INTERACTIVE TUTORIALS FOR UPPER LEVEL QUANTUM MECHANICS COURSES

by

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This thesis explores the ongoing need for intera ctive tutorials in the upper level undergraduate Quantum Mechanics co urse. It firs t summarizes the development and eval uation of tutorials at the introductory physics level by ot hers, and then challenges the be lief that upper level students do not need this type of intervention by citing research in stud ent dif ficulties in learn ing Quantum Mechanics. Physics Education res earch sho ws that there are common student misconceptions that persist even in the upper level undergraduate cour ses such as Quantum Mechanics. Cognitive research serv es as a guide for effective curriculu m design. A descriptio n of the iter ative p rocess f or developing and evalua ting the tuto rials is d iscussed. Th e development and evaluation of "The Ti me Evolution of a Wave Function" Quantum Interactive Learning Tutorial (QuIL T) is described in detail . Finally, the success of the QuILT in reducing the common misconceptions about time evolution is discussed.

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PREFACE

Thank you to Dr. Singh for encouraging m e to finish this degree. I began with great reluctance and the belief that writing a thesis about quantum m echanics could in no way benefit me professionally in m y current te aching position. I was wrong, as th e challenges of teaching ar e universal and independent of the clientele. Ev ery year I put a quote up in m y classroom that I refer to as my prof essional goal for the year – this year it will b e "counter the passive learning environment of a typi cal course, and secure a m ental comm itment from the student." (McDermott, 1991)

Thank you to my father who agreed to watch my three lovely children under the age of 5 while I wrote this thesis. Thank you to m y mom, who coordinated everyone 's schedule in true sergeant fashion, and also helped with the kids. Thank you to my husband for all the laundry and dishes support, and for the late night talks a bout education issues. Thank you to Aunt Cindy for the technical support.

And finally thank you to m y comm ittee m embers and all m y professors at Pitt who supported me throughout the years.

I dedicate this thesis to m y children; m ay I show by exa mple that what you start, you should finish.

1.0   INTRODUCTION

Our society increasingly relies on technology and our education system is feeling the pressure to produce a highly educated workforce. At the same time, education research shows that many of our traditional methods of teaching are ineffective. One of the main stumbling blocks to reforming our education system is the belief held by students, parents, instructors and administrators that the instructor should do most of the work for the student. Students at all levels of our education system often believe the instructor's primary role is to give information and answer questions in the form of lucid lectures. However, all the educational research states otherwise. The tutorials we are developing for undergraduate Quantum Mechanics courses change the instructor's role from a lecturer to a coach and the student's role from a passive learner to an active learner. In this thesis, I argue that to effect positive change in our educational system, we must employ research-based tools like the interactive Quantum tutorials.

Consider the following scenario: A professor lectures clearly and logically on new concepts, and works example problems which apply the concepts to a few situations. The students review lecture notes and complete homework. The professor writes an exam which requires the students to apply the concepts to new situations. The students come to the exam prepared and confident but leave wondering why the material on the evaluation seems so unfamiliar. The professor wonders why the students cannot think independently and apply the physics principles they learned in the lecture to new situations. This scenario is common

throughout our educational system . Some students, who are like professors and have learned to interpret an d apply con cepts in ne w situations and organize their knowle dge hierarchically o n their own, prevail while others who need help in   developing these skills struggle. Education research sh ows us that instruction can play a   critical role in helping students develop higher order thinking skills and can help them organize and extend their knowledge structure.

The goals of the course, instructional desi gn, and m ethod of evalua tion must be aligned with each other for learning to be meaningful. The desired outcomes can be broadly grouped into two areas which com pete for lim ited time during a sem ester: *Science as a Body of Knowledge* vs. *Science as a Process*. If the priority is to recall   definitions and reproduce proofs then instruction and evaluation should focu s on those. Likewise, if the go al is to help students learn to think like a physicist and in terpret and apply acquired knowledge flexibly in new situations, then instructional design and method of evaluation should reflect that. Unfortunately, traditional physics instruction in most in   troductory and advanced phys ics courses does not em   phasize *Science as a Process* and m ostly supports *Science as a Body of Knowledge.* The professor assumes that becau se th e studen ts h ave pass ively lis tened to the lectu res, they will be ab le to make a grea t cognitive leap and figure out how to   apply the concepts learned to new situations. Research has shown, however, that good teaching by an instructor in the form of a lecture alone is not enough for m eaningful le arning. S tudents need to be ac tively engaged in the learning process. They must be provided opportunities to practice the skills they must learn and they must be given feedback and scaffolding support as ne eded. Furtherm ore, this support m ust gradually decrease as the students develop self-reliance.

The goal of this thes is is to d iscuss the role of research based, inte ractive tutorials as an excellent tool for allowing student s to m ove from a passive to an active role. The tu torials help

students learn to reason system atically and bu ild a robust knowledge structure. They help students view *Science as a Process* rather than *Science as a Body of Knowledge.* The lecture and tutorial activities can be combined as par t of the instruction. During tu torials, students work in small groups and practice applying the co ncepts to new and unique situations *with support from their peers and instructor.* Common misconceptions and difficulties are explicitly brought out in the tutorials and students are provided gui dance and support to organize their knowledge hierarchically where there is less room for misconceptions. After discussing the role of research-based tutorials as an effective learning tool at all levels of physics instruction, I discuss the development of a tutorial for teach ing tim e-development of wave function in u ndergraduate Quantum Mechanics.

2.0   OVERVIEW OF EDUCATION RESEARCH

Educational Research shows that the traditiona l approach to teaching physics or "teaching by telling" does not work. In this section, we will cite educational re search to assert that no m atter how logical and engaging the lect ures are in the tradition al ap proach, exclus ive use of this method will not allow students to gain m eaningful knowledge. Novel teaching and learning methods such as in teractive tu torials which co mbine the tradition al and constructivist appro ach are needed to keep stud ents actively engaged in the learning process and help them construct their own understandings. One major advanta ge of the tu torial appro ach is that it does not require a m ajor change in th e instructor's teaching style and hence can be adopted in a widespread m anner. But before we discuss the tutorial appro ach in introductory physics and quantum mechanics, we will f irst re view the tr aditional vs. construc tivist approach a nd review some relevant prior research.

2.1   TRADITIONAL VERSUS CONSTRUCTIVIST APPROACH

In the Traditional teach ing approach , lecture is the prevalen t m ode of instruction. First, the formalism is discussed by the instructor. A good instructor may then illustrate how the formalism applies using straightforward examples. Later, more complex examples may be used to illustrate how different concepts fit together. Due to year s of experience and expertise teaching the topic,

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the instructor may even point out misconceptions and errors students have made in the past. The instructor may believe that this knowledge will prevent students from making similar mistakes. In the next class, the instructor may take students' questions on the homework assignments, and the class moves on to the next topic. This approach demands only deductive reasoning from the student; use of inductive and abstract thought processes, and making generalizations are not emphasized. Students tend to focus on finding an algorithm that works for problems of similar types, rather than focusing on the reasons for the steps in the algorithm (McDermott, 1990).

This traditional approach is acceptable to most physics majors and other students taking physics courses including non-majors (Tobias, 1990) because this is what they are used to. Moreover, the constructivist approach requires significantly more thinking on the students' part and humans try to minimize the cognitive load unless the topic is intrinsically interesting. Unfortunately, just like it is impossible to learn how to play the piano well by mostly listening to other people play, it is impossible for students to develop a solid grasp of physics concepts without having the opportunity to practice the skills with sufficient guidance and support from an instructor. Many successfully complete the course, but comment afterward that it wasn't interesting or relevant and they hardly remember anything. When students come out of their physics courses with the belief that physics is a collection of facts and formulas and is irrelevant for everyday life, the impact is felt through the whole educational system because many of these students will become future K-12 educators. Of course, the beauty of physics as a field of science is its ability to explain diverse phenomena in everyday experience in terms of a few basic principles. Unfortunately, traditional instruction does not provide adequate opportunity to students to organize their physics knowledge in such a way that they can see the coherence of physics and be able to apply the principles flexibly in different situations.

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In the c onstructivist approach, the in structor starts by presen ting a s ituation, and posing questions. The students observe, discuss, experiment, and ask for guidance. This bottom up instruction allows students to construct their ow n knowledge on the topic with guidance and support from the instructor based upon the studen ts' prior knowledge. Form alism is introduced after exploration and generalization occur. This approach focuses on the students and keeps them active in the learning process. Man y studies have shown that this appro ach is more effective for *all populations* of students (McDermott, 1990).

The tuto rial approa ch is a c ombination of the tra ditional and cons tructivist appr oach. Research shows that exclusiv ely traditional methods do not work but requiring students to discover all the m aterial in an introductory co urse on their own in the constructivist fashion would turn a two sem ester course into a four year nightmare! The question remains: within the constraints of the current post secondary education system , how do we take the best of both methods and com bine them into som ething that wo rks for the students? On e tool that has been developed by education research is the interac tive tutorial. It is a m iddle ground between the traditional a nd construc tivist appro aches allowing f or some top down instru ction and som e bottom up discovery.

2.2   A GROWING BODY OF RESEARCH INFORMATION

The interactive tutorial is an excellent teaching and learning tool, but it must be research based for it to work effectively. In the developm ent of any edu cational material, research from different fields is em ployed. These fields can be broadly classified in three categories: 1. Cognitive R esearch (C R), which s heds light on how people learn and solve p roblems. 2.

Education Research (ER), which is conducted by re searchers in the schools of education; it does not focus so m uch on the content but on the evaluati on of the instructional strategies in genera l. The focus is often on the learning and teaching t ool. 3. Physics Edu cation Rese arch (PER), which focuses (am ong other things) on the student 's understanding of specific topics in depth and on devising and evaluating strategies to si gnificantly reduce difficulties related to those topics. The following sections su mmarize the findings fro m the three fields that are re levant for the development and evaluation of the tutorials. Although presented here as three separate fields, they are intertwined; one cannot work in any one field without drawing upon the others.

2.2.1 Contributions from Cognitive Psychology

Relevant findings from cognitive research are labeled with di fferent letters for ease in referring to them later when we discuss the process of developing an interactive tutorial.

A. Piaget's stages of co gnitive development distinguish students who are at th e concrete operational stage from those who are at the formal operational stage (Ginsberg and Opper, 1969). Students at the concrete opera tional level can conserve numb er, length, liquid, m ass, weight, area, and v olume when the object is presen t and can be m anipulated. They can com bine, separate, and transf orm the quantities. W hen students m ove to the formal level, they can use symbols and m anipulate variable s using abstract thought without the presence of concrete objects. At this level they becom e interested in the actual process of thinking. A ccording to Piaget, students begin to operate at the form al level between 11 and 15 years. However, m ore recent research (Cole, 2000) shows that only 30 to 40 percent of students in their teen s and early twenties can solve prob lems at the form al operational level. The age at which transition occurs between the two levels is a bell shaped curve w ith a large standard devi ation (Siegler, 1996).

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The fact that many students have not made a transition from the concrete to the formal level even in college has im plications for the developm ent of effective teaching and learning tools such as tutorials.   Moreover, P iaget proposed that students should be provided "optim al mismatch" for cognitive g rowth with guidance and suppor   t for accommodation and assim   ilation of ne   w concepts. The interactive tuto   rials strive to provide "op   timal m ismatch" based upon prior investigation of student difficulties.

B. W hile Piaget's con tributions to cognitive s cience were f ocused on the individ ual, Vygotsky was a social educationa l psychologist and focused on the effect of social environm ent on learning. Vygotsky's notion of Zone of Proxim al Development (ZPD) is defined as "the gap between what a student can achieve individually versus what s/he can achieve with the help of an instructor who is fa miliar with student' s prior knowledge" (Vygotsky, 1978). When the gap between the prior knowledge of the student and th e instruction is large, no meaningful learning occurs. Ho wever, if the instructor builds on the exis ting knowledge of stude nts and takes sm all steps to stretch student's learni ng towards the desired goals of th e course, the student can m ake great gains in knowledge. This process is often   referred to as scaffolding, and is an integral component of the interactive tutorials.

C. Miller's groundbreaking work in the area of short term and long term memory has far reaching implications o n development of instruc tional too ls (Miller, 1 956). Miller's research showed that only five to nine "b its" of information can be stored and processed in the short term memory (STM). For e xample, a nine digit phone num ber would be c onsidered nine "bits" of information. Later research (Chase, 1973; Neves, 1981) shows that an expert in a field can store a m uch larger am ount of infor mation in a "b it" of STM by "chunking" different pieces of information together. A nine digit phone numb   er can be chunked into an area code, a local

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exchange, and four digits. Nine bits of information are reduced to six. Similarly, physics experts develop deep connections between different concepts in mechanics such as displacement, velocity, acceleration, force, momentum, etc. Due to this "chunking" of information by a physics expert, a large body of knowledge can be recalled together and takes up just one "bit" in STM. For the novice physics learner, the connections are not yet made, and the six mechanics concepts listed above each require their own "bit" in STM. This implies that the student can have cognitive overload while going through a long chain of reasoning requiring more cognitive resources than is available if different instructional units do not build on each other and give students an opportunity to chunk knowledge gradually. Interactive tutorials use tools such as flow charts and concept maps to give students the opportunity to practice chunking information to reduce cognitive load and to promote robust knowledge structure.

    D. Research on the effects of practice (Anderson, 1999) should make any teacher smile. It's what we've known intuitively all along – doing your homework helps! These results show that more practice yields faster and more accurate recall, and provides for better long term retention after the practice is stopped. Of course, the student must be practicing useful and appropriate skills – if the goal is to apply acceleration concepts in different contexts, reciting the definition of acceleration is not helpful. Just plugging numbers in equations is not helpful if the student does not take the time to think about the physics. Moreover, some approaches to practice prove to be more effective. Immediate recall was slightly better for a group that studied eight times in a day, compared to a group that studied twice a day for 4 days. However, long term retention was markedly higher for the latter group. Cramming the day before a test may give a slight edge on test day, but the recall drops quickly, and the information reviewed for the test is gone (Anderson, 1999). Gone with it is the opportunity to repair, extend and organize the LTM

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and make deep, rich connections. The spiral back method, where students apply a concept in several different contexts, and student elaboration, where students are given an opportunity to explain their reasoning, both prove to be effective ways of providing alternative pathways for recall (Anderson, 1999). Well designed interactive tutorials assure that the student is getting appropriate practice as opposed to mindless busywork.

E. We want our students to come to the table with ideas and observations about the world around them. But our work would be much easier if they came without some of their preconceptions. Whether a help or a hindrance, we must consider that "Students are not blank slates" (Shauble, 1990) and we must work to remove the wrong ideas and replace them with correct ones. It should be no surprise that students familiar with a world filled with friction think that there must be a constant force on an object to produce constant velocity. They have made a valid observation that in order to move an object across a table, they have to apply a force. Then they make an invalid conclusion that Newton's first law just does not apply in the presence of friction. The student's knowledge structure must be robust so that there is little room for misconceptions. Research has shown that simply showing students why the new ideas are applicable in a situation is not sufficient; students must learn why the old ideas do not apply (Posner, 1982). Interactive learning tutorials allow students opportunities to examine their misconceptions and then provide scaffolding support to build a robust knowledge structure.

One may wonder why students have a difficult time with physics courses in light of A through E above. In any given class, there may be students who may not have reached Piaget's formal level of thinking, students for whom the gap between what they know and the level at which instruction is targeted is insurmountable, students with cognitive overload because STM does not have resources available to process information, students who don't know how to study

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and practice effectively, and students with pr econceptions about the world around them which hinder learning. The ba rriers to learning are la rge but surmountable, so good instructional tools such as research-based tutorials are needed to help students.

2.2.2 Contributions from Education Research and Physics Education Research

A m ajor go al of Physics Education Research (PER) is to investigate th e kinds of difficulties students have in learning physics and to develop and evaluate strategies to help students. Written tests are adm inistered to a la rge population of students and in depth interviews are conducted with a subset of students. Infor mation is gath ered before, during, and after instruction, and can be used for two purposes: to assess the effectiveness of an instructional strategy and to guide the development or revision of a curriculum . Aft er the inte rvention is r evised, the process star ts again. This continuous, interactive, iterative process has produced a wealth of information about how students learn physics effectively.

For the sake of this discussion, *meaningful learning* is defined as the "ability to interpret and apply knowledge in sim ilar situations whic h a re no t identical to those in w hich it was initially acquired." After form al, traditional instruction, it is very common for students to have misconceptions about physics which are only detect ed when we check for m eaningful learning. It has also been shown that the ability to solve a complex quantitative problem does not equate to a good con ceptual und erstanding because stu dents can often m emorize algo rithm without understanding (McDermott, 1992) . In the physics field, *meaningful learning* occu rs when th e knowledge of physics concepts, th e skills to repr esent the concepts, an d the ability to reas on using the concepts is developed simultaneously (McDermott, 1991). Research shows that the concepts should be developed in stages, and a spiral back approach going back to the old

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concepts is effective. E ach topic should be applied to m ore than one situation, so that students don't com partmentalize. The pr actice of sim ply pointing out   common m isconceptions and reasoning errors is not e ffective: although the error m ay resolve itself in that context or for that problem, it simply resurfaces later in other contexts (Posner, 1982).

The process of inquiry has repeatedly been s hown to be superior to instructional methods that present inform ation as an inert body of   knowledge. (McDerm ott, 1991) It lends itself to more peer interaction, which has m any proven advantages: the articulation of an idea to a peer requires logic and organization. It forces the students to refine th eir thoughts. It allows students to rem ember concep ts by recalling   actual con versations. Students ch   allenge the logic of arguments and work harder to jus tify the ir own logic when they in teract with p eers rathe r than teachers. They are m ore inclin ed to draw a   picture or write an equation during their efforts (Rogoff, 1998). Research shows that peers ca   n "co-construct" knowledge, m eaning that two students working together could   get a correct answer, even   when neith er one h ad the co rrect answer working independently (Singh, 2005). The intera ctive tutorials are designed with these pedagogical issues in mind.

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3.0   THE PROCESS OF DEVELOPING AN INTERACTIVE TUTORIAL

In this ch apter, th e proc ess of developing a tuto rial will be discussed. The three p art pro cess consists of conducting investigations of st udent understanding, deve loping instructional strategies which directly addr ess s tudent d ifficulties and m isconceptions, and iterating the process of designing, testing, modifying and revising the tutorials.

3.1   IDENTIFYING COMMON STUDENT DIFFICULTIES

Several strategies are used to identify and characterize the types of difficulties student encounter. Often at the beginning of a topic, individua l students a re inte rviewed to determ ine their understanding of a phenomenon. Discussions are a udio taped for furt her analysis. Students are also monitored as they p articipate in labs a nd lecture discussions, and hom ework and exams are evaluated. Common difficulties ar e found for this g roup o f students. Several p rofessors and graduate students work togethe r to f ormulate questions whic h explore th e common dif ficulties. These questions are giv en to a larger group of students to determ ine if the common difficulties can be attributed to a local environment (such as professor's teachi ng style, textbook, or university) or if they re present a more perva sive problem. These common dif ficulties a re the basis for tutorial development.

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Research in introductory physics h as shown th at comm on difficulties exist for stu dents enrolled in calculus based, al gebra based, and non-sc ience m ajor courses. These common difficulties persis t after for mal tradition al ins truction for all th ree g roups, as th ey do with students enrolled in overseas un iversities (McDerm ott, 1992). A ll of this im plies that the difficulties are unive rsal. The skill sets and levels of preparation tha t students bring to th e introductory courses vary widely, an d we cannot assume that the studen ts will ov ercome these obstacles on their own. If we, as educators, want students to acquire *meaningful learning* in the field of physics, we must make our teaching student centered.

3.2 POSSIBLE REASONS FOR COMMON STUDENT DIFFICULTIES

Once common difficulties related to a topic have b een identified, a tutorial is developed in an attempt to help students build robust knowledge structure where there is less room for such difficulties. The tu torial m ust also address sp ecific m isconceptions, an d take in to account the issues discussed in section 2.2.

Common difficulties related to a particular topic arise for s everal reasons. When a robust knowledge structure related to a topic is lacking, st udents try to over gene ralize their knowledge resources th at a re appr opriate in s ome situa tions to cases where th ey are not applicable. Cognitive re search show s that the w ay people o ver gene ralize the ir kno wledge reso urces in a particular context based upon their experiences is very sim ilar and leads to comm on misconceptions and difficulties. Mo reover, students who are learning a topic for th e first tim e may lack a conceptual model especially because they may not have hands on experience with the phenomenon. Students m ay also lack experience or practice reasoning about the concept. In

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order to ov ercome the difficulties , they s hould be given opportunities to exp erience the phenomenon if possible, and be put in a situation where they can practice reasoning about it. It is the job of the curr iculum developer to cr eate these situations, and the job of the instructor to facilitate them. It is NOT the job of the instructor to answer every quest ion students ask in grea t detail. As a m atter of fact, doi ng so can in terfere with ab ility of students to cons truct their own knowledge. Providing appropriate scaffolding a nd guiding students with appropriate hints has been found to be more effective in helping students build a robust knowledge structure.

An interactive tutorial **based on research** should do the following things as appropriate:

- Start with a misconception which was identified through research.

- Provide m ultiple contexts for the st udent to experien ce the phenom enon and practice reasoning about the phenomenon.

- Provide scaffolding which helps student s start with their prior knowledge and gives them opportunity to build on their prior knowledge.

- Promote chunking of concepts by explori ng the interconnections between sim ilar ideas.

- Provide opportunities to replace preconceptions with a robust knowledge structure so that students understand why the preconceptions are wrong.

- Provide opportunities to interact with peers and co-construct knowledge.

The QuILT shown in the appendix , titled "The Tim e Evolution of a W ave Function" strives to accom plish these goals. A descripti on of its developm ent and an analysis of its effectiveness follow in chapter 5.

More than any other field of physics, Quan tum Mechanics fits the description of "no direct hands on experience" with m icroscopic wo rld and "no experience or practice reasoning

about it" outside of the class. It is a field   that would benefit greatly   from m ore interactive

learning tools. Chapter 4 discusses this need.

4.0 RATIONALE FOR INTERACTIVE TUTORIALS AT THE ADVANCED UNDERGRADUATE LEVEL

Quantum Mechanics is abstract. Like Physical Chemistry for chemistry majors, and Organic Chemistry for nursing majors, it is commonly referred to as the "weeding out" class. It is the course that causes many students to get off the PhD track, and head for degrees in related fields. Richard Feynman himself said "I think I can safely say that nobody understands quantum mechanics" (Feynman, 1945).

Commonly used textbooks include warnings for students of the difficulties associated with the course (Griffiths, 1995). "There is no general consensus as to what its fundamental principals are, how it should be taught, or what it really *means*." Griffiths goes on to say, "Not only is quantum theory conceptually rich, it is also technically difficult, and exact solutions … are few and far between." Midway through his preface, he invites students who want to know what quantum mechanics *means* instead of just what it *does*, to read chapter one and then skip to the afterward to get an idea of how mysterious it is. Now, like any good student, I made a mental note to do just that. But by the time I was introduced to the Schrödinger Equation, the three different statistical interpretations, and normalization of a wave function, all my bits of STM were full, and my good intention of reading the afterward to learn what quantum mechanics really *means* was forgotten. Besides, I had a long homework assignment in each of my classes and didn't have time for extracurricular pursuits – I had to get good at solving the Schrödinger

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equation for different potentials and grade my papers for my TA assignment. There was no time to think about what it really means – a pitfall into which many of our students fall.

In my textbook (Griffiths), the following quote is highlighted with a frowny face next to it, and the word "ugh" written in the margin: "…quantum mechanics is not, in my view, something that flows smoothly and naturally from earlier theories. On the contrary it represents an abrupt and revolutionary departure from classical ideas, calling forth a wholly new and radically counterintuitive way of thinking about the world. That, indeed, is what makes it such a fascinating subject." The counterintuitive nature of the course also makes it the perfect upper-level subject for the development of interactive learning tools to help students.

4.1   THE EXPERT'S VIEW OF QUANTUM MECHANICS

A professional physicist may make the case that doing Quantum Mechanics is actually easier than some other topics in upper-level undergraduate physics. Compared to Electricity and Magnetism (E&M), quantum mechanics may be easier because it deals with one field, and that field is a scalar field rather than a vector field. In Classical Mechanics, one solves Newton's Second Law for a given potential, applies the boundary conditions, and writes the solution. Quantum mechanics differs in the fact that one solves the Schrödinger Equation rather than Newton's Second Law. Many professional physicists feel that, with a solid background in differential equations, linear algebra, and probability and statistics, doing Quantum Mechanics can become routine. But there is a disparity between how difficult students perceive the material and how difficult the instructors believe it is to learn. Again, this makes it an excellent subject

area f or de veloping tu torials whic h will br ing th e ins tructor and the student togethe r in Vygotsky's Zone of Proximal Development.

4.2   SOURCES OF COMMON DIFFICULTIES IN QUANTUM MECHANICS

As m entioned in sectio n 3.2, there are s everal reasons for common difficulties related to a particular topic. The lack of a conceptual m odel due to a lack of hands on experience with the phenomenon and a lack of experience or practic e reasoning about the concepts in everyday experience m akes learning quantu m m echanics ev en m ore challenging. In general, students starting an introductory course have seen rainbo ws, used electricity, cooked on a stove, driven a car, and played contact sports.   These everyday experiences give them a context for learning electricity, heat, and kinematics.   There are no corresponding everyday experiences to familiarize students with quantum phenom enon. For t he introductory course, additional hands on experience can be easily created in the instru ctional setting with demonstrations.   For the quantum course, computer simulations are the main resources to help students visualize quantum mechanical concepts.

For example, consider th e question o f the m inimum safe spe ed for a train at the top of a roller coas ter loop. W hen you ask a group of st udents what they think this speed will depend upon, the mass of the object is always one of t he responses. After the traditional treatm ent of applying Newton's Laws and showing that m ass cancels on both sides of the equation, m any students will be convinced that m ass does not m atter in calculating the v elocity and they can use the prin ciples successfully to so lve quantitative problem s. For the students who don't quit e believe the algebra, and still feel in tuitively that mass should m atter, the instructor has the v ery

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effective op tion of asking the stud ent "Have y ou been on a coaster with som eone heavier or lighter than you? Did the m ass m atter?" The in structor can guide students and focus their attention back on the equation, and students m ay make the c onnection. The opportunity to practice reasoning about the concept, and to dr aw on their experiences is a very powerful teaching tool (Laws, 1997). In Quan tum Mechanics, the concrete stud ent experience, analogous to the roller coaster ride for introductory physics st udents, does not exist. The running joke with my classmates when I was taking quantum mechanics was, "What *is* the particle in the box? Can we have an in-class demonstration to visualize it?" An interactive research-based tutorial is not a demonstration but gives students an opportunity to build on what they know in a system atic manner and addresses the common misconceptions explicitly.

5.0   APPLYING RESEARCH FINDINGS TO
QUANTUM MECHANICS INSTRUCTION

Section 2.2.1 summarized major findings of cognitive research. In this section I will discuss how four of these five findings impact the learning of Quantum Mechanics.

A. As mentioned in section 2.2.1.A, as few as 30 to 40 % of thirteen to twenty-four year olds are able to solv e p roblems at the form al op erational le vel. This is a m ajor roadblock to teaching the introdu ctory course. My years o   f experience teach ing from m iddle school to undergraduate level lead m e to believe that ed   ucators waste a great deal of tim   e and m oney attempting to teach ab stract concepts to studen ts who are still at the conc rete level. One would think that any student who has successfully m ade it through an intermediate level undergraduate physics curriculum to the upper level course such as quantum mechanics *must* be operating at the abstract lev el. W hile most studen ts in the   quantum m echanics cou rse can do m ore abs tract reasoning than a typical introduc tory physics student, the level   of abstractness in a quantum mechanics course increases significantly com   pared to introductory c   ourses and presents a challenge for many students.

B. Finding of cognitive research in section   2.2.1.B discussed the no tion of the Zone of Proximal Developm ent. One of the m ost difficult cha llenges f or an instruc tor is to determ ine what the students know at the beginning of th   e course, and build on that knowledge. For the introductory course, there are two possibilities: either students had a high school physics course,

or they didn't. Certainly the differences in the skills tha t the students bring to the course are large in introductory courses, bu t research shows that spread in students ' preparedness for advanced courses remains large as the students move through the curriculum. Remembering my first Quantum course of about 15 students, half were undergraduates ta king the course for the first time. Of the graduate students, about half were foreign students w ho had already earned a masters degree in their hom e county – i.e. they had already com pleted a com parable course. Some had already taken Modern Physics, som e were taking it con currently, and two had never had Modern Physics. The m athematical b ackground of the students varied from no Mathematical Methods course to a few students who had completed two semesters. Six of the students were taking linear algebr a during the sam e se mester, a nd one was taking a statistics course. Any professor who faces such a diverse level of student p reparation can particula rly benefit from tools such as tutorials that help students start with what they know, and build on that knowledge.

C. Even the upper-level students in quantum mechanics courses can have issues with cognitive overload. There must be some cognitive discomfort during the learning process just as there must be som e muscular di scomfort during a weight lifting workout. However, if it hurts too much, you are impeding your progress. Similarly, cognitive overload can impede learning.

In an abstract cours e s uch a quantum m echanics, the organizing of infor mation into chunks and storing in LTM may be more challenging than introductory physics. As an example, compare the concept of "velocity" in introductory physics and the concept of "expectation value of momentum" in quantum mechanics. For introductory students , the concept of velocity m ay require different bits of STM for displacem ent, time, ratio, and appropriate units. Once this is sorted out in STM, connection s can be m ade to acceleration and dis placement. All of these

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concepts account for on ly six "b its" of STM, a nd students may still have th e capacity to reason about these topics.

Velocity is a funda mental idea at the introduc tory level. Advanced students in quantum mechanics may come to the class believing that velocity (and eventually momentum) should be a fundamental idea at the quantum level and it should be an attribute independent of the position of the particle. However, what they d iscover is that in quantum mechanics position and momentum (or velocity) are operators rather   than being determ inistic vari ables as in cl assical physics. Students learn to find the expectation value of momentum in a given state since the m omentum of a particle in a give   n state is n ot well-d efined and is probabilistic. To truly follow the treatment of this concep t of expectation value of m omentum, students must have room in STM for m omentum, wave function, probability densit   y, expectation value,   integration by parts, measurement and collapse of   wave function, and average of   measurements of m omentum on large number of identically prepared systems. Not only do these con cepts take up several "bits" of STM, these concepts are m   uch more com plicated th an the conc ept of velocity in c lassical physics. Even for advanced students it is di   fficult to chunk these ideas together without guidance. To add to the confusion, at the end of the treatment, students learn that the expectation value of velocity (or mom entum) i s not the velo city (or mom entum) of the partic le. Grif fith's notes in his quantum textbook: "Not hing we have seen so far woul d enable us to calculate the velocity of a particle – it's not even clear what velocity *means* in quantum mechanics" (Griffith, 1995). W hen asked what expectation value of mo mentum means, most st udents can reply, "it's what you get when you sandwich the momentum operator between Ψ^* and Ψ, and integrate over all space."   Their ability to reci te this fact, however, does   not m ean they fully understand expectation value. (Singh, 2008) For exam ple, many students had difficulty understanding that

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the expectation value of m omentum is the average of large num ber of measurements performed on identically prepared system.

  D. Effective practice is cri tical for meaningful learning and is the corner stone of a good course. Often during the process of developing tuto rials, it occurred to m e that perhaps all th at needs to be done is to slightly modify the exercises that already exist in popular textbooks. In an effort to evaluate th e extent to which exercises given to beginning Quantum Mechanics students promote *effective learning,* I randomly picked problem 1.7 on pg 13 of Griffith's text to analyze in terms of Anderson's cognitive research findings. Here is the problem:

At t = 0, a particle is represented by the wave function

$$\Psi = \frac{Ax}{a}, if\, 0 \le x \le a$$

$$\Psi = \frac{A(b-x)}{(b-a)}, if a \le x \le b$$

$$\Psi = 0, otherwise$$

Where A, a, and b are constants.

  a. *Normalize Ψ, that is, find A in terms of a and b.*

  b. *Sketch Ψ(x,0) as a function of x.*

  c. *Where is the particle most likely to be found, at t=0?*

  d. *What is the probability of finding the particle to the left of a? Check your result in the limiting case of b = a and b = 2a.*

  e. *What is the expectation value of x?*

  I set out to analyze the problem   in term s of Anderson's c riteria for effective p ractice. Long, quantitative exercises do not promote meaningful learning if one does not reflect upon the problem solving proces s. Qualitativ ely rich pro blems or quantitative problem s coupled with qualitative or conceptual prob   lems are m ore likely to prom   ote learning. For m   eaningful

learning, the need to develop mathematical skills must be balanced with the need to develop conceptual understanding. On this problem, I spent 2.5 hours working independently, with a peer, and independently again. I took advantage of my peer's strengths, and I checked to make sure my answer made sense to me in parts a, c, and e even though the problem did not instruct me to do so.

Because I took these actions, the exercise became both mathematically rigorous and qualitatively rich. However, my mindset while working this problem was most likely different from typical students actually enrolled in a course. Eighteen years of teaching has shown me that even the most conscientious student often does not take the extra step to make an existing quantitative exercise qualitatively rich. Students get the assignment done so that it can be turned in on time. This exercise has the potential to promote meaningful learning if one takes the time to reflect upon why certain physics principles were employed for the problem and how one can recognize in the future that another problem with different context should be solved using similar method However, most students are likely to miss the opportunity. The qualitatively interesting part of the problem chosen is the comparison between the answer for parts C and E. One common difficulty found at the quantum level is a student's inability to distinguish between individual measurement and expectation value. (Singh, 2008) This problem provides the framework to explore this further. Adding a "part F" that asks the student to explain how C and E are different may be a good start.

E. Ironically, well prepared students who have had extensive practice reasoning about introductory concepts may be at a disadvantage learning quantum mechanics! The great difficulty of removing student misconceptions discussed earlier is complicated by the fact that some ideas which are correct for macroscopic systems are incorrect for microscopic systems. It

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is not just a m atter of removing a misconception; students have to be convinced that the idea is correct in o ne situation, incorrect in another, and m ust be able to distinguish between the two! An exam ple is the phenom enon of tunneling at a potential barrier . In the m acroscopic world, when an object approaches a potential energy barrier with E < V, the probability of detecting the object on the other side of the barrier is zero. In the microscopic world, there are non-zero transmission and reflection pr obabilities, and the phenomenon of tunneling is exploited in modern technology.

Students need m ultiple opportunities to practice thes e non-intuitive concepts in different situations. A clever application of this idea (Griffith, 1995) uses an infinite well rather than a potential barrier, and asks students to consider the p robability that Quantum Mechanics will allow you and your car to bounce back from the edge of a cliff. Again, this exercise provides great material for an interactive tutorial.

6.0 THE TIME EVOLUTION OF A WAVE FUNCTION QUANTUM INTERACTIVE

LEARNING TUTORIAL (QUILT)

6.1 IDENTIFYING COMMON DIFFICULTIES

The first step of creating this interactive tutorial was researching the common student difficulties on the topic of time development. This research was done during the late 90's by Dr. Singh and her group at the University of Pittsburgh (Singh, 2001). The research included 89 students at 6 universities. The test covered quantum measurement, time development of wave functions, time dependence of expectation values, the statistical nature of quantum mechanics, and the Copenhagen interpretation. The questions were adapted from existing exams and homework. A preliminary version was given to a group of students at Duquesne University, and the test was modified after extensive class discussion and individual discussion with volunteers. This modified, 50 minute test is included in Appendix B.

An in-depth analysis of students' misconceptions was performed by interviewing nine students as they worked through the test using a think aloud protocol. In this protocol, students talked aloud while answering the questions without much interruptions and the interviewer asked them for clarification of the points they had not made clear only at the end. The common difficulties found stem from a tendency to over generalize and an inability to discriminate between related concepts. The common difficulties found in the interviews were the same as

those found by administering the written test to eighty-nine other students. They are summarized as follows:

- Students have difficulty distinguishing between individual m easurements of a physical quantity and its expectation value.

- Students often believe that a system in any eigenstate (even if it is not an energy eigenstate) does not evolve in time.

- Students believe that if the expectation value is zero in the initial s tate, then the expectation value cannot have any tim e dependence. This m isconception is similar to the m isconception in introducto ry physics that if the velocity of a particle is zero in an initial state then the rate of change of velocity is zero as well.

- Students are often confused about the ro le of eigenstates of the Ham iltonian operator (energy eigenstates) in the tim e evolution of the system and the eigenstates of other operators.

- Students believe that since the tim e e volution of a wave function is of the form $e^{\frac{-i\hat{H}t}{\hbar}}$, it cannot chang e the p robability of obtaining an ou tcome when an observable is measured.

We can broadly classify knowle dge deficiencies in three ca tegories: lack of knowledge, knowledge that can be retrieve d but not interpreted correctl y, and knowledge that can be retrieved an d interpr eted at the ba sic leve l, but canno t b e used to draw inferences. Th e difficulties above rep resent knowledge deficien cies at all three levels. Often instru ctors do no t provide the scaffolding needed to build m eaningful knowledge structure and students can end up with knowledge deficiency. If the curriculum is designed to address the three levels of

knowledge deficiency, the student s m ay develop robust understa nding of quantum m echanics concepts.

Most of the introductory physics misconceptions are actually *preconceptions* which are often formed by the student's interpretation of everyday phenomenon. These preconceptions are difficult to correct and reappear in different contexts if they ar e not addressed appropriately in the curriculum. In Qua ntum Mechanics, ther e are few preconceptions – students sim ply don't have everyday experiences with quantum ph enomenon. However, the m isconceptions in quantum mechanics could be due to the d ifficulty in re conciling th e quantum m echanical concepts with the classical concepts learned in classical physics cour ses or from everyday experiences. Students may also have difficulty in distinguishing between related concepts and over generalize the knowledge gained from a Modern Physics course or learned previously in the same course. For ex ample, students often over generalize and believe that pos ition eigenstates (in which the position of the particle has a defini te value) are the sam e as energy e igenstates (in which the energy of the particle has a definite value). Similarly, students over generalize Bohr model and of ten believe tha t even in the full quantum m echanical m odel, both th e energy an d position of an electron in a Hydrogen atom can be well-defined simultaneously.

6.2   INITIAL DESIGN OF THE TIME EVOLUTION OF A WAVE FUNCTION QuILT

With these common difficulties in m ind, and t he findings of Cognitive Research and Physics Education Research as a guide, we began the design of the **Time Evolution of a Wave Function QuILT**.   Many different versions were written, and iterations occurred after feedback from small groups of students, and after interviews with students using the think aloud protocol and working on the tutorials.

6.2.1   The First Version

The first version gives an initial wave function $\Psi(x,0) = Ae^{-ax^2}$, and asks the students to write the energy eigens tates, write the in itial wave function as a linear co mbination o f the energy eigenstates, describe th e process of finding the   expansion coefficien ts C_n 's, and write an expression that would have to be evaluated to find a $\Psi(x,t)$. The question is repeated three times in th ree dif ferent contexts: a n inf inite square we ll o r ISW (of c ourse for this potential energy, the initial state is not po ssible since the wave function must go to zero at the boundaries of the well but a Gaussian initial state only deca ys to zero at positive a nd negative infinity), a harmonic oscillator, and a free particle. Thes   e questions were designed to point out the importance of the initial state on the time evolution of wave function. When $\Psi(x,0)$ is an energy eigenstate o f the system , the tim e e volution is easy to express. W hen $\Psi(x,0)$ is not an en ergy eigenstate of the given system   , the tim e deve lopment of wave function involves a linear superposition of energy eigenstate s. After discussion, we determ ined that the confusion about

the role of energy eigenstates needed to be addr essed very early in the sem ester, and could not wait until students had learned about all the three different potentials above. The set of questions in the preliminary tutorial might make the basis for a good midterm exam, but was not suited for an early semester interactive tutorial.

The second section asked the student to consider an inf inite square well with two different initial wave functions: $\Psi(x,0) = A\sin(kx)$, $\Psi(x,0) = A\Psi_1 + B\Psi_2$, where $k = \dfrac{n\pi}{a}$, a is the width of the potential well, and Ψ_1 and Ψ_2 are the first and second energy eigenstates of the ISW. For each initial w ave function, students were asked to write down the energy eigens tates and energy eigenvalues, write the initial wave f unction as a linear combination o f the energy eigenstates, describe the process of finding the C_n's, and write an expression for $\Psi(x,t)$. The solutions for this potential were worked out in the lecture, and the low mathematical difficulty of ISW allows for m ore cognitive effort to be directed to resolv ing th e common difficulties surrounding energy eigenstates and tim e evolution. These quest ions eventually becam e the framework for the final version of the tutorial. They give stude nts an opportunity to overcom e the knowledge deficiencies related to lack of knowledge and lack of ability to apply th e knowledge properly. They set the stage for questions that provide practice in applying knowledge in different situations and making inferences.

The third section used the tim e-dependence of expectation value $\dfrac{d<Q>}{dt} = \dfrac{i}{h}<[H,Q]> + <\dfrac{dQ}{dt}>$ to explore whether or not the expectation value of an observable is tim e dependent for a given potential and initial wave function. This section was revised quite a bit in the second version.

6.2.2 The Second Version

The second version began with a summary of time evolution of wave function as shown below:

The time evolution of a system can be found for any initial wave function even if it is not an energy eigenstate (or stationary state or solution of Time Independent Schrödinger Equation) by using the following steps:

First: *Write the initial wave function as a linear combination of the energy eigenstates:*

$$\Psi(x,0) = \sum C_n \Psi_n$$

Second: *Solve for the expansion coefficients C_n by multiplying both sides by Ψ_m^{*}, and integrating over all space.*

Third: *Tack on the time dependence to each energy eigenstate as $e^{\frac{-iE_n t}{h}}$.*

Beginning this version with a sum mary of ti me evolution does several things for the studen t. It reduces the cognitive load and fr ees more STM to reason about th e concepts. This summ ary was expanded, and eventually became a flow chart in the latest version.

Unfortunately, students are often instructed to use an algorithm to solve a problem , and do not understand why the steps w ork. A section of the tutorial gives them the opportunity to clarify the reasoning. The st udents sta rt w ith the Tim e D ependent Schrodinger Equation (TDSE), and are in structed to "put the equation in a slightly different fo rm, by gathering all the time dependence on the right, and all the Ψ dependence on the left, and integrating both sides." This allows them to see the in tegral form of the TDSE, and asks them to describe in words how the Hamiltonian oper ator re lates to the evolution of the system in ti me. Next, the students are asked to co nsider two s ituations; the initial wave function is w ritten as a line ar combination of momentum eigenstates, and the initial wave function is written as a linear combination of energy

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eigenstates. The studen ts are to ld that both ex pressions are correct, b ut one is more useful f or

finding the tim e-dependence of wave function than the oth er. They are asked to ex plain why.

The students are gu ided and learn that the correct expression is w ritten for the wave function at

some later tim e easily w hen the wave f unction is w ritten as a linear superposition of energy

eigenstates (stationary states): $\Psi(x,t) = \sum C_n e^{\frac{-i\hat{H}t}{\hbar}} \Psi_n$. This type of open ended question brings

out many misconceptions. Although this series of que stions was not used verbatim in the latest

version, the student responses gathered during th is part of the developm ent of the QuILT were

useful in forming the multiple choice questions used in the present version.

The third s ection s tarts by stating that th e tim e dependence of expectation value is

$\frac{d<\hat{Q}>}{dt} = \frac{i}{\hbar}<[H,Q]> + <\frac{dQ}{dt}>$ and begins with a summary in the sam e style as the first

section. The following two im portant points are inferred from the tim e-dependence of

expectation value:

First: *If [H,Q] = 0, then d<Q>/dt= 0. This implies that expectation value <Q> does not evolve*

with time regardless of the initial state if the Hamiltonian commutes with Q. In this case, Q is a

constant of motion.

Second: *If [H,Q]≠0, there is* <u>*still*</u> *a situation in which d<Q>/dt= 0, and therefore <Q> does not*

evolve with time. This occurs when the particle is in an energy eigenstate of the system.

Similar to section one, the summ ary is desi gned to reduce the cognitive load and leave

resources available in STM to practice reasoning for different physical observable Q for different

initial s tates f or a given potentia l e nergy. The students le arn that th e initial sta te m atters. A

series of potentials, initial wave functions, and observables are gi ven. They are su mmarized in

the table below. $A\Psi_1 + B\Psi_2$ refers to the first and second energy eigenstate for the ISW, and a is

the width of the potential well. HO stands for harmonic oscillator potential.

Table 1. List of activities in the second version of the QuILT

Observable	Potential	Initial state
Momentum	ISW	$\Psi(x,0) = A\sin(\dfrac{3\pi x}{a})$
Momentum	ISW	$\Psi(x,0) = A\Psi_1 + B\Psi_2$
Momentum	HO	$\Psi(x,0) = A\sin(\dfrac{3\pi x}{a})$
Momentum	HO	$\Psi(x,0) = Ae^{-ax^2}$
Position	ISW	$\Psi(x,0) = A\sin(\dfrac{3\pi x}{a})$
Position	ISW	$\Psi(x,0) = A\Psi_1 + B\Psi_2$
Position	HO	$\Psi(x,0) = Ae^{-ax^2}$

The students are led through the process of using the expression to calculate the time-dependence of expectation value for each of the observables for the given potential energy and given initial state of the system and make sense of what they obtain. The following questions are asked for each case:

- What is the Hamiltonian for the system?

- Does the Hamiltonian commute with the operator corresponding to the observable?

- What is the energy eigenstate for the given potential?

- Is the initial wave function an energy eigenstate for the given potential?

- Is the expectation value of the observable time dependent or time independent?

This process of m odeling the thinking pro cess m ay be called "hand holding" by som e educators who feel that students should be able to develop these skills on their own. But a guided approach which provides appropriate scaffo lding is necessa ry to help stu dents build robust knowledge structure. Ther e is great debate in Educa tion Research about how m uch leading and m odeling students need to becom e proficient with thes e types of reas oning skills. Preliminary research (Singh, 2001) shows that traditional lecture on this material does not allow students to become proficient in reasoning about the time-dependence of expectation value.

6.2.3   The Third Version

In the previous versions, when students were instructed to "put the equation in a slightly different form, by gathering all the time dependence on the right, and all the Ψ dependence on the left, and integrating both sides," a very common response was $\Psi(x,t) = \Psi_0 e^{\frac{-i\hat{H}t}{\hbar}}$. This prom pted us to write three more questions:

- *If you wrote your answer in the form $\Psi(x,t) = \Psi_0 e^{\frac{-i\hat{H}t}{\hbar}}$, you have done something*

  wrong. What? (Hint: what is H? Why must it be written to the left of Ψ_0 ?)

- *Why is the Hamiltonian H called the time evolution operator?*

- *If you know the state of a system at t = 0, the state at time t is given*

  by $\Psi(x,t) = e^{\frac{-i\hat{H}t}{\hbar}}\Psi_0$. Can you think of a state Ψ_0 for which the operator

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$e^{\frac{-i\hat{H}t}{\hbar}}$ *acting on* Ψ_0 *will give a number times the same state? (Hint: Only for very special states will this operator acting of the state give a number. If the initial state is not the special state, we will not get a number times the same state. We will need to write the initial state as a linear superposition of those special states first if we do not want to write the wave function after time t in terms of the Hamiltonian operator.)*

This is an exam ple of directly ad dressing a misconception. Even a student who wrote the expression correctly the first tim e would benefit from going through this thought process.

6.3   THE CURRENT VERSION

6.3.1   Rationale for Changes

Education Research has shown that students' ability to reason about a concept develops faster when they are given opportu nity to think about different c ontexts in which the concept is applied and also given opportunity to think about the concept abstractly. Consider Part B, questions 1 through 4 of the curren t version in the Appendix. Th e responses to choose from do not specify a particular potential or a particular observable. The lack of context is by design. Part A gave a context for each pro blem, part B represen ts an attem pt to gradually decrease the guidance and let students generalize the concept.

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The Roman numeral choices for these four questions came from written student responses to earlier versions of the tutorial, and from interviews of students working through the tutorial using the think aloud protocol. The decision to move from open ended to multiple choice was made to take advantage of these common misconceptions. Students benefit from discussing why statements are incorrect just as much as discussing why statements are correct. It also allows for more objectivity in grading responses. We will eventually convert the tutorial to an online tool. In the multiple choice format, when students click a response, a help window will pop up. The text in the help window will address the reasoning errors and misconceptions that lead to the incorrect response.

Very few topics in Quantum Mechanics lend themselves to hands on experiences. The addition of a computer simulation in this tutorial is the next best thing. The tutorial directs the student to consider a wave function in an initial state and a specific potential. Students then make a prediction about the wave function's time evolution. The computer simulates the situation, and the students get immediate visual feedback on their prediction. The tutorial then guides students in reevaluating their thinking. For example, students often predict that the probability density after a time t for both stationary and non-stationary state wave functions will be time-independent because they forget that they must first write the non-stationary states as a linear superposition of stationary states before tacking on the time-dependence to each term in the expansion. Thus, the time-dependent cross terms will survive when probability density is found after time t for a non-stationary state wave function. When students look at the computer simulations for the time-dependence of probability density for two different initial wave functions, one of which is a stationary state wave function and the other is a non-stationary state wave function, they are in a state of disequilibrium. They observe that there is no time

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dependence to the probability density for the sta tionary state but there is tim e-dependence to the probability density for the non-stationary state. At this poin t, the tuto rial provides g uidance and helps students understand that since the Ham iltonian operator governs th e time evolution of the system, eigenstates of the Hamiltonian (stationary states or energy eig enstates) are s pecial when it comes to time-development issues. Then students learn the procedure for how to find the wave function after a tim e t for m any different initia l non-stationary state wave func tions that correspond to different potential energies.
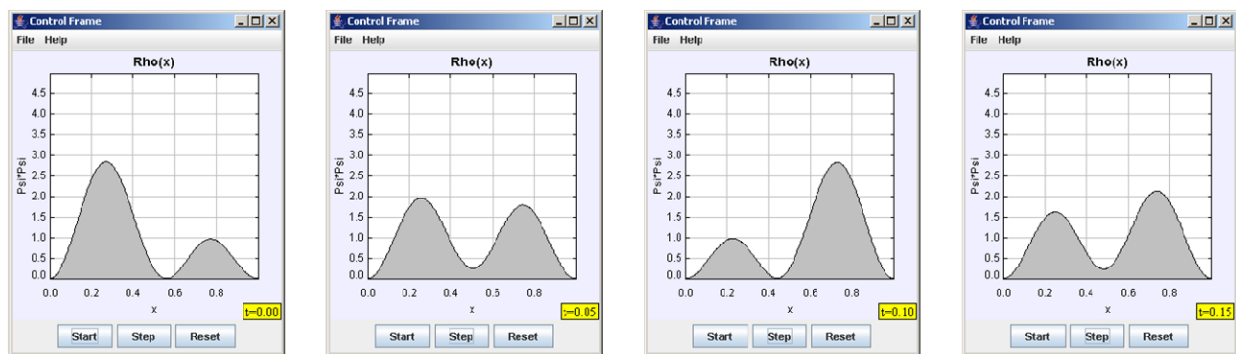
Figure 1: Computer simulation of time evolution of a wave function.

In the current version, the only context used fo r the pretest is the one dimensional infinite square well (ISW). This decision was m ade so that th e QuILT could address dif ficulties with time evolution earlier in the sem ester. These di fficulties are common and pe rsistent even at the graduate level (Singh, 2008). In most sequences used in the introductory quantum course, the stationary states are in troduced, and then severa l potentials are introdu ced in turn – the first usually being the ISW. Earlier inte rvention seemed to help students solidify the concept of time evolution, which in turn benefited the students for the rest of the semester.

Another major change is the addition of a flow chart for computing the time evolution of a state. This chart is at the end of Part B of the tutorial in the appendix, and summarizes the concepts used in Part A and B. It gives the formalism for the discrete and continuous states side by side in column format. Physics Education Research shows that when students construct their own concept maps and flow charts, they gain robust knowledge and relate ideas to allow for better chunking of material. To avoid cognitive overload, the decision was made in this case to create the flow chart for the students. The students can still benefit from using the chart.

6.4 IMPLEMENTATION AT THE UNIVERSITY OF PITTSBURGH

The final version shown in Appendix A was administered to 9 students in the undergraduate Quantum Mechanics course at the University of Pittsburgh. After traditional instruction, part A was given as a pretest. Students then worked in small groups to complete Part B. If students could not finish during the class period, they finished part B as homework. At the next class meeting, Part C was given as a posttest, and counted as a quiz grade for the class. The average pre to posttest scores improved from 53% to 85%. One of the most interesting results is from the two students who were absent the day of the pretest and tutorial – they scored 0% and 30% on the post test (Singh, 2008).

Along with the time development QuILT, 12 students also worked through a tutorial on the uncertainty principle and the Mach-Zehnder interferometer. The pretest to posttest improvements were 42% to 83% and 48% to 83%, respectively. It seems that the QuILTS help students learn. Moreover, the students like them! A survey was given to the class to determine

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how effective students found the time they spent on the tutorial. The responses were almost unanimously positive. Below are the questions and responses (Singh, 2008):

1. *Please rate the tutorials for their overall effectiveness where 1 means totally ineffective and 5 means very effective.* Average response 4.42

2. *How often did you complete the tutorial at home that you could not complete during the class? (1) never, (2) less than half the time, (3) often, (4) most of the time, or (5) always.* Average response 3.91

3. *How often were the hints provided for the tutorial useful?* Average response 4.52.

4. *Is it more helpful to do the tutorials in class or would you prefer to do them as homework? Please explain the advantages and disadvantages as you see it.* Ten students responded that doing the tutorials in class was more useful because the group discussion allowed them to focus on conceptual understanding. They saw the advantage of working with their peers and instructor. The two who preferred to work on them independently felt that they would put more time and effort into them at home.

5. *How frequently should the tutorials be administered in the class? Explain your reasoning.* All the students preferred once a week. Two reasons were given. The first was that the concepts learned in the tutorial helped them understand the homework problems. The second was that the tutorials helped them focus on concepts that were missed in other contexts, like lecture, homework or office hour discussions with the instructor.

6. *Do you prefer a multiple choice or open ended format for the tutorials? Explain your reasoning.* This was a question that elicited a varied response, but most

students saw the benefit of both types of questions even if they pref erred one over the other. The m ultiple ch oice questio ns allowed students to f ocus on common di fficulties an d funda mental conc epts, and they preferred multiple choice for the war m up. The open ende d questions stim ulated creative thought, allowed students to apply concepts and deepen understanding. Students preferred open ended for the main tutorial.

6.5   IMPLICATIONS FOR EDUCATION REFORM

Most educators actively working in the classroom, from K-12 to upper level graduate instructors, do not have the time to develop effective teaching tools. The way to reform our education system is to d evelop and evalu ate in teractive tools tha t can be ea sily im plemented in the class room without requiring a major change in the instru ctor's lecturing style. W e have developed research-based tutorials in quantum mechanics to help students develop a solid grasp of quantum mechanics. The tutorials can be used in conjunction with regular lectures. Prelim inary evaluations of the tim e-development of wave func tion QuILT shows that the tuto rial is effective in helping students in devel oping a functional understanding of quantum m echanics concepts related to the dynamics of wave function.

APPENDIX A

The Time Evolution of a Wave Function

- A "system" refers to an electron in a potential energy well, e.g., an electron in a one-dimensional infinite square well. The system is specified by a given Hamiltonian.
- Assume all systems are <u>isolated</u>.
- Assume all systems have a time-independent Hamiltonian operator $\hat{H}$.
- TISE and TDSE are abbreviations for the time-independent Schrödinger equation and the time-dependent Schrödinger equation, respectively.
- The symbol $\sum$ in all questions denotes a sum over a complete set of states.

PART A

- **Information for questions (I)-(VI)**

In the following questions, an electron is in a one-dimensional infinite square well of width L. (The stationary states are $\psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$, and the allowed energies are $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, where $n = 1, 2, 3...$)
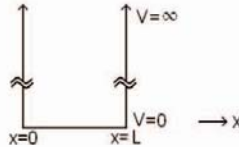
(I) Suppose the wave function for an electron at time $t = 0$ is given by $\psi(x, 0) = \sqrt{2/L}\sin(5\pi x/L)$. Which one of the following is the wave function at time t?

(a) $\psi(x,t) = \sqrt{\frac{2}{L}}\,\sin(5\pi x/L)\,\cos(E_5 t/\hbar)$

(b) $\psi(x,t) = \sqrt{\frac{2}{L}}\,\sin(5\pi x/L)\,e^{-iE_5 t/\hbar}$

(c) Both (a) and (b) above are appropriate ways to write the wave function.

(d) None of the above.

(II) The wave function for an electron at time $t = 0$ is given by $\psi(x, 0) = \sqrt{\frac{2}{L}}\,\sin(5\pi x/L)$. Which one of the following is true about the probability density, $|\psi(x,t)|^2$, after time t?

(a) $|\psi(x,t)|^2 = \frac{2}{L}\,\sin^2(5\pi x/L)\,\cos^2(E_5 t/\hbar)$.

(b) $|\psi(x,t)|^2 = \frac{2}{L}\,\sin^2(5\pi x/L)\,e^{-i2E_5 t/\hbar}$.

(c) $|\psi(x,t)|^2 = \frac{2}{L}\,\sin^2(5\pi x/L)$ which is time-independent.

(d) None of the above.

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(III) Now suppose that the wave function for an electron at time $t = 0$ is given by $\psi(x, 0) = A \sin^5(\pi x/L)$ where A is a suitable normalization constant. Which one of the following is the wave function at time t?

(a) $\psi(x, t) = A \sin^5(\pi x/L) \cos(E_5 t/\hbar)$

(b) $\psi(x, t) = A \sin^5(\pi x/L) e^{-iE_5 t/\hbar}$

(c) Both (a) and (b) above are appropriate ways to write the wave function.

(d) None of the above.

(IV) The wave function for an electron at time $t = 0$ is given by $\psi(x, 0) = A \sin^5(\pi x/L)$ where A is a suitable normalization constant. Which one of the following is true about the probability density $|\psi(x, t)|^2$ after time t?

(a) $|\psi(x, t)|^2 = |A|^2 \sin^{10}(\pi x/L) \cos^2(E_5 t/\hbar)$.

(b) $|\psi(x, t)|^2 = |A|^2 \sin^{10}(\pi x/L) e^{-i2E_5 t/\hbar}$.

(c) $|\psi(x, t)|^2 = |A|^2 \sin^{10}(\pi x/L)$ which is time-independent.

(d) None of the above.

Open the simulation by double-clicking the green arrow associated with this exercise. Shown is the wave function for an electron in the one-dimensional infinite square well at time $t = 0$ given by $\psi(x, 0) = \sqrt{\frac{2}{L}} \sin(5\pi x/L)$ for which in the simulation $\hbar = 2m = L = 1$. Click on the time-development to evolve the wave function in time.

(V) Does $|\psi(x, t)|^2$ for a given x change with time in the simulation?

(a) Yes.

(b) No.

(c) I do not know what I should be looking at in the simulation.

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Next, choose the simulation that shows the wave function for an electron in the one-dimensional infinite square well at time $t = 0$ given by $\psi(x, 0) = A \sin^5(\pi x / L)$.

Click on the time-development to evolve the wave function in time.

(VI) Does $|\psi(x, t)|^2$ for a given x change with time in the simulation?

(a) Yes.

(b) No.

(c) I do not know what I should be looking at in the simulation.

Try to reconcile your previous responses with what you observed in the simulation. Remember to begin the simulation by double-clicking the green arrow associated with this exercise.

You watched the time-development simulation for two initial wave functions $\psi(x, 0) = \sqrt{\frac{2}{L}} \sin(5\pi x / L)$ and $\psi(x, 0) = A \sin^5(\pi x / L)$ and noted that $|\psi(x, t)|^2$ for the first initial wave function does not depend on time, whereas it does depend on time for the second wave function.

Let's think systematically about time evolution of a wave function by working through the following questions.

(1) Choose all of the following statements that are correct about the time evolution of a general wave function:

     (I) The time evolution of a general wave function is governed by the Hamiltonian operator for that system.

     (II) The time evolution of a general wave function is governed by the time-independent Schrödinger equation (TISE): $\hat{H}\psi_n(x) = E_n\psi_n(x)$.

     (III) A general wave function $\psi(x,0)$ at time $t = 0$ will evolve into $\psi(x,t) = e^{-iEt/\hbar}\psi(x,0)$ at time t.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(2) Choose all of the following statements that are correct about the time evolution of a general wave function:

     (I) The time evolution of a wave function is governed by the time-dependent Schrödinger equation (TDSE).

     (II) Given $\Psi(x,0)$ and the Hamiltonian, $\hat{H}$, for a system, we can determine the wave function at time t by first calculating the momentum eigenstates.

     (III) Given $\Psi(x,0)$ and the Hamiltonian, $\hat{H}$, we can determine the wave function at time t if we first expand $\Psi(x,0)$ in terms of the energy eigenstates.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(3) Choose all of the following statements that are important steps in evaluating
the wave function at time t given $\Psi(x,0)$ at time $t = 0$:

> (I) Solve the TISE to obtain the stationary states $\psi_n(x)$ and the allowed energies E_n.
>
> (II) Solve the TDSE to obtain the stationary states $\psi_n(x)$ and the allowed energies E_n.
>
> (III) Write $\Psi(x,0)$ as a linear superposition of the eigenstates of any operator, $\hat{A}$, corresponding to a <u>physical observable</u>: $\Psi(x,0) = \sum c_n \psi_n^{(\hat{A})}(x)$.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (III) only

(e) (II) and (III) only

(4) Choose all of the following statements that are correct:

> (I) The stationary states for a Hamiltonian, $\hat{H}$, form a complete set of states.
>
> (II) Any allowed wave function can be written as a linear superposition of stationary states: $\Psi(x) = \sum c_n \psi_n(x)$
>
> (III) c_n in (II) above can be evaluated using orthogonality of wave functions.

Choose all of the above statements that are correct.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) All of the above.

ANSWERS

(1) Correct answer: (a)

Reasoning: (I) is correct because the Hamiltonian operator $\hat{H}$ governs the time evolution of a wave function.

(II) is incorrect because the time evolution of a wave function is governed by the time-dependent Schrödinger equation (TDSE): $i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t)$.

(III) is incorrect because only for a stationary state (which is a solution of $\hat{H}\psi_n(x) = E_n\psi_n(x)$) does the wave function evolve in time via a simple phase factor given by $\psi_n(x,t) = e^{-iE_nt/\hbar}\psi_n(x)$. E_n is the nth allowed energy for the electron. A general state at time $t = 0$ *may not be a stationary state*.

(2) Correct answer: (e)

Reasoning: (I) is correct because the Hamiltonian for a system governs the time evolution via the TDSE.

(II) is incorrect because in order to find $\Psi(x,t)$, we need to know the energy eigenstates (not the momentum eigenstates) because the time evolution of a wave function is governed by the Hamiltonian $\hat{H}$ and not the momentum operator. Only in special situations, e.g., for a free particle, momentum eigenstates are the same as energy eigenstates.

(III) is correct because once $\Psi(x,0)$ is written as a linear superposition of the energy eigenstates (stationary states), the wave function at time t can easily be determined by evolving each stationary state in time via an appropriate phase factor $e^{-iE_nt/\hbar}\psi_n(x)$. Note that each stationary state will have a different phase factor if the energy E_n for those states is different. Mathematically, $\Psi(x,t) = e^{-i\hat{H}t/\hbar}\Psi(x,0) = e^{-i\hat{H}t/\hbar}\left[\sum c_n \psi_n\right] = \sum c_n e^{-iE_nt/\hbar}\psi_n$ where the time-evolution operator $e^{-i\hat{H}t/\hbar}$ acting on each energy eigenstate yields the corresponding phase $e^{-iE_nt/\hbar}$. Note that $H\Psi \neq E\Psi$ although $H\psi_n = E_n\psi_n$.

(3) Correct answer: (a)

Reasoning: (I) is correct because solving the TISE, $\hat{H}\psi_n(x) = E_n\psi_n(x)$, gives the stationary states (states of definite energy) and the allowed energies E_n.

(II) is incorrect because stationary states (energy eigenstates) and the allowed energies are obtained by solving TISE.

(III) is incorrect because the time-development is governed by the $\hat{H}$ operator via TDSE. Therefore, the initial state should be expanded in terms of the stationary states (energy eigenstates) and not eigenstates of *any* operator corresponding to a physical observable.

(4) Correct answer: (e)

Reasoning: (I) is correct because stationary states are the eigenstates of the Hamiltonian (energy eigenstates).

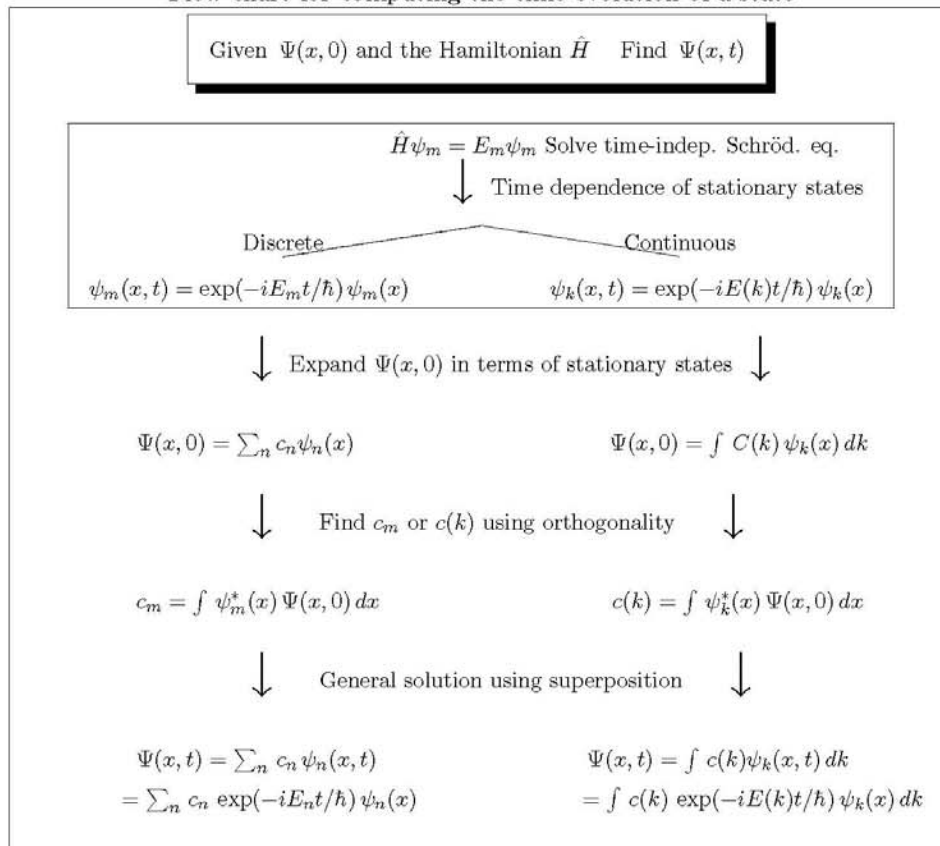(II) is correct because the wave function can be expanded in terms of a complete set of states.

(III) is correct because c_n is evaluated by multiplying both sides of $\Psi(x) = \sum c_n \psi_n(x)$ by $\psi_m^\star(x)$, integrating over all space, and using orthogonality of stationary states: $\int \psi_m^\star(x)\psi_n(x)dx = \delta_{mn}$. This yields $c_n = \int \psi_n^\ast(x)\,\Psi(x)\,dx$.

Summary: The following procedure can be used to calculate the wave function $\Psi(x, t)$ at time t given the wave function $\Psi(x, 0)$ at time $t = 0$ for an electron in a system with a given Hamiltonian $\hat{H}$.

- <u>FIRST:</u> Solve the TISE for that system to find the stationary states $\psi_n(x)$ and allowed energies E_n.

- <u>SECOND:</u> Write the given initial wave function as a linear superposition of the stationary states: $\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$.

- <u>THIRD:</u> To find c_n's for each $\psi_n(x)$, multiply both sides of $\Psi(x, 0) = \sum c_n \psi_n(x)$ by $\psi_m^*(x)$, integrate over all space, and use orthogonality of stationary states: $\int \psi_m^*(x)\psi_n(x)\, dx = \delta_{mn}$

- <u>FOURTH:</u> Tack on the time dependence with each stationary state component to obtain the time dependence of the general state: $\Psi(x, t) = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar} \psi_n(x)$.

Flow chart for computing the time evolution of a state

Given $\Psi(x, 0)$ and the Hamiltonian $\hat{H}$ Find $\Psi(x, t)$

$\hat{H}\psi_m = E_m \psi_m$ Solve time-indep. Schröd. eq.

Time dependence of stationary states

Discrete ——————— ————Continuous

$\psi_m(x, t) = \exp(-iE_m t/\hbar)\, \psi_m(x)$ $\psi_k(x, t) = \exp(-iE(k)t/\hbar)\, \psi_k(x)$

Expand $\Psi(x, 0)$ in terms of stationary states

$\Psi(x, 0) = \sum_n c_n \psi_n(x)$ $\Psi(x, 0) = \int C(k)\, \psi_k(x)\, dk$

Find c_m or $c(k)$ using orthogonality

$c_m = \int \psi_m^*(x)\, \Psi(x, 0)\, dx$ $c(k) = \int \psi_k^*(x)\, \Psi(x, 0)\, dx$

General solution using superposition

$\Psi(x, t) = \sum_n c_n \psi_n(x, t)$ $\Psi(x, t) = \int c(k)\psi_k(x, t)\, dk$
$= \sum_n c_n \exp(-iE_n t/\hbar)\, \psi_n(x)$ $= \int c(k) \exp(-iE(k)t/\hbar)\, \psi_k(x)\, dk$

• **Information for the next four questions**

In the following four questions, an electron is in a one-dimensional infinite square well of width L. (The stationary states are $\psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$, and the allowed energies are $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, where $n = 1, 2, 3...$)
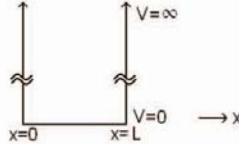
(1.1) The wave function of the electron at time $t = 0$ is given by $\Psi(x,0) = \sqrt{\frac{2}{7}} \psi_1(x) + \sqrt{\frac{5}{7}} \psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ are the ground state and first excited state. Which one of the following is the wave function $\Psi(x,t)$ at time t?

(a) $e^{-iEt/\hbar} \Psi(x,0)$

(b) $e^{-Et/\hbar} \Psi(x,0)$

(c) $e^{-ixt/\hbar} \Psi(x,0)$

(d) $\sqrt{\frac{2}{7}} \psi_1(x + \omega t) + \sqrt{\frac{5}{7}} \psi_2(x + \omega t)$ where $E = \hbar\omega$

(e) None of the above.

(1.2) Calculate $|\Psi(x,t)|^2$ for the above wave function. Does it depend on time? Explain your reasoning.

Now open the simulation (remember to double-click the green arrow) and choose the initial wave function $\Psi(x,0) = \sqrt{\frac{2}{7}} \psi_1(x) + \sqrt{\frac{5}{7}} \psi_2(x)$. Watch the time evolution of $|\Psi(x,t)|^2$. Is the time evolution of this wave function consistent with what you predicted earlier? Explain.

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(2) An electron is confined in a one-dimensional infinite square well. Choose one of the following statements that is correct about whether $\psi(x) = A \sin^2(\pi x/L)$ is an allowed wave function for the electron. (A is a suitable normalization constant):

(a) It is an allowed wave function.

(b) It is not allowed because it does not satisfy $\hat{H}\psi = E\psi$ where $\hat{H}$ is the Hamiltonian operator.

(c) It is not allowed because it is not a linear function but the Schrödinger equation is linear.

(d) It is not allowed because it is not an energy eigenfunction nor is it a linear superposition of energy eigenfunctions.

(e) $A \sin^2(\pi x/L)$ is an allowed wave function for two electrons but not for one electron.

(3) The initial wave function for an electron in a one-dimensional infinite square well at $t = 0$ is given by $\Psi(x,0) = A \sin^2(\pi x/L)$. Choose all of the following statements related to $\Psi(x,t)$ at time t that are correct.

$\quad$ (I) $\Psi(x,t) = A\, e^{-iE_n t/\hbar} \sin^2(\pi x/L)$

$\quad$ (II) $\Psi(x,t) = \sum c_n\, e^{-iE_n t/\hbar}\, \psi_n(x)$ where the expansion coefficients
$\quad\quad c_n = \int \psi_n^*(x)\, A \sin^2(\pi x/L)\, dx.$

$\quad$ (III) $\Psi(x,t) = \sum c_n \sin^2(\pi x/L)\, e^{-iE_n t/\hbar}\, \psi_n(x)$ where the expansion coefficients
$\quad\quad c_n = \int \psi_n^*(x)\, A \sin^2(\pi x/L)\, dx.$

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

(4) An electron is confined in a one-dimensional infinite square well. The wave function at time $t = 0$ is denoted by $\Psi(x,0)$. $|\Psi(x_0,t)|^2\, dx$ is the probability of finding the electron in the narrow range between x_0 and $x_0 + dx$ at time t. Choose all of the following statements that are correct:

$\quad$ (I) If $\Psi(x,0)$ is a stationary state, $|\Psi(x_0,t)|^2\, dx$ will not depend on time.

$\quad$ (II) If $\Psi(x,0)$ is localized in space around a position x_0, $|\Psi(x_0,t)|^2\, dx$ will
$\quad\quad$ not depend on time.

$\quad$ (III) If $\Psi(x,0)$ is in an energy eigenstate, $|\Psi(x_0,t)|^2\, dx$ will not depend on time.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (III) only

(e) All of the above.

ANSWERS

(1.1) Correct answer: (e)

Reasoning: Using the above flow chart, we should first expand $\Psi(x, 0)$ in terms of the stationary states: $\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ and determine the coefficients c_n. Fortunately, this $\Psi(x, 0)$ is already given in the form $\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sqrt{\frac{2}{7}} \psi_1(x) + \sqrt{\frac{5}{7}} \psi_2(x)$ and simply by observation, the coefficients $c_1 = \sqrt{\frac{2}{7}}$, $c_2 = \sqrt{\frac{5}{7}}$, and all the other c_n are zero. Since each stationary state $\psi_n(x)$ evolves independently via a phase factor $e^{-iE_n t/\hbar}$, $\Psi(x, t) = \sqrt{\frac{2}{7}} e^{-iE_1 t/\hbar} \psi_1(x) + \sqrt{\frac{5}{7}} e^{-iE_2 t/\hbar} \psi_2(x)$.

Note that the time-dependent phase factors <u>cannot</u> be factored out because they are different for different stationary states.

(2) Correct answer: (a)

Reasoning: (b) is incorrect because the allowed wave functions need not satisfy the TISE $\hat{H}\psi = E\psi$. Rather, any linear superposition of the stationary states (which are solutions of $\hat{H}\psi = E\psi$) is also an allowed state.

(c) $A \sin^2(\pi x/L)$ is a smooth function (no discontinuity or cusp) and satisfies the boundary condition for this system (i.e., $\psi(x = 0) = 0$ and $\psi(x = L) = 0$ as can be checked by plugging $x = 0$ and $x = L$ in $\psi(x) = A \sin^2(\pi x/L)$). Therefore, $A \sin^2(\pi x/L)$ can be written as a linear superposition of stationary states.

(d) $A \sin^2(\pi x/L)$ can be written as a linear superposition of stationary states (energy eigenstates).

(e) $A \sin^2(\pi x/L)$ is an allowed wave function for a single electron. A two-electron wave function must have different coordinates for the two electrons, e.g., $\Psi(x_1, x_2) = A \sin(\pi x_1/L) \sin(\pi x_2/L)$.

<u>Bottom line</u>: If the wave function is single-valued, normalizable, the wave function and its derivative are continuous everywhere, and it satisfies the boundary condition for a particular system, it is an allowed wave function.

(3) Correct answer: (b)

Reasoning: (I) is incorrect because $\Psi(x,0) = A\sin^2(\pi x/L)$ is not a stationary state with energy E_n.

(II) is correct because in evaluating $\Psi(x,t)$, $\Psi(x,0) = A\sin^2(\pi x/L)$ should only determine the coefficients $c_n = \int \psi_n^*(x)\, A\sin^2(\pi x/L)\, dx$.

(III) is incorrect because $\Psi(x,t) = \sum c_n\, e^{-iE_n t/\hbar}\, \psi_n(x)$.

(4) Correct answer: (d) Stationary state and energy eigenstate have the same meaning.

Reasoning: (I) is correct because the stationary states evolve via a simple phase factor: $\Psi(x,t) = e^{-iE_n t/\hbar}\, \psi_n(x)$ and $|\Psi(x_0,t)|^2\, dx = e^{-iE_n t/\hbar}\, \psi_n(x)\, e^{iE_n t/\hbar}\, \psi_n^*(x) = \psi_n(x)\psi_n^*(x)$ will not depend on time because the time-dependent phase factor will drop out due to complex conjugation.

(II) is incorrect because a wave function which is very localized in space around a position x_0 cannot be a stationary state. Therefore, we will have to expand the wave function in terms of the stationary states to find the wave function at time t and different stationary states will evolve via different phase factors: $\Psi(x,t) = \sum_n c_n \exp(-iE_n t/\hbar)\, \psi_n(x)$. Taking $|\Psi(x,t)|^2\, dx$ will not get rid of the time-dependent factors (as the cross terms do not necessarily vanish) and $|\Psi(x_0,t)|^2\, dx$ will change with time.

(III) is correct because a state with definite energy is a stationary state (solution of TISE). Because the stationary states evolve via a simple phase factor: $\Psi(x,t) = e^{-iE_n t/\hbar}\, \psi_n(x)$, $|\Psi(x_0,t)|^2\, dx$ will not depend on time (the time-dependent phase factor will drop out due to complex conjugation).

# APPENDIX B

QUANTUM MECHANICS TEST

Preliminary Quiz

(1) Write down the most fundamental equation of quantum mechanics.

In all of the problems below, assume that the measurement of all physical observables is ideal.

(2) The wave function of an electron in a one dimensional infinite square well of width a at time $t = 0$ is given by $\psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary state of the system. ($\phi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1,2,3...$)

Answer the following questions about this system:

- (a) Write down the wave function $\Psi(x,t)$ at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

- (b) You measure the energy of an electron at time $t = 0$. Write down the possible values of energy and the probability of measuring each.

- (c) Calculate the expectation value of energy in the state $\psi(x,t)$ above.

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- (d) If the energy measurement yields $4\pi^2\hbar^2/(2ma^2)$, write an expression for the wave function right after the measurement.

- (e) Sketch the above wave function in position space right after the measurement of energy for the previous question.

- (f) Immediately after the energy measurement, you measure the position of the electron. What are the possible values of position you can measure in a narrow range and the probability of measuring each?

(3) Which of the following wave functions are allowed for an electron in a one dimensional infinite square well of width a: $A \sin^3(\pi x/a)$, $A[\sqrt{2/5}\sin(\pi x/a) + \sqrt{3/5}\sin(2\pi x/a)]$ and $Ae^{-((x-a/2)/a)^2}$? In each of the three cases, A is a suitable normalization constant. You must provide a clear reasoning for each case.
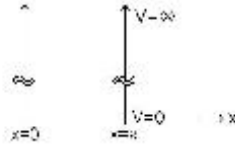

(4) Consider the following statement: "By definition, the Hamiltonian acting on *any* allowed state of the system ψ will give the same state back, i.e., $\hat{H}\psi = E\psi$" where E is the energy of the system. Explain why you agree or disagree with this statement.

(5) Notation: $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ represent the orthonormal eigenstates of $\hat{S}_z$ (the z component of the spin angular momentum) of the electron. SGA is an abbreviation for a Stern-Gerlach apparatus. Ignore the Lorentz force on the electron.

(a) A beam of electrons propagating along the y direction (into the page) in spin state $(|\uparrow_z\rangle + |\downarrow_z\rangle)/\sqrt{2}$ is sent through a SGA with a vertical magnetic field gradient in the $-z$ direction. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x-z plane. Explain your reasoning.
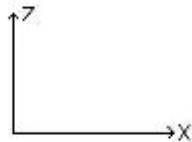
(b) A beam of electrons propagating along the y direction (into the page) in spin state $|\uparrow_z\rangle$ is sent through a SGA with a horizontal magnetic field gradient in the $-x$ direction. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x-z plane. Explain your reasoning.
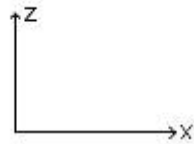
(6) The potential energy diagram for a finite square well of width a and depth $-V_0$ is shown below.
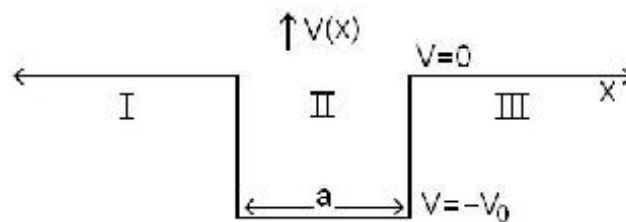
(a) Below, draw a qualitative sketch of the ground state wave function and comment on the shape of the wave function in all the three regions shown above.

(b) Below, draw a qualitative sketch of any one scattering state wave function (energy $E > 0$) and comment on the shape of the wave function in all the three regions shown above.

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