii), we must have $\lambda_2''(\bar{t}) = 0$ and $\lambda_2''(t) < 0$ for all $t < \bar{t}$. Then from equation (8), we can obtain the equation $3\lambda_2(\bar{t}) = \lambda_1(\bar{t}) + 2\bar{t}$. From equation (9), we have the equation $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \lambda_2(\bar{t})$. Combining these two equations, we obtain $3(\lambda_2(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \lambda_2(\bar{t})$. Notice that $\lambda_2'(\bar{t}) < \frac{4}{3}$ because $\lambda_2'(0) = \frac{4}{3}$ and $\lambda_2''(t) < 0$ for all $t < \bar{t}$. Then $\lambda_2(\bar{t}) < \frac{4}{3} \times 3(\lambda_2(\bar{t}) - \bar{t})$. So we must have $\frac{\lambda_2(\bar{t})}{\bar{t}} > \frac{4}{3}$, which is a contradiction to the fact that until $\bar{t}$, $\lambda_2(t)$ still lies below the line $\frac{4}{3}t$. So $\lambda_1(t)$ can not lie entirely above $2t$.

B. Suppose $\lambda_1(t)$ crosses $2t$ from above first. Suppose the crossing happens at the point $\hat{t}$, where $\hat{t} > 0$ and $\lambda_1'(\hat{t}) < 2$. Again there exists a $\bar{t}$ such that $0 < \bar{t} < \hat{t}$ and $\lambda_1''(\bar{t}) = 2$ and $\lambda_1'(\bar{t}) > 2$ for all $t < \bar{t}$. The above argument can be applied in exactly the same way here. So we can rule out this case too.

C. Suppose $\lambda_1(t)$ crosses $2t$ from below first at $\bar{t} > 0$. Then $\lambda_1'(\bar{t}) > 2$ and $\lambda_1(\bar{t}) = 2\bar{t}$. From the equation (8), we have $(\lambda_2(\bar{t}) - \bar{t})\lambda_1'(\bar{t}) = 2\bar{t} - \lambda_2(\bar{t})$. So we must have $\frac{\lambda_2(\bar{t})}{\bar{t}} < \frac{4}{3}$, i.e., $\lambda_2(t)$ goes below $\frac{4}{3}t$ at the point $\bar{t}$. As $\lambda_1'(t)$ must first decrease below 2, $\lambda_2'(t)$ will first increase above $\frac{4}{3}$ from relation (i). Then, there must exist a $\hat{t} < \bar{t}$ such that $\lambda_2(t)$ crosses the line $\frac{4}{3}t$ from the above at the point $\hat{t}$. So $\lambda_2'(\hat{t}) < \frac{4}{3}$. From equation (9), we have $\left(\lambda_1(\hat{t}) - \hat{t}\right)\lambda_2'(\hat{t}) = \frac{4}{3}\hat{t}$. So we obtain $\frac{\lambda_1(\hat{t})}{\hat{t}} > 2$, which is a contradiction to the fact that $\lambda_1(t)$ lies low the line $2t$ until $\bar{t}$. So this case is also impossible. To sum up the step A to C, we show that $2t > \lambda_1(t)$.

D. It is easy to obtain $\lambda_1(t) > \lambda_2(t)$ and their monotonicity from the differential equation system (8) and (9).

E. Suppose $\lambda_1(t)$ lies entirely under $2t$ and crosses $\frac{4}{3}t$ from above. Then $\lambda_2'(t)$ must rise above $\frac{4}{3}$ first and $\lambda_2(t)$ must cross $\frac{4}{3}t$ from above at least once due to relation (i). Let the crossing happen at $\bar{t}$. So $\lambda_2'(\bar{t}) < \frac{4}{3}$ and $\lambda_2(\bar{t}) = \frac{4}{3}\bar{t}$. From equation (9), we have $(\lambda_1(\bar{t}) - t)\lambda_2'(\bar{t}) = \frac{4}{3}\bar{t}$. Then we can obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which contradicts the fact that $\lambda_1(t)$

lies entirely under $2t$. So $\lambda_1(t)$ must lie above the line $\frac{3}{4}t$.

F. Suppose $\lambda_2(t)$ crosses $\frac{4}{3}t$ at least once. Since $2t > \lambda_1(t)$, there exits a neighborhood around zero such that $\lambda_1'(t) < 2$ for all t belong to this neighborhood. Then $\lambda_2''(t) > 0$ in this neighborhood from relation (i). Suppose the inf of the set of all crossing points of $\frac{4}{3}t$ is $\bar{t}$, where $\bar{t} > 0$ and $\lambda_2'(\bar{t}) < \frac{4}{3}$. From equation (9), we have $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \frac{4}{3}\bar{t}$. So we obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which is a contradiction to our obtained conclusion at the end of step C that $2t > \lambda_1(t)$. Then $\lambda_2(t)$ must lie entirely above $\frac{4}{3}t$.

G. Actually, the essence of the above argument can be used to show that $\lambda_2''(t) > 0$. Suppose not, then there must exist a line αt $(2 > \alpha > \frac{3}{4})$ from the origin cutting $\lambda_2(t)$ from below at such a $\bar{t}$, where $\bar{t} > 0$ and $\lambda_2'(\bar{t}) < \alpha$. Again from equation (9), we have $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \alpha\bar{t}$. So we obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which is a contradiction.

H. Similarly, suppose $\lambda_1''(t) < 0$ does not hold, then there must exist a line αt $(2 > \alpha > \frac{3}{4})$ from the origin cutting $\lambda_2(t)$ from above at such a $\bar{t}$, where $\bar{t} > 0$ and $\lambda_1'(\bar{t}) > \alpha$. From equation (8), we have $(\lambda_2(\bar{t}) - \bar{t})\lambda_1'(\bar{t}) = \alpha\bar{t} - \lambda_2(\bar{t})$, which gives us $\frac{\lambda_2(\bar{t})}{\bar{t}} < \frac{4}{3}$. This is a contradiction. So $\lambda_1''(t) > 0$.

I. Since $\lambda_1''(t) > 0$ and $\lambda_2''(t) < 0$, the result that $\frac{t}{t^*}$ separating $\lambda_1(t)$ and $\lambda_2(t)$ is immediate.

J. To sum up all the above steps, we prove property b) and c).

As to Property e), by replacing $f(v)$ with 1 and $F(v)$ with v in the general formula of $\beta(v)$ derived in the proof of Proposition 2, we obtain the unique solution:
$\beta(v) = \frac{1}{v}\int_0^v (t - \delta t + \delta\beta_2(t))dt = \frac{v}{2}(1-\delta) + \frac{\delta}{v}\int_0^v \beta_2(t)\,dt$. Using the fact that $\frac{3}{4}v > \beta_2(v)$, it is easy to see that $\beta(v)$ is smaller than $\frac{1}{2}v$, where $\frac{1}{2}v$ is the equilibrium bid function for a single auction.

Finally, we show Property d). Its upper bound is shown as follows. At the valuation 1, the second-stage equilibrium payoff for the bidder is $1 - t^*$. Given bidders bid truthfully in the first stage, it is necessary for the second-stage bid function to prevent any second-stage deviation. Let the second-stage payoff $\pi = (1-b)\lambda_2(b)$, where b is the choice of the bid. Then $\pi \leq 1 - t^*$ for all $b \in [0,1]$. Notice that $\pi > (1-b)\frac{4}{3}b$ because of the fact that

$\lambda_2(b) > \frac{4}{3}b$ and $\max(1-b)\frac{4}{3}b = \frac{1}{3}$ when $b = \frac{1}{2}$, so we must have $\frac{1}{3} \leq 1 - t^*$, i.e. $t^* \leq \frac{2}{3}$. The lower bound of t^* will be derived in the step B in the proof of Lemma 2 later. Q.E.D.

Proof of Proposition 4.

A. We start with the second-stage bid functions by assuming both bidders follow the same pure strategy bid function at the first stage. We will use bidder 1 to denote the generic bidder for our derivation of the equilibrium bid functions. If bidder 1 wins at the valuation v, then she randomly chooses a bid b on the support $(t_*, t^*]$ with p.d.f. $g^v(b)$ and c.d.f. $G^v(b)$. Her rival will bid according to $\beta_2^v(\hat{v})$, where $\hat{v}$ is the first-stage loser's valuation and the winner believes that $\hat{v}$ is in $[0, v)$ with density $\frac{1}{v}$. The optimality of the randomization requires $(v-b)\Pr(b > \beta_2^v(\hat{v}))$ to be a constant K^v for all $b \in (r_*, r^*)$. Let $\lambda_2^v(\hat{v}) = \beta_2^{v^{-1}}(\hat{v})$. We have $\Pr(b > \beta_2^v(\hat{v})) = \frac{\beta_2^{v^{-1}}(b)}{v} = \frac{\lambda_2^v(b)}{v}$ and $(v-b)\Pr(b > \beta_2^v(\hat{v})) = (v-b)\frac{\lambda_2^v(b)}{v}$. Hence we need that:

(10) $(v-b)\frac{\lambda_2^v(b)}{v} = K^v$

The optimality of bidder 1's rival's strategy requires that: $\beta_2^v(\hat{v}) = \arg\max_{\hat{b}} \left(\hat{v} - \hat{b}\right) \Pr\left(\hat{b} > b\right)$ for all $\hat{v} \in [0, v)$. Let $t = \beta_2^v(\hat{v})$. Since $\Pr(\hat{b} > b) = F^v(t)$, the above expression can be rewritten as $t = \arg\max(\hat{v} - t)\, G^v(t)$ for all $\hat{v} \in [0, v)$. The first-order condition leads to:

(11) $(\lambda_2^v(t) - t)\, f^v(t) = F^v(t)$

It is almost the same asymmetric auction case examined by Vickrey (1961). It is easy to check that both bidders should have the same ending (maximum) bid. Let the ending bid be t^*. From equation (10), we obtain $\lambda_2^v(b) = \frac{vK^v}{v-b}$. By symmetry, this implies that the bid function for bidder 1's rival is $\beta_2^v(\hat{v}) = v - \frac{vK^v}{\hat{v}}$. Notice that $\beta_2^v(K^v) = 0$. So in order to obtain a monotonic bid function, we assume that when $\hat{v} \in [0,\ K^v)$, bidder 1's rival will not bid at all. Similarly, $\lambda_2^v(t^*) = v$, so $\frac{vK^v}{v-t^*} = v$, then $t^* = v - K^v$. We adopt the stability refinement argument by Vickrey (1961) and choose the particular equilibrium where $\lambda_2^v(b) = \frac{vK^v}{v-b}$ is tangent to the 45 degree line. This implies that $K^v = \frac{v}{4}$. So $t^* = \frac{3v}{4}$ and $\lambda_2^v(b) = \frac{v^2}{4(v-b)}$ and $\beta_2^v(\hat{v}) = v - \frac{v^2}{4\hat{v}}$. Substitute the functional form of $\lambda_2^v(.)$ into equation (11), we have $\frac{g^v(t)}{G^v(t)} = \frac{4(v-t)}{(v-2t)^2}$. Integration on both sides leads to: $\ln G^v(t) = -\ln(2t-v) - \dfrac{v}{2t-v} + C$. For all values of C, we have $G^v(t) \to 0$ when $t \to \dfrac{v}{2}$. So $\dfrac{v}{2}$ is the lower bound of the support of the randomized bids. Also $G^v(t^*) = 1$, i.e., $\ln G^v(t^*) = 0$, so $-\ln(2 \times \dfrac{3v}{4} - v) - \dfrac{v}{2 \times \dfrac{3v}{4} - v} + C = 0$. Then $C = \ln v + 2 - \ln 2$.

So $G^v(t) = \dfrac{v}{2(2t-v)} e^{\frac{4t-3v}{2t-v}}$. We have shown that when $\hat{v} \in [0, K^v)$, bidder 1's rival will not bid. Since we find that $K^v = \dfrac{v}{4}$ and bidder 1 will randomize on the support $(\dfrac{v}{2}, \dfrac{3v}{4}]$, we can see that bidder 1's rival will not win at all when $\hat{v} \in [\ K^v, \dfrac{v}{2})$ either. With the bidder symmetry, by now we have found the second-stage bid functions.

B. Now we start to derive the first-stage bid function. Given a valuation v, bidder 1 bids $\beta(z)$ where $z > v$. So we first consider the situation when bidder 1 mimics a higher type and the other bidder truthfully follows $\beta(.)$. Then she wins with probability z and loses with probability $1 - z$. If bidder 1 wins, at the second stage, she believes that her rival has the valuation $\hat{v}$ distributed on $[0, z)$ with density $\dfrac{1}{z}$. Her rival believes that she faces a first-stage winner with valuation z, hence if i) $\hat{v} \in [0, \dfrac{z}{4})$, then she will not submit a bid. ii) if $\hat{v} \in [\dfrac{z}{4}, z)$, then she will bid according to the bid function $\beta_2^z(\hat{v}) = z - \dfrac{z^2}{4\hat{v}}$. Then the second-stage best response for the first-stage winner given she mimics type z will be to submit a bid b satisfying the following conditions. If $b = 0$, then her second-stage payoff is $\dfrac{v}{4}$. If she submits a nonzero bid, her payoff is $(v-b)\Pr(b > \beta_2^z(\hat{v}))$, which equals $\dfrac{(v-b)z}{4(z-b)}$. This term is maximized by choosing $b = 0$. In this case, her second-stage best payoff is again $\dfrac{v}{4}$. So we can see that under all instances, if bidder 1 mimics z at the first stage and wins, she can get $\dfrac{v}{4}$ at most in the second stage given the other bidder follows the specified equilibrium strategy. Now consider what if bidder 1 loses the first stage by mimicking type z. She knows exactly her rival's valuation $\hat{v}$, where $v < z < \hat{v}$. Her rival will randomly choose a bid b from the interval $(\dfrac{\hat{v}}{2}, \dfrac{3\hat{v}}{4}]$ according to the c.d.f. $G^{\hat{v}}(b) = \dfrac{\hat{v}}{2(2b-\hat{v})} e^{\frac{4b-3\hat{v}}{2b-\hat{v}}}$. Then the best response for the bidder will be i) if $v \in [0, \dfrac{\hat{v}}{4})$, then she will not submit a bid. ii) if $v \in [\dfrac{\hat{v}}{4}, \hat{v})$, then she will bid according to the bid function $\beta_2^{\hat{v}}(v) = \hat{v} - \dfrac{\hat{v}^2}{4v}$. Next we need to determine the appropriate integration regions for payoff functions under different values of v. First, if $v < \dfrac{z}{2}$, she will obtain zero second-stage payoff when she loses the first stage according to the stated strategies. Then we can be sure that she never needs to mimic such a z that $v < \frac{z}{2}$ because a) it does not bring her any higher second-stage payoff following winning the first stage due to the fact that a bidder can always obtain $\frac{v}{4}$ at the second stage given winning the first stage no matter what z she mimics and b) it does not bring her any higher second-stage payoff following losing the first stage either since other mimicking type can always bring her nonnegative second-stage payoff. Second, when $\frac{1}{2} > v > \frac{z}{2}$, she will not bid when $\hat{v} \in (2v, 1]$ and bid $\hat{v} - \frac{\hat{v}^2}{4v}$ when $\hat{v} \in (z, 2v]$. So her second-stage payoff when $\hat{v} \in (z, 2v]$ is $[v - (\hat{v} - \frac{\hat{v}^2}{4v})] \Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b)$. $\Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = F^{\hat{v}}(\hat{v} - \frac{\hat{v}^2}{4v}) = \frac{v}{2v - \hat{v}} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}$. Then $[v - (\hat{v} - \frac{\hat{v}^2}{4v})] \Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = \frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}$. So her overall expected second-stage payoff following she losing the first stage is $\pi_1(v, z) = \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}$. Finally, in the case where $\frac{1}{2} < v$, her overall expected second-stage payoff following losing the first stage is $\pi_z(v, z) = \int_z^1 (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}$. Notice that if we can find a first-stage bid function to implement truthful bidding, i.e. $z = v$, then at $v = \frac{1}{2}$, we will have $\pi_1(\frac{1}{2}) = \pi_2(\frac{1}{2})$ and $\pi_1^{'}(\frac{1}{2}) = \pi_2^{'}(\frac{1}{2})$ according to the above derived expressions of $\pi_1(.)$ and $\pi_2(.)$, which ensures the continuity of the first-stage bid function at $v = \frac{1}{2}$. Now we are ready to derive the first-stage bid function. The bidder's first-stage payoff if she observes a valuation v while mimics $z > v$ when the other bidder follows the equilibrium bid functions is:

i) $v \leq \frac{1}{2}$

$\Pi_1(v, z)$

$= z[(v - \beta(z)) + \delta\frac{v}{4}] + (1 - z)[\delta\pi_1(v, z)]$

$= z[(v - \beta(z)) + \delta\frac{v}{4}] + (1 - z)[\delta \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}]$

$= z[(v - \beta(z)) + \delta\frac{v}{4}] + \delta \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}) d\hat{v}$

$= z[(v - \beta(z)) + \delta\frac{v}{4}] - \delta \int_{2v}^{z} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}) d\hat{v}$

The first-order condition w.r.t. z yields:

(12) $v - \beta(z) - z\beta'(z) + \frac{1}{4}\delta v - \delta\frac{2v-z}{4}e^{\frac{2v-2z}{2v-z}} = 0$

In equilibrium $z = v$, so we can have:

(13) $v - \beta(v) - v\beta'(v) = 0$

The unique solution is $\beta(v) = \frac{1}{2}v$.

ii) $v > \frac{1}{2}$

$\Pi_1(v, z) = z[(v - \beta(z)) + \delta\frac{v}{4}] + (1-z)[\delta\pi_2(v, z)]$ and we can obtain the same bid function. The above is the solution when the bidder mimics a type bigger than her own valuation. Also, we need to check when the bidder mimics a lower type whether the bid function still remains the same. It is easy to check that it is indeed the case. Q.E.D.

Proof of Proposition 5.

Let $I_1(0)$ denote the inf of the randomized bids for type zero at the first stage where the subscript represents the number of the stage. We know that $I_1(v) > I_1(0)$ for all $v > 0$ from the monotonicity requirement. Once the zero type bids $I_1(0)$ in the first stage, she will lose with probability one and her zero valuation can be inferred with probability one too. Under this contingency, we have a subsequent second-stage auction where type zero competes with the type uniformly distributed over [0, 1]. In this subgame, suppose first that the zero type randomizes over zero. Let $S_2(0) > 0$ denote the sup of her randomized bids at the second stage. We claim that this zero type must be defeated with probability one. Otherwise, the zero type will win with positive probability bringing her negative payoff, which is impossible in equilibrium. Therefore, all her rival's bids must be above $S_2(0)$. However, this will not be optimal for those of her rivals whose types fall in the interval $[0, S_2(0)]$ since these types will obtain negative payoff with probability one in this particular subgame. Hence we obtain that the zero type can not randomizes above zero at the second stage. Then suppose $S_2(0) < 0$, the zero type's rivals can always win with negative bids, which is impossible for any normal auction rule. So finally we consider the only case left where $S_2(0) = 0$. In this case, we can argue that the zero type must be defeated with probability one as before. Then any of type zero's rivals who has a valuation $v > 0$ must bid above zero. But this makes the optimal bid non-existent because whenever the bidder bids $b > 0$, it is always better for her to bid $\frac{b}{2}$. So no equilibrium can exist in the subgame. Then we can conclude that there is no symmetric monotonic equilibrium for the whole two-stage first-price auctions with the announcement of the first-stage losing bid game. Q.E.D.

Proof of Lemma 1.

The equilibrium will be derived in the following step A and B, which at the same time yields the necessary equilibrium condition.

A. To support the second-stage equilibrium, we need to show that $k \leq \frac{1}{2}$. The loser definitely does not want to deviate. We need to further show that the winner does not want to decrease the bid. If the winner decrease the bid by $\varepsilon > 0$, her second-stage payoff will be $\pi = (\hat{v} - ((1-k)\,v + k\hat{v} - \varepsilon))\,\frac{(1-k)\,v + k\hat{v} - \varepsilon - v}{(1-k)\,v + k\hat{v} - v}$, where $\frac{(1-k)\,v + k\hat{v} - \varepsilon - v}{(1-k)\,v + k\hat{v} - v}$ is the probability of winning. We need $\frac{d\pi}{d\varepsilon} \leq 0$. So we have $(\hat{v} - v)\,(2k-1) \leq 2\varepsilon$. Since this inequality needs to hold no matter how small ε is, we then need $k \leq \frac{1}{2}$.

B. To support the first-stage equilibrium, we need to show that $k \geq \frac{\delta}{1+\delta}$. To show this, we have to derive the first-stage bid functions first. Let us assume bidder 1 observes a valuation v while mimics $z < v$. Then her first-stage payoff is $z\,(v - \beta\,(z))$. Conditional on bidder 1 wins the first stage, her rival will have valuation t uniform over $[0,\, z)$ with density $\frac{1}{z}$ and randomize over $[t,\, (1-k)\,t + kz)$. It is easy to check that the best response for bidder 1 will be to bid $(1-k)\,t + kz$ to win the object for sure. Then her payoff is $\int_0^z [v - ((1-k)\,t + kz)]\frac{1}{z}dt = v - \frac{z}{2} - \frac{kz}{2}$. Conditional on bidder 1 loses the first stage, her rival will have valuation t uniform over $[z,\, 1)$ with density $\frac{1}{1-z}$ and bid $(1-k)\,z + kt$. Then bidder 1's best response will be to bid $(1-k)\,z + kt + \varepsilon$ as long as $(1-k)\,z + kt < v$. The deviation of z will be analyzed in the following two categories.

First, we consider a z deviation such that $(1-k)\,z + kt > v$ when $t = 1$, i.e., $z \in (\frac{v-k}{1-k}, v)$, which represents a small deviation as bidder 1's rival may bid above v for some realizations of t. We assume bidder 1 can win the object only with $(1-k)\,z + kt$ at the second stage. If we can obtain an equilibrium under this assumption, the equilibrium will still remain valid under such assumption as bidder 1 needs to win with $(1-k)\,z + kt + \varepsilon$. Notice that bidder 1 will only bid when $(1-k)z+kt<v$, which implies that $t<\dfrac{v-(1-k)z}{k}$. So her second-stage payoff is $\int_z^{\frac{v-(1-k)z}{k}}[v-((1-k)z+kt)]\dfrac{1}{1-z}dt=\dfrac{(v-z)^2}{2k}\dfrac{1}{1-z}$.

Then her overall two-stage expected payoff is:
$\pi=z(v-\beta(z))+\delta z(v-\dfrac{z}{2}-\dfrac{kz}{2})+\delta(1-z)\dfrac{(v-z)^2}{2k(1-z)}$
$=zv-z\beta(z)+\delta(zv-\dfrac{z^2}{2}-\dfrac{kz^2}{2})+\delta\dfrac{(v-z)^2}{2k}$
Differentiate π w.r.t. z and set the first-order condition to zero, we obtain:
(14) $v-z\beta'(z)-\beta(z)+\delta(v-z-kz)-\dfrac{\delta(v-z)}{k}=0$
In equilibrium, $z=v$, so we have $v-v\beta'(v)-\beta(v)-\delta kv=0$. With usual boundary condition $\beta(0)=0$, we have $\beta(v)=\dfrac{(1-\delta k)v}{2}$.

Second, we then require the above derived bid functions can also prevent a large z deviation such that $(1-k)z+kt\leq v$ when $t=1$, i.e., $z\in[0,\dfrac{v-k}{1-k}]$ where bidder 1's rival will always bid under or equal to v. Following the above specified strategy and truthful bidding, a bidder's total payoff is $\dfrac{v^2}{2}+\dfrac{\delta v^2}{2}$, whose derivation is in the proof of Lemma 4. We next will show that bidder 1 will not deviate to zero at the first stage, which requires that the deviation profit $\delta\int_0^1(v-kt)dt\leq\dfrac{v^2}{2}+\dfrac{\delta v^2}{2}$. This inequality can be rearranged as $(v-\dfrac{\delta}{1+\delta})^2-(\dfrac{\delta}{1+\delta})^2+\dfrac{\delta k}{1+\delta}\geq 0$ for all v including $v=\dfrac{\delta}{1+\delta}$. Hence we must have $k\geq\dfrac{\delta}{1+\delta}$. We then need to show that under the condition $k\geq\dfrac{\delta}{1+\delta}$, all other big deviations can be prevented too. The total deviation profit is:
$\pi=(zv-\dfrac{(1-\delta k)z}{2})+(\delta zv-\dfrac{z^2}{2}-\dfrac{kz^2}{2})+\delta(1-z)\int_z^1(v-(1-k)z-kt)\dfrac{1}{1-z}dt$. Setting $\dfrac{d\pi}{dz}=0$, we find the best z deviation is $\dfrac{v+\delta k-\delta}{1+\delta k-\delta}$. While, we can check that this best deviation fails the large deviation constraint $(1-k)z+k<v$. This means that π is monotonic in z for all $z\in[0,\dfrac{v-k}{1-k}]$. It is straight forward to check that when $z=\dfrac{v-k}{1-k}$, the deviation profit π is smaller than the profit from truthful bidding. Therefore all large deviations can be prevented. Similarly, we can analyze the situation where bidder 1 mimics a type $z>v$ in the first stage, which yields the same equilibrium bid functions and equilibrium conditions. So we can conclude that the necessary condition for the above derived bid functions to be an equilibrium is $\dfrac{1}{2}\geq k\geq\dfrac{\delta}{1+\delta}$. Q.E.D.

Proof of Proposition 6.

It is straight forward to check that the derived bid functions in Lemma 1 constitute an equilibrium. Q.E.D.

Proof of Lemma 2.

The proof consists of the following three steps.

A. We need to find an appropriate expression for the revenue. The expected payment for one bidder in the first stage is $\int_0^1 [\frac{v^2}{2}(1-\delta)+\delta\int_0^v \beta_2(t)\,dt]dv$. The second-stage expected sum of payment can be rearranged as $\int_0^1 [\beta_1(v)\,\beta_2^{-1}(\beta_1(v))+\beta_2(v)\,\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]\}dv$. So the total revenue is:

$R_3 = 2\{\int_0^1 \frac{v^2}{2}(1-\delta)\,dv+\int_0^1 \delta[\int_0^v \beta_2(t)\,dt+\beta_1(v)\,\beta_2^{-1}(\beta_1(v))+\beta_2(v)\,\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]dv\}$.

Let $P = \int_0^1 [\int_0^v \beta_2(t)\,dt+\beta_1(v)\,\beta_2^{-1}(\beta_1(v))+\beta_2(v)\,\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]dv$.

Since $\int_0^1\int_0^v \beta_2(t)\,dtdv = \int_0^1 (1-v)\,\beta_2(v)\,dv$, P can then be rewritten as:

$P = \int_0^1 \{\beta_2(v)+\beta_1(v)\,\lambda_2((\beta_1(v))+\beta_2(v)\,\lambda_1(\beta_2(v))-2v\beta_2(v)\}dv$

B. We need to find a bound for P. As there is no closed form solution to the bid functions, our approach is to use their bounds to bound P. We claim that $P < 1-\frac{5}{3}t^*+t^{*2}-\frac{1}{3}t^{*4}+\frac{1}{3}t^{*5}$ and $t^* > \frac{5}{8}$, where t^* is the common end point of two second-stage bid functions. To show this, we will first obtain some auxiliary results. Here we abbreviate the terminal bid t^* as t.

Result 1.

$\int_0^t s\lambda_2(s)\,\lambda_2'(s)\,ds = s\lambda_2(s)\,\lambda_2(s)\,|_0^t - \int_0^t \lambda_2(s)\,(\lambda_2(s)+s\lambda_2'(s))\,ds$

$= t - \int_0^t \lambda_2(s)\,\lambda_2(s)\,ds - \int_0^t s\lambda_2(s)\,\lambda_2'(s)\,ds$

So $2\int_0^t s\lambda_2(s)\,\lambda_2'(s)\,ds = t - \int_0^t [\lambda_2(s)]^2 ds$

Result 2.

Multiply s to both sides of equation (9) and integrate, we have

$\int_0^t s\lambda_1(s)\,\lambda_2'(s)\,ds - \int_0^t s^2\lambda_2'(s)\,ds = \int_0^t s\lambda_2(s)\,ds$, which leads to:

$\int_0^t s\lambda_1(s)\,\lambda_2'(s)\,ds - [s^2\lambda_2(s)\,|_0^t - \int_0^t 2s\lambda_2(s)\,ds] = \int_0^t s\lambda_2(s)\,ds$

So $\int_0^t s\lambda_1(s)\,\lambda_2'(s)\,ds = t^2 - \int_0^t s\lambda_2(s)\,ds$

Result 3.

$\int_0^t \lambda_2(s)\,\lambda_1'(s)\,ds$

$= \lambda_2(s)\,\lambda_1(s)\,|_0^t - \int_0^t \lambda_2'(s)\,\lambda_1(s)\,ds = 1 - \int_0^t (\lambda_2(s)+s\lambda_2'(s))\,ds$

$= 1 - [\int_0^t \lambda_2(s)\,ds + \int_0^t s\lambda_2'(s)\,ds] = 1 - [\int_0^t \lambda_2(s)\,ds + s\lambda_2(s)\mid_0^t - \int_0^t \lambda_2(s)\,ds] = 1 - t$

It is easy to see that $\int_0^t \lambda_1(s)\,\lambda_2'(s)\,ds = t$.

Result 4.

$\lambda_1(s)\,\lambda_2'(s) + \lambda_2(s)\,\lambda_1'(s) = \lambda_1(s) + s\lambda_1'(s) + s\lambda_2'(s)$ by adding equation (6) and (7).

So $\int_0^t [\lambda_1(s) + s\lambda_1'(s) + s\lambda_2'(s)]ds = 1$

$\int_0^t \lambda_1(s)\,ds + s\lambda_1(s)\mid_0^t - \int_0^t \lambda_1(s)\,ds + s\lambda_2(s)\mid_0^t - \int_0^t \lambda_2(s)\,ds = 1$

Then $\int_0^t \lambda_2(s)\,ds = 2t - 1$

Result 5.

$\int_0^t s\lambda_2'(s)\,ds = s\lambda_2(s)\mid_0^t - \int_0^t \lambda_2(s)\,ds = t - \int_0^t \lambda_2(s)\,ds = 1 - t.$

Using the above results, we can find the new expression and the bound for P as follows.

I. $\int_0^1 \beta_2(v)\,dv = \int_0^t s\lambda_2'(s)\,ds = 1 - t$. Then $1 - t < \int_0^1 \frac{3}{4}vdv$ because $\beta_2(v) < \frac{3}{4}$, leading to $t > \frac{5}{8}$.

II. $\int_0^1 \beta_1(v)\,\lambda_2((\beta_1(v))\,dv < \int_0^1 tv^2dv = \frac{t}{3}$ because $\beta_1(v) < tv$ and $\lambda_2(\beta_1(v)) < v$.

III. $\int_0^1 [\beta_2(v)\,\lambda_1(\beta_2(v)) - 2v\beta_2(v)]dv = \int_0^t s\lambda_1(s)\,\lambda_2'(s)\,ds - \int_0^t 2s\lambda_2(s)\,\lambda_2'(s)\,ds$

$= t^2 - \int_0^t s\lambda_2(s)\,ds - t + \int_0^t [\lambda_2(s)]^2 ds < t^2 - t + \int_0^t ts\,(ts - s)\,ds = t^2 - t + \frac{1}{3}t^5 - \frac{1}{3}t^4$

IV. Therefore, $P < 1 - t + \frac{t}{3} + t^2 - t + \frac{1}{3}t^5 - \frac{1}{3}t^4$

C. We are ready to find the final bound. $P - \int_0^1 \frac{v^2}{2}dv < \frac{5}{6} - \frac{5}{3}t^* + t^{*2} - \frac{1}{3}t^{*4} + \frac{1}{3}t^{*5}$. The expression on the right hand side of the inequality is monotonically decreasing for $t^* \in [\frac{1}{2}, \frac{3}{4}]$. Using the fact that $t^* > \frac{5}{8}$, we obtain a upper bound of $P - \int_0^1 \frac{v^2}{2}dv$ as $\frac{8}{49}$. Since $R_3 = \frac{1}{3} + 2\delta(P - \int_0^1 \frac{v^2}{2}dv)$, we then have $R_3 < \frac{1}{3} + \frac{16}{49}\delta$. Q.E.D.

Proof of Lemma 3.

We need to show that the overall revenue has the following expression:

$R_4 = 2 \times \{\int_0^1 \frac{v^2}{2}(1+\delta)\,dv + \delta[\int_0^{\frac{1}{2}} (e^2 \int_0^{\frac{v}{2}} \frac{v^2 - t^2}{t} e^{-\frac{v}{t}} dt)dv + \int_{\frac{1}{2}}^1 (e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2 - t^2}{t} e^{-\frac{v}{t}} dt)dv]\}$

Once this result can be obtained, we can immediately reach the conclusion that $R_4 > \frac{1}{3} + \frac{1}{3}\delta$ because

$2 \times \int_0^1 \frac{v^2}{2}(1+\delta)\,dv = \frac{1}{3} + \frac{1}{3}\delta$ and $\int_0^{\frac{1}{2}} (e^2 \int_0^{\frac{v}{2}} \frac{v^2 - t^2}{t} e^{-\frac{v}{t}} dt)dv + \int_{\frac{1}{2}}^1 (e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2 - t^2}{t} e^{-\frac{v}{t}} dt)dv > 0$. Now we start to derive the expression of R_4. At the valuation v, if bidder 1 loses the first stage and the winner's valuation is inferred as $\hat{v}$, where $\hat{v}$ is uniformly distributed in $(v, 1]$ with density $\frac{1}{1-v}$, then she will bid $\beta_2^{\hat{v}}(v) = \hat{v} - \frac{\hat{v}^2}{4v}$. Her rival will randomize over $(\frac{\hat{v}}{2}, \frac{3\hat{v}}{4}]$ with c.d.f. $G^v(b) = \frac{\hat{v}}{2(2b-\hat{v})} e^{\frac{4b-3\hat{v}}{2b-\hat{v}}}$. If $v > \frac{1}{2}$, the probability for bidder 1 to win the second stage is $\Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = G^v(\hat{v} - \frac{\hat{v}^2}{4v}) = \frac{v}{2v-\hat{v}} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}}$. Her expected payment is $\int_v^1 \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$. Similarly if $v \leq \frac{1}{2}$, her expected payment is $\int_v^{2v} \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$. Next consider the case where bidder 1 wins the first stage at the valuation v. Then her rival will have valuation $\hat{v}$ distributed over the support $[0, v)$ with density $\frac{1}{v}$. The probability for bidder 1 to win the second stage is $\Pr(b > v - \frac{v^2}{4\hat{v}}) = \Pr(\hat{v} < \frac{v^2}{4(v-b)}) = \frac{v}{4(v-b)}$. So the expected payment for bidder 1 following her winning the first stage is:

$\int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}})$. We use $T_{v>\frac{1}{2}}$ and $T_{v<\frac{1}{2}}$ to denote the overall expected payments for bidder 1 at the second stage when $v > \frac{1}{2}$ and $v < \frac{1}{2}$ respectively. Then we have:

$$T_{v>\frac{1}{2}} = v \int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + (1-v) \int_v^1 \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$$

$$= v \int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + \int_v^1 \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} d\hat{v}$$

$T_{v>\frac{1}{2}}$ can be rearranged as:

$$T_{v>\frac{1}{2}} = \frac{e^2}{8} v^3 \int_0^{\frac{v}{2}} \frac{v+t}{t^3} e^{-\frac{v}{t}} dt + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$$

$$= \frac{e^2}{8} v^3 [\int_0^{\frac{v}{2}} \frac{v}{t^3} e^{-\frac{v}{t}} dt + \int_0^{\frac{v}{2}} \frac{1}{t^2} e^{-\frac{v}{t}} dt] + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$$

Notice that $\int_0^{\frac{v}{2}} \frac{v}{t^3} e^{-\frac{v}{t}} dt = -\frac{1}{v} \int_0^{\frac{v}{2}} -\frac{v}{t} e^{-\frac{v}{t}} d(-\frac{v}{t}) = -\frac{1}{v}(-\frac{v}{t} e^{-\frac{v}{t}} - e^{-\frac{v}{t}})|_0^{\frac{v}{2}} = \frac{3}{v} e^{-2}$ and $\int_0^{\frac{v}{2}} \frac{1}{t^2} e^{-\frac{v}{t}} dt = \frac{1}{v} \int_0^{\frac{v}{2}} e^{-\frac{v}{t}} d(-\frac{v}{t}) = \frac{1}{v}(e^{-\frac{v}{t}})|_0^{\frac{v}{2}} = \frac{1}{v} e^{-2}$. Therefore, $T_{v>\frac{1}{2}} = \frac{v^2}{2} + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$. With similar method, we can find $T_{v\leq\frac{1}{2}}$ as follows.

$$T_{v\leq\frac{1}{2}} = v \int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + (1-v) \int_v^{2v} \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$$

$= \dfrac{v^2}{2} + e^2 \int_0^{\frac{v}{2}} \dfrac{v^2-t^2}{t} e^{-\dfrac{v}{t}} dt$

To sum up, the expected payment T for bidder 1 at the second stage is:

$T = \dfrac{v^2}{2} + e^2 \int_0^{\frac{v}{2}} \dfrac{v^2-t^2}{t} e^{-\dfrac{v}{t}} dt \quad \text{when} \quad v \leq \dfrac{1}{2}$

$= \dfrac{v^2}{2} + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \dfrac{v^2-t^2}{t} e^{-\dfrac{v}{t}} dt \text{ when } \ v > \dfrac{1}{2}$

Then it is straight forward to obtain the desired expression for R_4. Q.E.D.

Proof of Lemma 4.

The expected payment for a bidder with valuation v at the second stage is: $Q_2 = v \int_0^v [(1-k)\,t + kv] \dfrac{1}{v} dt = \dfrac{1}{2} v^2 (1+k)$. The expected payment for a bidder with valuation v at the first stage is: $Q_1 = v[\dfrac{(1-\delta k)v}{2}] = \dfrac{(1-\delta k)v^2}{2}$. So $R_5 = 2 \times \int_0^1 (Q_1 + \delta Q_2) dv = 2 \times \int_0^1 (\dfrac{(1-\delta k)v^2}{2} + \dfrac{1}{2}\delta v^2 (1+k)) dv = \dfrac{1}{3} + \dfrac{1}{3}\delta$. Q.E.D.

Proofs for Chapter 4

Proof of Proposition 1.

Let bidder 1 be the generic bidder for our derivation of the equilibrium bid functions. Since bidders' signals are cross-period independent, while Θ_1 is correlated with Θ_2, the *only valuable* information the bidders will condition on is the inferred true value of θ_1. Then the second-stage bid function will take the form of $\beta_2 (x_2^1, \theta_1)$, where the first-stage information $p_1, x_1^1, x_1^2, ...x_1^N$ affects the bid only through the aggregated variable θ_1.

Let $v_2^1 (x_2^1, y_2^1, \theta_1) = E[P_2 | X_2^1 = x_2^1, Y_2^1 = y_2^1, \Theta_1 = \theta_1]$, where $Y_2^1 = \max\limits_{s \neq 1} X_2^s$ and y_2^1 is its realized value. The c.d.f. of Y_2^1 is $F^{N-1}(.)$ and the p.d.f of Y_2^1 is $(N-1)\,F^{N-2}(.)\,f(.)$. Then

$E[P_2 | X_2^1 = x_2^1, Y_2^1 = y_2^1, \Theta_1 = \theta_1]$

$= E[\alpha \Theta_2 + (1-\alpha)\,U(X_2^1, X_2^2, ...X_2^N) | X_2^1 = x_2^1, Y_2^1 = y_2^1, \Theta_1 = \theta_1]$

$= \alpha \theta_1 + (1-\alpha)\,E[U(x_2^1, X_t^2, ...X_t^N) | Y_2^i = y_2^1]$

Let us assume bidder 1 mimics type z at the second stage given her true signal is x_2^1 . Then her second-stage payoff is:

$\Pi_2 (z, x_2^1, \theta_1) = \int_0^z [v_2^1 (x_2^1, y_2^1, \theta_1) - \beta_2 (z, \theta_1)]\,(N-1)\,F^{N-2}(y_2^1)\,f(y_2^1)\,dy_2^1$

The first-order condition w.r.t. z leads to:

(1) $[v_2^1(x_2^1, z, \theta_1) - \beta_2(z, \theta_1)](N-1)F^{N-2}(z)f(z) - \beta_2'(z, \theta_1)F^{N-1}(z) = 0$

The equilibrium condition $z = x_2^1$ leads to:

(2) $[v_2^1(x_2^1, x_2^1, \theta_1) - \beta_2(x_2^1, \theta_1)](N-1)f(x_2^1) - \beta_2'(x_2^1, \theta_1)F(x_2^1) = 0.$

The solution to equation (2) is standard, which gives:

$\beta_2(x_2^1, \theta_1)$

$= \dfrac{1}{F^{N-1}(x_2^1)} \int_0^{x_2^1} v_2^1(y, y, \theta_1)\, dF^{N-1}(y)$

$= E(v_2^1(Y_2^1, Y_2^1, \theta_1)|Y_2^1 < x_2^1)$

$= \alpha\theta_1 + (1-\alpha)E\{E[U(Y_2^1, X_t^2, ...X_t^N)|Y_2^1]|Y_2^1 < x_2^1\}$

We can obtain the final equilibrium bid function by replacing θ_1 with $\dfrac{p_1}{\alpha} - \dfrac{(1-\alpha)U(x_1^1, x_1^2, ...x_1^N)}{\alpha}$.

Q.E.D.

Proof of Proposition 2.

Let us assume all bidders follow the monotonic bid functions $\beta_1(.)$ and $\beta_2(.)$ truthfully except bidder 1. At the signal x_1^1, let us assume bidder 1 mimics $z \neq x_1^1$ at the first stage. Let $v_1^1(x_1^1, y_1^1) = E[P_1|X_1^1 = x_1^1, Y_1^1 = y_1^1]$, where $Y_1^1 = \max\limits_{s \neq 1} X_1^s$ and y_1^1 is its realized value. Then her first-stage payoff is: $\Pi_1(x_1^1, z) = \int_0^z (v_1^1(x_1^1, y_1^1) - \beta_1(z))(N-1)F^{N-2}(y_2^1)f(y_2^1)\,dy_1^1$

At the second stage, all other bidders will follow $\beta_2\left(x_2^i, \hat{\theta}_1\right)$, where $i \neq 1$. Notice that the true $\theta_1 = \dfrac{p_1}{\alpha} - (\dfrac{1}{\alpha} - 1)U(x_1^1, x_1^2, ...x_1^N)$, where θ_1 is increasing in p_1 and decreasing in all other arguments. But bidder 1 mimics type z, so all other bidder will be induced to believe that the realized fundamental is $\hat{\theta}_1 = \dfrac{p_1}{\alpha} - (\dfrac{1}{\alpha} - 1)U(z, x_1^2, ...x_1^N) \neq \theta_1$. Therefore, bidder 1's second-stage payoff is: $\Pi_2\left(x_2^1, \tau, \theta_1, \hat{\theta}_1\right) = \int_0^\tau [v_2^1(x_2^1, y_2^1, \theta_1) - \beta_2(\tau, \hat{\theta}_1)](N-1)F^{N-2}(y_2^1)f(y_2^1)\,dy_2^1$

Notice that we let bidder 1 bid according to $\beta_2(., \hat{\theta}_1)$ rather than $\beta_2(., \theta_1)$ even though she knows the true value θ_1. This is because it is never optimal for bidder 1 to bid outside the support of $\beta_2(., \hat{\theta}_1)$, according to which all her rivals bid. Then Bidder 1's decision is simply to choose a bid within this particular support, which corresponds to a certain type $\tau \in [0, \omega]$. So $\tau^* = \arg\max\limits_{\tau} \Pi_2\left(x_2^1, \tau, \theta_1, \hat{\theta}_1\right)$. The FOC w.r.t. τ leads to:

(3) $[v_2^1(x_2^1, \tau, \theta_1) - \beta_2\left(\tau, \hat{\theta}_1\right)](N-1)f(\tau) - \beta_2'\left(\tau, \hat{\theta}_1\right)F(\tau) = 0$

Let $v_2^1(y, y, \theta_1)(N-1)F^{N-2}(y)f(y) = V(y, \theta_1)$. Then $\beta_2(x_2^1, \theta_1) = \dfrac{1}{F^{N-1}(x_2^1)} \int_0^{x_2^1} V(y, \theta_1)\, dy$.

So $\beta_2'(x_2^1,\theta_1) = \dfrac{V(x_2^1,\theta_1)}{F^{N-1}(x_2^1)} - \dfrac{(N-1)f(x_2^1)\beta_2(x_2^1,\theta_1)}{F(x_2^1)}$. Substituting the functional form of $\beta_2'(x_2^1,\theta_1)$ into the above equation (3), we have:

(4) $v_2^1(x_2^1,\tau,\theta_1) = v_2^1\left(\tau,\tau,\hat{\theta}_1\right)$

Equation (4) can be transformed into:

(5) $\alpha\theta_1 + \dfrac{1-\alpha}{F^{N-1}(\tau)}\int_0^\tau ... \int_0^\tau U\left(x_2^1,x_2^2...x_2^N\right) f(x_2^2)...f\left(x_2^N\right) dx_2^2...dx_2^N$

$= \alpha\hat{\theta}_1 + \dfrac{1-\alpha}{F^{N-1}(\tau)}\int_0^\tau ... \int_0^\tau U\left(\tau,x_2^2...x_2^N\right) f(x_2^2)...f\left(x_2^N\right) dx_2^2...dx_2^N$

Equation (5) can be reduced to:

(6) $-U(x_1^1,x_1^2,...x_1^N) + \dfrac{1}{F^{N-1}(\tau)}\int_0^\tau ... \int_0^\tau U\left(x_2^1,x_2^2...x_2^N\right) f(x_2^2)...f\left(x_2^N\right) dx_2^2...dx_2^N$

$= -U(z,x_1^2,...x_1^N) + \dfrac{1}{F^{N-1}(\tau)}\int_0^\tau ... \int_0^\tau U\left(\tau,x_2^2...x_2^N\right) f(x_2^2)...f\left(x_2^N\right) dx_2^2...dx_2^N$

Given that $U(.)$ is both monotonic and symmetric in each of its components, we can obtain that for any z together with the set of realized signals $\{x_2^1,x_1^1,x_1^2,...x_1^N\}$, there is a unique optimal $\tau^*\left(x_2^1,z,x_1^1,x_1^2,...x_1^N\right)$ that corresponds to z. Also, $\tau^*\left(x_2^1,z,x_1^1,x_1^2,...x_1^N\right)$ is monotonic in x_1^1. When $z \to x_1^1, \tau^* \to x_2^1$, so τ^* will not be a corner solution over the support $[0,\,\omega]$ with probability 1.

The overall two-stage payoff is: $\Pi = \Pi_1 + \delta E_{X_1^2,...,x_1^N,\Theta_1,X_2^1}\Pi_2$. Π_1 is bidder 1's first-stage payoff given she mimics type z. Π_2 is her optimal second-stage payoff by mimicking type $\tau^*\left(x_2^1,z,x_1^1,x_1^2,...x_1^N\right)$ after processing all the interperiod information. Then the FOC. w.r.t. z is:

(7) $\dfrac{\partial\Pi_1}{\partial z} + \delta\dfrac{dE_{X_1^2,...,x_1^N,\Theta_1,X_2^1}\Pi_2^*}{dz} = 0$

It is easy to see that $\dfrac{\partial\Pi_1}{\partial z} = [v_1^1(x_1^1,z) - \beta_1(z)](N-1)F^{N-2}(z)f(z) - \beta_1'(z)F^{N-1}(z)$ and $\dfrac{dE_{X_1^2,...,x_1^N,\Theta_1,X_2^1}\Pi_2}{dz} = E_{X_1^2,...,x_1^N,\Theta_1,X_2^1}(\dfrac{d\Pi_2}{dz})$. Using the envelope theorem without constraint (since we argue before that the optimal τ^* will not be a corner solution with probability 1), we have:

$\dfrac{d\Pi_2}{dz} = \int_0^{\tau^*} -\dfrac{\partial\beta_2(\tau^*,\hat{\theta}_1)}{\partial\hat{\theta}_1}\dfrac{\partial\hat{\theta}_1}{\partial z}(N-1)F^{N-2}(y_2^1)f(y_2^1)dy_2^1$

$= (1-\alpha)\dfrac{\partial U(z,x_1^2,...x_1^N)}{\partial z}F^{N-1}\left(\tau^*\left(x_2^1,z,x_1^1,x_1^2,...x_1^N\right)\right)$

So $\dfrac{d\Pi_2}{dz} > 0$ and $E_{X_1^2,...,x_1^N,\Theta_1,X_2^1}(\dfrac{d\Pi_2}{dz}) > 0$. Let $E_{X_1^2,...,x_1^N,\Theta_1,X_2^1}(\dfrac{d\Pi_2}{dz}) = H(z,x_1^1)$. Then we have

$H(x_1^1,x_1^1) = E_{X_1^2,...,x_1^N,\Theta_1,X_2^1}(\dfrac{d\Pi_2}{dz})|_{z=x_1^1}$

$= (1-\alpha)E_{X_2^1}\{E_{X_1^2,\ldots,x_1^N}[\frac{\partial U(z,x_1^2,\ldots x_1^N)}{\partial z}F^{N-1}\left(\tau^*\left(x_2^1,z,x_1^1,x_1^2,\ldots x_1^N\right)\right)]\}|_{z=x_1^1}$

From the above derivation, we know that $\tau^*\left(x_2^1,z,x_1^1,x_1^2,\ldots x_1^N\right)|_{z=x_1^1} = x_2^1$, $H\left(x_1^1,x_1^1\right)$ can thus be further simplified as:

$H\left(x_1^1,x_1^1\right)$

$= (1-\alpha)E_{X_2^1}\{E_{X_1^2,\ldots,x_1^N}[\frac{\partial U(z,x_1^2,\ldots x_1^N)}{\partial z}F^{N-1}\left(x_2^1\right)]\}$

$= (1-\alpha)E_{X_1^2,\ldots,x_1^N}[\frac{\partial U(z,x_1^2,\ldots x_1^N)}{\partial z}E_{X_2^1}F^{N-1}\left(x_2^1\right)]$

$= \frac{1-\alpha}{N}E_{X_1^2,\ldots,x_1^N}[\frac{\partial U(x_1^1,x_1^2,\ldots x_1^N)}{\partial z}]$

$= \frac{1-\alpha}{N}[\frac{\partial E_{X_1^2,\ldots,x_1^N}[U(x_1^1,x_1^2,\ldots x_1^N)]}{\partial x_1^1}]$

In equilibrium, we need $z = x_1^1$, so the FOC of equation (7) becomes:

(8) $[v_1^1\left(x_1^1,x_1^1\right)-\beta_1\left(x_1^1\right)]\left(N-1\right)F^{N-2}\left(x_1^1\right)f\left(x_1^1\right)-\beta_1'\left(x_1^1\right)F^{N-1}\left(x_1^1\right)+\delta H\left(x_1^1,x_1^1\right)=0$

Equation (8) can be rearranged as

(9) $[(v_1^1\left(x_1^1,x_1^1\right)+\frac{\delta H\left(x_1^1,x_1^1\right)}{\left(N-1\right)F^{N-2}\left(x_1^1\right)f\left(x_1^1\right)})-\beta_1\left(x_1^1\right)]\left(N-1\right)f\left(x_1^1\right)-\beta_1'\left(x_1^1\right)F\left(x_1^1\right)=0$

Let $\hat{V}\left(y_1^1,y_1^1\right) = [v_1^1\left(y_1^1,y_1^1\right)+\frac{\delta H\left(y_1^1,y_1^1\right)}{\left(N-1\right)F^{N-2}\left(y_1^1\right)f\left(y_1^1\right)}]$. Then $\beta_1\left(x_1^1\right) = E[\hat{V}\left(Y_1^1,Y_1^1\right)|Y_1^1 < x_1^1]$.

Replacing $v_1^1\left(y_1^1,y_1^1\right)$ with $\alpha E\Theta_1+\left(1-\alpha\right)E[U(Y_1^1,X_1^{-1})|Y_1^1=y_1^1]$, we can obtain the desired expression. Q.E.D.

Proof of Proposition 3.

The monotonicity of $\frac{[E_{X_1^{-i}}U(X_1^i,X_1^{-i})]'}{[F^{N-1}\left(X_1^i\right)]'}$ ensures the monotonicity of the derived first-stage bid function. Now we will check whether the bid functions derived in Proposition 1 and 2 indeed constitute an equilibrium. We need to show that given all other bidders follow these monotonic bid functions, bidder 1 has no incentive to mimic a false type. Let us assume bidder 1 observe a type y while decide to mimic type z in the first stage and let her total expected payoff (the first-stage payoff plus the expected second-stage optimal continuation payoff) be Π. Then $\frac{d\Pi}{dz} = [v_1^1\left(y,z\right)-\beta_1\left(z\right)]\left(N-1\right)F^{N-2}\left(z\right)f\left(z\right)-\beta_1'\left(z\right)F^{N-1}\left(z\right)+\delta H\left(z,y\right)$ according to equation (7) in the proof of Proposition 2. The right-hand side of the above equation can be rearranged as $\left(N-1\right)F^{N-2}\left(z\right)f\left(z\right)[v_1^1\left(y,z\right)+\frac{\delta H\left(z,y\right)}{\left(N-1\right)F^{N-2}\left(z\right)f\left(z\right)}-\beta_1\left(z\right)]-\beta_1'\left(z\right)F^{N-1}\left(z\right)$. We next need to show that the term $v_1^1\left(y,z\right)+\frac{\delta H\left(z,y\right)}{\left(N-1\right)F^{N-2}\left(z\right)f\left(z\right)}$ is monotonic in y. Since both $v_1^1(y,z)$ and $H(z,y)$ are monotonic in y as we can see from the proof of Proposition 2, the whole term is monotonic in y too. Then using equation (8) in the proof of Proposition 2, it is easy to show that setting $z<y$ makes $\frac{d\Pi}{dz}>0$ and $z>y$ makes $\frac{d\Pi}{dz}<0$. So Π is maximized by choosing $z=y$. The off-equilibrium path belief also eliminates a bidder's incentive to bid above $\beta_1(\omega)$ in the first stage because she will be inferred as zero type, which always makes her rivals bid more aggressively at the second stage hence lowering the bidder's overall payoff. Given that bidders bid truthfully in the first stage, the second-stage problem is a standard common-value auction with public information, whose equilibrium check is standard hence omitted.

A comparison of each stage's bid function shows that for the same signal, the first-stage bid function is bigger than the second-stage one by the term
$\frac{\delta(1-\alpha)}{N}E_{Y_1^i}\{\frac{\delta[E_{X_1^{-i}}U(Y_1^i,X_1^{-i})]'}{[F^{N-1}(Y_1^i)]'}|Y_1^i<x_1^i\}$. Q.E.D.

Proof of Proposition 4.

We will first consider the finite N-stage case. Again, we let bidder 1 be our generic bidder and assume a symmetric monotonic equilibrium exists. Then in equilibrium, i.e., given truthful bidding, we must have $E\beta_{N-1}(.)>E\beta_N(.)$ (N is the last stage) from our proof for the two-stage problem. Now at period $t+1$, where $t\in\{1,2,...N-1\}$, if bidder 1 observes a signal x_{t+1}^1 and mimics a type z, given all other bidders follow the equilibrium strategy, she can obtain a best continuation payoff $c(x_{t+1}^1,z,\ p_t,\ x_t^{-1})$. Let $v_t^1(x_t^1,y_t^1,\theta_{t-1})=E[P_t|X_t^1=x_t^1,Y_t^1=y_t^1,\Theta_{t-1}=\theta_{t-1}]$, where $Y_t^1=\max\limits_{s\neq 1}X_t^s$ and y_t^1 is its realized value. Then bidder 1's stage payoff of period $t+1$ is:
$\Pi_{t+1}\left(z,x_{t+1}^1,\theta_t\right)=\int_0^z[v_{t+1}^1\left(x_{t+1}^1,y_{t+1}^1,\theta_t\right)-\beta_{t+1}(z,\theta_t)](N-1)F^{N-2}\left(y_{t+1}^1\right)f\left(y_{t+1}^1\right)dy_{t+1}^1$
At period t, if bidder 1 observes a signal x_t^1 and mimics a type γ, given all other bidders follow the equilibrium strategy, she will mimic a type z at period $t+1$ given signal x_{t+1}^1 and obtain a best continuation payoff $c(x_{t+1}^1,z,\ p_t,\ x_t^{-1})$.[1] Bidder 1's stage payoff of period t is:
$\Pi_t(\gamma,x_t^1,\theta_{t-1})=\int_0^\gamma[v_t^1(x_t^1,y_t^1,\theta_{t-1})-\beta_t(\gamma,\theta_{t-1})](N-1)F^{N-2}(y_t^1)f(y_t^1)dy_t^1$
Given bidder 1 mimics γ at period t, her rivals' inference of $\hat{\theta}_t$ will be $\frac{p_t}{\alpha}-(\frac{1}{\alpha}-1)U(\gamma,x_t^{-1})$

[1] Note: the continuation payoff from period $t+1$ remains the same no matter there is mimicking in period t or not because the inference of z at period $t+1$ will not be affected.

and her stage payoff of period $t+1$ will become:

$\Pi_{t+1}\left(z, x_{t+1}^1, \theta_t, \hat{\theta}_t\right) = \int_0^z [v_{t+1}^1\left(x_{t+1}^1, y_{t+1}^1, \theta_t\right) - \beta_{t+1}\left(z, \hat{\theta}_t\right)] (N-1) F^{N-2}\left(y_{t+1}^1\right) f\left(y_{t+1}^1\right) dy_{t+1}^1$

Then bidder 1's total payoff starting from period t on is:

$\int_0^\gamma [v_t^1 (x_t^1, y_t^1, \theta_{t-1}) - \beta_t (\gamma, \theta_{t-1})] (N-1) F^{N-2} (y_t^1) f (y_t^1) dy_t^1$

$+\delta E_{,X_{t+1}^1, X_t^{-1}, P_t} \int_0^z [v_{t+1}^1\left(x_{t+1}^1, y_{t+1}^1, \theta_t\right) - \beta_{t+1}\left(z, \hat{\theta}_t\right)] (N-1) F^{N-2}\left(y_{t+1}^1\right) f\left(y_{t+1}^1\right) dy_{t+1}^1$

$+\delta^2 E_{X_{t+1}^1, X_t^{-1}, P_t} c(x_{t+1}^1, z, p_t, x_t^{-1})$

Then the FOC w.r.t. γ leads to:

(10) $[v_t^1 (x_t^1, \gamma, \theta_{t-1}) - \beta_t (\gamma, \theta_{t-1})] (N-1) F^{N-2} (\gamma) f (\gamma) - \beta_t' (\gamma, \theta_{t-1}) F^{N-1} (\gamma)$

$-\delta E_{X_{t+1}^1, X_t^{-1}, P_t} [\frac{\partial \beta_{t+1}(z, \hat{\theta}_t)}{\partial \hat{\theta}_t} \frac{\partial \hat{\theta}_t}{\partial \gamma} (N-1) F^{N-2}\left(y_{t+1}^1\right) f\left(y_{t+1}^1\right) dy_{t+1}^1]|_{z^*} = 0$

Equation (10) is a necessary condition for any equilibrium. Setting $\gamma = x_t^1$, then the optimal z^* must equal x_{t+1}^1 as in the proof of Proposition 2. So equation (10) can be transformed to:

(11) $[v_t^1 (x_t^1, x_t^1, \theta_{t-1}) - \beta_t (x_t^1, \theta_{t-1})] (N-1) F^{N-2} (x_t^1) f (x_t^1) - \beta_t' (x_t^1, \theta_{t-1}) F^{N-1} (x_t^1)$

$+\delta E_{X_{t+1}^1, X_t^{-1}, \Theta_t} [\frac{\partial \beta_{t+1}(z, \hat{\theta}_t)}{\partial \hat{\theta}_t} (\frac{1}{\alpha} - 1) \frac{\partial U(x_t^1, x_t^{-1})}{\partial x_t^1} F^{N-1}\left(x_{t+1}^1\right)] = 0$

In the finite N-stage case, $\beta_N(x_N^1, \theta_{N-1})$ can be obtained in the same way as in the proof of Proposition 1, where θ_{N-1} is linearly separable with coefficient α from all other parts of the expression of the bid function. So $\frac{\partial \beta_N(x_N^1, \hat{\theta}_{N-1})}{\partial \hat{\theta}_{N-1}}$ equals the constant α. Similarly, from equation (11) we can show that $\frac{\partial \beta_t(x_t^1, \hat{\theta}_{t-1})}{\partial \hat{\theta}_{t-1}} = \alpha$ for all $t < N$ with backward induction. This leads to the conclusion that $E\ \beta_t(.)$ will be the same for all period t except the last one since the last stage does not have a continuation payoff anymore.

We then consider the infinite-stage case. We assume there exists a stationary symmetric monotonic equilibrium, then the payoff from period t on under valuation x_t^1 with mimicking type γ equals:

$\int_0^\gamma [v_t^1 (x_t^1, y_t^1, \theta_{t-1}) - \beta_t (\gamma, \theta_{t-1})] (N-1) F^{N-2} (y_t^1) f (y_t^1) dy_t^1 + \delta E_{X_{t+1}^1, X_t^{-1}, P_t} c(x_{t+1}^1, \gamma, p_t, x_t^{-1})$.

Since the equilibrium is assumed to be stationary, the functional form of the expected continuation payoff $E_{X_{t+1}^1, X_t^{-1}, P_t} c(x_{t+1}^1, z, p_t, x_t^{-1})$ will be the same for any period t. Then standard FOC w.r.t. γ plus setting γ to x_t^1 method gives us a constant price path. Q.E.D.

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