THE K-EPSILON MODEL IN THE THEORY OF TURBULENCE

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We consider the $k-\varepsilon$ model in the theory of turbulence:

$$k_t = \alpha\left(\frac{k^2}{\varepsilon}k_x\right)_x - \varepsilon$$
$$\varepsilon_t = \beta\left(\frac{k^2}{\varepsilon}\varepsilon_x\right)_x - \gamma\frac{\varepsilon^2}{k}$$

where k is the turbulent kinetic energy, ε is the dissipation rate of the turbulent energy, and α, β, and γ are positive constants. After substituting

$$k = \frac{\widetilde{A}^2}{t^{2\mu}}f(\zeta), \quad \varepsilon = \frac{\widetilde{A}^2}{t^{2\mu+1}}g(\zeta), \quad \zeta = \frac{x}{\widetilde{A}t^{1-\mu}}$$

into the $k-\varepsilon$ model, where $\widetilde{A} > 0$ is a free scaling parameter, we examine the Barenblatt self-similar $k-\varepsilon$ model for turbulence:

$$\alpha\left(\frac{f^2}{g}f'\right)' + (1-\mu)\zeta f' + 2\mu f - g = 0, \qquad 0 < \zeta < 1$$
$$\beta\left(\frac{f^2}{g}g'\right)' + (1-\mu)\zeta g' + (1+2\mu)g - \gamma\frac{g^2}{f} = 0, \quad 0 < \zeta < 1,$$

along with the boundary conditions

$$f'(0) = 0, g'(0) = 0$$
$$f(1) = 0, g(1) = 0, \frac{f^2}{g}f'(1) = 0, \frac{f^2}{g}g'(1) = 0.$$

Under the assumptions $\beta > \alpha$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, we show the existence of μ for which there is a positive solution to the system and corresponding boundary conditions by proving a series of lemmas. We also include graphs of solutions (f, g) obtained by using XPPAUT 5.85.

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PREFACE

For my parents

Faith and Dick Scott

and my husband

Stu Pomerantz

I would also like to express my gratitude to Bryce McLeod, Peter Bushell, Stuart Hastings, and Tom Metzger.

1.0 INTRODUCTION

Two equation models for turbulence are a popular variety because

> ... although a great number of equations should in principle permit greater realism to be achieved, it has proved hard to demonstrate this advantage in practice [11].

The first two-equation model for predicting the behavior of turbulent flows was proposed in 1942 by A.N. Kolmogorov and used the variables b for fluctuation energy and ω for frequency. In 1968 Harlow and Nakayama [6] introduced the $k-\varepsilon$ model for turbulence:

$$\begin{aligned} k_t &= \alpha \left(\frac{k^2}{\varepsilon} k_x\right)_x - \varepsilon \\ \varepsilon_t &= \beta \left(\frac{k^2}{\varepsilon} \varepsilon_x\right)_x - \gamma \frac{\varepsilon^2}{k} \end{aligned} \qquad (KE)$$

where $k = k(x,t)$ is the turbulent kinetic energy, $\varepsilon = \varepsilon(x,t)$ is the rate of dissipation of the turbulent energy, and α, β, and γ are positive constants. For completeness a derivation [3, 17] of the model is included in Section 2.

Although the true development of the model is often credited to Jones and Launder [9], it should be noted that (KE) is sometimes referred to as the $b-\varepsilon$ model, in acknowledgement of Kolmogorov's original insight and the relationship between the variables used: $b = \frac{2}{3}k$ and ωb is proportional to ε [11].

In 1987 Barenblatt, Galerkina, and Luneva [2] found that for the special case of $\alpha = \beta = 1$ and $\gamma > \frac{3}{2}$, (KE) has a family of self-similar compactly supported solutions:

$$\begin{aligned} k(x,t) &= (\gamma-1)(t+t_0)^{1-\nu_\gamma} \left(C - C_\gamma (x-x_0)^2 (t+t_0)^{-2\mu_\gamma}\right)_+ \\ \varepsilon(x,t) &= (t+t_0)^{-\nu_\gamma} \left(C - C_\gamma (x-x_0)^2 (t+t_0)^{-2\mu_\gamma}\right)_- \end{aligned}$$

where $t > -t_0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, $\mu_\gamma = \frac{2\gamma-3}{3(\gamma-1)}$, $\nu_\gamma = \frac{5\gamma-3}{3(\gamma-1)}$, $C_\gamma = \frac{2\gamma-3}{6(\gamma-1)^3}$, $C > 0$, and $a_+ = \max\{a, 0\}$.

When $\alpha = \beta = 1$ and $\gamma \geq 1$, Bertsch, Dal Passo, and Kersner [3] proved an existence result for the Cauchy problem:

$$\begin{cases} k_t = \alpha\left(\frac{k^2}{\varepsilon}k_x\right)_x - \varepsilon, \text{ in } Q \\ \varepsilon_t = \beta\left(\frac{k^2}{\varepsilon}\varepsilon_x\right)_x - \gamma\frac{\varepsilon^2}{k}, \text{ in } Q \\ k(x,0) = k_0(x), \ \varepsilon(x,0) = \varepsilon_0(x), \text{ for } x \in \mathbb{R} \end{cases}$$

where $Q = \{(x,t)|x \in \mathbb{R}, t > 0\}$, and k_0 and ε_0 are given bounded, non-negative and continuous functions. They discovered if k and ε are initially bounded, the solutions remain bounded with respect to x for any $t > 0$. For $\gamma > \frac{3}{2}$ they showed that the constructed solutions behave like the self-similar solutions for large values of t.

In order to obtain physical solutions to (KE), however, it is usual to take $\alpha \neq \beta$ [3, 8]. For example when specifying $\alpha = .09$, $\beta = .07$, and $\gamma = 1.92$ in (KE), the resulting Standard $k-\varepsilon$ model is only useful in regions with turbulent, high Reynolds number flows. Hulshof [8] considered the existence of compactly supported self-similar solutions for $\alpha \neq \beta$, by use of the Barenblatt solutions, but his analysis applies only when α and β are sufficiently close to 1 and $\gamma > \frac{3}{2}$. Further, his approach proceeds by looking for a perturbation about the known solution when $\alpha = \beta$.

The Barenblatt self-similar solutions can be found by substituting

$$k = \frac{\widetilde{A}^2}{t^{2\mu}}f(\zeta), \ \ \varepsilon = \frac{\widetilde{A}^2}{t^{2\mu+1}}g(\zeta), \ \ \zeta = \frac{x}{\widetilde{A}t^{1-\mu}}$$

in (KE), where $\widetilde{A}$ is a free scaling positive parameter. We will consider the system of ordinary differential equations which results from the substitution

$$\alpha\left(\frac{f^2}{g}f'\right)' + (1-\mu)\,\zeta f' + 2\mu f - g = 0, \ \ 0 < \zeta < 1 \tag{1.0.1}$$

$$\beta\left(\frac{f^2}{g}g'\right)' + (1-\mu)\zeta g' + (1+2\mu)g - \gamma\frac{g^2}{f} = 0, \ \ 0 < \zeta < 1, \tag{1.0.2}$$

along with the boundary conditions

$$f'(0) = 0, g'(0) = 0 \tag{1.0.3}$$

$$f(1) = 0, g(1) = 0, \frac{f^2}{g}f'(1) = 0, \frac{f^2}{g}g'(1) = 0, \tag{1.0.4}$$

taken to ensure the symmetry and compactness of the support of solutions.

Given the positive parameters α, β, and γ, we aim to show the existence of μ for which there is a positive solution (f,g) to (1.0.1)-(1.0.4).

1.1 DERIVING THE K-EPSILON MODEL

The $k - \varepsilon$ model can be derived from the incompressible Navier-Stokes equations

$$\rho\left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \eta\nabla^2 u_i, \qquad (NS)$$

where $u(x,t)$ represents the velocity vector field, $p(x,t)$ is the pressure field, ρ is the density constant, η is the dynamic viscosity, and $\nu = \frac{\eta}{\rho}$ is the kinematic viscosity.

Noting (NS) are derived from the equations for conservation of mass, momentum, and energy, we have that

$$\frac{\partial \rho}{\partial t} + \sum_j u_j \frac{\partial \rho}{\partial x_j} = \rho \sum_j \frac{\partial u_j}{dx_j} = 0. \qquad (1.1.1)$$

Applying statistical averaging to (NS) produces the Reynolds equations:

$$\rho\frac{\partial \overline{u_i}}{\partial t} + \sum_j \left(\rho\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \rho\overline{\frac{\partial u_i'}{\partial x_j}u_j'}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \sum_j \frac{\partial \overline{\tau_{ij}}}{\partial x_j} \qquad (R)$$

with $u = \bar{u} + u'$ written in the mean plus fluctuation decomposition, $\overline{\tau_{ij}} = \eta\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$, $\eta\nabla^2 u_i = \sum_j \frac{\partial \tau_{ij}}{\partial x_j}$, and averaging satisfying the rules summarized as follows:

$$\begin{cases} \overline{v+w} = \bar{v} + \bar{w} \\ \bar{a}v = a\overline{v},\ a = \text{constant} \\ \bar{a} = a \\ \overline{\frac{\partial v}{\partial s}} = \frac{\partial \bar{v}}{\partial s},\ s = x_i \text{ or } s = t \\ \overline{\bar{v}w} = \bar{v}\bar{w} \end{cases} \qquad (1.1.2)$$

for arbitrary fields v and w.

Some consequences of (1.1.2) applied to u are

$$\begin{cases} \overline{u_i u_j} = \overline{u_i}\,\overline{u_j} + \overline{u_i' u_j'} \\ \overline{u_i u_j u_k} = \overline{u_i' u_j' u_k'} + \overline{u_i' u_j'}\,\overline{u_k} + \overline{u_j' u_k'}\,\overline{u_i} + \overline{u_k' u_i'}\,\overline{u_j} + \overline{u_i}\,\overline{u_j}\,\overline{u_k} \\ \overline{\frac{\partial u_i}{\partial t}u_i} - \frac{\partial \overline{u_i}}{\partial t}\overline{u_i} = \overline{\frac{\partial u_i'}{\partial t}u_i'}. \end{cases} \qquad (1.1.3)$$

Thus multiplying (NS) by u_i and averaging we find

$$\rho\overline{\frac{\partial u_i}{\partial t}u_i} + \rho\sum_j \overline{u_j\frac{\partial u_i}{\partial x_j}u_i} = -\overline{\frac{\partial p}{\partial x_i}u_i} + \sum_j \overline{\frac{\partial \tau_{ij}}{\partial x_j}u_i}. \qquad (1.1.4)$$

Multiplying (R) by $\overline{u_i}$ gives

$$\rho\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\sum_j\left(\rho\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}+\rho\,\overline{\underbrace{\frac{\partial u_i'}{\partial x_j}u_j'}_{\frac{\partial}{\partial x_j}\left(u_i'u_j'\right)}}\,\overline{u_i}\right)=-\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\sum_j\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i},$$

or equivalently

$$\rho\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\rho\sum_j\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}=-\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\sum_j\left(\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i}+\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)\tag{1.1.5}$$

with $T_{ij}=-\rho\overline{u_i'u_j'}$ representing the components of the Reynolds stress matrix T.

By subtracting (1.1.5) from (1.1.4) we have

$$\rho\overline{\frac{\partial u_i'}{\partial t}u_i'}+\rho\sum_j\left(\overline{u_j\frac{\partial u_i}{\partial x_j}u_i}-\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}\right)=-\overline{\frac{\partial p'}{\partial x_i}u_i'}+\sum_j\left(\underbrace{\overline{\frac{\partial\tau_{ij}'}{\partial x_j}u_i'}}_{\frac{\partial\left(\overline{\tau_{ij}'u_i'}\right)}{\partial x_j}-\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}}-\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)\tag{1.1.6}$$

where

$$\overline{\frac{\partial u_i}{\partial t}u_i}=\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\overline{\frac{\partial u_i'}{\partial t}u_i'}\tag{1.1.7}$$

$$\overline{\frac{\partial p}{\partial x_i}u_i}=\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\overline{\frac{\partial p'}{x_i}u_i'}\tag{1.1.8}$$

$$\overline{\frac{\partial\tau_{ij}}{\partial x_j}u_i}=\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i}+\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}\tag{1.1.9}$$

and

$$\overline{u_j\frac{\partial u_i}{\partial x_j}u_i}-\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}=\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\overline{u_j}+\overline{\frac{\partial u_i'}{\partial x_j}u_i'u_j'}+\overline{\frac{\partial u_i'}{\partial x_j}u_j'}\overline{u_i}+\overline{u_j'u_i'}\overline{\frac{\partial u_i'}{\partial x_j}}\tag{1.1.10}$$

by the averaging rules.

Since $\overline{\tau_{ij}'u_i'}$ represents the viscous transfer of turbulent energy, a very small quantity in contrast to the terms responsible for the turbulent transfer of turbulent energy in (1.1.6), it is neglected. Thus (1.1.6) becomes

$$\rho\overline{\frac{\partial u_i'}{\partial t}u_i'}+\rho\sum_j\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\overline{u_j}+\sum_j\left(\rho\overline{\frac{\partial u_i'}{\partial x_j}u_i'u_j'}+\frac{\partial\rho\overline{u_i'u_j'}}{\partial x_j}\overline{u_i}+\rho\overline{u_j'u_i'}\overline{\frac{\partial u_i'}{\partial x_j}}\right)=-\overline{\frac{\partial p'}{x_i}u_i'}-\sum_j\left(\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}+\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)$$

by using (1.1.10), or

$$\frac{\rho}{2}\left(\frac{\partial\overline{(u_i')^2}}{\partial t}+\sum_j\frac{\partial\overline{(u_i'^2)}}{\partial x_j}\overline{u_j}\right)=-\overline{\frac{\partial p'}{x_i}u_i'}-\frac{\rho}{2}\sum_j\frac{\partial}{\partial x_j}\overline{\left(u_i'^2u_j'\right)}-\sum_j\left(\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}+\underbrace{\rho\overline{u_j'u_i'}}_{-T_{ij}}\frac{\partial\overline{u_i'}}{\partial x_j}\right). \quad (1.1.11)$$

Summing over i, (1.1.11) becomes an energy balance equation of turbulent flow:

$$\rho\left(\frac{\partial k}{\partial t}+\sum_j\frac{\partial k}{\partial x_j}\overline{u_j}\right)=-\sum_j\frac{\partial}{\partial x_j}\left(\overline{p'u_j'}+\frac{\rho}{2}\sum_i\overline{u_i'^2u_j'}\right)+\sum_{i,j}\left(\overline{T_{ij}}\frac{\partial\overline{u_i'}}{\partial x_j}\right)-\rho\epsilon \quad (1.1.12)$$

where the turbulent kinetic energy is defined as

$$k=\frac{1}{2}\sum_i\overline{(u_i')^2} \quad (1.1.13)$$

and the rate of dissipation of the turbulent energy is

$$\varepsilon=\frac{1}{\rho}\sum_{i,j}\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}=\frac{\nu}{2}\sum_{i,j}\overline{\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)^2}. \quad (1.1.14)$$

Using the hypotheses from [3] for the class of fluid flow under consideration, the equation for turbulent energy balance reduces to

$$\frac{\partial k}{\partial t}=\frac{\partial}{\partial x}\left(c_k\frac{\partial k}{\partial x}\right)-\varepsilon \quad (K)$$

where c_k is the turbulent energy exchange coefficient. Similarly the equation for the balance of the turbulent energy dissipation rate for flows is

$$\frac{\partial \varepsilon}{\partial t}=\frac{\partial}{\partial x}\left(c_\varepsilon\frac{\partial \varepsilon}{\partial x}\right)-U \quad (E)$$

where c_ε is the turbulent energy dissipation rate exchange coefficient and $U>0$ is the rate of homogenization of the dissipation rate.

By Kolmogorov's similarity hypothesis, c_k, c_ε, and U can be expressed in terms of two kinematic quantities: $L=$ length and $V=$ characteristic velocity, where $T=LV^{-1}=$ time. By (1.1.13) and (1.1.14),

$$[k]=L^2T^{-2}$$

$$[\varepsilon]=L^2T^{-3}.$$

By (K),

$$\left[\frac{\partial}{\partial x}\left(c_k\frac{\partial k}{\partial x}\right)\right] = L^2T^{-3}$$

which implies

$$[c_k] = L^2T^{-1}.$$

Therefore, for dimensionless constants, $\alpha > 0$, and δ_1, δ_2,

$$c_k = \alpha k^{\delta_1}\varepsilon^{\delta_2}.$$

Equating powers of the fundamental dimensions:

$$\begin{cases} L: & 2 = 2\delta_1 + 2\delta_2 \\ T: & -1 = -2\delta_1 - 3\delta_2 \end{cases}$$

we find that $\delta_2 = -1$, $\delta_1 = 2$, and

$$c_k = \alpha\frac{k^2}{\varepsilon}.$$

Similarly

$$[c_\varepsilon] = L^2T^{-1} \quad \text{and}$$

$$\left[\frac{\partial\varepsilon}{\partial t}\right] = [U] = L^2T^{-4}.$$

Therefore, with constants β and γ, the dimensional analysis yields

$$c_\varepsilon = \beta\frac{k^2}{\varepsilon}$$

$$U = \gamma\frac{\varepsilon^2}{k}$$

and by applying (K) and (E), we have (KE). An alternative derivation of (KE) is cited in [5, 14].

2.0 NEW RESULTS

Using t in place of ζ and $a > 0$ a constant, we note that under the mapping

$$f \to af$$
$$g \to ag$$
$$t \to a^{\frac{1}{2}}t$$

(1.0.1) becomes

$$a\left(\alpha\left(\frac{f^2}{g}f'\right)' + (1-\mu)\,tf' + 2\mu f - g\right) = 0$$

and similarly for (1.0.2)

$$a\left(\beta\left(\frac{f^2}{g}g'\right)' + (1-\mu)\,tg' + (1+2\mu)g - \gamma\frac{g^2}{f}\right) = 0.$$

Indeed (1.0.1)-(1.0.2) are invariant under the mapping, with the constants α, β, γ, and μ unaltered. Thus without loss of generality we may take, for instance, $f(0) = 1$; and then we notice that (1.0.1)-(1.0.2) has a unique solutions once we choose μ and $\frac{f(0)}{g(0)}$. We will show that we can choose μ and $\frac{f(0)}{g(0)}$ so that boundary conditions (1.0.4) are satisfied at some positive point, t_0. Then we will need to rescale so that t_0 becomes 1 .

Thus we will have

$$f(t_0) = 0 \text{ and } g(t_0) = 0,$$

but using the mapping $t_0 \to a^{\frac{1}{2}}t_0 = 1$, we find

$$a^{\frac{1}{2}} = \frac{1}{t_0}$$

and so

$$a = \frac{1}{t_0^2}.$$

In rescaling t_0 we will also scale

$$f \to \frac{f}{t_0^2} \quad \text{and} \quad g \to \frac{g}{t_0^2}.$$

We further note that by multiplying (1.0.1) by $\frac{g}{\alpha}$, multiplying (1.0.2) by $\frac{f}{\beta}$, and forming their difference we have

$$\frac{d}{dt}\left(\frac{f^2}{g}\left(f'g - fg'\right)\right) + (1-\mu)\,t\left(\frac{gf'}{\alpha} - \frac{fg'}{\beta}\right) + fg\left(\frac{2\mu}{\alpha} - \frac{2\mu+1}{\beta}\right) - \frac{g^2}{\alpha} + \gamma\frac{g^2}{\beta} = 0$$

or equivalently

$$\frac{d}{dt}\left(f^2 g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} = g^2\left(\frac{2\mu+1}{\beta} - \frac{2\mu}{\alpha}\right)\left(\theta - \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha - 2\mu\beta}\right) + (1-\mu)tfg'\left(\frac{1}{\beta} - \frac{1}{\alpha}\right), \tag{2.0.1}$$

where θ is defined to be $\frac{f}{g}$.

Using (1.0.3) we can express (1.0.1)-(1.0.2) in integrated form as

$$\alpha\frac{f^2}{g}f' + (1-\mu)tf = \int_0^t \left(g - (3\mu-1)\,f\right)ds \tag{2.0.2}$$

$$\beta\frac{f^2}{g}g' + (1-\mu)tg = \int_0^t \gamma\frac{g}{f}\left(g - \frac{3\mu f}{\gamma}\right)ds. \tag{2.0.3}$$

2.1 ASSUMPTIONS ON THE CONSTANTS

Our particular assumptions on the parameters α, β, and γ are such that

$$\beta > \alpha, 3\alpha > 2\beta, \gamma > \frac{3}{2}. \tag{2.1.1}$$

We will later demonstrate numerically that solutions (f,g) exist only if $\frac{\alpha}{\beta}$ is neither too big nor too small.

We will be interested in the range of $\theta(0)$ for given μ defined by

$$\frac{\gamma}{2\mu+1} < \theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}, \tag{2.1.2}$$

where μ falls within

$$\max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right) < \mu < 1. \tag{2.1.3}$$

We begin by demonstrating why $\gamma > \frac{3}{2}$ is a natural condition, by looking at the case where $\alpha = \beta$. Assuming $\alpha = \beta$, it is clear that $\theta = \gamma - 1$ is a solution to (2.0.1). Thus by substituting $\theta = \frac{f}{g} = \gamma - 1$ into (1.0.2) we have

$$\beta(\gamma-1)^2\left(gg'\right)' + (1-\mu)tg' + \left(1+2\mu-\frac{\gamma}{\gamma-1}\right)g = 0 \quad 0 < t < 1. \tag{2.1.4}$$

since $f' = (\gamma-1)g'$. Integrating and applying (1.0.3)-(1.0.4) to (2.1.4),

$$(1-\mu)\int_0^1 tg'(t)dt + \left(1+2\mu-\frac{\gamma}{\gamma-1}\right)\int_0^1 g(t)dt \quad = \quad 0 \tag{2.1.5}$$

or

$$\left(3\mu - \frac{\gamma}{\gamma-1}\right)\int_0^1 g(t)dt = 0.$$

Therefore,

$$\mu = \frac{\gamma}{3(\gamma-1)} \tag{2.1.6}$$

and so by (2.1.4),

$$\beta(\gamma-1)^2\left(gg'\right)' + (1-\mu)tg' + \left(1-\frac{\gamma}{3(\gamma-1)}\right)g = 0$$

or

$$\beta\left(\gamma-1\right)^{2}\left(gg'\right)'+\left(1-\mu\right)\left(tg\right)'=0.$$

Thus integrating again and applying (1.0.3)-(1.0.4) gives

$$g'(t)=-\frac{(1-\mu)}{\beta(\gamma-1)^{2}}t$$

and so

$$g(t)=\frac{1-\mu}{2\beta(\gamma-1)^{2}}\left(1-t^{2}\right)$$

where $g>0$ if $1-\mu=\frac{2\gamma-3}{3(\gamma-1)}>0$. Thus by (2.1.6), if $\frac{3}{2}<\gamma<\infty$ then $\frac{1}{3}<\mu<1$.

Further note that $\gamma>\frac{3}{2}$ implies

$$\alpha\gamma-\beta>\frac{3}{2}\alpha-\beta>0,$$

by the assumptions on α and β; and also

$$\left(2\mu+1\right)\alpha-2\mu\beta>0$$

as long as $\mu<\frac{1}{2\left(\frac{\beta}{\alpha}-1\right)}$, which is true by $3\alpha>2\beta$ and (2.1.3). Thus (2.1.2) defines a positive interval under (2.1.1) and (2.1.3), so

$$\frac{\gamma}{2\mu+1}<\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$$

as long as

$$\mu>\frac{1}{2(\gamma-1)}. \tag{2.1.7}$$

Also under (2.1.7) we note that

$$\frac{1}{2\mu}<\frac{\gamma}{2\mu+1}. \tag{2.1.8}$$

Lastly the fact that $\mu<1$ means $\theta(0)$ can be bounded as

$$\frac{\gamma}{3}<\theta\left(0\right)<\frac{\alpha\gamma-\beta}{3\alpha-2\beta}.$$

2.2 BEHAVIOR OF THE DERIVATIVES

By taking $\theta(0) > \frac{\gamma}{2\mu+1}$, we have at $t = 0$ that

$$(1+2\mu)g - \gamma\frac{g^2}{f} > 0.$$

Then for $t > 0$ sufficiently small we have from (1.0.2)

$$\beta\left(\frac{f^2}{g}g'\right)' + (1-\mu)tg' < 0.$$

Thus g' is initially negative for sufficiently small $t > 0$.

Supposing that $\theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then by (2.0.1) we have

$$\frac{d}{dt}\left(f^2 g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} < 0$$

for $t > 0$, t sufficiently small; and so $\theta'(t)$ is initially negative.

By applying the above facts we have the following results:

Lemma 1. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\gamma}{2\mu+1}$, then there exists $t^- > 0$ such that*

$$g'(t^-) > 0 \tag{2.2.1}$$

$$while\ \theta'(t) < 0,\ \ for\ 0 < t \leq t^-. \tag{2.2.2}$$

Lemma 2. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then there exists $t^+ > 0$ such that*

$$\theta'(t^+) > 0 \tag{2.2.3}$$

$$while\ g'(t) < 0,\ \ for\ 0 < t \leq t^+. \tag{2.2.4}$$

Thus under the assumptions of Lemma 1, although g' is initially negative, it becomes positive before θ' becomes 0. Similarly in Lemma 2 we have that $\theta'(t)$, while initially negative, becomes positive at small t and before g' becomes 0.

Consider the quadrilateral Q bounded vertically by

$$\mu = \max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right) \text{ and } \mu = 1$$

and horizontally by

$$\theta(0) = \frac{\gamma}{2\mu+1} \text{ and } \theta(0) = \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}\ .$$

We define sets:

$$S^- = \{(\mu, \theta(0)) : \text{there exists } t^- \text{ for which (2.2.1) and (2.2.2) hold}\}$$

$$S^+ = \{(\mu, \theta(0)) : \text{there exists } t^+ \text{ for which (2.2.3) and (2.2.4) hold}\}\,.$$

By definition the top and bottom boundaries of Q are contained in sets S^+ and S^-, where S^+ and S^- are disjoint and, consequently, open relative to Q because of continuity of the solutions of a differential equation with respect to the initial data. By LEMMAS 1 and 2 the sets are also non-empty. Therefore, we have shown the existence of some $(\mu, \theta(0)) \notin S^+ \cup S^-$.

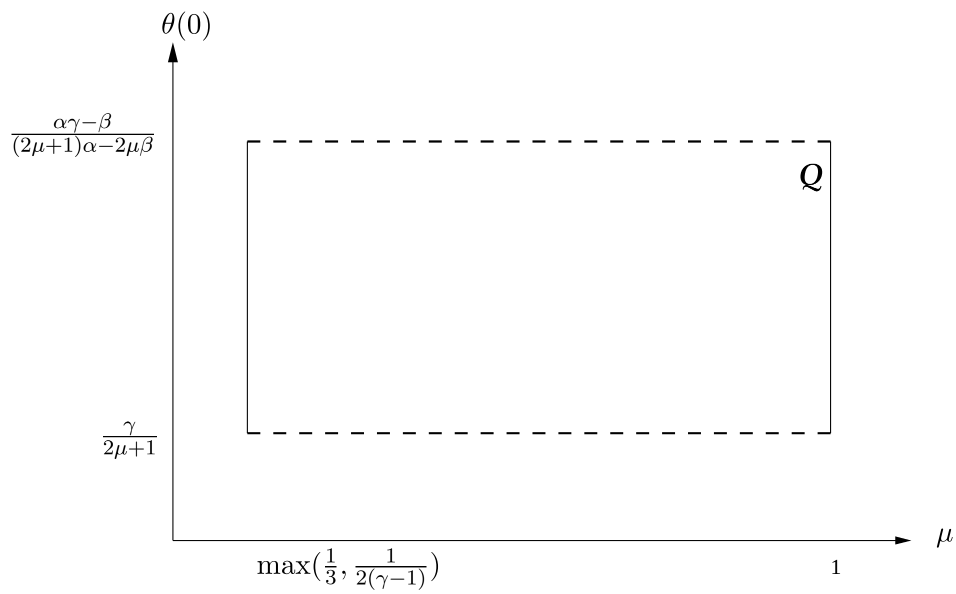

Figure 1: Quadrilateral Q with the top and bottom boundaries removed

We recall the following proposition from plane point set topology [15], a proof of which is included in Chapter 3.

Proposition. *Let I be the closed unit square $\{0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ in the (x, y) plane, and let S^- and S^+ be disjoint relatively open subsets of I, respectively containing the lines $y = 0$ and $y = 1$. Then the complement D of S^+ and S^- in I contains a continuum joining the lines $x = 0$ and $x = 1$.*

The proposition then gives the existence of a continuum $\mathscr{C}$ that lies entirely in $Q - (S^+ \cup S^-)$ and which joins a point on $\mu = \max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right)$ to a point on $\mu = 1$.

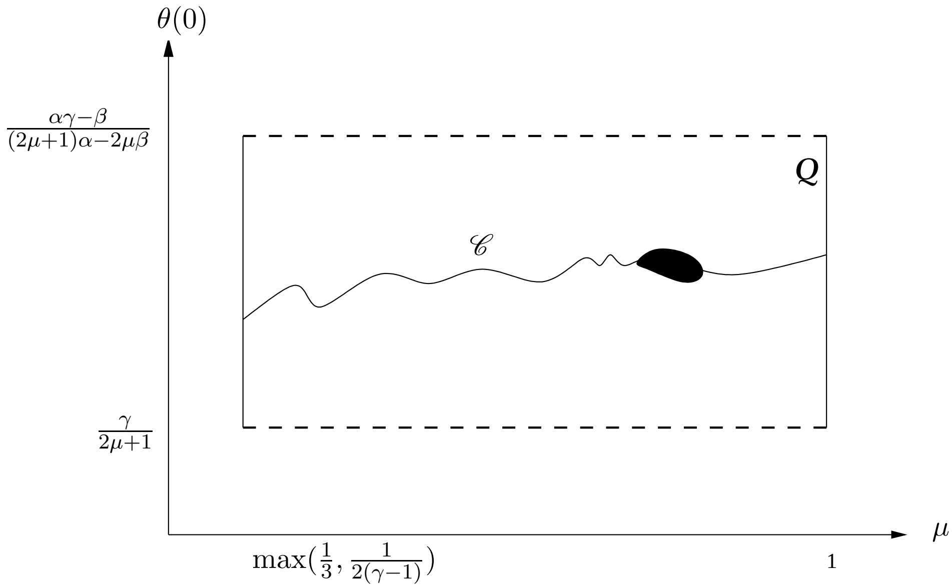

Figure 2: Plot of continuum

If $(\mu, \theta(0)) \in Q - (S^+ \cup S^-)$, then

$$\frac{\gamma}{2\mu+1} < \theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$$

and

$$\max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right) \leq \mu \leq 1.$$

By Lemmas 1 and 2, neither g' nor θ' must cross 0 first, and so g' and θ' must vanish simultaneously or neither crosses 0 at all.

The former cannot be true, otherwise we could find $t^* > 0$ such that

$$g'(t^*) = 0, \qquad \theta'(t^*) = 0, \tag{2.2.5}$$

while $g'(t) \leq 0$ and $\theta'(t) \leq 0$ for $0 \leq t \leq t^*$. Since $g''(t^*) \geq 0$, then by (1.0.2) and (2.2.5)

$$\beta \left(\frac{f^2}{g} g'(t^*) \right)' + (1-\mu) t^* g'(t^*) \geq 0, \quad \text{or}$$

$$(1+2\mu) g(t^*) - \gamma \frac{g(t^*)^2}{f(t^*)} \leq 0.$$

Therefore $\theta(t^*) \leq \frac{\gamma}{2\mu+1}$. Also $\theta''(t^*) \geq 0$, so that by (2.0.1) and (2.2.5), $\theta(t^*) \geq \frac{\alpha\gamma - \beta}{(2\mu+1)\alpha - 2\mu\beta}$. This implies

$$\frac{\alpha\gamma - \beta}{(2\mu+1)\alpha - 2\mu\beta} \leq \frac{\gamma}{2\mu+1},$$

a contradiction to (2.1.2).

Thus neither crosses zero and so $g' \leq 0$ and $\theta' = \frac{f'}{g} - \frac{fg'}{g^2} \leq 0$, and consequently $f' \leq 0$, for as long as the solution is defined. We lastly note that g cannot vanish before f, since θ is bounded.

2.3 DEVELOPING THE MAIN THEOREM

For $(\mu, \theta(0)) \in Q - (S^+ \cup S^-)$, we define

$$T = \begin{cases} t_0, \text{ if } t_0 \text{ is the first zero of } f, \\ \infty, \text{ if always } f > 0 \end{cases} \tag{2.3.1}$$

and from (2.0.2)-(2.0.3), consider the values of

$$I = \int_0^T (g - (3\mu - 1) f)\, dt, \quad J = \int_0^T \gamma \frac{g}{f} \left(g - \frac{3\mu}{\gamma} f \right) dt. \tag{2.3.2}$$

The existence of $(\mu, \theta(0))$ such that there is a positive solution to (1.0.1)-(1.0.4) will be proved by use of the following lemmas:

Lemma 3. *If $I > 0$, then the solution (f, g) exists on $[0, \infty)$, with $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$, and in fact $I = \infty$. Also, $J = \infty$. If $A = \{(\mu, \theta(0)) : I > 0\}$, then A is open in $\mathscr{C}$ and non-empty. Indeed, if*

$$\mu \leq \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 2) - \beta}$$

then $(\mu, \theta(0)) \in A$.

Lemma 4. $A \neq \mathscr{C}$

Lemma 5. *If $(\mu, \theta(0))$ is a point in $\mathscr{C}$ and on the boundary of A, then for the corresponding solution (f, g) there exists a finite point t_0 such that $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$ for $t < t_0$, while $f(t_0) = 0$, $g(t_0) = 0$, $\frac{f^2}{g} f'(t_0) = 0$, $\frac{f^2}{g} g'(t_0) = 0$.*

Now if we rescale t_0 to become 1, we have the following theorem:

Theorem. *With $\beta > \alpha$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, there exists a $\mu > 0$ such that the problem (1.0.1)-(1.0.4) possesses a solution, and the solution is such that $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$ in $(0, 1)$.*

3.0 PROOFS

3.1 LEMMA 1

Lemma 1. *If (2.1.1)-(2.1.3) hold, and* $\theta(0)$ *is sufficiently close to* $\frac{\gamma}{2\mu+1}$*, then there exists* $t^- > 0$ *such that*

$$g'(t^-) > 0$$

$$while\ \theta'(t) < 0,\ for\ 0 < t \leq t^-.$$

Proof.

If $\theta(0) = \frac{f(0)}{g(0)} = \frac{\gamma}{2\mu+1}$, then by (1.0.2) we have that $g'(0) = 0$ and $g''(0) = 0$. Differentiating (1.0.2) gives

$$\beta\left(\left(\frac{f^2}{g}\right)''g' + 2\left(\frac{f^2}{g}\right)'g'' + \left(\frac{f^2}{g}\right)g'''\right) + (1-\mu)g' + (1-\mu)tg'' + (1+2\mu)g' - \gamma\left(\frac{g^2}{f}\right)' = 0 \quad (3.1.1)$$

which evaluated at $t = 0$ reduces to $g'''(0) = 0$.

If we differentiate (3.1.1) and let $t = 0$ then

$$\beta\frac{f^2}{g}g^{(4)} + \gamma\frac{g^2}{f^2}f'' = 0. \quad (3.1.2)$$

At $t = 0$ we also know from (1.0.1) that

$$\alpha\frac{f^2}{g}f'' + 2\mu f - g = 0, \quad (3.1.3)$$

and so

$$\alpha\frac{f^2}{g}f'' = g - 2\mu f.$$

Now recalling (2.1.8), then it is clear that $\theta(0) > \frac{1}{2\mu}$ and so $g - 2\mu f < 0$ at $t = 0$. From (3.1.3) $f''(0) < 0$ and so by (3.1.2) $g^{(4)}(0) > 0$.

For $t > 0$ yet sufficiently close to zero, we have

$$\int_0^t g^{(4)}(\tau)d\tau = g'''(t) > 0,$$
$$\int_0^t g'''(\tau)d\tau = g''(t) > 0, \text{ and}$$
$$\int_0^t g''(\tau)d\tau = g'(t) > 0.$$

By (2.0.1), $\theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, and the observations made in Section 2.2, for such t

$$\frac{d}{dt}\left(f^2 g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} < 0.$$

Thus $\theta'(t)$ is initially negative. Consequently, when $\theta(0)$ is just greater than $\frac{\gamma}{2\mu+1}$, by continuity of the initial data $g'(t)$ becomes positive at small t and so before $\theta' = \frac{f'g-g'f}{g^2}$ becomes zero.

3.2 LEMMA 2

Lemma 2. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then there exists $t^+ > 0$ such that*

$$\theta'(t^+) > 0$$

$$\textit{while } g'(t) < 0, \ \textit{for } 0 < t \leq t^+.$$

Proof.

From (2.0.1) it follows that if $\theta(0) = \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then $\theta'(0) = 0$ and $\theta''(0) = 0$. Differentiating (2.0.1) we have

$$\begin{aligned} \left(f^2 g\theta'\right)'' + \tfrac{(1-\mu)}{\alpha}\left(tg^2\theta\right)' = 2gg'\left(\tfrac{2\mu+1}{\beta} - \tfrac{2\mu}{\alpha}\right)\left(\theta - \tfrac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}\right) + g^2\theta'\left(\tfrac{2\mu+1}{\beta} - \tfrac{2\mu}{\alpha}\right) + \\ +(1-\mu)\left(fg' + tf'g' + tfg''\right)\left(\tfrac{1}{\beta} - \tfrac{1}{\alpha}\right) \end{aligned} \tag{3.2.1}$$

and evaluating the result at $t = 0$ gives $\theta'''(0) = 0$. Differentiating (2.0.1) a second time and evaluating at $t = 0$ gives

$$f^2 g\theta^{(4)} = 2(1-\mu) fg''\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) > 0$$

since $\alpha < \beta$ and $g''(0) < 0$ from (1.0.2). Thus for $t > 0$ sufficiently close to zero,

$$\begin{aligned} \textstyle\int_0^t \theta^{(4)}(\tau)d\tau = \theta'''(t) > 0, \\ \textstyle\int_0^t \theta'''(\tau)d\tau = \theta''(t) > 0, \text{ and} \\ \textstyle\int_0^t \theta''(\tau)d\tau = \theta'(t) > 0. \end{aligned}$$

Now by (2.0.1), $\theta(0) > \frac{\gamma}{2\mu+1}$, and the observations made in Section 2.2, g' is initially negative for sufficiently small $t > 0$.

By continuity of the initial data, if $\theta(0)$ is just less than $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then $\theta'(t)$ becomes positive at small t and before g' becomes 0.

3.3 PROPOSITION

Proposition. *Let I be the closed unit square $\{0 \le x \le 1,\ 0 \le y \le 1\}$ in the (x, y) plane, and let S^- and S^+ be disjoint relatively open subsets of I, respectively containing the lines $y = 0$ and $y = 1$. Then the complement D of S^+ and S^- in I contains a continuum joining $x = 0$ and $x = 1$.*

Proof. (Based on the proof in [15].) By setting M to be the closure of the component of S^- that contains the line $y = 0$, N to be component of $I - M$ that contains the line $y = 1$, and Δ to be the intersection of M with the boundary of N, we aim to show the following about Δ:

- Δ is closed
- Δ contains a point on lines $x = 0$ and $x = 1$
- $\Delta \subset D$
- Δ is connected.

By construction $\Delta = M \cap \partial N$, the intersection of two closed sets, contains exactly the points on $x = 0$ and $x = 1$ that are furthest away from $y = 0$ in M. Further, if $P \in \Delta$, then $P \in M$ and there are points close to P in S^-; also $P \in \partial N$ and there are points close to P that are not in S^-. Since S^+ and S^- are open sets, if $P \in S^+$ or $P \in S^-$ then nearby points must also be in S^+ or S^-, which is clearly a contradiction. Consequently, Δ lies in D, the complement of S^+ and S^-.

If Δ is not connected then $\Delta = H \cup K$, where H, K are mutually separated, closed sets. Suppose that H and K are some positive distance, δ, away from each other.

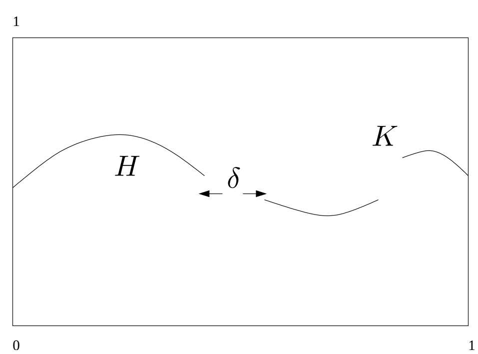

Figure 3: Sets H and K separated by distance delta

Setting down a grid of closed squares of length $\frac{\delta}{\sqrt{2}}$, consider just those squares which intersect K.

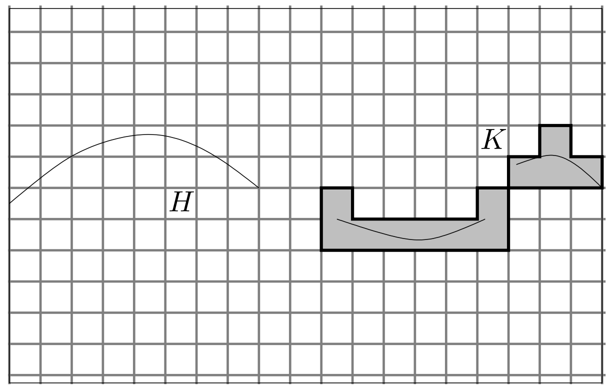

Figure 4: Grid of closed squares

The boundary of the union of the squares can be expressed as a finite number of simple closed curves. Let Q' be such a curve picked closest to a point $A \in H$, which by construction is disjoint from both H and K.

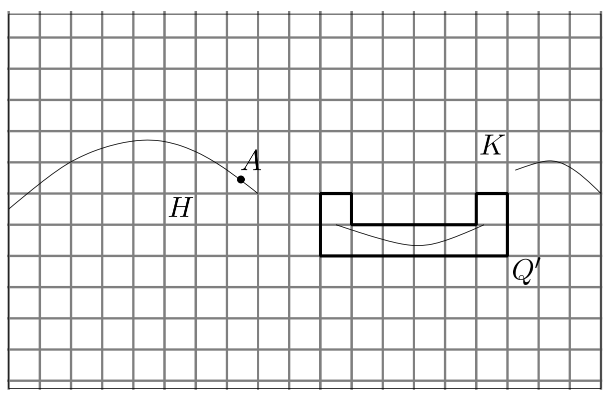

Figure 5: Curve Q' is disjoint from both H and K

If R is one of the points on Q' closest to A, and Q is the component of $Q' \cap I$ which contains R, then Q separates $A \in H \subset \Delta$ from some point, say B, in I, where $B \in K \subset \Delta$ lies in the square corresponding to R.

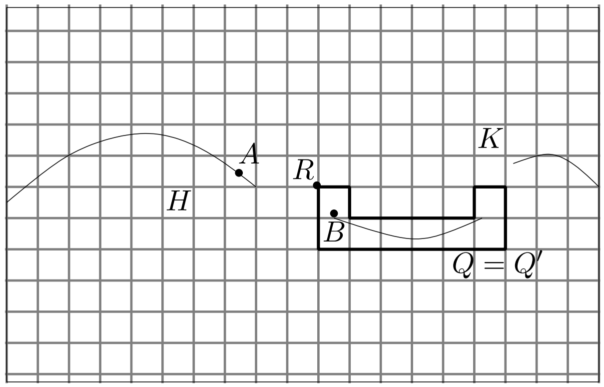

Figure 6: Simple curve Q separates A from B

Recalling that $\Delta = M \cup \partial N$, both of the regions separated by Q contain points in M and N. If $Q \cap M = \emptyset$, then $M = M_1 \cup M_2$, where $\overline{M_1} \cap M_2 = \emptyset$ and $M_1 \cap \overline{M_2} = \emptyset$, M_1 a region inside of Q, and M_2 a region outside of Q. Hence, M_1 and M_2 are contained in each of the complementary domains of Q, a contradiction to the assumption that M is connected. Similarly, if $Q \cap N = \emptyset$, then $N = N_1 \cup N_2$, the union of two mutually separated sets, each of which is contained in the region inside or outside of Q. As before, this contradicts the assumption that N is connected.

Therefore, $Q \cap M \neq \emptyset$ and $Q \cap N \neq \emptyset$, implying that we can find points $P_M \in M$ and $P_N \in N$ which are on Q. If we follow along Q from P_N to P_M we will reach $P_{\overline{N}} \in Q$, one of the last points in the closure of N.

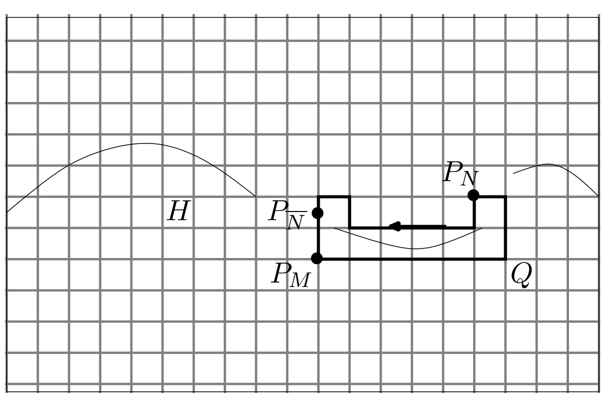

Figure 7: Existence of points on Q

By construction of sets M and N, $P_{\overline{N}} \in \partial N \cup M = \Delta$ implying that $\Delta \cap Q \neq \emptyset$, although it was assumed prior that Q is disjoint from both H and K, and thus from Δ. Hence Δ is connected.

3.4 LEMMA 3

Lemma 3. *If* $I > 0$*, then the solution* (f, g) *exists on* $[0, \infty)$*, with* $f' \leq 0$*,* $g' \leq 0$*,* $\theta' \leq 0$*, and in fact* $I = \infty$*. Also,* $J = \infty$*. If* $A = \{(\mu, \theta(0)) : I > 0\}$*, then* A *is open in* $\mathscr{C}$ *and non-empty. Indeed, if*

$$\mu \leq \frac{\alpha(\gamma+1) - \beta}{\alpha(3\gamma - 2) - \beta}$$

then $(\mu, \theta(0)) \in A$.

Proof. If T is finite then by definition (2.3.1), $f(T) = 0$, Thus from (2.0.2) we have

$$\frac{\alpha f^2}{g} f'(T) = \int_0^T (g - (3\mu - 1) f)\, dt.$$

However, since f is decreasing,

$$\liminf_{t \to T} f'(t) \leq 0,$$

from which (2.3.2) gives

$$I = \int_0^T (g - (3\mu - 1) f)\, dt \leq 0.$$

Therefore by the contrapositive, if we assume $I > 0$, then $T = \infty$ and so $f(t) > 0$ for all $t \in [0, \infty)$. Since $\theta = \frac{f}{g}$ is bounded, then $g(t) > 0$ for t in $[0, \infty)$.

It also must be true that

$$\lim_{t \to \infty} (g(t) - (3\mu - 1) f(t)) > 0. \tag{3.4.1}$$

Otherwise if

$$\lim_{t \to \infty} (g(t) - (3\mu - 1) f(t)) \leq 0 \tag{3.4.2}$$

, then

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} \geq \frac{1}{3\mu - 1}.$$

and so

$$\frac{f(t)}{g(t)} > \lim_{t\to\infty} \frac{f(t)}{g(t)} \geq \frac{1}{3\mu - 1}. \text{ for } t \in [0, \infty).$$

since $\frac{f}{g}$ is decreasing. This implies that

$$\int_0^t (g - (3\mu - 1) f) \, ds < 0$$

for all $t \in [0, \infty)$, which contradicts our assumption that $I > 0$.

Also, for all $t \in [0, \infty)$ we have that

$$(1 - \mu) t f(t) \geq \int_0^t (g - (3\mu - 1) f) \, ds \tag{3.4.3}$$

since $f' \leq 0$. By (3.4.1)

$$\int_0^t (g - (3\mu - 1) f) \, ds > 0$$

for t sufficiently large, and so

$$f(t) \geq \frac{K}{t},$$

for a constant $K > 0$. This implies that

$$\int^\infty f dt = \infty$$

and by $\theta = \frac{f}{g} \leq M$, where $M > 0$

$$\int^\infty g dt \geq \frac{1}{M} \int^\infty f dt = \infty.$$

Consequently, we find that $I = \infty$. Otherwise, if $0 < I < \infty$, then

$$I = \int_0^\infty f \left(\frac{g}{f} - (3\mu - 1) \right) dt = \int_0^{T^*} f \left(\frac{g}{f} - (3\mu - 1) \right) dt + \int_{T^*}^\infty f \left(\frac{g}{f} - (3\mu - 1) \right) dt < \infty$$

for $0 \leq T^* < \infty$ such that

$$\int_0^{T^*} f \left(\frac{g}{f} - (3\mu - 1) \right) dt \leq 0$$

and

$$f \left(\frac{g}{f} - (3\mu - 1) \right) > 0, \text{ for } t > T^*.$$

Thus

$$\int_{T^*}^\infty f \left(\frac{g}{f} - (3\mu - 1) \right) dt > \int_{T^*}^\infty K dt = \infty,$$

for some constant $K > 0$, a contradiction to I finite. Therefore, $I = \infty$.

From above $I > 0$ implies that $T = \infty$, and so $f > 0$ and $g > 0$ for all $t \in [0, \infty)$. Recall (2.0.3):

$$\beta \frac{f^2}{g} g' + (1-\mu)tg = \int_0^t \gamma \frac{g}{f}\left(g - \frac{3\mu}{\gamma} f\right) ds.$$

Since θ is bounded

$$\left|\frac{f^2}{g} g'\right| \leq |Mfg'| \leq |M^2 gg'| = O\left(\left(g^2\right)'\right).$$

Therefore, there exists a sequence $\{t_n\}$ such that

$$\frac{f^2}{g} g'(t_n) \to 0, \quad \text{as } t_n \to \infty. \tag{3.4.4}$$

Now it also must be that $\lim_{t\to\infty}\left(g(t) - \frac{3\mu}{\gamma} f(t)\right) > 0$ If not, then for all $t < \infty$,

$$\frac{f(t)}{g(t)} > \lim_{t\to\infty} \frac{f(t)}{g(t)} \geq \frac{\gamma}{3\ mu}$$

and so for all t,

$$\int_0^t \gamma \frac{g}{f}\left(g - \frac{3\mu}{\gamma} f\right) ds < 0$$

implying by (2.0.3) that

$$\beta \frac{f^2}{g} g' + (1-\mu)tg < 0$$

for all $t < \infty$, a contradiction to the assumption that (3.4.4) holds and $g > 0$. Hence we have

$$\lim_{t\to\infty} \theta(t) < \frac{\gamma}{3\mu}$$

and as done above for I,

$$J = \int_0^\infty \gamma \frac{g}{f}\left(g - \frac{3\mu}{\gamma} f\right) dt = \infty.$$

We now will prove that the set

$$A = \{(\mu, \theta(0)) : I > 0\}$$

is open. If we have a solution f_0 such that $f_0(T_0) = 0$, so that $(\mu_0, \theta_0(0)) \notin A$, and if we also consider a sequence $\{(\mu_n, \theta_n(0))\}$ tending to $(\mu_0, \theta_0(0))$, with solutions f_n such that $f_n(T_n) = 0$, then

$$\liminf_{n\to\infty} T_n \geq T_0.$$

This holds because $f_0 > 0$ for all $t < T_0$, and so while solutions f_n are close to f_0 for $t < T_0$, f_n cannot disappear before f_0 does as $n \to \infty$.

Consequently, if $(\mu_0, \theta_0(0)) \in A$ with corresponding solution (f_0, g_0), then a nearby solution, say (f, g), must either be in A (and so A is open) or will at least have the behavior that (f, g) is close to (f_0, g_0) in $[0, \widetilde{T}]$, where $f(\widetilde{T}) = 0$. By (3.4.1) we have that

$$\lim_{t\to\infty} \theta_0(t) < \frac{1}{3\mu - 1}$$

but if (f_0, g_0) is close enough to (f, g) over $[0, \widetilde{T}]$ there exists a finite value T^*, such that $\theta_0(T^*) < \frac{1}{3\mu-1}$, which implies that $\theta(T^*) < \frac{1}{3\mu-1}$. Now θ monotone implies that

$$\lim_{t\to\infty} \theta(t) < \theta(T^*) < \frac{1}{3\mu - 1}.$$

Then T^* can be chosen so that

$$\int_0^{T^*} (g_0 - (3\mu - 1) f_0)\, dt > 0, \text{ and } \int_0^{T^*} (g - (3\mu - 1) f)\, dt > 0,$$

and so $I > 0$ and $(\mu, \theta(0)) \in A$. Therefore, A is open.

Lastly we note that if $\theta(t) < \frac{1}{3\mu-1}$ for all $t \geq 0$, then $I = \int_0^t (g - (3\mu - 1) f)\, ds > 0$. Further from (2.1.2)

$$\theta(t) \leq \theta(0) < \frac{\alpha\gamma - \beta}{(2\mu + 1)\alpha - 2\mu\beta}$$

for all $t \geq 0$. If we could find appropriate conditions so that

$$\frac{\alpha\gamma - \beta}{(2\mu + 1)\,\alpha - 2\mu\beta} \leq \frac{1}{3\mu - 1}, \tag{3.4.5}$$

it would be sufficient for proving $I > 0$.

Thus in order for (3.4.5) to be true, then

$$(\alpha\gamma - \beta)\,(3\mu - 1) \leq (2\mu + 1)\,\alpha - 2\mu\beta,$$

and so

$$\mu \leq \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 2) - \beta}. \tag{3.4.6}$$

If (3.4.6) holds, then $(\mu, \theta(0)) \in A$.

We also note that new bound (3.4.6) for μ satisfies

$$\max\left(\frac{1}{3}, \frac{1}{2\,(\gamma - 1)}\right) < \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 1) + 2\beta} \leq 1. \tag{3.4.7}$$

Under the assumptions (2.1.1) and if in particular $\gamma > \frac{5}{2}$, then it is clear that

$$\max\left(\frac{1}{3}, \frac{1}{2\,(\gamma-1)}\right) = \frac{1}{3} < \frac{\alpha(\gamma+1)-\beta}{\alpha(3\gamma-1)+2\beta}.$$

In order to prove

$$\frac{\alpha(\gamma+1)-\beta}{\alpha(3\gamma-1)+2\beta} \leq 1,$$

we need only show that

$$\alpha(\gamma+1)-\beta \leq \alpha(3\gamma-1)+2\beta,$$

which again holds under (2.1.1). The remaining case of $\frac{3}{2} < \gamma \leq \frac{5}{2}$ is similar.

3.5 LEMMA 4

Lemma 4. $A \neq \mathscr{C}$

Proof. We will prove that there exists $(\mu, \theta(0)) \in \mathscr{C}$ that is not in A.

We note that when $\mu = 1$ the left-hand side of (2.0.2) becomes

$$\frac{f^2}{g} f' = \int_0^t (g - (3\mu - 1) f)\, ds$$

and since $f' \leq 0$ for all t for which the solution exists, then

$$\int_0^t (g - (3\mu - 1) f)\, ds \leq 0.$$

Thus $I \leq 0$ and so $(\mu, \theta(0)) \notin A$ using a result from the proof of Lemma 3. If we suppose $I = 0$,

$$\int_0^T (g - (3\mu - 1) f)\, ds = \int_0^T f \left(\frac{g}{f} - (3\mu - 1) \right) ds = 0, \tag{3.5.1}$$

then since $\frac{g}{f}$ is increasing,

$$\int_0^t (g - (3\mu - 1) f)\, ds < 0 \tag{3.5.2}$$

for t small enough, $0 < t < T$. We also notice from (2.0.3) with $\mu = 1$,

$$\beta \frac{f^2}{g} g' = \int_0^t \gamma \frac{g}{f} \left(g - \frac{3\mu f}{\gamma} \right) ds$$

and since $g' \leq 0$ for as long as the solution continues to exist, then $J \leq 0$.

However, we can express

$$J = \int_0^T \gamma \tfrac{g}{f} \left(g - \tfrac{3\mu f}{\gamma} \right) ds = \int_0^T \gamma \tfrac{g}{f} \left(g - \tfrac{3\mu f}{\gamma} - (3\mu - 1) f + (3\mu - 1) f \right) ds$$

$$= \int_0^T \gamma \tfrac{g}{f} (g - (3\mu - 1) f)\, ds + \int_0^T \gamma \tfrac{g}{f} \left((3\mu - 1) - \tfrac{3\mu}{\gamma} \right) f ds$$

$$= J_1 + J_2.$$

We notice when $\mu = 1$, the integrand of J_2 is

$$(3\mu - 1) - \frac{3\mu}{\gamma} = \left(2 - \frac{3}{\gamma} \right) > 0$$

since $\gamma > \frac{3}{2}$, and consequently $J_2 > 0$.

For J_1 we can integrate by parts to get

$$J_1 = \int_0^T \gamma \tfrac{g}{f} \left(g - (3\mu - 1)f\right) ds$$

$$= \gamma \tfrac{g}{f} \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau \right)\Big|_0^T - \int_0^T \gamma \left(\tfrac{g}{f}\right)' \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau \right) ds$$

$$= - \int_0^T \gamma \left(\tfrac{g}{f}\right)' \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau \right) ds$$

by (3.5.1). By $\frac{g}{f} > 0$ and (3.5.2), $J_1 > 0$ which implies that $J > 0$. This contradicts that above we found $J \leq 0$.

Therefore, it must be that $I < 0$ when $\mu = 1$, and so by (2.0.2)

$$\alpha \frac{f^2}{g} f' < -K$$

for some constant $K > 0$. This is equivalent to having

$$f' < -K \left(\frac{g}{f}\right) \frac{1}{f}$$

for a constant $K > 0$. As $\frac{g}{f}$ and $\frac{1}{f}$ increase, f' is increasingly negative, forcing the existence of a finite point $t^* > 0$ such that $f(t^*) = 0$.

3.6 LEMMA 5

Lemma 5. *If* $(\mu, \theta(0))$ *is a point in* $\mathscr{C}$ *and on the boundary of* A*, then for the corresponding solution* (f, g) *there exists a finite point* t_0 *such that* $f' \le 0$*,* $g' \le 0$*,* $\theta' \le 0$ *for* $t < t_0$*, while* $f(t_0) = 0$*,* $g(t_0) = 0$*,* $\frac{f^2}{g} f'(t_0) = 0$*,* $\frac{f^2}{g} g'(t_0) = 0$*.*

Proof. From Lemma 4, if we let $(\mu, \theta(0))$ be one of the first points in $\mathscr{C}$ that is not in A, then

$$I = \int_0^T (g - (3\mu - 1)f)\, dt \le 0. \tag{3.6.1}$$

Consequently the solution (f, g) cannot exist for all $t \in [0, \infty)$. If not, we take $f > 0$ for all t and use a technique from Lemma 3 that

$$\left| \frac{f^2}{g} f' \right| \le |M f f'| = O\left(\left(f^2\right)' \right)$$

means there exists a sequence $\{t_n\}$ such that

$$\frac{f^2}{g} f'(t_n) \to 0, \quad \text{as } t_n \to \infty. \tag{3.6.2}$$

Then

$$0 \le \alpha \frac{f^2}{g} f'(t_n) + (1 - \mu) t_n f(t_n) \le I \le 0,$$

so $I = 0$; and we have

$$g(T) - (3\mu - 1) f(T) = 0$$

so

$$\frac{f(t)}{g(t)} > \frac{f(T)}{g(T)} = \frac{1}{3\mu - 1}$$

for all $t < T$. By (1.0.2) for all $t < T$,

$$\int_0^t (g - (3\mu - 1)f)\, ds < 0$$

or

$$\alpha \frac{f^2}{g} f' + (1 - \mu) t f < 0.$$

Hence

$$f' < \frac{-(1 - \mu)t}{\alpha} \left(\frac{g}{f} \right),$$

which, for $\mu \neq 1$, becomes more and more negative as t and $\frac{g}{f}$ increase, a contradiction to the assumption that $f > 0$ for all t. When $\mu = 1$, we have from LEMMA 4 that f does not exist on all of $[0, \infty)$.

This proves there exists $T = t_0 < \infty$, for which $f(t_0) = 0$ and so

$$I = \int_0^{t_0} (g - (3\mu - 1)f)\, ds \leq 0. \tag{3.6.3}$$

Next we let

$$B = \{(\mu, \theta(0)) : I < 0\},$$

and we want to prove that B is open.

From LEMMA 3 if we have a solution f_0 such that $f_0(T_0) = 0$, so that $(\mu_0, \theta_0(0)) \in B$, and if we also consider a sequence $\{(\mu_n, \theta_n(0))\}$ tending to $(\mu_0, \theta_0(0))$, with solutions f_n such that $f_n(T_n) = 0$, then it must be that

$$\liminf_{n \to \infty} \; T_n \geq T_0.$$

If we can prove that indeed

$$\lim_{n \to \infty} T_n = T_0$$

then

$$\int_0^{T_n} (g_n - (3\mu - 1)\, f_n)\, ds \longrightarrow \int_0^{T_0} (g_0 - (3\mu - 1)\, f_0)\, ds < 0,$$

implying that nearby solutions are also in B.

Suppose for a contradiction that $T_n \to T^* > T_0$. We know, however, that f_n and f_0 are close for $t < T_0$, and since f_n is decreasing, $f_n \to 0$ in $[T_0, T_n]$.

Now by (2.0.2), for $t \in [T_0, T_n]$

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu)t f_n = \int_0^t (g_n - (3\mu - 1)\, f_n)\, ds = \int_0^{T_0} (g_0 - (3\mu - 1)\, f_0)\, ds + \int_{T_0}^t (g_n - (3\mu - 1)\, f_n)\, ds$$

and so

$$g_n = \left(\alpha \frac{f_n^2}{g_n} f_n' + (1 - \mu) t f_n + \int_{T_0}^t (3\mu - 1)\, f_n ds \right)'.$$

Since $g_n' \leq 0$, then

$$\left(\alpha \frac{f_n^2}{g_n} f_n' + (1 - \mu) t f_n + \int_{T_0}^t (3\mu - 1)\, f_n ds \right)' \leq 0. \tag{3.6.4}$$

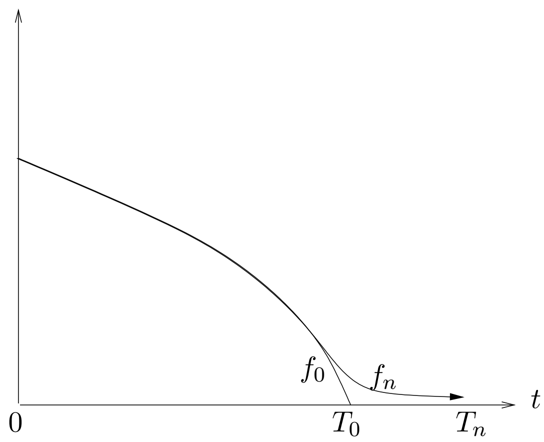

Figure 8: Plot over [T0,Tn]

Since

$$\left|\int_{T_0}^{T_n} \frac{f_n^2}{g_n} f_n' ds\right| = \left|\int_{T_0}^{T_n} \frac{f_n}{g_n} f_n f_n' ds\right| \leq \theta_n(T_0) \left|\int_{T_0}^{T_n} f_n f_n' ds\right| = \theta_n(T_0) \left|\int_{T_0}^{T_n} \left(\frac{f_n^2}{2}\right)' ds\right| \longrightarrow 0$$

then for $T_0 \leq t \leq T_n$,

$$\int_{T_0}^{t} \left(\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) s f_n + \int_{T_0}^{s} (3\mu - 1) f_n d\tau\right) ds \longrightarrow 0.$$

Thus by (3.6.4),

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) s f_n + \int_{T_0}^{s} (3\mu - 1) f_n d\tau \longrightarrow 0$$

almost everywhere in $[T_0, T_n]$, which means that

$$\int_{T_0}^{t} g_n ds \to 0$$

for $t \in [T_0, T_n]$. As above we write

$$\int_0^t (g_n - (3\mu - 1) f_n) ds = \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds + \int_{T_0}^{t} (g_n - (3\mu - 1) f_n) ds \to \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds$$

for $t \in [T_0, T_n]$. Since $(\mu_0, \theta_0(0)) \in B$, $I_0 = \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds < 0$, this contradicts

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) t f_n = \int_0^t (g_n - (3\mu - 1) f_n) ds \longrightarrow 0,$$

for $T_0 \leq t \leq T_n$. Therefore, $\lim_{n\to\infty} T_n = T_0$ and so B is open in $\mathscr{C}$.

Consequently, if $(\mu,\theta(0))$ is to be one of the first points on $\mathscr{C}$ that is not in A, then we also must restrict $(\mu,\theta(0)) \notin B$. Thus by (3.6.3), $(\mu,\theta(0))$ is such that

$$I=\int_0^{t_0}\left(g-(3\mu-1)\,f\right)ds=0$$

and so

$$\alpha\frac{f^2}{g}f'\left(t_0\right)+(1-\mu)t_0 f\left(t_0\right)=0$$

implying that

$$\alpha\frac{f^2}{g}f'\left(t_0\right)=0.$$

Letting $\{(f_n,g_n)\}$ be a sequence on $[0,\infty)$ that approximates (f,g) on $[0,t_0)$ where $f_n'\le 0$, then $f_n\to 0$ on $[t_0,\infty)$, and as similarly done in the proof of B being open, $\frac{f_n^2}{g_n}f_n'\to 0$ almost everywhere on $[t_0,\infty)$. So

$$\alpha\frac{f_n^2}{g_n}f_n'+(1-\mu)tf_n+\int_{t_0}^{t}\left(3\mu-1\right)f_n ds=\int_{t_0}^{t}g_n ds\longrightarrow 0$$

and then $g_n\to 0$ on $[t_0,\infty)$.

If we now assume

$$J=\int_0^{t_0}\gamma g\left(\frac{g}{f}-\frac{3\mu}{\gamma}\right)ds>0,$$

then at t_0

$$g\left(\frac{g}{f}-\frac{3\mu}{\gamma}\right)>0$$

and so

$$\theta\left(t_0\right)<\frac{\gamma}{3\mu}.$$

Thus there exists $T^*<t_0$, large enough and still close enough to t_0, for which

$$\int_0^{T^*}\gamma g\left(\frac{g}{f}-\frac{3\mu}{\gamma}\right)ds>0,\ \text{and}$$

$$\int_0^{T^*}\gamma g_n\left(\frac{g_n}{f_n}-\frac{3\mu}{\gamma}\right)ds>0.$$

For $t\ge T^*$

$$\frac{g_n(t)}{f_n(t)}\ge\frac{g_n(T^*)}{f_n(T^*)}>\frac{3\mu}{\gamma}$$

and so

$$\int_0^t \gamma g_n \left(\frac{g_n}{f_n} - \frac{3\mu}{\gamma} \right) ds > 0.$$

Thus by (2.0.3) for $t \in (t_0, \infty)$,

$$\beta \frac{f_n^2}{g_n} g_n' + (1-\mu) t g_n = \int_0^t \gamma g_n \left(\frac{g_n}{f_n} - \frac{3\mu}{\gamma} \right) ds > 0,$$

a contradiction to our results from above that $g_n \to 0$ and $g_n' \leq 0$. Therefore $J \leq 0$, which implies that for $t < t_0$, sufficiently small enough,

$$\beta \frac{f^2}{g} g' + (1-\mu) t g = \int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma} \right) ds < 0. \tag{3.6.5}$$

In order to prove $g(t_0) = 0$, we assume that $g(t_0) = g_0 > 0$ and hope for a contradiction. Thus for $t < t_0$

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f = \int_0^t (g - (3\mu - 1) f) \, ds < I = 0, \tag{3.6.6}$$

and so

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f < 0,$$

which implies that

$$\alpha \left(\frac{f^2}{2} \right)' < -(1-\mu) t g. \tag{3.6.7}$$

As $t \to t_0$, $g \to g_0$, so if we integrate both sides of (3.6.7) from t to t_0,

$$f^2 > K(t_0 - t),$$

for positive constant K, and so

$$f > K(t_0 - t)^{\frac{1}{2}}$$

for another constant $K > 0$.

Now from (3.6.6) and $I = 0$, as $t \to t_0$

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f \leq K(t_0 - t)$$

for some constant $K > 0$, and so

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f = O(t_0 - t)$$

as $t \to t_0$ which implies that

$$\alpha \frac{f^2}{g} f' \sim -(1-\mu) t f.$$

Since

$$\lim_{t\to t_0} \frac{\frac{f^2}{g} f'}{-(1-\mu)tf} = \lim_{t\to t_0} \frac{ff'}{-(1-\mu)tg} = \frac{\lim_{t\to t_0} ff'}{-(1-\mu)t_0 g_0},$$

then it is also true that

$$ff' \sim -(1-\mu)t_0 g_0. \tag{3.6.8}$$

Additionally by integrating up to t_0

$$\lim_{t\to t_0} \frac{\left(\frac{f^2}{2}\right)'}{-(1-\mu)t_0 g_0} = \lim_{t\to t_0} \frac{f^2}{K(t_0-t)}$$

for a positive constant K, and thus

$$f^2 \sim K(t_0 - t) \tag{3.6.9}$$

as $t \to t_0$. For $t < t_0$

$$\beta \frac{f^2}{g} g' < \beta \frac{f^2}{g} g' + (1-\mu)tg < 0,$$

then

$$g' < \frac{-Kg}{f^2}$$

for some $K > 0$, and so by (3.6.9)

$$g' < \frac{-K}{t_0 - t}$$

for another positive constant K. Integrating both sides up to t_0,

$$\int_t^{t_0} g' ds < \int_t^{t_0} \frac{K}{s - t_0} ds$$

we find that

$$g(t) - g_0 > \infty$$

for all $t < t_0$, which is not possible. Therefore it must be that $g(t_0) = 0$.

It remains to show that $J = 0$, which will be used to prove that $\frac{f^2}{g} g'(t_0) = 0$. We therefore suppose $J < 0$, in order to get a contradiction. That is from (3.6.5), for $t < t_0$

$$\beta \frac{f^2}{g} g' + (1-\mu)tg = \int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds < 0$$

and for $t \to t_0$

$$\lim_{t\to t_0} \frac{\beta \frac{f^2}{g} g' + (1-\mu)tg}{\int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds} \to \frac{\beta \frac{f^2}{g}}{J},$$

so that

$$\beta\frac{f^2}{g}g' \sim J \tag{3.6.10}$$

as $t \longrightarrow t_0$.

Now using that $\frac{g}{f}$ is increasing, as $t \longrightarrow t_0$

$$\frac{g}{f}(t_0 - t) \leq \int_t^{t_0} \frac{g}{f}ds = o\left(\int_t^{t_0} \frac{g}{f^2}ds\right). \tag{3.6.11}$$

That is

$$\lim_{t\to t_0}\frac{\int_t^{t_0} \frac{g}{f}ds}{\int_t^{t_0} \frac{g}{f^2}ds} = \lim_{t\to t_0}\frac{\frac{g}{f}}{\frac{g}{f^2}} = \lim_{t\to t_0} f = f(t_0) = 0.$$

Using (3.6.10), it is true that $\int_t^{t_0} \frac{g}{f^2}ds \asymp \int_t^{t_0} g' ds$, hence of the same order:

$$o\left(\int_t^{t_0} \frac{g}{f^2}ds\right) = o\left(\int_t^{t_0} g' ds\right)$$

as $t \longrightarrow t_0$ using that

$$\lim_{t\to t_0} \frac{\beta\frac{f^2}{g}g'}{J} \longrightarrow 1$$

and so equally

$$\lim_{t\to t_0} \frac{-\beta\frac{f^2}{g}g'}{-J} \longrightarrow 1.$$

When combined with (3.6.11) and

$$o\left(\int_t^{t_0} g' ds\right) = o(1),$$

this gives

$$\frac{g}{f}(t_0 - t) = o(1). \tag{3.6.12}$$

Thus we have that

$$\lim_{t\to t_0}\frac{\frac{g}{f}(t_0 - t)}{1} = 0$$

and

$$\lim_{t\to t_0}\frac{\frac{g}{f}}{\frac{1}{(t_0-t)}} = 0,$$

implying

$$\frac{g}{f} = o\left(\frac{1}{(t_0 - t)}\right)$$

as $t \longrightarrow t_0$.

From (2.0.2) with $t < t_0$,

$$\alpha\frac{f^2}{g}f' + (1-\mu)tf = \int_0^t (g - (3\mu - 1) f) \, ds < \int_0^{t_0} (g - (3\mu - 1) f) \, ds = I,$$

then

$$\alpha\frac{f^2}{g}f'+(1-\mu)tf=\int_0^{t_0}(g-(3\mu-1)f)\,ds-\int_t^{t_0}(g-(3\mu-1)f)\,ds=-\int_t^{t_0}(g-(3\mu-1)f)\,ds<0. \tag{3.6.13}$$

From (2.0.3)

$$\beta\frac{f^2}{g}g'+(1-\mu)tg=\int_0^t\gamma g\left(\frac{g}{f}-\frac{3\mu}{\gamma}\right)ds<J, \tag{3.6.14}$$

then we have

$$\beta\frac{f^2}{g}g'+(1-\mu)tg\sim J \tag{3.6.15}$$

as $t\to t_0$, and since

$$\lim_{t\to t_0}\frac{\beta\frac{f^2}{g}g'+(1-\mu)tg}{J}=\lim_{t\to t_0}\frac{\beta\frac{f^2}{g^2}g'+(1-\mu)t}{\frac{J}{g}},$$

then

$$\beta\frac{f^2}{g^2}g'+(1-\mu)t\sim\frac{J}{g}. \tag{3.6.16}$$

Similarly dividing (3.6.13) by f results in

$$\alpha\frac{f}{g}f'+(1-\mu)t=\frac{1}{f}\int_{t_0}^t(g-(3\mu-1)f)\,ds, \tag{3.6.17}$$

where

$$\lim_{t\to t_0}\frac{\int_{t_0}^t(g-(3\mu-1)f)\,ds}{f}=0$$

since

$$\frac{1}{f}\int_t^{t_0}gds\le\frac{g}{f}(t_0-t)=o(1)$$

and

$$\frac{3\mu-1}{f}\int_t^{t_0}fds=o(1)$$

as $t\to t_0$ by (3.6.12). Thus

$$\lim_{t\to t_0}\frac{\frac{f^2}{g}\left(\alpha\frac{f'}{f}-\beta\frac{g'}{g}\right)}{-\frac{J}{g}}=1$$

by subtracting (3.6.16) from (3.6.17), implying

$$\frac{f^2}{g}\left(\alpha\frac{f'}{f}-\beta\frac{g'}{g}\right)\sim-\frac{J}{g},$$

or equivalently

$$\frac{f^2}{g}\left(\log\left(\frac{f^\alpha}{g^\beta}\right)\right)'\sim-\frac{J}{g}>0.$$

Therefore the function $\log\left(\frac{f^\alpha}{g^\beta}\right)$ is increasing, which implies that $\left(\frac{f^\alpha}{g^\beta}\right)' > 0$. Hence for some positive constant K

$$\frac{f^\alpha}{g^\beta} > K,$$

or more simply

$$\frac{f^2}{g} > K g^{\frac{2\beta}{\alpha}-1}. \tag{3.6.18}$$

By (3.6.14) and (3.6.10)

$$-\beta \frac{f^2}{g} g' \sim -J, \tag{3.6.19}$$

so that

$$0 \leq -g' < K\frac{g}{f^2} \tag{3.6.20}$$

for a positive constant K. Combining (3.6.18) and (3.6.20) we have

$$-g' < K g^{1-\frac{2\beta}{\alpha}}$$

which implies

$$g^{\frac{2\beta}{\alpha}-1} g' > -K.$$

Thus

$$\left| g^{\frac{2\beta}{\alpha}-1} g' \right| < K.$$

Integrating up to t_0 gives

$$\left| \int_t^{t_0} \left(g^{\frac{2\beta}{\alpha}}\right)' ds \right| \leq \int_t^{t_0} \left| \left(g^{\frac{2\beta}{\alpha}}\right)' \right| ds < K(t_0 - t)$$

so that

$$g^{\frac{2\beta}{\alpha}} < K(t_0 - t)$$

or

$$g < K(t_0 - t)^{\frac{\alpha}{2\beta}}. \tag{3.6.21}$$

Multiplying (3.6.13) by g,

$$f^2 f' + (1-\mu) t f g = g \int_{t_0}^{t} \left(g - (3\mu - 1) f\right) ds$$

we find that

$$\left(\frac{f^3}{3}\right)' \asymp (1-\mu) t_0 \int_t^{t_0} f g ds + \frac{1}{2}\left(\left(\int_{t_0}^{t} g ds\right)^2\right)'.$$

Now integrating to t_0 we find using (3.6.21)

$$\frac{f^3}{3} \leq (1-\mu)t_0 f \int_t^{t_0} g ds + \frac{1}{2}\left(\int_t^{t_0} g^2 ds\right)^2 < K_1 f\,(t_0-t)^{\frac{\alpha}{2\beta}+1} + K_2\,(t_0-t)^{\frac{\alpha}{\beta}+2} \tag{3.6.22}$$

for positive constants K_1 and K_2.

If

$$K_1 f\,(t_0-t)^{\frac{\alpha}{2\beta}+1} > K_2\,(t_0-t)^{\frac{\alpha}{\beta}+2},$$

then by (3.6.22)

$$f^3 \leq K f\,(t_0-t)^{\frac{\alpha}{2\beta}+1},$$

or

$$f \leq K\,(t_0-t)^{\frac{\alpha}{4\beta}+\frac{1}{2}} \tag{3.6.23}$$

for some constant K. If, on the other hand,

$$K_2\,(t_0-t)^{\frac{\alpha}{\beta}+2} < K_1 f\,(t_0-t)^{\frac{\alpha}{2\beta}+1},$$

then

$$f^3 \leq K\,(t_0-t)^{\frac{\alpha}{\beta}+2}$$

and thus

$$f \leq K\,(t_0-t)^{\frac{\alpha}{3\beta}+\frac{2}{3}}$$

for some constant K. We note that if

$$f = O\left((t_0-t)^{\frac{\alpha}{3\beta}+\frac{2}{3}}\right)$$

as $t \to t_0$, then indeed it still is true that (3.6.23) holds and so

$$f = O\left((t_0-t)^{\frac{\alpha}{4\beta}+\frac{1}{2}}\right).$$

This is because $\frac{\alpha+2\beta}{3\beta} > \frac{\alpha+2\beta}{4\beta}$ and thus

$$(t_0-t)^{\frac{\alpha+2\beta}{3\beta}} < (t_0-t)^{\frac{\alpha+2\beta}{4\beta}}$$

for t close to t_0. Therefore, for either case we have that

$$f \leq K(t_0-t)^{\frac{\alpha+2\beta}{4\beta}}$$

and from (3.6.15) for $t \to t_0$,

$$\beta\frac{f^2}{g}g' \sim J$$

$$\frac{J}{f^2} < -K(t_0 - t)^{-\frac{\alpha+2\beta}{2\beta}}$$

and so

$$\log(g^\beta)' = \beta\frac{g'}{g} < -K(t_0 - t)^{-\frac{\alpha+2\beta}{2\beta}}.$$

By integrating both sides

$$g < e^{-K(t_0-t)^{-\frac{\alpha}{2\beta}}}$$

so that as $t \to t_0$

$$f \le Mg < Me^{-K(t_0-t)^{-\frac{\alpha}{2\beta}}}.$$

However, from (3.6.13) we had that

$$\alpha\frac{f^2}{g}f' + (1-\mu)tf < 0$$

$$\alpha f' < -(1-\mu)\frac{g}{f}t,$$

and so

$$f > K(t_0 - t).$$

We have a contradiction and therefore, it is not possible for $J < 0$, and thus $J = 0$. Now from (2.0.3)

$$\beta\frac{f^2}{g}g'(t_0) = \int_0^{t_0} \gamma g\left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) dt = 0,$$

we have that

$$\beta\frac{f^2}{g}g'(t_0) = 0.$$

4.0 PARTIAL NUMERICS

Using XPPAUT 5.85 we graph solutions (f, g) to (1.0.1)-(1.0.4), which are found using the following steps:

- fixing values of α, β, γ, and $\theta(0)$ subject to the conditions (2.1.1)-(2.1.2).
- integrating over a proper $\mu-$range determined from (2.1.3) and LEMMA 3.
- identifying μ and $t_0 < \infty$ such that $f' \leq 0$, $g' \leq 0$, for $t < t_0$, (1.0.3) hold, and $f(t_0) = 0$, $g(t_0) = 0$.
- rescaling t_0 to become 1.

The numerics will be partially incomplete, since we do not specify that the conditions $\frac{f^2}{g} f'(1) = 0$ and $\frac{f^2}{g} g'(1) = 0$ are satisfied. We begin by rewriting (1.0.1)-(1.0.2) as a first order system:

$$\begin{aligned} f_1' &= f_2 \\ f_2' &= \tfrac{g_1^2}{\alpha f_1^2} - \tfrac{2\mu g_1}{\alpha f_1} - \tfrac{(1-\mu) t f_2 g_1}{\alpha f_1^2} - \tfrac{2 f_2^2}{f_1} + \tfrac{f_2 g_2}{g_1} \\ g_1' &= g_2 \\ g_2' &= \tfrac{\gamma g_1^3}{\beta f_1^3} - \tfrac{(1+2\mu) g_1^2}{\beta f_1^2} - \tfrac{(1-\mu) t g_1 g_2}{\beta f_1^2} - \tfrac{2 f_2 g_2}{f_1} + \tfrac{g_2^2}{g_1} \end{aligned}$$

where $(f_1, f_2, g_1, g_2) = (f, f', g, g')$. From (2.1.2) we saw that

$$\frac{\gamma}{3} < \theta(0) < \frac{\alpha\gamma - \beta}{3\alpha - 2\beta}.$$

For instance if we pick $\alpha = 1$, $\beta = 1.3$, and $\gamma = 2$ subject to (2.1.1), then $f(0)$ and $g(0)$ should be chosen such that

$$\frac{2}{3} < \frac{f(0)}{g(0)} < \frac{7}{4}.$$

Indeed if $\alpha = .1$, $\beta = .13$, and $\gamma = 2$, the same bound for $\theta(0)$ still holds. We will choose $f(0) = 1.4$ and $g(0) = 1$. Additionally from (2.1.3) we find that

$$\frac{1}{3} < \mu < 1$$

and from LEMMA 3 that

$$\mu > \frac{\alpha\,(\gamma+1)-\beta}{\alpha\,(3\gamma-2)-\beta} = \frac{17}{27}.$$

However solving for μ in (2.1.2), we actually find that

$$\mu > \frac{\alpha\gamma-\beta-\theta(0)\alpha}{2\theta(0)(\alpha-\beta)} = \frac{35}{42}.$$

Thus

$$.8\bar{3} < \mu < 1. \tag{4.0.1}$$

The following are examples of the ode files used when $\alpha = 1$, $\beta = 1.3$, $\gamma = 2$, and $\theta(0) = 1.4$:

```
#bvp1.ode
par alpha=1, beta=1.3, gamma=2, mu=.834
f1'=f2
f2'=g1^2/(alpha*f1^2)-2*mu*g1/(alpha*f1)-(1-mu)*t*g1*f2/(alpha*f1^2)-(2*f2^2/f1)+(f2*g2)/g1
g1'=g2
g2'=gamma*g1^3/(beta*f1^3)-(1+2*mu)*g1^2/(beta*f1^2)-(1-mu)*t*g1*g2/(beta*f1^2)-(2*f2*g2)/f1+(g2^2/g1)
bndry f1'
bndry g1'
bndry f2
bndry g2
init f1=1.4
init f2=0
init g1=1
init g2 =0
@ dt=.001, bell=0, total=2, xhi=2, yhi=1
done
```

and

```
#bvp2.ode
par alpha=1, beta=1.3, gamma=2, mu=.834
f1'=f2
f2'=g1^2/(alpha*f1^2)-2*mu*g1/(alpha*f1)-(1-mu)*t*g1*f2/(alpha*f1^2)-(2*f2^2/f1)+(f2*g2)/g1
g1'=g2
g2'=gamma*g1^3/(beta*f1^3)-(1+2*mu)*g1^2/(beta*f1^2)-(1-mu)*t*g1*g2/(beta*f1^2)-(2*f2*g2)/f1+(g2^2/g1)
bndry f1'
bndry g1'
bndry f1'*f1'*f2'/g1'
bndry f1'*f1'*g2'/g1'
init f1=1.4
init f2=0
init g1=1
init g2 =0
@ dt=.001, bell=0, total=2, xhi=2, yhi=1
done
```

Note that a value of μ must be specified in the parameter declaration portion of the code. The value chosen above is simply the smallest possible value μ may attain based on (4.0.1). We then run the codes integrating over the range of possible $\mu-$values, with a time step of $dt = .001$. The code produces the following results when varying $.84 < \mu < .9$:

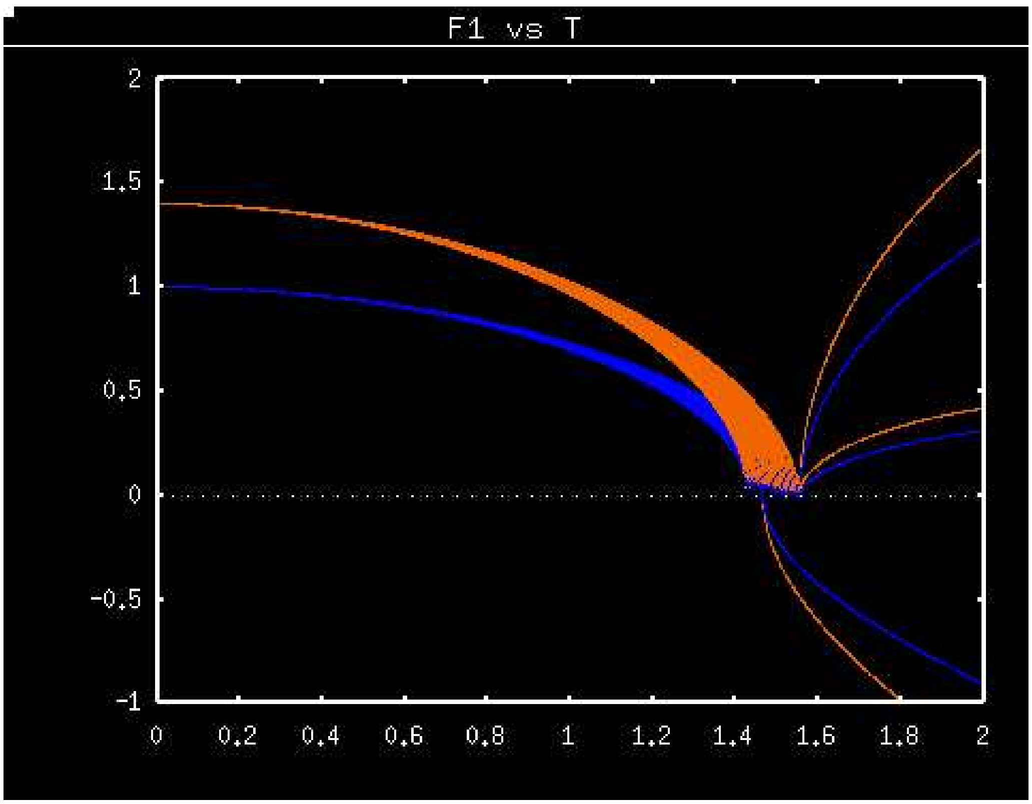

Figure 9: Result after integrating over 0.84 < mu < 0.9

It is particularly easy to identify the curves (f, g) resulting when $\mu = .88$:

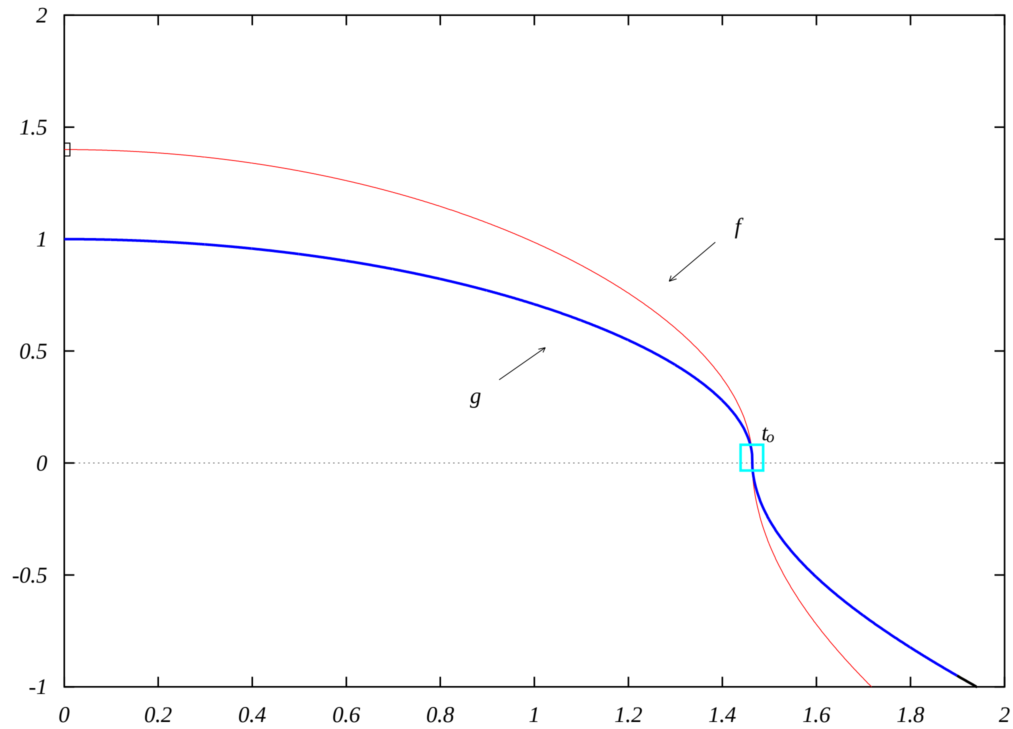

Figure 10: Setting mu = 0.88

Where the graphs approach zero, the data is summarized by the following:

Table 1: Data for mu = 0.88

t	f	f'	g	g'	$\frac{f^2}{g}f'$	$\frac{f^2}{g}g'$
1.46	.0956	-12.2155	.0757	-8.7903	-1.4728	-1.0598
1.461	.0824	-14.3608	.0662	-10.3379	-1.47059	-1.05863
1.462	.0663	-18.2643	.0547	-13.1537	-1.4681	-1.0573
1.4630001	.0439	-29.3803	.0385	-21.1726	-1.4675	-1.0576
1.464	-.0401	-36.1895	-.02206	-26.06894	2.6421	1.8891

We rescale by noting that $t_0 \approx 1.46375$, but as $t_0 \to a^{\frac{1}{2}} t_0 = 1$ then $a = \frac{1}{t_0^2}$. Since $f \to af$ and $g \to ag$, then the mapping affects the initial condition $\theta(0)$ by

$$f(0) \to 1.4a \text{ and } g(0) \to a$$

with α, β, γ, and μ unchanged. The following is the plot which results from rescaling the initial conditions:

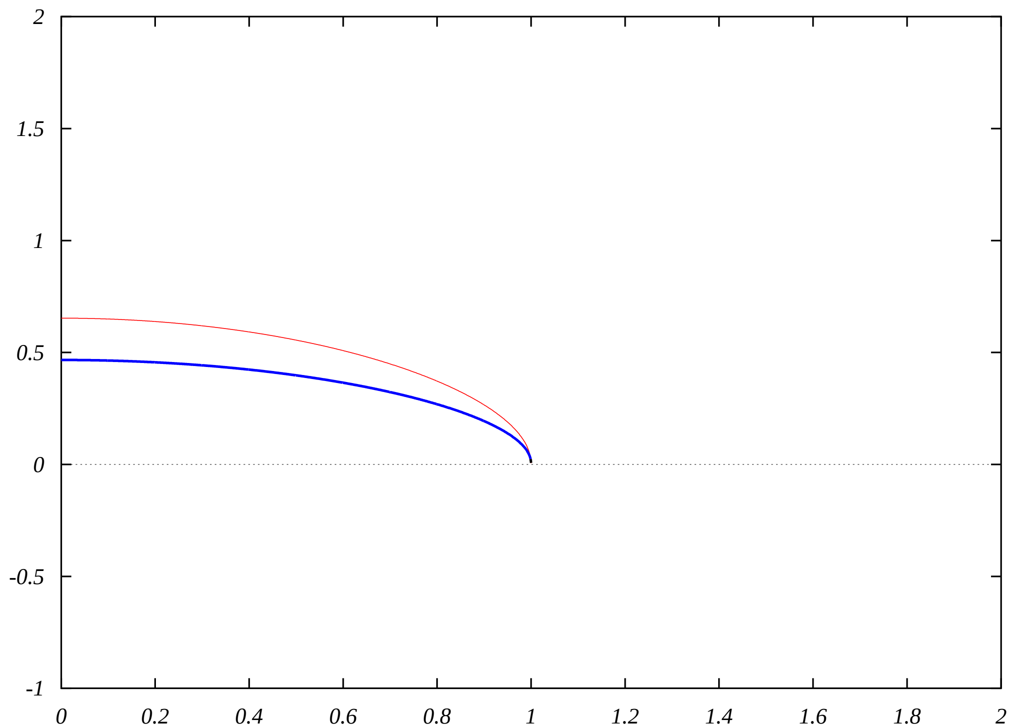

Figure 11: Rescaled solutions

Also included is a plot of the rescaled data together with the unscaled curves:

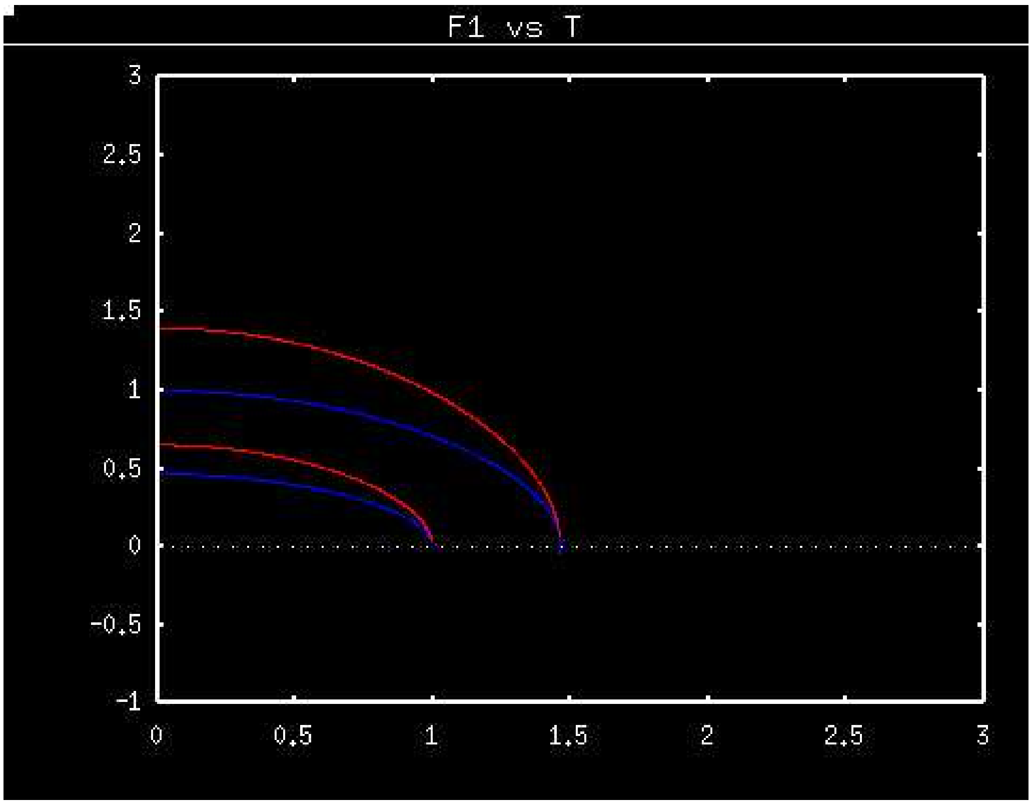

Figure 12: Rescaled and original graphs

Additionally we demonstrate the requirement of $\frac{\alpha}{\beta}$ being neither too big nor too small, by fixing α, γ, $\theta(0)$, and μ as above and integrating over an interval of $\beta-$values. To illustrate what happens when we vary, for example, $\beta \in [0.1,\ 3]$ in the prior example, the results are:

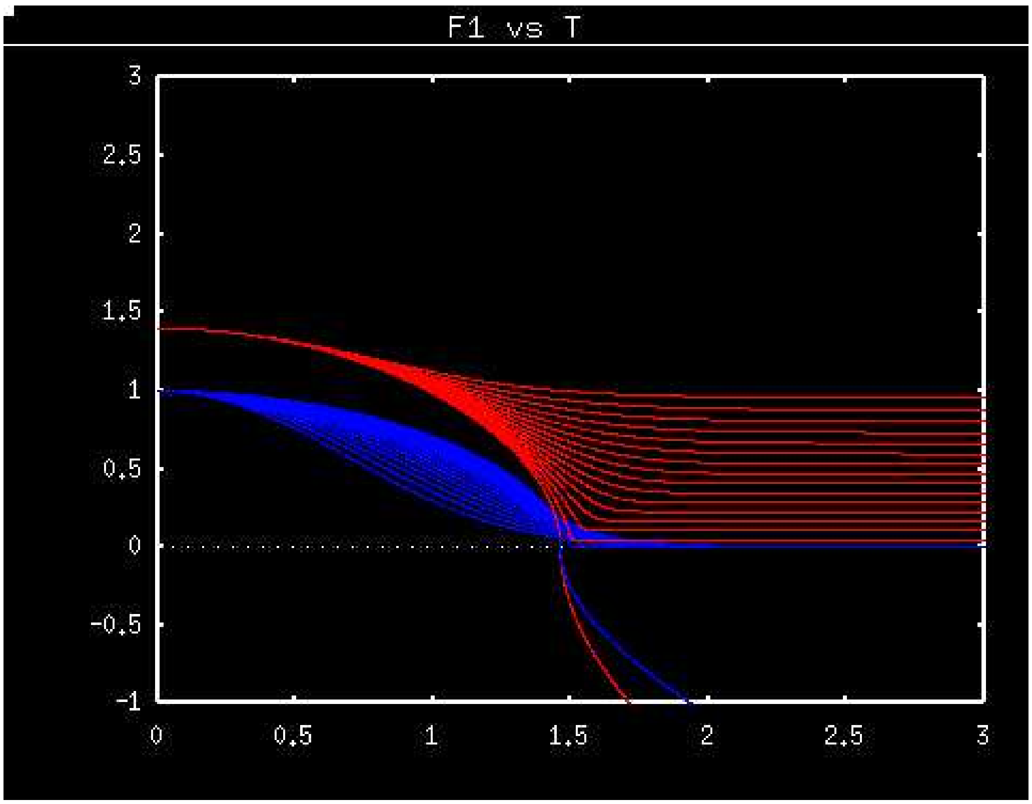

Figure 13: Result from varying beta over [0.1,1.3]

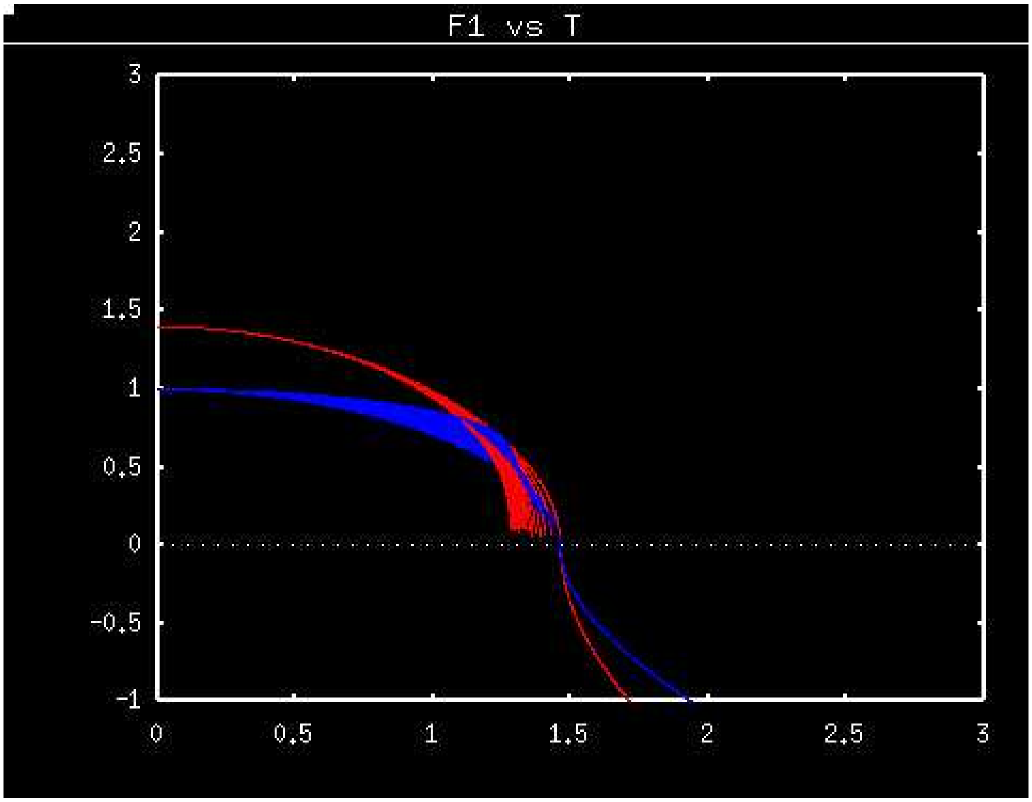

Figure 14: Result from varying beta over [1.3,3]

It is clear from the varied behavior in the blue and red curves that solutions (f, g) to (1.0.1)-(1.0.4) do not exist when we vary β too far from 1.3.

5.0 CONCLUSION

In summary, for the case of $\alpha < \beta$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, we have proven the existence of a solution (f, g) to (1.0.1)-(1.0.4) by finding $(\mu, \theta(0))$ using a shooting technique developed through a series of lemmas. Additionally, we have provided graphs of (f, g) obtained by using numerical shooting with XPPAUT.

To complete the case of $\alpha \neq \beta$, it remains to consider $\alpha > \beta$.

Bibliography

[1] G.I. Barenblatt, *Self-similar turbulence propagation from an instantaneous plane source,* in Nonlinear Dynamics and Turbulence, G.I. Barenblatt, G. Iooss, D.D. Joseph, eds., Pitman, Boston, 1983, 48-60.

[2] G.I. Barenblatt, N.L. Galerkina, and M.V. Luneva, *Evolution of a turbulent burst,* Inzherno-Fizicheskii Zh., 53, 1987, 773-740.

[3] M. Bertsch, R. Dal Passo, and R. Kersner, *The evolution of turbulent bursts: the $b-\varepsilon$ model,* European J. Appl. Math., 5, 1994, 537-557.

[4] A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Fluid Dynamics,* Springer-Verlag, 1979.

[5] K. Hanjalič and B.E. Lauder, *A Reynolds stress model of turbulence and its applications to thin shear flows,* J. Fluid Mech., 52, 1974, 609-638.

[6] F.H. Harlow and P.I. Nakayama, *Transport of turbulence energy decay rate,* Los Alamos Sci. Lab., LA-3854, 1968.

[7] S.P. Hastings and L.A. Peletier, *On a self-similar solution for the decay of turbulent bursts,* European. J. Appl. Math., 3, 1992, 319-341.

[8] Josephus Hulshof, *Self-similar solutions of Barenblatt's model for turbulence,* SIAM J. Math. Anal., 28, No.1, 1997, 33-48.

[9] W.P. Jones and B.E. Launder, *The prediction of laminarization with a two-equation model of turbulence*, Int. J. of Heat Mass Tran., Vol 15, 1972, 301-304.

[10] A.N. Kolmogorov, *Equations of turbulent motion of an incompressible field,* Izv. Akad. Nauk SSSR, Seria fizicheska, VI, 1942, 55-58. Translated from the original Russian by D.B. Spalding, Proc. Roy. Soc. London Ser. A., 434, 1991, 214-215

[11] B.E. Launder, *Kolmogorov's two-equation model of turbulence*, Proc. Roy. Soc. London Ser. A., 434, 1991, 214-214.

[12] B.E. Launder, A.P. Morse, W. Rodi, and D.B. Spalding, *Prediction of free shear flows-a comparison of six turbulence models,* NASA SP, 321, 1972.

[13] B.E. Launder and D.B. Spalding, *Lectures in Mathematical Models of Turbulence,* Academic Press, 1972.

[14] B.E. Launder and D.B. Spalding, *The numerical computation of turbulent flows,* Computer Math. in Appl. Mech. Eng., 3, 1974, 269-289.

[15] J.B. McLeod and James Serrin, *The existence of similar solutions for some laminar boundary layer problems*, Arch. Rational Mech. Anal., 31, 1968/1969, 288-303.

[16] B. Mohammadi and O. Pironneau, *Analysis of the K-Epsilon Turbulence Model,* John Wiley & Sons, 1994.

[17] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics*, *vol. 1*, 2, MIT Press, Cambridge, Mass., 1971, 1975.