# GENERALIZED TOPOLOGICAL SEMANTICS FOR FIRST-ORDER MODAL LOGIC

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Submitted to the Graduate Faculty of

the Department of Philosophy in partial fulfillment

of the requirements for the degree of

**Doctor of Philosophy**

University of Pittsburgh

2010

UNIVERSITY OF PITTSBURGH

DEPARTMENT OF PHILOSOPHY

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November 17th, 2010

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This dissertation provides a new semantics for first-order modal logic. It is philosophically motivated by the epistemic reading of modal operators and, in particular, three desiderata in the analysis of epistemic modalities.

(i) The semantic modelling of epistemic modalities, in particular verifiability and falsifiability, cannot be properly achieved by Kripke's relational notion of accessibility. It requires instead a more general, topological notion of accessibility.

(ii) Also, the epistemic reading of modal operators seems to require that we combine modal logic with fully classical first-order logic. For this purpose, however, Kripke's semantics for quantified modal logic is inadequate; its logic is free logic as opposed to classical logic.

(iii) More importantly, Kripke's semantics comes with a restriction that is too strong to let us semantically express, for instance, that the identity of Hesperus and Phosphorus, even if metaphysically necessary, can still be a matter of epistemic discovery.

To provide a semantics that accommodates the three desiderata, I show, on the one hand, how the desideratum (i) can be achieved with topological semantics, and more generally neighborhood semantics, for propositional modal logic. On the other hand, to achieve (ii) and (iii), it turns out that David Lewis's counterpart theory is helpful at least technically. Even though Lewis's own formulation is too liberal—in contrast to Kripke's being too restrictive—to achieve our goals, this dissertation provides a unification of the two frameworks, Kripke's and Lewis's. Through a series of both formal and conceptual comparisons of their ontologies and semantic ideas, it is shown that structures called sheaves are needed to unify the ideas and achieve the desiderata (ii) and (iii). In the end, I define a category of sheaves over a neighborhood frame with certain properties, and show that it provides a semantics that naturally unifies neighborhood semantics for propositional modal logic, on the one hand, and semantics for first-order logic on the other. Completeness theorems are proved.

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# I.0 MATHEMATICAL INTRODUCTION

In this short chapter, I briefly lay out, without proofs, the principal mathematical results of this dissertation. The precise, full exposition of them is found in later chapters, mostly in Chapter VI, along with proofs.

## I.1 NEIGHBORHOOD SEMANTICS FOR PROPOSITIONAL MODAL LOGIC

To describe it in mathematical terms, the chief result of this dissertation is to extend neighborhood semantics for propositional modal logic to first-order modal logic. In this section, we lay out neighborhood semantics for propositional modal logic to prepare ourselves for the extension.

### I.1.1 Basic Definition

Let us fix a propositional modal language $\mathcal{L}$, that is, a language obtained by adding unary sentential operators $\Box$ and $\Diamond$, called *modal operators*, to any language of classical propositional logic.

Neighborhood semantics can be regarded as a kind of possible-world semantics, in the sense that it interprets $\mathcal{L}$ with a structure that consists of

- a set $X \neq \varnothing$, and
- a map $[\![-]\!] : \mathrm{sent}(\mathcal{L}) \to \mathcal{P}X$, where $\mathrm{sent}(\mathcal{L})$ is the set of sentences of $\mathcal{L}$,

among other things. We may call points in $X$ *possible worlds*, and subsets of $X$ *propositions*, so that we can read $w \in [\![\varphi]\!]$ as meaning that $\varphi$ is true at $w$. In a manner coherent to this reading, we define validity in $[\![-]\!]$ (or in a suitable tuple such as $(X, [\![-]\!])$) in the following manner. Note that we take binary sequents as units of validity; so, accordingly, we will consider formulations of logic in which a logic or theory proves binary sequents.

- A binary sequent $\varphi \vdash \psi$ is valid in $[\![-]\!]$ if $[\![\varphi]\!] \subseteq [\![\psi]\!]$. By the validity of a sentence $\varphi$, we mean the validity of $\top \vdash \varphi$, where $[\![\top]\!] = X$—that is, $\varphi$ is valid in $[\![-]\!]$ if $[\![\varphi]\!] = X$.
- An inference $(\Gamma, (\varphi \vdash \psi))$—deriving a sequent $\varphi \vdash \psi$ from premises $\Gamma$ of sequents—is valid in $[\![-]\!]$ if it preserves validity, that is, if either $[\![\varphi_i]\!] \nsubseteq [\![\psi_i]\!]$ for some sequent $\varphi_i \vdash \psi_i$ in $\Gamma$ or $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

$$\text{Propositional logic} \rightsquigarrow X$$
$$\varphi \rightsquigarrow [\![\varphi]\!] \subseteq X$$

We can extend $[\![-]\!]$ to interpret sentential operators, so that, for each $n$-ary operator $\otimes$, we have $[\![\otimes]\!] : (\mathcal{P}X)^n \to \mathcal{P}X$ and then

$$[\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]) = [\![\otimes(\varphi_1, \ldots, \varphi_n)]\!].$$

For the logic to have its non-modal base classical, we set

$$\begin{aligned}
[\![\neg]\!] &= X \setminus -, &\quad \text{so that} \quad && [\![\neg\varphi]\!] &= X \setminus [\![\varphi]\!];\\
[\![\wedge]\!] &= \cap, &\quad \text{so that} \quad && [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\varphi]\!];\\
[\![\vee]\!] &= \cup, &\quad \text{so that} \quad && [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\varphi]\!];\\
[\![\to]\!] &= \to, &\quad \text{so that} \quad && [\![\varphi \to \psi]\!] &= [\![\varphi]\!] \to [\![\varphi]\!].
\end{aligned}$$ [1]

What is characteristic of neighborhood semantics is to further equip $X$ with

- a map $\mathcal{N} : X \to \mathcal{P}\mathcal{P}X$,

called a *neighborhood function*. Such a map $\mathcal{N}$ is mathematically equivalent to

- an operation **int** : $\mathcal{P}X \to \mathcal{P}X$,

[1] The binary operation $\to : (\mathcal{P}X)^2 \to \mathcal{P}X$ is such that $A \to B = (X \setminus A) \cup B$.

called an *interior operation*, via the correspondence

$$w \in \mathbf{int}(A) \iff A \in \mathcal{N}(w) \tag{I.1}$$

for every $A \subseteq X$ and $w \in X$. We assume no constraint at all for $\mathcal{N}$ or **int**, though we will consider a few in Subsection I.1.2 (and some turn out essential for the extension of neighborhood semantics to the first-order modal logic). Any pair $(X, \mathcal{N})$ of the type above is called a *neighborhood frame*.

Over such a structure $(X, \mathcal{N})$, the modal operator $\Box$ is interpreted by the interior operation **int** defined by $\mathcal{N}$. That is,

$$[\![\Box]\!] = \mathbf{int}, \qquad \text{so that} \qquad [\![\Box\varphi]\!] = \mathbf{int}([\![\varphi]\!]),$$

which means, by (I.1), that

$$w \in [\![\Box\varphi]\!] \iff [\![\varphi]\!] \in \mathcal{N}(w);$$

thus, when $\Box$ is read as "necessarily", $\mathcal{N}(w)$ amounts to the family of propositions necessarily true at $w$. The operator $\Diamond$ is interpreted by a dual of **int**, the *closure* operation $\mathbf{cl} : \mathcal{P}X \to \mathcal{P}X$, such that

$$\mathbf{cl}(A) = X \setminus \mathbf{int}(X \setminus A);$$

that is,

$$[\![\Diamond]\!] = \mathbf{cl}, \qquad \text{so that} \qquad [\![\Diamond\varphi]\!] = X \setminus \mathbf{int}(X \setminus [\![\varphi]\!]).$$

Hence, with $\neg$ interpreted classically, that is, with $[\![\neg]\!] = X \setminus -$, $\Diamond$ can simply be defined as $\neg\Box\neg$. Any neighborhood frame equipped with $[\![-]\!]$ satisfying these conditions is called a *neighborhood model*, and neighborhood semantics is given by the class of all neighborhood models.

To describe the logic of neighborhood semantics, write E for the following rule.

$$\text{E} \qquad \frac{\varphi \vdash \psi \qquad \psi \vdash \varphi}{\Box\varphi \vdash \Box\psi}$$

This is valid in neighborhood semantics because, trivially, $[\![\varphi]\!] = [\![\psi]\!]$ implies $\mathbf{int}([\![\varphi]\!]) = \mathbf{int}([\![\psi]\!])$. Therefore modal logic **E** obtained by adding E to classical propositional logic is sound with respect to neighborhood semantics; and, indeed, it is also complete, in the following strong form:

**Theorem** (Scott [37], Montague [32], Segerberg [39])**.** *For any consistent theory $\mathbb{T}$ containing* **E**, *there exists a neighborhood model $(X, \mathcal{N}, [\![-]\!])$ that validates all and only theorems of $\mathbb{T}$.*

### I.1.2 Some Conditions on Neighborhood Frames

Though any set $X$ can be paired with any arbitrary map $\mathcal{N} : X \to \mathcal{PP}X$ and $(X, \mathcal{N})$ forms a neighborhood frame, we may consider conditions that $\mathcal{N}$ should satisfy. Many of them are directly reflected in the modal logic of the class of frames satisfying them.

For instance, consider

$$A \subseteq B \subseteq X \text{ and } A \in \mathcal{N}(w) \implies B \in \mathcal{N}(w), \tag{I.2}$$

$$A, B \in \mathcal{N}(w) \implies A \cap B \in \mathcal{N}(w), \tag{I.3}$$

$$X \in \mathcal{N}(w), \tag{I.4}$$

$$A \in \mathcal{N}(w) \implies w \in A, \tag{I.5}$$

$$A \in \mathcal{N}(w) \implies \mathbf{int}(A) \in \mathcal{N}(w). \tag{I.6}$$

It is easy to see that these are the case iff

$$A \subseteq B \implies \mathbf{int}(A) \subseteq \mathbf{int}(B),$$

$$\mathbf{int}(A) \cap \mathbf{int}(B) \subseteq \mathbf{int}(A \cap B),$$

$$\mathbf{int}(X) = X,$$

$$\mathbf{int}(A) \subseteq A,$$

$$\mathbf{int}(A) \subseteq \mathbf{int}(\mathbf{int}(A)),$$

respectively. This immediately gives a correspondence result: For each of (I.2)–(I.6), a neighborhood frame $(X, \mathcal{N})$ satisfies it iff its corresponding rule or axiom below is valid in all models

$(X, \mathcal{N}, [\![-]\!])$ over $(X, \mathcal{N})$.[2]

M $$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

C $$\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$$

N $$\frac{\vdash \varphi}{\vdash \Box\varphi}$$

T $$\Box\varphi \vdash \varphi$$

4 $$\Box\varphi \vdash \Box\Box\varphi$$

We should observe that (I.2)–(I.6) together characterize topology, in the sense that the topological spaces are exactly the neighborhood frames satisfying (I.2)–(I.6). To describe the details, on the one hand, every topological space $(X, \mathcal{O}X)$, where $\mathcal{O}X \subseteq \mathcal{P}X$ is the family of its open sets, comes with an interior operation $\mathbf{int}_{\mathcal{O}X} : \mathcal{P}X \to \mathcal{P}X$ and a neighborhood function $\mathcal{N}_{\mathcal{O}X} : X \to \mathcal{P}\mathcal{P}X$, by

$$A \in \mathcal{N}_{\mathcal{O}X}(w) \overset{\text{(I.1)}}{\iff} w \in \mathbf{int}_{\mathcal{O}X}(A) \iff w \in U \subseteq A \text{ for some } U \in \mathcal{O}X.$$

And (I.2)–(I.6) for $\mathcal{N}_{\mathcal{O}X}$ follow straightforwardly from the assumption that $\mathcal{O}X$ is a topology. On the other hand, for any neighborhood frame $(X, \mathcal{N})$ satisfying (I.2)–(I.6), it is easy to show that the family of images of the corresponding **int**, that is,

$$\mathcal{O}_{\mathcal{N}}X = \{\, \mathbf{int}(A) \mid A \subseteq X \,\},$$

is a topology. Moreover, these operations $(X, \mathcal{O}X) \mapsto (X, \mathcal{N}_{\mathcal{O}X})$ and $(X, \mathcal{N}) \mapsto (X, \mathcal{O}_{\mathcal{N}}X)$ are inverse to each other.[3] This correspondence extends to semantics, because topological semantics interprets $\Box$ with topological interior operations (and $\Diamond$ with closure operations); thus, topological semantics is subsumed by neighborhood semantics, being just neighborhood semantics with (I.2)–(I.6).

For the purpose of this dissertation, (I.2) and (I.3) are the most crucial conditions. Soundness and completeness results extend to the logics **MC** and **S4** obtained by adding M, C and M, C, N, T, 4, respectively, to classical propositional logic.

[2] We let $\vdash \varphi$ be short for $\top \vdash \varphi$.

[3] This extends to an isomorphism between the categories of topological spaces and of neighborhood frames that satisfy (I.2)–(I.6), once we define continuous maps between neighborhood frames in Subsection I.4.2.

**Theorem 1** (Segerberg [39]). *For any consistent theory $\mathbb{T}$ extending* **MC***, there exists a neighborhood model* $(X, \mathcal{N}, [\![-]\!])$ *with* $\mathcal{N}$ *satisfying* (I.2) *and* (I.3) *that validates all and only theorems of* $\mathbb{T}$.

**Theorem 2** (McKinsey-Tarski [30]). *For any consistent theory $\mathbb{T}$ extending* **S4***, there exists a topological model* $(X, \mathcal{O}X, [\![-]\!])$ *that validates all and only theorems of* $\mathbb{T}$.

## I.2 SEMANTICS FOR FIRST-ORDER LOGIC

This dissertation aims at extending neighborhood semantics to first-order modal logic. In this section, we introduce a notation for the standard semantics of first-order non-modal logic that will be convenient for the purpose of this extension.

### I.2.1 Denotational Interpretation

Fix any classical first-order language $\mathcal{L}$; it has primitive predicates $R_i$ ($i \in I$), function symbols $f_j$ ($j \in J$), and (individual) constants $c_k$ ($k \in K$). Then, as usual, an *$\mathcal{L}$ structure* $\mathfrak{M} = (D, R_i^{\mathfrak{M}}, f_j^{\mathfrak{M}}, c_k^{\mathfrak{M}})_{i \in I, j \in J, k \in K}$ consists of the following.

- a set $D$, called the *domain of individuals*;
- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq D^n$ of the $n$-fold Cartesian product of the domain $D$;
- for each $n$-ary function symbol $f$, a map $f^{\mathfrak{M}} : D^n \to D$; and
- for each constant $c$, an individual $c^{\mathfrak{M}} \in D$.

Given such a structure $\mathfrak{M}$, we recursively define the the relation of satisfaction as usual, so that

$$\mathfrak{M} \models_{[a_1,\dots,a_n/x_1,\dots,x_n]} \varphi$$

means that, in $\mathfrak{M}$, an open sentence $\varphi$ is true of elements $a_1, \dots, a_n \in D$, with $a_i$ in place of the free variable $x_i$. This notation makes sense only if no variables occur freely in $\varphi$ except $x_1, \dots, x_n$. We will write $\bar{a}$ and $\bar{x}$ for tuples (that are $n$-ary, unless noted otherwise).

Now we extend the "denotational" point of view to first-order languages. Whereas we gave an interpretation $[\![\varphi]\!]$ to sentences $\varphi$ in Section I.1, here for first-order logic we give an interpretation also to formulas containing free variables; so we extend the notation to include interpretations

$$[\![\, \bar{x} \mid \varphi \,]\!]$$

of all sentences, closed or open. Again, this notation makes sense only if no variables occur freely in $\varphi$ except $\bar{x}$; but not all of $\bar{x}$ may actually occur in $\varphi$. We also give interpretation $[\![\, \bar{x} \mid t \,]\!]$ to a term $t(\bar{x})$ built up from constants and variables with function symbols.

$$\begin{aligned} \text{First-order logic} &\rightsquigarrow \mathfrak{M} \\ \varphi(\bar{x}) &\rightsquigarrow [\![\, \bar{x} \mid \varphi \,]\!] \subseteq D^n \end{aligned}$$

The interpretation of an open sentence $\varphi$ is essentially the subset of the model $\mathfrak{M}$ defined by $\varphi$:

$$[\![\, \bar{x} \mid \varphi \,]\!] = \{\, \bar{a} \in D^n \mid \mathfrak{M} \vDash_{[\bar{a}/\bar{x}]} \varphi \,\} \subseteq D^n.$$

That is, the set of tuples satisfying $\varphi$. Then the following properties are easily derived:

$$\begin{aligned}
[\![\, \bar{x} \mid R\bar{x} \,]\!] &= R^{\mathfrak{M}} && \text{for } n\text{-ary primitive predicate } R \text{, and} \\
[\![\, x, y \mid x = y \,]\!] &= \{\, (a, a) \in D \times D \mid a \in D \,\} && \text{in particular;} \\
[\![\, \bar{x} \mid \top \,]\!] &= D^n; \\
[\![\, \bar{x} \mid \neg\varphi \,]\!] &= D^n \setminus [\![\, \bar{x} \mid \varphi \,]\!] && (\text{that is, } [\![\neg]\!] = D^n \setminus -); \\
[\![\, \bar{x} \mid \varphi \wedge \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \cap [\![\, \bar{x} \mid \psi \,]\!] && (\text{that is, } [\![\wedge]\!] = \cap); \\
[\![\, \bar{x} \mid \varphi \vee \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \cup [\![\, \bar{x} \mid \psi \,]\!] && (\text{that is, } [\![\vee]\!] = \cup); \\
[\![\, \bar{x} \mid \varphi \to \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \to [\![\, \bar{x} \mid \psi \,]\!] && (\text{that is, } [\![\to]\!] = \to); \\
[\![\, \bar{x} \mid \forall y \,.\, \varphi \,]\!] &= \{\, \bar{a} \in D^n \mid (\bar{a}, b) \in [\![\, \bar{x}, y \mid \varphi \,]\!] \text{ for every } b \in D \,\}; \\
[\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] &= \{\, \bar{a} \in D^n \mid (\bar{a}, b) \in [\![\, \bar{x}, y \mid \varphi \,]\!] \text{ for some } b \in D \,\}.
\end{aligned}$$

These properties can also be used as conditions to *define* the interpretation recursively, skipping $\vDash$ altogether. In doing so, we need to define $[\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!] \subseteq D^{n+1}$ also for a sentence $\varphi(\bar{x})$ in which

$y$ does not actually occur freely, so that we can define, for instance, $[\![\, \bar{x}, y \mid \varphi(\bar{x}) \wedge \psi(\bar{x}, y) \,]\!] \subseteq D^{n+1}$ as the intersection of $[\![\, \bar{x}, y \mid \psi(\bar{x}, y) \,]\!] \subseteq D^{n+1}$ with $[\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!]$. Yet it can be done simply by

$$\begin{aligned} [\![\, \bar{x}, y \mid \varphi \,]\!] &= \{\, (\bar{a}, b) \in D^{n+1} \mid \mathfrak{M} \vDash_{[\bar{a}/\bar{x}]} \varphi \,\} \\ &= [\![\, \bar{x} \mid \varphi \,]\!] \times D. \end{aligned}$$

Similarly, when a term $t(\bar{x})$ has $n$ arguments, its interpretation $[\![\, \bar{x} \mid t \,]\!]$ is the function $f : D^n \to D$ recursively defined from $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$ in the expected way.

This definition covers the case of $n = 0$ naturally, with $D^0 = \{*\}$, any one-element set. That is, while an open sentence $\varphi$ is interpreted with a subset $[\![\, \bar{x} \mid \varphi \,]\!]$ of $D^n$, the interpretation of a closed sentence $\sigma$ is in a similar manner given as a subset $[\![\sigma]\!]$ of $D^0$ (a "truth value") as follows.

$$[\![\sigma]\!] = \{\, * \in D^0 \mid \mathfrak{M} \vDash \sigma \,\} = \begin{cases} 1 = \{*\} = D^0 & \text{if } \mathfrak{M} \vDash \sigma, \\ 0 = \varnothing \subseteq D^0 & \text{if } \mathfrak{M} \nvDash \sigma. \end{cases}$$

We define validity in $\mathfrak{M}$ in a manner similar to the definition in Section I.1. That is, $\varphi \vdash \psi$ is valid in $\mathfrak{M}$ iff $[\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$, where no variables occur freely in $\varphi$ or $\psi$ except $\bar{x}$. In particular, $\varphi$ is valid in $\mathfrak{M}$ iff $[\![\, \bar{x} \mid \varphi \,]\!] = D^n$. An inference is valid iff it preserves validity of sequents. Now, in terms of $[\![-]\!]$, the usual soundness and completeness of first-order logic are expressed as follows.

**Theorem.** *Given a language $\mathcal{L}$ of first-order logic, for any pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$ in which no variables occur freely except $\bar{x}$,*

$$\varphi \vdash \psi \textit{ is provable} \iff \textit{every } \mathcal{L} \textit{ structure } \mathfrak{M} \textit{ has } [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!].$$

### I.2.2 Interpretation and Images

We saw in Subsection I.2.1 that Boolean connectives can be interpreted with Boolean operations on sets, such as $[\![\wedge]\!] = \cap$. We can extend this insight by observing that other syntactic operations can be interpreted with images of maps. We sum up this fact in this subsection, because it will later play a crucial role.

First let us introduce some notation for images. Given a map $f : X \to Y$ and subsets $A \subseteq X$ and $B \subseteq Y$, the *direct image* of $A$ and the *inverse image* of $B$ under $f$ shall be written as

$$f[A] = \{\, f(a) \in Y \mid a \in A \,\},$$
$$f^{-1}[B] = \{\, a \in X \mid f(a) \in B \,\}.$$

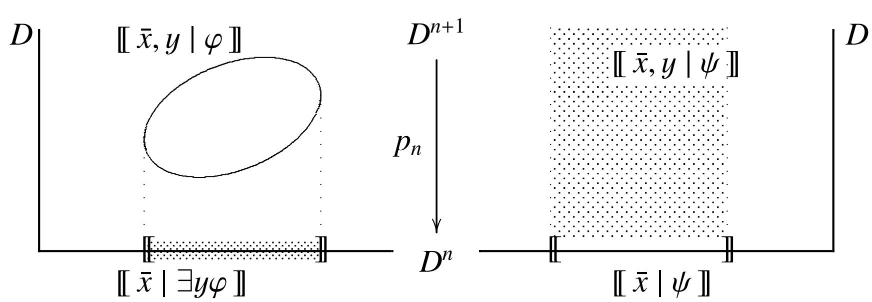
respectively. We also define, for each $n$, the *projection*

$$p_n : D^{n+1} \to D^n :: (\bar{a}, b) \mapsto \bar{a};$$

in particular, $p_0 : D \to D^0 = \{*\}$ has $p_0(b) = *$ for all $b \in D$.

Then we have

$$[\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] = \{\, \bar{a} \in D^n \mid (\bar{a}, b) \in [\![\, \bar{x}, y \mid \varphi \,]\!] \text{ for some } b \in D \,\} = p_n[[\![\, \bar{x}, y \mid \varphi \,]\!]],$$
$$[\![\, \bar{x}, y \mid \psi \,]\!] = [\![\, \bar{x} \mid \psi \,]\!] \times D = p_n{}^{-1}[[\![\, \bar{x} \mid \psi \,]\!]].$$



For instance, $[\![\, y \mid \varphi \,]\!]$ and its direct image under the projection $p_0$, that is, $p_0[[\![\, y \mid \varphi \,]\!]] = [\![\exists y \,.\, \varphi]\!]$, are in the relation illustrated as:

$$\begin{array}{ccc} [\![\, y \mid \varphi \,]\!] \neq \varnothing & \Longleftrightarrow & p_0[[\![\, y \mid \varphi \,]\!]] = [\![\exists y \,.\, \varphi]\!] = \{*\} \neq \varnothing \\ \Updownarrow & & \Updownarrow \\ \mathfrak{M} \vDash_{[b/y]} \varphi \text{ for some } b \in M & \Longleftrightarrow & \mathfrak{M} \vDash \exists y \,.\, \varphi \end{array}$$

Observe that, for any map $f : X \to Y$, the direct-image operation under $f$ is left adjoint to the inverse-image operation; that is, $f$ always has

$$\frac{f[A] \subseteq B}{A \subseteq f^{-1}[B]},$$

where we draw a double line for equivalence. Therefore, as an instance, we have

$$\frac{[\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] = p_n[[\![\, \bar{x}, y \mid \varphi \,]\!]] \subseteq [\![\, \bar{x} \mid \psi \,]\!]}{[\![\, \bar{x}, y \mid \varphi \,]\!] \subseteq p_n{}^{-1}[[\![\, \bar{x} \mid \psi \,]\!]] = [\![\, \bar{x}, y \mid \psi \,]\!]},$$

which corresponds to the (two-way) rule of first-order logic that

$$\frac{\exists y \,.\, \varphi \vdash \psi}{\varphi \vdash \psi}.$$

Here the “eigenvariable” condition that $y$ does not occur freely in $\psi$ is expressed by $[\![\, \bar{x} \mid \psi \,]\!]$ making sense. Thus, we interpret $\exists$ with the direct-image operation under a suitable projection $p$, and this operation can be characterized as a (unique) left adjoint to the inverse-image operation $p^{-1}$ under $p$. In addition, $p^{-1}$ also has a (unique) right adjoint, and we can interpret $\forall$ with it.[4]

Moreover, substitution of terms can also be interpreted by inverse images. For instance, given a sentence $\varphi(z)$ with only one free variable $z$ and a term $t(\bar{y})$ with $m$ variables $\bar{y}$, using the obvious notation for substitution we have

$$\begin{aligned} [\![\, \bar{y} \mid \varphi(t(\bar{y})) \,]\!] &= \{\, \bar{b} \in D^m \mid \mathfrak{M} \vDash_{[\bar{b}/\bar{y}]} \varphi(t(\bar{y})) \,\} \\ &= \{\, \bar{b} \in D^m \mid \mathfrak{M} \vDash_{[[\![\, \bar{y} \mid t \,]\!](\bar{b})/z]} \varphi(z) \,\} \\ &= \{\, \bar{b} \in D^m \mid [\![\, \bar{y} \mid t \,]\!](\bar{b}) \in [\![\, z \mid \varphi(z) \,]\!] \,\} \\ &= [\![\, \bar{y} \mid t \,]\!]^{-1}[[\![\, z \mid \varphi(z) \,]\!]], \end{aligned}$$

[4] The insight that $\exists$ and $\forall$ are left and right adjoints to an inverse-image operation is due to Lawvere [21].

where $[\![\, \bar{y} \mid t \,]\!] : D^m \to D$ is the interpretation of $t$. More generally, more variables may occur freely in $\varphi$; so, assume $\bar{x}$, $\bar{y}$, $z$ are disjoint, and we have

$$[\![\, \bar{x}, \bar{y} \mid [t/z]\varphi \,]\!] = (1_{D^n} \times [\![\, \bar{y} \mid t \,]\!])^{-1}[[\![\, \bar{x}, z \mid \varphi \,]\!]],$$

where $[t/z]$ denotes the substitution of $t$ for $z$ and we define

$$1_{D^n} \times [\![\, \bar{y} \mid t \,]\!] : D^{n+m} \to D :: (\bar{a}, \bar{b}) \mapsto (\bar{a}, [\![\, \bar{y} \mid t \,]\!](\bar{b})).$$

We have another type of substitution of terms, namely, to obtain $\varphi(y, y)$ from $\varphi(y, z)$, and this can also be interpreted by inverse images. Let $\Delta$ be the "diagonal map", that is,

$$\Delta : D \to D^2 :: a \mapsto (a, a).$$

Then we have

$$[\![\, y \mid \varphi(y, y) \,]\!] = \{\, a \in D \mid (a, a) \in [\![\, y, z \mid \varphi(y, z) \,]\!] \,\} = \Delta^{-1}[[\![\, y, z \mid \varphi(y, z) \,]\!]],$$

and, more generally,

$$[\![\, \bar{x}, y \mid [y/z]\varphi \,]\!] = (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \varphi \,]\!]].$$

It is worth noting that we can write

$$[\![\, x, y \mid x = y \,]\!] = \{\, (a, a) \in D \times D \mid a \in D \,\} = \Delta[D];$$

indeed, since for each $A \subseteq D$ we have $\Delta[A] = {p_1}^{-1}[A] \cap \Delta[D]$, it follows that

$$\Delta[[\![\, y \mid \varphi \,]\!]] = {p_1}^{-1}[[\![\, y \mid \varphi \,]\!]] \cap \Delta[D] = [\![\, y, z \mid \varphi \wedge y = z \,]\!];$$

$$(1_{D^n} \times \Delta)[[\![\, \bar{x}, y \mid \varphi \,]\!]] = [\![\, \bar{x}, y, z \mid \varphi \wedge y = z \,]\!].$$

Therefore, by the adjunction of the direct-image and inverse-image operations, we have

$$\frac{[\![\, \bar{x}, y, z \mid \varphi \wedge y = z \,]\!] = (1_{D^n} \times \Delta)[[\![\, \bar{x}, y \mid \varphi \,]\!]] \subseteq [\![\, \bar{x}, y, z \mid \psi \,]\!]}{[\![\, \bar{x}, y \mid \varphi \,]\!] \subseteq (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \psi \,]\!]] = [\![\, \bar{x}, y \mid [y/z]\psi \,]\!]}$$

for a sentence $\varphi$ in which $z$ does not occur freely; and this corresponds to the rule

(I.7) $$\frac{\varphi \wedge x = y \vdash \psi}{\varphi \vdash [x/y]\psi} \ (y \text{ does not occur freely in } \varphi)$$

of first-order logic, from which we can derive the (more familiar) axioms on identity as follows.[5]

$$\frac{x = y \vdash x = y}{\vdash x = x} \qquad \frac{[x/y]\varphi \vdash [x/y]\varphi}{[x/y]\varphi \wedge x = y \vdash \varphi}$$

[5]This insight is also due to Lawvere; see his [22].

## I.3 TOPOLOGICAL SEMANTICS FOR FIRST-ORDER MODAL LOGIC

In extending the semantics reviewed in Section I.1 to first-order logic, the chief idea is given by the notion of a sheaf over a topological space. In this section, we show how topological sheaves provide semantics for first-order modal logic, as a preparation for the more general extension we will give in Section I.4.2 of neighborhood semantics to the first-order modal logic.

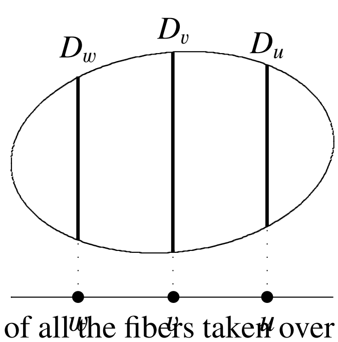
### I.3.1 Domain of Possible Individuals

On one hand, as we reviewed in Section I.1, we use more than one possible world to interpret modality. On the other hand, as in Section I.2, we equip a model—or a world—with a domain of individuals to interpret the first-order vocabulary. In this subsection, we lay out how to unify these two ideas (setting aside the interpretation of modality).

The unification is done by considering a map in the following way. Given any map $\pi : D \to X$, each $w \in X$ has its inverse image

$$D_w = \pi^{-1}[\{w\}] \subseteq D,$$

called the *fiber* over $w$, for the reason that should be obvious from the following picture.




$$\begin{array}{ccccccc} D & = & D_w & \cup & D_v & \cup & \cdots \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & \\ X & = & \{w\} & \cup & \{v\} & \cup & \cdots \end{array}$$

$D$ is then the "bundle" of all the fibers taken over $X$, meaning that $D$ is the disjoint union of all $D_w$. To indicate this bundle idea, we use the "sum" notation and write

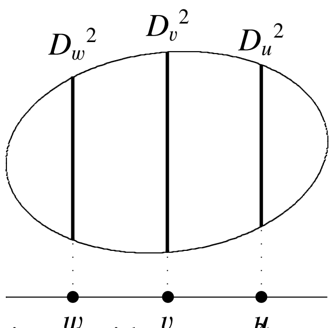
$$D = \sum_{w \in X} D_w.$$

Using this picture, we can regard each $w \in X$ as a possible world, and the fiber $D_w$ as the domain of individuals that live in $w$. Then $D$ is the set of "possible individuals" that live in some world or other. Indeed, each individual $a \in D$ lives in a unique world $\pi(a) \in X$; in this sense, we can call $\pi$ a *residence* map.

The bundle idea can be extended to give the set of "all possible pairs". For any $\pi : D \to X$, we define the (two-fold) *product of $D$ over $X$* by

$$D \times_X D = \sum_{w \in X} (D_w \times D_w),$$

that is, by first taking the product $D_w \times D_w$ of $D_w$ for each $w$ and then bundling up all of them.




$$\begin{array}{ccccccc} D^2 & = & D_w{}^2 & + & D_v{}^2 & + & \cdots \\ \big\downarrow{\scriptstyle \pi^2} & & \big\downarrow & & \big\downarrow & & \\ X & = & \{w\} & + & \{v\} & + & \cdots \end{array}$$

$D \times_X D$ is naturally equipped with a map $\pi^2 : D \times_X D \to X$; it sends $(a, b) \in D_w \times D_w$ to $w$.

The point of introducing the product $D \times_X D$ over $X$, as opposed to the usual Cartesian product $D \times D$, is as follows. Note that we can also describe $D \times_X D$ as

$$D \times_X D = \{\, (a, b) \in D \times D \mid \pi(a) = \pi(b) \,\};$$

that is, in terms of residence, $D \times_X D$ is the set of pairs $(a, b)$ of possible individuals that live in the same worlds $\pi(a) = \pi(b)$. In our semantics, we will use $R \subseteq D \times_X D$, rather than any $R \subseteq D \times D$, to interpret a binary relation, say "$x$ and $y$ are friends" for instance. By doing so, we rule that the sentence "$x$ and $y$ are friends" makes sense only when $x$ and $y$ refer to a pair from the same world.

We have just taken the two-fold product $D \times_X D$ over $X$; let us write $D^2$ for it (instead of for the Cartesian product of $D$). This obviously extends to general $D^n$, the $n$-fold product over $X$ or the set of "all possible $n$-tuples", by taking

$$D^n = \sum_{w \in X} D_w{}^n.$$

In particular, we have

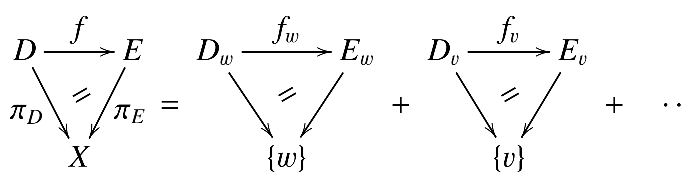
$$D^0 = \sum_{w \in X} D_w{}^0 \cong \sum_{w \in X} \{w\} = X;$$

that is, the set $X$ of possible worlds can be written as a product over $X$ itself.

With the bundle idea we can also take a map over $X$. Given maps $\pi_D : D \to X$ and $\pi_E : E \to X$, we say that a map $f : D \to E$ is *over* $X$ if

$$f = \sum_{w \in X} (f_w : D_w \to E_w).$$

Or, equivalently, $f$ is over $X$ if it has $\pi_E \circ f = \pi_D$, making the triangle to the left below commute, by bundling up the trivially commutative triangles to the right.




The point of taking a map over $X$ is as follows. In our semantics, we will use a map $f : D^n \to D$ over $X$, rather than just any map, to interpret a function symbol, say "the father of $x$". By doing so, we rule that the father of $a$ must be found in the same world $\pi(a)$ in which $a$ lives.

Let us write **Sets** for the category of sets. Then, given a fixed set $X$, the kinds of structures we reviewed in this subsection form a category **Sets**/$X$, the slice category of **Sets** over $X$; its objects are maps $\pi : D \to X$ and arrows from $\pi_D : D \to X$ to $\pi_E : E \to X$ are maps $f : D \to E$ over $X$. Products over $X$ are just products in **Sets**/$X$. Therefore, what we laid out in this subsection can be summarized by saying that we can regard **Sets**/$X$ as the category of domains of, sets of tuples of, and functions among, possible individuals, over the set $X$ of possible worlds.

### I.3.2 Interpreting First-Order Logic

With the bundle representation of possible individuals we introduced in Subsection I.3.1, we can formulate the non-modal part of our semantics in the following way. Given a first-order language $\mathcal{L}$, a model $\mathfrak{M}$ consists of:

- a surjection $\pi$; let us write $D$ and $X$ for its domain and codomain, so that $\pi : D \twoheadrightarrow X$;[6]
- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq D^n$ of the $n$-fold product of $D$ over $X$;
- for each $n$-ary function symbol $f$, a map $f^{\mathfrak{M}} : D^n \to D$ over $X$; and

[6]We require $\pi$ to be surjective, so that $D_w \neq \varnothing$ for every $w \in X$.

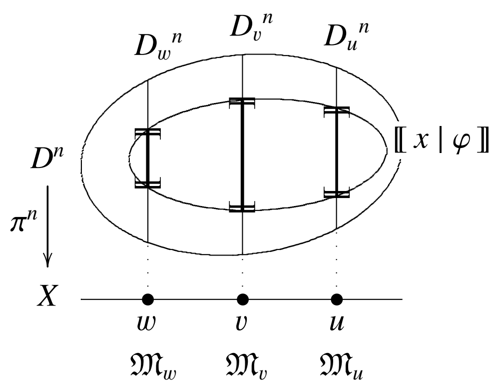
- for each constant $c$, a map $c^{\mathfrak{M}} : D^0 \to D$ over $X$, that is, a map $c^{\mathfrak{M}} : X \to D$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

Then, restricted to each fiber $D_w$,

$$\mathfrak{M}_w = (D_w, (R_i{}^{\mathfrak{M}})_w, (f_j{}^{\mathfrak{M}})_w, (c_k{}^{\mathfrak{M}})_w)_{i \in I, j \in J, k \in K}$$

is a standard $\mathcal{L}$ structure, just as we reviewed in Section I.2. Therefore we interpret first-order logic by first interpreting it in each $\mathfrak{M}_w$ and then bundling up all of them. That is, with each $\mathcal{L}$ structure $\mathfrak{M}_w$ interpreting a sentence $\varphi$ with $[\![\, \bar{x} \mid \varphi \,]\!]_w \subseteq D_w{}^n$, the entire model $\mathfrak{M}$ interprets $\varphi$ with

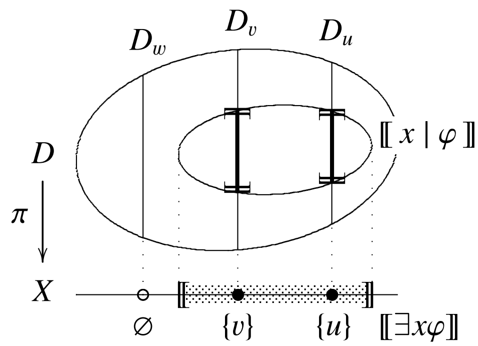
$$[\![\, \bar{x} \mid \varphi \,]\!] = \sum_{w \in X} [\![\, \bar{x} \mid \varphi \,]\!]_w \subseteq \sum_{w \in X} D_w{}^n = D^n.$$




Then the definition of validity we gave before extends straightforwardly; that is, $\varphi \vdash \psi$ is valid in $\mathfrak{M}$ iff $[\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$, and an inference is valid iff it preserves validity.

Observe moreover that the interpretations of classical operators reviewed in Section I.2 simply carry over to this setting involving many worlds, because they all commute with $\sum_{w \in X}$. For instance, given $[\![\, x \mid \varphi \,]\!] \subseteq D$, we have

$$[\![\exists x \,.\, \varphi]\!] = \sum_{w \in X} [\![\exists x \,.\, \varphi]\!]_w = \sum_{w \in X} \pi[[\![\varphi]\!]_w] = \pi[\sum_{w \in X} [\![\varphi]\!]_w] = \pi[[\![\varphi]\!]];$$




that is, $\exists$ is again interpreted by the direct-image operation under a suitable projection $p : D^{n+1} \to D^n$ (with $n = 0$ in the example above). Hence we set as follows. Here $\Delta$ is again the diagonal map; note that it is of the type $\Delta : D \to D \times_X D$ and is over $X$.

$$
\begin{aligned}
[\![ \bar{x} \mid R\bar{x} ]\!] &= R^{\mathfrak{M}} \qquad \text{for } n\text{-ary primitive predicate } R \text{, and} \\
[\![ x, y \mid x = y ]\!] &= \Delta[D] \qquad \text{in particular;} \\
[\![ \bar{x} \mid \top ]\!] &= D^n; \\
[\![ \bar{x} \mid \neg\varphi ]\!] &= D^n \setminus [\![ \bar{x} \mid \varphi ]\!] \qquad (\text{that is, } [\![\neg]\!] = D^n \setminus -); \\
[\![ \bar{x} \mid \varphi \wedge \psi ]\!] &= [\![ \bar{x} \mid \varphi ]\!] \cap [\![ \bar{x} \mid \psi ]\!] \qquad (\text{that is, } [\![\wedge]\!] = \cap); \\
&\vdots \\
[\![ \bar{x} \mid \exists y \,.\, \varphi ]\!] &= p[[\![ \bar{x}, y \mid \varphi ]\!]]; \\
[\![ \bar{x}, y \mid \varphi(\bar{x}) ]\!] &= p^{-1}[[\![ \bar{x} \mid \varphi(\bar{x}) ]\!]]; \\
[\![ \bar{x}, \bar{y} \mid [t/z]\varphi ]\!] &= (1_{D^n} \times [\![ \bar{y} \mid t ]\!])^{-1}[[\![ \bar{x}, z \mid \varphi ]\!]]; \\
[\![ \bar{x}, y \mid [y/z]\varphi ]\!] &= (1_{D^n} \times \Delta)^{-1}[[\![ \bar{x}, y, z \mid \varphi ]\!]].
\end{aligned}
$$

This is how first-order logic is interpreted in the category **Sets**$/X$. And then, as one may expect, the upshot of our semantics is to interpret $\Box$ with interior operations of suitable topologies on the structure; in particular, we interpret $[\![ \bar{x} \mid \varphi ]\!] \mapsto [\![ \bar{x} \mid \Box\varphi ]\!]$—that is, $\Box$ operating on $n$-ary formulas—with the interior operation $\mathbf{int}_{D^n} : \mathcal{P}(D^n) \to \mathcal{P}(D^n)$ of a suitable topology on the $n$-fold product $D^n$ over $X$. For this purpose, we need to define with what topology $D^n$ should be equipped.

### I.3.3 Sheaves over a Topological Space

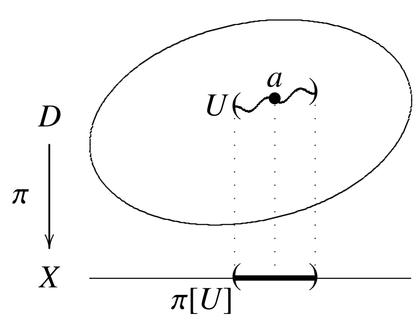
In Section I.1, we showed how to interpret propositional modal logic by interpreting modal operators with interior and closure operations on a topological space $X$ of possible worlds. In Subsection I.3.1, we showed how to equip the set $X$ with a domain $D$ of possible individuals by using a residence map $\pi : D \to X$, and then, in Subsection I.3.2, we showed how to interpret first-order logic in the category **Sets**$/X$ of such structures. We are not yet ready, however, to interpret modal operators, because we have not given any topology to those structures. In this subsection, we show how to equip $D$, and $D^n$ in general, with suitable topologies, so that, in Subsection I.3.4, we can finally give a semantics for first-order modal logic.

Let us first recall that, given any pair of topological spaces $X$ and $Y$,[7] we say a map $f : Y \to X$ is

- *continuous* if $f^{-1}[U] \in \mathcal{O}Y$ for every $U \in \mathcal{O}X$ (that is, if $f : Y \to X$ pulls open sets of $X$ back to open sets of $Y$), and
- a *homeomorphism* if $f$ is a continuous bijection with a continuous inverse (or, equivalently, if $X$ and $Y$ share the same topological structure, with points renamed by $f$).

Then the topological notion of a sheaf is defined as follows.

**Definition.** Given topological spaces $X$ and $D$, a map $\pi : D \to X$ is called a *local homeomorphism* if every $a \in D$ has some $U \in \mathcal{O}D$ such that $a \in U$, $\pi[U] \in \mathcal{O}D$, and the restriction $\pi\restriction U : U \to \pi[U]$ of $\pi$ to $U$ is a homeomorphism.



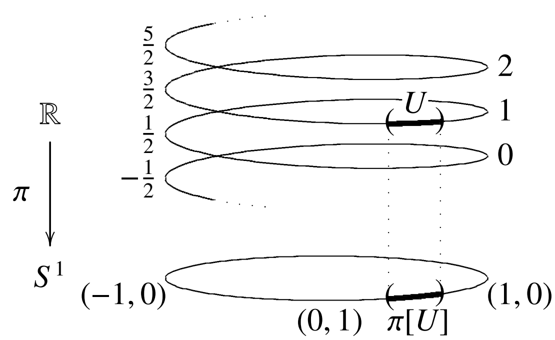

[7] For the sake of simplicity, from this subsection on we write $X$ for topological spaces $(|X|, \mathcal{O}X)$; we write $|X|$, when we would like it explicit that we mean underlying sets. We write $f : Y \to X$ for any maps, not necessarily continuous, from $|Y|$ to $|X|$.

When this is the case, we say that the pair $(D, \pi)$ is a *sheaf over the space* $X$, and also that $X$, $D$, and $\pi$ are respectively the *base space*, *total space*, and *projection* of the sheaf.[8]

Taking a concrete example, $\mathbb{R}$ (with its usual topology) and $\pi : \mathbb{R} \to S^1$ such that

$$\pi(a) = e^{i2\pi a} = (\cos 2\pi a, \sin 2\pi a)$$

form a sheaf over the circle $S^1$ (with the subspace topology in $\mathbb{R}^2$). As in the picture below, we may say that $\mathbb{R}$ draws a helix over $S^1$; indeed, for every $a \in \mathbb{R}$, a small enough open set $U$ around $a$ is homeomorphic to its image $\pi[U]$.




Given two sheaves $(D, \pi_D : D \to X)$ and $(E, \pi_E : E \to X)$, we say a map $f : D \to E$ is a *map of sheaves over* $X$ if $f$ is continuous and over the set $|X|$. Therefore, sheaves and maps of sheaves over $X$ form a full subcategory of $\mathbf{Top}/X$—the category $\mathbf{Top}$ of topological spaces and continuous maps over $X$—since local homeomorphisms are continuous maps. Moreover, we can show that maps of sheaves are themselves local homeomorphisms; due to this fact, the category of sheaves and maps of sheaves is just $\mathbf{LH}/X$, the category $\mathbf{LH}$ of topological spaces and local homeomorphisms over $X$. (This fact turns out crucial for the purpose of providing semantics for first-order modal logic.) This is how we add topological structures to objects and maps in $\mathbf{Sets}/|X|$.

The category $\mathbf{Top}/X$ of topological spaces and continuous maps over a topological space $X$ has finite products, because for any finite collection of spaces $(D_i, \pi_i : D_i \to X)$ over $X$ $(i = 1, \ldots, n)$, its product can be defined explicitly in $\mathbf{Top}/X$ as follows. First take the product of the sets $|D_i|$ over $|X|$, that is,

$$|D| = |D_1| \times_{|X|} \cdots \times_{|X|} |D_n| = \{ (a_1, \ldots, a_n) \in |D_1| \times \cdots \times |D_n| \mid \pi_1(a_1) = \cdots = \pi_n(a_n) \};$$

[8] The notion of a sheaf is sometimes defined in terms of the notion of a functor, in which case the version used here is called an étale space. The functorial notion is equivalent (in the category-theoretical sense) to the version here.

this comes with a projection

$$\pi = \pi_1 \times_X \cdots \times_X \pi_n : |D_1| \times_{|X|} \cdots \times_{|X|} |D_n| \to |X| :: (a_1, \ldots, a_n) \mapsto \pi_1(a_1).$$

Then, because $|D|$ is a subset of the Cartesian product $|D_1| \times \cdots \times |D_n|$, on which the product space $D_1 \times \cdots \times D_n$ is defined, we simply let $D$ be the subspace of $D_1 \times \cdots \times D_n$; that is,

$$\begin{aligned} U \in \mathcal{O}D \iff & U = \bigcup_{i \in I} B_i \cap |D| \text{ for a collection } \{B_i\}_{i \in I} \text{ such that, for each } i \in I, \\ & B_i = V_1 \times \ldots \times V_n \text{ for some } V_1 \in \mathcal{O}D_1, \ldots, V_n \in \mathcal{O}D_n \\ \iff & U = \bigcup_{i \in I} B_i \text{ for a collection } \{B_i\}_{i \in I} \text{ such that, for each } i \in I, \\ & B_i = V_1 \times_{|X|} \ldots \times_{|X|} V_n \text{ for some } V_1 \in \mathcal{O}D_1, \ldots, V_n \in \mathcal{O}D_n. \end{aligned}$$

Then $\pi$ is continuous, as are the projections $p_i : D \to D_i$. Indeed, $(D, \pi)$ moreover serves as the product of $(D_i, \pi_i)$ in $\mathbf{LH}/X$ as well: We can show that, if $\pi_i$ are all local homeomorphisms, $\pi$ is a local homeomorphism, that is, $(D, \pi)$ is a sheaf over $X$; it follows that $p_i$ are maps of sheaves. And, as we can also show, it is the product in $\mathbf{LH}/X$ of $(D_i, \pi_i)$. The $n$-fold product in $\mathbf{LH}/X$ of the same sheaf, which we will use to interpret logic, is just a special case of this definition.

### I.3.4 Topological-Sheaf Semantics for First-Order Modal Logic

In Subsection I.3.2 we showed how to interpret first-order logic with a map $\pi$. Now that we have added a nice topological structure to $\pi$ in Subsection I.3.3, we can further add a topological interpretation of modal operators to the interpretation with $\pi$ of first-order logic.

Let us fix any first-order modal language $\mathcal{L}$, that is, a language obtained by adding $\Box$ and $\Diamond$ to a classical first-order language. About this addition, we should make a remark (that will be crucial later) that, syntactically, we treat $\Box$, $\Diamond$ as unary sentential operators just like $\neg$; in particular, we have $[t/z](\Box\varphi) = \Box([t/z]\varphi)$. Then recall from Subsection I.3.2 that we take the following type of structures to semantically interpret the non-modal part of $\mathcal{L}$.

- a surjection $\pi$; let us write $|D|$ and $|X|$ for its domain and codomain, so that $\pi : |D| \twoheadrightarrow |X|$;
- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq |D|^n$ of the $n$-fold product of $|D|$ over $|X|$;
- for each $n$-ary function symbol $f$, a map $f^{\mathfrak{M}} : |D|^n \to |D|$ over $|X|$;

- for each constant $c$, a map $c^{\mathfrak{M}} : |D|^0 \to |D|$ over $|X|$, that is, a map $c^{\mathfrak{M}} : |X| \to |D|$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

Now, rather than just any surjection $\pi$, we take a surjective local homeomorphism to further interpret modal operators. Then, to interpret a primitive predicate, we may take any arbitrary subset (of the type above). By contrast, to interpret function symbols and constants, we need to take maps of sheaves over $X$ rather than just any maps over $|X|$. So, we enter:

**Definition.** Given a first-order modal language $\mathcal{L}$, by a *topological-sheaf model* for $\mathcal{L}$ we mean a structure $\mathfrak{M} = (\pi, R_i{}^{\mathfrak{M}}, f_j{}^{\mathfrak{M}}, c_k{}^{\mathfrak{M}})_{i \in I, j \in J, k \in K}$ consisting of

- a surjective local homeomorphism $\pi$; let us write $X$ and $D$ for its base and total spaces, so that $\pi : D \twoheadrightarrow X$;

- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq |D|^n$ of the $n$-fold product of $|D|$ over $|X|$;

- for each $n$-ary function symbol $f$, a continuous map $f^{\mathfrak{M}} : D^n \to D$ over $X$; and

- for each constant $c$, a continuous map $c^{\mathfrak{M}} : X \to D$ over $X$, that is, such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

On such a structure, we interpret the non-modal part of $\mathcal{L}$ as we did before in Subsection I.3.2, and moreover $\Box$, $\Diamond$ with the interior operation of the corresponding space $D^n$.

**Definition.** Given a first-order modal language $\mathcal{L}$, by a *topological-sheaf interpretation* for $\mathcal{L}$ we mean a pair $(\mathfrak{M}, [\![-]\!])$ of a topological-sheaf model $\mathfrak{M}$ with a map $[\![-]\!]$ (of the suitable type) defined inductively by

$$
\begin{aligned}
[\![\, \bar{x} \mid R\bar{x} \,]\!] &= R^{\mathfrak{M}} && \text{for } n\text{-ary primitive predicate } R \text{, and} \\
[\![\, x, y \mid x = y \,]\!] &= \Delta[D] && \text{in particular;} \\
[\![\, \bar{x} \mid \top \,]\!] &= D^n; \\
[\![\, \bar{x} \mid \neg\varphi \,]\!] &= D^n \setminus [\![\, \bar{x} \mid \varphi \,]\!] && \text{(that is, } [\![\neg]\!] = D^n \setminus - \text{);} \\
[\![\, \bar{x} \mid \varphi \wedge \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \cap [\![\, \bar{x} \mid \psi \,]\!] && \text{(that is, } [\![\wedge]\!] = \cap \text{);} \\
&\ \vdots \\
[\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] &= p[[\![\, \bar{x}, y \mid \varphi \,]\!]]; \\
[\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!] &= {p_n}^{-1}[[\![\, \bar{x} \mid \varphi(\bar{x}) \,]\!]]; \\
[\![\, \bar{x}, \bar{y} \mid [t/z]\varphi \,]\!] &= (1_{D^n} \times [\![\, \bar{y} \mid t \,]\!])^{-1}[[\![\, \bar{x}, z \mid \varphi \,]\!]]; \\
[\![\, \bar{x}, y \mid [y/z]\varphi \,]\!] &= (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \varphi \,]\!]]; \\
[\![\, \bar{x} \mid \Box\varphi \,]\!] &= \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) && \text{(that is, } [\![\Box]\!] = \mathbf{int}_{D^n} \text{);} \\
[\![\, \bar{x} \mid \Diamond\varphi \,]\!] &= \mathbf{cl}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) && \text{(that is, } [\![\Diamond]\!] = \mathbf{cl}_{D^n} \text{).}
\end{aligned}
$$

The class of such interpretations constitutes topological-sheaf semantics for first-order modal logic. To figuratively illustrate how the semantics works, recall our pictures of sheaves. On the one hand, the first-order part of a first-order modal language is interpreted by the "vertical" aspect of a sheaf, that is, within each fiber as a world, as in the picture on p. 15. On the other hand, the modal part is interpreted by the "horizontal" aspect, that is, as in the picture on p. 17, with open sets of $X$ and neighborhoods $U$ in $D$ that are locally homeomorphic to open sets of $X$. To take a sheaf is to take a "product" of these two directions, and then, correspondingly, the logic of topological-sheaf semantics—which we lay out in Subsection I.3.5—is a "product" of the two logics, first-order and modal.

### I.3.5 First-Order Modal Logic FOS4

The semantics we reviewed in Subsection I.3.2 is a semantics for first-order logic, while topological semantics is a semantics for (propositional) modal logic **S4**, as we mentioned in Subsection I.1.2. Topological-sheaf semantics, which we just laid out in Subsection I.3.4, unifies these two semantics naturally, in the sense that it gives rise to a logic that is a simple union of first-order logic and **S4**. More precisely, let us enter:

**Definition.** First-order modal logic **FOS4** consists of the following two sorts of axioms and rules.

1. All axioms and rules of (classical) first-order logic.
2. The rules and axioms of propositional modal logic **S4**; that is, M, C, N, T, 4.

We should emphasize that, in this logic, first-order axioms and rules are $\Box$- (and $\Diamond$-) insensitive, in the sense that, in applying schemes, sentences containing modal operators and ones not are *not* distinguished. For instance, in the following axiom of identity, $\varphi$ may contain modal operators.

$$x = y \vdash [x/z]\varphi \to [y/z]\varphi. \tag{I.8}$$

Also, modal axioms and rules are insensitive to the first-order structure of sentences. This is why we call **FOS4** a simple union of first-order logic and **S4**.

To illustrate this point, let us take some examples of proofs in **FOS4**. To instantiate (I.8), take $\Box(x = z)$ for $\varphi$; this is allowed by the $\Box$-insensitivity. Then (I.8) yields the left sequent in the middle below. The top sequent to the right is another axiom on =; the first inference after that is by N, whereas the last inference is by a kind of cut.

$$\dfrac{x = y \vdash \Box(x = x) \to \Box(x = y) \qquad \dfrac{\vdash x = x}{\vdash \Box(x = x)}}{x = y \vdash \Box(x = y)}$$

Thus $x = y \vdash \Box(x = y)$ is provable in **FOS4**. Also, the so-called converse Barcan formula and its $\exists$ variant are provable as follows.

$$\dfrac{\dfrac{\forall x.\varphi \vdash \varphi}{\Box\forall x.\varphi \vdash \Box\varphi}}{\Box\forall x.\varphi \vdash \forall x\Box\varphi} \qquad\qquad \dfrac{\dfrac{\varphi \vdash \exists x.\varphi}{\Box\varphi \vdash \Box\exists x.\varphi}}{\exists x\Box\varphi \vdash \Box\exists x.\varphi}$$

In each proof, the first sequent is an axiom on $\forall$ or $\exists$, and the first inference is by M. The second inference is justified by the rule on $\forall$ or $\exists$, because $x$ occurs freely neither in $\Box\forall x.\varphi$ nor in $\Box\exists x.\varphi$ (and, again, because the rule is $\Box$-insensitive).

By contrast,

$$x \neq y \vdash \Box(x \neq y)$$

$$\forall x \Box\varphi \vdash \Box\forall x\,.\,\varphi$$

$$\Box\exists x\,.\,\varphi \vdash \exists x \Box\varphi$$

are not theorems of **FOS4**. For the Barcan formula $\forall x \Box\varphi \vdash \Box\forall x\,.\,\varphi$ and its $\exists$ variant, we will give a countermodel to illustrate their invalidity in Subsection I.3.6.

Using an axiom more characteristic of **S4**, we can extend the proof above of $\exists x \Box\varphi \vdash \Box\exists x\,.\,\varphi$ as follows. As before, the first inference to the right is by N, and the last is by the rule on $\exists$. Then the instance $\Box\varphi \vdash \Box\Box\varphi$ of axiom 4 yields the second inference by the transitivity of $\vdash$.

$$\dfrac{\dfrac{\Box\varphi \vdash \Box\Box\varphi \qquad \dfrac{\Box\varphi \vdash \exists x \Box\varphi}{\Box\Box\varphi \vdash \Box\exists x \Box\varphi}}{\Box\varphi \vdash \Box\exists x \Box\varphi}}{\exists x \Box\varphi \vdash \Box\exists x \Box\varphi}$$

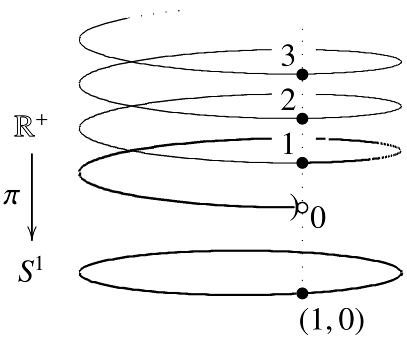
Combined with the instance $\Box\exists x \Box\varphi \vdash \exists x \Box\varphi$ of axiom T, this means that $\exists x \Box\varphi$ and $\Box\exists x \Box\varphi$ are provably equivalent in **FOS4**.

It can be checked straightforwardly that **FOS4** is sound with respect to topological-sheaf semantics. It is moreover complete, in the strong form that exactly extends Theorem 2 (Subsection I.1.2), the completeness **S4** for propositional modal logic. This is one of the chief results of this dissertation.

**Theorem** (Awodey-Kishida [5])**.** *For any consistent theory $\mathbb{T}$ of first-order modal logic extending* **FOS4***, there exists a topological-sheaf interpretation $(\pi, [\![-]\!])$ that validates all and only theorems of $\mathbb{T}$.*

### I.3.6 An Example of Interpretation

Recall the example of a sheaf given in Subsection I.3.3, that is, the infinite helix over the circle $S^1$ with the projection $\pi : \mathbb{R} \to S^1 :: a \mapsto (\cos 2\pi a, \sin 2\pi a)$. Let us now take $D = \mathbb{R}^+ = \{ a \in \mathbb{R} \mid 0 < a \}$, the positive reals, instead of $\mathbb{R}$, as a total space; so we have a helix infinitely continuing upward but with an open lower end at 0.




This is also a sheaf. Observe that each fiber $D_w$ is of the form $\{ n + a_w \mid n \in \mathbb{N} \}$ for the unique $a_w$ such that $0 < a_w \leqslant 1$ and $\pi(a_w) = w$. Then let a topological-sheaf model $\mathfrak{M} = (\pi, \leqslant^{\mathfrak{M}})$ interpret the binary primitive predicate $\leqslant$ with the usual $\leqslant$ relation of real numbers restricted to $D$; that is, for all $a, b \in \mathbb{R}$,

$$(a, b) \in \leqslant^{\mathfrak{M}} = [\![ x, y \mid x \leqslant y ]\!] \iff 0 < a \leqslant b \text{ and } \pi(a) = \pi(b),$$

where $[\![ - ]\!]$ is the topological-sheaf interpretation on $\mathfrak{M}$.

Then consider the truth of the following sentences under this interpretation:

| | | |
|---|---|---|
| (I.9) | $\exists x \forall y . x \leqslant y$ | "Some $x$ is the least number." |
| (I.10) | $\exists x \Box \forall y . x \leqslant y$ | "Some $x$ is necessarily the least number." |

By looking at each fiber $D_w = \{ n + a \mid n \in \mathbb{N} \}$, we can see that $[\![ x \mid \forall y . x \leqslant y ]\!]_w = \{a_w\}$, the least point in $D_w$; so, bundling up all fibers, we have

$$[\![ x \mid \forall y . x \leqslant y ]\!] = \{ a \in \mathbb{R} \mid 0 < a \leqslant 1 \} = (0, 1].$$

Therefore, by applying the direct-image operation $[\![\exists x]\!]$ under $\pi$ to this, we have

$$[\![\, \exists x\, \forall y \,.\, x \leqslant y \,]\!] = \pi[[\![\, x \mid \forall y \,.\, x \leqslant y \,]\!]] = \pi[(0,1]] = S^1;$$

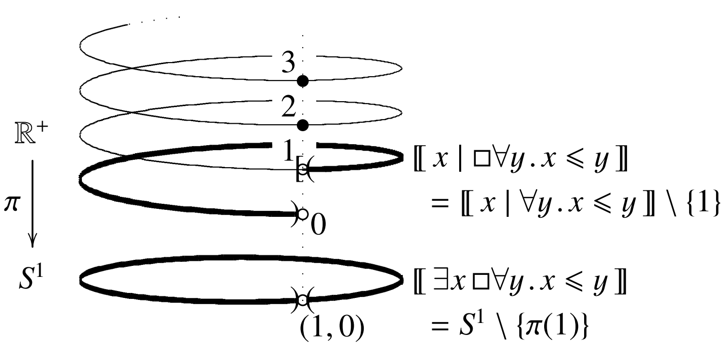
that is, (I.9) is valid in $(\mathfrak{M}, [\![-]\!])$. On the other hand, by applying the interior operation $\mathbf{int}_{\mathbb{R}^+} = [\![\Box]\!]$, we have

$$[\![\, x \mid \Box\forall y \,.\, x \leqslant y \,]\!] = \mathbf{int}_{\mathbb{R}^+}([\![\, x \mid \forall y \,.\, x \leqslant y \,]\!]) = \mathbf{int}_{\mathbb{R}^+}((0,1]) = \{\, a \in \mathbb{R} \mid 0 < a < 1 \,\} = (0,1).$$

This is why, by again applying the direct-image operation $[\![\exists x]\!]$ under $\pi$, we have

$$[\![\, \exists x \Box \forall y \,.\, x \leqslant y \,]\!] = \pi[(0,1)] = S^1 \setminus \{\pi(1)\} \neq S^1;$$

that is, (I.10) is *not* valid in $(\mathfrak{M}, [\![-]\!])$.




In other words, $1 \in D = \mathbb{R}^+$ is "actually the least" in its fiber $D_{\pi(1)} = \{\, n+1 \mid n \in \mathbb{N} \,\}$ but not "necessarily the least". Speaking in terms of worlds and individuals, the individual 1 is the least number in its world $\pi(1)$, but any neighborhood of $\pi(1)$, no matter how small a one we may take, contains some world $w$ (with $D_w = \{\, n + \varepsilon \mid n \in \mathbb{N} \,\}$ for $\varepsilon > 0$) in which (the counterpart of) 1 is no longer the least. Note the notion of a counterpart we used here. Even though 1 only lives in the world $\pi(1)$, it still makes intuitive sense to talk about "1 in worlds near by" because, due to the local homeomorphism property of $\pi$, if you take a small enough neighborhood $U$ around 1 then $a \in U$ corresponds one-to-one to $\pi(a)$ and therefore can be called "(the counterpart of) 1 in the world $\pi(a)$".

Finally, let us observe that $(\mathfrak{M}, [\![-]\!])$ is a countermodel to the formulas of the Barcan sort which we claimed were invalid in Subsection I.3.5. First, because $[\![\exists x \forall y . x \leqslant y]\!] = S^1$, we have

$$[\![\Box \exists x \, \forall y \, . \, x \leqslant y]\!] = \mathbf{int}_{S^1}([\![\exists x \, \forall y \, . \, x \leqslant y]\!]) = \mathbf{int}_{S^1}(S^1) = S^1.$$

This means, since $[\![\exists x \Box \forall y \, . \, x \leqslant y]\!] = S^1 \setminus \{\pi(1)\}$, that the instance

$$\Box \exists x \, \forall y \, . \, x \leqslant y \vdash \exists x \Box \forall y \, . \, x \leqslant y,$$

of the $\exists$ variant of Barcan formula, "$\Box \exists \vdash \exists \Box$", is not valid in $(\mathfrak{M}, [\![-]\!])$. Also, observe that

$$[\![\, x, y \mid \Box(x \leqslant y) \,]\!] = \mathbf{int}_{D^2}([\![\, x, y \mid x \leqslant y \,]\!]) = [\![\, x, y \mid x \leqslant y \,]\!].$$

While it is not hard to see this by formally checking that $[\![\, x, y \mid x \leqslant y \,]\!]$ is open, we can intuitively see it by taking an arbitrary pair $(a, b) \in [\![\, x, y \mid x \leqslant y \,]\!]$ and "sliding" it a little bit; around the world $\pi(a) = \pi(b)$, there is a neighborhood in which the counterpart of $a$ is always no greater than that of $b$, which means that $a$ is necessarily no greater than $b$. Then it follows that

$$[\![\, x \mid \forall y \Box (x \leqslant y) \,]\!] = [\![\, x \mid \forall y \, . \, x \leqslant y \,]\!] = (0, 1],$$

and therefore, again because $[\![\, x \mid \Box \forall y \, . \, x \leqslant y \,]\!] = (0, 1)$, that

$$\forall y \Box (x \leqslant y) \vdash \Box \forall y \, . \, x \leqslant y$$

is not valid in $(\mathfrak{M}, [\![-]\!])$; and this provides a countermodel to the Barcan formula, "$\forall \Box \to \Box \forall$".

## I.4 NEIGHBORHOOD SEMANTICS FOR FIRST-ORDER MODAL LOGIC

As its most mathematically significant result, this dissertation extends topological-sheaf semantics of Section I.3 to a more general semantics, namely, a semantics for first-order modal logic in terms of an extended notion of sheaves over a more general neighborhood frame.

### I.4.1 Why Sheaves are Needed

For the purpose of obtaining neighborhood semantics for first-order modal logic, we need to analyze the topological notion of sheaves and identify an aspect of sheaves that is essential in providing semantics for the unification of first-order and modal logics, so that we can preserve it as we move to a more general notion of sheaves.

Although we used a standard definition of local homeomorphisms in Subsection I.3.3, it is helpful for our purpose to rewrite it in terms more directly related to logic. The notion crucial for this rewriting is openness of maps. Given topological spaces $X$ and $Y$, we say that a map $f : Y \to X$ is *open* if $f[V] \in \mathcal{O}X$ for every $V \in \mathcal{O}Y$, that is, if it sends open sets to open sets.[9]

To give an example of the connection between openness of maps and logic, recall the fact we saw in Subsection I.3.5 that, in **FOS4**, $\Box\exists x\Box\varphi$ and $\exists x\Box\varphi$ are equivalent; or, to put it semantically with a topological interpretation,

$$\mathbf{int}(p[\mathbf{int}(A)]) = [\![\Box]\!][\![\exists x]\!][\![\Box]\!](A) = [\![\exists x]\!][\![\Box]\!](A) = p[\mathbf{int}(A)].$$

Because a set $U$ is open iff $U = \mathbf{int}(A)$ for some $A$ and also iff $\mathbf{int}(U) = U$, this means that the direct image of an open set under $p$ is always open; that is, projections $p_n : D^{n+1} \to D^n$, and in particular $p_0 = \pi : D \to X$, are open maps.

Then sheaves can be described in terms of openness of maps in the following way.

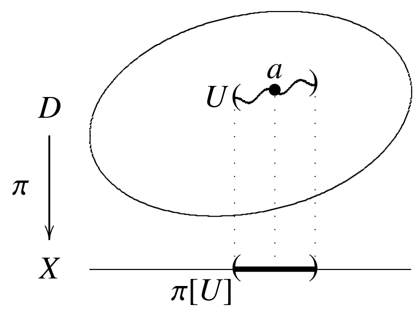
**Fact 1.** For any topological spaces $X$ and $D$ and any map $\pi : D \to X$, the following are equivalent:

- $\pi$ is a local homeomorphism (as defined in Subsection I.3.3).
- $\pi$ satisfies (i) and (ii) below.
- $\pi$ satisfies (i) and (iii) below.

(i) $\pi$ is continuous and open.

(ii) For every $a \in D$ there is $U \in \mathcal{O}D$ such that $a \in U$ and $\pi{\restriction}U : U \to \pi[U]$ is bijective.

[9] In the usual terminology, only continuous maps can be open. We adopt a terminology, however, in which open maps may not be continuous, because openness (in our sense) by itself has consequences for logic.




(iii) The diagonal map $\Delta : D \to D^2$ is open.

Note that the diagonal map $\Delta$ is continuous by definition. Also, recall a fact we mentioned in Subsection I.3.3, namely that maps of sheaves are themselves local homeomorphisms. Therefore we can summarize the fact above by saying that, in topological-sheaf semantics, all the maps we use to interpret the first-order part of first-order modal logic—projections $\pi$ and $p_n$, interpretations $[\![ \bar{y} \mid t ]\!]$ of terms, and the diagonal map $\Delta$—are continuous and open, and indeed that, in order for this to be the case, we must take a sheaf.

Let us further analyze why this should be the case for the purpose of interpreting logic. For this analysis, it is particularly helpful to redefine continuous maps and open maps in terms of interior operations—rather than in terms of open sets as in the common definition—because it is interior operations that are directly connected to logic via the interpretation of $\Box$. So let us observe that, given topological spaces $X$ and $Y$, a map $f : Y \to X$ is continuous iff

$$f^{-1}[\mathbf{int}_X(B)] \subseteq \mathbf{int}_Y(f^{-1}[B])$$

for all $B \subseteq X$, and open iff

$$\mathbf{int}_Y(f^{-1}[B]) \subseteq f^{-1}[\mathbf{int}_X(B)]$$

for all $B \subseteq X$. That is, open continuous maps $f : Y \to X$ are characterized by

$$f^{-1}[\mathbf{int}_X(B)] = \mathbf{int}_Y(f^{-1}[B]),$$

the commutation of its inverse-image operation with the interior operations **int**. This should make it obvious what it means to use open continuous maps to interpret logic, once we recall what are interpreted by inverse-image operations and interior operations. That is, given our interpretations

$$\begin{aligned} [\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!] &= {p_n}^{-1}[[\![\, \bar{x} \mid \varphi(\bar{x}) \,]\!]], \\ [\![\, \bar{y} \mid [t/z]\varphi \,]\!] &= [\![\, \bar{y} \mid t \,]\!]^{-1}[[\![\, z \mid \varphi \,]\!]], \\ [\![\, y \mid [y/z]\varphi \,]\!] &= \Delta^{-1}[[\![\, y, z \mid \varphi \,]\!]] \end{aligned}$$

on the one hand and

$$[\![\, \bar{x} \mid \Box\varphi \,]\!] = \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!])$$

on the other, taking a sheaf means that we assume that these operations—adding a vacuous variable to the context of free variables, and substituting and duplicating terms—all commute with $\Box$.

Let us consider this commutation more closely. For instance, given an $n$-ary formula $\varphi$, we can regard $\varphi$, and moreover $\Box\varphi$, as $(n+1)$-ary formulas; and, accordingly, we *need*—for the reason we gave in Subsection I.2.2—to obtain $[\![\, \bar{x}, y \mid \Box\varphi \,]\!]$ from $[\![\, \bar{x} \mid \varphi \,]\!]$. Nonetheless, there are two ways to do so, as in the following diagram, the commutation of which exactly means the openness of $p_n$.

$$\begin{array}{ccc} [\![\, \bar{x} \mid \varphi \,]\!] & \xmapsto{\ \mathbf{int}_{D^n}\ } & [\![\, \bar{x} \mid \Box\varphi \,]\!] \\ {\scriptstyle {p_n}^{-1}}\big\downarrow & \neq & \big\downarrow{\scriptstyle {p_n}^{-1}} \\ [\![\, \bar{x}, y \mid \varphi \,]\!] & \xmapsto[\ \mathbf{int}_{D^{n+1}}\ ]{} & [\![\, \bar{x}, y \mid \Box\varphi \,]\!] \end{array}$$

In this way, the well-definedness of the semantics requires that projections $p_n$ be open.[10]

[10] That is, on the assumption that we interpret $[\![\, \bar{x}, y \mid \varphi \,]\!] \mapsto [\![\, \bar{x}, y \mid \Box\varphi \,]\!]$ with $\mathbf{int}_{D^{n+1}}$. This is a non-trivial assumption. Even when we adopt the general idea that we interpret $\Box$ with interior operators, it is possible to implement that idea with a "non-uniform" interpretation of $\Box$; that is, instead of the single operation $\mathbf{int}_{D^{n+m}}$, we may use a family of operations (each of which may be induced by interior operations) to define

$$[\![\, \bar{x}, \bar{y} \mid \varphi \,]\!] \mapsto [\![\, \bar{x}, \bar{y} \mid \Box\varphi \,]\!],$$

so that what interpretation is given to the application of $\Box$ to $\varphi$ depends on what free variables actually occur in $\varphi$. To given an example of a non-uniform interpretation, we may set

$$[\![\, \bar{x}, \bar{y} \mid \Box\varphi \,]\!] = \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \times D^m,$$

where all of $\bar{x}$ actually occur freely in $\varphi$ whereas none of $\bar{y}$ does; and the square in question, with this interpretation in place of $\mathbf{int}_{D^{n+m}}$, commutes trivially, regardless of whether projections are open or not.

The other cases of commutation, for $f^{\mathfrak{M}}$ and $\Delta$, are also required by the well-definedness of the semantics. As we noted, the syntax of first-order modal language we adopt has the feature that, given any variables $y$, $z$ and sentence $\varphi(y, z)$ in which only $y$ and $z$ occur freely,

- $\Box([y/z]\varphi)$, the sentence obtained by first substituting $y$ for $z$ in $\varphi$ and then applying $\Box$,
- $[y/z](\Box\varphi)$, the sentence obtained by first applying $\Box$ to $\varphi$ and then substituting $t$ for $z$,

are identical; if you write down these two sentences unpacking the defined operation $[y/z]$, in both cases you just have $\Box\varphi(y, y)$—taking $y = z$ as an instance of $\varphi$, it is just $\Box(y = y)$.[11] Corresponding to these two orders of applying syntactic operations, we semantically need

$$\begin{array}{ccc} [\![\, y, z \mid \varphi(y,z) \,]\!] & \xmapsto{\ \mathbf{int}_{D^2}\ } & [\![\, y, z \mid \Box\varphi(y,z) \,]\!] \\ {\scriptstyle \Delta^{-1}} \big\downarrow & \circlearrowleft & \big\downarrow {\scriptstyle \Delta^{-1}} \\ [\![\, y \mid \varphi(y,y) \,]\!] & \xmapsto[\ \mathbf{int}_{D}\ ]{} & [\![\, y \mid \Box\varphi(y,y) \,]\!] \end{array}$$

to commute in order for $[\![\, y \mid \Box\varphi(y, y) \,]\!]$ to be well-defined.

Similarly, given any sentence $\varphi$ (in which only $z$ occurs freely) and term $t$ (that is free for $z$ in $\varphi$), $\Box([t/z]\varphi)$ and $[t/z](\Box\varphi)$ are identical; it is just the sentence $\Box\varphi(t)$. Therefore,

$$\begin{array}{ccc} [\![\, z \mid \varphi \,]\!] & \xmapsto{\ \mathbf{int}_{D}\ } & [\![\, z \mid \Box\varphi \,]\!] \\ {\scriptstyle [\![\, \bar{y} \mid t \,]\!]^{-1}} \big\downarrow & \circlearrowleft & \big\downarrow {\scriptstyle [\![\, \bar{y} \mid t \,]\!]^{-1}} \\ [\![\, \bar{y} \mid [t/z]\varphi \,]\!] & \xmapsto[\ \mathbf{int}_{D^m}\ ]{} & [\![\, \bar{y} \mid \Box([t/z]\varphi) \,]\!] = [\![\, \bar{y} \mid [t/z](\Box\varphi) \,]\!] \end{array}$$

needs to commute for $[\![\, \bar{y} \mid \Box\varphi(t) \,]\!]$ to be well-defined. These are how, under certain assumptions on syntax and semantics,[12] Fact 1 implies that the sheaf property is needed to make the semantics well defined.

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One cost of the non-uniformity in this sense is that we would have to give up

E $$\frac{\varphi \vdash \psi \qquad \psi \vdash \varphi}{\Box\varphi \vdash \Box\psi}.$$

This may fail because, even when $[\![\, \bar{x} \mid \varphi \,]\!] = [\![\, \bar{x} \mid \psi \,]\!]$, under a non-uniform interpretation of $\Box$ the application of $\Box$ to $\varphi$ and to $\psi$ may be interpreted differently, if different sets of free variables are in $\varphi$ and $\psi$, so that $[\![\, \bar{x} \mid \Box\varphi \,]\!] \neq [\![\, \bar{x} \mid \Box\psi \,]\!]$. We will give a thorough analysis of non-uniformity and variable-sensitivity in Chapter IV. Here we choose to save E (and M, C, K, and so on) by interpreting $\Box$ uniformly.

[11] In other words, if you need to distinguish the two orders of applying the two syntactic operations, then you need to treat the substitution operation as a primitive syntactic operation of the language, rather than as a derived one as in the usual language.

[12] In particular, that the syntax comes with the usual substitution, and that $\Box$ is interpreted uniformly (see footnote 10).

### I.4.2 Sheaves over a Neighborhood Frame

In Subsection I.4.1, we saw an aspect of topological sheaves that is essential in interpreting first-order modal logic. In this subsection, we extend this aspect and obtain a generalized notion of sheaves over more general neighborhood frames.

This extension can be done with a straightforward idea because, even though the notion of open sets may not make sense any more in general neighborhood frames, the notions of continuous and open maps can be defined without open sets, but with interior operations and hence, equivalently, with neighborhood functions. (The non-trivial part of the extension is to make sure that the desired property of topological sheaves still obtains with our generalized definition of sheaves, as well as that a completeness result is available.) Recall, as we saw in Subsection I.4.1, that a map $f : Y \to X$ between topological spaces $Y$, $X$ is continuous iff

$$f^{-1}[\mathbf{int}_X(B)] \subseteq \mathbf{int}_Y(f^{-1}[B])$$

and open iff

$$\mathbf{int}_Y(f^{-1}[B]) \subseteq f^{-1}[\mathbf{int}_X(B)].$$

Rewriting these relations in terms of neighborhood functions, we enter:

**Definition.** Given any pair of neighborhood frames $X$ and $Y$,[13] a map $f : Y \to X$ is said to be *continuous* if

$$B \in \mathcal{N}_X(f(x)) \implies f^{-1}[B] \in \mathcal{N}_Y(x)$$

for every $x \in Y$ and $B \subseteq X$, and *open* if

$$f^{-1}[B] \in \mathcal{N}_Y(x) \implies B \in \mathcal{N}_X(f(x))$$

for every $x \in Y$ and $B \subseteq X$.

[13] Just like our notation for topological spaces, we write $X$ for neighborhood frames $(|X|, \mathcal{N}_X)$.

Clearly, continuous maps and open maps are both composable. Thus neighborhood frames and these maps (continuous maps, open maps, or both) form subcategories of **Sets**; in particular, we consider the category **Nb** of continuous maps. And we take the slice category **Nb**$/X$ over a fixed neighborhood frame $X$, which is a subcategory of **Sets**$/|X|$, for the sake of interpreting first-order logic. Indeed, not just the category **Nb** of all neighborhood frames, we also have full subcategories of it with constraints on frames (**Top** is an example of such a category). In particular, let us say that a neighborhood frame $(X, \mathcal{N})$ is *MC* (after the logical rule M and axiom C, to which (I.2) and (I.3) correspond) if it satisfies

(I.2) $$A \subseteq B \subseteq X \text{ and } A \in \mathcal{N}(w) \implies B \in \mathcal{N}(w),$$

(I.3) $$A, B \in \mathcal{N}(w) \implies A \cap B \in \mathcal{N}(w);$$ [14]

we can combine (I.2) and (I.3) together into the following, equivalent condition:

$$\mathbf{int}(A \cap B) = \mathbf{int}(A) \cap \mathbf{int}(B),$$

that is, that the interior operation preserves binary meets (and hence all finite meets, except possibly the empty meet $X$). And let us write **MCNb** for the category of MC frames and continuous maps.

It is crucial to distinguish **MCNb** from **Nb** for several reasons. One is that, given an MC frame $X$, **Nb**$/X$ and **MCNb**$/X$ have different products. In **MCNb**$/X$, products are defined in essentially the same way they are in **Top**$/X$; that is, given MC frames $(D_i, \pi_{D_i} : D_i \to X)$ over $X$, their product in **MCNb**$/X$ is $D_1 \times_X \cdots \times_X D_n$ equipped with a neighborhood function $\mathcal{N}$ such that

$$U \in \mathcal{N}(x_1, \ldots, x_n) \iff U_1 \times_X \cdots \times_X U_n \subseteq U \text{ for some } U_1 \in \mathcal{N}_{D_1}(x_1), \ldots, U_n \in \mathcal{N}_{D_n}(x_n)$$

for every $(x_1, \ldots, x_n) \in D_1 \times_X \cdots \times_X D_n$, and with the projection

$$\pi : D_1 \times_X \cdots \times_X D_n \to X :: (x_1, \ldots, x_n) \mapsto \pi_{D_1}(x_1) = \cdots = \pi_{D_n}(x_n).$$

Then all the projections $p_i : D_1 \times_X \cdots \times_X D_n \to D_i$ are continuous and open. Also, the continuity of all $\pi_i$ implies that $\pi$ is continuous. Moreover, this definition guarantees that the diagonal map $\Delta : D \to D^2$ is continuous.

With these notions, we can extend Fact 1 as a definition of topological sheaves to sheaves over general neighborhood frames.

[14] We could instead say such $(X, \mathcal{N})$ is *quasifiltered*, since that $(X, \mathcal{N})$ is MC means that each $\mathcal{N}(w)$ is closed under supersets and binary meets, and therefore is a quasifilter. But we opt for the shorter name.

**Definition.** Given neighborhood frames $X$ and $D$, we say that a map $\pi : D \to X$ is a *local isomorphism* if

(i) $\pi$ is continuous and open, and

(ii) for every $a \in D$ such that $\mathcal{N}_D(a) \neq \varnothing$, there is $U \in \mathcal{N}_D(a)$ such that $\pi \restriction U : U \to \pi[U]$ is bijective.

We say that the pair $(D, \pi : D \to X)$ is a *neighborhood sheaf* over $X$ if $\pi$ is a local isomorphism.

And, as we did before, we define maps of sheaves over $X$ to be continuous maps over $X$, so that the category of sheaves and maps of sheaves over $X$ is a full subcategory of $\mathbf{MCNb}/X$. Then all the nice properties of the category of topological sheaves we mentioned in Subsections I.3.3 and I.4.1 carry over to the category of sheaves over an MC neighborhood frame $X$. In particular,

**Fact.** Maps of sheaves are local isomorphisms; hence the category of sheaves over a given MC neighborhood frame $X$ is $\mathbf{LI}/X$, the category of local isomorphisms over $X$.

**Fact.** For any MC neighborhood frames $X$ and $D$ and any continuous and open map $\pi : D \to X$ (that is, that satisfies (i) in the definition above), (ii) is the case iff

(iii) The diagonal map $\Delta : D \to D^2$ is open.

That is, in the same way as we did with topological sheaves, we have all relevant maps continuous and open if and only if we take sheaves. This completes our preparation of semantic structures needed for extending topological-sheaf semantics to neighborhood-sheaf semantics.

### I.4.3 Neighborhood-Sheaf Semantics for First-Order Modal Logic

Now we are ready to extend sheaf semantics to more general, MC neighborhood frames and to provide a semantics for first-order modal logic that is more general than **FOS4**. We can take a straightforward extension of the semantics in Subsection I.4.1, because in Subsection I.4.2 we extended all the relevant notions to the category $\mathbf{LI}/X$ of neighborhood sheaves.

Let us again fix any first-order modal language $\mathcal{L}$. Then we enter:

**Definition.** Given a first-order modal language $\mathcal{L}$, by a *neighborhood-sheaf model* for $\mathcal{L}$ we mean a structure $\mathfrak{M} = (\pi, R_i^{\mathfrak{M}}, f_j^{\mathfrak{M}}, c_k^{\mathfrak{M}})_{i \in I, j \in J, k \in K}$ consisting of

- a surjective local isomorphism $\pi$; let us write $X$ and $D$ for its base and total spaces, so that $\pi : D \twoheadrightarrow X$;
- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq |D|^n$ of the $n$-fold product of $|D|$ over $|X|$;
- for each $n$-ary function symbol $f$, a continuous map $f^{\mathfrak{M}} : D^n \to D$ over $X$; and
- for each constant $c$, a continuous map $c^{\mathfrak{M}} : X \to D$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

**Definition.** Given a first-order modal language $\mathcal{L}$, by a *neighborhood-sheaf interpretation* for $\mathcal{L}$ we mean a pair $(\mathfrak{M}, [\![-]\!])$ of a neighborhood-sheaf model $\mathfrak{M}$ with a map $[\![-]\!]$ (of the suitable type) that satisfies

$$
\begin{aligned}
[\![\, \bar{x} \mid R\bar{x} \,]\!] &= R^{\mathfrak{M}} \qquad \text{for } n\text{-ary primitive predicate } R, \text{ and} \\
[\![\, x, y \mid x = y \,]\!] &= \Delta[D] \qquad \text{in particular;} \\
[\![\, \bar{x} \mid \top \,]\!] &= D^n; \\
[\![\, \bar{x} \mid \neg\varphi \,]\!] &= D^n \setminus [\![\, \bar{x} \mid \varphi \,]\!] \qquad (\text{that is, } [\![\neg]\!] = D^n \setminus -); \\
[\![\, \bar{x} \mid \varphi \wedge \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \cap [\![\, \bar{x} \mid \psi \,]\!] \qquad (\text{that is, } [\![\wedge]\!] = \cap); \\
&\vdots \\
[\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] &= p[[\![\, \bar{x}, y \mid \varphi \,]\!]]; \\
[\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!] &= {p_n}^{-1}[[\![\, \bar{x} \mid \varphi(\bar{x}) \,]\!]]; \\
[\![\, \bar{x}, \bar{y} \mid [t/z]\varphi \,]\!] &= (1_{D^n} \times [\![\, \bar{y} \mid t \,]\!])^{-1}[[\![\, \bar{x}, z \mid \varphi \,]\!]]; \\
[\![\, \bar{x}, y \mid [y/z]\varphi \,]\!] &= (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \varphi \,]\!]]; \\
[\![\, \bar{x} \mid \Box\varphi \,]\!] &= \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Box]\!] = \mathbf{int}_{D^n}); \\
[\![\, \bar{x} \mid \Diamond\varphi \,]\!] &= \mathbf{cl}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Diamond]\!] = \mathbf{cl}_{D^n}).
\end{aligned}
$$

The class of such interpretations constitutes neighborhood-sheaf semantics for first-order modal logic. In the same way topological-sheaf semantics unified classical first-order logic and **S4**, the new semantics unifies classical first-order logic and **MC**.

**Definition.** First-order modal logic **FOMC** consists of the following two sorts of axioms and rules.

1. All axioms and rules of (classical) first-order logic.
2. The rule and axiom of propositional modal logic **MC**; that is, M and C.

The converse Barcan formula and its $\exists$ variant are provable in **FOMC**, with the same proofs we saw in Subsection I.3.5. By contrast,

$$x = y \vdash \Box(x = y)$$

is no longer provable, for its proof needs N. Instead, we can use M in place of N to prove

$$\dfrac{x = y \vdash \Box(x = x) \to \Box(x = y) \qquad \dfrac{\varphi \vdash x = x}{\Box\varphi \vdash \Box(x = x)}}{\Box\varphi \wedge x = y \vdash \Box(x = y)},$$

a theorem that says "If anything is necessary, identity is necessary (though it may be that nothing is necessary)".

Again, it can be checked straightforwardly that **FOMC** is sound with respect to neighborhood-sheaf semantics. Moreover, as the principal result of this dissertation, it is complete in the following form that extends Theorem 1 (Subsection I.1.2).[15]

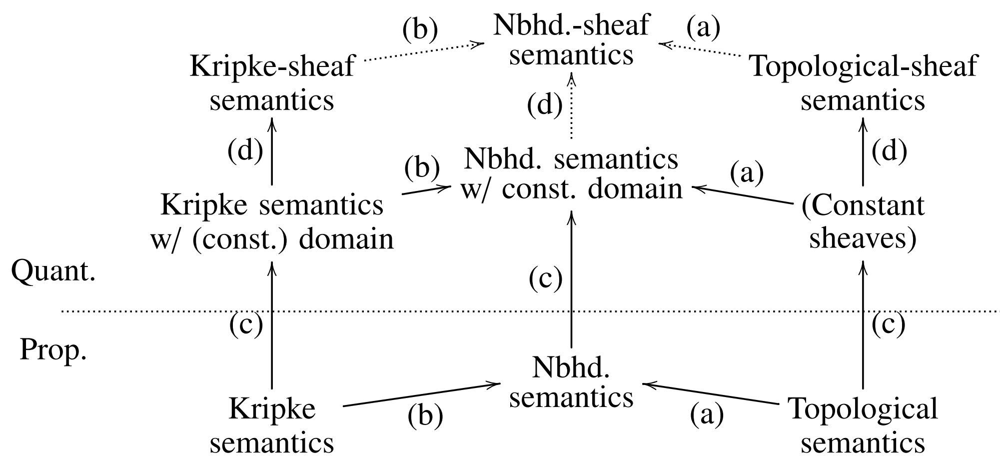
**Theorem.** *For any consistent theory* $\mathbb{T}$ *of first-order modal logic extending* **FOMC**, *there exists a neighborhood-sheaf interpretation* $(\pi, [\![-]\!])$ *that validates all and only theorems of* $\mathbb{T}$.

### I.4.4 Comparison to Other Semantics for First-Order Modal Logic

We close this chapter by comparing the semantics we reviewed with other frameworks of semantics for first-order modal logic—in particular, *neighborhood semantics with constant domains* and

[15] I gave a completeness proof for **FOS4** with Awodey in [5]. It was inspired by completeness proofs of McKinsey and Tarski [30], Segerberg [39], and Butz and Moerdijk [8, 9, 31]. My completeness proof for **FOMC**, which generalizes this proof for **FOS4**, will be given in Section VI.3.

*Kripke-sheaf semantics*. The relations among relevant frameworks can be summarized by the following diagram:




Here the labels with alphabets indicate semantic ideas explained in the following, and the dotted arrows indicate the unification offered in this dissertation.

Among several frameworks of semantics for propositional modal logic—that is, on the bottom level in the diagram—neighborhood semantics extends topological semantics and Kripke semantics by the following ideas, respectively:

(a) Neighborhood semantics extends topological semantics by considering interior operations that are more general than topological ones.

(b) Neighborhood semantics generalizes accessibility relations in Kripke semantics with the neighborhood notion of accessibility.[16]

This is how neighborhood semantics subsumes and unifies topological semantics and Kripke semantics for propositional modal logic.

To extend his Kripke semantics for propositional modal logic to the level of quantified modal logic, Kripke [19] took advantage of the following idea:

(c) Interpret the first-order part of the language with a domain of all possible individuals.

[16] See Subsection II.1.1.

This idea gives rise to Kripke semantics with domains, and in particular with *constant* domains—thereby bringing semantics to the middle level in the diagram above.

*Neighborhood semantics with constant domains* was given by Arló-Costa and Pacuit [2], who showed how to combine the ideas (b) and (c). This semantics has constant domains of all possible individuals, but interprets modal operators in terms of neighborhoods rather than accessibility relations.

This extension to the quantified case based on the idea (c), however, is not general enough for treating the necessity and contingency of identity of individuals; in particular, it forces the identity and non-identity of individuals to always be necessary. By contrast, *Kripke-sheaf semantics* [12, 13, 17] can model the contingency of non-identity, by extending the idea (c) further with

(d) Instead of taking a domain of all possible individuals, take a sheaf over a structured set of possible worlds (a Kripke-sheaf over a Kripke frame, for instance).

Then, from the viewpoint of sheaves, constant domains are subsumed as constant sheaves.

Neighborhood-sheaf semantics applies this idea, (d), to neighborhood semantics; it subsumes neighborhood semantics with constant domains as a subclass (namely, with constant sheaves), and brings it up to the sheaf level—that is, the top level in the diagram above.[17]

Moreover, neighborhood-sheaf semantics subsumes Kripke-sheaf semantics, because Kripke-sheaves are simply neighborhood-sheaves over Kripke frames as regarded as neighborhood frames. Thus, on the sheaf level, neighborhood-sheaf semantics subsumes and unifies not only topological-sheaf semantics via (a) but also Kripke-sheaf semantics via (b), in just the same way neighborhood semantics subsumes and unifies topological semantics and Kripke semantics for propositional modal logic.

[17] Neighborhood-sheaf semantics fails to entirely subsume neighborhood semantics with constant domains, because the former requires certain conditions on neighborhood structures, which the latter does not.

# II.0 PHILOSOPHICAL INTRODUCTION

## II.1 QUESTIONS THAT THIS DISSERTATION TRIES TO ANSWER

### II.1.1 Epistemic Logic and Topological Semantics

Modal logic has many applications, as modal operators can be read in many ways. While it is not a goal of this dissertation to discuss any of such particular readings, the epistemic reading provides the driving force for this dissertation. In this subsection, we briefly lay out a possible-world interpretation of propositional epistemic logic. This interpretation, as it will turn out, gives rise to topological semantics for modal logic; in fact, we give an epistemic interpretation of topology in terms of verifiability and falsifiability. And this interpretation will show that Kripke's semantics in terms of an accessibility relation is inadequate in representing the verifiability and falsifiability reading of modal operators.

By a possible-world semantics, let us refer to a semantics equipped with a (nonempty) set of points in which subsets of the set can represent propositions; so, whereas Kripke's semantics with an accessibility relation among possible worlds is surely a possible-world semantics, not every possible-world semantics is equipped with an accessibility relation. Indeed, while we are going to lay out a semantics for modal logic (propositional, in this subsection), we give an interpretation of modal operators that does not presuppose—but even precludes, as we will argue—an accessibility relation.

Let us take a set $W \neq \varnothing$ and regard it as a set of possible worlds, in the sense that we represent propositions with subsets of $W$. Then assume that some subsets of $W$ represent *observable propositions*. A typical example is the following: Consider an infinite series of coin tosses (the first toss, the second, ..., *ad infinitum*) and assume that, for each toss, we can observe its outcome. That is, when we introduce an atomic sentence

$$p_n \qquad \text{for} \qquad \text{“The } n\text{th toss comes up heads”}$$

for each $n \in \mathbb{N}$ (for the sake of simplicity, let us say the series of tosses starts with the "0th" toss), it seems plausible that each $p_n$ expresses an observable proposition. Then we provide a possible-world semantics, for these sentences $p_n$, with the set of all possible histories, each of which is an infinite sequence of coin-toss outcomes; formally, with 0 and 1 standing for heads and tails, each history is of the form $w : \mathbb{N} \to 2$, so that $W = 2^{\mathbb{N}}$, the Cantor set. So we interpret each sentence $p_n$ and its negation $\neg p_n$ with the propositions

$$[\![p_n]\!] = \{ w : \mathbb{N} \to 2 \mid w(n) = 0 \} \subseteq W,$$
$$[\![\neg p_n]\!] = \{ w : \mathbb{N} \to 2 \mid w(n) = 1 \} \subseteq W,$$

and we assume both $[\![p_n]\!]$ and $[\![\neg p_n]\!]$ to be observable for each $n \in \mathbb{N}$. In this way, we have a set $W$ of possible worlds along with a special family of observable propositions.

We can extend this to a possible-world semantics for classical propositional logic by interpreting the Boolean connectives $\neg$, $\wedge$, $\vee$, $\to$ with the corresponding Boolean operations on $\mathcal{P}(W)$, that is, $W \setminus -$, $\cap$, $\cup$, and $\to$.[1] Furthermore, we interpret the modal operators $\Box$ and $\Diamond$. In particular, we are interested in the epistemic reading of $\Box$ in which, for each sentence $\varphi$, we read

$$\Box\varphi \qquad \text{as} \qquad \text{“It is verifiably true that } \varphi\text{”, or “We can verify that } \varphi\text{”.}$$

Let us take the notion of verification in a way that to verify something is to observe something that entails it. This idea seems to yield the truth condition that $\Box\varphi$ is true at $w$—that is, $\varphi$ is verifiably true at $w$—iff

- There is a proposition $B \subseteq W$ such that
  - $B$ is observable,
  - $B$ is true at $w$, and
  - $B$ entails $\varphi$.

[1] We define the Boolean operation $\to : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$ so that $A \to B = (W \setminus A) \cup B$ for every $A, B \subseteq W$.

More formally, writing $\mathcal{B} \subseteq \mathcal{P}(W)$ for the family of observable propositions, we set

$$w \in [\![\Box\varphi]\!] \iff w \in B \subseteq [\![\varphi]\!] \text{ for some } B \in \mathcal{B}. \tag{II.1}$$

In the example of coin tosses above, consider the sentence "At least one toss comes up heads", that is,

$$\bigvee_{n\in\mathbb{N}} p_n,$$

and the world $w_{\text{tails}}$ such that $w_{\text{tails}}(n) = 1$ for all $n \in \mathbb{N}$—that is, in which all tosses come up tails. Then $\Box\bigvee_{n\in\mathbb{N}} p_n$ is true at every $w \in W$ except $w_{\text{tails}}$, since if $w(m) = 0$ for some $m \in \mathbb{N}$—that is, if some toss, say the $m$th, comes up heads in $w$—then

$$w \in [\![p_m]\!] \subseteq \bigcup_{n\in\mathbb{N}} [\![p_n]\!] = [\![\bigvee_{n\in\mathbb{N}} p_n]\!]$$

for the observable proposition $[\![p_m]\!]$—that is, observing the $m$th toss coming up heads verifies $\varphi$ at $w$. By contrast, consider the sentence "All tosses come up heads", that is,

$$\bigwedge_{n\in\mathbb{N}} p_n.$$

Then $\Box\bigwedge_{n\in\mathbb{N}} p_n$ is true at no $w \in W$, not even at the world $w_{\text{heads}}$ at which $\bigwedge_{n\in\mathbb{N}} p_n$ is actually true (that is, such that $w_{\text{heads}}(n) = 0$ for all $n \in \mathbb{N}$). Conceptually, it is because, in any sense of observability good enough to express the problem of induction, we can never observe the outcomes of *all* tosses (although, by a crucial contrast, we can observe the outcome of *each* toss). Indeed, in this setting of infinite coin tosses, we can formalize the problem of induction, in one of its forms, by the fact that, at $w_{\text{heads}}$ for instance, $p_n$ and $\Box p_n$ are true for every $n \in \mathbb{N}$ and $\bigwedge_{n\in\mathbb{N}} p_n$ is true as well, but nonetheless $\Box\bigwedge_{n\in\mathbb{N}} p_n$ is not true. For the rest of this subsection, by the problem of induction we mean this form of it.

A formal proof that $\Box\bigwedge_{n\in\mathbb{N}} p_n$ is not true at $w_{\text{heads}}$ depends on a formal definition of $\mathcal{B}$ (note that, although we have already assumed $[\![p_n]\!], [\![\neg p_n]\!] \in \mathcal{B}$ for all $n \in \mathbb{N}$, we have not said anything about what is *not* in $\mathcal{B}$). We might set, for instance,

$$\mathcal{B} = \{\, [\![\varphi_n]\!] \mid n \in \mathbb{N} \text{ and } \varphi_n \in \{p_n, \neg p_n\} \,\},$$

assuming we can only observe the outcomes of single tosses.[2] Then, for any $B \in \mathcal{B}$, say $B = [\![p_m]\!]$, there is $w \in W$ at which $p_m$ is true but $p_k$ is not (for some $k \neq m$), that is, $w \in B$ but $w \notin [\![\bigwedge_{n\in\mathbb{N}} p_n]\!]$; thus $B \subseteq [\![\bigwedge_{n\in\mathbb{N}} p_n]\!]$ for no $B \in \mathcal{B}$ and therefore $w_{\text{heads}} \notin [\![\Box \bigwedge_{n\in\mathbb{N}} p_n]\!]$. Put intuitively, this proof says that any observation $B$ is consistent with the possibility $w$ that the hypothesis $\bigwedge_{n\in\mathbb{N}} p_n$ ("All tosses comes up heads") eventually turns out false, thereby capturing the problem of induction.

Instead of defining $[\![\Box\varphi]\!]$ only for sets $[\![\varphi]\!]$ interpreting sentences with (II.1), let us more generally define an operation $\mathbf{int} : \mathcal{P}(W) \to \mathcal{P}(W)$ (called an "interior" operation for the reason that we will clarify shortly) such that

$$w \in \mathbf{int}(A) \iff w \in B \subseteq A \text{ for some } B \in \mathcal{B}, \tag{II.2}$$

and interpret $\Box$ with $\mathbf{int}$ by setting $[\![\Box\varphi]\!] = \mathbf{int}([\![\varphi]\!])$; this enables us to investigate the structure of observability and verifiability on the set $W$ of possible worlds that obtains independently of a particular interpretation $[\![-]\!]$ of sentences.

This operation $\mathbf{int}$ is a monotone operation, that is,

$$A_0 \subseteq A_1 \implies \mathbf{int}(A_0) \subseteq \mathbf{int}(A_1), \tag{II.3}$$

because if $A_0 \subseteq A_1$ then

$$w \in \mathbf{int}(A_0) \overset{\text{(II.2)}}{\implies} w \in B \subseteq A_0 \subseteq A_1 \text{ for some } B \in \mathcal{B} \overset{\text{(II.2)}}{\implies} w \in \mathbf{int}(A_1).$$

Also, by (II.2), $w \in \mathbf{int}(A)$ entails $w \in A$; hence

$$\mathbf{int}(A) \subseteq A. \tag{II.4}$$

It is important to observe

$$\mathbf{int}(A) \subseteq \mathbf{int}(\mathbf{int}(A)). \tag{II.5}$$

[2]This assumption seems too strong, and we will weaken it by assuming a condition (ii) for $\mathcal{B}$ shortly.

This holds because

$$\begin{aligned} w \in \mathbf{int}(A) &\overset{\text{(II.2)}}{\Longrightarrow} \text{there is } B \in \mathcal{B} \text{ such that } w \in B \subseteq A \\ &\Longrightarrow \text{there is } B \in \mathcal{B} \text{ such that } w \in B \text{ and } w' \in B \subseteq A \text{ for every } w' \in B \\ &\overset{\text{(II.2)}}{\Longrightarrow} \text{there is } B \in \mathcal{B} \text{ such that } w \in B \text{ and } w' \in \mathbf{int}(A) \text{ for every } w' \in B \\ &\Longrightarrow \text{there is } B \in \mathcal{B} \text{ such that } w \in B \subseteq \mathbf{int}(A) \\ &\overset{\text{(II.2)}}{\Longrightarrow} w \in \mathbf{int}(\mathbf{int}(A)). \end{aligned}$$

These two properties (II.4) and (II.5) justify calling **int** an *interior* operation on $\mathcal{P}(W)$. Moreover, when we say a binary sequent $\varphi \vdash \psi$ is valid in a model $(W, \mathcal{B}, [\![-]\!])$ if $[\![\varphi]\!] \subseteq [\![\psi]\!]$ (and in particular that $\vdash \varphi$ is valid if $[\![\varphi]\!] = W$), (II.3)–(II.5) translate respectively into the validity of the rule and axioms

M $$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

T $$\Box\varphi \vdash \varphi$$

4 $$\Box\varphi \vdash \Box\Box\varphi$$

of modal logic.[3]

With a few assumptions on $\mathcal{B}$, we can also show

(II.6) $$\mathbf{int}(W) = W,$$

(II.7) $$\mathbf{int}(A_0) \cap \mathbf{int}(A_1) = \mathbf{int}(A_0 \cap A_1).$$

To have (II.6), we should assume

(i) For every $w \in \mathbb{W}$, there is $B \in \mathcal{B}$ such that $w \in B$;

that is, in every world something is observable (in the sense of being observable and true in that world). Then it follows that $W \subseteq \mathbf{int}(W)$, and hence (II.6), because

$$w \in W \overset{\text{(i)}}{\Longrightarrow} w \in B \subseteq W \text{ for some } B \in \mathcal{B} \overset{\text{(II.2)}}{\Longrightarrow} w \in \mathbf{int}(W).$$

For (II.7), we may assume

[3] In this chapter, we formulate logic in terms of binary sequents. $\vdash \varphi$ is short for $\top \vdash \varphi$ with $\top$ for the truth.

- $B_0 \cap B_1 \in \mathcal{B}$ for every $B_0, B_1 \in \mathcal{B}$.

This roughly means that a combination of *finitely* many observations is itself an observation. Think of tossing a coin $n$ times; not only can we observe the outcome of *each* toss, we can observe *all* the outcomes throughout (since the series of $n$ tosses ends in a finite amount of time). So, for the example of infinite coin tosses, we set

$$\mathcal{B} = \{ \bigcap_{n \in J} [\![\varphi_n]\!] \mid J \text{ is a finite (nonempty) subset of } \mathbb{N} \text{ and } \varphi_n \in \{p_n, \neg p_n\} \text{ for each } n \in J \}.$$

(With this $\mathcal{B}$, essentially the same argument as the one on p. 41 shows $w_{\text{heads}} \notin [\![\Box \bigwedge_{n \in \mathbb{N}} p_n]\!]$.) For a more general setting, however, it may be plausible to weaken the assumption above to

(ii) For $B_0, B_1 \in \mathcal{B}$, if $w \in B_0 \cap B_1$ then $w \in B_2 \subseteq B_0 \cap B_1$ for some $B_2 \in \mathcal{B}$,

since how observations are combined may depend on each world, so that different $w, w' \in B_0 \cap B_1$ may have different $B_2, B_3 \in \mathcal{B}$ such that $w \in B_2 \subseteq B_0 \cap B_1$ and $w' \in B_3 \subseteq B_0 \cap B_1$. With this weaker assumption (ii), (II.7) is derived as follows.

$$\begin{aligned} w \in \mathbf{int}(A_0), \mathbf{int}(A_1) &\overset{\text{(II.2)}}{\Longrightarrow} w \in B_0 \subseteq A_0 \text{ and } w \in B_1 \subseteq A_1 \text{ for some } B_0, B_1 \in \mathcal{B} \\ &\overset{\text{(ii)}}{\Longrightarrow} w \in B_2 \subseteq B_0 \cap B_1 \subseteq A_0 \cap A_1 \text{ for some } B_2 \in \mathcal{B} \\ &\overset{\text{(II.2)}}{\Longrightarrow} w \in \mathbf{int}(A_0 \cap A_1). \end{aligned}$$

(II.6) and (II.7) correspond respectively to the following rule and axiom.

N $$\frac{\vdash \varphi}{\vdash \Box\varphi}$$

C $$\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$$

In this way, the semantics given by $(W, \mathcal{B}, [\![-]\!])$ satisfying (II.2) along with (i) and (ii) validates M, T, 4, N, C (in addition to all the rules and axioms of classical propositional logic). This means that, with respect to this semantics, modal logic **S4** is sound.[4]

[4] **S4** is often formulated with T, 4, N and

K $$\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$$

instead of M and C, but it is easy to show that, on classical logic, M and C entail K, while N and K entail M and C.

As is immediately implied by the classical result of McKinsey and Tarski [30], propositional **S4** is sound and complete with respect to topological semantics, which interprets modal logic with

- a topological space, that is, a set $W$ equipped with a topological interior operation $\mathbf{int} : \mathcal{P}(W) \to \mathcal{P}(W)$ that satisfies the axioms (II.3)–(II.7); and
- a map $[\![-]\!]$, sending sentences to subsets of $W$, such that $[\![\Box\varphi]\!] = \mathbf{int}([\![\varphi]\!])$.

Indeed, what we have done with $(W, \mathcal{B}, [\![-]\!])$ is to use (II.2), which formally expresses how verifiability is related to observability, to define a topological interior operation **int** (with the assumption of (i) and (ii)), thereby deriving topological semantics as well as **S4** from the notions of observability and verifiability.[5]

While we have laid out the verifiability interpretation of **int** and $\Box$, other notions from topology are susceptible of epistemic interpretations as well. In particular, let us lay out interpretations for open sets, closed sets, and the closure operation.

Every topological space $(W, \mathbf{int})$ comes with two privileged families of subsets, the *open* and the *closed* subsets. Open sets are defined as fixed points of **int**; or, because **int** is idempotent by (II.4) and (II.5), open sets can also be defined as the images of **int**. Thus, writing $\mathcal{O} \subseteq \mathcal{P}(W)$ for the family of open sets, we have

$$\mathcal{O} = \{\, A \subseteq W \mid \mathbf{int}(A) = A \,\} = \{\, \mathbf{int}(A) \mid A \subseteq W \,\}.$$

Openness of subsets of $W$, or propositions, can be interpreted epistemically as follows. While **int** represents one sense of verifiability by interpreting $\Box$ as read as "It is verifiably true that ...", there is a closely related but crucially different notion of verifiability, namely, verifiability as a property of propositions. Taking the example of infinite coin tosses again, let $\varphi$ be short for

$$p_2 \vee \bigwedge_{n \in \mathbb{N}} \neg p_n,$$

that is, "Either the second toss comes up heads or no toss does". We can verify $\varphi$ if we are lucky; that is, if the second toss comes up heads and so $p_2$ is true at a world $w$, then $w \in [\![p_2]\!] \subseteq [\![\varphi]\!]$ for $[\![p_2]\!] \in \mathcal{B}$ and hence $w \in [\![\Box\varphi]\!]$. On the other hand, if we are unlucky and $p_2$ false, we cannot verify

[5] Any family of subsets that satisfies (i) and (ii) is called a *basis* for a topology, and "generates" a topology via (II.2). Therefore, what we have shown is that, when we take the family of observable propositions as a basis, it generates a topology of verifiability.

$\varphi$ even if it is true. Consider the world $w_{\text{tails}}$ at which all tosses comes up tails and $p_n$ is false for all $n \in \mathbb{N}$. At $w_{\text{tails}}$, even though $\varphi$ is true (since $\bigwedge_{n\in\mathbb{N}} \neg p_n$ is true), $\Box\varphi$ is not (since for every $B \in \mathcal{B}$ such that $w_{\text{tails}} \in B$ there is $w \in B$ at which $p_2$ is false but $p_m$ is true for some $m$, which means $B \not\subseteq [\![\varphi]\!]$). In this way, whether $\varphi$ is verifiably true or not (verifiability in the sense we laid out above) depends contingently on worlds, since $\Box\varphi$ can be false even if $\varphi$ is true; and, to describe this contingency, we say that the proposition $[\![\varphi]\!]$ is not verifiable by itself. In other words, for a proposition to be verifiable in the second, world-independent sense, we require that it be verifiably true (verifiable in the first, world-dependent sense) whenever it is true. So, formally, we say

$$[\![\varphi]\!] \text{ is verifiable if } [\![\varphi]\!] \subseteq [\![\Box\varphi]\!], \text{ or generally}$$
$$A \subseteq W \text{ is verifiable if } A \subseteq \mathbf{int}(A);$$

but this, combined this with (II.4), amounts to saying that $A \subseteq W$ is verifiable iff $\mathbf{int}(A) = A$, that is, iff $A$ is open.

Closed sets are defined as the complements of open sets; that is, $A \subseteq W$ is closed if $W \setminus A$ is open—or we can read this epistemically, with help of the classical interpretation $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$, as saying $A$ is closed if its negation is verifiable. Hence we can interpret closedness of a proposition as representing its falsifiability, as a world-independent property of propositions. To make this more obvious, let us further unpack the definition and we can see that $A$ is closed iff

- For every $w \in W$, if $w \notin A$ then there is $B \in \mathcal{B}$ such that $w \in B \subseteq W \setminus A$,

where we can read $w \in B \subseteq W \setminus A$ as observable $B$ falsifying $A$ at $w$. Thus we interpret closedness with falsifiablity, so that a proposition is falsifiable iff we can falsify it whenever it is false.

In any topological space $(W, \mathbf{int})$, the interior operation $\mathbf{int}$ has a dual, the *closure* operation $\mathbf{cl} : \mathcal{P}(W) \to \mathcal{P}(W)$, which is defined as

$$\mathbf{cl}(A) = W \setminus \mathbf{int}(W \setminus A).$$

Let us interpret the modal operator $\Diamond$ with $\mathbf{cl}$, so that

$$[\![\Diamond\varphi]\!] = \mathbf{cl}([\![\varphi]\!]);$$

this, combined with $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$, implies

$$[\![\Diamond\varphi]\!] = W \setminus \mathbf{int}(W \setminus [\![\varphi]\!]) = [\![\neg\Box\neg\varphi]\!].$$

It should be obvious from this that we can read

$\Diamond\varphi$ as "It is not verifiably false that $\varphi$", or "We cannot falsify the hypothesis that $\varphi$".

To make the reading clearer, let us observe that

$$w \in \mathbf{cl}(A) \iff w \notin \mathbf{int}(W \setminus A) \iff w \in B \subseteq W \setminus A \text{ for no } B \in \mathcal{B},$$

and again take the world $w_{\text{tails}}$ in the example of infinite coin tosses; at $w_{\text{tails}}$, $p_n$ is false for all $n \in \mathbb{N}$—all tosses comes up tails—and so $\bigvee_{n\in\mathbb{N}} p_n$—"At least one toss comes up heads"—is false. Observe that, nonetheless, $\Diamond\bigvee_{n\in\mathbb{N}} p_n$—"We cannot falsify the hypothesis that at least one toss comes up heads"—is true at $w_{\text{tails}}$, since any $B \in \mathcal{B}$ contains some $w \in W$ at which some $p_m$ is true and so $B \nsubseteq W \setminus [\![\bigvee_{n\in\mathbb{N}} p_n]\!]$. It is worth noting that $w_{\text{tails}}$ constitutes a counterexample to $[\![\Diamond\varphi]\!] \subseteq [\![\varphi]\!]$ for $\varphi = \bigvee_{n\in\mathbb{N}} p_n$. Indeed, dually to the discussion for verifiability above, falsifiability of propositions can be characterized by saying

$$A \subseteq W \text{ is falsifiable iff } \mathbf{cl}(A) \subseteq A,$$

which is another way to formally express the idea (which we saw in the previous paragraph) that we can falsify a falsifiable proposition unless it is true.

Let us compare the semantics we have laid out with Kripke semantics for **S4**, that is, a possible-world semantics with a reflexive and transitive accessibility relation among possible worlds. As is shown in Kripke [19], propositional **S4** is sound and complete with respect to **S4** Kripke semantics. This fact may appear to mean that the difference between topological and **S4** Kripke semantics is not semantically or logically significant. That is not correct, since the difference becomes logically significant once the language is extended to have infinitary conjunction. Recall that, as we showed above using the example of infinite coin tosses,

(II.8) $$\bigwedge_{i\in I} \Box\varphi_i \vdash \Box \bigwedge_{i\in I} \varphi_i \qquad (I \text{ may be infinite})$$

is not valid in topological semantics; this invalidity indeed represents the problem of induction. By contrast, we should observe, Kripke semantics manages to validate (II.8), thereby preventing us from representing the problem of induction.

(II.8) is valid in Kripke semantics for the following reason. Recall Kripke's truth condition for $\Box$, that is, $\Box\varphi$ is true at a world $w$ iff $\varphi$ is true at all worlds accessible from $w$. It follows from this that the following are equivalent:

- $\bigwedge_{i \in I} \Box\varphi_i$ is true at $w$;
- $\Box \bigwedge_{i \in I} \varphi_i$ is true at $w$;
- $\varphi_i$ is true at $u$, for all $i \in I$ and all $u$ accessible from $w$.

Thus, not only does it validate (II.8), Kripke semantics equates $\bigwedge_{i \in I} \Box\varphi_i$ and $\Box \bigwedge_{i \in I} \varphi_i$, thereby precluding the verifiability reading of $\Box$.

It is more instructive to observe this preclusion from a viewpoint of our observablity semantics. Given a set $W$ of possible worlds and an accessibility relation $R$ on $W$, write, for each $w \in W$,

$$\overrightarrow{R}(w) = \{\, u \in W \mid Rwu \,\}$$

for the set of worlds accessible from $w$. Then Kripke's truth condition for $\Box$ can be written as

$$w \in [\![\Box\varphi]\!] \iff \overrightarrow{R}(w) \subseteq [\![\varphi]\!].$$

Assuming $R$ to be **S4**, and reflexive in particular, we have $w \in \overrightarrow{R}(w)$ for each $w \in W$; hence we can also write

$$w \in [\![\Box\varphi]\!] \iff w \in \overrightarrow{R}(w) \subseteq [\![\varphi]\!].$$

Compare this to

(II.1) $$w \in [\![\Box\varphi]\!] \iff w \in B \subseteq [\![\varphi]\!] \text{ for } \textit{some } B \in \mathcal{B},$$

which states our idea that $\varphi$ is verifiably true at $w$ iff *some or another* observable proposition true at $w$ verifies (by entailing) $\varphi$; then it should be obvious that, in Kripke semantics, $\overrightarrow{R}(w)$ serves as *the* observable proposition for $w$, verifying everything verifiably true at $w$.[6] This is why the problem of induction is not expressible (or at least not straightforwardly) in Kripke semantics. In topological semantics, different propositions can be verified by different observable propositions; in the example of infinite coin tosses, $[\![p_1]\!]$ verifies that some toss comes up heads, but it cannot verify, and we need some $[\![\neg p_n]\!]$ to verify, that some other toss comes up tails. Moreover, although we can refine observations in a finitary manner, we cannot generally obtain a single, universal observation that encompasses all the other observations. This is how we distinguish between observing *each* and observing *all at once*, a distinction essential for the problem of induction. And this distinction is not available when $\overrightarrow{R}(w)$ is given the privilege of verifying everything verifiably true. Therefore, even though the difference between topological and **S4** Kripke semantics has no logical role when the language is finitary (and so the problem of induction cannot be expressed even syntactically), Kripke's notion of accessibility as a relation among possible worlds has conceptual shortcomings in semantically representing the epistemic reading of $\Box$ and $\Diamond$.

This observation provides a motivation for the project of this dissertation. We have seen that, in a certain reading of modal operators, they should be interpreted in terms of a generalized notion of accessibility more general than a relation among worlds—formally, by a family $\mathcal{B}$ (in the case of topological semantics) rather than by single $\overrightarrow{R}(w)$. One of the questions this dissertation tries to answer is how we can combine this insight with quantification and extend it to the first-order level. This is a not only technically but also philosophically significant question, because, as we will argue, we should assume a certain parallelism between accessibility among worlds and transworld identity—or *transworld identification*, to render it coherent with the epistemic reading of modal operators—of possible individuals; therefore, in so far as we generalize the notion of accessibility by replacing *the* observable proposition $\overrightarrow{R}(w)$ with a family $\mathcal{B}$ of observable propositions, we need also to seek a good conception of (perhaps a family of) transworld identifications to generalize *the* transworld identity.

[6] More formally, given any **S4** Kripke frame $(W, R)$, we can define a topological space $(W, \mathbf{int})$ by setting

$$\mathcal{B} = \{ \bigcap_{w \in J} \overrightarrow{R}(w) \mid J \text{ is a finite (nonempty) subset of } W \}$$

and using (II.2). Then, for each $w \in W$, $\overrightarrow{R}(w)$ is the smallest $B \in \mathcal{B}$ such that $w \in B$.

# III.0 SEMANTICS FOR FIRST-ORDER LOGIC REVISITED

# III.1 MORE GENERAL LANGUAGES OF FIRST-ORDER LOGIC

## III.1.1 Standard Semantics for Classical First-Order Logic

In this subsection, we review one formulation of standard semantics for classical first-order logic, so that we can later extend it to obtain semantics for first-order modal logic.

We start with a brief definition of first-order language (we assume the reader is familiar enough with the terminology and the ideas involved).[1]

**Definition 1.** A *(purely) classical first-order language* is a language given by the following:[2]

- any number (at least one) of primitive predicates (perhaps 0-ary);
- individual terms given by infinitely many variables and any number (perhaps none) of function and constant symbols; and
- the following sentential operators, called the *classical operators*: a unary connective $\neg$; binary connectives $\wedge$, $\vee$, $\rightarrow$; and quantifiers $\forall x$ and $\exists x$ for all individual variables $x$ (but not for any other variables $x$).

Given a classical first-order language $\mathcal{L}$, by an *atomic sentence* of $\mathcal{L}$ we mean a result of combining (in a manner allowed by the grammar of $\mathcal{L}$) an $n$-ary primitive predicate of $\mathcal{L}$ with $n$ individual terms of $\mathcal{L}$. And then, from the atomic sentences of $\mathcal{L}$, we define the set of sentences of $\mathcal{L}$, written sent($\mathcal{L}$), recursively with the sentential operators of $\mathcal{L}$. We also write var($\mathcal{L}$) for the set of variables of $\mathcal{L}$.

[1] By a language, we mean a purely grammatical entity independent of any proof theory or semantics.

[2] Languages or operators being classical is purely a matter of grammar, and not semantic at all, or not even proof-theoretic. For example, even when we consider intuitionistic axioms or semantics for the operator $\rightarrow$, we nonetheless say $\rightarrow$ itself, as a grammatical entity, is classical.

We call such a language a *classical* first-order language; we will discuss in Subsection III.1.3 a first-order language that is not classical, that is, that has sentential operators other than $\neg$, $\wedge$, $\vee$, $\to$, and $\forall x$, $\exists x$ (for all $x \in \mathrm{var}(\mathcal{L})$), so that we can deal with $\Box$ and $\Diamond$.

To interpret a classical first-order language, in the standard semantics for first-order logic, the notion of truth of a sentence is relativized to a *model*. More precisely,

**Definition 2.** Given a first-order language $\mathcal{L}$, an $\mathcal{L}$ *structure* $\mathfrak{M}$ is a tuple

$$\begin{aligned}\mathfrak{M} = (\,|\mathfrak{M}|, F^{\mathfrak{M}}, f^{\mathfrak{M}}, c^{\mathfrak{M}} \mid\, & F \text{ is a primitive predicate of } \mathcal{L},\\ & f \text{ is a primitive function symbol of } \mathcal{L},\\ & c \text{ is a individual constant symbol of } \mathcal{L}\,)\end{aligned}$$

such that

- $|\mathfrak{M}|$ is a nonempty set;
- $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$ for each $n$-ary primitive predicate $F$ of $\mathcal{L}$;
- $=^{\mathfrak{M}} = \{\,(a, a) \mid a \in |\mathfrak{M}|\,\}$ for the binary primitive predicate $=$, if $\mathcal{L}$ has it;
- $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$ for each $n$-ary primitive function symbol $f$ of $\mathcal{L}$;
- $c^{\mathfrak{M}} \in |\mathfrak{M}|$ for each individual constant symbol $c$ of $\mathcal{L}$.

Instead of the notation above, which makes explicit that $\mathfrak{M}$ is equipped with $F^{\mathfrak{M}}$, $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$ for each $F$, $f$, $c$, we will simply write, when it causes no confusion,

$$\mathfrak{M} = (|\mathfrak{M}|, F^{\mathfrak{M}}, f^{\mathfrak{M}}, c^{\mathfrak{M}})$$

for $\mathcal{L}$ structures.

An $\mathcal{L}$ structure $\mathfrak{M}$ interprets a primitive predicate $F$ of $\mathcal{L}$ by assigning to it an extension $F^{\mathfrak{M}}$. In particular, when $\mathcal{L}$ has the binary relation symbol $=$, it is always interpreted by what may be called the *diagonal line* (on the plane $|\mathfrak{M}|^2 = |\mathfrak{M}| \times |\mathfrak{M}|$), that is,

$$=^{\mathfrak{M}} = \{\,(a, a) \mid a \in |\mathfrak{M}|\,\}.$$

When $F$ is 0-ary, $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^0$. Note that $|\mathfrak{M}|^0$ is a singleton $\{*\}$. Let us regard its subsets $\mathbf{1} = \{*\}$ and $\mathbf{0} = \varnothing$ as the truth values, so that $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\} = \mathcal{P}(\{*\})$. Then we can regard $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^0 = \{*\}$ as

$F^{\mathfrak{M}} \in \mathcal{P}(\{*\}) = \mathbf{2}$; and $F^{\mathfrak{M}} = \{*\} = \mathbf{1}$ and $F^{\mathfrak{M}} = \varnothing = \mathbf{0}$ respectively mean that $F$ is true and that $F$ is false (in $\mathfrak{M}$).

Also, though $\mathcal{L}$ may have no individual function or constant symbols, if it does, an $\mathcal{L}$ structure $\mathfrak{M}$ interprets a constant symbol $c$ with its referent $c^{\mathfrak{M}} \in |\mathfrak{M}|$, and an $n$-ary function symbol $f$ with an $n$-ary function $f^{\mathfrak{M}}$ on $|\mathfrak{M}|$, so that, when individual terms $t_1, \ldots, t_n$ refer to $a_1, \ldots, a_n \in |\mathfrak{M}|$, the term $ft_1, \ldots, t_n$ refers to $f^{\mathfrak{M}}(a_1, \ldots, a_n) \in |\mathfrak{M}|$. It is worth noting that the case of function symbols subsumes that of constant symbols, by regarding a constant symbol $c$ as a 0-ary function symbol and its interpretation as a map $c^{\mathfrak{M}} : |\mathfrak{M}|^0 \to |\mathfrak{M}|$ with $|\mathfrak{M}|^0 = \{*\}$, so that $c$ refers to $c^{\mathfrak{M}}(*)$.

In such a structure, a given sentence is either true or false, if it is closed.[3] Generally, however, as sentences in first-order logic may contain free (individual) variables, their truth is also relativized to an assignment of objects to variables. For example, the truth of the sentence "$x$ is a logician" depends on the object to which the variable $x$ refers; it is true when $x$ refers to, say, Russell. To formally implement this idea of assignment, each structure $\mathfrak{M}$ is equipped with a set $|\mathfrak{M}|$ of objects that can be assigned to variables, so that an assignment is a map from variables to elements in $|\mathfrak{M}|$. Let us call $|\mathfrak{M}|$ the *domain of individuals* and its elements *individuals*, in the sense that

- the *domain of individuals* is the range of assignments, and
- *individuals* are values of assignments.

In other words, an assignment $\alpha$ is a map from variables to individuals in the domain $|\mathfrak{M}|$; here, following an idea due to Tarski, we let an assignment assign individuals in $|\mathfrak{M}|$ to *all* variables of $\mathcal{L}$, so that $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, where $\mathrm{var}(\mathcal{L})$ is the set of all individual variables of $\mathcal{L}$.[4] We also write $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ for the set $\mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, that is, the set of all assignments. Then, for an $\mathcal{L}$ structure $\mathfrak{M}$, an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, and a sentence $\varphi$ of $\mathcal{L}$, we write

$$\mathfrak{M} \vDash_\alpha \varphi$$

to mean that, when $x_1, \ldots, x_n$ are the free variables occurring in $\varphi$,

- $\varphi$ is true, in $\mathfrak{M}$, of the (sequence of) individuals $\alpha(x_1), \ldots, \alpha(x_n) \in |\mathfrak{M}|$ in place of $x_1, \ldots, x_n$.

[3]This is a desideratum rather than something we simply assume for the formal semantics. For it to hold with the formal semantics as we are going to define, satisfaction relations (Definition 4) need to satisfy the property called local determination (Definition 7). See p. 56.

[4](Draft: historical remarks to be filled in.)

We may equivalently and interchangeably say that

- (The sequence) $\alpha(x_1), \ldots, \alpha(x_n)$ satisfies $\varphi$ in $\mathfrak{M}$.[5]

This reading of $\vDash$, together with the reading of $F^{\mathfrak{M}}$ as the extension in $\mathfrak{M}$ of an $n$-ary primitive predicate $F$, should make the following truth condition natural:

$$\text{(III.1)} \quad \mathfrak{M} \vDash_\alpha Fx_1 \cdots x_n \iff (\alpha(x_1), \ldots, \alpha(x_n)) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

This serves as the basis clause for the recursive definition of the $\mathfrak{M} \vDash_\alpha \varphi$ relation along the construction of $\varphi$. Among the inductive clauses, those regarding the (classical) sentential connectives simply carry over from propositional logic:

$$\text{(III.2)} \qquad \mathfrak{M} \vDash_\alpha \neg\varphi \iff \mathfrak{M} \nvDash_\alpha \varphi,$$

$$\text{(III.3)} \qquad \mathfrak{M} \vDash_\alpha \varphi \wedge \psi \iff \mathfrak{M} \vDash_\alpha \varphi \text{ and } \mathfrak{M} \vDash_\alpha \psi,$$

$$\text{(III.4)} \qquad \mathfrak{M} \vDash_\alpha \varphi \vee \psi \iff \mathfrak{M} \vDash_\alpha \varphi \text{ or } \mathfrak{M} \vDash_\alpha \psi,$$

$$\text{(III.5)} \qquad \mathfrak{M} \vDash_\alpha \varphi \to \psi \iff \mathfrak{M} \nvDash_\alpha \varphi \text{ or } \mathfrak{M} \vDash_\alpha \psi.$$

To lay out truth conditions for $\forall x$ and $\exists x$, observe that our intuitive understanding of what they mean makes the following (semi-intuitive) equivalences desirable:

- $\forall x . \varphi$ is true in $\mathfrak{M}$, iff
- $\varphi$ is true in $\mathfrak{M}$ of *each thing* (in place of $x$),

and

- $\exists x . \varphi$ is true in $\mathfrak{M}$, iff
- $\varphi$ is true in $\mathfrak{M}$ of *something* (in place of $x$).

We cash out the notions of "each thing" and "something" here with "each $a \in |\mathfrak{M}|$" and "some $a \in |\mathfrak{M}|$"; in other words, we take $|\mathfrak{M}|$ as the *domain of quantification*, in the sense that

- the *domain of quantification* is the set over which "thing" as in "each thing" and "something" (or, formally, the variable $x$ of a quantifier $\forall x$ or $\exists x$) ranges.[6]

[5] We may keep the order of variables $x_1, \ldots, x_n$ implicit if it is obvious.

[6] Compare this notion to that of domain of individuals introduced on p. 51; these two notions are based on two ideas that are in principle different. Indeed, whereas they refer to the same set in this subsection, we will distinguish them in Subsection III.2.1 and on.

To formally express this idea, it is helpful to introduce the notation that, when $a \in |\mathfrak{M}|$, $x \in \mathrm{var}(\mathcal{L})$, and $\alpha$ is an assignment, $[a/x]\alpha$ is the assignment that assigns $a$ to $x$ but agrees with $\alpha$ on all other variables; that is, $[a/x]\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that

$$([a/x]\alpha)(y) = \begin{cases} a & \text{if } y = x, \\ \alpha(y) & \text{otherwise.} \end{cases}$$

Then we set

$$\text{(III.6)} \qquad \mathfrak{M} \vDash_\alpha \forall x . \varphi \iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}|,$$

$$\text{(III.7)} \qquad \mathfrak{M} \vDash_\alpha \exists x . \varphi \iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for some } a \in |\mathfrak{M}|,$$

so that these yield

- $\forall x . \varphi$ is true in $\mathfrak{M}$ of $\alpha(x_1), \ldots, \alpha(x_n)$ (in place of $x_1, \ldots, x_n$), iff
- $\varphi$ is true in $\mathfrak{M}$ of $\alpha(x_1), \ldots, \alpha(x_n)$ and *every* $a \in |\mathfrak{M}|$ (in place of $x_1, \ldots, x_n$ and $x$),

and

- $\exists x . \varphi$ is true in $\mathfrak{M}$ of $\alpha(x_1), \ldots, \alpha(x_n)$ (in place of $x_1, \ldots, x_n$), iff
- $\varphi$ is true in $\mathfrak{M}$ of $\alpha(x_1), \ldots, \alpha(x_n)$ and *some* $a \in |\mathfrak{M}|$ (in place of $x_1, \ldots, x_n$ and $x$),

which subsume our desired equivalences above (with $|\mathfrak{M}|$ the domain of quantification).

We reviewed so far how to settle the truth—relative to an $\mathcal{L}$ structure and an assignment—of sentences without function or constant symbols. We extend this to all sentences, containing not only variables but also function and constant symbols, in the following manner. First we extend the interpretion of variables $x$ in terms of $\alpha(x)$ to all individual terms.

**Definition 3.** Fix a first-order language $\mathcal{L}$, an $\mathcal{L}$ structure $\mathfrak{M}$ and an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$. Given an individual term $t$, its interpretation $t^{\mathfrak{M},\alpha}$, relative to $\mathfrak{M}$ and $\alpha$, is given recursively as follows:

$$x^{\mathfrak{M},\alpha} = \alpha(x) \qquad \text{for a variable } x,$$

$$(f t_1, \ldots, t_n)^{\mathfrak{M},\alpha} = f^{\mathfrak{M}}({t_1}^{\mathfrak{M},\alpha}, \ldots, {t_n}^{\mathfrak{M},\alpha}) \qquad \text{for an } n\text{-ary function symbol } f.$$

Note that the latter subsumes the case of $n = 0$, that is,

$$c^{\mathfrak{M},\alpha} = c^{\mathfrak{M}} \qquad \text{for a constant symbol } c.$$

With the interpretation of variables extended in this way to all terms, we simply extend (III.1) to

(III.8) $\quad \mathfrak{M} \vDash_\alpha F t_1 \cdots t_n \iff ({t_1}^{\mathfrak{M},\alpha}, \ldots, {t_n}^{\mathfrak{M},\alpha}) \in F^{\mathfrak{M}}$ $\quad$ for an $n$-ary primitive predicate $F$,

which clearly subsumes (III.1). Then combining the new base clause (III.8) with the inductive clauses (III.2)–(III.7) extends the recursive definition of the semantic relation $\vDash$ to all sentences.

This semantic relation (for each $\mathfrak{M}$) provides the classical semantics for first-order logic; so let us simply define *the* classical semantics, regarded as a formal object, to be the class of relations that satisfy the truth conditions (III.2)–(III.8).

**Definition 4.** Given a first-order language $\mathcal{L}$, a *classical-type satisfaction relation for* $\mathcal{L}$ is a pair of an $\mathcal{L}$ structure $\mathfrak{M}$ and any relation $(\mathfrak{M} \vDash_{-} -)$, as in $\mathfrak{M} \vDash_\alpha \varphi$, of

- an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, and
- a sentence $\varphi$ of $\mathcal{L}$;

in other words, it is a pair $(\mathfrak{M}, \vDash)$ of $\mathfrak{M}$ and any subset $\vDash$ of $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$, where, we should recall, $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ is the set $\mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and $\mathrm{sent}(\mathcal{L})$ is the set of sentences of $\mathcal{L}$. We say a classical-type satisfaction relation for $\mathcal{L}$ is *on* $\mathfrak{M}$ if its first coordinate is an $\mathcal{L}$ structure $\mathfrak{M}$.

One might find the first coordinate $\mathfrak{M}$ in the pair $(\mathfrak{M}, \vDash)$ above redundant, but it is needed for the following purpose. Suppose a pair of $\mathcal{L}$ structures $\mathfrak{M}_0$, $\mathfrak{M}_1$ has $|\mathfrak{M}_0| = |\mathfrak{M}_1|$ but $F^{\mathfrak{M}_0} \neq F^{\mathfrak{M}_1}$, and fix any $\vDash \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$. (III.8) may hold with $\mathfrak{M}_0$ in place of $\mathfrak{M}$ (and with the set $\vDash$ in the place denoted by "$\vDash$" in (III.8)), but then it cannot hold with $\mathfrak{M}_1$ in place of $\mathfrak{M}$. Hence we need to relativize satisfaction relations to $\mathcal{L}$ structures and say that $(\mathfrak{M}_0, \vDash)$ satisfies (III.8) whereas $(\mathfrak{M}_1, \vDash)$ does not.

**Definition 5.** Given a classical first-order language $\mathcal{L}$, a classical-type satisfaction relation for $\mathcal{L}$ is said to be *classical*, and called simply a *classical satisfaction relation for* $\mathcal{L}$, if it satisfies (III.2)–(III.8). The class of all the classical satisfaction relations for $\mathcal{L}$ is called the *classical semantics for* $\mathcal{L}$.

Indeed, an $\mathcal{L}$ structure $\mathfrak{M}$ corresponds one-to-one to a classical satisfaction relation for $\mathcal{L}$ on $\mathfrak{M}$ (under the assumption that all the sentential operators of $\mathcal{L}$ are first-order—which will not hold generally in Subsection III.1.3), as follows:

**Fact 2.** If $\mathcal{L}$ is a *classical* first-order language, then on each $\mathcal{L}$ strucure $\mathfrak{M}$ there is a unique classical satisfaction relation for $\mathcal{L}$.

*Proof.* By induction on the construction of $\varphi$ (we need to use the assumption that $\mathcal{L}$ is classical). □

We can define the notion of validity with respect to classical satisfaction relations. Indeed, the definition can be independent of classical semantics; it works with any satisfaction relation for a first-order language.

**Definition 6.** Given a first-order language $\mathcal{L}$, we say, for each classical-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$,

- a sentence $\varphi$ of $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash \varphi$ (with a slight abuse of notation), meaning that $\mathfrak{M} \vDash_\alpha \varphi$ for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$; and
- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash \varphi$.[7]

Given a class of classical-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is *valid in* that class if it is valid in every member of that class.

So, in particular, a sentence or inference is valid in the classical semantics for $\mathcal{L}$ if it is valid in every classical satisfaction relation for $\mathcal{L}$.

Let us close this subsection by introducing an "overscore" notation to use for a sequence. For example, $\bar{a}$ is short for $(a_1, \dots, a_n)$; also, $\alpha(\bar{x})$ is short for $(\alpha(x_1), \dots, \alpha(x_n))$; then (III.8) becomes

$$\mathfrak{M} \vDash_\alpha F\bar{t} \iff \bar{t}^{\mathfrak{M},\alpha} \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

The length of the sequence is typically assumed to be $n$, unless otherwise noted.

[7] An inference in $\mathcal{L}$ is a pair of a set of sentences of $\mathcal{L}$ and a sentence of $\mathcal{L}$.

### III.1.2 The Forgotten Trio

In this subsection, we review three properties of classical semantics. They seem so obvious and natural that logicians often take them for granted and rarely mention them explicitly;[8] but they are essential in conceptually connecting the semantics as a technical machinery and what we take it as expressing. Also, as will be shown in Subsection III.1.3, they are essential in characterizing classical semantics, once the language is extended beyond the classical one.

First, we consider the notion of *local determination*.

**Definition 7.** Let $(\mathfrak{M}, \vDash)$ be a classical-type satisfaction relation for a first-order language $\mathcal{L}$. We say a sentence $\varphi$ of $\mathcal{L}$ is *locally determined* in $(\mathfrak{M}, \vDash)$ if it satisfies, for every pair of assignments $\alpha, \beta : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

(III.9) $$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_\beta \varphi \qquad \text{if } \alpha(x) = \beta(x) \text{ for every free variable } x \text{ in } \varphi.$$

We also say $(\mathfrak{M}, \vDash)$ is locally determined if every sentence of $\mathcal{L}$ is locally determined in it, and that a class of classical-type satisfaction relations for $\mathcal{L}$ is locally determined if all its members are locally determined.

In short, local determination means that the truth of a sentence $\varphi$ does not depend on the referent of a variable that does not occur freely in $\varphi$. It is a property with various imports, both technical and conceptual. To list two,[9] local determination is needed to make sure that the truth of a closed sentence is independent of assignments; indeed, this independence amounts exactly to the local determination of the closed sentence. Also, without local determination, it is hard to maintain the connection between the syntactic and semantic conceptions of an $n$-ary predicate. It surely makes perfect sense, without local determination, to say that $\varphi$ is a unary predicate, when $\varphi$ contains (at most) one free variable, say $x$—this is a syntactic conception of a unary predicate. In contrast, it seems to make sense to say that $\varphi$ is a unary predicate true (in $\mathfrak{M}$) of an individual $a$, only if any assignment $\alpha$ with $\alpha(x) = a$ has $\mathfrak{M} \vDash_\alpha \varphi$. Otherwise, if assignments $\alpha$ and $\beta$ had

[8] A respectable exception is Belnap, [6], to whom I owe the notions (and their names) of local determination and semantics of substitution.

[9] Another import, which we will discuss on p. 65, is that local determination is needed to validate a rule of classical first-order logic.

$\alpha(x) = \beta(x) = a$ but $\mathfrak{M} \vDash_\alpha \varphi$ and $\mathfrak{M} \nvDash_\beta \varphi$, then we could no longer sensibly say either that $\varphi$ is true of $a$ or that it is not.

Fortunately, classical semantics is locally determined. To prove it, we need to first show that the interpretation of terms is locally determined (so to speak). It is worth noting that the statement of Fact 3 depends on $\mathcal{L}$ structures but not on any satisfaction relation.

**Fact 3.** Suppose $\mathfrak{M}$ is an $\mathcal{L}$ structure for a first-order language $\mathcal{L}$. Then, for every term $t$ of $\mathcal{L}$ and pair of assignments $\alpha, \beta : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$t^{\mathfrak{M},\alpha} = t^{\mathfrak{M},\beta} \qquad \text{if } \alpha(x) = \beta(x) \text{ for every (free) variable } x \text{ in } t.$$

*Proof.* By induction on the construction of $t$. □

**Fact 4.** If $\mathcal{L}$ is a *classical* first-order language, every classical satisfaction relation for $\mathcal{L}$ is locally determined; this means that the classical semantics for $\mathcal{L}$ is locally determined.

*Proof.* By induction on the construction of sentences (we need to use the assumption that $\mathcal{L}$ is classical). Use Fact 3 for the base case. □

Next, we review how *substitution of terms* works semantically in classical semantics. First let us introduce a notation for substitution of terms.

**Definition 8.** Given any term $t$, variable $x$, and sentence $\varphi$, we say $t$ is *free for $x$ in $\varphi$* if $t$ contains no variable $y$ such that $x$ occurs freely in the scope of either $\forall y$ or $\exists y$ in $\varphi$.

**Definition 9.** Given a first-order language, let $x$ be an individual variable, let $t$, $t_0$ be terms, and let $\varphi$ be a sentence in which $t$ is free for $x$. Then

$$[t/x]t_0, \qquad\qquad [t/x]\varphi$$

respectively stand for the term and the sentence obtained by substituting $t$ for all the free occurrences of $x$ in $t_0$ and $\varphi$, respectively. More rigorously, $[t/x]t_0$ and $[t/x]\varphi$ are recursively defined as follows:

$$[t/x]y = \begin{cases} t & \text{if } x = y, \\ y & \text{if } x \neq y, \end{cases}$$

$$[t/x]ft_1 \cdots t_n = f([t/x]t_1) \cdots ([t/x]t_n) \quad \text{for any } n\text{-ary function symbol } f,$$

$$[t/x]Ft_1 \cdots t_n = F([t/x]t_1) \cdots ([t/x]t_n) \quad \text{for any } n\text{-ary primitive predicate } F,$$

$$[t/x]\otimes(\varphi_1, \ldots, \varphi_n) = \begin{cases} \otimes(\varphi_1, \ldots, \varphi_n) & \text{if } \otimes \text{ binds } x, \\ \otimes([t/x]\varphi_1, \ldots, [t/x]\varphi_n) & \text{otherwise} \end{cases} \quad \text{for any } n\text{-ary sentential operator } \otimes.$$

With this $[t/x]$ notation, we define the following notion, the *SoS property*, with SoS short for "semantics-of-substitution-respecting".

**Definition 10.** Let $(\mathfrak{M}, \vDash)$ be a classical-type satisfaction relation for a first-order language $\mathcal{L}$. We say $(\mathfrak{M}, \vDash)$ is *SoS for* a sentence $\varphi$ of $\mathcal{L}$ if, for every assignment $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$, variable $x$, and individual term $t$,

$$\text{(III.10)} \qquad \mathfrak{M} \vDash_\alpha [t/x]\varphi \iff \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/x]\alpha} \varphi \qquad \text{if } t \text{ is free for } x \text{ in } \varphi.$$

We also say $(\mathfrak{M}, \vDash)$ is SoS if it is SoS for every sentence of $\mathcal{L}$, and that a class of classical-type satisfaction relations for $\mathcal{L}$ is SoS if all its members are SoS.

Like local determination, the SoS property also has both technical and philosophical imports.[10] Conceptually, SoS means that whether or not an individual has a given property does not depend on what individual term we use to refer to the individual. For example, to express that an individual $a \in |\mathfrak{M}|$ has a (unary) property $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|$ (in $\mathfrak{M}$) with a sentence of a given language, we can write $\mathfrak{M} \vDash_\alpha Fx$ for a pair of a variable $x$ and an assignment $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(x) = a$. In this expression, however, the choice of $x$ and $\alpha$ should not be significant: we should be able to express the same thing with $\mathfrak{M} \vDash_\beta Fy$, as long as $\beta(y) = a$, even when $\beta = [a/y]\alpha$. To put it differently, if $\mathfrak{M} \vDash_\alpha Fx$ and $\mathfrak{M} \vDash_{[a/y]\alpha} Fy$ did not coincide, we could no longer use either of them to express that $a$ satisfies $F^{\mathfrak{M}}$. That they coincide is guaranteed by the SoS property of $Fy$ (since $Fx = [x/y]Fy$ and $[a/y]\alpha = [x^{\mathfrak{M},\alpha}/y]\alpha$).

[10]The SoS property is needed to validate a rule of classical first-order logic, as we will discuss on p. 66.

Again, fortunately, classical semantics is SoS. To show it, we need—as we did in the case of local determination—to first show that the interpretation of terms is SoS (so to speak); this fact, again, depends on $\mathcal{L}$ structures but not on any satisfaction relation.

**Fact 5.** Suppose $\mathfrak{M}$ is an $\mathcal{L}$ structure for a first-order language $\mathcal{L}$. Then, for every pair of terms $t$, $t_0$, variable $x$, and assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$([t/x]t_0)^{\mathfrak{M},\alpha} = {t_0}^{\mathfrak{M},[t^{\mathfrak{M},\alpha}/x]\alpha}$$

*Proof.* By induction on the construction of $t_0$. □

**Fact 6.** If $\mathcal{L}$ is a *classical* first-order language, every classical satisfaction relation for $\mathcal{L}$ is SoS; this means that the classical semantics for $\mathcal{L}$ is SoS.

*Proof.* By induction on the construction of sentences (we need to use the assumption that $\mathcal{L}$ is classical). Fixing any classical satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$, use Fact 5 for the base case. The inductive case for $\forall$ goes as follows. Because $[t/x](\forall x\,.\,\varphi) = \forall x\,.\,\varphi$, local determination of $\forall x\,.\,\varphi$ in $(\mathfrak{M}, \vDash)$ (by Fact 4) entails

$$\mathfrak{M} \vDash_\alpha [t/x](\forall x\,.\,\varphi) \iff \mathfrak{M} \vDash_\alpha \forall x\,.\,\varphi \iff \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/x]\alpha} \forall x\,.\,\varphi.$$

On the other hand, if $y \neq x$, then $[t/x](\forall y\,.\,\varphi) = \forall y\,([t/x]\varphi)$ entails the equivalence marked with ! below. The one with $*$ is by the induction hypothesis. Fact 3 implies $t^{\mathfrak{M},[a/y]\alpha} = t^{\mathfrak{M},\alpha}$ and hence the equivalence with †, because $y$ does not occur in $t$, due to the assumption of the notation $[t/x](\forall y\,.\,\varphi)$ that $t$ is free for $x$ in $\forall y\,.\,\varphi$. And, finally, $x \neq y$ entails $[t^{\mathfrak{M},\alpha}/x][a/y]\alpha = [a/y][t^{\mathfrak{M},\alpha}/x]\alpha$ and hence the equivalence with ‡.

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha [t/x](\forall y\,.\,\varphi) &\overset{!}{\iff} \mathfrak{M} \vDash_\alpha \forall y\,([t/x]\varphi) \\
&\overset{\text{(III.6)}}{\iff} \mathfrak{M} \vDash_{[a/y]\alpha} [t/x]\varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{*}{\iff} \mathfrak{M} \vDash_{[t^{\mathfrak{M},[a/y]\alpha}/x][a/y]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\dagger}{\iff} \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/x][a/y]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\ddagger}{\iff} \mathfrak{M} \vDash_{[a/y][t^{\mathfrak{M},\alpha}/x]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\text{(III.6)}}{\iff} \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/x]\alpha} \forall y\,.\,\varphi.
\end{aligned}$$

□

Lastly, we introduce the notion of *alpha-equivalence*.

**Definition 11.** We say a sentence $\varphi_1$ is an *alpha-conversion of* a sentence $\varphi_0$, and write $\alpha_0 \propto_0 \alpha_1$, if $\varphi_1$ is obtained by replacing a subformula $\forall x\,.\,\psi$ of $\varphi_0$ with $\forall y\,([y/x]\psi)$, or a subformula $\exists x\,.\,\psi$ of $\varphi_0$ with $\exists y\,([y/x]\psi)$, for any pair of variables $x$, $y$ such that $y$ does not occur freely in $\psi$ and $y$ is free for $x$ in $\psi$. More precisely, $\propto_0$ is the smallest binary relation $R$ on sent($\mathcal{L}$) such that

(i) $R(\forall x\,.\,\varphi, \forall y\,([y/x]\varphi))$ if $y$ does not occur freely in $\varphi$ and $y$ is free for $x$ in $\varphi$;

(ii) $R(\exists x\,.\,\varphi, \exists y\,([y/x]\varphi))$ if $y$ does not occur freely in $\varphi$ and $y$ is free for $x$ in $\varphi$;

(iii) for every $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, $R(\otimes(\bar{\varphi}), \otimes(\bar{\psi}))$ if $R(\varphi_i, \psi_i)$ for exactly one $i \leqslant n$ and $\varphi_j = \psi_j$ for all the other $j \leqslant n$.

Moreover, we write $\propto$ for the reflextive and transitive closure of $\propto_0$; we say $\psi$ is *alpha-equivalent to* $\varphi$ if $\varphi \propto \psi$.[11]

To see the point of this definition, suppose $\varphi_1$ is an alpha-conversion of $\varphi_0$ obtained by replacing a subformula $\forall x\,.\,\psi$ of $\varphi_0$ with $\forall y\,([y/x]\psi)$. For the sake of explanation, let us call these occurrences of $\forall x$ (in $\varphi_0$) and $\forall y$ (in $\varphi_1$) their *principal* occurrences. Then observe:

- Every free occurrence of $x$ in $\psi$, which was originally bound by the principal occurrence of $\forall x$ in $\varphi_0$, is replaced by a new occurrence of $y$ but then bound in $\varphi_1$, because it occurs within $\forall y\,([y/x]\psi)$. (So, there is no variable that was bound in $\varphi_0$ but is no longer bound in $\varphi_1$.)
- These new occurrences of $y$ are bound by the principal occurrence of $\forall y$ in $\varphi_1$, due to the requirement that $y$ is free for $x$ in $\psi$.
- Moreover, the requirement that $y$ does not occur freely in $\psi$ guarantees that no free occurrence of $y$ in $\varphi_0$ is newly bound in $\varphi_1$.

In short, $\varphi_0$ and $\varphi_1$ share the same variable structure, while the bound variable $x$ in $\varphi_0$—not just an occurrence but the occurrence in the principal occurrence of $\forall x$ and all the occurrences it binds—is replaced with $y$ in $\varphi_1$. And this property extends to the case of alpha-equivalence in general; that is, alpha-equivalence between $\varphi$ and $\psi$ means that $\varphi$ and $\psi$ share the same variable structure with possibly different bound variables (but the same free variables hold the same places).

It is easy to observe:

[11] In this definition, we assume that $\forall x$ and $\exists x$ are the only sentential operators of the language that bind variables. It should be clear how to extend the definition to languages with more operators that bind variables.

**Fact 7.** $\propto_0$ is a symmetric relation.

*Proof.* Write $R(\varphi, \psi)$ to mean that both $\varphi \propto_0 \psi$ and $\psi \propto_0 \varphi$; then, to show $\propto_0$ symmetric, it is enough to show that $R$ satisfies (i)–(iii) of Definition 11, because then $\propto_0 \subseteq R$. To show (i), suppose $y$ does not occur freely in $\varphi$ and $y$ is free for $x$ in $\varphi$. Then $\forall x . \varphi \propto_0 \forall y\,([y/x]\varphi)$. Yet it also follows that $x$ does not occur freely in $[y/x]\varphi$ and $x$ is free for $y$ in $[y/x]\varphi$, and therefore $\forall y\,([y/x]\varphi) \propto_0 \forall x\,([x/y][y/x]\varphi)$, while it moreover follows that $[x/y][y/x]\varphi = \varphi$. So, $\forall y\,([y/x]\varphi) \propto_0 \forall x . \varphi$ as well. Thus $R$ satisfies (i), and similarly (ii). (iii) for $R$ is straightforward. □

Hence the reflexive and transitive closure $\propto$ of $\propto_0$, that is, alpha-equivalence, is an equivalence relation.

Our ordinary conception of bound variables implies that alpha-equivalent sentences should be treated as equivalent semantically. Given an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, different variables $x$ and $y$ generally refer to different individuals $\alpha(x)$ and $\alpha(y)$, and then different sentences $Fx$ and $Fy$ make different claims: $Fx$ claims that $\alpha(x)$ satisfies $F$, whereas $Fy$ claims that $\alpha(y)$ satisfies $F$. We should however note here that, in order for $Fx$ and $Fy$ to make different claims, it is essential that the occurrences of $x$ and $y$ are free. By contrast, we regard $\forall x\, Fx$ and $\forall y\, Fy$, for example, as making the same claim. This is because bound variables are mere labels, or *placeholders*, and do not refer to anything significantly—we read $\forall x . \varphi$ as "*Regardless of* to what $x$ may refer, $\varphi$ is true of it *in place of* $x$". The only significant role bound variables play instead is to indicate the binding structure, that is, which quantifier binds which occurrence of variables. To extract the conceptual content of $\forall x . \varphi$, it is "Regardless of to what $-$ may refer, $\varphi$ is true of it in place of $-$", which is invariant whether we formally use $x$, $y$, or any variable in the place indicated by "$-$". This is why our technical semantics should treat alpha-equivalent sentences as equivalent.

Classical semantics indeed treats alpha-equivalent sentences as equivalent, as stated in Fact 8. We call the property the *AE property*, with AE short for "alpha-equivalence-respecting".

**Definition 12.** Let $(\mathfrak{M}, \vDash)$ be a classical-type satisfaction relation for a first-order language $\mathcal{L}$. We say $(\mathfrak{M}, \vDash)$ is *AE* if, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and sentences $\varphi$, $\psi$ of $\mathcal{L}$,

$$\text{(III.11)} \qquad \mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_\alpha \psi \qquad \text{if } \varphi \propto \psi.$$

We also say a class of classical-type satisfaction relations for $\mathcal{L}$ is AE if all its members are AE.

**Fact 8.** If $\mathcal{L}$ is a *classical* first-order language, every classical satisfaction relation for $\mathcal{L}$ is AE, which means that the classical semantics for $\mathcal{L}$ is AE.

To prove Fact 8, let us first observe the following, more general lemma (the proof is straightforward and we omit it).

**Fact 9.** Suppose $\mathcal{L}$ is a *classical* first-order language. Fix any classical satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$, let us write $R(\varphi, \psi)$ to mean that

$$\text{(III.12)} \qquad \mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_\alpha \psi \qquad \text{for every } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|.$$

Then, for every $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, if $R(\varphi_i, \psi_i)$ for all $i \leqslant n$ then $R(\otimes(\bar{\varphi}), \otimes(\bar{\psi}))$.

Using this, we give:

*Proof for Fact 8.* Fix $(\mathfrak{M}, \vDash)$ and write $R(\varphi, \psi)$ as in Fact 9. Then, clearly, to prove Fact 8 it is enough to show that if $\varphi \propto_0 \psi$ then $R(\varphi, \psi)$. To show it, then, it is enough to show that $R$ satisfies (i)–(iii) of Definition 11; but (iii) for $R$ is immediate from Fact 9. To show (i), suppose $y$ does not occur freely in $\varphi$ and $y$ is free for $x$ in $\varphi$. Then, for every $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha \forall y\,([y/x]\varphi) &\overset{\text{(III.6)}}{\iff} \mathfrak{M} \vDash_{[a/y]\alpha} [y/x]\varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{*}{\iff} \mathfrak{M} \vDash_{[a/x][a/y]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\dagger}{\iff} \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\text{(III.6)}}{\iff} \mathfrak{M} \vDash_\alpha \forall x\,.\,\varphi.
\end{aligned}$$

Here the equivalence marked with $*$ holds by Fact 6, because $y^{\mathfrak{M},[a/y]\alpha} = a$; the one with $\dagger$ follows from Fact 4, because $[a/x][a/y]\alpha$ and $[a/x]\alpha$ agree on all variables except $y$, which does not occur freely in $\varphi$. Thus $R$ satisfies (i), and similarly (ii). □

### III.1.3 What If the Language is not Pure

So far we have discussed semantics for languages that are *classical* first-order—that is, languages whose only sentential operators are $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$ (for all variables $x$). In this subsection, we expand our semantics to a wider class of languages, to include other operators; the typical example we have is a language with the modal operators $\Box$ and $\Diamond$. This generality will be useful in later chapters, where we discuss quantified modal logic.

Let us give a more general definition of a first-order language than we did in Definition 1. The generalization consists in the introduction of sentential operators beyond the classical $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$.

**Definition 13.** A *first-order language* is a language given by the following:

- any number (at least one) of primitive predicates (perhaps 0-ary);
- individual terms given by infinitely many variables and any number (perhaps none) of function and constant symbols; and
- a number of sentential operators including all the classical ones—but no $\forall x$ or $\exists x$ unless $x$ is an individual variable—and perhaps ones that are not classical; such operators are called *non-classical*.

Modal operators $\Box$ and $\Diamond$ are the typical examples of non-classical operators (we will say that a first-order modal language is a first-order language that is not classical but modal).

The introduction of non-classical operators gives rise to a new notion of atomic sentence: Let us say a sentence is *classically atomic* if none of the classical operators is its major operator. Then, whereas all atomic sentences in the sense we defined before—that is, results of combining an $n$-ary primitive predicate of $\mathcal{L}$ with $n$ individual terms of $\mathcal{L}$—are classically atomic, there can be a classically atomic sentences of $\mathcal{L}$ that are not atomic in this sense, for example, $\Box\forall x\,(\varphi \rightarrow \psi)$; let us say the former kind of classically atomic sentences are *primitive*, while the latter are *non-primitive*. We can put this differently as follows: Given a first-order language $\mathcal{L}$, by a primitive classically atomic sentence, or atomic sentence for short, of $\mathcal{L}$ we mean a result of combining an $n$-ary primitive predicate of $\mathcal{L}$ with $n$ individual terms of $\mathcal{L}$. We define the set of sentences of $\mathcal{L}$ recursively from the atomic sentences of $\mathcal{L}$ with the sentential operators of $\mathcal{L}$. Then, among sentences of $\mathcal{L}$ that are not atomic, those whose major operators are non-classical are called non-primitive classically atomic sentences, or non-primitive atomic sentences for short. We also say that a sentence is *non-classical* if it contains non-classical operators and is *purely classical* otherwise.

Given a first-order language $\mathcal{L}$, let us write $\mathrm{ca}(\mathcal{L})$ for the set of classically atomic sentences of $\mathcal{L}$. Regarding this set of sentences, it is crucial to make:

**Observation 1.** Let $\mathcal{L}$ be a first-order language. Its sentences could be recursively defined from $\mathrm{ca}(\mathcal{L})$ with classical operators. More precisely, when we write $R(\varphi, \psi)$ for sentences $\varphi, \psi$ of $\mathcal{L}$ iff

- $\psi = \otimes(\varphi_1, \ldots, \varphi_n)$ and $\varphi = \varphi_i$ for some $\varphi_1, \ldots, \varphi_n$, $i \leqslant n$, and $n$-ary *classical* operator $\otimes$,

the transitive closure of $R$ is a well-founded relation on $\mathrm{sent}(\mathcal{L})$.

For a first-order language $\mathcal{L}$ in the general sense as above, we can use the same definition of $\mathcal{L}$ structures (Definition 2). Yet we need to note that, while $\mathcal{L}$ structures interpret primitive predicates $F$, thereby interpreting atomic sentences $F\bar{t}$ of $\mathcal{L}$ (with the help of (III.8), of course), they by no means interpret other classically atomic sentences, namely non-primitive ones, such as $\Box\varphi$.

The definitions of assignments $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$, interpretation of terms $t^{\mathfrak{M},\alpha}$ (Definition 3), classical-type semantic relations $(\mathfrak{M}, \vDash)$ (Definition 4), validity (Definition 6), and local determination (Definition 7) all work fine as they were. Then, using the same truth conditions (III.2)–(III.8), we might try (though we will give up) keeping the same definition for classical semantic relations (the first half of Definition 5), namely,

- A classical-type satisfaction relation $(\mathfrak{M}, \vDash)$ for a first-order language $\mathcal{L}$ (that, in general, may not be classical) is called a *classical satisfaction relation* if it satisfies (III.2)–(III.8).

Here the generalization starts to make difference: Under this definition of classical satisfaction relations, (the consequent of) Fact 2 no longer holds. That is, given an $\mathcal{L}$ structure $\mathfrak{M}$ for a *non-classical* first-order language $\mathcal{L}$, there is more than one classical satisfaction relation for $\mathcal{L}$ on $\mathfrak{M}$. This is because, as mentioned above, $\mathfrak{M}$ does not interpret non-primitive atomic sentences—for example, $\Box\varphi$—and the definition above simply gives no constraints on how classical satisfaction relations for $\mathcal{L}$ on $\mathfrak{M}$ should evaluate the truth of these sentences. This can be formally expressed by the following generalization of Fact 2. For a first-order language $\mathcal{L}$, write $\mathrm{npa}(\mathcal{L})$ for the set of non-primitive atomic sentences of $\mathcal{L}$; then:

**Fact 10.** Given a first-order language $\mathcal{L}$, for every $\mathcal{L}$ structure $\mathfrak{M}$ there is a bijection $e$ such that

- $e$ is from $\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L}))$ to the set of classical-type satisfaction relations for $\mathcal{L}$ on $\mathfrak{M}$ that satisfy (III.2)–(III.8), and
- for each $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L})$, $e(A) = (\mathfrak{M}, \vDash)$ satisfies

  (III.13) $$\mathfrak{M} \vDash_{\alpha} \varphi \iff (\alpha, \varphi) \in A$$

  for every assignment $\alpha$ and non-primitive atomic sentence $\varphi$ of $\mathcal{L}$.

Fact 10 subsumes Fact 2 because, if $\mathcal{L}$ is classical then $\mathrm{npa}(\mathcal{L}) = \varnothing$, which implies $\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L}))$ is a singleton and, by $e$ being bijective, so is the set of classical satisfaction relations for $\mathcal{L}$ on $\mathfrak{M}$ (as defined in Definition 5).

*Proof for Fact 10.* Fix any $\mathcal{L}$ structure $\mathfrak{M}$ and write $C$ for the set of classical-type satisfaction relations for $\mathcal{L}$ on $\mathfrak{M}$ that satisfies (III.2)–(III.8). Then define an operation $r : C \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L}))$ so that $r(\mathfrak{M}, \vDash) = {\vDash} \cap (|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L}))$ for every $(\mathfrak{M}, \vDash) \in C$; that is, $r(\mathfrak{M}, \vDash)$ is the set $A$ that satisfies (III.13).

Fix any $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L})$. By Observation 1, induction on the construction of sentences of $\mathcal{L}$ from $\mathrm{ca}(\mathcal{L})$ with the classical operators enables us to define $e(A)$ as the unique $(\mathfrak{M}, \vDash) \in C$ that satisfies (III.13). This uniqueness entails $e \circ r = 1$. Moreover, clearly, $r \circ e = 1$. Therefore $e$ is bijective. □

This wild behavior regarding non-classical sentences, according to the proposed definition of classical satisfaction relations, moreover prevents (III.9)–(III.11)—that is, local determination, SoS property, and AE property—from holding. This is a serious issue, conceptually because, as we argued in Subsection III.1.2, these properties are essential in connecting the technical with the conceptual, but also, technically, because the failure of these properties results in the violation of some rules of classical first-order logic, as follows.

Assuming that a first-order language $\mathcal{L}$ has a sentential operator $\Box$, pick an $\mathcal{L}$ structure $\mathfrak{M}$ such that $|\mathfrak{M}| = \{a, b\}$ with $a \neq b$, a variable $x$, and an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(x) = a$. It follows that $[b/x]\alpha \neq \alpha$; hence, for a sentence $\varphi$ in which $x$ does *not* occur freely, there is a set $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L})$ such that $(\alpha, \Box\varphi) \in A$ but $([b/x]\alpha, \Box\varphi) \notin A$. Then Fact 10 yields a "classical satisfaction relation" for $\mathcal{L}$ (as in the definition above), $(\mathfrak{M}, \vDash) = e(A)$, such that

$$\mathfrak{M} \vDash_{\alpha} \Box\varphi, \qquad \mathfrak{M} \nvDash_{[b/x]\alpha} \Box\varphi.$$

These mean, since $\alpha$ and $[b/x]\alpha$ only differ at $x$, which does not occur freely in $\Box\varphi$, that $\Box\varphi$ violates (III.9); that is, it is not locally determined in $(\mathfrak{M}, \models)$. Moreover, it follows immediately from (III.5) that

$$\mathfrak{M} \models_\beta \Box\varphi \to \Box\varphi$$

for every assignment $\beta : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$. On the other hand, $\mathfrak{M} \not\models_{[b/x]\alpha} \Box\varphi$ entails

$$\mathfrak{M} \not\models_\alpha \forall x \, \Box\varphi$$

by (III.6), and hence $\mathfrak{M} \models_\alpha \Box\varphi$ entails

$$\mathfrak{M} \not\models_\alpha \Box\varphi \to \forall x \, \Box\varphi$$

by (III.5). Thus $\Box\varphi \to \Box\varphi$ is valid in $(\mathfrak{M}, \models)$ but $\Box\varphi \to \forall x \, \Box\varphi$ is not, even though $x$ does not occur freely in $\varphi$; therefore the rule

$$\frac{\vdash \psi_0 \to \psi_1}{\vdash \psi_0 \to \forall x \,.\, \psi_1} \ (x \text{ does not occur freely in } \psi_0)$$

of classical first-order logic is not valid in $(\mathfrak{M}, \models)$.

Also, for the same $\mathcal{L}$, pick an $\mathcal{L}$ structure $\mathfrak{M}$ such that, to make the example simple, $|\mathfrak{M}| = \{a\}$; so there is exactly one assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$—namely, the one that maps all variables to $a$. Fix a variable $x$, a sentence $\varphi$ in which $x$ occurs freely, and a term $t \neq x$ that is free for $x$ in $\varphi$. Then $\Box\varphi$ and $[t/x](\Box\varphi)$ are not identical; hence there is $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{npa}(\mathcal{L})$ such that $(\alpha, \Box\varphi) \in A$ but $(\alpha, [t/x](\Box\varphi)) \notin A$. Then, again, Fact 10 yields a "classical satisfaction relation" $(\mathfrak{M}, \models) = e(A)$ for $\mathcal{L}$ such that

$$\mathfrak{M} \models_\alpha \Box\varphi, \qquad\qquad \mathfrak{M} \not\models_\alpha [t/x](\Box\varphi).$$

This straightforwardly violates (III.10) (note that $[t^{\mathfrak{M},\alpha}/x]\alpha = \alpha$ because $|\mathfrak{M}| = \{a\}$). Moreover, $\Box\varphi$ is valid in $(\mathfrak{M}, \models)$ (because $\alpha$ is the only assignment) whereas $[t/x](\Box\varphi)$ is not; therefore the rule

$$\frac{\vdash \psi}{\vdash [t/x]\psi}$$

of classical first-order logic is not valid in $(\mathfrak{M}, \models)$.

This is why, to maintain classical first-order logic with a non-classical first-order language, we need to rule out satisfaction relations that violate (III.9) or (III.10). Also, for a reason we will explain later, we need to assume (III.11). Therefore we define classical satisfaction relations to be satisfaction relations satisfying not only (III.2)–(III.8) but also (III.9)–(III.11).

**Definition 14.** Given a first-order language $\mathcal{L}$, a classical-type satisfaction relation for $\mathcal{L}$ is said to be *classical*, and called simply a *classical satisfaction relation for* $\mathcal{L}$, if it satisfies (III.2)–(III.11). The class of all the classical satisfaction relations for $\mathcal{L}$ is called the *classical semantics for* $\mathcal{L}$.

This definition subsumes the case of classical languages in Definition 5, due to Facts 4, 6, and 8. Indeed, to fill in the gap from satisfying (III.2)–(III.8) to being classical, we only need to assume (III.9)–(III.11) for non-primitive atomic sentences, due to the following generalization of Facts 4, 6, and 8 (it subsumes Facts 4, 6, and 8 because a classical first-order language $\mathcal{L}$ has $\text{npa}(\mathcal{L}) = \varnothing$, that is, it has no non-primitive atomic sentences).

**Fact 11.** Given any first-order language $\mathcal{L}$, suppose $(\mathfrak{M}, \models)$ is a classical-type satisfaction relation for $\mathcal{L}$ that satisfies (III.2)–(III.8) for every $\varphi \in \text{sent}(\mathcal{L})$. Then

(i) $(\mathfrak{M}, \models)$ satisfies (III.9) for every $\varphi \in \text{sent}(\mathcal{L})$, if it satisfies (III.9) for every $\varphi \in \text{npa}(\mathcal{L})$.

(ii) $(\mathfrak{M}, \models)$ satisfies (III.10) for every $\varphi \in \text{sent}(\mathcal{L})$, if it satisfies (III.9) for every $\varphi \in \text{sent}(\mathcal{L})$ and (III.10) for every $\varphi \in \text{npa}(\mathcal{L})$.

(iii) $(\mathfrak{M}, \models)$ satisfies (III.11) for every $\varphi, \psi \in \text{sent}(\mathcal{L})$, if it satisfies (III.9), (III.10) for every $\varphi \in \text{sent}(\mathcal{L})$ and (III.11) for every $\varphi, \psi \in \text{npa}(\mathcal{L})$.

*Proof.* By Observation 1, induction on the construction of sentences from $\text{ca}(\mathcal{L})$ with classical operators proves (i) and (ii). The induction goes in the same way as in the proofs for Facts 4 and 6 except that we now have two base cases, one for atomic sentences and the other for non-primitive atomic sentences; but the latter is simply assumed to be the case.

To show (iii), assume its antecedent and write $R(\varphi, \psi)$ to mean that both $\varphi \propto_0 \psi$ and (III.12). Then we claim that $R$ satisfies (iii) of Definition 11. This is proven by induction on the usual construction of sentences—from primitive atomic sentences with all sentential operators—in a manner similar to the proof for Fact 9, except that in case $\otimes$ is non-classical we use the assumption that (III.11) holds for every $\varphi, \psi \in \text{npa}(\mathcal{L})$. Because we can show that $R$ satisfies (i), (ii) of

Definition 11 in a manner similar to the proof for Fact 8, it follows that $\propto_0 \subseteq R$. Therefore $\varphi \propto \psi$ for the transitive closure $\propto$ of $\propto_0$ entails the equivalence relation (III.12); thus (III.11). □

Thus, the definition of classical semantics assumes (III.9)–(III.11), at least for non-primitive atomic sentences. As argued above, this assumption is needed to validate certain rules of classical first-order logic: To assume more conditions for classical satisfaction relations is to make smaller the class of those relations; and (III.9)–(III.11) make that class small enough to validate the rules at issue. They are, nonetheless, not just assumed *ad hoc* to patch up the validity; rather, (III.9)–(III.11) are essential properties of classical semantics, in the sense that they make the class of classical satisfaction relations the right size, as in:

**Lemma 1.** *For every first-order language $\mathcal{L}$, there exist*

- *a* purely classical *first-order language $\mathcal{L}'$ such that* $\mathrm{var}(\mathcal{L}) = \mathrm{var}(\mathcal{L}')$*,*
- *a surjection* $* : \mathrm{sent}(\mathcal{L}) \twoheadrightarrow \mathrm{sent}(\mathcal{L}')$ *from the sentences of $\mathcal{L}$ onto those of $\mathcal{L}'$, and*
- *a (class-sized) bijection* $* : (\mathfrak{M}, \models) \mapsto (\mathfrak{M}^*, \models^*)$ *from the class of classical satisfaction relations* $(\mathfrak{M}, \models)$ *for $\mathcal{L}$—that is, classical-type satisfaction relations for $\mathcal{L}$ that satisfy* (III.9)–(III.11) *in addition to* (III.2)–(III.8)*—to the class of classical satisfaction relations* $(\mathfrak{M}^*, \models^*)$ *for $\mathcal{L}'$—that is, classical-type satisfaction relations for $\mathcal{L}'$ that satisfy* (III.2)–(III.8)*—such that, for each classical satisfaction relation* $(\mathfrak{M}, \models)$ *for $\mathcal{L}$,*
  - $|\mathfrak{M}| = |\mathfrak{M}^*|$ *(which implies that $\mathfrak{M}$ and $\mathfrak{M}^*$ share the same set* $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} = |\mathfrak{M}^*|^{\mathrm{var}(\mathcal{L}')}$ *of assignments) and,*
  - *for every assignment* $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ *and sentence $\varphi$ of $\mathcal{L}$,*

$$\mathfrak{M} \models_\alpha \varphi \iff \mathfrak{M}^* \models^*_\alpha \varphi^*.$$

We can say $\mathcal{L}'$ as above is a purely classical or "purified" version of $\mathcal{L}$; hence we will call this lemma the "purification lemma". This lemma means that the classical semantics for a non-classical first-order language $\mathcal{L}$ is, when defined with (III.9)–(III.11), equivalent to the classical semantics for a purified version of $\mathcal{L}$, with which (III.9)–(III.11) are just derivable. In other words, when a given language is not classical, (III.9)–(III.11) are exactly what we need to have essentially the same semantics as for classical languages. Thus, for example, the following corollary follows from Lemma 1 (and the soundness and completeness of classical logic for classical first-order languages):

**Corollary 1.** Given a first-order language $\mathcal{L}$ (whether classical or not), the classical logic for $\mathcal{L}$ is sound and complete with respect to the classical semantics for $\mathcal{L}$.

Hence the upshot of Lemma 1 is that, as long as the logic of classical semantics—validity in it or soundness and completeness with respect to it—is concerned, we can restrict our attention to classical first-order languages. We close this section by providing a construction and a proof for the lemma. Although its statement above only mentions the syntactic and semantics aspects of "purification", we will also show the proof-theoretic aspect as well, as Fact 13.

To construct a purified version $\mathcal{L}'$ of $\mathcal{L}$ and operations $*$ as in Lemma 1, we need the following definitions and observations. First, fixing a first-order language $\mathcal{L}$, let us write $\varphi \precsim \psi$, for sentences $\varphi$ and $\psi$ of $\mathcal{L}$, to mean that

- $\psi = [t_n/x_n]\cdots[t_1/x_1]\varphi$ for some terms $t_1, \ldots, t_n$ and variables $x_1, \ldots, x_n$ (perhaps $n = 0$, to allow $\varphi \precsim \varphi$).[12]

That is, $\varphi \precsim \psi$ if $\psi$ can be obtained from $\varphi$ by substituting terms. $\precsim$ is clearly a preorder, that is, a reflexive and transitive binary relation. It is also worth noting that, writing $\varphi \sim \psi$ to mean that both $\varphi \precsim \psi$ and $\psi \precsim \varphi$, we have $\varphi \sim \psi$ iff $\varphi$ and $\psi$ share the same variable structure with possibly different free variables. Then, recalling that $\varphi \propto \psi$—alpha-equivalence between $\varphi$ and $\psi$—means that $\varphi$ and $\psi$ share the same variable structure with possibly different bound variables, write $\approx$ for the transitive closure of $\propto \cup \sim$; that is, $\varphi \approx \psi$ if

- there is a sequence of sentences $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_{n-1}, \varphi_n = \psi$ (perhaps $n = 0$) such that, for each $i < n$, either $\varphi_i \propto \varphi_{i+1}$ or $\varphi_i \sim \varphi_{i+1}$.

Thus $\varphi \approx \psi$ means that $\varphi$ and $\psi$ share the same variable structure with possibly different variables, bound or free. $\approx$ is an equivalence relation because $\propto$ and $\sim$ are so.

Moreover, let us introduce:

**Definition 15.** We say a sentence $\varphi$ of $\mathcal{L}$ is *minimally termed* if $\varphi$ is $\precsim$-minimal, that is, if $\varphi \sim \psi$ whenever $\psi \precsim \varphi$ for a sentence $\psi$ of $\mathcal{L}$.

[12] To make the scopes of operations explicit, $[t_n/x_n]\cdots[t_1/x_1]\varphi$ is $[t_n/x_n](\cdots([t_2/x_2]([t_1/x_1]\varphi))\cdots)$.

Here is a more concrete description of being minimally termed.

**Observation 2.** A sentence $\varphi$ of $\mathcal{L}$ is minimally termed iff the following are the case.

- no variable occurs freely in $\varphi$ more than once, and,
- moreover, if $\varphi$ contains a term $t$ that is not a variable, then $t$ contains some variable that is bound in $\varphi$.

For a minimally termed sentence $\varphi$ of $\mathcal{L}$, let us say $\varphi$ is *n-ary* if $\varphi$ contains exactly $n$ free variables, and then write $\varphi(x_1, \ldots, x_n)$ to refer to $\varphi$ with the assumption that $x_1, \ldots, x_n$ occur freely in $\varphi$ in that order. Observe that every sentence $\psi$ of $\mathcal{L}$ can be written as $[t_n/x_n] \cdots [t_1/x_1]\varphi$ for some minimally termed $\varphi(x_1, \ldots, x_n)$ (for some $n$) such that none of variables $\bar{x}$ occurs in any of terms $\bar{t}$; in particular, if $\psi$ is non-primitive atomic, then $\psi = [t_n/x_n] \cdots [t_1/x_1]\varphi$ for some minimally termed, non-primitive atomic $\varphi(x_1, \ldots, x_n)$ (such that none of $\bar{x}$ occurs in any of $\bar{t}$).

Then write mnpa($\mathcal{L}$) for the set of minimally termed, non-primitive atomic sentences of $\mathcal{L}$. So we write mnpa($\mathcal{L}$)$/\approx$ for the quotient of mnpa($\mathcal{L}$) by $\approx$; that is, writing $[\varphi]$ for the equivalence class

$$[\varphi] = \{\, \psi \in \mathrm{mnpa}(\mathcal{L}) \mid \varphi \approx \psi \,\}$$

under $\approx$ to which $\varphi \in$ mnpa($\mathcal{L}$) belongs,

$$\mathrm{mnpa}(\mathcal{L})/{\approx} = \{\, [\varphi] \mid \varphi \in \mathrm{mnpa}(\mathcal{L}) \,\}$$

is the set of equivalence classes under $\approx$ of minimally termed, non-primitive atomic sentences of $\mathcal{L}$.

Using this set, we give the following language as a language $\mathcal{L}'$ as required in Lemma 1:

**Definition 16.** Given a first-order language $\mathcal{L}$, its *purification*, written $\mathcal{L}^{\mathrm{pc}}$, is the purely classical first-order language given by the following:

- the primitive predicates of $\mathcal{L}$ together with the elements of mnpa($\mathcal{L}$)$/\approx$ regarded as new primitive predicates, so that the set of primitive predicates of $\mathcal{L}^{\mathrm{pc}}$ is the union of the set of those of $\mathcal{L}$ and mnpa($\mathcal{L}$)$/\approx$;
- the same individual variables, function symbols, and constant symbols as $\mathcal{L}$ has; and
- the classical operators, but no other sentential operators.

By definition, $\mathrm{var}(\mathcal{L}) = \mathrm{var}(\mathcal{L}^{\mathrm{pc}})$ as required in Lemma 1. We then further define a surjection $* : \mathrm{sent}(\mathcal{L}) \twoheadrightarrow \mathrm{sent}(\mathcal{L}^{\mathrm{pc}})$ as in Lemma 1, by the following induction:

$$\varphi^* = \varphi \qquad \text{for an atomic sentence } \varphi \text{ of } \mathcal{L},$$

$$\psi^* = [\varphi]t_1 \cdots t_n \qquad \text{for an non-primitive atomic sentence } \psi \text{ of } \mathcal{L} \text{ such that}$$
$$\psi = [t_n/x_n] \cdots [t_1/x_1]\varphi \text{ for some } \varphi(x_1, \ldots, x_n) \in \mathrm{mnpa}(\mathcal{L}),$$
$$\text{where none of variables } \bar{x} \text{ occurs in any of terms } \bar{t},$$

$$(\neg\varphi)^* = \neg\varphi^* \qquad (\text{similarly for } \forall x \text{ and } \exists x),$$

$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* \qquad (\text{similarly for } \vee \text{ and } \rightarrow).$$

With this definition, the following need checking, but it is easy to check them:

- The induction defines $\varphi^*$ for all $\varphi \in \mathrm{sent}(\mathcal{L})$, because all sentences of $\mathcal{L}$ are constructed from classically atomic sentences of $\mathcal{L}$ with the classical operators.
- $*$ is well-defined, because, for each non-primitive atomic sentence $\psi$ of $\mathcal{L}$, if

$$\psi = [t_n/x_n] \cdots [t_1/x_1]\varphi_0 = [t'_n/y_n] \cdots [t'_1/y_1]\varphi_1$$

  for $\varphi_0(\bar{x}), \varphi_1(\bar{y}) \in \mathrm{mnpa}(\mathcal{L})$ then $\varphi_0 \sim \varphi_1$, which implies $[\varphi_0] = [\varphi_1]$, and $t_i = t'_i$ for every $i \leqslant n$.
- $*$ is surjective since, for each atomic sentence $F\bar{t}$ of $\mathcal{L}^{\mathrm{pc}}$, if $F$ is a primitive predicate of $\mathcal{L}$ then $F\bar{t} = (F\bar{t})^*$, whereas $F\bar{t} = ([t_n/x_n] \cdots [t_1/x_1]\varphi)^*$ if $F = [\varphi]$ for $\varphi(\bar{x}) \in \mathrm{mnpa}(\mathcal{L})$.

So far, we have "purified" $\mathcal{L}$ syntactically. It is worth observing that the "purification" extends to proof-theory; Fact 13 expresses the proof-theoretic aspect of the "purification". Let us first observe

**Fact 12.** For every pair of sentences $\varphi, \psi$ of $\mathcal{L}$, if $\varphi^* = \psi^*$ then $\varphi \propto \psi$.

*Proof.* By induction on the construction of sentences of $\mathcal{L}$ from classically atomic sentences with classical operators. □

Then we have

**Fact 13.** If a theory $\mathbb{T}$ in $\mathcal{L}$ respects alpha-equivalence, in the sense that

$$\varphi \propto \psi \implies \mathbb{T} \text{ proves } \varphi \text{ iff it proves } \psi$$

for every pair of sentences $\varphi, \psi$ of $\mathcal{L}$, then there is a theory $\mathbb{T}^{\mathrm{pc}}$ in $\mathcal{L}^{\mathrm{pc}}$ such that

$$\text{(III.14)} \qquad \mathbb{T}^{\mathrm{pc}} \text{ proves } \varphi^* \iff \mathbb{T} \text{ proves } \varphi$$

for every sentence $\varphi$ of $\mathcal{L}$.

*Proof.* Given any theory $\mathbb{T}$ in $\mathcal{L}$, define a theory $\mathbb{T}^{\mathrm{pc}}$ as

$$\mathbb{T}^{\mathrm{pc}} \text{ proves } \varphi \iff \mathbb{T} \text{ proves } \varphi_0 \text{ for some sentence } \varphi_0 \text{ of } \mathcal{L} \text{ such that } {\varphi_0}^* = \varphi$$

for every sentence $\varphi$ of $\mathcal{L}^{\mathrm{pc}}$; in other words, $\mathbb{T}^{\mathrm{pc}}$ is the direct image of $\mathbb{T}$ under $*$. Then "$\Leftarrow$" of (III.14) is trivial. On the other hand, if $\mathbb{T}$ respects alpha-equivalence, it implies the last entailment below, while Fact 12 implies the second:

$$\begin{aligned} \mathbb{T}^{\mathrm{pc}} \text{ proves } \varphi^* &\implies \mathbb{T} \text{ proves } \varphi_0 \text{ for some sentence } \varphi_0 \text{ of } \mathcal{L} \text{ such that } {\varphi_0}^* = \varphi^* \\ &\implies \mathbb{T} \text{ proves } \varphi_0 \text{ for some sentence } \varphi_0 \text{ of } \mathcal{L} \text{ such that } \varphi_0 \propto \varphi \\ &\implies \mathbb{T} \text{ proves } \varphi. \end{aligned}$$

□

Finally, we show the semantic aspect of the "purification", by constructing a bijective operation $* : (\mathfrak{M}, \vDash) \mapsto (\mathfrak{M}^*, \vDash^*)$ as in Lemma 1, as follows.

**Definition 17.** Given any classical satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$, define $\mathfrak{M}^*$ as the expansion

$$\mathfrak{M}^* = (|\mathfrak{M}|, F^{\mathfrak{M}}, [\varphi]^{\mathfrak{M}^*}, f^{\mathfrak{M}}, c^{\mathfrak{M}})$$

of $\mathcal{L}$ structure $\mathfrak{M}$ to $\mathcal{L}^{\mathrm{pc}}$ such that, for each $\varphi(\bar{x}) \in \mathrm{mnpa}(\mathcal{L})$,

$$[\varphi]^{\mathfrak{M}^*} = \{\, \bar{a} \in |\mathfrak{M}|^n \mid \mathfrak{M} \vDash_\alpha \varphi \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n \,\},$$

and define $\vDash^*$ as the relation such that $(\mathfrak{M}^*, \vDash^*)$ is the classical satisfaction relation for $\mathcal{L}^{\mathrm{pc}}$ (which is unique because $\mathcal{L}^{\mathrm{pc}}$ is classical).

We need to check and prove:

**Remark 1.** $[\varphi]^{\mathfrak{M}^*}$ in Definition 17 is well-defined.

To prove this, it is helpful to show:

**Remark 2.** If $\varphi \approx \psi$ for $\varphi(\bar{x}), \psi(\bar{y}) \in \text{mnpa}(\mathcal{L})$ then, for every $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n]\cdots[\alpha(x_1)/y_1]\alpha} \psi.$$

*Proof.* Suppose $\varphi \approx \psi$ for $\varphi(\bar{x}), \psi(\bar{y}) \in \text{mnpa}(\mathcal{L})$. Then there is a sequence of sentences

$$\varphi(\bar{x}) = \varphi_0(y_1^0, \ldots, y_n^0), \varphi_1(y_1^1, \ldots, y_n^1), \ldots, \varphi_{m-1}(y_1^{m-1}, \ldots, y_n^{m-1}), \varphi_m(y_1^m, \ldots, y_n^m) = \psi(\bar{y})$$

such that, for each $i < m$, $\varphi_i \in \text{mnpa}(\mathcal{L})$ and either

(i) $\varphi_i \propto \varphi_{i+1}$, with $y_j^i = y_j^{i+1}$ for all $j \leqslant n$, or

(ii) $\varphi_{i+1} = [y_k^{i+1}/y_k^i]\varphi_i$ for some $k \leqslant n$, where $y_k^{i+1}$ does not occur freely in $\varphi_i$, with $y_j^i = y_j^{i+1}$ for all $j \leqslant n$ except $k$.

Hence, fixing $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$, we show by induction on this sequence that

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i]\alpha} \varphi_i \tag{III.15}$$

for each $i \leqslant n$. This is trivial for $i = 0$, since $\varphi = \varphi_0$ and $[\alpha(x_n)/y_n^0]\cdots[\alpha(x_1)/y_1^0]\alpha = \alpha$. Suppose (III.15) holds for $i$; we want to show that it holds for $i+1$. We have two cases corresponding to Cases (i) and (ii) described above:

(i) In Case (i), that $y_j^i = y_j^{i+1}$ for all $j \leqslant n$ trivially entails

$$[\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha = [\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i]\alpha$$

and hence, by (III.11), $\varphi_i \propto \varphi_{i+1}$ entails

$$\mathfrak{M} \vDash_{[\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i]\alpha} \varphi_i \iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha} \varphi_{i+1}.$$

(ii) In Case (ii), we have

$$\begin{aligned}\Big[\big([\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha\big)(y_k^{i+1})/y_k^i\Big][\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha \\ = [\alpha(x_k)/y_k^i][\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha \\ = [\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i][\alpha(x_k)/y_k^{i+1}]\alpha\end{aligned}$$

because $y_j^i = y_j^{i+1}$ for all $j \leqslant n$ except $k$ (and because $y_1^i, \ldots, y_n^i, y_k^{i+1}$ are all distinct); and hence, by (III.9) and (III.10), $\varphi_{i+1} = [y_k^{i+1}/y_k^i]\varphi_i$ entails

$$\begin{aligned}\mathfrak{M} \vDash_{[\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha} \varphi_{i+1} &\iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha} [y_k^{i+1}/y_k^i]\varphi_i \\ &\overset{\text{(III.10)}}{\iff} \mathfrak{M} \vDash_{[([\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha)(y_k^{i+1})/y_k^i][\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha} \varphi_i \\ &\iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i][\alpha(x_k)/y_k^{i+1}]\alpha} \varphi_i \\ &\overset{\text{(III.9)}}{\iff} \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i]\alpha} \varphi_i,\end{aligned}$$

where the last equivalence holds by (III.9) because $y_k^{i+1}$ does not occur freely in $\varphi_i$.

Therefore, in either case, (III.15) for $i$ implies

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^i]\cdots[\alpha(x_1)/y_1^i]\alpha} \varphi_i \iff \mathfrak{M} \vDash_{[\alpha(x_n)/y_n^{i+1}]\cdots[\alpha(x_1)/y_1^{i+1}]\alpha} \varphi_{i+1},$$

that is, that (III.15) holds for $i + 1$. □

*Proof for Remark 1.* For $n$-ary $\varphi(\bar{x}), \psi(\bar{y}) \in \mathrm{mnpa}(\mathcal{L})$, suppose $[\varphi] = [\psi]$. This means $\varphi \approx \psi$, and hence Remark 2 implies, for every $\bar{a} \in |\mathfrak{M}|^n$,

$$\begin{aligned}\bar{a} \in [\varphi]^{\mathfrak{M}^*} &\implies \mathfrak{M} \vDash_\alpha \varphi \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n \\ &\implies \mathfrak{M} \vDash_{[\alpha(x_n)/y_n]\cdots[\alpha(x_1)/y_1]\alpha} \psi \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \\ &\qquad\qquad \text{such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n \\ &\implies \mathfrak{M} \vDash_{[a_n/y_n]\cdots[a_1/y_1]\alpha} \psi \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \\ &\implies \mathfrak{M} \vDash_\beta \psi \text{ for some } \beta : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \beta(y_i) = a_i \text{ for each } i \leqslant n \\ &\implies \bar{a} \in [\psi]^{\mathfrak{M}^*},\end{aligned}$$

and, symmetrically, $\bar{a} \in [\varphi]^{\mathfrak{M}^*}$ if $\bar{a} \in [\psi]^{\mathfrak{M}^*}$. Thus $[\varphi]^{\mathfrak{M}^*} = [\psi]^{\mathfrak{M}^*}$. □

We have two more things left to prove to establish Lemma 1.

**Remark 3.** For each classical satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$, assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and sentence $\varphi$ of $\mathcal{L}$,

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M}^* \vDash^*_\alpha \varphi^*.$$

*Proof.* By induction on the construction of $\varphi$ from classically atomic sentences of $\mathcal{L}$ with the classical operators. If $\varphi = F\bar{t}$ for an $n$-ary primitive predicate $F$ of $\mathcal{L}$, then $\varphi^* = F\bar{t}$ and

$$\mathfrak{M} \vDash_\alpha \varphi \iff \vec{t}^{\mathfrak{M},\alpha} \in F^{\mathfrak{M}} \iff \mathfrak{M}^* \vDash^*_\alpha \varphi^*$$

for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$. If $\varphi$ is an non-primitive atomic sentence of $\mathcal{L}$, then $\varphi = [t_n/x_n] \cdots [t_1/x_1]\psi$ for some $\psi(x_1, \ldots, x_n) \in \mathrm{mnpa}(\mathcal{L})$ such that none of variables $\bar{x}$ occurs in any of terms $\bar{t}$, and hence $\varphi^* = [\psi]\bar{t}$ and, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha \varphi &\iff \mathfrak{M} \vDash_\alpha [t_n/x_n] \cdots [t_1/x_1]\psi \\
&\overset{\text{(III.10)}}{\iff} \mathfrak{M} \vDash_{[{t_n}^{\mathfrak{M},\alpha}/x_n]\alpha} [t_{n-1}/x_{n-1}] \cdots [t_1/x_1]\psi \\
&\overset{\text{(III.10)}}{\iff} \mathfrak{M} \vDash_{[{t_{n-1}}^{\mathfrak{M},[{t_n}^{\mathfrak{M},\alpha}/x_n]\alpha}/x_{n-1}][{t_n}^{\mathfrak{M},\alpha}/x_n]\alpha} [t_{n-2}/x_{n-2}] \cdots [t_1/x_1]\psi \\
&\overset{\dagger}{\iff} \mathfrak{M} \vDash_{[{t_{n-1}}^{\mathfrak{M},\alpha}/x_{n-1}][{t_n}^{\mathfrak{M},\alpha}/x_n]\alpha} [t_{n-2}/x_{n-2}] \cdots [t_1/x_1]\psi \\
&\ \ \vdots \\
&\iff \mathfrak{M} \vDash_{[{t_1}^{\mathfrak{M},\alpha}/x_1]\cdots[{t_n}^{\mathfrak{M},\alpha}/x_n]\alpha} \psi \\
&\overset{\ddagger}{\iff} \mathfrak{M} \vDash_{[{t_n}^{\mathfrak{M},\alpha}/x_n]\cdots[{t_1}^{\mathfrak{M},\alpha}/x_1]\alpha} \psi \\
&\iff \vec{t}^{\mathfrak{M},\alpha} \in [\psi]^{\mathfrak{M}^*} \\
&\iff \mathfrak{M}^* \vDash^*_\alpha [\psi]\bar{t} \\
&\iff \mathfrak{M}^* \vDash^*_\alpha \varphi^*.
\end{aligned}$$

Here the equivalence marked with † holds by Fact 3 since none of variables $\bar{x}$ occurs in any of terms $\bar{t}$, which also yields the equivalence marked with ‡.

Now, given sentences $\varphi$ and $\psi$ of $\mathcal{L}$, suppose

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M}^* \vDash^*_\alpha \varphi^*, \qquad \mathfrak{M} \vDash_\alpha \psi \iff \mathfrak{M}^* \vDash^*_\alpha \psi^*$$

for every assignment $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$. Then, for every $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha \varphi \wedge \psi &\iff \mathfrak{M} \vDash_\alpha \varphi \text{ and } \mathfrak{M} \vDash_\alpha \psi \\
&\iff \mathfrak{M}^* \vDash^*_\alpha \varphi^* \text{ and } \mathfrak{M}^* \vDash^*_\alpha \psi^* \\
&\iff \mathfrak{M}^* \vDash^*_\alpha \varphi^* \wedge \psi^* \\
&\iff \mathfrak{M}^* \vDash^*_\alpha (\varphi \wedge \psi)^*
\end{aligned}$$

because $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, and also

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha \forall x . \varphi &\iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| \\
&\iff \mathfrak{M}^* \vDash^*_{[a/x]\alpha} \varphi^* \text{ for every } a \in |\mathfrak{M}| \\
&\iff \mathfrak{M}^* \vDash^*_\alpha \forall x . \varphi^* \\
&\iff \mathfrak{M}^* \vDash^*_\alpha (\forall x . \varphi)^*
\end{aligned}$$

because $(\forall x . \varphi)^* = \forall x . \varphi^*$. Similarly for $\neg$, $\vee$, $\to$, and $\exists x$. □

**Remark 4.** The operation $* : (\mathfrak{M}, \vDash) \mapsto (\mathfrak{M}^*, \vDash^*)$ defined in Definition 17 is bijective.

*Proof.* To show $*$ injective, fix two distinct classical satisfaction relations $(\mathfrak{M}, \vDash)$, $(\mathfrak{M}', \vDash')$ for $\mathcal{L}$. If $\mathfrak{M} \neq \mathfrak{M}'$ then $\mathfrak{M}^* \neq \mathfrak{M}'^*$ and hence $(\mathfrak{M}^*, \vDash^*) \neq (\mathfrak{M}'^*, \vDash'^*)$. Assume $\mathfrak{M} = \mathfrak{M}'$, on the other hand. Then $\vDash \neq \vDash'$, which means $\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M}' \nvDash'_\alpha \varphi$ for some sentence $\varphi$ of $\mathcal{L}$. Therefore Remark 3 implies $\mathfrak{M}^* \vDash^*_\alpha \varphi^* \iff \mathfrak{M}'^* \nvDash'^*_\alpha \varphi^*$, which means $(\mathfrak{M}^*, \vDash^*) \neq (\mathfrak{M}'^*, \vDash'^*)$. Thus $*$ is injective.

To show $*$ surjective, fix a classical satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}^{\text{pc}}$. Let $\mathfrak{M}'$ be the restriction of $\mathfrak{M}$ to $\mathcal{L}$, and let $\vDash' \subseteq |\mathfrak{M}|^{\text{var}(\mathcal{L})} \times \text{sent}(\mathcal{L})$ be such that

$$\mathfrak{M}' \vDash'_\alpha \varphi \iff \mathfrak{M} \vDash_\alpha \varphi^*$$

for every $\alpha : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$ and $\varphi \in \text{sent}(\mathcal{L})$. We first claim that $(\mathfrak{M}', \vDash')$ is a classical satisfaction relation for $\mathcal{L}$, by showing that it satisfies (III.2)–(III.11). It satisfies (III.3) because

$$\begin{aligned}
\mathfrak{M}' \vDash'_\alpha \varphi \wedge \psi &\iff \mathfrak{M} \vDash_\alpha (\varphi \wedge \psi)^* \\
&\iff \mathfrak{M} \vDash_\alpha \varphi^* \wedge \psi^* \\
&\iff \mathfrak{M} \vDash_\alpha \varphi^* \text{ and } \mathfrak{M} \vDash_\alpha \psi^* \\
&\iff \mathfrak{M}' \vDash'_\alpha \varphi \text{ and } \mathfrak{M}' \vDash'_\alpha \psi;
\end{aligned}$$

similarly for (III.2), (III.4), (III.5). $(\mathfrak{M}', \vDash')$ satisfies (III.6) because

$$\begin{aligned}
\mathfrak{M}' \vDash'_\alpha \forall x . \varphi &\iff \mathfrak{M} \vDash_\alpha (\forall x . \varphi)^* \\
&\iff \mathfrak{M} \vDash_\alpha \forall x . \varphi^* \\
&\iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi^* \text{ for every } a \in |\mathfrak{M}| \\
&\iff \mathfrak{M}' \vDash'_{[a/x]\alpha} \varphi \text{ for every } a \in |\mathfrak{M}| = |\mathfrak{M}'|;
\end{aligned}$$

similarly for (III.7). $(\mathfrak{M}', \vDash')$ satisfies (III.8) because

$$\mathfrak{M}' \vDash'_\alpha F\bar{t} \iff \mathfrak{M} \vDash_\alpha (F\bar{t})^* \iff \mathfrak{M} \vDash_\alpha F\bar{t} \iff \bar{t}^{\mathfrak{M},\alpha} \in F^{\mathfrak{M}} = F^{\mathfrak{M}'}.$$

To show that $(\mathfrak{M}', \vDash')$ satisfies (III.9)–(III.11), it is enough, by Fact 11, to show just that it satisfies (III.9)–(III.11) for every non-primitive atomic sentence of $\mathcal{L}$. Fix any non-primitive atomic $\varphi$; then $\varphi = [t_n/x_n] \cdots [t_1/x_1]\psi$ for some $\psi(x_1, \ldots, x_n) \in \text{mnpa}(\mathcal{L})$ such that none of variables $\bar{x}$ occurs in any of terms $\bar{t}$, and hence $\varphi^* = [\psi]\bar{t}$.

Now, fix any assignments $\alpha, \beta : \text{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(y) = \beta(y)$ for every free variable $y$ in $\varphi$. Then $\alpha(y) = \beta(y)$ for every variable $y$ that occurs in any of $\bar{t}$ and hence for every free variable $y$ in $\varphi^* = [\psi]\bar{t}$. Therefore

$$\mathfrak{M}' \vDash'_\alpha \varphi \iff \mathfrak{M} \vDash_\alpha \varphi^* \iff \mathfrak{M} \vDash_\beta \varphi^* \iff \mathfrak{M}' \vDash'_\beta \varphi$$

because $(\mathfrak{M}, \vDash)$ satisfies (III.9). Thus $(\mathfrak{M}', \vDash')$ satisfies (III.9) for $\varphi$.

Fix a variable $y$ that occurs freely in $\varphi$ (which implies $y$ is not any of $\bar{x}$) and a term $t$ that is free for $y$ in $\varphi$. Then $y$ does not occur freely in $\psi$ and hence

$$[t/y]\varphi = [t/y]([t_n/x_n]\cdots[t_1/x_1]\psi) = [[t/y]t_n/x_n]\cdots[[t/y]t_1/x_1]\psi,$$
$$([t/y]\varphi)^* = [\psi]([t/y]t_1)\cdots([t/y]t_1) = [t/y]([\psi]\bar{t}).$$

Therefore, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\begin{aligned}
\mathfrak{M}' \vDash'_\alpha [t/y]\varphi &\iff \mathfrak{M} \vDash_\alpha ([t/y]\varphi)^* \\
&\iff \mathfrak{M} \vDash_\alpha [t/y]([\psi]\bar{t}) \\
&\iff \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/y]\alpha} [\psi]\bar{t} \\
&\iff \mathfrak{M} \vDash_{[t^{\mathfrak{M},\alpha}/y]\alpha} \varphi^* \\
&\iff \mathfrak{M}' \vDash'_{[t^{\mathfrak{M},\alpha}/y]\alpha} \varphi
\end{aligned}$$

because $(\mathfrak{M}, \vDash)$ satisfies (III.10). Thus $(\mathfrak{M}', \vDash')$ satisfies (III.10) for $\varphi$.

Fix any sentence $\varphi'$ of $\mathcal{L}$ such that $\varphi' \propto \varphi$. Then, clearly, $\varphi' = [t_n/y_n]\cdots[t_1/y_1]\psi'$ for some $\psi'(y_1, \ldots, y_n) \in \mathrm{mnpa}(\mathcal{L})$ such that none of variables $\bar{y}$ occurs in any of $\bar{t}$ and such that $\psi' \propto \psi$, which means $[\psi'] = [\psi]$. Hence $\varphi'^* = [\psi']\bar{t} = [\psi]\bar{t} = \varphi^*$ and, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\mathfrak{M}' \vDash'_\alpha \varphi \iff \mathfrak{M} \vDash_\alpha \varphi^* \iff \mathfrak{M} \vDash_\alpha \varphi'^* \iff \mathfrak{M}' \vDash'_\alpha \varphi'.$$

Thus $(\mathfrak{M}', \vDash')$ satisfies (III.11) for $\varphi$. Therefore $(\mathfrak{M}', \vDash')$ is a classical satisfaction relation for $\mathcal{L}$.

Lastly, we claim $(\mathfrak{M}, \vDash) = (\mathfrak{M}'^*, \vDash'^*)$. Fix any $\varphi(\bar{x}) \in \mathrm{mnpa}(\mathcal{L})$; then $\varphi^* = [\varphi]\bar{x}$. Hence

$$\begin{aligned}
\bar{a} \in [\varphi]^{\mathfrak{M}^*} &\iff \mathfrak{M} \vDash_\alpha [\varphi]\bar{x} \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n \\
&\iff \mathfrak{M} \vDash_\alpha \varphi^* \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n \\
&\iff \mathfrak{M}' \vDash'_\alpha \varphi \text{ for some } \alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}| \text{ such that } \alpha(x_i) = a_i \text{ for each } i \leqslant n.
\end{aligned}$$

Therefore $(\mathfrak{M}, \vDash) = (\mathfrak{M}'^*, \vDash'^*)$ by Definition 17. Thus $*$ is bijective. □

Thus we have proven:

**Lemma 1** (Purification lemma)**.** *For any first-order language $\mathcal{L}$ and its purification $\mathcal{L}^{\mathrm{pc}}$, which is given by (perhaps) adding new primitive predicates to $\mathcal{L}$, there exist*

- *a surjection* $* : \mathrm{sent}(\mathcal{L}) \twoheadrightarrow \mathrm{sent}(\mathcal{L}^{\mathrm{pc}})$ *such that*
  - *if a theory $\mathbb{T}$ in $\mathcal{L}$ respects alpha-equivalence, then there is a theory $\mathbb{T}^{\mathrm{pc}}$ in $L^{\mathrm{pc}}$ such that, for every sentence $\varphi$ of $\mathcal{L}$,*

$$\mathbb{T}^{\mathrm{pc}} \text{ proves } \varphi^* \iff \mathbb{T} \text{ proves } \varphi,$$

- *a (class-sized) bijective operation* $* : (\mathfrak{M}, \vDash) \mapsto (\mathfrak{M}^*, \vDash^*)$ *from the class of classical satisfaction relations for $\mathcal{L}$ to the class of those for $\mathcal{L}^{\mathrm{pc}}$ such that, for each classical satisfaction relation* $(\mathfrak{M}, \vDash)$ *for $\mathcal{L}$,*
  - *$\mathfrak{M}^*$ is an expansion of the $\mathcal{L}$ structure $\mathfrak{M}$ to $\mathcal{L}^{\mathrm{pc}}$;*
  - $(\mathfrak{M}^*, \vDash^*)$ *is the unique classical satisfaction relation for $\mathcal{L}^{\mathrm{pc}}$ on $\mathfrak{M}^*$; and,*
  - *moreover, for every assignment* $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ *and sentence $\varphi$ of $\mathcal{L}$,*

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M}^* \vDash^*{}_\alpha \varphi^*.$$

## III.2 OPERATIONAL SEMANTICS FOR FIRST-ORDER FREE LOGIC

### III.2.1 Existence and Two Notions of Domain

Recall that in Subsection III.1.1 we introduced two terms, *domain of individuals* (on p. 51) and *domain of quantification* (on p. 52), and that, although we did let them refer to the same set $|\mathfrak{M}|$ (given an $\mathcal{L}$ structure $\mathfrak{M}$), we associated with those two terms two different ideas:

(i) the *domain of individuals* is the range of assignments (p. 51);

(ii) the *domain of quantification* is the set over which the variable $x$ of a quantifier $\forall x$ or $\exists x$ ranges (p. 52).

Henceforth we distinguish the two notions by, given an $\mathcal{L}$ structure $\mathfrak{M}$, writing $|\mathfrak{M}|$ for its domain of individuals and $\forall^{\mathfrak{M}}$ for its domain of quantification. So, to expand the ideas,

(i) The assignments are exactly the maps from $\mathrm{var}(\mathcal{L})$ to $|\mathfrak{M}|$, so that we regard the notation $\mathfrak{M} \vDash_\alpha \varphi$ as making sense for any $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$. In other words, for any $a \in |\mathfrak{M}|$, it makes sense to ask whether or not $a$ satisfies a given property $F$ in $\mathfrak{M}$ (via taking an assignment $\alpha$ such that $\alpha(x) = a$ and asking whether $\mathfrak{M} \vDash_\alpha Fx$ or not).

(ii) The variable $x$ in quantifiers $\forall x$ and $\exists x$ ranges over $\forall^{\mathfrak{M}}$, in the sense that

$$\mathfrak{M} \vDash_\alpha \forall x.\varphi \iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in \forall^{\mathfrak{M}}, \tag{III.16}$$

$$\mathfrak{M} \vDash_\alpha \exists x.\varphi \iff \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for some } a \in \forall^{\mathfrak{M}}, \tag{III.17}$$

replacing (III.6) and (III.7).

It should be clear that we need to set $\forall^{\mathfrak{M}} \subseteq |\mathfrak{M}|$ because, for $\mathfrak{M} \vDash_{[a/x]\alpha} \varphi$ to make sense, we need to have $[a/x]\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and hence $a = ([a/x]\alpha)(x) \in |\mathfrak{M}|$. In Subsection III.1.1, we further set $\forall^{\mathfrak{M}} = |\mathfrak{M}|$; but this is a stipulation, however natural it may be. In this subsection, we discuss both the technical and conceptual import of this stipulation.

Before starting the discussion, let us introduce the following terminology, because our discussion in this subsection, and for the most part of the next subsection, ignores function and constant symbols.

**Definition 18.** A *quantified language* is a first-order language (Definition 13) that has no function or constant symbols.

To start discussing the import of the stipulation $\forall^{\mathfrak{M}} = |\mathfrak{M}|$, then, let us define what the semantics would look like without the stipulation. The essential idea is merely to add $\forall^{\mathfrak{M}} \subseteq |\mathfrak{M}|$ to structures and to replace (III.6) and (III.7) with (III.16) and (III.17).

**Definition 19.** Given a quantified language $\mathcal{L}$, we call a tuple $\mathfrak{M} = (|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}})$ a *two-domain $\mathcal{L}$ structure* if $(|\mathfrak{M}|, F^{\mathfrak{M}})$ is an $\mathcal{L}$ structure (Definition 2) and $\varnothing \neq \forall^{\mathfrak{M}} \subseteq \mathfrak{M}$.

**Definition 20.** Given a quantified language $\mathcal{L}$, a *two-domain-type satisfaction relation for $\mathcal{L}$* is a pair of a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ and any relation $(\mathfrak{M} \vDash_{-} -) \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$, as in $\mathfrak{M} \vDash_\alpha \varphi$. We say a two-domain-type satisfaction relation for $\mathcal{L}$ is *on* $\mathfrak{M}$ if its first coordinate is a two-domain $\mathcal{L}$ structure $\mathfrak{M}$.

**Definition 21.** Given a quantified language $\mathcal{L}$, we say, for each two-domain-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$,

- a sentence $\varphi$ of $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash \varphi$, meaning that $\mathfrak{M} \vDash_\alpha \varphi$ for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$; and
- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash \varphi$.

Given a class of two-domain-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is *valid in that class* if it is valid in every member of that class.

**Definition 22.** Given a quantified language $\mathcal{L}$, a two-domain-type satisfaction relation for $\mathcal{L}$ is called a *two-domain satisfaction relation for* $\mathcal{L}$ if it satisfies (III.2)–(III.5), (III.8)–(III.11), (III.16), (III.17) (in which $\mathfrak{M}$ now ranges over two-domain $\mathcal{L}$ structures).[13] The class of all the two-domain satisfaction relations for $\mathcal{L}$ is called the *two-domain semantics for* $\mathcal{L}$.

Here we take (III.9)–(III.11)—local determination, SoS property, and AE property—as part of the definition of two-domain satisfaction relations, rather than as derived properties, for the same reason we discussed in Subsection III.1.3. And, in a similar manner to our proofs in Subsection III.1.3, we can prove the two-domain versions not only of Fact 11 but also of Lemma 1, the purification lemma.

**Lemma 2.** *For every first-order language* $\mathcal{L}$ *and its purification* $\mathcal{L}^{\mathrm{pc}}$ *(Definition 16), there exist a surjection* $* : \mathrm{sent}(\mathcal{L}) \twoheadrightarrow \mathrm{sent}(\mathcal{L}^{\mathrm{pc}})$ *and an bijective operation* $* : (\mathfrak{M}, \vDash) \mapsto (\mathfrak{M}^*, \vDash^*)$ *from the class of two-domain satisfaction relations for* $\mathcal{L}$ *to the class of those for* $\mathcal{L}^{\mathrm{pc}}$ *such that, for each two-domain satisfaction relation* $(\mathfrak{M}, \vDash)$ *for* $\mathcal{L}$*,*

- $\mathfrak{M}^*$ *is an expansion of* $\mathcal{L}$ *structure* $\mathfrak{M}$ *to* $\mathcal{L}^{\mathrm{pc}}$*;*
- $(\mathfrak{M}^*, \vDash^*)$ *is the (unique) two-domain satisfaction relation for* $\mathcal{L}^{\mathrm{pc}}$ *on* $\mathfrak{M}^*$*; and,*
- *moreover, for every assignment* $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ *and sentence* $\varphi$ *of* $\mathcal{L}$*,*

$$\mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M}^* \vDash^*_\alpha \varphi^*.$$

Now that we have defined the semantics without the stipulation $\forall^{\mathfrak{M}} = |\mathfrak{M}|$, let us discuss the import of that stipulation. It should be clear that, given a quantified language $\mathcal{L}$, every $\mathcal{L}$ structure $\mathfrak{M} = (|\mathfrak{M}|, F^{\mathfrak{M}})$ can be identified with the two-domain $\mathcal{L}$ structure $(|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}})$ with $\forall^{\mathfrak{M}} = |\mathfrak{M}|$, for which (III.16) and (III.17) coincide with (III.6) and (III.7). So it follows that the

[13] As the quantified language $\mathcal{L}$ has no function or constant symbols, we could take the simpler (III.1) in place of (III.8); yet the definition with (III.8) extends to the general case of first-order languages with function and constant symbols.

class of classical satisfaction relations for $\mathcal{L}$ as in Subsection III.1.1 is just the class of two-domain satisfaction relations for $\mathcal{L}$ restricted to the class of $\mathcal{L}$ structures. Therefore, the technical import of the stipulation $\forall^{\mathfrak{M}} = |\mathfrak{M}|$ can be captured by the axioms and rules that are valid in the classical semantics but not in the two-domain semantics: the stipulation is required to validate those axioms and rules.

In this sense, the stipulation is required to validate, in particular, the axioms

$$\forall x . \varphi \to \varphi, \qquad \varphi \to \exists x . \varphi.$$

To see this (that is, to show these invalid in the two-domain semantics), fix a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ such that $\forall^{\mathfrak{M}} \subseteq F^{\mathfrak{M}} \subset |\mathfrak{M}|$ for a unary predicate $F$, so that we can pick $b \in |\mathfrak{M}| \setminus F^{\mathfrak{M}}$ and $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(x) = b$. Then (III.8) implies

$$\mathfrak{M} \nvDash_{\alpha} Fx, \qquad \mathfrak{M} \vDash_{\alpha} \neg Fx$$

because $\alpha(x) = b \notin F^{\mathfrak{M}}$. On the other hand, (III.16) and (III.17) imply

$$\mathfrak{M} \vDash_{\alpha} \forall x\, Fx, \qquad \mathfrak{M} \nvDash_{\alpha} \exists x \neg Fx$$

because each $a \in \forall^{\mathfrak{M}}$ has $([a/x]\alpha)(x) = a \in \forall^{\mathfrak{M}} \subseteq F^{\mathfrak{M}}$ and hence

$$\mathfrak{M} \vDash_{[a/x]\alpha} Fx, \qquad \mathfrak{M} \nvDash_{[a/x]\alpha} \neg Fx.$$

Therefore, by (III.5),

$$\mathfrak{M} \nvDash_{\alpha} \forall x\, Fx \to Fx, \qquad \mathfrak{M} \nvDash_{\alpha} \neg Fx \to \exists x \neg Fx.$$

To describe this invalidity in less rigorous terms, $Fx$ is true of "everything" but not of $b = \alpha(x)$, and $\neg Fx$ is true of $b$ but not of "something"; in this sense, $b \notin \forall^{\mathfrak{M}}$ means that $b$ does not count as a "thing" as in "everything" and "something". Or one may find it better to say $b$ is a *non-existing* individual. For example, we might think it makes some sense to ask whether Sherlock Holmes is a logician or not; but, even if Holmes is a logician, it does not imply that a logician exists, since Holmes does not exist. Thus we can regard $\forall^{\mathfrak{M}}$ as the set of existing individuals, with $|\mathfrak{M}|$ the set of all individuals, existing or not. This reading of stipulating (or not stipulating) $\forall^{\mathfrak{M}} = |\mathfrak{M}|$ turns out to be conceptually significant in the context of Kripke's semantics for quantified modal logic, which we will review in Section IV.1.

It is interesting (and will be relevant later in Subsection IV.1.2) to note that if $\forall^{\mathfrak{M}} \subset |\mathfrak{M}|$ then, even though the axioms $\forall x.\varphi \to \varphi$ and $\varphi \to \exists x.\varphi$ are not valid, their universally quantified versions, namely

$$\forall x\,(\forall x.\varphi \to \varphi), \qquad\qquad \forall x\,(\varphi \to \exists x.\varphi),$$

are still valid. Whereas we discussed above what is not valid in two-domain semantics, let us also discuss what is valid.

To show the sentences above to be valid in the two-domain semantics for a given quantified language $\mathcal{L}$, fix any two-domain satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$, assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and $a \in \forall^{\mathfrak{M}}$. Then, if $\mathfrak{M} \vDash_{[a/x]\alpha} \forall x.\varphi$, then (III.16) implies $\mathfrak{M} \vDash_{[a/x]([a/x]\alpha)} \varphi$ because $a \in \forall^{\mathfrak{M}}$, but then $\mathfrak{M} \vDash_{[a/x]\alpha} \varphi$ since $[a/x]([a/x]\alpha) = [a/x]\alpha$; thus

$$\mathfrak{M} \vDash_{[a/x]\alpha} \forall x.\varphi \to \varphi$$

by (III.5). Also, if $\mathfrak{M} \vDash_{[a/x]\alpha} \varphi$, then $\mathfrak{M} \vDash_{[a/x]([a/x]\alpha)} \varphi$ because $[a/x]\alpha = [a/x]([a/x]\alpha)$, and hence $a \in \forall^{\mathfrak{M}}$ implies $\mathfrak{M} \vDash_{[a/x]\alpha} \exists x.\varphi$ by (III.17); thus (III.5) implies

$$\mathfrak{M} \vDash_{[a/x]\alpha} \varphi \to \exists x.\varphi.$$

Because these hold for every $a \in \forall^{\mathfrak{M}}$, (III.16) implies

$$\mathfrak{M} \vDash_{\alpha} \forall x\,(\forall x.\varphi \to \varphi), \qquad\qquad \mathfrak{M} \vDash_{\alpha} \forall x\,(\varphi \to \exists x.\varphi).$$

The moral of this proof is that, even though the individuals $b \notin \forall^{\mathfrak{M}}$ outside $\forall^{\mathfrak{M}}$ may not validate classical quantifier logic, it does not prevent the individuals $a \in \forall^{\mathfrak{M}}$ in $\forall^{\mathfrak{M}}$ from validating it. This observation can be formally stated as Theorem 3 below, but to state it we need some definitions. First, let us introduce a subclass of assignments:

**Definition 23.** Given a quantified language $\mathcal{L}$ and a two-domain $\mathcal{L}$ structure $\mathfrak{M}$, we mean by a *domain-of-quantification assignment*, or *DoQ-assignment* for short, any map $\alpha : \mathrm{var}(\mathcal{L}) \to \forall^{\mathfrak{M}}$.

Using this notion, we can add a new notion of validity.

**Definition 24.** We rename the notion of validity in Definition 21 *all-assignment validity*, or *AA-validity* for short, and introduce a new notion, *domain-of-quantification validity*, or *DoQ-validity* for short: Given a quantified language $\mathcal{L}$, we say, for each two-domain-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$,

- a sentence $\varphi$ of $\mathcal{L}$ is *DoQ-valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash_{\forall} \varphi$, meaning that $\mathfrak{M} \vDash_{\alpha} \varphi$ for every DoQ-assignment $\alpha : \mathrm{var}(\mathcal{L}) \to \forall^{\mathfrak{M}}$; and
- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *DoQ-valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash_{\forall} \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash_{\forall} \varphi$.

Given a class of two-domain-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is *DoQ-valid in that class* if it is DoQ-valid in every member of that class.

Note the following, immediate consequence of (III.9), that is, of local determination.

**Fact 14.** In a two-domain satisfaction relation for a quantified language $\mathcal{L}$, any closed sentence $\varphi$ of $\mathcal{L}$ is AA-valid if and only if it is DoQ-valid.

The observation above that classical quantifier logic is valid within $\forall^{\mathfrak{M}}$, though not valid outside $\forall^{\mathfrak{M}}$, is formally incorporated in:

**Theorem 3.** *For a quantified language $\mathcal{L}$, classical quantifier logic is sound and complete with respect to the DoQ-validity in the two-domain semantics for $\mathcal{L}$.*

This together with Fact 14 entails:

**Corollary 2.** For a quantified language $\mathcal{L}$, if a sentence $\varphi$ of $\mathcal{L}$ is a theorem of classical quantifier logic and if $x_1, \ldots, x_n$ are the only free variables in $\varphi$, then $\forall x_1 \cdots \forall x_n . \varphi$ is valid in the two-domain semantics for $\mathcal{L}$.

To prove Theorem 3, the completeness of classical quantifier logic with respect to the two-domain semantics is immediate from its completeness with respect to the classical semantics, since the latter semantics is a subclass of the former. The soundness is more significant; even though we can prove it by fixing an axiomatic system of classical quantifier logic and checking that its axioms and rules are DoQ-valid, the soundness follows from a conceptually more interesting property of the two-domain semantics for quantifier logic. Intuitively put, this property is that, in the two-domain semantics, whatever holds outside the domain of quantification does not make any difference to what holds within the domain. In this sense, we may adopt the slogan that the domain of quantification is "autonomous" in the two-domain semantics.

It is, however, a tricky problem how to formally express this intuitive idea. In order to solve this problem, it is essential to have available a different notation to the semantics we reviewed so far.

### III.2.2 Operational Semantics: A First Step

As defined in Subsection III.2.1, a two-domain satisfaction relation $(\mathfrak{M}, \vDash)$ for a first-order language $\mathcal{L}$ consists of a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ and a relation $\vDash \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$ that satisfies certain conditions. In this subsection, we first rewrite this relation and interpret sentences with their "extensions", and then extend the notation to also interpret other parts of the vocabulary—not only terms but also sentential operators. In Subsection III.2.4, this new notation will serve the purpose of formally expressing the idea that the domain of quantification is "autonomous" in the two-domain semantics.

Given any sets $X$ and $Y$, a relation $R \subseteq X \times Y$ is mathematically equivalent to its "left transpose" $\overleftarrow{R} : Y \to \mathcal{P}(X)$, which is defined by

$$\overleftarrow{R}(b) = \{\, a \in X \mid Rab \,\} \subseteq X.$$

In other words, when we identify the relation $R$ with its characteristic function

$$R : X \times Y \to \mathbf{2},$$

where $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ is the set of truth values, the left transpose $\overleftarrow{R}$ is the map

$$\overleftarrow{R} : Y \to (X \to \mathbf{2}).$$

With $X = |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ and $Y = \mathrm{sent}(\mathcal{L})$, we take the left transpose of

$$\vDash \; \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L}), \qquad \text{or} \qquad \vDash \; : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L}) \to \mathbf{2};$$

that is, we define

$$[\![-]\!] : \mathrm{sent}(\mathcal{L}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}), \qquad \text{or} \qquad [\![-]\!] : \mathrm{sent}(\mathcal{L}) \to (|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}),$$

by

$$[\![\varphi]\!] = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid \mathfrak{M} \vDash_\alpha \varphi \,\} \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}.$$

In other words, $[\![\varphi]\!]$ is the set of assignments relative to which $\varphi$ is true in $\mathfrak{M}$.[14] So, in this notation,

$$\varphi \text{ is valid in } (\mathfrak{M}, \vDash) \iff [\![\varphi]\!] = |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$$

for classical-type satisfaction relations; and, for two-domain-type satisfaction relations,

$$\varphi \text{ is AA-valid in } (\mathfrak{M}, \vDash) \iff [\![\varphi]\!] = |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})},$$
$$\varphi \text{ is DoQ-valid in } (\mathfrak{M}, \vDash) \iff (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})} \subseteq [\![\varphi]\!],$$

where $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ is the set $\mathrm{var}(\mathcal{L}) \to \forall^{\mathfrak{M}}$ of DoQ-assignments for a two-domain $\mathcal{L}$ structure $\mathfrak{M}$.

In this notation, the truth condition (III.3), for example, amounts to

$$\alpha \in [\![\varphi \wedge \psi]\!] \iff \alpha \in [\![\varphi]\!] \text{ and } \alpha \in [\![\psi]\!],$$

and hence

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!].$$

We express this fact by saying that the operation $\cap$ interprets the operator $\wedge$, and write (with abuse, or extension, of notation) that

$$[\![\wedge]\!] = \cap_{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}} : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \times \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}),$$

so that $[\![\varphi \wedge \psi]\!] = [\![\wedge]\!]([\![\varphi]\!], [\![\psi]\!])$;[15] or, more tellingly,

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!][\![\wedge]\!][\![\psi]\!].$$

[14] $[\![-]\!]$ as the left transpose of $\vDash$ is determined solely by $\vDash$, and is not dependent on $\mathfrak{M}$ (except that the type of $\vDash$ depends on the domain $|\mathfrak{M}|$ of individuals). We will, however, extend the $[\![-]\!]$ notation to interpret terms, and then it will depend partly on $\mathfrak{M}$.

[15] Strictly speaking, (III.3) defines the interpretation of $\wedge$ only on subsets of $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ of the form $[\![\varphi]\!]$. In this sense, $[\![\wedge]\!] = \cap_{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}}$ is stronger than (III.3). This difference is, however, not significant for our purpose.

Similarly, (III.2) and (III.4) are expressed by

$$[\![\neg]\!] = |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \setminus - : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}),$$

$$[\![\vee]\!] = \cup_{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}} : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \times \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}).$$

(III.5) can be expressed by saying that

$$[\![\to]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \times \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$$

satisfies, for any $A, B \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$,

$$[\![\to]\!](A, B) = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid \text{either } \alpha \notin A \text{ or } \alpha \in B \,\}.$$

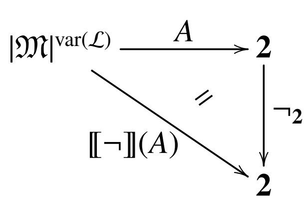
It is worth noting that $[\![\neg]\!]$ can be alternatively defined as follows. Let us write $\neg_{\mathbf{2}} : \mathbf{2} \to \mathbf{2}$ for the truth function—that is, the operation on $\mathbf{2}$—such that

$$\neg_{\mathbf{2}}(\mathbf{1}) = \mathbf{0}, \qquad \neg_{\mathbf{2}}(\mathbf{0}) = \mathbf{1}.$$

Also write $\circ$ for the composition of two maps. Then, identifying $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ with $A : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$, we can see (III.2) means that

$$[\![\neg]\!](A) = \neg_{\mathbf{2}} \circ A : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2},$$

as in the following commutative diagram:




In short, $[\![\neg]\!]$ is given by the "postcomposition" with $\neg_{\mathbf{2}}$, so that

(III.18) $$[\![\neg]\!] = \neg_{\mathbf{2}} \circ -.$$

To define $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\rightarrow]\!]$ in a similar vein, write $\wedge_{\mathbf{2}}, \vee_{\mathbf{2}}, \rightarrow_{\mathbf{2}} : \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ for the truth functions such that

$$\begin{aligned} \wedge_{\mathbf{2}}(\mathbf{0},\mathbf{0}) = \wedge_{\mathbf{2}}(\mathbf{0},\mathbf{1}) = \wedge_{\mathbf{2}}(\mathbf{1},\mathbf{0}) &= \mathbf{0}, & \wedge_{\mathbf{2}}(\mathbf{1},\mathbf{1}) &= \mathbf{1}, \\ \vee_{\mathbf{2}}(\mathbf{0},\mathbf{0}) &= \mathbf{0}, & \vee_{\mathbf{2}}(\mathbf{0},\mathbf{1}) = \vee_{\mathbf{2}}(\mathbf{1},\mathbf{0}) = \vee_{\mathbf{2}}(\mathbf{1},\mathbf{1}) &= \mathbf{1}, \\ \rightarrow_{\mathbf{2}}(\mathbf{1},\mathbf{0}) &= \mathbf{0}, & \rightarrow_{\mathbf{2}}(\mathbf{0},\mathbf{0}) = \rightarrow_{\mathbf{2}}(\mathbf{0},\mathbf{1}) = \rightarrow_{\mathbf{2}}(\mathbf{1},\mathbf{1}) &= \mathbf{1}. \end{aligned}$$

Also, let us introduce the notation that, given maps $f_1 : X \rightarrow Y_1, \ldots, f_n : X \rightarrow Y_n$ of the same domain $X$, $\langle f_1, \ldots, f_n \rangle$ is the map $\langle f_1, \ldots, f_n \rangle : X \rightarrow Y_1 \times \cdots \times Y_n$ such that

$$\langle f_1, \ldots, f_n \rangle(a) = (f_1(a), \ldots, f_n(a)).$$

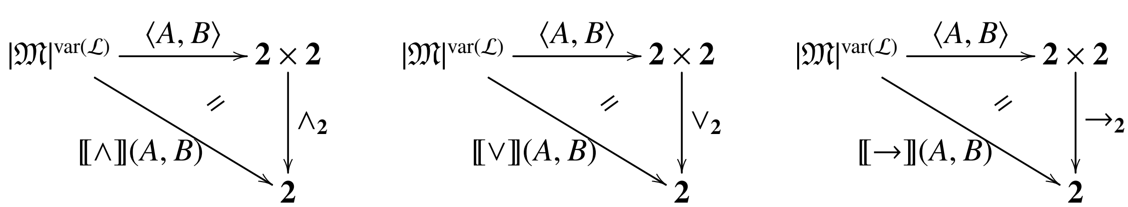
Then, given $A, B : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \rightarrow \mathbf{2}$, (III.3)–(III.5) mean that

$$[\![\wedge]\!](A,B) = \wedge_{\mathbf{2}} \circ \langle A, B \rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \rightarrow \mathbf{2}, \qquad [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ -, \tag{III.19}$$

$$[\![\vee]\!](A,B) = \vee_{\mathbf{2}} \circ \langle A, B \rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \rightarrow \mathbf{2}, \qquad [\![\vee]\!] = \vee_{\mathbf{2}} \circ -, \tag{III.20}$$

$$[\![\rightarrow]\!](A,B) = \rightarrow_{\mathbf{2}} \circ \langle A, B \rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \rightarrow \mathbf{2}, \qquad [\![\rightarrow]\!] = \rightarrow_{\mathbf{2}} \circ -, \tag{III.21}$$

as in:



In general, let us adopt:

**Definition 25.** We say an ($n$-ary) operation $f : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \rightarrow \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ is *truth-functional* if it is a postcomposition $f = f_{\mathbf{2}} \circ -$ with some ($n$-ary) truth function $f_{\mathbf{2}} : \mathbf{2}^n \rightarrow \mathbf{2}$.

To interpret the quantifiers with $[\![\forall x]\!], [\![\exists x]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ so that

$$[\![\forall x\,.\,\varphi]\!] = [\![\forall x]\!][\![\varphi]\!], \qquad\qquad [\![\exists x\,.\,\varphi]\!] = [\![\exists x]\!][\![\varphi]\!],$$

consider the conditions that

$$[\![\forall x]\!](A) = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for every } a \in \forall^{\mathfrak{M}} \,\}, \tag{III.22}$$

$$[\![\exists x]\!](A) = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for some } a \in \forall^{\mathfrak{M}} \,\}. \tag{III.23}$$

(III.16) and (III.17) mean (III.22) and (III.23), respectively, because we have

$$\begin{array}{ccc} \alpha \in [\![\forall x\,.\,\varphi]\!] = [\![\forall x]\!][\![\varphi]\!] & \overset{\text{(III.22)}}{\Longleftrightarrow} & [a/x]\alpha \in [\![\varphi]\!] \text{ for every } a \in \forall^{\mathfrak{M}} \\ \Updownarrow & & \Updownarrow \\ \mathfrak{M} \vDash_{\alpha} \forall x\,.\,\varphi & \underset{\text{(III.16)}}{\Longleftrightarrow} & \mathfrak{M} \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in \forall^{\mathfrak{M}} \end{array}$$

for $\forall x$, and similarly for $\exists x$.[16] We will provide a further analysis of such $[\![\forall x]\!]$ and $[\![\exists x]\!]$ in Subsection III.2.3.

Before expressing the truth condition (III.8) for atomic sentences in the $[\![-]\!]$ notation, we need to extend that notation to interpret terms. This requires fixing an $\mathcal{L}$ structure;[17] but once we fix $\mathfrak{M}$, we can interpret terms simply by taking $[\![t]\!] : \alpha \mapsto t^{\mathfrak{M},\alpha}$ (Definition 3); that is,

$$[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$$

such that $[\![t]\!](\alpha) = t^{\mathfrak{M},\alpha}$. Indeed, by observing that the recursive definition of $t^{\mathfrak{M},\alpha}$ in Definition 3 amounts to

$$\begin{aligned} [\![x]\!](\alpha) &= x^{\mathfrak{M},\alpha} = \alpha(x), \\ [\![ft_1,\ldots,t_n]\!](\alpha) &= (ft_1,\ldots,t_n)^{\mathfrak{M},\alpha} = f^{\mathfrak{M}}({t_1}^{\mathfrak{M},\alpha},\ldots,{t_n}^{\mathfrak{M},\alpha}) = f^{\mathfrak{M}}([\![t_1]\!](\alpha),\ldots,[\![t_n]\!](\alpha)), \end{aligned}$$

[16] We can also interpret the quantifiers $\forall$ and $\exists$ themselves—as opposed to $\forall x$ and $\exists x$—with $[\![\forall]\!], [\![\exists]\!] : \mathrm{var}(\mathcal{L}) \to (\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}))$ such that $[\![\forall]\!](x) = [\![\forall x]\!]$ and $[\![\exists]\!](x) = [\![\exists x]\!]$, though they are not useful for our purpose.

[17] We have not discussed how to interpret function and constant symbols in two-domain $\mathcal{L}$ structures. Nonetheless, as long as a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ interprets $f$ with a map of the type $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$ (perhaps subject to certain restrictions)—as we will see it does—the following remarks on how to rewrite $t^{\mathfrak{M},\alpha}$ with $[\![t]\!]$ extend straightforwardly to the two-domain case.

we can define $[\![t]\!]$ as follows. First let us introduce the notation that, given maps $f_1 : X \to Y_1$, $\dots$, $f_n : X \to Y_n$ (note that they share the same domain $X$), we write $\langle f_1, \dots, f_n \rangle$ for the map $\langle f_1, \dots, f_n \rangle : X \to Y_1 \times \cdots \times Y_n$ such that

$$\langle f_1, \dots, f_n \rangle(a) = (f_1(a), \dots, f_n(a)).$$

So, $\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|^n$ is such that

$$\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle(\alpha) = ([\![t_1]\!](\alpha), \dots, [\![t_n]\!](\alpha)).$$

Then the observation above on $[\![f t_1, \dots, t_n]\!](\alpha)$ amounts to

$$[\![f t_1, \dots, t_n]\!](\alpha) = f^{\mathfrak{M}} \circ \langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle(\alpha),$$

that is,

$$\overbrace{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \xrightarrow[\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle]{} |\mathfrak{M}|^n \xrightarrow[f^{\mathfrak{M}}]{} |\mathfrak{M}|}^{[\![f t_1, \dots, t_n]\!] \; (=)},$$

Hence we can define $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$ recursively with

$$[\![x]\!] : \alpha \mapsto \alpha(x);$$
$$[\![f t_1, \dots, t_n]\!] = f^{\mathfrak{M}} \circ \langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle.$$

It is worth noting that the latter clause subsumes the case of constant symbols, with $n = 0$. Recall $c^{\mathfrak{M}} : |\mathfrak{M}|^0 \to |\mathfrak{M}|$, where $|\mathfrak{M}|^0 = \{*\}$, so $c$ refers to $c^{\mathfrak{M}}(*)$. Then, with $n = 0$ for $\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|^n$, write $\langle\rangle : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|^0$ (so that $\langle\rangle(\alpha) = *$ for all $\alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$). This yields $[\![c]\!] = c^{\mathfrak{M}} \circ \langle\rangle$ as in

$$\overbrace{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \xrightarrow[\langle\rangle]{} |\mathfrak{M}|^0 \xrightarrow[c^{\mathfrak{M}}]{} |\mathfrak{M}|}^{[\![c]\!] \; (=)},$$

and, for each $\alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$,

$$[\![c]\!](\alpha) = c^{\mathfrak{M}} \circ \langle\rangle(\alpha) = c^{\mathfrak{M}}(*).$$

Now that we have rewritten the interpretation of terms, we can rewrite the truth condition (III.8) for atomic sentences. It is helpful to regard $[\![\varphi]\!] \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$, for a sentence $\varphi$ in general, as a map

$$[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}.$$

In a similar vein, the interpretation $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$ (in $\mathfrak{M}$) of an $n$-ary primitive predicate of $\mathcal{L}$ is a map

$$F^{\mathfrak{M}} : |\mathfrak{M}|^n \to \mathbf{2}.$$

Then (III.8) can be expressed by

$$[\![Ft_1 \cdots t_n]\!] = F^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle \tag{III.24}$$

as in

$$\begin{array}{c} \overset{[\![Ft_1, \ldots, t_n]\!]}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}} \\ \shortparallel \\ |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \xrightarrow[\langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle]{} |\mathfrak{M}|^n \xrightarrow[F^{\mathfrak{M}}]{} \mathbf{2}, \end{array}$$

because we have the following:

$$\begin{array}{ccc} [\![F\bar{t}]\!](\alpha) = 1 & \overset{\text{(III.24)}}{\Longleftrightarrow} & F^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle(\alpha) = 1 \\ & & \Updownarrow \\ \Updownarrow & & \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle(\alpha) \in F^{\mathfrak{M}} \\ & & \Updownarrow \\ \mathfrak{M} \models_\alpha F\bar{t} & \underset{\text{(III.8)}}{\Longleftrightarrow} & \bar{t}^{\mathfrak{M},\alpha} \in F^{\mathfrak{M}} \end{array}$$

Let us summarize the observations so far into the following form of definition.

**Definition 26.** Given a first-order language $\mathcal{L}$, a *two-domain-type interpretation for* $\mathcal{L}$ is a pair of a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ and a map $[\![-]\!]$ that assigns, to each term $t$, sentence $\varphi$, and $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, maps

$$[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|,$$
$$[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2},$$
$$[\![\otimes]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$$

that satisfy

$$[\![x]\!] : \alpha \mapsto \alpha(x),$$
$$[\![f t_1 \cdots t_n]\!] = f^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle,$$
$$[\![F t_1 \cdots t_n]\!] = F^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle,$$
$$[\![\otimes(\varphi_1, \ldots, \varphi_n)]\!] = [\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]).$$

We say a two-domain-type interpretation for $\mathcal{L}$ is *on* $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$. Moreover, we say it is a *classical-type interpretation for* $\mathcal{L}$ if it is on an $\mathcal{L}$ structure (that is, on $\mathfrak{M}$ with $\forall^{\mathfrak{M}} = |\mathfrak{M}|$).

Note that, whereas every two-domain-type (or, respectively, classical-type) interpretation for $\mathcal{L}$ gives rise to a two-domain-type (or, respectively, classical-type) satisfaction relation for $\mathcal{L}$ via transposition, not every satisfaction relation arises in that way. This is because the clause (III.8) for atomic sentences and Fact 9 are part of the definition for interpretations—(III.8) is expressed by (III.24), and Fact 9 simply means that $[\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]) = [\![\otimes]\!]([\![\psi_1]\!], \ldots, [\![\psi_n]\!])$ if $[\![\varphi_i]\!] = [\![\psi_i]\!]$ for all $i \leqslant n$—but not for satisfaction relations. Nonetheless, when $\mathcal{L}$ is classical, the following subclass of two-domain-type (or, respectively, classical-type) interpretations for $\mathcal{L}$ is equivalent to the class of two-domain (or, respectively, classical) satisfaction relations for $\mathcal{L}$.

**Definition 27.** Given a classical first-order language $\mathcal{L}$, a two-domain-type interpretation for $\mathcal{L}$ on $\mathfrak{M}$ is said to be a *two-domain interpretation for* $\mathcal{L}$, if it satisfies (III.18)–(III.23):

$$\text{(III.18)} \qquad [\![\neg]\!] = \neg_{\mathbf{2}} \circ -,$$

$$\text{(III.19)} \qquad [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ -,$$

$$\text{(III.20)} \qquad [\![\vee]\!] = \vee_{\mathbf{2}} \circ -,$$

$$\text{(III.21)} \qquad [\![\rightarrow]\!] = \rightarrow_{\mathbf{2}} \circ -,$$

$$\text{(III.22)} \qquad [\![\forall x]\!] : A \mapsto \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for every } a \in \forall^{\mathfrak{M}} \,\},$$

$$\text{(III.23)} \qquad [\![\exists x]\!] : A \mapsto \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for some } a \in \forall^{\mathfrak{M}} \,\}.$$

Moreover, by a *classical interpretation* for $\mathcal{L}$ we mean a classical-type two-domain interpretation for $\mathcal{L}$ (that is, a two-domain interpretation for $\mathcal{L}$ on $\mathfrak{M}$ with $\forall^{\mathfrak{M}} = |\mathfrak{M}|$).

The remark above, just before Definition 27, can be put more rigorously, as follows:

**Fact 15.** Let $\mathcal{L}$ be a quantified language. Given any two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, define a relation $\vDash \subseteq W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$ by transposition

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff (w, \alpha) \in [\![\varphi]\!].$$

This gives an operation from the class of two-domain-type interpretations $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ to the class of two-domain-type satisfaction relations $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$. Restricted to the class of classical-type interpretations for $\mathcal{L}$, this operation is to the class of classical-type satisfaction relations. Moreover, if $\mathcal{L}$ is classical, this operation is bijective when restricted to the class of two-domain interpretations for $\mathcal{L}$ (or to the class of classical interpretations for $\mathcal{L}$).

Therefore, when $\mathcal{L}$ is classical, the classical semantics and the two-domain semantics for $\mathcal{L}$—which are simply the classes of classical and two-domain satisfaction relations—are given by the classes of classical and two-domain interpretations for $\mathcal{L}$.[18]

[18] When $\mathcal{L}$ is not classical, we need to further assume (III.9)–(III.11), as we did in Definition 14.

### III.2.3 A Bit Categorical Preliminary

In this subsections, we give a more algebraic analysis of two-domain-type interpretations, with the help of some notions and insights from category theory and topos theory. This makes available some observations that will be useful later in proofs, as well as technical tools essential in expressing the autonomy of a domain of quantification.

First let us observe that any map $f : X \to Y$ induces three operations $f^*$, $\exists_f$, $\forall_f$ of the types

$$X \xrightarrow{f} Y \qquad\qquad \exists_f : \mathcal{P}(X) \to \mathcal{P}(Y), \quad f^* : \mathcal{P}(Y) \to \mathcal{P}(X), \quad \forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

by the definitions that $f^*$ is the "precomposition" $- \circ f$ with $f$, that is,

$$f^*(B) = f^{-1}[B] = \{\, a \in X \mid f(a) \in B \,\} = B \circ f : X \to \mathbf{2}$$

for every $B : Y \to \mathbf{2}$, and that, for every $A \subseteq X$,

$$\exists_f(A) = \{\, b \in X \mid a \in A \text{ for some } a \text{ such that } b = f(a) \,\} = f[A],$$
$$\forall_f(A) = \{\, b \in X \mid a \in A \text{ for every } a \text{ such that } b = f(a) \,\};$$

in other words, $f^*$ and $\exists_f$ are the inverse- and direct-image operations under $f$; we may sometimes write $f_!$ for $\exists_f$. Take, as a concrete example, the inclusion map $i : X \hookrightarrow Y$ for sets $X \subseteq Y$. Then $i^*$ and $\exists_i$ are such that, for $A \subseteq Y$ and $B \subseteq X \subseteq Y$,

$$i^*(A) = A \cap X,$$
$$\exists_i(B) = B.$$

The maps $f^*$, $\exists_f$, and $\forall_f$ between powersets are obviously monotone in the sense that they preserve $\subseteq$; that is, in the case of $f^*$ for example, $B \subseteq B' \subseteq Y$ entails $f^*(B) \subseteq f^*(B')$.

It is important that the following hold for every $A \subseteq X$ and $B \subseteq Y$, where double lines signify equivalence (in contrast to single lines, which mean one-way entailment).

$$\frac{\exists_f(A) \subseteq B}{A \subseteq f^*(B)}\,, \qquad\qquad \frac{f^*(B) \subseteq A}{B \subseteq \forall_f(A)}\,.$$

To refer to this property, we write $\exists_f \dashv f^* \dashv \forall_f$. In general, given two monotone operators between powersets $\ell : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : r$, we write $\ell \dashv r$, and say that $\ell$ is a *left adjoint* to $r$ and that $r$ is a *right adjoint* to $\ell$, if

$$\frac{\ell(A) \subseteq B}{\overline{A \subseteq r(B)}}$$

for every $A \subseteq X$ and $B \subseteq Y$.[19] Adjoints are unique, because if $\ell \dashv r$ and $\ell' \dashv r$ for $\ell, \ell' : \mathcal{P}(X) \rightrightarrows \mathcal{P}(Y)$ and $r : \mathcal{P}(Y) \to \mathcal{P}(X)$ then for every $A \subseteq X$ we have $\ell(A) = \ell'(A)$ by

$$\begin{array}{c} \ell(A) \subseteq \ell(A) \\ \hline\hline A \subseteq r \circ \ell(A) \\ \hline\hline \ell'(A) \subseteq \ell(A) \end{array}\ , \qquad \begin{array}{c} \ell'(A) \subseteq \ell'(A) \\ \hline\hline A \subseteq r \circ \ell'(A) \\ \hline\hline \ell(A) \subseteq \ell'(A) \end{array}\ .$$

Now, fixing a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a given first-order language $\mathcal{L}$, let us consider the following three maps: Since $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ is clearly isomorphic to $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times |\mathfrak{M}|$ and $\forall^{\mathfrak{M}} \subseteq |\mathfrak{M}|$, there is an obvious injection

$$i : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}} \rightarrowtail |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} :: (\beta, a) \longmapsto \beta \cup \{(x, a)\}.$$

Also, let us write $p$ for the projection

$$p : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}} \twoheadrightarrow |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} :: (\beta, a) \longmapsto \beta,$$

which is clearly surjective. Finally, write $r$ for the restriction

$$r : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \twoheadrightarrow |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} :: \alpha \longmapsto \alpha{\restriction}(\mathrm{var}(\mathcal{L}) \setminus \{x\}),$$

[19] Although we define the notion of adjoints only for monotone maps between powersets here, it is defined in more general terms for functors between categories. See [3] and [28].

which is also clearly surjective. These three maps induce the following nine operations in the way described above. (We will show shortly that they are injective or surjective as indicated.)

$$
\begin{array}{c}
|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}\\
i \big\uparrow\\
|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}}\\
p \big\downarrow\\
|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}}\\
r \big\uparrow\\
|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}
\end{array}
\qquad\qquad
\begin{array}{c}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})\\
\exists_i \big\uparrow \dashv \; i^* \big\downarrow \dashv \big\uparrow \forall_i\\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}})\\
\exists_p \big\downarrow \dashv \; p^* \big\uparrow \dashv \big\downarrow \forall_p\\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}})\\
\exists_r \big\uparrow \dashv \; r^* \big\downarrow \dashv \big\uparrow \forall_r\\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})
\end{array}
$$

To more concretely describe $\exists_p$ and $\forall_p$, in particular, they are such that, for every $\beta : \mathrm{var}(\mathcal{L})\setminus\{x\} \to |\mathfrak{M}|$ and $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}}$,

$$
\begin{aligned}
\beta \in \exists_p(A) &\iff (\beta, a) \in A \text{ for some } a \in \forall^{\mathfrak{M}},\\
\beta \in \forall_p(A) &\iff (\beta, a) \in A \text{ for every } a \in \forall^{\mathfrak{M}}.
\end{aligned}
$$

Then $[\![\forall x]\!]$ and $[\![\exists x]\!]$ satisfying (III.22) and (III.23) can be analyzed with:

**Observation 3.** $[\![\forall x]\!]$ satisfies (III.22) and $[\![\exists x]\!]$ satisfies (III.23), respectively, if and only if

$$
\begin{aligned}
[\![\forall x]\!] &= r^* \circ \forall_p \circ i^*,\\
[\![\exists x]\!] &= r^* \circ \exists_p \circ i^*.
\end{aligned}
$$

*Proof.* For every $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$,

$$
\begin{aligned}
\alpha \in r^* \circ \forall_p \circ i^*(A) &\iff \alpha\restriction(\mathrm{var}(\mathcal{L}) \setminus \{x\}) = r(\alpha) \in \forall_p \circ i^*(A)\\
&\iff (\alpha\restriction(\mathrm{var}(\mathcal{L}) \setminus \{x\}), a) \in i^*(A) \text{ for every } a \in \forall^{\mathfrak{M}}\\
&\iff [a/x]\alpha = i(\alpha\restriction(\mathrm{var}(\mathcal{L}) \setminus \{x\}), a) \in A \text{ for every } a \in \forall^{\mathfrak{M}};
\end{aligned}
$$

that is, $r^* \circ \forall_p \circ i^*$ satisfies (III.22) in place of $[\![\forall x]\!]$. Hence $[\![\forall x]\!] = r^* \circ \forall_p \circ i^*$ iff (III.22). The similar argument with “some” in place of “every” above shows that $[\![\exists x]\!] = r^* \circ \exists_p \circ i^*$ iff (III.23). □

In this way, $[\![\exists x]\!]$ and $[\![\forall x]\!]$ can be defined uniquely by precompositions and their adjoints. It is also worth observing that $[\![\exists x]\!]$ and $[\![\forall x]\!]$ are themselves adjoints. Note that the adjunctions are "composable", in the sense that

$$\mathcal{P}(X) \underset{r_0}{\overset{\ell_0}{\underset{\bot}{\rightleftarrows}}} \mathcal{P}(Y) \underset{r_1}{\overset{\ell_1}{\underset{\bot}{\rightleftarrows}}} \mathcal{P}(Z) \qquad \text{entails} \qquad \mathcal{P}(X) \underset{r_0 \circ r_1}{\overset{\ell_1 \circ \ell_0}{\underset{\bot}{\rightleftarrows}}} \mathcal{P}(Z)\,,$$

because, for every $A \subseteq X$ and $B \subseteq Z$,

$$\begin{array}{c} \ell_1 \circ \ell_0(A) \subseteq B \\ \hline\hline \ell_0(A) \subseteq r_1(B) \\ \hline\hline A \subseteq r_0 \circ r_1(B) \end{array}\,.$$

So we have $[\![\exists x]\!] = r^* \circ \exists_p \circ i^* \dashv \forall_i \circ p^* \circ \forall_r$ and $\exists_i \circ p^* \circ \exists_r \dashv r^* \circ \forall_p \circ i^* = [\![\forall x]\!]$ by composing the adjunctions (twice each).

Let us make some more observations on the induced operations $\exists_f \dashv f^* \dashv \forall_f$ and adjunctions. First, for any $f : X \to Y$ and $g : Y \to Z$, we have $(g \circ f)^* = f^* \circ g^* : \mathcal{P}(Z) \to \mathcal{P}(X)$, because $(g \circ f)^*(C) = C \circ g \circ f = g^*(C) \circ f = f^*(g^*(C))$. Note that, by composing the adjunctions $\exists_f \dashv f^*$ and $\exists_g \dashv g^*$, we have $\exists_g \circ \exists_f \dashv f^* \circ g^*$; therefore $\exists_{g \circ f} = \exists_g \circ \exists_f : \mathcal{P}(X) \to \mathcal{P}(Z)$, since $(g \circ f)^* = f^* \circ g^*$ has a unique left adjoint. Similarly $\forall_{g \circ f} = \forall_g \circ \forall_f : \mathcal{P}(X) \to \mathcal{P}(Z)$.

Note also that, if $f$ is surjective, then $f^*$ is injective because $B \circ f = B' \circ f$ implies $B = B'$. On the other hand, if $f$ is injective, then $\exists_f$ is injective because $f[A] = f[A']$ implies $A = A'$ by

$$a \in A \iff f(a) \in f[A] = f[A'] \iff a \in A'.$$

So, for example, $\exists_i$, $p^*$, and $r^*$ are all injective.

Moreover, if $\ell \dashv r$ for monotone $\ell : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : r$ then $\ell \circ r \circ \ell = \ell$ because, for every $A \subseteq X$,

$$\begin{array}{c} \ell(A) \subseteq \ell(A) \\ \hline\hline A \subseteq r \circ \ell(A) \\ \hline \ell(A) \subseteq \ell \circ r \circ \ell(A) \end{array}\,, \qquad\qquad \begin{array}{c} r \circ \ell(A) \subseteq r \circ \ell(A) \\ \hline\hline \ell \circ r \circ \ell(A) \subseteq \ell(A) \end{array}\,;$$

we can symmetrically show $r \circ \ell \circ r = r$. To put this in other words, suppose that either $m \dashv e$ or $e \dashv m$ is the case for $m : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : e$. Then $m \circ e \circ m = m$ and $e \circ m \circ e = e$. It follows that

$e \circ m = 1$ if either $m$ is injective or $e$ is surjective, while $e \circ m = 1$ implies both that $m$ is injective and that $e$ is surjective.

Therefore, for example, $i^* \circ \exists_i \circ i^* = i^* \circ \forall_i \circ i^* = i^*$. This implies

$$i^* \circ \exists_i = i^* \circ \forall_i = 1$$

since $\exists_i$ is injective; it follows that $i^*$ is surjective and hence $\forall_i$ is injective. Also, $f^* \circ \exists_f \circ f^* = f^* \circ \forall_f \circ f^* = f^*$ for $f = p, r$ implies

$$\exists_p \circ p^* = \forall_p \circ p^* = 1,$$
$$\exists_r \circ r^* = \forall_r \circ r^* = 1,$$

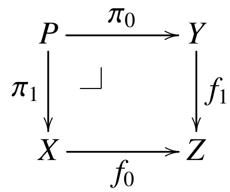
because $p^*$ and $r^*$ are injective; therefore $\exists_p, \forall_p, \exists_r, \forall_r$ are surjective.

Next let us list some properties of the category-theoretic notion of pullbacks. While the reader should consult [3] for instance for a general definition of pullbacks, we can take the following fact as providing a definition for the particular case of the category **Sets** of sets.

**Fact 16.** Given any maps $f_0 : X \to Z$ and $f_1 : Y \to Z$, a set $P$ with maps $\pi_0 : P \to Y$ and $\pi_1 : P \to X$ is a pullback of $f_0$ and $f_1$ if and only if

- $f_0 \circ \pi_1 = f_1 \circ \pi_0$ and,
- moreover, for every $x \in X$ and $y \in Y$ such that $f_0(x) = f_1(y)$, there is an unique element of $P$, written $\langle x, y \rangle$, such that $x = \pi_1(\langle x, y \rangle)$ and $y = \pi_0(\langle x, y \rangle)$.

When $P$ with $\pi_0$ and $\pi_1$ is a pullback of $f_0$ and $f_1$ as above, we indicate it with the diagram




and also say $\pi_0$ is a pullback of $f_0$ *along* $f_1$ (and symmetrically that $\pi_1$ is a pullback of $f_1$ along $f_0$).

In **Sets**, pullbacks always exist, as they can be constructed as follows.

**Definition 28.** Given any maps $f_0 : X \to Z$ and $f_1 : Y \to Z$, the *fibered product* of $X$ and $Y$ is the set

$$X \times_Z Y = \{\,(x, y) \in X \times Y \mid f_0(x) = f_1(y)\,\}$$

together with the "projections"

$$\pi_1 : X \times_Z Y \to X :: (x, y) \mapsto x, \qquad \pi_0 : X \times_Z Y \to Y :: (x, y) \mapsto y.$$

Then the fibered product $X \times_Z Y$ with $p_0$ and $p_1$ is a pullback of $f_0$ and $f_1$.

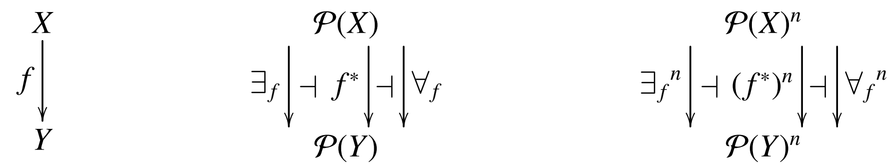
$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_0} & Y \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array}$$

The following lemma states that the category **Sets** of sets satisfies the "Beck-Chevalley condition". See [27], 205, for a general definition of the condition and a proof that it holds of categories called "elementary topoi". (A proof for the case of **Sets** is straightforward and we omit it.)

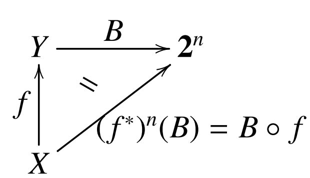
**Lemma 3.** *For every pullback as in the diagram to the left below, we have $\exists_{\pi_1} \circ {\pi_0}^* = {f_0}^* \circ \exists_{f_1}$ and $\forall_{\pi_1} \circ {\pi_0}^* = {f_0}^* \circ \forall_{f_1}$, that is, the two diagrams to the right both commute.*

$$\begin{array}{ccc} P & \xrightarrow{\pi_0} & Y \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array} \qquad \begin{array}{ccc} \mathcal{P}(P) & \xleftarrow{{\pi_0}^*} & \mathcal{P}(Y) \\ {\scriptstyle \exists_{\pi_1}}\downarrow & \approx & \downarrow{\scriptstyle \exists_{f_1}} \\ \mathcal{P}(X) & \xleftarrow[{f_0}^*]{} & \mathcal{P}(Z) \end{array} \qquad \begin{array}{ccc} \mathcal{P}(P) & \xleftarrow{{\pi_0}^*} & \mathcal{P}(Y) \\ {\scriptstyle \forall_{\pi_1}}\downarrow & \approx & \downarrow{\scriptstyle \forall_{f_1}} \\ \mathcal{P}(X) & \xleftarrow[{f_0}^*]{} & \mathcal{P}(Z) \end{array}$$

So far we have observed how a map $f : X \to Y$ induces operations between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ and some properties that hold among the induced operations. To close this subsection, let us observe that these observations extend to the case of $\mathcal{P}(X)^n$ and $\mathcal{P}(Y)^n$.




That is, for every $n$ and $f : X \to Y$, we also have operations $(f^*)^n$, ${\exists_f}^n$, ${\forall_f}^n$ of the types above. To describe $(f^*)^n : \mathcal{P}(Y)^n \to \mathcal{P}(X)^n$ concretely, it maps $(B_1, \ldots, B_n)$ to $(f^*(B_1), \ldots, f^*(B_n))$; but it is more simply described as the precomposition $- \circ f$ with $f$, as in the following:




Also, whereas ${\exists_f}^n, {\forall_f}^n : \mathcal{P}(X)^n \to \mathcal{P}(Y)^n$ are such that

$$ {\exists_f}^n(A_1, \ldots, A_n) = (\exists_f(A_1), \ldots, \exists_f(A_n)), \qquad {\forall_f}^n(A_1, \ldots, A_n) = (\forall_f(A_1), \ldots, \forall_f(A_n)), $$

we can simply say they are the left and right adjoints to $(f^*)^n$. Everything we observed above for the case of $n = 1$ extends to the general case of $n$. We will omit the superscript $n$ and write simply $f^*$, $\exists_f$, $\forall_f$ for $(f^*)^n$, ${\exists_f}^n$, ${\forall_f}^n$, unless it causes confusion.

### III.2.4 Autonomy of Domain of Quantification

Using the operational formulation of semantics we introduced in Subsection III.2.2, we can formally express the intuitive idea we mentioned in Subsection III.2.1 that, in the two-domain semantics, the domain of quantification is "autonomous", in the sense that whatever holds outside the domain of quantification does not make any difference to what holds within the domain.

As defined above in Definition 26, a two-domain-type interpretation for a first-order language $\mathcal{L}$ is a pair $(\mathfrak{M}, [\![-]\!])$, and consists of sets $|\mathfrak{M}|$ and $\forall^{\mathfrak{M}}$ and maps

| | |
|---|---|
| $F^{\mathfrak{M}} : |\mathfrak{M}|^n \to \mathbf{2}$ | for each $n$-ary primitive predicate $F$, |
| $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$ | for each $n$-ary function symbol $f$, |
| $c^{\mathfrak{M}} : |\mathfrak{M}|^0 \to |\mathfrak{M}|$ | for each constant symbol $c$, |
| $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$ | for each term $t$, |
| $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ | for each sentence $\varphi$, |
| $[\![\otimes]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ | for each $n$-ary sentential operator $\otimes$ of $\mathcal{L}$. |

While the set $|\mathfrak{M}|$ defines the types of these maps, it contains existing individuals but perhaps some non-existing individuals as well. We are going to lay out in what sense we can (or cannot) ignore non-existing individuals and restrict our attention to the set $\forall^{\mathfrak{M}} \subseteq |\mathfrak{M}|$ of existing individuals.

Let us first discuss the most trivial case of restricting our attention to $\forall^{\mathfrak{M}}$. Suppose we want to ask whether or not a given existing individual $a \in \forall^{\mathfrak{M}}$ has a property $F$; this is to ask whether $a \in F^{\mathfrak{M}}$ or not. Suppose we indeed ask the same question for all the existing individuals; this is, in effect, to ask what set $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ is. And we should note the following truism: To tell whether given $a \in \forall^{\mathfrak{M}}$ has $F$ or not, this piece of information of what set $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ is is sufficient and it is irrelevant whether any *non-existing* individual $b \notin \forall^{\mathfrak{M}}$ has $F$ or not. In this sense, it is by restricting $F^{\mathfrak{M}}$ to $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ that we restrict our attention to $\forall^{\mathfrak{M}}$ and ignore non-existing individuals.

Let us observe that, from the point of view of $F^{\mathfrak{M}}$ as a map $F^{\mathfrak{M}} : |\mathfrak{M}| \to \mathbf{2}$, what we have just seen is to restrict the map $F^{\mathfrak{M}}$ to the map $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}} : \forall^{\mathfrak{M}} \to \mathbf{2}$, which is of the same type as $F^{\mathfrak{M}}$, except that $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ takes $\forall^{\mathfrak{M}}$ in place of $|\mathfrak{M}|$.

$$|\mathfrak{M}| \xrightarrow{F^{\mathfrak{M}}} \mathbf{2}$$

$$\forall^{\mathfrak{M}} \xrightarrow{F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}} \mathbf{2}$$

Indeed, the fact that the second map is the intersection of $F^{\mathfrak{M}}$ with $\forall^{\mathfrak{M}}$ can be expressed by saying that, when we connect the "vertices" above with "edges", the obtained square

$$\begin{array}{ccc} |\mathfrak{M}| & \xrightarrow{F^{\mathfrak{M}}} & \mathbf{2} \\ {\scriptstyle i}\big\uparrow & = & \big\| \\ \forall^{\mathfrak{M}} & \xrightarrow[F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}]{} & \mathbf{2} \end{array}$$

commutes, where we write $i$ for the inclusion map.

This idea of "drawing a square by connecting vertices" extends to all the other types of maps we use to interpret $\mathcal{L}$. For instance, it extends straightforwardly to a map $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$. That an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ lies in $[\![\varphi(\bar{x})]\!]$ means that a sentence $\varphi$ is true of the tuple $\alpha(\bar{x})$ of individuals. But if we want to know, in particular, which tuples of *existing* individuals satisfy $\varphi$, we can restrict our attention to DoQ-assignments $\alpha : \mathrm{var}(\mathcal{L}) \to \forall^{\mathfrak{M}}$ in place of just any assignments; that is, we take the following commutative square with the inclusion map $i^{\mathrm{var}(\mathcal{L})}$.[20]

$$\begin{array}{ccc} |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} & \xrightarrow{[\![\varphi]\!]} & \mathbf{2} \\ {\scriptstyle i^{\mathrm{var}(\mathcal{L})}}\big\uparrow & = & \big\| \\ (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})} & \xrightarrow[[\![\varphi]\!] \cap \forall^{\mathfrak{M}}]{} & \mathbf{2} \end{array}$$

These cases, namely of $F^{\mathfrak{M}}$ and $[\![\varphi]\!]$, are trivial cases of restriction, in the sense that the restrictions $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ and $[\![\varphi]\!] \cap \forall^{\mathfrak{M}}$ are always available. It is because, from the viewpoint of maps, their

[20] It may be worth noting that, given an inclusion map $i : D \hookrightarrow X$ (for instance, $D = \forall^{\mathfrak{M}}$ and $X = |\mathfrak{M}|$), any set $V$ (for instance, $\mathrm{var}(\mathcal{L})$) induces another inclusion map $i^V : D^V \hookrightarrow X^V$ by the postcomposition $i \circ -$, as in

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & D \\ & {\scriptstyle i \circ \alpha} \searrow \; = & \big\downarrow {\scriptstyle i} \\ & & X \end{array} \qquad\qquad \begin{array}{c} D^V \\ \big\downarrow {\scriptstyle i^V = i \circ -} \\ X^V \end{array}$$

restriction is defined by the precomposition $- \circ i$ with the inclusion map $i$. In general, given any set $X$ and subset $D \subseteq X$, the operation of restricting subsets of $X$ to $D$, that is,

$$(- \cap D)^n : \mathcal{P}(X)^n \to \mathcal{P}(D)^n :: (A_1, \ldots, A_n) \mapsto (A_1 \cap D, \ldots, A_n \cap D),$$

can be written as the precomposition $i^* = - \circ i$ with the inclusion map $i : D \hookrightarrow X$, as in:

$$\begin{array}{ccc} X & \xrightarrow{\langle A_1, \ldots, A_n \rangle} & \mathbf{2}^n \\ {\scriptstyle i}\uparrow & \overset{=}{} \nearrow{\scriptstyle \langle A_1 \cap D, \ldots, A_n \cap D \rangle} & \\ D & & \end{array} \qquad \begin{array}{c} \mathcal{P}(X)^n \\ \downarrow {\scriptstyle i^* = \langle -, \ldots, - \rangle \circ i = (- \cap D)^n} \\ \mathcal{P}(D)^n \end{array}$$

By contrast, with $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$, $[\![t]\!]$, $[\![\otimes]\!]$, the restriction to $\forall^{\mathfrak{M}}$ is not trivial in this sense. Let us take $f^{\mathfrak{M}}$ for instance. For a map $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$, we can always define its restriction to $\forall^{\mathfrak{M}}$ in the usual sense, namely,

$$f^{\mathfrak{M}} \restriction (\forall^{\mathfrak{M}})^n : (\forall^{\mathfrak{M}})^n \to |\mathfrak{M}|.$$

This, however, does not generally serve our purpose of restricting a semantics to the domain of quantification $\forall^{\mathfrak{M}}$, as long as the semantics works recursively, by building up interpretations of compound expressions from interpretations of primitive expressions. To see this, suppose we want to know which existing individuals satisfy a sentence $Ffx$ (for unary $F$ and $f$). As we have seen, once we already know $[\![Ffx]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$, it is the only relevant piece of information. But the point of a recursive semantics is to show how to obtain $[\![Ffx]\!]$ (or $[\![Ffx]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$), by the composition of $F^{\mathfrak{M}} \circ f^{\mathfrak{M}} \circ [\![x]\!]$. Therefore, if

$$f^{\mathfrak{M}} \restriction \forall^{\mathfrak{M}} : \forall^{\mathfrak{M}} \to |\mathfrak{M}|$$

sends an *existing* individual $a \in \forall^{\mathfrak{M}}$ to *non-existing* $b \in |\mathfrak{M}| \setminus \forall^{M}$ (which may well be the case!), we cannot restrict the semantics to $\forall^{\mathfrak{M}}$, because then whether *existing* $a$ satisfies $Ffx$ or not depends on whether *non-existing* $b$ is in $F^{\mathfrak{M}}$ or not.

This is why, for our purpose, $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$ needs—non-trivially—to be restrictable to a map of the type $(\forall^{\mathfrak{M}})^n \to \forall^{\mathfrak{M}}$; that is, for $f^{\mathfrak{M}}$, there needs to be a map ${f^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}$ making

$$\begin{array}{ccc} |\mathfrak{M}|^n & \xrightarrow{\quad f^{\mathfrak{M}} \quad} & |\mathfrak{M}| \\ {\scriptstyle i^n}\uparrow & \circlearrowleft & \uparrow{\scriptstyle i} \\ (\forall^{\mathfrak{M}})^n & \underset{{f^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}}{\dashrightarrow} & \forall^{\mathfrak{M}} \end{array}$$

commute. In other words, since the usual restriction $f^{\mathfrak{M}} \restriction (\forall^{\mathfrak{M}})^n$ is simply $f^{\mathfrak{M}} \circ i_0$, we can say that $f^{\mathfrak{M}}$ is restrictable to $\forall^{\mathfrak{M}}$ (in the sense we need) if $f^{\mathfrak{M}} \restriction (\forall^{\mathfrak{M}})^n$ factors through $\forall^{\mathfrak{M}}$. And similar things can be said for the restriction of $c^{\mathfrak{M}}$ and $[\![t]\!]$ as well. So let us enter, in general:

**Definition 29.** Given sets $X$, $V$, $U$ and a subset $D \subseteq X$, a map

$$f : X^V \to X^U$$

is said to be *restrictable (from X) to D* if there is a map $g : D^V \to D^U$ that makes

$$\begin{array}{ccc} X^V & \xrightarrow{\quad f \quad} & X^U \\ {\scriptstyle i^V}\uparrow & \circlearrowleft & \uparrow{\scriptstyle i^U} \\ D^V & \underset{g}{\dashrightarrow} & D^U \end{array}$$

commute.

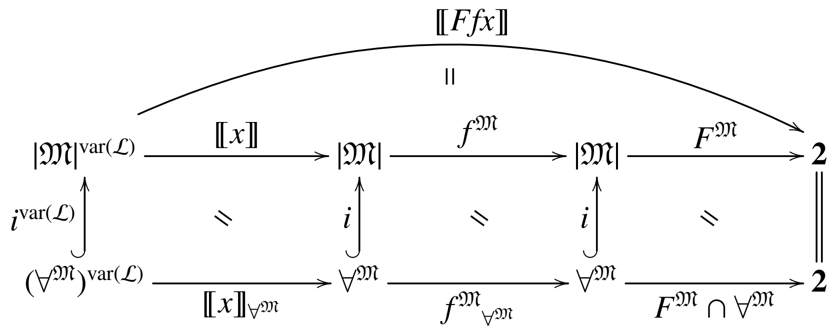
**Fact 17.** If $f : X^V \to X^U$ is restrictable to $D \subseteq X$, then $g$ as in Definition 29 is unique, so we can call it *the* restriction of $f$ to $D$, written $f_D$.

*Proof.* Since $i^U$ is an injection, $i^U \circ g = f \circ i^V = i^U \circ g'$ implies $g = g'$. □

Then, by the autonomy of $\forall^{\mathfrak{M}}$ regarding $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$, or $[\![t]\!]$, we mean the restrictability of $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$, or $[\![t]\!]$ to $\forall^{\mathfrak{M}}$. Note that $\forall^{\mathfrak{M}}$ is trivially autonomous regarding interpretations of terms if $\mathcal{L}$ has no function symbols or constants, due to

**Fact 18.** In a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a given first-order language $\mathcal{L}$, for every variable $x$ of $\mathcal{L}$ the map $[\![x]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$ is restrictable to any $D \subseteq |\mathfrak{M}|$.

Restrictability of $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$, $[\![t]\!]$ in this sense enables us to restrict the semantics to $\forall^{\mathfrak{M}}$ regarding (primitive) atomic sentences. Taking $[\![Ffx]\!]$ as an example again, observe that if $f^{\mathfrak{M}}$ is restrictable to $\forall^{\mathfrak{M}}$—while $F^{\mathfrak{M}}$ and $[\![x]\!]$ are trivially restrictable—it gives us three commutative squares that can be composed as follows.



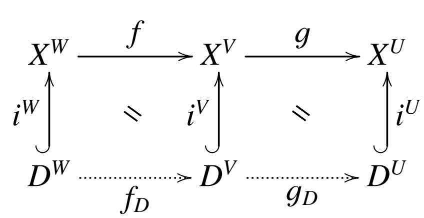
This commutative diagram means that

$$[\![Ffx]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})} = [\![Ffx]\!] \circ i^{\mathrm{var}(\mathcal{L})}$$

can be obtained by composing the restrictions $[\![x]\!]_{\forall^{\mathfrak{M}}}$, ${f^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}$, and $F^{\mathfrak{M}} \cap \forall^{\mathfrak{M}}$ of $[\![x]\!]$, $f^{\mathfrak{M}}$, and $F^{\mathfrak{M}}$ to $\forall^{\mathfrak{M}}$. More intuitively put, when we want to know which existing individual satisfies the sentence $Ffx$, we can compute that piece of information by ignoring non-existing individuals from the outset and building up interpretations without any regard, at any stage of interpretation, to whatever is the case outside the domain of quantification $\forall^{\mathfrak{M}}$. This point can be formally captured in general by

**Fact 19.** Given sets $X$, $W$, $V$, $U$ and a subset $D \subseteq X$, if maps $f : X^W \to X^V$ and $g : X^V \to X^U$ are both restrictable to $D$, then so is $g \circ f : X^W \to X^U$.

*Proof.* $g_D \circ f_D$ gives $(g \circ f)_D$. □



The insight so far is about how we can restrict to $\forall^{\mathfrak{M}}$ the process of building up interpretations of atomic sentences from interpretations of terms and primitive predicates. This insight, expressed by commutative squares, can indeed be extended to the process of building up interpretations of compound sentences from interpretations of atomic ones; in other words, restrictability to $\forall^{M}$ of the interpretation

$$[\![\otimes]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$$

of a sentential operator $\otimes$ can be defined in terms of a commutative square. So, let us introduce the following definition; its point will be clarified shortly by Corollary 3 and Fact 21.

**Definition 30.** Given a set $X$ and a subset $D \subseteq X$ with the inclusion map $i : D \hookrightarrow X$, an operation

$$f : \mathcal{P}(X)^n \to \mathcal{P}(X)^m$$

is said to be *restrictable to $D$* if there is an operation

$$g : \mathcal{P}(D)^n \to \mathcal{P}(D)^m$$

that makes the following diagrams commute:

$$\begin{array}{ccc} \mathcal{P}(X)^n & \xrightarrow{\ f\ } & \mathcal{P}(X)^m \\ {\scriptstyle i^*}\downarrow & /\!\!/ & \downarrow{\scriptstyle i^*} \\ \mathcal{P}(D)^n & \underset{g}{\dashrightarrow} & \mathcal{P}(D)^m \end{array} \qquad \begin{array}{ccc} (A_1, \ldots, A_n) & \xmapsto{\ f\ } & (B_1, \ldots, B_m) \\ {\scriptstyle i^*}\downarrow & /\!\!/ & \downarrow{\scriptstyle i^*} \\ (A_1 \cap D, \ldots, A_n \cap D) & \underset{g}{\dashrightarrow} & (B_1 \cap D, \ldots, B_m \cap D) \end{array}$$

Let us note that, since $i^*$ is surjective, if $f : \mathcal{P}(X)^m \to \mathcal{P}(X)^n$ is restrictable to $D$ then $g$ such that $g \circ i^* = i^* \circ f$ is unique, so we can call it *the* restriction of $f$ to $D$, written $f_D$. Indeed, a more concrete definition of $f_D$ is available.

**Fact 20.** Given a set $X$ and a subset $D \subseteq X$ with the inclusion map $i : D \hookrightarrow X$, a map $f : \mathcal{P}(X)^n \to \mathcal{P}(X)^m$ is restrictable to $D$ for some $f_D$, if and only if $i^* \circ f \circ i_! \circ i^* = i^* \circ f$ in:

$$\begin{array}{ccc} \mathcal{P}(X)^n & \xrightarrow{\ f\ } & \mathcal{P}(X)^m \\ {\scriptstyle i_!}\uparrow \dashv \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathcal{P}(D)^n & \underset{g}{\dashrightarrow} & \mathcal{P}(D)^m \end{array}$$

It follows that the restriction $f_D$ of $f$ to $D$, if it exists, is $i^* \circ f \circ i_!$.

*Proof.* The "if" direction is by definition: $i^* \circ f \circ i_! \circ i^* = i^* \circ f$ means that $i^* \circ f \circ i_!$ serves as $g$ in Definition 30. On the other hand, for the "only if" direction, suppose $f$ is restrictable to $D$, that is, $g \circ i^* = i^* \circ f$ for some $g$. This entails the equalities marked with ! below, while $i^* \circ i_! = 1$ entails the one marked with †:

$$i^* \circ f \circ i_! \circ i^* \overset{!}{=} g \circ i^* \circ i_! \circ i^* \overset{\dagger}{=} g \circ i^* \overset{!}{=} i^* \circ f. \qquad \square$$

Rewriting Fact 20 in terms of intersections with $D$ rather than precompositions with $i$, we have the following description of restrictable maps and their restrictions, in the case of $m = 1$.

**Corollary 3.** Given a set $X$ and a subset $D \subseteq X$, a map $f : \mathcal{P}(X)^n \to \mathcal{P}(X)$ is restrictable to $D$ if and only if

$$f(A_1 \cap D, \ldots, A_n \cap D) \cap D = f(A_1, \ldots, A_n) \cap D$$

for every tuple $(A_1, \ldots, A_n) \in \mathcal{P}(X)^n$. When this is the case, the restriction of $f$ to $D$ is the map $f_D : \mathcal{P}(D)^n \to \mathcal{P}(D)$ such that, for every $(B_1, \ldots, B_n) \in \mathcal{P}(D)^n$,

$$f_D(B_1, \ldots, B_n) = f(B_1, \ldots, B_n) \cap D.$$

Hence, for instance, the restrictability of $[\![\wedge]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^2 \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ to $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ (which we will prove shortly for two-domain interpretations) means the following. Suppose we want to know which DoQ-assignments are in $[\![\varphi \wedge \psi]\!]$; then it is sufficient to figure out what sets $[\![\varphi]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ and $[\![\psi]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ are, and we can ignore anything outside $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$, because we can compute $[\![\varphi \wedge \psi]\!] \cap (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ from those two sets. Or, more intuitively, suppose we want to know what tuples of *existing* individuals satisfy $\varphi \wedge \psi$; then we can ignore any *non-existing* individuals and we only need to know what tuples of *existing* individuals satisfy $\varphi$ and $\psi$.

And the restrictability of $[\![\otimes]\!]$ extends to the restrictability of interpretations of all sentences, due to

**Fact 21.** Given a set $X$ and a subset $D \subseteq X$, if maps $f : \mathcal{P}(X)^n \to \mathcal{P}(X)^m$ and $g : \mathcal{P}(X)^m \to \mathcal{P}(X)^k$ are both restrictable to $D$, then so is $g \circ f : \mathcal{P}(X)^n \to \mathcal{P}(X)^k$.

*Proof.* $g_D \circ f_D$ gives $(g \circ f)_D$. ◻

$$\begin{array}{ccccc}
\mathcal{P}(X)^n & \xrightarrow{f} & \mathcal{P}(X)^m & \xrightarrow{g} & \mathcal{P}(X)^k \\
\downarrow i^* & \not\parallel & \downarrow i^* & \not\parallel & \downarrow i^* \\
\mathcal{P}(D)^n & \xrightarrow[f_D]{} & \mathcal{P}(D)^m & \xrightarrow[g_D]{} & \mathcal{P}(D)^k
\end{array}$$

For instance, consider the interpretation of the sentence $\forall x . \varphi \to \varphi$, that is,

$$[\![\forall x . \varphi \to \varphi]\!] = [\![\to]\!] \circ \langle [\![\forall x]\!], 1 \rangle ([\![\varphi]\!]).$$

Then the restrictability of the operations $[\![\forall x]\!]$ and $[\![\to]\!]$ to $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ implies that the composition

$$\begin{array}{ccccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\langle [\![\forall x]\!], 1 \rangle} & \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^2 & \xrightarrow{[\![\to]\!]} & \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \\
[\![\varphi]\!] & \longmapsto & ([\![\forall x . \varphi]\!], [\![\varphi]\!]) & \longmapsto & [\![\forall x . \varphi \to \varphi]\!]
\end{array}$$

can be restricted to $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$, as in

$$\begin{array}{ccccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\langle [\![\forall x]\!], 1 \rangle} & \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^2 & \xrightarrow{[\![\to]\!]} & \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \\
\downarrow i^* & \not\parallel & \downarrow i^* & \not\parallel & \downarrow i^* \\
\mathcal{P}((\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}) & \xrightarrow[\langle [\![\forall x]\!]_{(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}}, 1_{\mathcal{P}((\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})})} \rangle]{} & \mathcal{P}((\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})})^2 & \xrightarrow[[\![\to]\!]_{(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}}]{} & \mathcal{P}((\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})})
\end{array}$$

It is crucial to observe here that, given this commutative diagram, the DoQ-validity of the scheme $\forall x . \varphi \to \varphi$—that is, the fact that

$$(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})} \subseteq [\![\to]\!] \circ \langle [\![\forall x]\!], 1 \rangle (A)$$

is the case for every $A \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$—is equivalent to the condition that

$$(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})} = [\![\to]\!]_{(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}} \circ \langle [\![\forall x]\!]_{(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}}, 1_{\mathcal{P}((\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})})} \rangle (B)$$

is the case for every $B \subseteq (\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$, which expresses the validity of the same scheme in terms of the classical interpretations of $\to$ and $\forall x$, that is, the restrictions of $[\![\to]\!]$ and $[\![\forall x]\!]$ to $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$. This is how Theorem 3 follows from the autonomy of the domain of quantification $\forall^{\mathfrak{M}}$, in the sense of the restrictability of all the maps $F^{\mathfrak{M}}$, $f^{\mathfrak{M}}$, $c^{\mathfrak{M}}$, $[\![t]\!]^{\mathfrak{M}}$, $[\![\varphi]\!]^{\mathfrak{M}}$, $[\![\otimes]\!]^{\mathfrak{M}}$.

Let us finally enter

**Definition 31.** Given any two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a first-order language $\mathcal{L}$ on a two-domain $\mathcal{L}$ structure $\mathfrak{M} = (|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}}, f^{\mathfrak{M}}, c^{\mathfrak{M}})$, we say that its interpretation

| | |
|---|---|
| $F^{\mathfrak{M}} : |\mathfrak{M}|^n \to \mathbf{2}$ | of an $n$-ary primitive predicate $F$, |
| $f^{\mathfrak{M}} : |\mathfrak{M}|^n \to |\mathfrak{M}|$ | of an $n$-ary function symbol $f$, |
| $c^{\mathfrak{M}} : |\mathfrak{M}|^0 \to |\mathfrak{M}|$ | of a constant symbol $c$, |
| $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$ | of a term $t$, |
| $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ | of a sentence $\varphi$, or |
| $[\![\otimes]\!] : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ | of an $n$-ary sentential operator $\otimes$ of $\mathcal{L}$ |

is *DoQ-restrictable* if it is restrictable to $\forall^{\mathfrak{M}}$ (or, strictly speaking, to $(\forall^{\mathfrak{M}})^{\mathrm{var}(\mathcal{L})}$ in the case of $[\![\otimes]\!]$); in that case we refer to the restriction ${F^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}$, etc., by the *DoQ-restriction* of $F^{\mathfrak{M}}$, etc. Moreover, extending this to the entire interpretation $(\mathfrak{M}, [\![-]\!])$, we say $(\mathfrak{M}, [\![-]\!])$ is DoQ-restrictable if all the maps above are DoQ-restrictable; in that case, by the DoQ-restriction of $(\mathfrak{M}, [\![-]\!])$ we mean the pair of

- the $\mathcal{L}$ structure $(\forall^{\mathfrak{M}}, {F^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}, {f^{\mathfrak{M}}}_{\forall^{\mathfrak{M}}}, c_{\mathfrak{M}\forall^{\mathfrak{M}}})$, and
- the map $[\![-]\!]_{\forall^{\mathfrak{M}}}$.

It is easy to check the definition to see that the DoQ-restriction of any two-domain-type interpretation for $\mathcal{L}$ is a classical-type interpretation for $\mathcal{L}$.

Then we can prove that, if $\mathcal{L}$ is a classical quantified language, that is, if $\mathcal{L}$ has no non-classical sentential operators and no function symbols or constants, then every two-domain interpretation for $\mathcal{L}$ is DoQ-restrictable. We show this more generally by taking any set $D$ such that $\forall^{\mathfrak{M}} \subseteq D \subseteq |\mathfrak{M}|$ for a given two-domain $\mathcal{L}$ structure $\mathfrak{M} = (|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}})$. Then, in the two-domain interpretation on $\mathfrak{M}$, $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, and $[\![\to]\!]$ are restrictable (from $|\mathfrak{M}|$) to $D$, because:

**Fact 22.** Every truth-functional operator $f \circ - : \mathcal{P}(X)^n \to \mathcal{P}(X)$ is restrictable to any $D \subseteq X$, with the same postcomposition $f \circ -$ (defined on $\mathcal{P}(D)^n$) being the restriction $(f \circ -)_D$.

*Proof.* Writing $i : D \hookrightarrow X$ for the inclusion map, we have $(f \circ A) \circ i = f \circ (A \circ i)$ for every $A : X \to \mathbf{2}^n$, which means that the diagram below commutes. □

$$
\begin{array}{ccc}
\mathcal{P}(X)^n & \xrightarrow{\ f \circ - \ } & \mathcal{P}(X) \\
{\scriptstyle i^* = - \circ i}\downarrow & /\!/ & \downarrow{\scriptstyle i^* = - \circ i} \\
\mathcal{P}(D)^n & \xrightarrow[\ f \circ - \ ]{} & \mathcal{P}(D)
\end{array}
$$

Moreover, we have

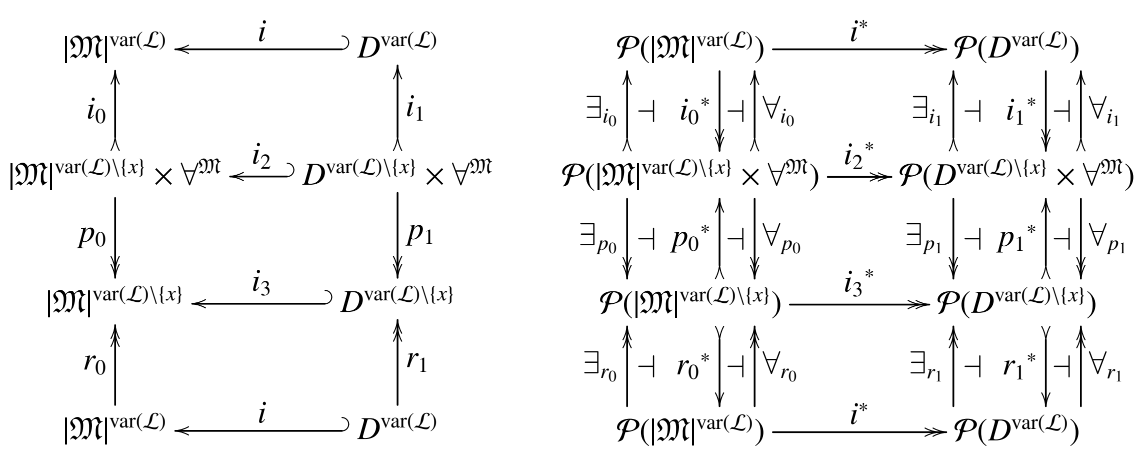
**Fact 23.** Suppose that a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a given first-order language $\mathcal{L}$ interprets $\forall x$ and $\exists x$ respectively with (III.22) and (III.23), that is, with

$$[\![\forall x]\!] = {r_0}^* \circ \forall_{p_0} \circ {i_0}^*, \qquad [\![\exists x]\!] = {r_0}^* \circ \exists_{p_0} \circ {i_0}^*$$

for the inclusion $i_0$, projection $p_0$, and restriction $r_0$ as below. Then $[\![\forall x]\!]$ and $[\![\exists x]\!]$ are restrictable (from $|\mathfrak{M}|$) to any set $D$ such that $\forall^{\mathfrak{M}} \subseteq D \subseteq |\mathfrak{M}|$, with the restrictions

$$[\![\forall x]\!]_D = {r_1}^* \circ \forall_{p_1} \circ {i_1}^*, \qquad [\![\exists x]\!]_D = {r_1}^* \circ \exists_{p_1} \circ {i_1}^*$$

for the similar $i_1$, $p_1$, $r_1$ as below.



*Proof.* The diagram to the left above clearly commutes; therefore the top and bottom squares below commute. Note that the middle square to the left above is a pullback (indeed, all the squares to the left are pullbacks); hence, by Lemma 3, the middle square below commutes as well.

$$\begin{array}{ccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\;i^*\;} & \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \\
{\scriptstyle {i_0}^*}\downarrow & & \downarrow{\scriptstyle {i_1}^*} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}}) & \xrightarrow{\;{i_2}^*\;} & \mathcal{P}(D^{\mathrm{var}(\mathcal{L})\setminus\{x\}} \times \forall^{\mathfrak{M}}) \\
{\scriptstyle \forall_{p_0}}\downarrow & & \downarrow{\scriptstyle \forall_{p_1}} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})\setminus\{x\}}) & \xrightarrow{\;{i_3}^*\;} & \mathcal{P}(D^{\mathrm{var}(\mathcal{L})\setminus\{x\}}) \\
{\scriptstyle {r_0}^*}\downarrow & & \downarrow{\scriptstyle {r_1}^*} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\;i^*\;} & \mathcal{P}(D^{\mathrm{var}(\mathcal{L})})
\end{array}$$

Thus the outermost square commutes, that is, $[\![\forall x]\!]_D \circ i^* = i^* \circ [\![\forall x]\!]$ for $[\![\forall x]\!]_D = {r_1}^* \circ \forall_{p_1} \circ {i_1}^*$. Similarly for $[\![\exists x]\!]_D = {r_1}^* \circ \exists_{p_1} \circ {i_1}^*$. □

These two facts together establish:

**Lemma 4.** *Given any classical quantified language $\mathcal{L}$, every two-domain interpretation for $\mathcal{L}$ is DoQ-restrictable and, moreover, its DoQ-restriction is a classical interpretation for $\mathcal{L}$. Clearly, this defines a (class-sized) surjection from the two-domain semantics for $\mathcal{L}$ to the classical semantics for $\mathcal{L}$.*

This lemma then entails Theorem 3.

*Proof for Theorem 3.* Any sentence of (or inference in) $\mathcal{L}$ is DoQ-valid in a two-domain $\mathcal{L}$ structure $\mathfrak{M}$ if and only if it is valid in the DoQ-restriction $\mathfrak{M}_{\forall^{\mathfrak{M}}}$ of $\mathfrak{M}$. Therefore, by Lemma 4, the sentence (or inference) is DoQ-valid in the two-domain semantics for $\mathcal{L}$ if (because the DoQ-restriction is surjective) and only if it is valid in the classical semantics for $\mathcal{L}$. □

This proof is just an instance of the following conceptual upshot of the autonomy of the domain of quantification. Given any two-domain $\mathcal{L}$ structure $\mathfrak{M} = (|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}})$, suppose $x$ is the free variable of a unary predicate $\varphi$ of $\mathcal{L}$. Then any object $b$ such that $b \notin \forall^{\mathfrak{M}}$ has three possibilities regarding whether $\varphi$ is true of $b$:

- $b \in |\mathfrak{M}|$ and $\mathfrak{M} \vDash_\alpha \varphi$ for an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(x) = b$, and so $\varphi$ is true of $b$;
- $b \in |\mathfrak{M}|$ and $\mathfrak{M} \vDash_\alpha \neg\varphi$ for an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ such that $\alpha(x) = b$, and so $\varphi$ is false of $b$;
- $b \notin |\mathfrak{M}|$, so it simply does not make sense either to say $\varphi$ is true or to say $\varphi$ is false of $b$.

One may find it a philosophically interesting, significant, or difficult question which of these possibilities is the case; nonetheless, the DoQ-restrictability of the two-domain semantics guarantees that these possibilities do not matter, as far as we are concerned with the logic of DoQ-validity. We can simply ignore any $b$ outside $\forall^{\mathfrak{M}}$, or, formally speaking, take the $\mathcal{L}$ structure $\mathfrak{M}_{\forall^{\mathfrak{M}}}$ in place of the two-domain $\mathcal{L}$ structure $\mathfrak{M}$, to see what the logic of DoQ-validity looks like. In this sense, DoQ-restrictability allows us to focus on the logic satisfied by existing individuals, without settling on the philosophical question of whether or not it makes sense to say that a non-existing individual has a certain property, and, if it does, whether it is true or false that that individual has that property.

### III.2.5 Local Determination

Recall our discussions in Subsection III.1.2 that local determination is essential to the expression of $n$-ary properties of ($n$-tuples of) individuals in terms of $n$-ary predicates. In this subsection, we investigate how local determination can be expressed in the operational formulation of semantics we laid out in Subsection III.2.2.

Recall from Definition 7 that a sentence $\varphi$ of a given first-order language $\mathcal{L}$ is locally determined in a classical-type, or, more generally, two-domain-type, satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ if, for every pair of assignments $\alpha, \beta : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$,

$$\text{(III.9)} \qquad \mathfrak{M} \vDash_\alpha \varphi \iff \mathfrak{M} \vDash_\beta \varphi \qquad \text{if } \alpha(x) = \beta(x) \text{ for every free variable } x \text{ in } \varphi.$$

This translates, in the $[\![-]\!]$ notation, as

$$\alpha \in [\![\varphi]\!] \iff \beta \in [\![\varphi]\!] \qquad \text{if } \alpha(x) = \beta(x) \text{ for every free variable } x \text{ in } \varphi,$$

or even as

$$[\![\varphi]\!](\alpha) = [\![\varphi]\!](\beta) \qquad \text{if } \alpha(x) = \beta(x) \text{ for every free variable } x \text{ in } \varphi$$

when we write $[\![\varphi]\!] \subseteq |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ as $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$. This straightforwardly generalizes to cover local determination of terms as well, since the condition

$$t^{\mathfrak{M},\alpha} = t^{\mathfrak{M},\beta} \qquad \text{if } \alpha(x) = \beta(x) \text{ for every variable } x \text{ in } \varphi$$

translates as the following, for $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$.

$$[\![t]\!](\alpha) = [\![t]\!](\beta) \qquad \text{if } \alpha(x) = \beta(x) \text{ for every variable } x \text{ in } \varphi.$$

So, more generally, let us say $f : X^V \to Y$ (typically, $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ and $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$) is *determined by* $U \subseteq V$ (typically, $\{x_1, \ldots, x_n\} \subseteq \mathrm{var}(\mathcal{L})$) if, for every pair $\alpha, \beta : V \to X$,

$$f(\alpha) = f(\beta) \qquad \text{if } \alpha(x) = \beta(x) \text{ for every } x \in U.$$

Then to say that $\varphi$ and $t$ are locally determined in $(\mathfrak{M}, [\![-]\!])$ is to say that $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ and $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$ are determined by the sets of free variables in $\varphi$ and $t$, respectively.

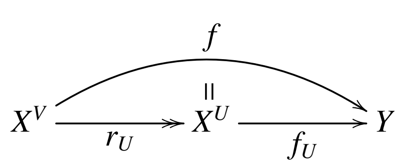
This can be expressed more operationally. Note that, given $U \subseteq V$, maps $\alpha, \beta : V \to X$ have $\alpha(x) = \beta(x)$ for every $x \in U$ if and only if they have the same restriction to $U$, that is, $\alpha{\upharpoonright}U = \beta{\upharpoonright}U$. So, determination of $f : X^V \to Y$ by $U \subseteq V$ amounts to the condition that

$$f(\alpha) = f(\beta) \qquad \text{for every pair } \alpha, \beta : V \to X \text{ such that } \alpha{\upharpoonright}U = \beta{\upharpoonright}U,$$

and hence to the condition that

- There exists $f_U : X^U \to Y$ such that, for every $\alpha : V \to X$, $f(\alpha) = f_U(\alpha{\upharpoonright}U)$.

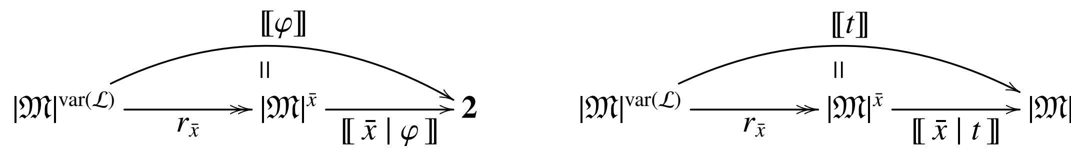
Thus, a map $f : X^V \to Y$ is determined by $U \subseteq V$ if and only if it "factors through" the restriction surjection $r_U = -{\upharpoonright}U : X^V \twoheadrightarrow X^U$, that is, if and only if there is $f_U : X^U \to Y$ such that $f = f_U \circ r_U$, making




commute. Observe that $f_U$, if it exists, is unique because $r_U$ is surjective.

Therefore we have:

**Definition 32.** Given a first-order language $\mathcal{L}$, suppose $\bar{x}$ are the free variables in a sentence $\varphi$ and a term $t$ of $\mathcal{L}$. Then we say that $\varphi$ and $t$, respectively, are locally determined in a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, if their interpretations $[\![\varphi]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ and $[\![t]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$, respectively, factor through $r_{\bar{x}} = - \restriction \bar{x}$, as in:

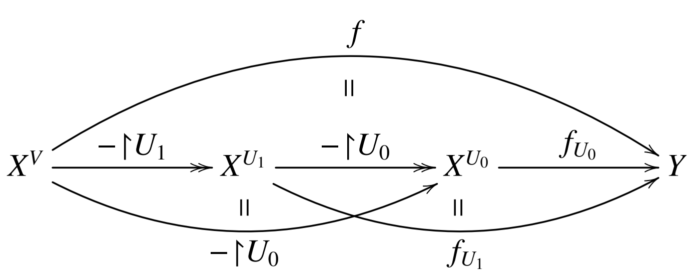



If this is the case, we write $[\![\, \bar{x} \mid \varphi \,]\!] : |\mathfrak{M}|^{\bar{x}} \to \mathbf{2}$ and $[\![\, \bar{x} \mid t \,]\!] : |\mathfrak{M}|^{\bar{x}} \to |\mathfrak{M}|$ for the unique maps such that $[\![\varphi]\!] = [\![\, \bar{x} \mid \varphi \,]\!] \circ (- \restriction \bar{x})$ and $[\![t]\!] = [\![\, \bar{x} \mid t \,]\!] \circ (- \restriction \bar{x})$, as above. Moreover, we say a two-domain-type interpretation for $\mathcal{L}$ is locally determined if every sentence and term are locally determined in it. We also say a class of two-domain-type interpretations for $\mathcal{L}$ is locally determined if every member of that class is locally determined.

Let us make two observations:

**Fact 24.** Supersets determine more. That is, if $f : X^V \to Y$ factors through $- \restriction U_0 : X^V \twoheadrightarrow X^{U_0}$ and $U_0 \subseteq U_1 \subseteq V$, then $f$ factors through $- \restriction U_1 : X^V \twoheadrightarrow X^{U_1}$.

*Proof.* Because $- \restriction U_0 : X^V \twoheadrightarrow X^{U_0}$ is composed of $- \restriction U_1 : X^V \twoheadrightarrow X^{U_1}$ and $- \restriction U_0 : X^{U_1} \twoheadrightarrow X^{U_0}$, $f = f_{U_0} \circ (- \restriction U_0)$ yields $f_{U_1}$ as in the commutative diagram below. □




**Fact 25.** If $f_1, \ldots, f_n : X^V \to Y$ all factor through $- \restriction U : X^V \twoheadrightarrow X^U$, then $\langle f_1, \ldots, f_n \rangle : X^V \to Y^n$ factors through $- \restriction U : X^V \twoheadrightarrow X^U$ as well.

*Proof.* $f_i = (f_i)_U \circ (- \restriction U)$ for all $i \leqslant n$ implies $\langle f_1, \ldots, f_n \rangle = \langle (f_1)_U, \ldots, (f_n)_U \rangle \circ (- \restriction U)$. □

These observations help us to prove:

**Fact 26.** Every atomic sentence and term of a given first-order language $\mathcal{L}$ is locally determined in any two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$.

*Proof.* First, $[\![x]\!]$ is determined by $\{x\}$ for every variable $x$, as in

$$\overbrace{|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \underset{-\restriction\{x\}}{\twoheadrightarrow} |\mathfrak{M}|^{\{x\}} \xrightarrow[{[\![\, x \mid x \,]\!]}]{} |\mathfrak{M}|}^{[\![x]\!]} \;,$$

because we can set $[\![\, x \mid x \,]\!](\alpha\restriction\{x\}) = \alpha(x) = [\![x]\!](\alpha)$.

Now suppose $t_1, \ldots, t_n$ are all locally determined in $(\mathfrak{M}, [\![-]\!])$, and $\bar{x} = (x_1, \ldots, x_m)$ are the variables occurring in at least one of $t_1, \ldots, t_n$. Then, by Fact 24, $[\![t_1]\!], \ldots, [\![t_1]\!]$ are determined by $\bar{x}$, and so is $\langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle = \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle \circ (-\restriction\bar{x})$ by Fact 25. It follows that $f\bar{t}$, for any $n$-ary function symbol $f$, is determined by $\bar{x}$, since $[\![\, \bar{x} \mid f\bar{t} \,]\!]$ can be given by $f^{\mathfrak{M}} \circ \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle$, as in:

$$|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \xrightarrow{-\restriction\bar{x}} |\mathfrak{M}|^{\bar{x}} \xrightarrow{\langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle} |\mathfrak{M}|^n \xrightarrow{f^{\mathfrak{M}}} |\mathfrak{M}|$$

$$[\![f\bar{t}]\!] = f^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle, \qquad \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle = \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle \circ (-\restriction\bar{x}), \qquad [\![\, \bar{x} \mid f\bar{t} \,]\!] = f^{\mathfrak{M}} \circ \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle$$

Thus all terms are locally determined in $(\mathfrak{M}, [\![-]\!])$. Then, by substituting $F$ for $f$ in the argument above, $F\bar{t}$, for any $n$-ary primitive predicate, is determined by its variables $\bar{x}$ as follows. ◻

$$|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \xrightarrow{-\restriction\bar{x}} |\mathfrak{M}|^{\bar{x}} \xrightarrow{\langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle} |\mathfrak{M}|^n \xrightarrow{F^{\mathfrak{M}}} \mathbf{2}$$

$$[\![F\bar{t}]\!] = F^{\mathfrak{M}} \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle, \qquad \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle = \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle \circ (-\restriction\bar{x}), \qquad [\![\, \bar{x} \mid F\bar{t} \,]\!] = F^{\mathfrak{M}} \circ \langle [\![\, \bar{x} \mid t_1 \,]\!], \ldots, [\![\, \bar{x} \mid t_n \,]\!] \rangle$$

Therefore whether all sentences, not just atomic sentences, are locally determined depends on whether the interpretations of sentential operators "preserve" local determination. Let us take $\neg$ as an example. Suppose $\bar{x}$ are the free variables in locally determined $\varphi$, which means that $[\![\varphi]\!]$ factors through $-\restriction\bar{x}$, so that $[\![\, \bar{x} \mid \varphi \,]\!]$ is defined. Then local determination of $\neg\varphi$ means that $[\![\neg\varphi]\!]$ factors through $-\restriction\bar{x}$ as well, and hence $[\![\, \bar{x} \mid \neg\varphi \,]\!]$ is defined. In this way, $[\![\neg]\!]$ preserves local determination by inducing an operation $[\![\, \bar{x} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \neg\varphi \,]\!]$; let us write $[\![\, \bar{x} \mid \neg \,]\!] : \mathcal{P}(|\mathfrak{M}|^{\bar{x}}) \to \mathcal{P}(|\mathfrak{M}|^{\bar{x}})$ for this operation, so that $[\![\, \bar{x} \mid \neg\varphi \,]\!] = [\![\, \bar{x} \mid \neg \,]\!][\![\, \bar{x} \mid \varphi \,]\!]$.

To see in precise terms how $[\![\neg]\!]$ induces $[\![\, \bar{x} \mid \neg \,]\!]$, let us observe that the definition of $[\![\, \bar{x} \mid \varphi]\!]$ and $[\![\, \bar{x} \mid \neg\varphi]\!]$, if they exist, as the unique maps such that

$$
\begin{aligned}
[\![\varphi]\!] &= [\![\, \bar{x} \mid \varphi]\!] \circ (- \restriction \bar{x}), \\
[\![\neg\varphi]\!] &= [\![\, \bar{x} \mid \neg\varphi]\!] \circ (- \restriction \bar{x})
\end{aligned}
$$

entails the equality marked with ! below.

$$
\begin{array}{ccc}
[\![\, \bar{x} \mid \neg\varphi \,]\!] \circ (- \restriction \bar{x}) = [\![\neg\varphi]\!] & = & [\![\neg]\!][\![\varphi]\!] \\
\| & & \| \\
([\![\, \bar{x} \mid \neg \,]\!][\![\, \bar{x} \mid \varphi \,]\!]) \circ (- \restriction \bar{x}) & \underset{!}{=} & [\![\neg]\!]([\![\, \bar{x} \mid \varphi]\!] \circ (- \restriction \bar{x})).
\end{array}
$$

In short, $[\![\, \bar{x} \mid \neg \,]\!]$ and $[\![\neg]\!]$ commute with factorizations. Or, in terms of a commutative diagram, $[\![\, \bar{x} \mid \neg \,]\!]$ is a map that makes the following diagram commute, where we write $r_{\bar{x}}$ for the restriction surjection $- \restriction \bar{x} : \alpha \mapsto \alpha \restriction \bar{x}$.

$$
\begin{array}{ccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;- \circ r_{\bar{x}}\;} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}}) \\
{\scriptstyle [\![\neg]\!]} \downarrow & \circlearrowright & \downarrow {\scriptstyle [\![\, \bar{x} \mid \neg \,]\!]} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;- \circ r_{\bar{x}}\;]{} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}})
\end{array}
$$

$[\![\, \bar{x} \mid \neg \,]\!]$ as above, if it exists, is unique since $- \circ r_{\bar{x}}$ is injective. Thus we can define $[\![\, \bar{x} \mid \neg \,]\!]$ as the unique map, if it exists, that makes the diagram above commute, and we say $[\![\neg]\!]$ preserves local determination if $[\![\, \bar{x} \mid \neg \,]\!]$ exists for every $\bar{x}$.

To extend this to the case of $\wedge$, suppose $\bar{x}$ are the variables that occur freely in either $\varphi$ or $\psi$, and that $\varphi$ and $\psi$ are locally determined. This means by Fact 24 that $[\![\varphi]\!]$ and $[\![\psi]\!]$ factor through $r_{\bar{x}}$, and by Fact 25 that $\langle [\![\varphi]\!], [\![\psi]\!] \rangle$ factors through $r_{\bar{x}}$. Therefore $[\![\wedge]\!]$ preserves local determination if, for each $\bar{x}$, there is a unique $[\![\, \bar{x} \mid \wedge \,]\!]$ that makes the diagram below commute. (Similarly for $\vee$ and $\rightarrow$.)

$$
\begin{array}{ccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^2 & \xleftarrow{\;- \circ r_{\bar{x}}\;} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}})^2 \\
{\scriptstyle [\![\wedge]\!]} \downarrow & \circlearrowright & \downarrow {\scriptstyle [\![\, \bar{x} \mid \wedge \,]\!]} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;- \circ r_{\bar{x}}\;]{} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}})
\end{array}
$$

For extension to the case of $\forall y$, suppose $\bar{x}$ are the free variables in locally determined $\varphi$; hence $[\![\varphi]\!]$ factors through $r_{\bar{x}} = -\restriction\bar{x}$. Then local determination of $\forall y\,.\,\varphi$ means that $[\![\forall y\,.\,\varphi]\!]$ factors through $r_{\bar{x}\setminus\{y\}} = -\restriction(\bar{x}\setminus\{y\})$. Here the factorization must be through $-\restriction(\bar{x}\setminus\{y\})$, and not just through $-\restriction\bar{x}$, because $\forall y$ binds $y$. Therefore $[\![\forall y]\!]$ preserves local determination (relative to the binding of $y$) if, for each $\bar{x}$, there is a unique $[\![\,\bar{x}\mid\forall y\,]\!]$ that makes the diagram below commute. (Similarly for $\exists y$.)

$$
\begin{array}{ccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;-\circ\, r_{\bar{x}}\;} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}}) \\
{\scriptstyle [\![\forall y]\!]}\big\downarrow & & \big\downarrow{\scriptstyle [\![\,\bar{x}\mid\forall y\,]\!]} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\circ\, r_{\bar{x}\setminus\{y\}}\;]{} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}\setminus\{y\}})
\end{array}
$$

Summarizing and generalizing these, we have:

**Definition 33.** Let $\mathcal{L}$ be a first-order language and let $\mathfrak{M}$ be a two-domain $\mathcal{L}$ structure. Then, for any set $\bar{y}$ of variables of $\mathcal{L}$, we say an operation $f : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ *preserves local determination with the binding of* $\bar{y}$ if, for every finite set $\bar{x}$ of variables of $\mathcal{L}$, there is an operation $f_{\bar{x}} : \mathcal{P}(|\mathfrak{M}|^{\bar{x}})^n \to \mathcal{P}(|\mathfrak{M}|^{\bar{x}\setminus\bar{y}})$ such that, for every $B : |\mathfrak{M}|^{\bar{x}} \to \mathbf{2}^n$,

$$f_{\bar{x}}(B)\circ r_{\bar{x}\setminus\bar{y}} = f(B\circ r_{\bar{x}}) : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

(where $r_{\bar{x}} = -\restriction\bar{x}$ and $r_{\bar{x}\setminus\bar{y}} = -\restriction(\bar{x}\setminus\bar{y})$), that is, that makes the following diagram commute.

$$
\begin{array}{ccc}
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;-\circ\, r_{\bar{x}}\;} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}}) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_{\bar{x}}} \\
\mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\circ\, r_{\bar{x}\setminus\bar{y}}\;]{} & \mathcal{P}(|\mathfrak{M}|^{\bar{x}\setminus\bar{y}})
\end{array}
$$

We also say $f : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ preserves local determination *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and if $f$ preserves local determination with the binding of the variables that $\otimes$ binds. Moreover, we say a two-domain-type interpretation for $\mathcal{L}$ preserves local determination if it interprets every sentential operator $\otimes$ of $\mathcal{L}$ with an operation that preserves local determination for $\otimes$, and that a class of two-domain-type interpretation for $\mathcal{L}$ preserves local determination if all of its members do.

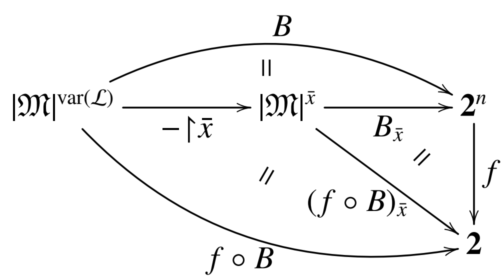
It should be obvious from the discussion so far that we have:

**Fact 27.** A two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a given first-order language $\mathcal{L}$ is locally determined if it preserves local determination.

Let us now prove that the two-domain semantics for *classical* first-order logic preserves local determination.

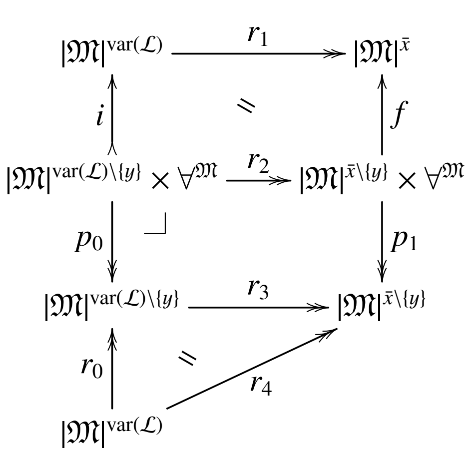
**Fact 28.** Given a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a first-order language $\mathcal{L}$, every truth-functional operator $f \circ - : \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ for $f : \mathbf{2}^n \to \mathbf{2}$ (in particular, $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\to]\!]$ that satisfy (III.18)–(III.21), respectively) preserves local determination with the binding of $\varnothing$ (and hence for $\neg$, $\wedge$, $\vee$, $\to$).

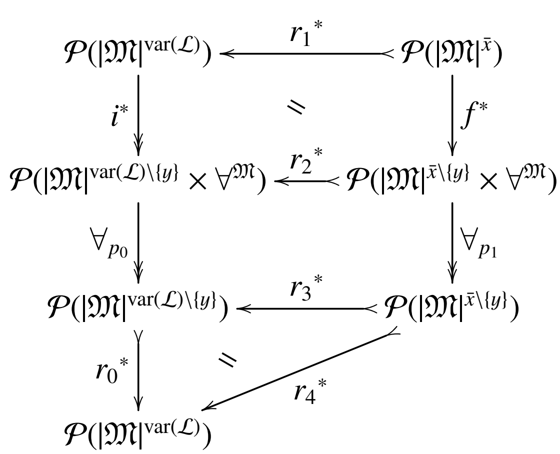
*Proof.* It is immediate with $(f \circ -)_{\bar{x}} : \mathcal{P}(|\mathfrak{M}|^{\bar{x}})^n \to \mathcal{P}(|\mathfrak{M}|^{\bar{x}})$ as follows. $\square$



**Fact 29.** Given a first-order language $\mathcal{L}$ and a two-domain-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, the operations $[\![\forall y]\!]$ and $[\![\exists y]\!]$ that satisfy (III.22) and (III.23), respectively, preserve local determination with the binding of $y$ (and hence for $\forall y$ and $\exists y$).

*Proof.* Fixing any variables $\bar{x}$, consider the maps and operations as in





where $i$ is (what essentially is) the inclusion, $p_i$ are the projections, $r_i$ are the restrictions, and

$$f = \begin{cases} \text{inclusion } (\beta, a) \mapsto \beta \cup \{(y, a)\} & \text{if } y \in x, \\ \text{projection } p_1 : (\beta, a) \mapsto \beta & \text{if } y \notin x \text{ (and hence } \bar{x} \setminus \{y\} = \bar{x}). \end{cases}$$

While $[\![\forall y]\!] = {r_0}^* \circ \forall_{p_0} \circ i^*$, let us write $[\![\, \bar{x} \mid \forall y \,]\!] = \forall_{p_1} \circ f^*$. The commutativity of the top square and the triangle to the left above implies that the top square and the triangle to the right commute as well. Also, the middle square to the right commutes by Lemma 3 because the middle square to the left is a pullback. Therefore ${r_4}^* \circ [\![\, \bar{x} \mid \forall y \,]\!] = [\![\forall y]\!] \circ {r_1}^*$; similarly ${r_4}^* \circ [\![\, \bar{x} \mid \exists y \,]\!] = [\![\exists y]\!] \circ {r_1}^*$. □

It is worth noting that, if $y \notin x$ above, $f = p_1$ and hence

$$[\![\, \bar{x} \mid \forall y \,]\!] = \forall_{p_1} \circ p^* = 1 \qquad\qquad [\![\, \bar{x} \mid \exists y \,]\!] = \exists_{p_1} \circ p^* = 1.$$

**Corollary 4.** If $\mathcal{L}$ is a *classical* first-order language, every two-domain interpretation (and hence the two-domain semantics) for $\mathcal{L}$ preserves locally determined.

# IV.0 KRIPKEAN SEMANTICS FOR QUANTIFIED MODAL LOGIC

# IV.1 KRIPKE SEMANTICS FOR QUANTIFIED MODAL LOGIC

## IV.1.1 Kripke's Ontology and Semantics

In this subsection, we review the semantics that Kripke [19] proposed for quantified modal logic.

We should first define a language of quantified modal logic; but the generality of Definitions 13 (on p. 63) and 18 (on p. 18) makes it simple. Let us say a language is *modal* if it has unary sentential operators $\Box$ and $\Diamond$; then we can simply say that a *quantified modal language* is just a quantified language that is modal (and hence is not purely first-order). For the rest of the chapter, we will only deal with quantified languages—that is, we will not deal with function or constant symbols. Also, we will assume that $\Box$ and $\Diamond$ are the only non-classical operators. Let us sum these up as follows:

**Definition 34.** A *quantified modal language* is a language given by the following:

- any number (at least one) of primitive predicates (perhaps 0-ary);
- infinitely many individual variables, but no function or constant symbols; and
- sentential operators that consist of the first-order operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\forall x$ and $\exists x$ for all individual variables $x$, and the modal operators, $\Box$ and $\Diamond$, but no other.

Given a quantified modal language $\mathcal{L}$, by an *atomic sentence* of $\mathcal{L}$ we mean a result of combining (in a manner allowed by the grammar of $\mathcal{L}$) an $n$-ary primitive predicate of $\mathcal{L}$ with $n$ individual variables of $\mathcal{L}$. And, from the atomic sentences of $\mathcal{L}$, we define the set of sentences of $\mathcal{L}$, written sent($\mathcal{L}$), recursively with the sentential operators of $\mathcal{L}$.

Recall that a Kripke frame $\mathfrak{F}$ for *propositional* modal logic consists of a set $W$ of worlds and an accessibility relation $R$ on $W$, that a Kripke model $\mathfrak{M}$ over $\mathfrak{F}$ interprets each atomic sentence $p$ with its proposition $p^{\mathfrak{M}} \subseteq W$, and that the truth of a sentence is relativized not just to a model $\mathfrak{M}$ but also to a world $w \in W$, so that the semantic relation is of the form

$$\mathfrak{M}, w \models \varphi,$$

and satisfies the following truth conditions:

$$\mathfrak{M}, w \models p \iff w \in p^{\mathfrak{M}} \qquad \text{for atomic } p, \tag{IV.1}$$

$$\mathfrak{M}, w \models \neg\varphi \iff \mathfrak{M}, w \not\models \varphi, \tag{IV.2}$$

$$\mathfrak{M}, w \models \varphi \wedge \psi \iff \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi, \tag{IV.3}$$

$$\mathfrak{M}, w \models \varphi \vee \psi \iff \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi, \tag{IV.4}$$

$$\mathfrak{M}, w \models \varphi \to \psi \iff \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi, \tag{IV.5}$$

$$\mathfrak{M}, w \models \Box\varphi \iff \mathfrak{M}, u \models \varphi \text{ for every } u \text{ such that } Rwu, \tag{IV.6}$$

$$\mathfrak{M}, w \models \Diamond\varphi \iff \mathfrak{M}, u \models \varphi \text{ for some } u \text{ such that } Rwu. \tag{IV.7}$$

To interpret variables and quantification on top of this framework, Kripke equips a frame with a domain of *possible individuals*, so that the truth of sentences is further relativized to an assignment of individuals to variables. So, using the notation for assignments we reviewed in Subsection III.1.1, Kripke's idea can be expressed as follows. Let $\alpha$ be a map from $\mathrm{var}(\mathcal{L})$ to a domain of individuals (which we will review shortly in more detail);[1] then the semantic relation is of the form

$$\mathfrak{M}, w \models_{\alpha} \varphi,$$

[1] It is worth noting that, in [19], Kripke does not take assignments as maps from all of $\mathrm{var}(\mathcal{L})$. Even though he does not give a full definition explicitly, he considers assignments of individuals to a finite set of variables containing all those occurring freely in the sentence to be evaluated.

to mean that, in the model $\mathfrak{M}$, the sentence $\varphi$ is true, at the world $w$, of individuals $\alpha(\bar{x})$ in place of the free variables $\bar{x}$ in $\varphi$. With an assignment added to the semantic relation, Kripke lets (IV.2)–(IV.7) simply carry over:

$$\mathfrak{M}, w \models_\alpha \neg\varphi \iff \mathfrak{M}, w \not\models_\alpha \varphi, \tag{IV.8}$$

$$\mathfrak{M}, w \models_\alpha \varphi \wedge \psi \iff \mathfrak{M}, w \models_\alpha \varphi \text{ and } \mathfrak{M}, w \models_\alpha \psi, \tag{IV.9}$$

$$\mathfrak{M}, w \models_\alpha \varphi \vee \psi \iff \mathfrak{M}, w \models_\alpha \varphi \text{ or } \mathfrak{M}, w \models_\alpha \psi, \tag{IV.10}$$

$$\mathfrak{M}, w \models_\alpha \varphi \to \psi \iff \mathfrak{M}, w \not\models_\alpha \varphi \text{ or } \mathfrak{M}, w \models_\alpha \psi, \tag{IV.11}$$

$$\mathfrak{M}, w \models_\alpha \Box\varphi \iff \mathfrak{M}, u \models_\alpha \varphi \text{ for every } u \text{ such that } Rwu, \tag{IV.12}$$

$$\mathfrak{M}, w \models_\alpha \Diamond\varphi \iff \mathfrak{M}, u \models_\alpha \varphi \text{ for some } u \text{ such that } Rwu. \tag{IV.13}$$

With the satisfaction relation relativized to assignments, Kripke gives truth conditions to the quantifiers in the following manner. First, he equips his models with a map $D_{\_}$ that assigns a set $D_w \neq \varnothing$ to each world $w \in W$. Then, saying "Intuitively $D_w$ is the set of all individuals existing in $w$",[2] Kripke adopts

$$\mathfrak{M}, w \models_\alpha \forall x . \varphi \iff \mathfrak{M}, w \models_{[a/x]\alpha} \varphi \text{ for every } a \in D_w, \tag{IV.14}$$

$$\mathfrak{M}, w \models_\alpha \exists x . \varphi \iff \mathfrak{M}, w \models_{[a/x]\alpha} \varphi \text{ for some } a \in D_w. \tag{IV.15}$$

As he explains, "the restriction $a \in D_w$ means that, in $w$, we quantify only over the objects actually existing in $w$".[3] So, recalling the discussion in Subsection III.2.1, we should be justified in calling $D_w$ the *domain of quantification for* $w$. Then Kripke takes the union $D$ of all $D_w$, that is,

$$D = \bigcup_{w \in W} D_w,$$

and permits every map $\alpha : \mathrm{var}(\mathcal{L}) \to D$ to this set $D$ to serve as an assignment as in $\mathfrak{M}, w \models_\alpha \varphi$, for any $w$ and $\varphi$. In this sense, we can call $D$ the domain of individuals (of a given model $\mathfrak{M}$) in our terminology. Or we may call it the *domain of possible individuals*, since it is the set of individuals that exist in some possible world or other.

[2][19], 65; I have changed Kripke's notations $\psi(\mathbf{H})$ and $\mathbf{H}$ into $D_w$ and $w$, respectively.
[3][19], 67; again, I have changed Kripke's $\psi(\mathbf{H})$ and $\mathbf{H}$ into $D_w$ and $w$.

Let us take the example of Sherlock Holmes again, as we did in Subsection III.2.1, to recall the two notions of domains and their connection to quantification. Holmes may exist in world $w$ but not in $u$. He may be a logician in $w$; in notation, $\mathfrak{M}, w \vDash_{[a/x]\alpha} \varphi$ for any assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$, where we write $a$ for Holmes and $\varphi$ for "$x$ is a logician". Then this implies that a logician exists in $w$—that is, $\mathfrak{M}, w \vDash_{\alpha} \exists x\,.\,\varphi$, because Holmes exists in $w$, that is, $a \in D_w$. In contrast, he may be a logician in $u$ as well; in notation, $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$. This, nonetheless, fails to imply that a logician exists in $u$—that is, $\mathfrak{M}, u \vDash_{\alpha} \exists x\,.\,\varphi$—because Holmes does not exist in $u$—that is, $a \notin D_u$.

Note that, in this example, $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$ makes sense—that is, it makes sense to ask whether Holmes is a logician or not in $u$—even though $a \notin D_u$—that is, Holmes does not exist in $u$. The technical reason why it *does* make sense in the setting laid out above is that, since $a \in D_w \subseteq D$, the map $[a/x]\alpha : \mathrm{var}(\mathcal{L}) \to D$ is an assignment. To put it a little more intuitively, Holmes, who exists in some possible world (namely $w$), is in the domain of possible individuals, and, in so far as he is in that domain, it makes sense to ask whether or not he has some property in whatever world $u$.

We should further note that, in Kripke's semantics, it is semantically significant that $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$ makes sense. The reason lies in Kripke's truth conditions for the modal operators, namely (IV.12) and (IV.13). To see this, let us assume in the example above that $Rwu$ and that Holmes is necessarily a logician in $w$—that is, $\mathfrak{M}, w \vDash_{[a/x]\alpha} \Box\varphi$. Then (IV.12) implies that $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$, that is, that Holmes is a logician in $u$ whether he exists in $u$ or not. In this way, (IV.12) and (IV.13) provide a semantic reason why we *should* deem $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$ to make sense even when $a \notin D_u$. And this semantic role that $\mathfrak{M}, u \vDash_{[a/x]\alpha} \varphi$ plays has significant import for the logic of Kripke's semantics; we will come back to this shortly, after fully defining Kripke semantics.

To complete our review of Kripke semantics, let us discuss the last of Kripke's truth conditions, namely, the one for atomic sentences. We should emphasize that the question of what truth condition atomic sentences should have is really a question of what type of interpretation models should have. To illustrate this point, take as an example the truth conditions for atomic sentences (with no function or constant symbols) in the classical semantics, namely,

(III.1) $$\mathfrak{M} \vDash_{\alpha} F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

The left-hand side is simply required so that (III.1) is, in the first place, a truth condition for atomic sentences; hence the conceptually significant part of this condition is that we let $F^{\mathfrak{M}}$ be a subset of $|\mathfrak{M}|^n$; in other words, that an $\mathcal{L}$ structure $\mathfrak{M}$ should interpret $F$ with a subset of $|\mathfrak{M}|^n$, the $n$-fold product of the domain of individuals. In a similar vein, in Kripke semantics, we need to discuss with what type of sets we should interpret primitive predicates.

The type of interpretation used in classical semantics, namely $F^{\mathfrak{M}} \subseteq D^n$, where $D$ is the domain of possible individuals, does not work in the Kripke setting. For it would give

$$\mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F,$$

which would then entail

$$\mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \iff \mathfrak{M}, u \vDash_\alpha F\bar{x}$$

for any pair of worlds $w$, $u$. For example, if we interpret the predicate "$x$ is a logician" with the property of being-a-logician that possible individuals may or may not have, but which is independent of worlds, then Sherlock Holmes is or is not a logician independently of worlds, so he is a logician either at all worlds or at no worlds, and then (IV.12) implies that he is either necessarily a logician or necessarily not a logician.

This is why we should relativize $F^{\mathfrak{M}}$ to worlds. In the example above, a world $w$ should have its own set of logicians, or the *extension* of "$x$ is a logician", and another world $u$ may well have a different extension; then Holmes may be a logician at $w$ but may not at $u$. In other words, we should use the property of being-a-logician-at-$w$ rather than being-a-logician *simpliciter*. So, for each $w \in W$, let us write $\overrightarrow{F^{\mathfrak{M}}}(w)$ for the extension of $F$ (in $\mathfrak{M}$) at $w$—which stands for the property of being-a-logician-at-$w$, in the example—so that $\overrightarrow{F^{\mathfrak{M}}}(w) \subseteq D^n$ and

$$\text{(IV.16)} \qquad \mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in \overrightarrow{F^{\mathfrak{M}}}(w) \qquad \text{for an } n\text{-ary primitive predicate } F.$$

Thus we can interpret $F$ with the family of $\overrightarrow{F^{\mathfrak{M}}}(w)$ for all $w$.

It is helpful to note that the map $\overrightarrow{F^{\mathfrak{M}}} : W \to \mathcal{P}(D^n)$, or $\overrightarrow{F^{\mathfrak{M}}} : W \to (D^n \to \mathbf{2})$, is mathematically equivalent to a set $F^{\mathfrak{M}} \subseteq W \times D^n$, or $F^{\mathfrak{M}} : W \times D^n \to \mathbf{2}$, via

$$(w, \bar{a}) \in F^{\mathfrak{M}} \iff \bar{a} \in \overrightarrow{F^{\mathfrak{M}}}(w).$$

So, plugging this biconditional in to (IV.16), the truth condition for atomic sentences is

$$\text{(IV.17)} \qquad \mathfrak{M}, w \vDash_\alpha F\bar{x} \iff (w, \alpha(\bar{x})) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

Whereas $F^{\mathfrak{M}}$ subsumes the case of $n = 0$ easily, by setting $p^{\mathfrak{M}} \subseteq W = W \times D^0$ for a propositional variable $p$,[4] $\overrightarrow{F^{\mathfrak{M}}}(w)$ is easier to grasp conceptually, or at least easier to express in English—in the example, $F^{\mathfrak{M}}$ is the set of world-individual pairs such that the individual is a logician at the world. Although we use $F^{\mathfrak{M}}$ in our official definition,[5] we will use both $F^{\mathfrak{M}}$ and $\overrightarrow{F^{\mathfrak{M}}}$ in later discussions.

Now that we have completed the review of Kripke's semantic ideas, we can give

**Definition 35.** A *Kripke frame with domains* is a tuple $\mathfrak{F} = (W, R, D_{-})$ such that

- $(W, R)$ is a Kripke frame, namely, $W$ is a set and $R \subseteq W \times W$; we call $W$ the *set of worlds* of $\mathfrak{F}$, and $R$ the *accessibility relation* of $\mathfrak{F}$.
- $D_{-}$ is a map that assigns to each $w \in W$ a set $D_w \neq \varnothing$, called the *domain of quantification for* $w$; we also call $D = \bigcup_{w \in W} D_w$ the *domain of possible individuals* of $\mathfrak{F}$.

**Definition 36.** Let $\mathcal{L}$ be a language $\mathcal{L}$ of quantified modal logic. A *Kripke model* for $\mathcal{L}$ is a tuple $\mathfrak{M} = (W, R, D_{-}, F^{\mathfrak{M}})$ such that

- $(W, R, D_{-})$ is a Kripke frame with domains as in Definition 35; by the set of worlds, accessibility relation, and domain of possible individuals of $\mathfrak{M}$, we mean those of the frame $(W, R, D_{-})$;
- $\mathfrak{M}$ is equipped with $F^{\mathfrak{M}} \subseteq D^n \times W$ for each $n$-ary primitive predicate of $\mathcal{L}$.

We say $\mathfrak{M} = (W, R, D_{-}, F^{\mathfrak{M}})$ is a Kripke model for $\mathcal{L}$ *over* the frame $(W, R, D_{-})$.

[4] Although $D^0 = \{*\}$ may suggest $W \times D^0 = \{ (w, *) \mid w \in W \}$, we take $W \times D^0 = W$ instead, as suggested by the sequence

$$\begin{aligned} &\vdots \\ (w, a_1, \ldots, a_n) &\in W \times D^n, \\ &\vdots \\ (w, a_1, a_2) &\in W \times D^2, \\ (w, a_1) &\in W \times D^1, \\ w &\in W \times D^0. \end{aligned}$$

On the other hand, in terms of $\overrightarrow{F^{\mathfrak{M}}}$, the case of $n = 0$ is treated as $\overrightarrow{p^{\mathfrak{M}}} : W \to \mathcal{P}(D^0)$, which agrees with $p^{\mathfrak{M}} \subseteq W$ since $D^0 = \{*\}$ and implies $\mathcal{P}(D^0) = \mathcal{P}(\{*\}) = \mathbf{2}$.

[5] Kripke [19] uses $\overrightarrow{F^{\mathfrak{M}}}$ instead of $F^{\mathfrak{M}}$ as primitive.

**Definition 37.** Given a quantified modal language $\mathcal{L}$, a *Kripke-type satisfaction relation for* $\mathcal{L}$ is a pair $(\mathfrak{M}, \vDash)$ of a Kripke model $\mathfrak{M}$ for $\mathcal{L}$ and any relation $(\mathfrak{M}, - \vDash_{-} -) \subseteq W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$, as in $\mathfrak{M}, w \vDash_\alpha \varphi$, where $W$ and $D$ are the set of worlds and domain of possible individuals of $\mathfrak{M}$. We say a Kripke-type satisfaction relation for $\mathcal{L}$ is *on* $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$.

**Definition 38.** Given a quantified modal language $\mathcal{L}$, for each Kripke-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ with $W$ and $D$ the set of worlds and domain of possible individuals of $\mathfrak{M}$, we say

- a sentence $\varphi$ of $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash \varphi$, meaning that $\mathfrak{M}, w \vDash_\alpha \varphi$ for every $w \in W$ and assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$; and
- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash \varphi$.

Given a class of Kripke-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is *valid in* that class if it is valid in every member of that class.

**Definition 39.** A Kripke-type satisfaction relation for a quantified modal language $\mathcal{L}$ is called a *Kripke satisfaction relation* for $\mathcal{L}$ if it satisfies (IV.8)–(IV.17). By *Kripke semantics* for $\mathcal{L}$, we mean the class of all Kripke satisfaction relations for $\mathcal{L}$.

Due to our assumption that $\Box$ and $\Diamond$ are the only non-classical operators of a quantified modal language $\mathcal{L}$, we have a one-to-one correspondence between Kripke models and Kripke satisfaction relations on them, as follows.

**Fact 30.** Given a quantified modal language $\mathcal{L}$ the only non-classical operators of which are $\Box$ and $\Diamond$, every Kripke model for $\mathcal{L}$ has a unique Kripke satisfaction relation on it.

Thus Kripke models correspond one-to-one to Kripke satisfaction relations on them, and hence the notion of validity makes sense not only regarding the latter but also regarding the former.

**Definition 40.** Given a quantified modal language $\mathcal{L}$, we say a sentence of $\mathcal{L}$ or inference in $\mathcal{L}$ is Kripke-valid in a Kripke model $\mathfrak{M}$ if it is valid in the Kripke satisfaction relation on $\mathfrak{M}$; we say it is Kripke-valid (with respect to $\mathcal{L}$) in a Kripke frame $\mathfrak{F}$ with domains if it is Kripke-valid in every Kripke model for $\mathcal{L}$ over $\mathfrak{F}$; we say it is Kripke-valid (with respect to $\mathcal{L}$) in a given class of Kripke frames with domains if it is Kripke-valid (with respect to $\mathcal{L}$) in every member of that class. When a sentence or an inference is Kripke-valid (with respect to $\mathcal{L}$) in a Kripke model, a Kripke frame with domains, or a class thereof, we also say that the latter *Kripke-validates* the former.

We extend the "forgotten trio" of Subsection III.1.2 straightforwardly to the Kripke setting, as follows.

**Definition 41.** Given any Kripke-type satisfaction relation $(\mathfrak{M}, \vDash)$ for a quantified modal language $\mathcal{L}$, with $W$ and $D$ the set of worlds and domain of possible individuals of $\mathfrak{M}$, we say $(\mathfrak{M}, \vDash)$ is

- locally determined if, for every sentence $\varphi$ of $\mathcal{L}$ and pair of assignments $\alpha, \beta : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha(x) = \beta(x)$ for every free variable $x$ of $\varphi$,

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff \mathfrak{M}, w \vDash_\beta \varphi; \tag{IV.18}$$

- SoS if, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$, sentence $\varphi$ and variables $x$, $y$ such that $y$ is free for $x$ in $\varphi$,

$$\mathfrak{M}, w \vDash_\alpha [y/x]\varphi \iff \mathfrak{M}, w \vDash_{[\alpha(y)/x]\alpha} \varphi; \tag{IV.19}$$

- AC if, for every assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$ and pair of sentences $\varphi$, $\psi$ such that $\varphi \propto \psi$,

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff \mathfrak{M}, w \vDash_\alpha \psi. \tag{IV.20}$$

These three properties hold of Kripke semantics, again due to our assumption that $\Box$ and $\Diamond$ are the only non-classical operators of a quantified modal language $\mathcal{L}$.

**Fact 31.** Given a quantified modal language $\mathcal{L}$ the only non-classical operators of which are $\Box$ and $\Diamond$, every Kripke satisfaction relation for $\mathcal{L}$ is locally determined, SoS, and AC.

*Proof.* The proof is similar to those for Facts 4, 6, and 8, by induction on the construction of sentences (we need to use the assumption that the only non-classical operators of $\mathcal{L}$ are $\Box$ and $\Diamond$). □

Let us close the section by noting that the logic **K**, which is given by adding the axiom

K $$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

and the rule

N $$\frac{\vdash \varphi}{\vdash \Box\varphi}$$

to classical propositional logic, is sound with respect to the Kripke semantics, in the sense that K and N as well as the axioms and rules of classical propositional logic are valid in the semantics.

**Fact 32.** **K** is sound with respect to the Kripke semantics.

### IV.1.2 Separation of Modal and Classical

To illustrate how his semantics as we reviewed in Subsection IV.1.1 works, Kripke takes as examples two sentence schemes,

$$\forall x\, \Box\varphi \to \Box\forall x\,.\,\varphi,$$
$$\Box\forall x\,.\,\varphi \to \forall x\, \Box\varphi,$$

which are called the *Barcan formula* and *converse Barcan formula*, respectively. Following Kripke, let us say, given any Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with domains (or any Kripke model $\mathfrak{M}$),

- $\mathfrak{F}$ has a *decreasing domain* if $D_u \subseteq D_w$ for each $u, w \in W$ such that $Rwu$, and that
- $\mathfrak{F}$ has an *increasing domain* if $D_w \subseteq D_u$ for each $u, w \in W$ such that $Rwu$.

Then it is easy to see that a Kripke frame with domains Kripke-validates the Barcan formula if and only if it has a decreasing domain. (For the sake of brevity, let us omit "Kripke" in "Kripke-valid" for the rest of this subsection.) It is also easy to see that a frame validates the converse Barcan formula if and only if it has an increasing domain. For the purpose of our discussion, however, we need a more refined analysis of how the converse Barcan formula may or may not be valid.

Let us first review how a frame can fail to validate the converse Barcan formula. Kripke uses the following counterexample.[6] Let $F$ be a unary primitive predicate, and let $\mathfrak{F} = (W, R, D_-)$ and $\mathfrak{M} = (W, R, D_-, F^{\mathfrak{M}})$ be given by

$$W = \{w, u\}, \qquad R = \{(w, w), (w, u)\},$$
$$D_w = \{a, b\}, \text{ where } a \neq b, \qquad D_u = \{a\},$$
$$\overrightarrow{F^{\mathfrak{M}}}(w) = \{a, b\}, \qquad \overrightarrow{F^{\mathfrak{M}}}(u) = \{a\}.$$

Note that $\mathfrak{M}$ does not have an increasing domain. Fixing any assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$, consider whether $\mathfrak{M}, w \vDash_\alpha \Box\forall x . Fx$ or not; it holds because of the following chain of equivalences (ignore the underline on the second line for the moment):

$$\begin{aligned}
\mathfrak{M}, w \vDash_\alpha \Box\forall x . Fx &\overset{\text{(IV.12)}}{\iff} \mathfrak{M}, w \vDash_\alpha \forall x . Fx \text{ and } \mathfrak{M}, u \vDash_\alpha \forall x . Fx \\
&\overset{\text{(IV.14)}}{\iff} \underline{\mathfrak{M}, w \vDash_{[a/x]\alpha} Fx, \mathfrak{M}, w \vDash_{[b/x]\alpha} Fx, \text{ and } \mathfrak{M}, u \vDash_{[a/x]\alpha} Fx} \\
&\overset{\text{(IV.16)}}{\iff} [a/x]\alpha(x) \in \overrightarrow{F^{\mathfrak{M}}}(w), [b/x]\alpha(x) \in \overrightarrow{F^{\mathfrak{M}}}(w), \text{ and } [a/x]\alpha(x) \in \overrightarrow{F^{\mathfrak{M}}}(u) \\
&\iff a, b \in \overrightarrow{F^{\mathfrak{M}}}(w) \text{ and } a \in \overrightarrow{F^{\mathfrak{M}}}(u).
\end{aligned}$$

On the other hand, $\mathfrak{M}, w \vDash_\alpha \forall x \Box Fx$ is not the case, because

$$\begin{aligned}
\mathfrak{M}, w \vDash_\alpha \forall x \Box Fx &\overset{\text{(IV.14)}}{\iff} \mathfrak{M}, w \vDash_{[a/x]\alpha} \Box Fx \text{ and } \mathfrak{M}, w \vDash_{[b/x]\alpha} \Box Fx \\
&\overset{\text{(IV.12)}}{\iff} \underline{\mathfrak{M}, w \vDash_{[a/x]\alpha} Fx, \mathfrak{M}, u \vDash_{[a/x]\alpha} Fx, \mathfrak{M}, w \vDash_{[b/x]\alpha} Fx}, \text{ and } \mathfrak{M}, u \vDash_{[b/x]\alpha} Fx \\
&\overset{\text{(IV.16)}}{\iff} a, b \in \overrightarrow{F^{\mathfrak{M}}}(w) \text{ and } a, b \in \overrightarrow{F^{\mathfrak{M}}}(u),
\end{aligned}$$

where in fact $b \notin \overrightarrow{F^{\mathfrak{M}}}(u)$. Thus $\mathfrak{M}, w \nvDash_\alpha \Box\forall x . Fx \to \forall x \Box Fx$; therefore $\mathfrak{M}$ fails to, and hence $\mathfrak{F}$ also fails to, validate the instance of the converse Barcan formula, $\Box\forall x . Fx \to \forall x \Box Fx$.

Comparing the two chains of equivalences above, and noting that the two underlined parts are equivalent, we can see that the difference between the right-hand sides for $\mathfrak{M}, w \vDash_\alpha \Box\forall x . Fx$ and $\mathfrak{M}, w \vDash_\alpha \forall x \Box Fx$ is that

$$\mathfrak{M}, u \vDash_{[b/x]\alpha} Fx \tag{IV.21}$$

[6] [19], 67f.

occurs in the latter but not in the former. So it is easy to see how $\mathfrak{M}$ would validate the converse Barcan formula $\Box\forall x\,.\,Fx \to \forall x\,\Box Fx$ if it had an increasing domain: If $D_u$ were $\{a, b\}$ (so $\mathfrak{M}$ had an increasing domain), the application of (IV.14) in the first chain of equivalences would also yield (IV.21) to the right.

More importantly, what we should emphasize about the key clause (IV.21) is that $b \notin D_u$, that is, that $b$ does not exist in $u$. In Kripke semantics, it makes sense to say that $b$ has the property $F$-at-$u$ even when $b$ does not exist in $u$; and, as we mentioned on p. 123, its making sense has semantic consequences. The invalidity of the converse Barcan formula above is one of such consequences. Indeed, against the near-orthodoxy of modal logicians that whether a frame validates the converse Barcan formula or not is a matter of whether it has an increasing domain or not, I propose a refinement—namely, whether a frame validates the converse Barcan formula or not is a matter of whether non-existent beings have semantic significance or not. An argument for this refinement will be laid out in Subsection IV.2.4.

The invalidity of the converse Barcan formula is just an instance of a more general invalidity in Kripke semantics. Consider the following derivation of the converse Barcan formula.

$$\dfrac{\dfrac{\vdash \forall x\,.\,\varphi \to \varphi \quad \text{(i)}}{\vdash \Box(\forall x\,.\,\varphi \to \varphi)}\,\text{(N)} \qquad \vdash \Box(\forall x\,.\,\varphi \to \varphi) \to (\Box\forall x\,.\,\varphi \to \Box\varphi) \quad \text{(K)}}{\dfrac{\vdash \Box\forall x\,.\,\varphi \to \Box\varphi}{\vdash \Box\forall x\,.\,\varphi \to \forall x\,\Box\varphi}\,\text{(iii)}}\,\text{(ii)}$$

The sentence marked with (i) is an axiom of classical quantified logic; the next step, marked with (N), is an application of rule N. The sentence (K) is an instance of axiom K. (ii) is *modus ponens*, the rule of classical propositional logic. (iii) is a rule of classical quantified logic, that allows the following inference:

$$\dfrac{\vdash \varphi \to \psi}{\vdash \varphi \to \forall x\,.\,\psi}\,(x \text{ does not occur freely in } \varphi)$$

In this way, even though its conclusion, the converse Barcan formula, is invalid in Kripke semantics, the derivation above is justified by the axioms and rules of **K** and classical quantified logic.

Then recall Fact 32—**K** is sound with respect to Kripke semantics. So, the upshot is that classical quantified logic is not sound with respect to Kripke semantics.

Looking further into what of classical quantified logic is invalid, it is easy to see that the rule above (the one used for (iii)) is valid. The invalidity lies in the axiom $\forall x . \varphi \to \varphi$ (which is why we cannot apply N after (i) above and the derivation does not go). And the reason should be obvious, once we observe that

$$\mathfrak{M}_w = (D, D_w, \overrightarrow{F^{\mathfrak{M}}}(w))$$

is a two-domain $\mathcal{L}$ structure (Definition 19) and, moreover, that

$$(\mathfrak{M}_w, (\mathfrak{M}, w \vDash_{\_} -))$$

is a two-domain satisfaction relation (Definition 22 on p. 81) for $\mathcal{L}$, due to Fact 31. As we saw in Subsection III.2.1, $\forall x . \varphi \to \varphi$ is not AA-valid (Definition 24 on p. 84) in the two-domain semantics, which is exactly why it is not valid in Kripke semantics either.

Given the unsoundness of classical quantified logic with respect to his semantics, Kripke considers two options on how to combine quantified logic with modal logic. One is to give up classical quantified logic, being content with the logic of (AA-validity in) the two-domain semantics;[7] it should be obvious from the observation above that that logic is sound with respect to Kripke semantics. The other option is to restrict our attention to closed sentences, as opposed to all sentences. This allows us to ignore the difference between classical quantified logic and the logic of the two-domain semantics, because these two logics coincide when restricted to closed sentences, as in Corollary 2 (on p. 84).[8]

While I find the first option more illuminating,[9] the motivation for the second option seems well grounded: As he says, Kripke chose it "since [he] wished to show that the difficulty can be solved without revising quantification theory or modal propositional logic".[10] For the rest of this section and Section IV.2, we further pursue the possibility of recovering classical quantified logic

[7] Kripke mentions this option in his footnote 13 on p. 68 of [19].

[8] Kripke lays out an axiomatization in this option on p. 69 of [19].

[9] There are several reasons why I find the first option more illuminating. One is that it illustrates more explicitly why certain sentences or inferences—the converse Barcan formula, for example—are invalid. Another reason is that, at least in our semantics, we do not give up dealing with open sentences. It seems difficult to find a coherent account of sentences according to which open sentences are not admissible proof-theoretically but admissible semantically.

[10] [19], 68, footnote 13.

in a more genuine manner (than the second option's simply ignoring the non-classical part of the logic of two-domain semantics).

To recover classical quantified logic, we should recall Theorem Theorem 3 (on p. 84); that is, although classical quantified logic is not sound with respect to AA-validity in the two-domain semantics, it is sound with respect to DoQ-validity (Definition 24 on p. 84). This result can be extended to the setting of Kripke semantics, by extending the notion of DoQ-validity to Kripke-type satisfaction relations in the following manner. First recall Definition 23 (on p. 83): For a quantified language $\mathcal{L}$ and a two-domain $\mathcal{L}$ structure $\mathfrak{M} = (|\mathfrak{M}|, \forall^{\mathfrak{M}}, F^{\mathfrak{M}})$, a DoQ-assignment is just a map $\alpha : \mathrm{var}(\mathcal{L}) \to \forall^{\mathfrak{M}}$. This should be the case with the two-domain $\mathcal{L}$ structure $\mathfrak{M}_w = (D, D_w, \overrightarrow{F^{\mathfrak{M}}}(w))$ given by a Kripke model $\mathfrak{M}$ for a quantified modal language $\mathcal{L}$ and a world $w \in W$; that is, for this $\mathfrak{M}_w$, a DoQ-assignment is a map $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$. So we have

**Definition 42.** Given a quantified modal language $\mathcal{L}$, a Kripke frame $\mathfrak{M} = (W, R, D_-)$, and $w \in W$, we mean by a *DoQ-assignment for* $w$, or a $w$*-DoQ-assignment*, any map $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$.

A sentence $\varphi$ of $\mathcal{L}$ is DoQ-valid in a two-domain-type satisfaction relation $(\mathfrak{M}_w, (\mathfrak{M}, w \vDash_- -))$ if $\mathfrak{M}, w \vDash_\alpha \varphi$ for all the DoQ-assignments for $\mathfrak{M}_w$, that is, all the $w$-DoQ-assignments. Extending this, we say $\varphi$ is DoQ-valid in a Kripke-type satisfaction relation $(\mathfrak{M}, (\mathfrak{M}, - \vDash_- -))$ if $\varphi$ is DoQ-valid in $(\mathfrak{M}_w, (\mathfrak{M}, w \vDash_- -))$ for all $w \in W$; that is,

**Definition 43.** We rename the notion of validity in Definition 38 *AA-validity*, and introduce *DoQ-validity* for the Kripke setting: Given a quantified modal language $\mathcal{L}$, we say, for each Kripke-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ on $\mathfrak{M} = (W, R, D_-, F^{\mathfrak{M}})$,

- a sentence $\varphi$ of $\mathcal{L}$ is *DoQ-valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash_\forall \varphi$, meaning that $\mathfrak{M}, w \vDash_\alpha \varphi$ for every pair of $w \in W$ and $w$-DoQ-assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$; and
- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *DoQ-valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash_\forall \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash_\forall \varphi$.

Given a class of Kripke-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is DoQ-valid in that class if it is DoQ-valid in every member of that class.

Clearly, classical quantified logic, which is sound with respect to the DoQ-validity in the two-domain semantics (Theorem 3 on p. 84), is also sound with respect to the DoQ-validity in Kripke semantics.

Therefore, if **K** were also sound with respect to DoQ-validity in Kripke semantics, that validity could provide semantics for the union of **K** with classical quantified logic. Unfortunately, this is not the case. On one hand, the axiom K of **K** is DoQ-valid in Kripke semantics; indeed, any AA-valid sentence is DoQ-valid as well, because the AA-validity (truth for all worlds and assignments) of a sentence entails its DoQ-validity (truth for all worlds and DoQ-assignments). On the other hand, the AA-validity of rules, including the rule

N $$\frac{\vdash \varphi}{\vdash \Box\varphi}$$

of **K**, fails to entail their DoQ-validity, and N is in fact DoQ-invalid.

Let us see how N can fail to be DoQ-valid, that is, how $\Box\varphi$ can fail to be DoQ-valid while $\varphi$ is. Note that the DoQ-validity of $\varphi$ does not imply its AA-validity (if it did, then, by the AA-validity of N, $\Box\varphi$ would be AA-valid and hence DoQ-valid, thereby making N DoQ-valid). For instance, $Fx$ is DoQ-valid but not AA-valid in (any Kripke-type satisfaction relation on) the Kripke model we considered on p. 129. It is because, intuitively put, everything that exists in $w$, namely $a$ and $b$, satisfies $Fx$ at $w$, whereas everything that exists in $u$, namely $a$, satisfies $Fx$ at $u$. Or, in terms of assignments, it is because any assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$, whether $\alpha(x) = a$ or $\alpha(x) = b$, satisfies $\mathfrak{M}, w \vDash_\alpha Fx$, whereas every $\alpha$ such that $\alpha(x) = a$—and hence every $u$-DoQ-assignment—satisfies $\mathfrak{M}, u \vDash_\alpha Fx$. Thus $Fx$ is DoQ-valid.

Note that, although $\mathfrak{M}, u \vDash_\alpha Fx$ fails for any $\alpha$ such that $\alpha(x) = b$, it does not keep $Fx$ from being DoQ-valid, since such $\alpha$ is not a $u$-DoQ-assignment; or, intuitively put, the failure of $b$ to satisfy $Fx$ at $u$, in which it does not exist, is irrelevant to the DoQ-validity of $Fx$. It is, nonetheless, this failure that makes $\Box Fx$ DoQ-invalid (in the Kripke satisfaction relation on the Kripke model). That is, for $\alpha$ such that $\alpha(x) = b$, $\mathfrak{M}, u \nvDash_\alpha Fx$ entails $\mathfrak{M}, w \nvDash_\alpha \Box Fx$ by (IV.12), even though $\alpha$ is a $w$-DoQ-assignment. Intuitively put, though the failure of $b$ to satisfy $F$ at $u$ is irrelevant to the DoQ-validity of $Fx$, it implies that $b$ fails to satisfy $\Box Fx$ at $w$, in which $b$ exists, and therefore that $\Box Fx$ is not DoQ-valid, whereas $Fx$. The rule N is DoQ-invalid in this way.

This observation suggests that the DoQ-invalidity of N is due to the non-autonomy of domains of quantification in Kripke semantics, that is, the fact that the truth values of sentences—modal ones, in particular—within the domains of quantification depends on what is the case outside the domains of quantification. In the example above, whether or not $Fx$ is true of $b$ at a world in which $b$ does not exist was crucial to whether or not $\Box Fx$ is true of $b$ at a world in which $b$ exists, and hence whether or not the DoQ-validity of $Fx$ implies that of $\Box Fx$. This is why, in an attempt to unify modal logic with quantified logic, which is sound with respect not to AA-validity but to DoQ-validity, we will try in Section IV.2 to modify and replace Kripke's truth conditions (IV.12) for $\Box$ and (IV.13) for $\Diamond$ with DoQ-restrictable versions (in the sense extending the definition we laid out in Subsection III.2.4), so that the new versions make N DoQ-valid.

## IV.2 AUTONOMOUS DOMAINS OF QUANTIFICATION FOR THE KRIPKEAN SETTING

### IV.2.1 Operational Form of Kripkean Semantics: A First Step

In this subsection, we first lay out what types of operations are needed to give an operational formulation to Kripke semantics, and then extend the notion of DoQ-restrictability to the Kripke setting. (Kripke's truth conditions will be reformulated operationally in Subsection IV.2.2).

First we import Definition 26 (on p. 92) of two-domain-type interpretations for a quantified modal language $\mathcal{L}$ from the two-domain setting to the Kripke setting. Clearly, we need to replace two-domain $\mathcal{L}$ structures with Kripke models for $\mathcal{L}$. So let us fix a Kripke model $\mathfrak{M}$ for $\mathcal{L}$, with $W$ the set of worlds and $D$ the domain of possible individuals of $\mathfrak{M}$. Then recall that an interpretation consists of certain maps. The map $[\![x]\!]$ that interprets a variable $x$ can keep its type and definition, that is,

$$[\![x]\!] : D^{\mathrm{var}(\mathcal{L})} \to D \qquad \text{and} \qquad [\![x]\!](\alpha) = \alpha(x)$$

as before (except that we replace $|\mathfrak{M}|$ with $D$), since assignments assign individuals to variables in the same way as before.

In contrast, the type of the map $[\![\varphi]\!]$ that interprets a sentence $\varphi$ needs modifying, because the truth of a sentence is now relativized to not only assignments but also to worlds. For modification, recall the type of $\vDash$ in a Kripke-type satisfaction relation, that is,

$$(\mathfrak{M}, - \vDash_{-} -) \subseteq W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L}), \qquad \text{or} \qquad (\mathfrak{M}, - \vDash_{-} -) : W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L}) \to \mathbf{2},$$

and note that $\vDash$ is therefore equivalent to

$$[\![-]\!] : \mathrm{sent}(\mathcal{L}) \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}), \qquad \text{or} \qquad [\![-]\!] : \mathrm{sent}(\mathcal{L}) \to (W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}).$$

This is why we have

$$[\![\varphi]\!] \subseteq W \times D^{\mathrm{var}(\mathcal{L})}, \qquad \text{or} \qquad [\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2},$$

for each sentence $\varphi$, and, accordingly, for each $n$-ary sentential operator $\otimes$ of $\mathcal{L}$,

$$[\![\otimes]\!] : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}),$$

with

$$[\![\otimes(\varphi_1, \ldots, \varphi_n)]\!] = [\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]).$$

The clause for the interpretation $[\![Fx_1 \cdots x_n]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ of an atomic sentence must be modified, since $F^{\mathfrak{M}}$ now has a different type, that is, $F^{\mathfrak{M}} \subseteq W \times D^n$. Yet it is simple to see how to modify it, because (IV.17) implies

$$[\![Fx_1 \cdots x_n]\!](w, \alpha) = \mathbf{1} \iff (w, \alpha(x_1), \ldots, \alpha(x_n)) \in F^{\mathfrak{M}},$$

where

$$\begin{aligned} (w, \alpha(x_1), \ldots, \alpha(x_n)) &= (w, [\![x_1]\!](\alpha), \ldots, [\![x_n]\!](\alpha)) \\ &= (w, \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle(\alpha)) \\ &= 1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle (w, \alpha); \end{aligned}$$

that is,

$$[\![Fx_1 \cdots x_n]\!](w, \alpha) = F^{\mathfrak{M}} \circ (1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle)(w, \alpha),$$

and hence

$$W \times D^{\mathrm{var}(\mathcal{L})} \xrightarrow{1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle} W \times D^n \xrightarrow{F^{\mathfrak{M}}} \mathbf{2}, \qquad [\![Fx_1 \cdots x_n]\!] = F^{\mathfrak{M}} \circ (1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle).$$

To sum these up, we have

**Definition 44.** Given a quantified modal language $\mathcal{L}$, a *Kripke-type interpretation for* $\mathcal{L}$ is a pair of a Kripke model $\mathfrak{M}$ for $\mathcal{L}$ and a map $[\![-]\!]$ that assigns, to each variable $x$, sentence $\varphi$, and $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, maps

$$
\begin{aligned}
&[\![x]\!] : D^{\mathrm{var}(\mathcal{L})} \to D, \\
&[\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}, \\
&[\![\otimes]\!] : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})
\end{aligned}
$$

that satisfy

$$
\begin{aligned}
[\![x]\!] &: \alpha \mapsto \alpha(x), \\
[\![Fx_1 \cdots x_n]\!] &= F^{\mathfrak{M}} \circ (1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle), \\
[\![\otimes(\varphi_1, \ldots, \varphi_n)]\!] &= [\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]).
\end{aligned}
$$

We say a Kripke-type interpretation for $\mathcal{L}$ is *on* a Kripke model $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$, and is *over* a Kripke frame $\mathfrak{F}$ with domains if it is on a Kripke model over $\mathfrak{F}$.

We should make a remark—exactly similar to the one we made after Definition 26 (on p. 92)—that, whereas every Kripke-type interpretation gives rise to a Kripke-type satisfaction relation via transposition, not every Kripke-type satisfaction relation arises in that way, since the definition for Kripke-type interpretations incorporates the truth condition (IV.17) for atomic sentences. Nevertheless, once we rewrite the conditions (IV.8)–(IV.15) for sentential operators, we can define the subclass of Kripke-type interpretations for $\mathcal{L}$ that is equivalent to the class of Kripke satisfaction relations for $\mathcal{L}$. We will summarize these facts in Fact 34, after reformulating (IV.8)–(IV.15) operationally in Subsection IV.2.2.

Now that we have given an operational formulation to the Kripke setting, we can extend notions we expressed in Section III.2 for two-domain semantics to this setting. Before discussing the notion of DoQ-restrictability, let us first describe how to express the notions of local determination and its preservation in terms of Kripke-type interpretations. Because a term $t$ is interpreted by the same type of map $[\![t]\!] : D^{\mathrm{var}(\mathcal{L})} \to D$ as it was before in two-domain semantics, local determination of $t$

is expressed in the same way as before, that is, by the factorization of $[\![t]\!]$ through the restriction surjection $-\restriction\bar{x} : \alpha \mapsto \alpha\restriction\bar{x}$ for the set $\bar{x}$ of (free) variables in $t$, as in:

$$\begin{array}{c} \overset{[\![t]\!]}{\overbrace{\qquad\qquad\qquad\qquad\qquad}} \\ \shortparallel \\ D^{\mathrm{var}(\mathcal{L})} \xrightarrow[-\restriction\bar{x}]{} D^{\bar{x}} \xrightarrow[[\![\,\bar{x}\mid t\,]\!]]{} D \end{array}$$

In a quantified language $\mathcal{L}$, every term $x$ is a variable and hence locally determined trivially.

The expression of local determination of sentences needs some modification since sentences are now interpreted by a different type of maps, but the modification is straightforward once we observe that $A : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ can be called determined by variables $\bar{x}$, in a slightly generalized sense, if $A$ factors through the (generalized) restriction surjection $1_W \times (-\restriction\bar{x}) : (w, \alpha) \mapsto (w, \alpha\restriction\bar{x})$, as in the following commutative diagram for some $A_{\bar{x}}$:

$$\begin{array}{c} \overset{A}{\overbrace{\qquad\qquad\qquad\qquad\qquad}} \\ \shortparallel \\ W \times D^{\mathrm{var}(\mathcal{L})} \xrightarrow[1_W \times (-\restriction\bar{x})]{} W \times D^{\bar{x}} \xrightarrow[A_{\bar{x}}]{} \mathbf{2} \end{array}$$

So, let us write $r_{\bar{x}}$ for the restriction $1_W \times (-\restriction\bar{x})$, and just replace $|\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ and $|\mathfrak{M}|^{\bar{x}}$ in Definitions 32 (on p. 114) and 33 (on p. 117) with $W \times D^{\mathrm{var}(\mathcal{L})}$ and $W \times D^{\bar{x}}$, respectively, so that we enter the following (only local determination of sentences is significant for local determination of a Kripke-type interpretation, because every term of a quantified $\mathcal{L}$ is locally determined trivially).

**Definition 45.** Given a quantified modal language $\mathcal{L}$, suppose $\bar{x}$ are the free variables in a sentence $\varphi$ of $\mathcal{L}$. Then we say that $\varphi$ is locally determined in a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, with $D$ the domain of individuals of $\mathfrak{M} = (W, R, D_{-})$, if its interpretation $[\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ factors through $r_{\bar{x}} = 1_W \times (-\restriction\bar{x})$, as in

$$\begin{array}{c} \overset{[\![\varphi]\!]}{\overbrace{\qquad\qquad\qquad\qquad\qquad}} \\ \shortparallel \\ W \times D^{\mathrm{var}(\mathcal{L})} \xrightarrow[r_{\bar{x}}]{} W \times D^{\bar{x}} \xrightarrow[[\![\,\bar{x}\mid\varphi\,]\!]]{} \mathbf{2} \end{array}$$

If this is the case, we write $[\![\,\bar{x} \mid \varphi\,]\!] : W \times D^{\bar{x}} \to \mathbf{2}$ for the unique map such that $[\![\varphi]\!] = [\![\,\bar{x} \mid \varphi\,]\!] \circ r_{\bar{x}}$. Moreover, we say a Kripke-type interpretation for $\mathcal{L}$ is locally determined if every sentence of $\mathcal{L}$ is locally determined in it. We also say a class of Kripke-type interpretations for $\mathcal{L}$ is locally determined if every member of that class is locally determined.

Now, using the precomposition operation ${r_{\bar{x}}}^* = - \circ r_{\bar{x}}$ with $r_{\bar{x}}$, that is,

$$ {r_{\bar{x}}}^*(B) = B \circ r_{\bar{x}} : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}^n $$

for every $B : W \times D^{\bar{x}} \to \mathbf{2}^n$ (see Subsection III.2.3 for details), we enter:

**Definition 46.** Let $\mathcal{L}$ be a quantified modal language and let $(W, R, D_{-})$ be a Kripke frame with domains, with a domain $D$ of possible individuals. Then, for variables $\bar{y}$ of $\mathcal{L}$, we say an operation $f : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ *preserves local determination with the binding of* $\bar{y}$ if, for every finite set $\bar{x}$ of variables of $\mathcal{L}$, there is an operation $f_{\bar{x}} : \mathcal{P}(W \times D^{\bar{x}})^n \to \mathcal{P}(W \times D^{\bar{x} \setminus \bar{y}})$ such that, for every $B : W \times D^{\bar{x}} \to \mathbf{2}^n$,

$$ f_{\bar{x}}(B) \circ r_{\bar{x} \setminus \bar{y}} = f(B \circ r_{\bar{x}}), $$

that is, that makes the following diagram commute.

$$ \begin{array}{ccc} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;{r_{\bar{x}}}^*\;} & \mathcal{P}(W \times D^{\bar{x}})^n \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle f_{\bar{x}}} \\ \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;{r_{\bar{x} \setminus \bar{y}}}^*\;]{} & \mathcal{P}(W \times D^{\bar{x} \setminus \bar{y}}) \end{array} $$

We also say $f : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ preserves local determination *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and if $f$ preserves local determination with the binding of the variables that $\otimes$ binds. Moreover, we say a Kripke-type interpretation for $\mathcal{L}$ preserves local determination if it interprets every sentential operator $\otimes$ of $\mathcal{L}$ with an operation that preserves local determination for $\otimes$.

Since atomic sentences are locally determined in any Kripke-type interpretation, we have:

**Fact 33.** Any Kripke-type interpretation for a given quantified modal language $\mathcal{L}$ is locally determined if it preserves local determination.

Let us finally discuss how to express the notion of DoQ-restrictability in terms of Kripke-type interpretations. Fix a quantified modal language $\mathcal{L}$, a Kripke model $\mathfrak{M} = (W, R, D_{-}, F^{\mathfrak{M}})$ for $\mathcal{L}$ with the domain $D$ of possible individuals, and a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ on $\mathfrak{M}$. We need to decide when the following four types of maps (only four, since $\mathcal{L}$ has no function or constant symbols) are said to be restrictable to the domain of quantification:

$$
\begin{aligned}
&F^{\mathfrak{M}} : W \times D^n \to \mathbf{2},\\
&[\![x]\!] : D^{\mathrm{var}(\mathcal{L})} \to D,\\
&[\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2},\\
&[\![\otimes]\!] : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}).
\end{aligned}
$$

This question is trickier than it was with the two-domain semantics because, in Kripke semantics, domains $D_w$ of quantification vary with worlds $w$.

Since a different world $w$ has a different domain $D_w$ of quantification, it has a different DoQ-restriction of $[\![x]\!] : D^{\mathrm{var}(\mathcal{L})} \to D$. Nonetheless, as before, $[\![x]\!]$ is always restrictable to any $D_w \subseteq D$, as in:

(IV.22)
$$
\begin{array}{ccc}
D^{\mathrm{var}(\mathcal{L})} & \xrightarrow{\;[\![x]\!]\;} & D \\
\uparrow & \approx & \uparrow \\
{D_w}^{\mathrm{var}(\mathcal{L})} & \xrightarrow[\;[\![x]\!]_{D_w}\;]{} & D_w
\end{array}
$$

So let us call $[\![x]\!]_{D_w}$ as above the *$w$-DoQ-restriction* of $[\![x]\!]$.

To restrict $F^{\mathfrak{M}} : W \times D^n \to \mathbf{2}$ and $[\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ to proper types of maps, observe that some $(w, \bar{a}) \in W \times D^n$ may be such that each $a_i$ exists in $w$—that is, such that $\bar{a}$ lies in the $n$-fold product ${D_w}^n$ of the domain $D_w$ of quantification for $w$—but another $(w, \bar{b}) \in W \times D^n$ may not be, and also that some $(w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})}$ may be such that $\alpha$ is a $w$-DoQ-assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$ but another $(w, \beta) \in W \times D^{\mathrm{var}(\mathcal{L})}$ may not be. So let us introduce

**Definition 47.** Given a Kripke frame $\mathfrak{F} = (W, R, D_{-})$, we say a pair $(w, \bar{a}) \in W \times D^n$ is a *world-tuple DoQ-pair*, or a *world-individual DoQ-pair* when $n = 1$, if $\bar{a} \in {D_w}^n$, and we also say a pair $(w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})}$ is a *world-assignment DoQ-pair* if $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$. We call them simply *DoQ-pairs* when it causes no confusion, and call other pairs *non-DoQ-pairs*.

And consider the sets of DoQ-pairs, that is, the subsets

$$\{\, (w, \bar{a}) \in W \times D^n \mid \bar{a} \in {D_w}^n \,\}, \qquad \{\, (w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid \alpha : \mathrm{var}(\mathcal{L}) \to D_w \,\}$$

of $W \times D^n$ and $W \times D^{\mathrm{var}(\mathcal{L})}$; indeed, they can be written, with $\sum$ for disjoint union, as

$$\sum_{w\in W} {D_w}^n = \bigcup_{w\in W} (\{w\} \times {D_w}^n) = \{\, (w, \bar{a}) \in W \times D^n \mid \bar{a} \in {D_w}^n \,\},$$
$$\sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})} = \bigcup_{w\in W} (\{w\} \times {D_w}^{\mathrm{var}(\mathcal{L})}) = \{\, (w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid \alpha : \mathrm{var}(\mathcal{L}) \to D_w \,\}.$$

Then let us take, as our candidates for the DoQ-restrictions ${F^{\mathfrak{M}}}_{\mathrm{DoQ}}$ of $F^{\mathfrak{M}}$ and $[\![\varphi]\!]_{\mathrm{DoQ}}$ of $[\![\varphi]\!]$, their restrictions to the sets above, as in

(IV.23)

$$\begin{array}{ccc} W \times D^n & \xrightarrow{\;F^{\mathfrak{M}}\;} & \mathbf{2} \\ {\scriptstyle i}\uparrow & \approx & \| \\ \sum_{w\in W} {D_w}^n & \xrightarrow[\;{F^{\mathfrak{M}}}_{\mathrm{DoQ}}\;]{} & \mathbf{2} \end{array}$$

and the following.

(IV.24)

$$\begin{array}{ccc} W \times D^{\mathrm{var}(\mathcal{L})} & \xrightarrow{\;[\![\varphi]\!]\;} & \mathbf{2} \\ {\scriptstyle i}\uparrow & \approx & \| \\ \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})} & \xrightarrow[\;[\![\varphi]\!]_{\mathrm{DoQ}}\;]{} & \mathbf{2} \end{array}$$

Again, $F^{\mathfrak{M}}$ and $[\![\varphi]\!]$ are trivially DoQ-restrictable, since they are restrictable to any subset of $W \times D^n$ and $W \times D^{\mathrm{var}(\mathcal{L})}$, respectively.

These proposed types of the DoQ-restrictions of $[\![x]\!]$, $F^{\mathfrak{M}}$, and $[\![\varphi]\!]$ fit together nicely, since from them it follows that the diagram

$$W \times D^{\mathrm{var}(\mathcal{L})} \xrightarrow[\;1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle\;]{} W \times D^n \xrightarrow[\;F^{\mathfrak{M}}\;]{} \mathbf{2}, \qquad = \; [\![F x_1 \cdots x_n]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

for interpreting atomic sentences $F\bar{x}$ restricts to the domains of quantification, as in

$$
\begin{array}{ccccc}
 & & \llbracket Fx_1 \cdots x_n \rrbracket & & \\
 & & \parallel & & \\
W \times D^{\mathrm{var}(\mathcal{L})} & \xrightarrow[1_W \times \langle \llbracket x_1 \rrbracket, \ldots, \llbracket x_n \rrbracket \rangle]{} & W \times D^n & \xrightarrow[F^{\mathfrak{M}}]{} & \mathbf{2} \\
\uparrow i & & \uparrow i & & \Vert \\
\sum_{w\in W} D_w{}^{\mathrm{var}(\mathcal{L})} & \xrightarrow{\sum_{w\in W} \langle \llbracket x_1 \rrbracket_{D_w}, \ldots, \llbracket x_n \rrbracket_{D_w} \rangle} & \sum_{w\in W} D_w{}^n & \xrightarrow{F^{\mathfrak{M}}{}_{\mathrm{DoQ}}} & \mathbf{2}, \\
 & & \parallel & & \\
 & & \llbracket Fx_1 \cdots x_n \rrbracket_{\mathrm{DoQ}} & &
\end{array}
$$

where the map $\sum_{w\in W} \langle \llbracket x_1 \rrbracket_{D_w}, \ldots, \llbracket x_n \rrbracket_{D_w} \rangle$ maps DoQ-pairs $(w, \alpha)$ to DoQ-pairs

$$(w, \langle \llbracket x_1 \rrbracket_{D_w}, \ldots, \llbracket x_n \rrbracket_{D_w} \rangle(\alpha)) = (w, \llbracket x_1 \rrbracket_{D_w}(\alpha), \ldots \llbracket x_n \rrbracket_{D_w}(\alpha)) = (w, \alpha(x_1), \ldots, \alpha(x_n)).$$

The left inner square above commutes due to (IV.22), while the right one is just (IV.23); and these entail the commutation of the outer square, which is an atomic instance of (IV.24).

Finally, the type of the DoQ-restriction of $\llbracket \varphi \rrbracket$, that is, $\llbracket \varphi \rrbracket_{\mathrm{DoQ}} : \sum_{w\in W} D_w{}^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$, settles the type of the DoQ-restriction $\llbracket \otimes \rrbracket_{\mathrm{DoQ}}$ of $\llbracket \otimes \rrbracket : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$; that is, it has to be

$$
\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xrightarrow{\llbracket \otimes \rrbracket} & \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^m \\
i^* \downarrow & /\!/ & \downarrow i^* \\
\mathcal{P}\left(\sum_{w\in W} D_w{}^{\mathrm{var}(\mathcal{L})}\right)^n & \xrightarrow[\llbracket \otimes \rrbracket_{\mathrm{DoQ}}]{} & \mathcal{P}\left(\sum_{w\in W} D_w{}^{\mathrm{var}(\mathcal{L})}\right)^m
\end{array}
$$

where $i^*$ is the precomposition $- \circ i$ with the inclusion $i : \sum_{w\in W} D_w{}^{\mathrm{var}(\mathcal{L})} \hookrightarrow W \times D^{\mathrm{var}(\mathcal{L})}$.

Summarizing these observations, we have the following definition. Only the DoQ-restrictability of $\llbracket \otimes \rrbracket$ plays a role in the definition, because $F^{\mathfrak{M}}$, $\llbracket x \rrbracket$, and $\llbracket \varphi \rrbracket$ are always DoQ-restrictable.

**Definition 48.** Let $\mathcal{L}$ be a quantified modal language and $\mathfrak{M} = (W, R, D_-, F^{\mathfrak{M}})$ be a Kripke model for $\mathcal{L}$ with a domain $D$ of possible individuals. Then we say an operation $f : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^m$ (for any $n$ and $m$) is *DoQ-restrictable* if it is restrictable to the set $\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}$ of DoQ-pairs, as in:

$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xrightarrow{\quad f \quad} & \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^m \\
{\scriptstyle i^*}\big\downarrow & /\!\!/ & \big\downarrow{\scriptstyle i^*} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)^n & \xrightarrow[\quad f_{\mathrm{DoQ}} \quad]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)^m
\end{array}$$

We also say a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ on $\mathfrak{M}$ is *DoQ-restrictable* if $[\![\otimes]\!] : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ is DoQ-restrictable for each $n$-ary sentential operator $\otimes$ of $\mathcal{L}$.

The following is worth noting. Recall that the equivalence (IV.27) between subsets $A = \sum_{w \in W} A_w \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ and families $\langle A_w \rangle_{w \in W}$ of subsets of $D^{\mathrm{var}(\mathcal{L})}$ gives rise to the isomorphism

$$\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \cong \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}).$$

Then note that this isomorphism restricts to

$$\mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) \cong \prod_{w \in W} \mathcal{P}({D_w}^{\mathrm{var}(\mathcal{L})}),$$

because $A \subseteq \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}$ iff $A_w \subseteq {D_w}^{\mathrm{var}(\mathcal{L})}$ for all $w \in W$.

### IV.2.2 Kripke's Operations

In Subsection IV.2.1 we laid out how to give an operational formulation of semantics to Kripke's setting of possible worlds and possible individuals, and how to express the notions of local determination, its preservation, and DoQ-restrictability. In this subsection, we first operationally reformulate Kripke's truth conditions (IV.8)–(IV.15) for sentential operators, and then show that the operations given by them preserve local determination.

It should be clear that, by (IV.8)–(IV.11), $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\rightarrow]\!]$ are truth-functional postcompositions $\neg_{\mathbf{2}} \circ -$, $\wedge_{\mathbf{2}} \circ -$, $\vee_{\mathbf{2}} \circ -$, $\rightarrow_{\mathbf{2}} \circ -$ as before (although working on maps of different types), for example:

$$\begin{array}{ccc} W \times D^{\mathrm{var}(\mathcal{L})} & \xrightarrow{\langle A, B\rangle} & \mathbf{2} \times \mathbf{2} \\ & \searrow^{[\![\wedge]\!](A,B)} \; /\!/ & \downarrow \wedge_{\mathbf{2}} \\ & & \mathbf{2} \end{array} \qquad\qquad \begin{array}{c} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^2 \\ \downarrow [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ - \\ \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \end{array}$$

For the interpretation of quantifiers, since (IV.14) means

$$(w, \alpha) \in [\![\forall x]\!][\![\varphi]\!] \iff (w, [a/x]\alpha) \in [\![\varphi]\!] \text{ for every } a \in D_w,$$

we should set

$$\text{(IV.25)} \qquad [\![\forall x]\!](A) = \{\, (w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid (w, [a/x]\alpha) \in A \text{ for every } a \in D_w \,\}.$$

This map $[\![\forall x]\!]$ can be further analyzed, taking advantage of the intuitive observation that, in Kripke semantics, the truth of $\forall x . \varphi$ is determined within a world, or "world-wise"; that is, in deciding whether or not $\forall x . \varphi$ is true at $w$ (with respect to an assignment), only the truth of $\varphi$ at the world $w$ (with respect to certain other assignments) is relevant.

To express this formally, given any $A \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ and $w \in W$, let us write $A_w$ for $\overrightarrow{A}(w)$, that is,

$$A_w = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid (w, \alpha) \in A \,\}.$$

For example, $[\![\varphi]\!]_w$ is the set of assignments with respect to which $\varphi$ is true at $w$. $A$ amounts to the disjoint union of $A_w$ with indexing with $w \in W$; that is,

$$A = \sum_{w \in W} A_w = \bigcup_{w \in W} (\{w\} \times A_w). \tag{IV.26}$$

Note that the map $\overrightarrow{A}$, to which the set $A$ is equivalent, is defined by the family of its values $\overrightarrow{A}(w) = A_w$ for all $w \in W$; in other words, any $A \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ is equivalent to the family of $A_w \subseteq D^{\mathrm{var}(\mathcal{L})}$, or, in notation,

$$A \cong \langle A_w \rangle_{w \in W}. \tag{IV.27}$$

Thus the correspondence above between $A = \sum_{w \in W} A_w$ and $\langle A_w \rangle_{w \in W}$ gives an isomorphism

$$\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \cong \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}).$$

In particular, the entire set $W \times D^{\mathrm{var}(\mathcal{L})}$ corresponds to $\langle D^{\mathrm{var}(\mathcal{L})} \rangle_{w \in W}$, with $(W \times D^{\mathrm{var}(\mathcal{L})})_w = D^{\mathrm{var}(\mathcal{L})}$ constant for all $w \in W$, so that

$$W \times D^{\mathrm{var}(\mathcal{L})} = \sum_{w \in W} D^{\mathrm{var}(\mathcal{L})} \cong \langle D^{\mathrm{var}(\mathcal{L})} \rangle_{w \in W}.$$

In this $-_w$ notation, $A_w$ and $([\![\forall x]\!](A))_w$ are related as follows. Recall that, as noted in Subsection IV.1.2, for each world $w \in W$, $\mathfrak{M}_w = (D, D_w, \overrightarrow{F^{\mathfrak{M}}})$ is a two-domain $\mathcal{L}$ structure, with $D$ the domain of individuals and $D_w$ the domain of quantification, and $(\mathfrak{M}_w, (\mathfrak{M}, w \vDash_{-} -))$ is a two-domain satisfaction relation for $\mathcal{L}$. Therefore $(\mathfrak{M}_w, [\![-]\!]_w)$ is a two-domain interpretation for $\mathcal{L}$. Let $[\![\forall x]\!]_w : \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(D^{\mathrm{var}(\mathcal{L})})$ be its interpretation of $\forall x$, as defined in (III.22) (on p. 89); that is, for each $B \subseteq D^{\mathrm{var}(\mathcal{L})}$,

$$[\![\forall x]\!]_w(B) = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in B \text{ for every } a \in D_w \,\}.$$

For example, since $[\![\varphi]\!]_w$ is the set of assignments with respect to which $\varphi$ is true at $w$, $[\![\forall x]\!]_w([\![\varphi]\!]_w)$ is supposed to be the set of assignments with respect to which $\forall x.\varphi$ is true at $w$, that is, $[\![\forall x.\varphi]\!]_w$. Indeed, (IV.25) implies in general that

$$(w, \alpha) \in [\![\forall x]\!](A) \iff [a/x]\alpha \in A_w \text{ for every } a \in D_w \iff \alpha \in [\![\forall x]\!]_w(A_w),$$

that is

$$(\text{IV.28}) \qquad ([\![\forall x]\!](A))_w = [\![\forall x]\!]_w(A_w),$$

and hence

$$[\![\forall x]\!](A) \overset{\text{(IV.26)}}{=} \sum_{w\in W}([\![\forall x]\!](A))_w \overset{\text{(IV.28)}}{=} \sum_{w\in W}([\![\forall x]\!]_w(A_w)) \overset{\text{(IV.26)}}{=} \bigcup_{w\in W}(\{w\} \times [\![\forall x]\!]_w(A_w)).$$

Or, in the $A \cong \langle A_w\rangle_{w\in W}$ notation,

$$[\![\forall x]\!](A) \overset{\text{(IV.27)}}{\cong} \langle([\![\forall x]\!](A))_w\rangle_{w\in W} \overset{\text{(IV.28)}}{=} \langle[\![\forall x]\!]_w(A_w)\rangle_{w\in W} = \prod_{w\in W}[\![\forall x]\!]_w(\langle A_w\rangle_{w\in W}) \overset{\text{(IV.27)}}{\cong} \prod_{w\in W}[\![\forall x]\!]_w(A),$$

where the $\prod$ notation generalizes the "product map" notation for $f_1 \times \cdots \times f_n : (a_1, \ldots, a_n) \mapsto (f_1(a_1), \ldots, f_n(a_n))$; so we simply write

$$[\![\forall x]\!] = \prod_{w\in W}[\![\forall x]\!]_w : \prod_{w\in W}\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \to \prod_{w\in W}\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}),$$

to indicate that the operation $[\![\forall x]\!]$ is given by taking the product of operations $[\![\forall x]\!]_w : \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \to \mathcal{P}(D^{\mathrm{var}(\mathcal{L})})$ for all $w \in W$. Similarly,

$$\begin{aligned}[\![\exists x]\!](A) &= \sum_{w\in W}([\![\exists x]\!](A))_w = \sum_{w\in W}([\![\exists x]\!]_w(A_w)) = \bigcup_{w\in W}(\{w\} \times [\![\exists x]\!]_w(A_w)) \\ &= \{\, (w,\alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid (w, [a/x]\alpha) \in A \text{ for some } a \in D_w \,\}\end{aligned}$$

and

$$[\![\exists x]\!] = \prod_{w\in W}[\![\exists x]\!]_w : \prod_{w\in W}\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \to \prod_{w\in W}\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}).$$

Lastly, for the interpretation of modal operators, since (IV.12) means

$$(w,\alpha) \in [\![\Box]\!][\![\varphi]\!] \iff (u,\alpha) \in [\![\varphi]\!] \text{ for every } u \in W \text{ such that } Rwu,$$

we should set

$$(\text{IV.29}) \qquad [\![\Box]\!](A) = \{\, (w,\alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid (u,\alpha) \in A \text{ for every } u \in W \text{ such that } Rwu \,\},$$

and similarly, by (IV.13), we should set

(IV.30) $$[\![\Diamond]\!](A) = \{\,(w,\alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid (u,\alpha) \in A \text{ for some } u \in W \text{ such that } Rwu\,\}.$$

These maps $[\![\Box]\!]$ and $[\![\Diamond]\!]$ can be analyzed in a manner similar to the analysis of $[\![\forall x]\!]$ above, since (IV.29) and (IV.30) determine the truth of $\Box\varphi$ and $\Diamond\varphi$ "assignment-wise". In a manner symmetric to the $-_w$ notation above, let us introduce the $-_\alpha$ notation as follows: Given any $A \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ and $\alpha \in D^{\mathrm{var}(\mathcal{L})}$, we write $A_\alpha$ for $\overleftarrow{A}(\alpha)$, that is,

$$A_\alpha = \{\, w \in W \mid (w,\alpha) \in A\,\}.$$

For example, $[\![\varphi]\!]_\alpha$ is the set of worlds at which $\varphi$ is true with respect to $\alpha$. Again, $A$ is the disjoint union of $A_\alpha$ with indexing with $\alpha \in D^{\mathrm{var}(\mathcal{L})}$, as in

$$A = \sum_{\alpha \in D^{\mathrm{var}(\mathcal{L})}} A_\alpha = \bigcup_{\alpha \in D^{\mathrm{var}(\mathcal{L})}} (A_\alpha \times \{\alpha\}),$$

and is equivalent to the family of $A_\alpha$ for all $\alpha : \mathrm{var}(\mathcal{L}) \to D$, as in

$$A \cong \langle A_\alpha \rangle_{\alpha \in D^{\mathrm{var}(\mathcal{L})}}.$$

It follows that, using the right transpose $\overrightarrow{R}$ of $R$, which has $\overrightarrow{R}(w) = \{\, u \in W \mid Rwu\,\}$, we can rewrite (IV.29) and (IV.30) as

(IV.31) $$(w,\alpha) \in [\![\Box]\!](A) \iff (u,\alpha) \in A \text{ for every } u \in \overrightarrow{R}(w) \iff \overrightarrow{R}(w) \subseteq A_\alpha,$$

(IV.32) $$(w,\alpha) \in [\![\Diamond]\!](A) \iff (u,\alpha) \in A \text{ for some } u \in \overrightarrow{R}(w) \iff \overrightarrow{R}(w) \cap A_\alpha \neq \varnothing.$$

These can be rewritten further in terms of the interior and closure operations $\mathbf{int}_R, \mathbf{cl}_R : \mathcal{P}(W) \to \mathcal{P}(W)$ associated with the accessibility relation $R$, as defined for the case of propositional modal logic; that is, for each $U \subseteq W$,

$$\mathbf{int}_R(U) = \{\, w \in W \mid \overrightarrow{R}(w) \subseteq U\,\} = \{\, w \in W \mid u \in U \text{ for every } u \in W \text{ such that } Rwu\,\},$$
$$\mathbf{cl}_R(U) = \{\, w \in W \mid \overrightarrow{R}(w) \cap U \neq \varnothing\,\} = \{\, w \in W \mid u \in U \text{ for some } u \in W \text{ such that } Rwu\,\}.$$

Thus, (IV.31) and (IV.32) amount to

(IV.33) $$(w,\alpha) \in [\![\Box]\!](A) \iff w \in \mathbf{int}_R(A_\alpha), \qquad (w,\alpha) \in [\![\Diamond]\!](A) \iff w \in \mathbf{cl}_R(A_\alpha),$$

that is,

$$([\![\Box]\!](A))_\alpha = \mathbf{int}_R(A_\alpha), \qquad ([\![\Diamond]\!](A))_\alpha = \mathbf{cl}_R(A_\alpha),$$

and hence

$$[\![\Box]\!](A) = \sum_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} ([\![\Box]\!](A))_\alpha = \sum_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{int}_R(A_\alpha) = \bigcup_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} (\mathbf{int}_R(A_\alpha)\times\{\alpha\}),$$
$$[\![\Diamond]\!](A) = \sum_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} ([\![\Diamond]\!](A))_\alpha = \sum_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{cl}_R(A_\alpha) = \bigcup_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} (\mathbf{cl}_R(A_\alpha)\times\{\alpha\}),$$

and moreover

$$[\![\Box]\!] = \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{int}_R : \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathcal{P}(W) \to \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathcal{P}(W),$$
$$[\![\Diamond]\!] = \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{cl}_R : \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathcal{P}(W) \to \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathcal{P}(W).$$

From these, we have

**Definition 49.** Given a quantified modal language $\mathcal{L}$, a Kripke-type interpretation for $\mathcal{L}$ on $\mathfrak{M}$ is said to be a *Kripke interpretation for* $\mathcal{L}$, if it satisfies (IV.34)–(IV.41):

(IV.34) $\quad [\![\neg]\!] = \neg_{\mathbf{2}} \circ -,$

(IV.35) $\quad [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ -,$

(IV.36) $\quad [\![\vee]\!] = \vee_{\mathbf{2}} \circ -,$

(IV.37) $\quad [\![\rightarrow]\!] = \rightarrow_{\mathbf{2}} \circ -,$

(IV.38) $\quad [\![\Box]\!] = \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{int}_R$, where $\mathbf{int}_R(A) = \{\, w \in W \mid \overrightarrow{R}(w) \subseteq A \,\}$,

(IV.39) $\quad [\![\Diamond]\!] = \prod_{\alpha\in D^{\mathrm{var}(\mathcal{L})}} \mathbf{cl}_R$, where $\mathbf{cl}_R(A) = \{\, w \in W \mid \overrightarrow{R}(w) \cap A \neq \varnothing \,\}$,

(IV.40) $\quad [\![\forall x]\!] = \prod_{w\in W} [\![\forall x]\!]_w$, where $[\![\forall x]\!]_w(A) = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A$ for every $a \in D_w \,\}$,

(IV.41) $\quad [\![\exists x]\!] = \prod_{w\in W} [\![\exists x]\!]_w$, where $[\![\exists x]\!]_w(A) = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A$ for some $a \in D_w \,\}$.

Then we have:

**Fact 34.** Let $\mathcal{L}$ be a quantified modal language. Given any Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, define a relation $\vDash \subseteq W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$ by transposition

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff (w, \alpha) \in [\![\varphi]\!],$$

so that we have a pair $(\mathfrak{M}, \vDash)$. This gives an operation from the class of Kripke-type interpretations for $\mathcal{L}$ to the class of Kripke-type satisfaction relations for $\mathcal{L}$. Moreover, this operation restricted to the class of Kripke interpretations for $\mathcal{L}$ is bijective to the class of Kripke satisfaction relations for $\mathcal{L}$.

We close this subsection by showing that Kripke semantics preserves local determination and hence is locally determined by Fact 33.

**Fact 35.** Any Kripke interpretation for a given quantified modal language $\mathcal{L}$ preserve local determination.

**Corollary 5.** Any Kripke interpretation for a given quantified modal language $\mathcal{L}$ is locally determined.

To prove Fact 35, the following notation is useful. In the same way we define $A_\alpha = \overleftarrow{A}(\alpha) : W \to \mathbf{2}$ from $A : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ and $\alpha \in D^{\mathrm{var}(\mathcal{L})}$, for given $\bar{x}$ let us define $B_\beta = \overleftarrow{B}(\beta) : W \to \mathbf{2}$ from $B : W \times D^{\bar{x}} \to \mathbf{2}$ and $\beta \in D^{\bar{x}}$, so that

$$w \in B_\beta \iff (w, \beta) \in B.$$

Then it follows that

$$B_{\alpha \restriction \bar{x}} = ({r_{\bar{x}}}^*(B))_\alpha \tag{IV.42}$$

for every $B \subseteq W \times D^{\bar{x}}$ and $\alpha : \mathrm{var}(\mathcal{L}) \to D$, because

$$w \in B_{\alpha \restriction \bar{x}} \iff r_{\bar{x}}(w, \alpha) = (w, \alpha \restriction \bar{x}) \in B \iff (w, \alpha) \in {r_{\bar{x}}}^*(B) \iff w \in ({r_{\bar{x}}}^*(B))_\alpha.$$

We use this in the last step of the following proof, where we prove that $[\![\Box]\!]$ and $[\![\Diamond]\!]$ preserve local determination for $\Box$ and $\Diamond$.

*Proof for Fact 35.* Fix a quantified modal language $\mathcal{L}$ and a Kripke interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ over a Kripke frame $(W, R, D_{-})$ with a domain $D$ of possible individuals. Then $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\rightarrow]\!]$ preserve local determination for $\neg$, $\wedge$, $\vee$, $\rightarrow$ because they are postcompositions.

That $[\![\forall x]\!]$ preserves local determination for $\forall y$ follows from the fact that it operates worldwise, in the following manner. Recall that, as in Fact 29 (on p. 118), $[\![\forall y]\!]_w$, for each $w \in W$, preserves local determination for $\forall y$ (in the two-domain sense of Definition 33 on p. 117), as in the following, where $r'_{\bar{x}} : \alpha \mapsto \alpha \restriction \bar{x}$ and $r'_{\bar{x}\setminus\{y\}} : \alpha \mapsto \alpha \restriction (\bar{x} \setminus \{y\})$.

$$\begin{array}{ccc}
\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;{r'_{\bar{x}}}^{*}\;} & \mathcal{P}(D^{\bar{x}}) \\
{\scriptstyle [\![\forall y]\!]_w}\big\downarrow & \gtrdot & \big\downarrow{\scriptstyle [\![\, \bar{x} \mid \forall y \,]\!]_w} \\
\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;{r'_{\bar{x}\setminus\{y\}}}^{*}\;]{} & \mathcal{P}(D^{\bar{x}\setminus\{y\}})
\end{array}$$

Observe that $r_{\bar{x}}^{*} = \prod_{w \in W} {r'_{\bar{x}}}^{*}$ and similarly that ${r_{\bar{x}\setminus\{y\}}}^{*} = \prod_{w \in W} {r'_{\bar{x}\setminus\{y\}}}^{*}$, because

$$\begin{aligned}
(u, \alpha) \in {r_{\bar{x}}}^{*}(B) \iff & \; r_{\bar{x}}(u, \alpha) = (u, \alpha \restriction \bar{x}) = (u, r'_{\bar{x}}(\alpha)) \in B \\
\iff & \; r'_{\bar{x}}(\alpha) \in B_u \iff \alpha \in {r'_{\bar{x}}}^{*}(B_u) \iff (u, \alpha) \in \langle {r'_{\bar{x}}}^{*}(B_w) \rangle_{w \in W}.
\end{aligned}$$

Hence the product $\prod_{w \in W} [\![\, \bar{x} \mid \forall y \,]\!]_w$ of all $[\![\, \bar{x} \mid \forall y \,]\!]_w$ as above makes the diagram below commute.

$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \cong \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;{r_{\bar{x}}}^{*} = \prod_{w \in W} {r'_{\bar{x}}}^{*}\;} & \prod_{w \in W} \mathcal{P}(D^{\bar{x}}) \cong \mathcal{P}(W \times D^{\bar{x}}) \\
{\scriptstyle [\![\forall y]\!] = \prod_{w \in W} [\![\forall y]\!]_w}\big\downarrow & \gtrdot & \big\downarrow{\scriptstyle \prod_{w \in W} [\![\, \bar{x} \mid \forall y \,]\!]_w} \\
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \cong \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;{r_{\bar{x}\setminus\{y\}}}^{*} = \prod_{w \in W} {r'_{\bar{x}\setminus\{y\}}}^{*}\;]{} & \prod_{w \in W} \mathcal{P}(D^{\bar{x}\setminus\{y\}}) \cong \mathcal{P}(W \times D^{\bar{x}\setminus\{y\}})
\end{array}$$

Thus $\prod_{w \in W} [\![\, \bar{x} \mid \forall y \,]\!]_w$ serves as $[\![\, \bar{x} \mid \forall y \,]\!]$; hence $[\![\forall y]\!]$ preserves local determination for $\forall y$. Similarly for $[\![\exists x]\!]$.

For $[\![\Box]\!]$ to preserve local determination for $\Box$, for each $\bar{x}$ we need $[\![\, \bar{x} \mid \Box \,]\!]$ that makes

$$\begin{array}{ccc} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;{r_{\bar{x}}}^{*}\;} & \mathcal{P}(W \times D^{\bar{x}}) \\ {\scriptstyle [\![\Box]\!]}\downarrow & \rotatebox{45}{$\Leftarrow$} & \downarrow{\scriptstyle [\![\, \bar{x} \mid \Box \,]\!]} \\ \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;{r_{\bar{x}}}^{*}\;]{} & \mathcal{P}(W \times D^{\bar{x}}) \end{array}$$

commute. So, fixing $\bar{x}$, define $[\![\, \bar{x} \mid \Box \,]\!]$ so that, for each $(w,\beta) \in W \times D^{\bar{x}}$ and $B \subseteq W \times D^{\bar{x}}$,

$$(w,\beta) \in [\![\, \bar{x} \mid \Box \,]\!](B) \iff w \in \mathbf{int}_R(B_\beta).$$

This entails the equivalence marked with $*$ in

$$\begin{aligned} (w,\alpha) \in [\![\Box]\!]({r_{\bar{x}}}^{*}(B)) &\overset{\text{(IV.33)}}{\iff} w \in \mathbf{int}_R(({r_{\bar{x}}}^{*}(B))_\alpha) \\ &\overset{\text{(IV.42)}}{\iff} w \in \mathbf{int}_R(B_{\alpha\restriction\bar{x}}) \\ &\overset{*}{\iff} r_{\bar{x}}(w,\alpha) = (w,\alpha\restriction\bar{x}) \in [\![\, \bar{x} \mid \Box \,]\!](B) \\ &\iff (w,\alpha) \in {r_{\bar{x}}}^{*}([\![\, \bar{x} \mid \Box \,]\!](B)); \end{aligned}$$

hence the diagram above commutes. Therefore $[\![\Box]\!]$ preserves local determination for $\Box$. Similarly, $[\![\, \bar{x} \mid \Diamond \,]\!] : \mathcal{P}(W \times D^{\bar{x}}) \to \mathcal{P}(W \times D^{\bar{x}})$ such that

$$(w,\beta) \in [\![\, \bar{x} \mid \Diamond \,]\!](B) \iff w \in \mathbf{cl}_R(B_\beta).$$

lets $[\![\Diamond]\!]$ preserve local determination for $\Diamond$. $\Box$

### IV.2.3 Autonomy of Kripkean Domains of Quantification

In Subsections IV.2.1 and IV.2.2 we reformulated Kripke semantics in an operational form and discussed how to express the notion of DoQ-restrictability in this setting. In this subsection, we first observe that in Kripke semantics—in particular, under Kripke's truth conditions (IV.12) and (IV.13)—the interpretations of $\Box$ and $\Diamond$ are not DoQ-restrictable, which, as we saw in Subsection IV.1.2, explains why Kripke semantics cannot combine modal logic **K** with classical first-order. Then we propose revisions of (IV.12) and (IV.13) that are DoQ-restrictable, so that classical first-order logic can be combined with modal logic in the new revised semantics.

Let us first show how Kripke semantics fails to be DoQ-restrictable. On the one hand, the DoQ-restrictability of two-domain semantics straightforwardly extends to the the DoQ-restrictability of the classical-language part of Kripke semantics, as in:

**Fact 36.** For every Kripke interpretation $(\mathfrak{M}, \vDash)$ for a given quantified modal language $\mathcal{L}$, $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\rightarrow]\!]$, $[\![\forall x]\!]$, $[\![\exists x]\!]$ are DoQ-restrictable.

*Proof.* $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, $[\![\rightarrow]\!]$ are DoQ-restrictable because they are postcompositions. To show $[\![\forall x]\!]$ to be DoQ-restrictable, recall that, as stated in Fact 23 (on p. 110), $[\![\forall x]\!]_w$, for each $w \in W$, is DoQ-restrictable (in the two-domain sense of Definition 31 on p. 109), as in:

$$\begin{array}{ccc}
\mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\;[\![\forall x]\!]_w\;} & \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \\
{\scriptstyle i^*}\big\downarrow & = & \big\downarrow{\scriptstyle i^*} \\
\mathcal{P}({D_w}^{\mathrm{var}(\mathcal{L})}) & \xrightarrow[\;([\![\forall x]\!]_w)_{D_w}\;]{} & \mathcal{P}({D_w}^{\mathrm{var}(\mathcal{L})})
\end{array}$$

Hence the DoQ-restriction of $[\![\forall x]\!]$ is given by taking the product $\prod_{w \in W} ([\![\forall x]\!]_w)_{D_w}$ of all $([\![\forall x]\!]_w)_{D_w}$ as above, as in:

$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \cong \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{\;[\![\forall x]\!] = \prod_{w \in W} [\![\forall x]\!]_w\;} & \prod_{w \in W} \mathcal{P}(D^{\mathrm{var}(\mathcal{L})}) \cong \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \\
{\scriptstyle i^*}\big\downarrow & = & \big\downarrow{\scriptstyle i^*} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) \cong \prod_{w \in W} \mathcal{P}({D_w}^{\mathrm{var}(\mathcal{L})}) & \xrightarrow[\;\prod_{w \in W} ([\![\forall x]\!]_w)_{D_w}\;]{} & \prod_{w \in W} \mathcal{P}({D_w}^{\mathrm{var}(\mathcal{L})}) \cong \mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)
\end{array}$$

Similarly for $[\![\exists x]\!]$. $\Box$

The modal part of Kripke semantics, on the other hand, is not DoQ-restrictable.

**Fact 37.** For a Kripke interpretation $(\mathfrak{M}, \vDash)$ for a given quantified modal language $\mathcal{L}$, $[\![\Box]\!]$ and $[\![\Diamond]\!]$ are *not* in general DoQ-restrictable.

We prove this in terms of world-assignment pairs, but attach parenthesized "subtitles" translating the proof into more intuitive terms of world-individual pairs, as follows:

*Proof.* Consider the Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with domains such that

$$W = \{u, v\}, \qquad R = \{(u, v)\},$$
$$D_u = \{a, b\}, \text{ where } a \neq b, \qquad D_v = \{a\}.$$

Fix any Kripke model $\mathfrak{M}$ for $\mathcal{L}$ over $\mathfrak{F}$, and any Kripke interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ on $\mathfrak{M}$. Then, fixing a variable $x$ of $\mathcal{L}$, let

$$A = \{\, (v, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid \alpha(x) = b \,\} \subseteq W \times D^{\mathrm{var}(\mathcal{L})}.$$

(We may think of $A$ as the property that holds exactly of $b$ at $v$—or, in terms of world-individual pairs, that holds of $(v, b)$ but no other pairs.) It follows, since no $(v, \alpha) \in A$ is a DoQ-pair, that

$$i^*(A) = A \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} = \varnothing = \varnothing \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} = i^*(\varnothing)$$

for the inclusion map $i : \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} \hookrightarrow W \times D^{\mathrm{var}(\mathcal{L})}$. (Think of $\varnothing$ as the property that holds of no world-individual pairs; then, since $A$ holds only of the non-DoQ-pair $(v, b)$, the properties $A$ and $\varnothing$ are equivalent for the DoQ-pairs.)

Now pick any $\beta : \mathrm{var}(\mathcal{L}) \to D_u$ such that $\beta(x) = b$, which implies $(v, \beta) \in A$. Then we have:

- Since $v$ is the only $w \in W$ with $Ruw$, $(v, \beta) \in A$ implies $(u, \beta) \in [\![\Box]\!](A)$ by (IV.29). Hence $(u, \beta) \in i^*([\![\Box]\!](A))$, because $(u, \beta) \in \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}$.
- Since $Ruv$, $(v, \beta) \notin \varnothing$ implies $(u, \beta) \notin [\![\Box]\!](\varnothing)$ by (IV.29). Hence $(u, \beta) \notin i^*([\![\Box]\!](\varnothing))$.

(That is, the DoQ-pair $(u, b)$ satisfies the property "necessarily-$A$" but not "necessarily-$\varnothing$".) Thus $i^*([\![\Box]\!](A)) \neq i^*([\![\Box]\!](\varnothing))$, whereas $i^*(A) = i^*(\varnothing)$. (When restricted to the DoQ-pairs, the properties $A$ and $\varnothing$ are equivalent, but "necessarily-$A$" and "necessarily-$\varnothing$" are not.) Therefore there can be no operation $f : \mathcal{P}\left(\sum_{w \in W} D_w{}^{\mathrm{var}(\mathcal{L})}\right) \to \mathcal{P}\left(\sum_{w \in W} D_w{}^{\mathrm{var}(\mathcal{L})}\right)$ such that $f \circ i^* = i^* \circ [\![\Box]\!]$; this means that $[\![\Box]\!]$ is not DoQ-restrictable. (The difference between "necessarily-$A$" and "necessarily-$\varnothing$" in terms of DoQ-pairs hinges on the difference between $A$ and $\varnothing$ in terms of non-DoQ pairs.)

Similarly, $[\![\Diamond]\!]$ is not DoQ-restrictable, because we have

- Since $Ruv$, $(v, \beta) \in A$ implies $(u, \beta) \in [\![\Diamond]\!](A)$ by (IV.30). Hence $(u, \beta) \in i^*([\![\Diamond]\!](A))$, because $(u, \beta) \in \sum_{w \in W} D_w{}^{\mathrm{var}(\mathcal{L})}$.
- Since $v$ is the only $w \in W$ with $Ruw$, $(v, \beta) \notin \varnothing$ implies $(u, \beta) \notin [\![\Diamond]\!](\varnothing)$ by (IV.30). Hence $(u, \beta) \notin i^*([\![\Diamond]\!](\varnothing))$. □

This is how Kripke's truth conditions (IV.12) for $\Box$ and (IV.13) for $\Diamond$—or their operational versions (IV.29) and (IV.30)—prevent Kripke semantics from being DoQ-restrictable. The upshot of the proof is that, due to (IV.29) and (IV.30), whether or not $[\![\Box]\!](A)$ and $[\![\Diamond]\!](A)$ (or $[\![\Box]\!](\varnothing)$ and $[\![\Diamond]\!](\varnothing)$) contain the DoQ-pair $(u, \beta)$ depends on whether or not $A$ (or $\varnothing$) contains the *non*-DoQ pair $(v, \beta)$.

For the rest of this subsection, we pursue a DoQ-restrictable revision of Kripke semantics. The upshot just laid out of the proof above indicates that the reason (IV.29) and (IV.30) fail to provide $\Box$ and $\Diamond$ with DoQ-restrictable interpretations is their reference to non-DoQ pairs. More precisely, once we recall that Kripke's truth conditions (IV.12) and (IV.13) can be reformulated as

(IV.31) $$(w, \alpha) \in [\![\Box]\!](A) \iff \overrightarrow{R}(w) \subseteq A_\alpha,$$

(IV.32) $$(w, \alpha) \in [\![\Diamond]\!](A) \iff \overrightarrow{R}(w) \cap A_\alpha \neq \varnothing$$

for $\overrightarrow{R}(w) = \{\, u \in W \mid Rwu \,\}$, it is obvious that $\overrightarrow{R}(w)$ may contain $u \in W$ regardless of whether $(u, \alpha)$ is a DoQ-pair or not.

Let us observe this more conceptually. It helps to divide Kripke's semantic idea into the following two aspects:

(A) In determining the truth of $\Box\varphi$ (and $\Diamond\varphi$, respectively) at $w$ with respect to $\alpha$, we refer to some set $U$ of worlds, and consider whether $\varphi$ is true at every $u \in U$ (and some $u \in U$) with respect to $\alpha$.

(B) Then we take, as this set $U$ of "reference worlds", $\overrightarrow{R}(w)$, that is, the set of worlds accessible from $w$, independent of $\alpha$.

Then $U = \overrightarrow{R}(w)$ may contain $u$ such that $(u, \alpha)$ is not a DoQ-pair, in which case we make reference to that non-DoQ-pair $(u, \alpha)$ by considering whether $\varphi$ is true with respect to it. This observation seems to suggest that we should revise the second aspect of Kripke's idea by taking, as the set $U$ of reference worlds, a set that can never contain such $u$. So let us try taking

$$\overrightarrow{R}_\alpha(w) = \{\, u \in W \mid Rwu \text{ and } \alpha : \mathrm{var}(\mathcal{L}) \to D_u \,\}, \tag{IV.43}$$

so that $(u, \alpha)$ is always a DoQ-pair for every $u \in \overrightarrow{R}_\alpha(w)$, and using it as the set of reference worlds (for $w$ and $\alpha$); that is, we modify (IV.31) and (IV.32) with

$$(w, \alpha) \in [\![\Box]\!](A) \iff \overrightarrow{R}_\alpha(w) \subseteq A_\alpha, \tag{IV.44}$$

$$(w, \alpha) \in [\![\Diamond]\!](A) \iff \overrightarrow{R}_\alpha(w) \cap A_\alpha \neq \varnothing, \tag{IV.45}$$

so as to rule out the reference to non-DoQ-pairs. Then (IV.44) and (IV.45) certainly make $[\![\Box]\!]$ and $[\![\Diamond]\!]$ DoQ-restrictable (we omit a proof).

The conditions (IV.44) and (IV.45) cause, however, a serious problem: $[\![\Box]\!]$ and $[\![\Diamond]\!]$ that satisfy them do not in general preserve local determination. To see how this problem arises, fix a Kripke frame $\mathfrak{F} = (W, R, D_-)$ with domains and a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for a given language $\mathcal{L}$ over $\mathfrak{F}$, and note that a sentence $\varphi$ of $\mathcal{L}$ is locally determined in $(\mathfrak{M}, [\![-]\!])$ iff the following holds for every $\alpha, \beta : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha(x) = \beta(x)$ for all free variables $x$ in $\varphi$:

- $(w, \alpha) \in [\![\varphi]\!]$ iff $(w, \beta) \in [\![\varphi]\!]$ for all $w \in W$; that is,
- $[\![\varphi]\!]_\alpha = [\![\varphi]\!]_\beta$.

Then it is obvious that, under (IV.31) of Kripke semantics on one hand, $[\![\Box]\!]$ preserves local determination, because $[\![\varphi]\!]_\alpha = [\![\varphi]\!]_\beta$ entails the second equivalence below:

$$(w, \alpha) \in [\![\Box\varphi]\!] \overset{\text{(IV.31)}}{\iff} \overrightarrow{R}(w) \subseteq [\![\varphi]\!]_\alpha \iff \overrightarrow{R}(w) \subseteq [\![\varphi]\!]_\beta \overset{\text{(IV.31)}}{\iff} (w, \beta) \in [\![\Box\varphi]\!].$$

On the other hand, even if $\alpha$ and $\beta$ agree on the free variables in $\varphi$, they may not agree on other variables, and so, given $u \in W$, we may have $\alpha : \text{var}(\mathcal{L}) \to D_u$ while $\beta : \text{var}(\mathcal{L}) \not\to D_u$; therefore, generally, $\overrightarrow{R}_\alpha \neq \overrightarrow{R}_\beta$. It follows that the conditions that $[\![\varphi]\!]_\alpha = [\![\varphi]\!]_\beta$ and that $\alpha(x) = \beta(x)$ for every free variable $x$ in $\varphi$ fail under (IV.44) to imply that $(w, \alpha) \in [\![\Box\varphi]\!]$ iff $(w, \beta) \in [\![\Box\varphi]\!]$ because, generally,

$$(w, \alpha) \in [\![\Box\varphi]\!] \overset{\text{(IV.44)}}{\iff} \overrightarrow{R}_\alpha(w) \subseteq [\![\varphi]\!]_\alpha = [\![\varphi]\!]_\beta \not\iff \overrightarrow{R}_\beta(w) \subseteq [\![\varphi]\!]_\alpha = [\![\varphi]\!]_\beta \overset{\text{(IV.44)}}{\iff} (w, \beta) \in [\![\Box\varphi]\!].$$ [11]

The moral of our failed trial with (IV.43)–(IV.45) is as follows. In determining the truth of $\Box\varphi$ and $\Diamond\varphi$ with respect to a DoQ-pair $(w, \alpha)$, we tried taking $\overrightarrow{R}_\alpha(w)$ instead of $\overrightarrow{R}(w)$ as the set of reference worlds, because the reference to non-DoQ-pairs—the obstacle to DoQ-restrictability—was made in the latter but not in the former. We have just learned, however, that if $\alpha$ and $\beta$ agree on the free variables in $\varphi$ then we must give them the same set of reference worlds in order to preserve local determination. Our trial of defining the set $\overrightarrow{R}_\alpha(w)$ of reference worlds for $\alpha$ (and $w$) by (IV.43) fails to meet this demand—it allows $\overrightarrow{R}_\alpha(w) \neq \overrightarrow{R}_\beta(w)$ even if $\alpha$ and $\beta$ agree on all free variables in $\varphi$—since whether or not $u \in \overrightarrow{R}_\alpha(w)$ depends on whether or not $\alpha(x) \in D_u$ for all variables $x \in \text{var}(\mathcal{L})$, even including those that do not occur freely in $\varphi$.

This is a technical reason (in addition to the intuitive motivation we will spell out shortly) why, in determining the truth of $\Box\varphi$ and $\Diamond\varphi$ in which $\bar{x}$ are all and only the free variables, we should take the set of reference worlds not relative to a *global* assignment $\alpha : \text{var}(\mathcal{L}) \to D$ but relative to a *local* assignment $\beta : \bar{x} \to D$. Hence we propose that, instead of $\overrightarrow{R}_\alpha(w)$ for a *global* assignment $\alpha : \text{var}(\mathcal{L}) \to D$, we take, for a *local* assignment $\beta : \bar{x} \to D$,

(IV.46) $$\overrightarrow{R}_\beta(w) = \{\, u \in W \mid Rwu \text{ and } \beta : \bar{x} \to D_u \,\},$$

[11] For example, let $\mathfrak{F}$ be as in the proof for Fact 37, and, fixing $x \in \text{var}(\mathcal{L})$, assume $\alpha(y) = a$ for every $y \in \text{var}(\mathcal{L})$, $\beta(x) = a$ and $\beta(y) = b$ for some $y \in \text{var}(\mathcal{L})$. Then $\alpha : \text{var}(\mathcal{L}) \to D_v$ and hence $\overrightarrow{R}_\alpha(u) = \{v\}$, while $\beta : \text{var}(\mathcal{L}) \not\to D_v$ entails $\overrightarrow{R}_\beta(u) = \varnothing$; thus $\overrightarrow{R}_\alpha(u) \nsubseteq \varnothing$ whereas $\overrightarrow{R}_\beta(u) \subseteq \varnothing$. Now, for a unary primitive predicate $F$ of $\mathcal{L}$, assume $F^{\mathfrak{M}} = \varnothing$, which implies $[\![Fx]\!]_\alpha = [\![Fx]\!]_\beta = \varnothing$. Then (IV.44) implies that $(u, \alpha) \notin [\![\Box Fx]\!]$ while $(u, \beta) \in [\![\Box Fx]\!]$, even though $\alpha$ and $\beta$ agree on the free variable $x$ in $\Box Fx$, thereby violating local determination.

and then modify (IV.44) and (IV.45) so that, for every $(w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})}$,[12]

(IV.47) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$(w, \alpha) \in [\![\Box\varphi]\!] \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \subseteq [\![\varphi]\!]_\alpha,$$

(IV.48) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$(w, \alpha) \in [\![\Diamond\varphi]\!] \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \cap [\![\varphi]\!]_\alpha \neq \varnothing.$$

In terms of satisfaction relations, we can write these conditions as follows:

(IV.49) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi \text{ for every } u \text{ such that } Rwu \text{ and } \alpha\restriction\bar{x} : \bar{x} \to D_u,$$

(IV.50) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$\mathfrak{M}, w \vDash_\alpha \Diamond\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi \text{ for some } u \text{ such that } Rwu \text{ and } \alpha\restriction\bar{x} : \bar{x} \to D_u.$$

It is worth noting that (IV.49) and (IV.50) (or (IV.47) and (IV.48)) coincide with Kripke's (IV.12) and (IV.13) (or (IV.29) and (IV.30)), respectively, for closed $\varphi$, that is, if the set of free variables in $\varphi$ is $\bar{x} = \varnothing$, because $\alpha\restriction\varnothing : \varnothing \to D_u$ is trivially the case for any $\alpha$ and $D_u$.

Let us give intuitive readings to these conditions, so that a conceptual reason for our proposing them is clearer. We can read (IV.49) intuitively as follows: When $\bar{x}$ are all and only the free variables in $\varphi$,

- $\Box\varphi$ is true of $\alpha(x_1), \ldots, \alpha(x_n)$ at $w$, iff
- $\varphi$ is true of $\alpha(x_1), \ldots, \alpha(x_n)$ at every $u \in W$ such that $Rwu$ and in which $\alpha(x_1), \ldots, \alpha(x_n)$ all exist.

Similarly, we can read (IV.50) intuitively as follows: When $\bar{x}$ are all and only the free variables in $\varphi$,

- $\Diamond\varphi$ is true of $\alpha(x_1), \ldots, \alpha(x_n)$ at $w$, iff
- $\varphi$ is true of $\alpha(x_1), \ldots, \alpha(x_n)$ at some $u \in W$ such that $Rwu$ and in which $\alpha(x_1), \ldots, \alpha(x_n)$ all exist.

Or, to put these even more intuitively,

[12] Rather than general subsets $A$ and $[\![\Box]\!](A)$, $[\![\Diamond]\!](A)$, we focus on interpretations $[\![\varphi]\!]$, $[\![\Box\varphi]\!]$, $[\![\Diamond\varphi]\!]$ of sentences $\varphi$, $\Box\varphi$, $\Diamond\varphi$, partly because we need to fix the free variables $\bar{x}$ in $\varphi$, but also because of technical issues involved with the general case of general subsets $A$. We will discuss these issues shortly in Subsection IV.3.1.

- An $n$-tuple $\bar{a}$ necessarily satisfies an $n$-ary property $\varphi$ iff $\bar{a}$ satisfies $\varphi$ in every accessible world where $\bar{a}$ exists (rather than just every accessible world);
- An $n$-tuple $\bar{a}$ possibly satisfies an $n$-ary property $\varphi$ iff $\bar{a}$ satisfies $\varphi$ in some accessible world where $\bar{a}$ exists (rather than just some accessible world).

As is obvious from these readings, the key idea of our modification with (IV.46)–(IV.50) is that, in distinguishing DoQ-pairs from non-DoQ-pairs (in order to rule out the reference to non-DoQ-pairs from truth conditions, and thereby to attain DoQ-restrictability), we refer to world-*tuple* DoQ-pairs rather than world-*assignment* DoQ-pairs. This idea is in accord with the conceptual motivation behind local determination that sentences should be true (or not) of tuples of individuals rather than (global) assignments. In determining the truth of $\Box\varphi$ or $\Diamond\varphi$ in which $\bar{x}$ are all and only the free variables, the only semantically relevant aspect of an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$ should be its values on $\bar{x}$. Accordingly, even though $\alpha$ may fail to be a $u$-DoQ-assignment due to the fact that $\alpha(y) \notin D_u$, this fact should be relevant to the truth of $\Box\varphi$ or $\Diamond\varphi$ only if $y$ occurs freely in $\varphi$.

So we enter:

**Definition 50.** Given a quantified modal language $\mathcal{L}$, a Kripke-type satisfaction relation for $\mathcal{L}$ is called a *DoQ-autonomous Kripkean satisfaction relation* for $\mathcal{L}$ if it satisfies (IV.8)–(IV.11), (IV.14)–(IV.17), (IV.49) and (IV.50). Moreover, by the *DoQ-autonomous Kripkean semantics* for $\mathcal{L}$, we mean the class of all DoQ-autonomous Kripkean satisfaction relations for $\mathcal{L}$.

The adjective "DoQ-autonomous" means that domains of quantification for such relations are autonomous, whereas "Kripkean" indicates that the semantic idea behind such relations is based on, though not quite the same as, Kripke's idea—in particular, we keep the aspect (A) of his idea, while revising (B) (see p. 154). This semantics is indeed DoQ-restrictable, in a certain sense; but our technical definition of DoQ-restrictability involves formulating the semantics operationally, which is, however, not straightforward, and requires modification of the definition of DoQ-restrictability. We will lay out an operational formulation of the semantics, as well as the definition and proof of its DoQ-restrictability, in Section IV.3.

The following facts can be proven similarly to Facts 30 and 31.

**Fact 38.** Given a quantified modal language $\mathcal{L}$ (the only non-classical operators of which are $\Box$ and $\Diamond$), every Kripke model for $\mathcal{L}$ has a unique DoQ-autonomous Kripkean satisfaction relation on it.

**Fact 39.** Given a quantified modal language $\mathcal{L}$, every DoQ-autonomous Kripkean satisfaction relation is locally determined, SoS, and AC (see Definition 41).

### IV.2.4 Autonomy of Domains and Converse Barcan Formula

In Subsection IV.1.2, we made a remark that the Kripke-validity of the converse Barcan formula in a Kripke frame hinges, not on the increase of its domain as maintained as a near-orthodoxy by modal logicians, but on the semantic insignificance of non-existent individuals, that is, the autonomy of domains of quantification. Now that we have given a DoQ-autonomous revision of Kripke semantics, we are finally prepared to precisely state this near-heterodoxy.

We should first note that, although Kripke semantics in general is not DoQ-restrictable, Kripke semantics *with increasing domains* is DoQ-restrictable, by Fact 36 and the following.

**Fact 40.** Given any quantified modal language $\mathcal{L}$, suppose a Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with a domain $D$ of possible individuals has an increasing domain. Then, if a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ over $\mathfrak{F}$ interprets $\Box$ and $\Diamond$ with $[\![\Box]\!]$ and $[\![\Diamond]\!]$ satisfying (IV.31) and (IV.32), respectively, then $[\![\Box]\!]$ and $[\![\Diamond]\!]$ are DoQ-restrictable.

To prove this, it is easy but useful to observe the following. First, for every $\alpha : \mathrm{var}(\mathcal{L}) \to D$ and $A, B \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$, we have

$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha, \tag{IV.51}$$

because

$$w \in (A \cap B)_\alpha \iff (w, \alpha) \in A \cap B \iff (w, \alpha) \in A, B \iff w \in A_\alpha, B_\alpha.$$

Also, observe that if the domain is increasing then, for every $(u, \alpha) \in \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}$, we have

$$\overrightarrow{R}(u) \subseteq \left( \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} \right)_\alpha ; \tag{IV.52}$$

this is because that the domain is increasing implies that if $\alpha : \mathrm{var}(\mathcal{L}) \to D_u$ and $Ruv$ then $\alpha(x) \in D_u \subseteq D_v$ for all $x \in \mathrm{var}(\mathcal{L})$, and hence that, for every $(u, \alpha) \in \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$,

$$v \in \overrightarrow{R}(u) \iff Ruv \implies \alpha : \mathrm{var}(\mathcal{L}) \to D_v \iff (v, \alpha) \in \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})} \iff v \in \left(\sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)_\alpha .$$

Using these observation, we give:

*Proof for Fact 40.* That $[\![\Box]\!]$ is DoQ-restrictable means that there is $[\![\Box]\!]_{\mathrm{DoQ}}$ making

$$\begin{array}{ccc} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xrightarrow{[\![\Box]\!]} & \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \\ {\scriptstyle i^*}\downarrow & /\!/ & \downarrow{\scriptstyle i^*} \\ \mathcal{P}\left(\sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) & \xrightarrow[{[\![\Box]\!]_{\mathrm{DoQ}}}]{} & \mathcal{P}\left(\sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) \end{array}$$

commute. So, let us define $[\![\Box]\!]_{\mathrm{DoQ}}$ so that, for each $(u, \alpha) \in \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$ and $A \subseteq \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$,

(IV.53) $$(u, \alpha) \in [\![\Box]\!]_{\mathrm{DoQ}}(A) \iff \overrightarrow{R}(u) \subseteq A_\alpha.$$

Then, for every $(u, \alpha) = i(u, \alpha) \in \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$ and $B \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$, we have

$$\begin{aligned} (u, \alpha) \in i^*([\![\Box]\!](B)) &\iff (u, \alpha) = i(u, \alpha) \in [\![\Box]\!](B) \\ &\overset{\text{(IV.31)}}{\iff} \overrightarrow{R}(u) \subseteq B_\alpha \\ &\overset{\text{(IV.51)}}{\iff} \overrightarrow{R}(u) \subseteq B_\alpha \cap \left(\sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)_\alpha \\ &\overset{\text{(IV.52)}}{\iff} \overrightarrow{R}(u) \subseteq \left(B \cap \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right)_\alpha \\ &\overset{\text{(IV.53)}}{\iff} (u, \alpha) \in [\![\Box]\!]_{\mathrm{DoQ}}\left(B \cap \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) = [\![\Box]\!]_{\mathrm{DoQ}}(i^*(B)); \end{aligned}$$

hence $i^* \circ [\![\Box]\!] = [\![\Box]\!]_{\mathrm{DoQ}} \circ i^*$, making the diagram above commute. Thus $[\![\Box]\!]$ is DoQ-restrictable. Similarly, $[\![\Diamond]\!]$ is DoQ-restrictable, with $[\![\Diamond]\!]_{\mathrm{DoQ}}$ such that

(IV.54) $$(u, \alpha) \in [\![\Diamond]\!]_{\mathrm{DoQ}}(A) \iff \overrightarrow{R}(u) \cap A_\alpha \neq \varnothing$$

for every $(u, \alpha) \in \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$ and $A \subseteq \sum_{w\in W} {D_w}^{\mathrm{var}(\mathcal{L})}$. □

**Corollary 6.** Given a quantified modal language $\mathcal{L}$, Kripke semantics for $\mathcal{L}$ with increasing domains is DoQ-restrictable.

Next we observe that, when a domain is increasing, Kripke's truth conditions (IV.12) and (IV.13) for $\Box$ and $\Diamond$ coincide with their DoQ-autonomous revisions (IV.49) and (IV.50) "up to DoQ-pairs". Indeed, this coincidence characterizes the increase of a domain, in the following sense.

**Fact 41.** Let $\mathfrak{F} = (W, R, D_{-})$ be a Kripke frame with a domain $D$ of possible individuals. Then the following are equivalent:

(i) $\mathfrak{F}$ has an increasing domain.

(ii) Given any quantified modal language $\mathcal{L}$, for every $w \in W$, $\alpha : \mathrm{var}(\mathcal{L}) \to D$, and finite set $\bar{x}$ of variables of $\mathcal{L}$ such that $\alpha \restriction \bar{x} : \bar{x} \to D_w$,

$$\overrightarrow{R}(w) = \overrightarrow{R}_{\alpha \restriction \bar{x}}(w).$$

(iii) Given any quantified modal language $\mathcal{L}$, fix a sentence $\varphi$ of $\mathcal{L}$ in which $\bar{x}$ are all and only the free variables. Then, in every Kripke-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ over $\mathfrak{F}$, every pair of $w \in W$ and $\alpha : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha \restriction \bar{x} : \bar{x} \to D_w$ satisfies (IV.12) iff (IV.49), and (IV.13) iff (IV.50).

*Proof.* Suppose (i) and fix any $\mathcal{L}$, $w$, $\alpha$, $\bar{x}$ as in (ii). Then, for any $u \in W$, $Rwu$ entails $\alpha(x_i) \in D_w \subseteq D_u$ for each $i \leqslant n$ (by (i)) and hence $\alpha \restriction \bar{x} : \bar{x} \to D_u$. That is,

$$u \in \overrightarrow{R}(w) \iff Rwu \iff Rwu \text{ and } \alpha \restriction \bar{x} : \bar{x} \to D_u \iff u \in \overrightarrow{R}_{\alpha \restriction \bar{x}}(w)$$

for every $u \in W$. Thus (ii) follows from (i).

Suppose (ii) and fix any $\mathcal{L}$, $\varphi$, $\bar{x}$, $(\mathfrak{M}, \vDash)$, $w$, $\alpha$ as in (iii). Then (ii) implies

$$Rwu \iff Rwu \text{ and } \alpha \restriction \bar{x} : \bar{x} \to D_u$$

for any $u \in W$, and hence immediately implies that (IV.12) iff (IV.49):

(IV.12) $\quad \mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi$ for every $u \in W$ such that $Rwu$;

(IV.49) $\quad$ (If $\bar{x}$ are all and only the free variables in $\varphi$, then)

$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi \text{ for every } u \in W \text{ such that } Rwu \text{ and } \alpha \restriction \bar{x} : \bar{x} \to D_u.$$

Similarly (IV.13) iff (IV.50). Thus (ii) entails (iii).

To show that (iii) entails (i), suppose (i) is not the case; this means that $Rwu$ and $a \notin D_u$ for some $w, u \in W$ and $a \in D_w$. Fixing a unary primitive predicate $F$ of $\mathcal{L}$, pick a DoQ-autonomous Kripkean satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ over $\mathfrak{F}$ that interprets $F$ as a kind of "existence predicate", that is, with the set of world-individual DoQ-pairs, so that

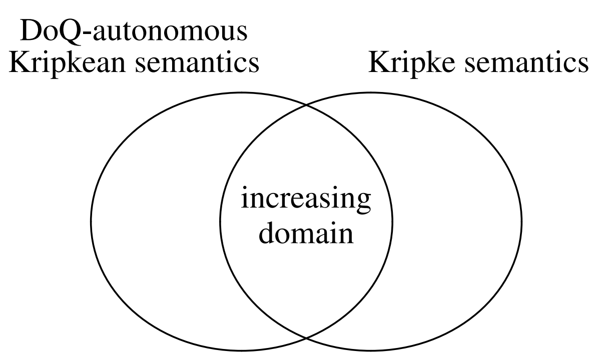
$$(v, b) \in F^{\mathfrak{M}} \iff b \in D_v.$$ [13]

$(\mathfrak{M}, \vDash)$ satisfies (IV.17) and (IV.49) by definition. Now fix $x \in \mathrm{var}(\mathcal{L})$ and pick any $\alpha : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha(x) = a$. Then, for every $v \in W$,

$$Rwv \text{ and } \alpha\restriction\{x\} : \{x\} \to D_v \implies \alpha(x) \in D_v \implies (v, \alpha(x)) \in F^{\mathfrak{M}} \overset{\text{(IV.17)}}{\implies} \mathfrak{M}, v \vDash_\alpha Fx;$$

this means $\mathfrak{M}, w \vDash_\alpha \Box Fx$ by (IV.49). On the other hand, $\alpha(x) = a \notin D_u$ implies $(u, \alpha(x)) \notin F^{\mathfrak{M}}$ and hence $\mathfrak{M}, u \nvDash_\alpha Fx$ by (IV.17), even though $Rwu$. Therefore $(w, \alpha)$, which has $\alpha\restriction\{x\} : \{x\} \to D_w$ since $\alpha(x) = a \in D_w$, does not satisfy (IV.12) for $\varphi = Fx$, whereas it does (IV.49). Thus (iii) is not the case. □

Fact 41 means that the Kripke frames with increasing domains are exactly the intersection, up to DoQ-pairs, of the DoQ-autonomous Kripkean semantics and Kripke semantics, and then Corollary 6 justifies identifying Kripke interpretations with increasing domains with DoQ-autonomous Kripkean interpretations with increasing domains. The situation can be illustrated by the following "class diagram":




[13] Kripke discusses this interpretation of an existence predicate on p. 70 of [19].

On the Basis of this observation, we can express our near-heterodoxy—that is, that the validity of the converse Barcan formula derives not from the increase of domains but from the autonomy of domains of quantification—by the fact that, as we will show as Fact 42 below, the converse Barcan formula is valid in the DoQ-autonomous Kripkean semantics (the left circle above) and not just in the semantics with increasing domains (the intersection).

**Fact 42.** Given any quantified modal language $\mathcal{L}$, its converse Barcan formula $\Box\forall y\,.\,\varphi \to \forall y\,\Box\varphi$ is AA-valid (and hence DoQ-valid) in DoQ-autonomous Kripkean semantics for $\mathcal{L}$.

*Proof.* Fix any DoQ-autonomous Kripkean satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ over a Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with a domain $D$ of possible individuals, any $w \in W$, and any $\alpha : \mathrm{var}(\mathcal{L}) \to D$, and suppose $\mathfrak{M}, w \vDash_{\alpha} \Box\forall y\,.\,\varphi$. Then this implies, when we write $\bar{x}$ for the set of free variables in $\forall y\,.\,\varphi$ (which implies $y \notin \bar{x}$), that

(IV.55) $\mathfrak{M}, u \vDash_{[a/y]\alpha} \varphi$ for every $u \in W$ such that $Rwu$ and $\alpha{\restriction}\bar{x} : \bar{x} \to D_u$ and every $a \in D_u$,

by the following chain of equivalences:

$$\mathfrak{M}, w \vDash_{\alpha} \Box\forall y\,.\,\varphi \overset{\text{(IV.49)}}{\iff} \mathfrak{M}, u \vDash_{\alpha} \forall y\,.\,\varphi \text{ for every } u \in W \text{ such that } Rwu \text{ and } \alpha{\restriction}\bar{x} : \bar{x} \to D_u \overset{\text{(IV.14)}}{\iff} \text{(IV.55)}.$$

Fix any $a \in D_w$; we want to show $\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi$. We have two cases depending on whether $y$ occurs freely in $\varphi$.

- Suppose $y$ occurs freely in $\varphi$; this means that $\bar{x} \cup \{y\}$ is the set of free variables in $\varphi$. Fix any $u \in W$ such that $Rwu$ and moreover

  (IV.56) $$([a/y]\alpha){\restriction}(\bar{x} \cup \{y\}) : (\bar{x} \cup \{y\}) \to D_u.$$

  This implies $a = ([a/y]\alpha)(y) \in D_u$ as well as that

  $$\alpha{\restriction}\bar{x} = (([a/y]\alpha){\restriction}(\bar{x} \cup \{y\})){\restriction}\bar{x} : \bar{x} \to D_u.$$

  Hence (IV.55) implies $\mathfrak{M}, u \vDash_{[a/y]\alpha} \varphi$. Therefore we have $\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi$ by (IV.49), as in:

  $$\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi \overset{\text{(IV.49)}}{\iff} \mathfrak{M}, u \vDash_{[a/y]\alpha} \varphi \text{ for every } u \in W \text{ such that } Rwu \text{ and (IV.56)}.$$

- Suppose $y$ does not occur freely in $\varphi$; this means that $\bar{x}$ is the set of free variables in $\varphi$. Fix any $u \in W$ such that $Rwu$ and $([a/y]\alpha)\restriction\bar{x} : \bar{x} \to D_u$, which implies $\alpha\restriction\bar{x} : \bar{x} \to D_u$ since $y \notin \bar{x}$. Also pick any $b \in D_u \neq \emptyset$. Then (IV.55) entails $\mathfrak{M}, u \vDash_{[b/y]\alpha} \varphi$. This implies by local determination of $\varphi$ (Fact 39) that $\mathfrak{M}, u \vDash_{[a/y]\alpha} \varphi$. Therefore we have $\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi$ by (IV.49), as in:

$$\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi \overset{\text{(IV.49)}}{\iff} \mathfrak{M}, u \vDash_{[a/y]\alpha} \varphi \text{ for every } u \in W \text{ such that } Rwu \text{ and } \alpha\restriction\bar{x} : \bar{x} \to D_u.$$

Thus $\mathfrak{M}, w \vDash_{[a/y]\alpha} \Box\varphi$ for every $a \in D_w$, and hence $\mathfrak{M}, w \vDash_{\alpha} \forall y\, \Box\varphi$ by (IV.14). Therefore (IV.11) implies $\mathfrak{M}, w \vDash_{\alpha} \Box\forall y\,.\,\varphi \to \forall y\, \Box\varphi$. □

In this sense, the converse Barcan formula corresponds to the autonomy of domains of quantification rather than to the increase of domains.

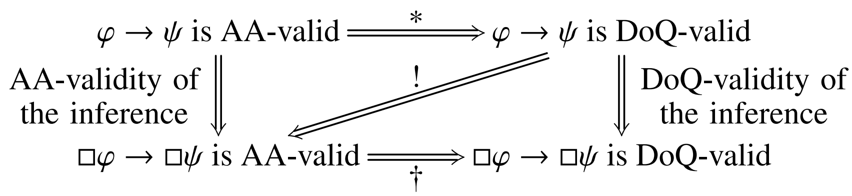
Whereas we have just shown the validity of the converse Barcan formula in a purely semantic manner, it is instructive—with regard to the discussion at the end of Subsection IV.1.2—to show it in a more axiomatic manner. That is, although the following Corollary 7 is an immediate consequence of Fact 42, we can also prove it by virtue of the DoQ-validity of (a weaker version of) N, as in Fact 43 below, in combination with the soundness of classical quantified logic with respect to DoQ-validity in DoQ-autonomous Kripkean semantics.

**Corollary 7.** Given any quantified modal language $\mathcal{L}$, its converse Barcan formula $\Box\forall x\,.\,\varphi \to \forall x\, \Box\varphi$ is DoQ-valid in DoQ-autonomous Kripkean semantics for $\mathcal{L}$.

**Fact 43.** Given any quantified modal language $\mathcal{L}$, the following inference is both AA- and DoQ-valid in DoQ-autonomous Kripkean semantics for $\mathcal{L}$, when every free variable in $\varphi$ occurs freely in $\psi$ as well:

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ \text{(every free variable in } \varphi \text{ occurs freely in } \psi \text{ as well)}$$

*Proof.* Fix any sentences $\varphi$, $\psi$ of $\mathcal{L}$ such that $\bar{x} \subseteq \bar{y}$ for the sets $\bar{x}$, $\bar{y}$ of free variables in $\varphi$, $\psi$, respectively. It is enough to show the entailment marked with ! below, since those marked with $*$ and † are trivial (then $! \circ *$ and $\dagger \circ !$ mean the AA-validity and DoQ-validity of the inference).




So, fixing any DoQ-autonomous Kripkean satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ over a Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with a domain $D$ of possible individuals, suppose $\varphi \to \psi$ is DoQ-valid in $(\mathfrak{M}, \vDash)$. Then fix any $w \in W$ and $\alpha : \mathrm{var}(\mathcal{L}) \to D$ and suppose $\mathfrak{M}, w \vDash_\alpha \Box\varphi$. To show $\mathfrak{M}, w \vDash_\alpha \Box\psi$, fix any $u \in W$ such that $Rwu$ and $\alpha{\upharpoonright}\bar{y} : \bar{y} \to D_u$. This implies $\alpha{\upharpoonright}\bar{x} : \bar{x} \to D_u$ by $\bar{x} \subseteq \bar{y}$. Hence $\mathfrak{M}, w \vDash_\alpha \Box\varphi$ and (IV.49) entail $\mathfrak{M}, u \vDash_\alpha \varphi$. Now, $\alpha{\upharpoonright}\bar{y} : \bar{y} \to D_u$ can be extended to some $\beta : \mathrm{var}(\mathcal{L}) \to D_u$, so that $\alpha{\upharpoonright}\bar{y} = \beta{\upharpoonright}\bar{y}$. Then $\mathfrak{M}, u \vDash_\alpha \varphi$ implies $\mathfrak{M}, u \vDash_\beta \varphi$ by local determination of $\varphi$ (Fact 39), since $\alpha{\upharpoonright}\bar{x} = \beta{\upharpoonright}\bar{x}$. Also, DoQ-validity of $\varphi \to \psi$ entails $\mathfrak{M}, u \vDash_\beta \varphi \to \psi$. Therefore $\mathfrak{M}, u \vDash_\beta \psi$ by (IV.11). Then this implies $\mathfrak{M}, u \vDash_\alpha \psi$ by local determination and $\alpha{\upharpoonright}\bar{y} = \beta{\upharpoonright}\bar{y}$. Thus $\mathfrak{M}, w \vDash_\alpha \Box\psi$ by (IV.49). In this way, by (IV.11), every $w \in W$ and $\alpha : \mathrm{var}(\mathcal{L}) \to D$ satisfy $\mathfrak{M}, w \vDash_\alpha \Box\varphi \to \Box\psi$. □

*Alternative proof for Corollary 7.* By soundness of classical quantified logic with respect to DoQ-validity in DoQ-autonomous Kripkean semantics and by Fact 43,

$$\dfrac{\dfrac{\forall x.\varphi \vdash \varphi}{\Box\forall x.\varphi \vdash \Box\varphi}\ \text{(every free variable in } \forall x.\varphi \text{ occurs freely in } \varphi \text{ as well)}}{\Box\forall x.\varphi \vdash \forall x\,\Box\varphi}$$

proves the converse Barcan formula. □

## IV.3 OPERATIONAL FORM OF KRIPKEAN SEMANTICS: A SECOND STEP

In Subsection IV.2.3 we laid out a DoQ-autonomous revision of Kripke semantics with (IV.46)–(IV.50); but we are yet to give a proof for its DoQ-autonomy. And, even though we formulated our revision in terms of satisfaction relations, our technical definition of DoQ-autonomy involves an operational formulation. In operationally formulating (IV.46)–(IV.50), however, we face three technical issues and their solution requires a revision of the operational formulation we gave in Subsection IV.2.1.

### IV.3.1 Free-Variable-Sensitive Interpretation of Operators

The largest issue concerning (IV.46)–(IV.50) is that, from (IV.46)–(IV.48), we cannot uniquely define operations $[\![\Box]\!]$ and $[\![\Diamond]\!]$ interpreting $\Box$ and $\Diamond$. We may well have $[\![\varphi_0]\!] = [\![\varphi_1]\!]$ for sentences $\varphi_0$ and $\varphi_1$ with different sets of free variables; say, $\bar{x}$ are the ones in $\varphi_0$, whereas $\bar{y}$ are the ones in $\varphi_1$. Then we generally have $R_{\alpha\restriction\bar{x}}(w) \neq R_{\alpha\restriction\bar{y}}(w)$, and moreover

$$
\begin{aligned}
(w, \alpha) \in [\![\Box\varphi_0]\!] &\overset{\text{(IV.47)}}{\iff} R_{\alpha\restriction\bar{x}}(w) \subseteq [\![\varphi_0]\!]_\alpha = [\![\varphi_1]\!]_\alpha \\
&\quad\not\Leftrightarrow\ R_{\alpha\restriction\bar{y}}(w) \subseteq [\![\varphi_0]\!]_\alpha = [\![\varphi_1]\!]_\alpha \overset{\text{(IV.47)}}{\iff} (w, \alpha) \in [\![\Box\varphi_1]\!]
\end{aligned}
$$

by (IV.47).[14] Thus, $[\![\varphi_0]\!] = [\![\varphi_1]\!]$ does not entail $[\![\Box\varphi_0]\!] = [\![\Box\varphi_1]\!]$, even though it would if there were an operation $[\![\Box]\!]$ such that $[\![\Box]\!][\![\psi]\!] = [\![\Box\psi]\!]$ for all sentences $\psi$. In other words, there can be no operation $[\![\Box]\!]$ that satisfies both $[\![\Box]\!][\![\psi]\!] = [\![\Box\psi]\!]$ (for all $\psi$) and (IV.47); similarly, no operation $[\![\Diamond]\!]$ can satisfy both $[\![\Diamond]\!][\![\psi]\!] = [\![\Diamond\psi]\!]$ (for all $\psi$) and (IV.48).

As this observation shows, the adoption of (IV.46)–(IV.48) requires us to give up having an operation $[\![\Box]\!]$ such that $[\![\Box]\!][\![\psi]\!] = [\![\Box\psi]\!]$ or $[\![\Diamond]\!]$ such that $[\![\Diamond]\!][\![\psi]\!] = [\![\Diamond\psi]\!]$. Instead, even when sentences $\varphi_0$ and $\varphi_1$ are interpreted by the same set $[\![\varphi_0]\!] = [\![\varphi_1]\!]$, as long as they have different sets

[14] To construct an example, let $\mathfrak{F}$ be as in the proof for Fact 37, and, fixing a variable $x \in \mathrm{var}(\mathcal{L})$, pick an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha(x) = b$. It follows that $\alpha\restriction\{x\} : \{x\} \not\to D_v$ and hence $\overrightarrow{R}_{\alpha\restriction\{x\}}(u) = \varnothing$, whereas $\overrightarrow{R}(u) = \{v\}$. Now take a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ that has $F^{\mathfrak{M}} = \{(u, a), (u, b)\}$ for a unary primitive predicate $F$ of $\mathcal{L}$. Then it is straightforward to see that

$$[\![Fx]\!] = [\![\forall x\,.\,Fx]\!] = \{u\} \times D^{\mathrm{var}(\mathcal{L})},$$

which implies $[\![Fx]\!]_\alpha = [\![\forall x\,.\,Fx]\!]_\alpha = \{u\}$, and hence $\overrightarrow{R}_{\alpha\restriction\{x\}}(u) \subseteq [\![Fx]\!]_\alpha$, whereas $\overrightarrow{R}(u) \not\subseteq [\![\forall x\,.\,Fx]\!]_\alpha$. Therefore (IV.47) implies $(u, \alpha) \in [\![\Box Fx]\!]$ and $(u, \alpha) \notin [\![\Box\forall x\,.\,Fx]\!]$; thus $[\![\Box Fx]\!] \neq [\![\Box\forall x\,.\,Fx]\!]$, even though $[\![Fx]\!] = [\![\forall x\,.\,Fx]\!]$.

of free variables we need to give different interpretations to the application of $\Box$ (or $\Diamond$) to $\varphi_0$ and that to $\varphi_1$. To implement this idea formally, we have two options:

(i) Modifying the syntax: Instead of a single operator $\Box$, the language has different $\Box^{\bar{x}}$ for each (finite) set $\bar{x}$ of variables. Regarding this infinite family of $\Box^{\bar{x}}$, the language has the unconventional rule of grammar that $\Box^{\bar{x}}$ can be applied to a sentence $\varphi$ if and only if $\bar{x}$ is the set of free variables in $\varphi$. We regard $\Box\varphi$ as a short-hand notation for $\Box^{\bar{x}}\varphi$, for the set $\bar{x}$ of free variables in $\varphi$. And semantics is modified accordingly: Different operators $\Box^{\bar{x}}$ are interpreted by (possibly) different operations $[\![\Box^{\bar{x}}]\!]$. Yet the condition is retained that $[\![\Box^{\bar{x}}\varphi]\!] = [\![\Box^{\bar{x}}]\!][\![\varphi]\!]$ whenever $\Box^{\bar{x}}\varphi$ is a "well-formed" sentence.

(ii) Keeping the syntax intact: There is no change to the language and its grammar. On the other hand, the semantic condition $[\![\Box\varphi]\!] = [\![\Box]\!][\![\varphi]\!]$ is dropped. Instead, $\Box$ is interpreted by the family of operators $[\![\Box]\!]^{\bar{x}}$ for all (finite) sets $\bar{x}$ of variables, with the new rule that $[\![\Box\varphi]\!] = [\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ for the set $\bar{x}$ of free variables in $\varphi$.

In either option, for each sentence $\varphi$ there are $[\![\Box^{\bar{x}}]\!][\![\varphi]\!]$, $[\![\Box^{\bar{y}}]\!][\![\varphi]\!]$, $\ldots$, or $[\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$, $[\![\Box]\!]^{\bar{y}}[\![\varphi]\!]$, $\ldots$, but only one of them—namely, the one indexed with the set of free variables in $\varphi$—is $[\![\Box\varphi]\!]$. Even if $[\![\varphi_0]\!] = [\![\varphi_1]\!]$, $[\![\Box\varphi_0]\!]$ and $[\![\Box\varphi_1]\!]$ can be the values on $[\![\varphi_0]\!] = [\![\varphi_1]\!]$ of different operations, namely $[\![\Box^{\bar{x}}]\!]$ and $[\![\Box^{\bar{y}}]\!]$, or $[\![\Box]\!]^{\bar{x}}$ and $[\![\Box]\!]^{\bar{y}}$, for the sets $\bar{x}$ and $\bar{y}$ of free variables in $\varphi_0$ and $\varphi_1$, respectively; this is how the technical issue described above for (IV.46)–(IV.48) is resolved.

While both options (i) and (ii) work equally well regarding the issue above, we need to decide which of them to adopt. Here we opt for (ii), simply because it seems to require a smaller change. Semantically, either in (i) or in (ii) we have to admit an infinite family of operations, $[\![\Box^{\bar{x}}]\!]$ or $[\![\Box]\!]^{\bar{x}}$, for all finite sets $\bar{x}$ of variables. Yet what we give up according to (i)—the grammatical condition that $\Box^{\bar{x}}\varphi$ is always "well-formed" for a sentence $\varphi$ and a unary sentential operator $\Box^{\bar{x}}$—seems much more serious than what we give up according to (ii)—the semantic condition that $[\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ always interprets the sentence $\Box\varphi$, no matter what set of free variables is in $\varphi$.

Therefore, while we do not need to generalize Definition 37 of a Kripke-type satisfaction relation $(\mathfrak{M}, \vDash)$, we generalize Definition 44 of a Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ along the option (ii) as follows. The point of generalization is that a sentential operator $\otimes$ is no longer interpreted by a single operation $[\![\otimes]\!]$ but instead by a family of operations $[\![\otimes]\!]^{\bar{x}}$ for all finite sets $\bar{x}$

of variables.

**Definition 51.** Given a quantified modal language $\mathcal{L}$, a *general Kripke-type interpretation for* $\mathcal{L}$ is a pair of a Kripke model $\mathfrak{M}$ for $\mathcal{L}$ and a map $[\![-]\!]$ that assigns,

- to each variable $x$, a map

$$[\![x]\!] : D^{\mathrm{var}(\mathcal{L})} \to D$$

  that satisfies

$$[\![x]\!] : \alpha \mapsto \alpha(x),$$

- to each sentence $\varphi$, a map

$$[\![\varphi]\!] : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

  that satisfies

$$[\![Fx_1 \cdots x_n]\!] = F^{\mathfrak{M}} \circ (1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle),$$

- and, to each $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, a family of maps

$$[\![\otimes]\!]^{\bar{x}} : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$$

  for all finite sets $\bar{x}$ of variables of $\mathcal{L}$, such that

$$[\![\otimes(\varphi_1, \ldots, \varphi_n)]\!] = [\![\otimes]\!]^{\bar{x}}([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!])$$

  for the set $\bar{x}$ of variables that occur freely in at least one of $\varphi_1, \ldots, \varphi_n$.

We say a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ interprets a sentential operator $\otimes$ of $\mathcal{L}$ *uniformly* if the family $[\![\otimes]\!]^{\bar{x}}$ is constant, that is, if there is a unique operation $f$ such that $[\![\otimes]\!]^{\bar{x}} = f$ for every $\bar{x}$; then we simply write $[\![\otimes]\!]$ for $[\![\otimes]\!]^{\bar{x}}$. We also say a general Kripke-type interpretation for $\mathcal{L}$ is *on* a Kripke model $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$, and is *over* a Kripke frame $\mathfrak{F}$ with domains if it is on a Kripke model over $\mathfrak{F}$.

Clearly, any general Kripke-type interpretation for $\mathcal{L}$ is a Kripke-type interpretation for $\mathcal{L}$ (as defined in Definition 44) iff it uniformly interprets every sentential operator $\otimes$ of $\mathcal{L}$. So let us refer to such interpretations by *uniform* Kripke-type interpretations when we need to contrast them with *general* Kripke-type interpretations. It should also be noted that Definition 45 of local determination for uniform Kripke-type interpretations simply applies to general Kripke-type interpretations, since both types of interpretations interpret sentences with the same types of maps.

In terms of general Kripke-type interpretations, our semantic idea of

$$(w, \alpha) \in [\![\Box\varphi]\!] \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \subseteq [\![\varphi]\!]_\alpha, \tag{IV.47}$$

$$(w, \alpha) \in [\![\Diamond\varphi]\!] \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \cap [\![\varphi]\!]_\alpha \neq \varnothing, \tag{IV.48}$$

can be formulated as follows: We consider general Kripke-type interpretations that interpret $\Box$ and $\Diamond$ respectively with the families of operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ for all finite sets $\bar{x}$ of variables such that

$$(w, \alpha) \in [\![\Box]\!]^{\bar{x}}(A) \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \subseteq A_\alpha, \tag{IV.57}$$

$$(w, \alpha) \in [\![\Diamond]\!]^{\bar{x}}(A) \iff \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \cap A_\alpha \neq \varnothing. \tag{IV.58}$$

### IV.3.2 Preservation of Local Determination Generalized

The second technical issue on our semantic idea in terms of (IV.46)–(IV.50) concerns the preservation of local determination. The issue arises because, under (IV.57) and (IV.58), $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ fail to preserve local determination in the sense we gave in Definition 46.

To see how they fail, let us consider the following example, taking again the same Kripke frame $\mathfrak{F} = (W, R, D_{-})$ with domains as in the proof of Fact 37; that is,

$$W = \{u, v\}, \qquad R = \{(u, v)\},$$
$$D_u = \{a, b\}, \text{ where } a \neq b, \qquad D_v = \{a\}.$$

Then, for a given quantified modal language $\mathcal{L}$, take a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ over $\mathfrak{F}$ that satisfies (IV.57) and $F^{\mathfrak{M}} = \varnothing$ for a unary primitive predicate $F$ of $\mathcal{L}$; indeed, we may consider the sentence $x \neq x$ in place of $Fx$, so that, in $(\mathfrak{M}, [\![-]\!])$, no individual is distinct from itself. Then observe that the individual $a$ has a world accessible from $u$ in which it exists, namely, $v$; to put it in terms of assignments, fixing a variable $x \in \mathrm{var}(\mathcal{L})$, any assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D$ such that $\alpha(x) = a$ has $\alpha{\restriction}\{x\} : \{x\} \to D_v$ and hence $\overrightarrow{R}_{\alpha\restriction\{x\}}(u) = \{v\}$. On the other hand, the individual $b$ has no accessible world from $u$ in which it exists; in terms of assignments, any $\beta : \mathrm{var}(\mathcal{L}) \to D$ such that $\beta(x) = b$ has $\overrightarrow{R}_{\beta\restriction\{x\}}(u) = \varnothing$ because $\beta{\restriction}\{x\} : \{x\} \not\to D_v$. Therefore, at $u$, $a$ does not necessarily satisfy $x \neq x$ whereas $b$ does trivially, since $a$ fails to satisfy $x \neq x$ at the world accessible from $u$ in which it exists (namely, at $v$), while $b$ has no such world. In terms of assignments, $(u, \alpha) \notin [\![\Box]\!]^{\{x\}}([\![x \neq x]\!])$ since $\overrightarrow{R}_{\alpha\restriction\{x\}}(u) \nsubseteq \varnothing = [\![x \neq x]\!]_\alpha$, whereas $(u, \beta) \in [\![\Box]\!]^{\{x\}}([\![x \neq x]\!])$ since $\overrightarrow{R}_{\beta\restriction\{x\}}(u) \subseteq \varnothing = [\![x \neq x]\!]_\beta$.

Now note that, according to Definition 46, for $[\![\Box]\!]^{\bar{x}}$ to preserve local determination, $[\![\Box]\!]^{\bar{x}}(A)$ must be determined by $\bar{y}$ whenever $A \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ is. In the example above, however, even though $\varnothing \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ is determined by $\varnothing \subseteq \mathrm{var}(\mathcal{L})$, we have just shown $[\![\Box]\!]^{\{x\}}(\varnothing)$ is not, since it contains $(u, \beta)$ but not $(u, \alpha)$. This is how the operation $[\![\Box]\!]^{\bar{x}}$ under (IV.57) fails to preserve local determination in the sense of Definition 46 (and similarly for $[\![\Diamond]\!]^{\bar{x}}$ under (IV.58)).

Nonetheless, this failure should not be taken as showing that (IV.57) and (IV.58) (and therefore our idea of (IV.46)–(IV.50)) are to blame; rather, it suggests that, in accordance with our generalization in terms of general Kripke-type interpretations, the definition of preservation of local determination should also be generalized. The example above of the failure of preservation can be summarized as follows: Even though $x \neq x$ is false (in every world) no matter what the referent of $x$ is, the truth of $\Box(x \neq x)$ depends on the referent of $x$ (and worlds). Nothing in this summary goes against the conceptual import of local determination—that is, sentences are true or false of the referents of free variables, rather than of assignments. Indeed, formally speaking as well, local determination is not violated (though its preservation, as defined in Definition 46, is) in the example.

Let $\otimes$ be a sentential operator that binds variables $\bar{z}$ but not $x$. As in the example above, even if $x$ freely occurs in a sentence $\varphi$, the interpretation $[\![\varphi]\!]$ of $\varphi$ may be independent of $x$, meaning that $[\![\varphi]\!]$ is determined by variables $\bar{y}$ such that $x \notin \bar{y}$. Then, in the case of *uniform* Kripke-type interpretations, Definition 46 requires that, for the operation $[\![\otimes]\!]$ to preserve local determination for $\otimes$, $[\![\otimes\varphi]\!] = [\![\otimes]\!][\![\varphi]\!]$ should also be independent of $x$, although $x$ occurs freely in $\otimes\varphi$. This is because we may have $[\![\varphi]\!] = [\![\psi]\!]$ for a sentence $\psi$ in which $x$ does not occur freely (in the example above, we indeed have $[\![x \neq x]\!] = [\![\exists x\,.\, x \neq x]\!]$); if this is the case, the local determination of $\otimes\psi$ requires—though that of $\otimes\varphi$ does not—that $[\![\otimes\varphi]\!] = [\![\otimes\psi]\!]$ should be independent of $x$. In the case of *general* Kripke-type interpretations, by contrast, we generally have $[\![\otimes\varphi]\!] \neq [\![\otimes\psi]\!]$ even though $[\![\varphi]\!] = [\![\psi]\!]$, since that $x$ occurs freely in $\varphi$ but not in $\psi$ entails $\bar{x} \neq \bar{y}$ for the sets $\bar{x}$ and $\bar{y}$ of free variables in $\varphi$ and $\psi$, and therefore, generally, $[\![\otimes\varphi]\!] = [\![\otimes]\!]^{\bar{x}}[\![\varphi]\!] \neq [\![\otimes]\!]^{\bar{y}}[\![\psi]\!] = [\![\otimes\psi]\!]$. This is why $[\![\otimes\psi]\!] = [\![\otimes]\!]^{\bar{y}}[\![\psi]\!]$ being independent of $x$ is consistent with $[\![\otimes\varphi]\!] = [\![\otimes]\!]^{\bar{x}}[\![\varphi]\!]$ being dependent on $x$. Indeed, since $[\![\otimes]\!]^{\bar{x}}[\![\varphi]\!]$ generally interprets $\otimes\varphi$ only if $\bar{x}$ are the free variables in $\varphi$, dependence of $[\![\otimes]\!]^{\bar{x}}[\![\varphi]\!]$ on $x \in \bar{x} \setminus \bar{z}$ never forms a threat to local determination. That is, to formulate a right definition of preservation of local determination for general Kripke-type interpretations, it is too strong to require $[\![\otimes]\!]^{\bar{x}}(A)$ be independent of $x \in \bar{x} \setminus \bar{z}$, whether or not $A$ is independent of $x$. What is required instead is simply that if $A$ is determined by $\bar{x}$—regardless of whether or not $A$ depends on all of $\bar{x}$—then $[\![\otimes]\!]^{\bar{x}}(A)$ is determined by $\bar{x} \setminus \bar{z}$. To put this operationally, if $A : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}^n$ is of the form $B \circ r_{\bar{x}}$ for the restriction $r_{\bar{x}} : W \times D^{\mathrm{var}(\mathcal{L})} \twoheadrightarrow W \times D^{\bar{x}}$ and some $B : W \times D^{\bar{x}} \to \mathbf{2}^n$, then $[\![\otimes]\!]^{\bar{x}}(A) : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$ is of the form $C \circ r_{\bar{x}\setminus\bar{z}}$ for the restriction $r_{\bar{x}\setminus\bar{z}} : W \times D^{\mathrm{var}(\mathcal{L})} \twoheadrightarrow W \times D^{\bar{x}\setminus\bar{z}}$ and some $C : W \times D^{\bar{x}\setminus\bar{z}} \to \mathbf{2}$. Even more operationally this means, since $- \circ r_{\bar{x}}$ and $- \circ r_{\bar{x}\setminus\bar{z}}$ are injective, that there uniquely exists an operation $[\![\, \bar{x} \mid \otimes \,]\!]^{\bar{x}}$ that makes the diagram below commute, so that, when $A = B \circ r_{\bar{x}}$ for unique $B$, $[\![\, \bar{x} \mid \otimes \,]\!]^{\bar{x}}(B)$ is the unique $C$ such that $[\![\otimes]\!]^{\bar{x}}(A) = C \circ r_{\bar{x}\setminus\bar{z}}$.

$$
\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;- \circ r_{\bar{x}}\;} & \mathcal{P}(W \times D^{\bar{x}})^n \\
{\scriptstyle [\![\otimes]\!]^{\bar{x}}}\downarrow & \circlearrowright & \downarrow{\scriptstyle [\![\, \bar{x} \mid \otimes \,]\!]^{\bar{x}}} \\
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;- \circ r_{\bar{x}\setminus\bar{z}}\;]{} & \mathcal{P}(W \times D^{\bar{x}\setminus\bar{z}})
\end{array}
$$

Therefore we enter:

**Definition 52.** Let $\mathcal{L}$ be a quantified modal language and let $\mathfrak{M} = (W, R, D_{-})$ be a Kripke model for $\mathcal{L}$ with a domain $D$ of individuals. Then, for variables $\bar{y}$ of $\mathcal{L}$, we say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ for all finite sets $\bar{x}$ of variables *preserves local determination with the binding of* $\bar{y}$ if, for every finite set $\bar{x}$ of variables of $\mathcal{L}$, there is an operation $f^{\bar{x}}_{\bar{x}} : \mathcal{P}(W \times D^{\bar{x}})^n \to \mathcal{P}(W \times D^{\bar{x}\setminus\bar{y}})$ such that, for every $B : W \times D^{\bar{x}} \to \mathbf{2}^n$,

$$f^{\bar{x}}_{\bar{x}}(B) \circ r_{\bar{x}\setminus\bar{y}} = f^{\bar{x}}(B \circ r_{\bar{x}}) : W \times D^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

(where $r_{\bar{x}} : (w, \alpha) \mapsto (w, \alpha \restriction \bar{x})$ and $r_{\bar{x} \setminus \bar{y}} : (w, \alpha) \mapsto (w, \alpha \restriction (\bar{x} \setminus \bar{y}))$), that is, that makes the following diagram commute.

$$\begin{array}{ccc} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;-\,\circ\, r_{\bar{x}}\;} & \mathcal{P}(W \times D^{\bar{x}})^n \\ {\scriptstyle f^{\bar{x}}} \big\downarrow & & \big\downarrow {\scriptstyle f^{\bar{x}}_{\bar{x}}} \\ \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\,\circ\, r_{\bar{x} \setminus \bar{y}}\;]{} & \mathcal{P}(W \times D^{\bar{x} \setminus \bar{y}}) \end{array}$$

We also say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ (for a fixed $n$) preserves local determination *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and if the family preserves local determination with the binding of the variables that $\otimes$ binds. Moreover, we say a general Kripke-type interpretation for $\mathcal{L}$ preserves local determination if it interprets every sentential operator $\otimes$ of $\mathcal{L}$ with a family of operations that preserves local determination for $\otimes$.

This definition works as desired, in the following sense.

**Fact 44.** Any general Kripke-type interpretation for a given quantified modal language $\mathcal{L}$ is locally determined if it preserves local determination.

*Proof.* By induction on the construction of sentences of $\mathcal{L}$. $\square$

Obviously, when a constant family $[\![\otimes]\!]^{\bar{x}} = [\![\otimes]\!]$ interprets $\otimes$ uniformly, the family $[\![\otimes]\!]^{\bar{x}}$ preserves local determination for $\otimes$ in the sense of Definition 52 iff the operation $[\![\otimes]\!]$ does so in the sense of Definition 46. Thus Definition 52 subsumes Definition 46, and the following immediately follows from Fact 35.

**Fact 45.** Given a quantified modal language $\mathcal{L}$ and a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, if the families $[\![\neg]\!]^{\bar{x}}$, $[\![\wedge]\!]^{\bar{x}}$, $[\![\vee]\!]^{\bar{x}}$, $[\![\rightarrow]\!]^{\bar{x}}$, $[\![\Box]\!]^{\bar{x}}$, $[\![\Diamond]\!]^{\bar{x}}$, $[\![\forall x]\!]^{\bar{x}}$, $[\![\exists y]\!]^{\bar{x}}$ uniformly interpret $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\Box$, $\Diamond$, $\forall x$, $\exists x$ following (IV.34)–(IV.41), respectively, then they preserve local determination for $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\Box$, $\Diamond$, $\forall x$, $\exists x$, respectively.

Moreover, as desired, we have:

**Fact 46.** Given a quantified modal language $\mathcal{L}$ and a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, if the families $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ interpret $\Box$ and $\Diamond$ following (IV.57) and (IV.58), respectively, then they preserves local determination for $\Box$ and $\Diamond$, respectively.

*Proof.* That the family $[\![\Box]\!]^{\bar{x}}$ satisfying (IV.57) to preserve local determination for $\Box$ means that, for every finite set $\bar{x}$ of variables, there is $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ making

$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;-\circ r_{\bar{x}}\;} & \mathcal{P}(W \times D^{\bar{x}}) \\
{\scriptstyle [\![\Box]\!]^{\bar{x}}} \downarrow & \simeq & \downarrow {\scriptstyle [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}} \\
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\circ r_{\bar{x}}\;]{} & \mathcal{P}(W \times D^{\bar{x}})
\end{array}$$

commute. So let us define $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ so that, for every $(w, \beta) \in W \times D^{\bar{x}}$ and $B \subseteq W \times D^{\bar{x}}$,

$$(w, \beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \iff \overrightarrow{R}_{\beta}(w) \subseteq B_{\beta}. \tag{IV.59}$$

Then the diagram above commutes, because (IV.42) $(B \circ r_{\bar{x}})_{\alpha} = B_{\alpha \restriction \bar{x}}$ entails

$$\begin{aligned}
(w, \alpha) \in [\![\Box]\!]^{\bar{x}}(B \circ r_{\bar{x}}) &\overset{\text{(IV.57)}}{\iff} \overrightarrow{R}_{\alpha \restriction \bar{x}}(w) \subseteq (B \circ r_{\bar{x}})_{\alpha} = B_{\alpha \restriction \bar{x}} \\
&\overset{\text{(IV.59)}}{\iff} r_{\bar{x}}(w, \alpha) = (w, \alpha \restriction \bar{x}) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \\
&\iff (w, \alpha) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \circ r_{\bar{x}}.
\end{aligned}$$

(IV.42) similarly implies that $[\![\Diamond]\!]^{\bar{x}}$ preserves local determination for $\Diamond$, with $[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}$ such that

$$(w, \beta) \in [\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}(B) \iff \overrightarrow{R}_{\beta}(w) \cap B_{\beta} \neq \varnothing. \qquad \Box$$

### IV.3.3 DoQ-Restrictability Generalized

The last of the technical issues we discuss regarding out proposal of (IV.46)–(IV.50) concerns DoQ-restrictability; it arises because (IV.57) and (IV.58) fail to make $[\![\Box]\!]^{\bar{x}}$ or $[\![\Diamond]\!]^{\bar{x}}$ DoQ-restrictable, thereby failing to meet our goal in designing (IV.46)–(IV.50), that is, to achieve DoQ-restrictability.

But, one may ask, how is it possible that (IV.57) and (IV.58) fail DoQ-restrictability, despite the fact that, by using $\overrightarrow{R}_{\alpha\restriction\bar{x}}$, we rule out from (IV.47) and (IV.48)—and hence from (IV.57) and (IV.58)—any reference to world-tuple non-DoQ-pairs $(u, \alpha(\bar{x}))$? The conceptual answer to this is that the non-reference to world-tuple non-DoQ-pairs makes $[\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ and $[\![\Diamond]\!]^{\bar{x}}[\![\varphi]\!]$ behave in a DoQ-restrictable fashion only if the truth conditions for $\Box\varphi$ and $\Diamond\varphi$ in terms of world-*tuple* pairs make sense, that is, only if $\varphi$ is locally determined. The conceptual idea of not referring to world-*tuple* non-DoQ-pairs is a good idea, but works technically only when the conceptual and the technical are connected through local determination.

To see this observe that, according to (IV.57), a world-*assignment* DoQ-pair $(w, \alpha)$ is in $[\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ iff every $(u, \alpha)$ such that $u \in \overrightarrow{R}_{\alpha\restriction\bar{x}}(w)$ is in $[\![\varphi]\!]$. In other words, whether or not $(w, \alpha) \in [\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ depends on whether or not $(u, \alpha) \in [\![\varphi]\!]$ for $u \in \overrightarrow{R}_{\alpha\restriction\bar{x}}(w)$, even if $(u, \alpha)$ is a world-*assignment* DoQ-pair (since what matters for $\overrightarrow{R}_{\alpha\restriction\bar{x}}(w)$ is that $(u, \alpha(\bar{x}))$ is a world-*tuple* DoQ-pair). Thus, technically, (IV.57) and similarly (IV.58) manage to make reference to world-*assignment* non-DoQ-pairs. This reference is, however, not essential if $\varphi$ is locally determined, that is, if $[\![\varphi]\!]$ is determined by $\bar{x}$ (recall that, generally, $[\![\Box]\!]^{\bar{x}}[\![\varphi]\!]$ is semantically significant only when $\bar{x}$ is the set of free variables in $\varphi$). This is because, even if $(u, \alpha)$ is not a DoQ-pair, the determination of $[\![\varphi]\!]$ by $\bar{x}$ implies that $(u, \alpha) \in [\![\varphi]\!]$ iff $(u, \beta) \in [\![\varphi]\!]$ for any DoQ-pair $(u, \beta)$ such that $\beta(\bar{x}) = \alpha(\bar{x})$ (that is, $\beta$ serves the purpose, as well as $\alpha$ does, of expressing $\varphi$ being true of the tuple $\alpha(\bar{x})$), thereby enabling us to replace the reference to the world-assignment *non-DoQ*-pair $(u, \alpha)$ with that to the world-assignment *DoQ*-pair $(u, \beta)$. On the other hand, if local determination fails, the reference to the world-assignment non-DoQ-pair $(u, \alpha)$ may be essential, because then we may have $(u, \alpha) \in [\![\varphi]\!]$ while no DoQ-pair $(u, \beta)$ is in $[\![\varphi]\!]$, or $(u, \alpha) \notin [\![\varphi]\!]$ while all DoQ-pairs $(u, \beta)$ are in $[\![\varphi]\!]$.[15]

We can distill two upshots here:

[15] $(v, \alpha)$ and $B$ in the following example give an instance of $(u, \alpha) \notin [\![\varphi]\!]$ with all DoQ-pairs $(u, \beta)$ lying in $[\![\varphi]\!]$. Let us again take the frame $\mathfrak{F} = (W, R, D_{-})$ as in the proof of Fact 37; that is,

$$W = \{u, v\}, \qquad R = \{(u, v)\}, \qquad D_u = \{a, b\}, \text{ where } a \neq b, \qquad D_v = \{a\}.$$

(i) Under (IV.57) and (IV.58), operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ are DoQ-restrictable in terms of world-*tuple* pairs, even though they are not in terms of world-*assignment* pairs.

(ii) Therefore, as long as local determination holds, it makes sense to say that interpretations given by (IV.57) and (IV.58) are DoQ-restrictable in the sense involving world-*tuple* pairs.

Hence we first express (i) formally with:

**Definition 53.** Let $\mathcal{L}$ be a quantified modal language and $\mathfrak{F} = (W, R, D_{-})$ be a Kripke frame with a domain $D$ of possible individuals. Then, for any finite sets $\bar{x}$ and $\bar{y}$ of variables of $\mathcal{L}$, we say an operation

$$f : \mathcal{P}(W \times D^{\bar{x}})^n \to \mathcal{P}(W \times D^{\bar{y}})^m$$

(note how the type involves $\bar{x}$ and $\bar{y}$) is *DoQ-restrictable* if it is restrictable to the sets $\sum_{w \in W} {D_w}^{\bar{x}}$ and $\sum_{w \in W} {D_w}^{\bar{y}}$ of DoQ-pairs, in the sense that there is $f_{\mathrm{DoQ}}$ that makes the diagram below commute.

$$\begin{array}{ccc} \mathcal{P}(W \times D^{\bar{x}})^n & \xrightarrow{\quad f \quad} & \mathcal{P}(W \times D^{\bar{y}})^m \\ {\scriptstyle - \circ i}\downarrow & /\!/ & \downarrow{\scriptstyle - \circ i} \\ \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right)^n & \xrightarrow[\quad f_{\mathrm{DoQ}} \quad]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{y}}\right)^m \end{array}$$

---

Then pick any $\alpha : \mathrm{var}(\mathcal{L}) \to D_u$ (not just $\alpha : \mathrm{var}(\mathcal{L}) \to D$) such that $\alpha(x) = a$ and $\alpha(y) = b$. This means that $(u, \alpha)$ is a DoQ-pair but $(v, \alpha)$ is not, since $\alpha(y) \notin D_v$. Moreover, $\overrightarrow{R}_{\alpha \restriction \{x\}}(u) = \{v\}$ because $\alpha \restriction \{x\} : \{x\} \to D_v$. Using this $\alpha$, let

$$A = W \times D^{\mathrm{var}(\mathcal{L})}, \qquad B = (W \times D^{\mathrm{var}(\mathcal{L})}) \setminus \{(v, \alpha)\}.$$

Note that $B$ is not determined by $\{x\}$ (or indeed by any finite set of variables). Then, since $A$ and $B$ are only different at the non-DoQ-pair $(v, \alpha)$, we have

$$A \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} = B \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}.$$

Therefore the DoQ-restriction ${[\![\Box]\!]^{\{x\}}}_{\mathrm{DoQ}}$ of $[\![\Box]\!]^{\{x\}}$, if it exists, must have

$$[\![\Box]\!]^{\{x\}}(A) \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} = {[\![\Box]\!]^{\{x\}}}_{\mathrm{DoQ}}\left(A \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) = {[\![\Box]\!]^{\{x\}}}_{\mathrm{DoQ}}\left(B \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) = [\![\Box]\!]^{\{x\}}(B) \cap \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}.$$

This nonetheless cannot be the case (and hence $[\![\Box]\!]^{\{x\}}$ is not DoQ-restrictable), because the difference between $A$ and $B$ at the non-DoQ-pair $(v, \alpha)$—that is, $\overrightarrow{R}_{\alpha \restriction \{x\}}(u) = \{v\} \subseteq W = A_\alpha$ and $\overrightarrow{R}_{\alpha \restriction \{x\}}(u) = \{v\} \nsubseteq \{u\} = B_\alpha$—implies that the DoQ-pair $(u, \alpha)$ is in $[\![\Box]\!]^{\{x\}}(A)$ but not in $[\![\Box]\!]^{\{x\}}(B)$. Thus the reference is essentially made to the non-DoQ-pair $(v, \alpha)$.

And then express (ii) with the following, which makes the notion of DoQ-restrictability partly dependent on that of local determination:

**Definition 54.** Let $\mathcal{L}$ be a quantified modal language and $\mathfrak{F} = (W, R, D_{-})$ be a Kripke frame with a domain $D$ of possible individuals. Then we say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ is *DoQ-restrictable with the binding of variables $\bar{y}$* if

- the family $f^{\bar{x}}$ preserves local determination with the binding of $\bar{y}$, and, moreover,
- for each finite set $\bar{x}$ of variables, the operator $f^{\bar{x}}_{\bar{x}}$ that makes

$$\begin{array}{ccc} \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;-\circ r_{\bar{x}}\;} & \mathcal{P}(W \times D^{\bar{x}})^n \\ {\scriptstyle f^{\bar{x}}}\downarrow & = & \downarrow{\scriptstyle f^{\bar{x}}_{\bar{x}}} \\ \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\circ r_{\bar{x}\setminus\bar{y}}\;]{} & \mathcal{P}(W \times D^{\bar{x}\setminus\bar{y}}) \end{array}$$

  commute (which uniquely exists since the family $f^{\bar{x}}$ preserves local determination with the binding of $\bar{y}$) is DoQ-restrictable.

We also say that a family of operations $f^{\bar{x}} : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ is DoQ-restrictable *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and the family $f^{\bar{x}}$ is DoQ-restrictable with the binding of the variables that $\otimes$ binds. Moreover, we say a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ is *DoQ-restrictable* if it interprets each sentential operator $\otimes$ of $\mathcal{L}$ with a family of operators that is DoQ-restrictable for $\otimes$.

This definition is weaker than Definition 48 in the following sense:

**Fact 47.** Given a quantified modal language $\mathcal{L}$ and a Kripke frame $(W, R, D_{-})$ with a domain $D$ of possible individuals, suppose an operation $f : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ preserves local determination with the binding of variables $\bar{y}$. Then, if $f$ is DoQ-restrictable as an operation (as in Definition 48) then it is DoQ-restrictable with the binding of $\bar{y}$, as a constant family of operations $f^{\bar{x}}$ (as in Definition 54).

This immediately entails the following by Facts 36 and 35.

**Corollary 8.** Given a quantified modal language $\mathcal{L}$ and any general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, if the families $[\![\neg]\!]^{\bar{x}}$, $[\![\wedge]\!]^{\bar{x}}$, $[\![\vee]\!]^{\bar{x}}$, $[\![\to]\!]^{\bar{x}}$, $[\![\forall x]\!]^{\bar{x}}$, $[\![\exists y]\!]^{\bar{x}}$ uniformly interpret $\neg$, $\wedge$, $\vee$,

$\to$, $\forall x$, $\exists x$ following (IV.34)–(IV.37), (IV.40), (IV.41), respectively, then they are DoQ-restrictable for $\neg$, $\wedge$, $\vee$, $\to$, $\forall x$, $\exists x$, respectively.

We postpone the proof of Fact 47 until the end of this subsection. Let us show first that operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ under (IV.57) and (IV.58) are DoQ-restrictable for each $\bar{x}$, and hence that DoQ-autonomous Kripkean semantics—our new semantics the satisfaction-relation version of which we laid out in Subsection IV.2.3—is DoQ-restrictable.

The proof of the DoQ-restrictability of operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ under (IV.57) and (IV.58) is similar to the proof for Fact 40. First observe that, for every $\beta : \bar{x} \to D$ and $B, C \subseteq W \times D^{\bar{x}}$, we have

(IV.60) $$(B \cap C)_\beta = B_\beta \cap C_\beta$$

exactly similarly to (IV.51). Also, again similarly to (IV.52), for every $(u, \beta) \in \sum_{w \in W} {D_w}^{\bar{x}}$, we have

(IV.61) $$\overrightarrow{R}_\beta(u) \subseteq \left(\sum_{w \in W} {D_w}^{\bar{x}}\right)_\beta,$$

because

$$v \in \overrightarrow{R}_\beta(u) \implies \beta : \bar{x} \to D_v \iff (v, \beta) \in \sum_{w \in W} {D_w}^{\bar{x}} \iff v \in \left(\sum_{w \in W} {D_w}^{\bar{x}}\right)_\beta.$$

Then, using these, we can prove:

**Fact 48.** Given a quantified modal language $\mathcal{L}$ and a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, if the families $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ interpret $\Box$ and $\Diamond$ following (IV.57) and (IV.58), respectively, then, for each $\bar{x}$ the operations $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ and $[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}$, which exist by Fact 46, are DoQ-restrictable.

*Proof.* A proof that $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ is DoQ-restrictable amounts to showing that there is $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}$ making

$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\bar{x}}) & \xrightarrow{[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}} & \mathcal{P}(W \times D^{\bar{x}}) \\
{\scriptstyle - \circ i}\downarrow & /\!/ & \downarrow{\scriptstyle - \circ i} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right) & \xrightarrow[{[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}}]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right)
\end{array}$$

commute. So let us define $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}$ so that, for every $(u,\beta) \in \sum_{w \in W} D_w{}^{\bar{x}}$ and $B \subseteq \sum_{w \in W} D_w{}^{\bar{x}}$,

$$(u,\beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}(B) \iff \overrightarrow{R}_{\beta}(u) \subseteq B_{\beta}. \tag{IV.62}$$

Then, for every $(u,\beta) \in \sum_{w \in W} D_w{}^{\bar{x}}$ and $B \subseteq W \times D^{\bar{x}}$, we have

$$\begin{aligned}
(u,\beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \cap \sum_{w \in W} D_w{}^{\bar{x}} &\iff (u,\beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \\
&\overset{\text{(IV.59)}}{\iff} \overrightarrow{R}_{\beta}(u) \subseteq B_{\beta} \\
&\overset{\text{(IV.60)}}{\iff} \overrightarrow{R}_{\beta}(u) \subseteq B_{\beta} \cap \left( \sum_{w \in W} D_w{}^{\bar{x}} \right)_{\beta} \\
&\overset{\text{(IV.61)}}{\iff} \overrightarrow{R}_{\beta}(u) \subseteq \left( B \cap \sum_{w \in W} D_w{}^{\bar{x}} \right)_{\beta} \\
&\overset{\text{(IV.62)}}{\iff} (u,\beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}} \left( B \cap \sum_{w \in W} D_w{}^{\bar{x}} \right);
\end{aligned}$$

thus $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \circ i = [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}(B \circ i)$, making the diagram above commute. Hence $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ is DoQ-restrictable. Similarly, $[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}$ is DoQ-restrictable, with $[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}$ such that

$$(u,\beta) \in [\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}{}_{\mathrm{DoQ}}(B) \iff \overrightarrow{R}_{\beta}(u) \cap B_{\beta} \neq \varnothing \tag{IV.63}$$

for every $(u,\beta) \in \sum_{w \in W} D_w{}^{\bar{x}}$ and $B \subseteq \sum_{w \in W} D_w{}^{\bar{x}}$. □

So, let us sum up the definition for the operational formulation of DoQ-autonomous Kripkean semantics; then it is equivalent to the satisfaction-relation formulation (we omit a proof), whereas the preservation of local determination and DoQ-restrictability follow from previous facts.

**Definition 55.** Given a quantified modal language $\mathcal{L}$, a general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ is said to be a *DoQ-autonomous Kripkean interpretation for* $\mathcal{L}$, if it interprets $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$ uniformly with the constant families of operations

(IV.34) $\quad [\![\neg]\!] = \neg_{\mathbf{2}} \circ -,$

(IV.35) $\quad [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ -,$

(IV.36) $\quad [\![\vee]\!] = \vee_{\mathbf{2}} \circ -,$

(IV.37) $\quad [\![\rightarrow]\!] = \rightarrow_{\mathbf{2}} \circ -,$

(IV.40) $\quad [\![\forall x]\!] = \prod_{w \in W} [\![\forall x]\!]_w$, where $[\![\forall x]\!]_w(A) = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for every } a \in D_w \,\}$,

(IV.41) $\quad [\![\exists x]\!] = \prod_{w \in W} [\![\exists x]\!]_w$, where $[\![\exists x]\!]_w(A) = \{\, \alpha \in D^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A \text{ for some } a \in D_w \,\}$,

respectively, and if it interprets $\Box$ and $\Diamond$ with the families of operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ satisfying (IV.57) and (IV.58), respectively, that is, with

(IV.64) $\quad [\![\Box]\!]^{\bar{x}} : A \mapsto \{\, (w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \subseteq A_\alpha \,\},$

(IV.65) $\quad [\![\Diamond]\!]^{\bar{x}} : A \mapsto \{\, (w, \alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \mid \overrightarrow{R}_{\alpha\restriction\bar{x}}(w) \cap A_\alpha \neq \varnothing \,\}.$

**Fact 49.** Let $\mathcal{L}$ be a quantified modal language. Given any general Kripke-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, define a relation $\vDash\ \subseteq W \times D^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$ by transposition

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff (w, \alpha) \in [\![\varphi]\!],$$

so that we have a pair $(\mathfrak{M}, \vDash)$. This gives an operation from the class of general Kripke-type interpretations for $\mathcal{L}$ to the class of Kripke-type satisfaction relations for $\mathcal{L}$. Moreover, this operation is bijective to the class of Kripke satisfaction relations for $\mathcal{L}$ when restricted to the class of Kripke interpretations for $\mathcal{L}$, whereas bijective to the class of DoQ-autonomous Kripkean satisfaction relations for $\mathcal{L}$ when restricted to the class of DoQ-autonomous Kripkean interpretations for $\mathcal{L}$.

So, not just the class of DoQ-autonomous Kripkean satisfaction relations for $\mathcal{L}$ but also that of DoQ-autonomous Kripkean interpretations for $\mathcal{L}$ can also be called the DoQ-autonomous Kripkean semantics for $\mathcal{L}$.

**Fact 50.** The DoQ-autonomous Kripkean semantics for any quantified modal language preserves local determination.

*Proof.* By Facts 45 and 46. □

**Fact 51.** The DoQ-autonomous Kripkean semantics for any quantified modal language is DoQ-restrictable.

*Proof.* By Corollary 8 and Fact 48. □

We are going to close this subsection by proving Fact 47. But for our proof it is useful to first make the following two observations. Given a finite set $\bar{x}$ of variables and the restriction surjection $r_{\bar{x}} : W \times D^{\mathrm{var}(\mathcal{L})} \to W \times D^{\bar{x}}$, write $(r_{\bar{x}})_*$ and $(r_{\bar{x}})^*$ respectively for the direct-image and inverse-image (that is, precomposition) operations under $r_{\bar{x}}$, that is, the operations of the types

$$\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) \underset{(r_{\bar{x}})_*}{\overset{(r_{\bar{x}})^*}{\rightleftarrows}} \mathcal{P}(W \times D^{\bar{x}})$$

such that, for every $A \subseteq W \times D^{\mathrm{var}(\mathcal{L})}$ and $B \subseteq W \times D^{\bar{x}}$,

$$(r_{\bar{x}})_*(A) = r_{\bar{x}}[A], \qquad (r_{\bar{x}})^*(B) = r_{\bar{x}}^{-1}[B] = B \circ r_{\bar{x}}.$$

Then $r_{\bar{x}}$ being surjective implies:

**Fact 52.** $(r_{\bar{x}})_* \circ (r_{\bar{x}})^* = 1$ for any restriction surjection $r_{\bar{x}} : W \times D^{\mathrm{var}(\mathcal{L})} \to W \times D^{\bar{x}}$.

*Proof.* For every $(u, \beta) \in W \times D^{\bar{x}}$ and $B \subseteq W \times D^{\bar{x}}$, the "only if" direction of the equivalence marked with ! below holds since $r_{\bar{x}}$ is surjective:

$$\begin{aligned} (u,\beta) \in B &\overset{!}{\iff} \text{there is } (u,\alpha) \in W \times D^{\mathrm{var}(\mathcal{L})} \text{ such that } r_{\bar{x}}(u,\alpha) = (u,\beta) \in B \\ &\iff \text{there is } (u,\alpha) \in (r_{\bar{x}})^*(B) \text{ such that } r_{\bar{x}}(u,\alpha) = (u,\beta) \\ &\iff (u,\beta) \in (r_{\bar{x}})_* \circ (r_{\bar{x}})^*(B). \end{aligned}$$

□

Let us also consider the precompositions $(r'_{\bar{x}})^*$, $i^*$, $i'^*$ with the restriction surjection

$$r'_{\bar{x}} : \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} \twoheadrightarrow \sum_{w \in W} {D_w}^{\bar{x}},$$

which maps $(u, \alpha)$ to $(u, \alpha \restriction \bar{x})$, and with the inclusion maps

$$i : \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})} \hookrightarrow W \times {D_w}^{\mathrm{var}(\mathcal{L})}, \qquad i' : \sum_{w \in W} {D_w}^{\bar{x}} \hookrightarrow W \times {D_w}^{\bar{x}}.$$

Then we have:

**Fact 53.** The diagram below commutes for the precompositions with suitable types of restriction surjections and inclusion maps.

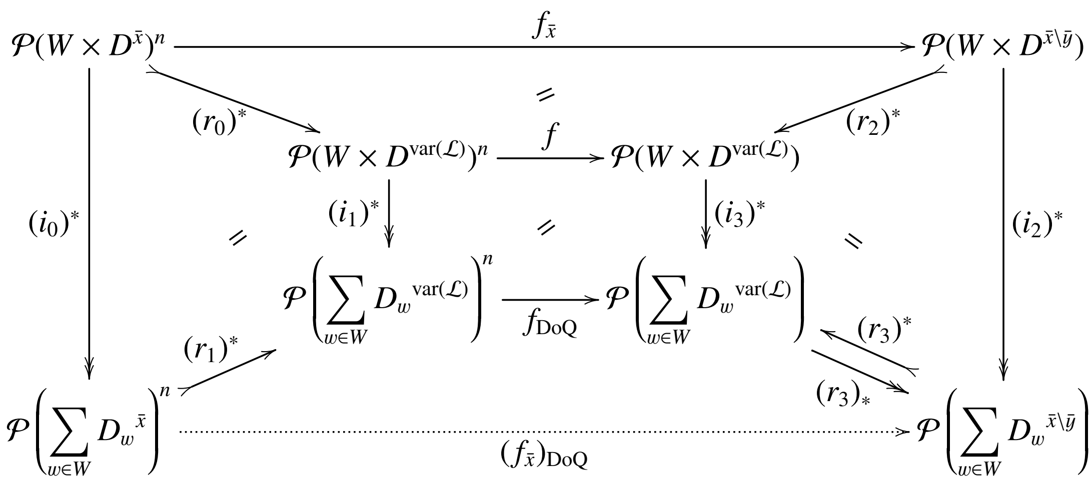
$$\begin{array}{ccc}
\mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{(r_{\bar{x}})^*} & \mathcal{P}(W \times D^{\bar{x}}) \\
{\scriptstyle i^*} \downarrow & \searrow\!\!\!\!\nwarrow & \downarrow {\scriptstyle i'^*} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}\right) & \xleftarrow[(r'_{\bar{x}})^*]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right)
\end{array}$$

*Proof.* $r_{\bar{x}} \circ i = i' \circ r'_{\bar{x}}$ because, for every $(u, \alpha) \in \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L})}$,

$$r_{\bar{x}} \circ i(u, \alpha) = r_{\bar{x}}(u, \alpha) = (u, \alpha \restriction \bar{x}) = i'(u, \alpha \restriction \bar{x}) = i' \circ r'_{\bar{x}}(u, \alpha).$$

Therefore $i^* \circ (r_{\bar{x}})^* = - \circ r_{\bar{x}} \circ i = - \circ i' \circ r'_{\bar{x}} = (r'_{\bar{x}})^* \circ i'^*$. □

*Proof for Fact 47.* Fix a DoQ-restrictable operation $f : \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times D^{\mathrm{var}(\mathcal{L})})$ that preserves local determination with the binding of variables $\bar{y}$, and fix any finite set $\bar{x}$ of variables of $\mathcal{L}$. Then let $i_0, \ldots, i_3$ and $r_1, \ldots, r_3$ be the inclusion maps and restriction surjections with which the precompositions have the types in the following diagram.




Here the existence of $f_{\mathrm{DoQ}}$ and $f_{\bar{x}}$ such that

(IV.66) $\quad f_{\mathrm{DoQ}} \circ (i_1)^* = (i_3)^* \circ f \quad$ (the middle square commutes),

(IV.67) $\quad (r_2)^* \circ f_{\bar{x}} = f \circ (r_0)^* \quad$ (the top square commutes)

is implied respectively by the DoQ-restrictability of $f$ and the preservation of local determination by $f$ with the binding of $\bar{y}$. A proof that $f$ is DoQ-restrictable, as a constant family of operations, with the binding of $\bar{y}$ amounts to showing that there is $(f_{\bar{x}})_{\mathrm{DoQ}}$ as above such that $(f_{\bar{x}})_{\mathrm{DoQ}} \circ (i_0)^* = (i_2)^* \circ f_{\bar{x}}$ (that is, the outer square commutes). To show it, note that Fact 53 implies

(IV.68) $\quad (r_1)^* \circ (i_0)^* = (i_1)^* \circ (r_0)^* \quad$ (the left square commutes),

(IV.69) $\quad (r_3)^* \circ (i_2)^* = (i_3)^* \circ (r_2)^* \quad$ (the right square without $(r_3)_*$ commutes),

whereas Fact 52 implies

(IV.70) $$(r_3)_* \circ (r_3)^* = 1.$$

From these it follows that $(r_3)_* \circ f_{\mathrm{DoQ}} \circ (r_1)^*$ serves as $(f_{\bar{x}})_{\mathrm{DoQ}}$, because

$$\begin{aligned}
(r_3)_* \circ f_{\mathrm{DoQ}} \circ (r_1)^* \circ (i_0)^* &\overset{\text{(IV.68)}}{=} (r_3)_* \circ f_{\mathrm{DoQ}} \circ (i_1)^* \circ (r_0)^* \\
&\overset{\text{(IV.66)}}{=} (r_3)_* \circ (i_3)^* \circ f \circ (r_0)^* \\
&\overset{\text{(IV.67)}}{=} (r_3)_* \circ (i_3)^* \circ (r_2)^* \circ f_{\bar{x}} \\
&\overset{\text{(IV.69)}}{=} (r_3)_* \circ (r_3)^* \circ (i_2)^* \circ f_{\bar{x}} \\
&\overset{\text{(IV.70)}}{=} (i_2)^* \circ f_{\bar{x}}.
\end{aligned}$$

Thus the constant family of operations $f$ is DoQ-restrictable with the binding of $\bar{y}$. □

## V.0 ACCESSIBILITY AND COUNTERPARTS

## V.1 DAVID LEWIS'S COUNTERPART THEORY

David Lewis [24] puts forward an ontology of possible individuals that he calls *counterpart theory*, and that differs from the ontology of Kripke's semantics, which we discussed in Chapter IV, in a crucial manner.[1] Lewis's primary purpose is to describe the ontology of possible objects in an extensional language, rather than to provide a semantics for quantified modal logic. Nonetheless, we can regard his theory as providing a semantics for quantified modal logic since, as Lewis himself shows, we can translate a modal language into his extensional language; then the translation of a modal sentence $\varphi$ can serve as the truth condition of $\varphi$. The difference between the two ontologies is philosophically significant, and reflected in the difference between the logics arising from the two semantics.

### V.1.1 Disjoint Ontology of Possible Individuals and the Notion of Counterparts

Counterpart theory has four primitive predicates, $Wx$, $Ixy$, $Cxy$, and $Ax$, whose intended interpretations are as follows (the range of $x$ and $y$ is intended to be unrestricted):

| | |
|---|---|
| $Wx$ | $x$ is a possible world, |
| $Ixy$ | $x$ is *in* a possible world $y$, |
| $Cxy$ | $y$ is a *counterpart* of $x$,[2] |
| $Ax$ | $x$ is *actual*. |

[1] Lewis has different formulations of counterpart theory; in this chapter we only discuss the version in [24].

And, for these predicates, Lewis assumes eight postulates. A first postulate is about $W$ and $I$:

$$\forall x \forall y \, (Ixy \to Wy). \tag{P1}$$

While Lewis reads this as "Nothing is in anything except a world",[3] it seems a little more accurate to take it as follows. Things can be in other things in many ways; e.g., a cat is in a box, or Pittsburgh is in the United States. Among those different senses of "is in", we intend the predicate $Ixy$ to refer to the particular kind of "is in" relation between a thing and a possible world; then this interpretation entails P1, that is, that the second argument $y$ of $Ixy$ must be a world. (It should be noted that Lewis and we use the word "things" to include worlds, and not just things in worlds. Also note that, instead of Lewis's "is in", I often use "live in"—which, of course, does not presuppose that its subject is animate.)

The following may be the most crucial postulate of counterpart theory:

$$\forall x \forall y \forall z \, (Ixy \wedge Ixz \to y = z). \tag{P2}$$

This states that each thing lives in at most one world; in Lewis's words, "things in different worlds are *never* identical".[4] We may say the ontology of counterpart theory is *disjoint*, in the sense that it assumes the set of things in a world to be disjoint from those in a different world. The disjointness in this sense is one of the most crucial divergences from Kripke's ontology, which is not disjoint, that is, in which a single possible individual can live in various different worlds.

One may be tempted to say that Lewis's counterpart-theoretic ontology is merely a special case of Kripke's ontology since it is just the version of Kripke's ontology—in which different possible worlds may have different sets of possible individuals that exist, or live, in them—gained by further assuming disjointness, that is, that each possible individual lives in at most one world. This is half right, but half wrong, as long as these ontologies are meant to serve as bases for semantics of modal logic. To show this point, let us suppose a possible individual $a$ exists in a world $w$, and consider whether $a$ satisfies a (unary) property $F$ necessarily at $w$, that is, whether or not $\mathfrak{M}, w \vDash_\alpha \Box Fx$ for an assignment $\alpha$ such that $\alpha(x) = a$. The truth condition (IV.49) stipulates that $\mathfrak{M}, w \vDash_\alpha \Box Fx$ if and

[2] Lewis [24] uses $x$ and $y$ in the opposite order; that is, he reads $Cxy$ as "$x$ is a counterpart of $y$". I adopt the opposite notation above for several reasons; one is the analogy, which will turn out to be important, between the counterpart relation and the accessibility relation, which is usually denoted by $Rxy$ for "$y$ is accessible from $x$".

[3] [24], 114.

[4] [24], 114.

only if $\mathfrak{M}, u \vDash_\alpha Fx$ for all worlds $u$ that are accessible from $w$ and in which $a$ exists. Let us further assume disjointness, which means that $w$ is the only world in which $a$ exists. Then it follows that either:

- $w$ is accessible from $w$ itself, and $\mathfrak{M}, w \vDash_\alpha \Box Fx$ iff $\mathfrak{M}, w \vDash_\alpha Fx$, or
- $w$ is not accessible from $w$ itself, and $\mathfrak{M}, w \vDash_\alpha \Box Fx$, no matter what $F$ is.

To sum this up, the combination of (IV.49) and disjointness trivializes the behavior of $\Box$; similarly, the combination of (IV.50) and disjointness trivializes the behavior of $\Diamond$. This is why the conception of the counterpart-theoretic ontology as Kripke's ontology combined with disjointness cannot be extended to the level of semantics.

To see how one can accommodate disjointness to the semantic ideas behind (IV.49) and (IV.50)—which are essentially the same ideas as Lewis adopts—let us extract from them the following ideas (see Chapter IV for more detail). Here $w$ is a world, $\bar{a}$ is an $n$-tuple of individuals that exist in $w$, and $\varphi$ is a sentence that has exactly $n$ free variables.

(i) That $\bar{a}$ satisfies $\varphi$ necessarily (or possibly) at $w$ means that $\bar{a}$ satisfies $\varphi$ at all (or some) $u \in U_{w,\bar{a}}$ for a certain set $U_{w,\bar{a}}$ of worlds (which may depend on $w$ and $\bar{a}$).

(ii) In general, if not all of individuals $\bar{b}$ exist in a world $u$, then what the tuple $\bar{b}$ satisfies at $u$ has no semantic significance to what $\bar{a}$ satisfies at $\bar{w}$ (in which all of $\bar{a}$ exist).

On the other hand, to repeat what it means to say an ontology is disjoint, it is that

(iii) Each possible individual exists in at most one world.

Then the observation in the previous paragraph can be put as follows. If follows from (i) that, for each $u \in U_{w,\bar{a}}$, what $\bar{a}$ satisfies at $u$ is significant to what $\bar{a}$ satisfies at $w$. Then (ii) implies that all of $\bar{a}$ exist in each $u \in U_{w,\bar{a}}$. Combining this with (iii), however, it follows that $U_{w,\bar{a}}$ contains at most one world. This, by (i), makes $\Box$ and $\Diamond$ trivial. This is why Lewis, who adopts (iii), as well as (ii) as we will discuss shortly, needs a way to reconcile (ii) and (iii) with the intuition behind (i).[5] The conceptual device he introduces for this purpose is the notion of *counterparts* of something, which allows us to distinguish the following two ways of expressing the intuition behind (i).

[5]By contrast, Kripke adopts (i) but rejects (ii), which saves $\Box$ and $\Diamond$ from triviality even when (iii) is assumed.

(i′) That $\bar{a}$ satisfies $\varphi$ necessarily (or possibly) at $w$ means that the same tuple $\bar{a}$ itself satisfies $\varphi$ at all (or some) $u \in U_{w,\bar{a}}$.

(i″) That $\bar{a}$ satisfies $\varphi$ necessarily (or possibly) at $w$ means that, at each (or some) $u \in U_{w,\bar{a}}$, a tuple $(b_1, \ldots, b_n)$ of counterparts $b_i$ of $a_i$ in $u$ satisfies $\varphi$.

(i′) together with (ii) and (iii) implies (more precisely) that $\Box$ and $\Diamond$ behave trivially; but, with the help of the notion of counterparts, Lewis adopts (i″) instead, and, as we will see shortly, it yields nontrivial $\Box$ and $\Diamond$ even on the basis of (ii) and (iii).[6]

Let us then list the postulates about the *counterpart relation*, denoted by $C$. The first set is

(P3) $$\forall x \forall y\, (Cxy \to \exists z . Iyz),$$

(P4) $$\forall x \forall y\, (Cxy \to \exists z . Ixz).$$

These state that only things in worlds are (by P3) or have (by P4) counterparts; in other words, the counterpart relation is a relation among things in worlds. This may make our understanding of the counterpart relation simpler, but it is of little consequence to the logic given by counterpart theory. In contrast, the following set is more significant to the logic:

(P5) $$\forall x \forall y \forall z\, (Ixy \wedge Izy \wedge Cxz \to x = z),$$

(P6) $$\forall x \forall y\, (Ixy \to Cxx).$$

To read these let us say, given things $x$, $y$, $z$, that $z$ is a *counterpart of $x$ in $y$* if $z$ is a counterpart of $x$ and is in $y$, that is, if $Cxz$ and $Izy$. Then P6 and P5 state that anything in any world is the one (by P6) and only (by P5) counterpart of itself in that world. In other words, the counterpart relation, when restricted to things in a single world, coincides with the identity relation.

Finally, the last two postulates are about the notion of actuality, expressed by $A$:

(P7) $$\exists x\, (Wx \wedge \forall y\, (Iyx \equiv Ay)),$$

(P8) $$\exists x\, Ax.$$

[6] Lewis seems ([24], 114) to think that disjointness saves the ontology from the charge that the identity of possible individuals across different worlds is obscure. Needless to say, his notion of counterpart may well face the charge that it is obscure when things are counterparts of other things; but it is open to question whether the obscurity of the counterpart relation is as threatening to the status of ontology as the obscurity of the identity of individuals is, since one may argue that the former is about us being unable to know which particular model to choose from all the models the ontology gives, while the latter is about each model the ontology gives being not well defined.

P7 states that there is a world such that anything is actual if and only if it lives in that world. We may call such a world an *actual world* (we should note that an actual world itself does not satisfy $A$). Then P2 and P8, which states something or other is actual, imply that an actual world exists uniquely; so we call it *the* actual world and write @ for it.

Let us close this subsection by introducing two closely connected notions.[7]

**Definition 56.** We say a tuple $(X, W, I, C, A)$ is a *counterpart structure* if

- $X$ is a set,
- $W, A \subseteq X$,
- $I, C \subseteq X \times X$,

and if, furthermore, $W$, $I$, $C$, and $A$ satisfy P1–P8 (or their set-theoretic versions). We say a tuple $(X, W, I, C, A, @)$ is a counterpart structure as well, if $(X, W, I, C, A)$ is a counterpart structure as above and if

- $@ \in X$ satisfies $(x, @) \in I$ iff $x \in A$ for every $x \in X$.

**Definition 57.** When a (classical) first-order language $\mathcal{L}$ contains unary primitive predicates $I$, $C$ and binary primitive predicates $W$, $A$ (possibly as well as a constant @), a (classical) $\mathcal{L}$ structure $\mathfrak{M}$ is called a *model of counterpart theory* if it validates P1–P8, or equivalently if $(|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}})$ (or $(|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}}, @^{\mathfrak{M}})$) is a counterpart structure.

### V.1.2 Counterpart Translation of a Modal Language

Using the notions reviewed in Subsection V.1.1, Lewis gives a scheme for translating a quantified modal language into the language of counterpart theory. We review this translation scheme in this subsection; the review will suggest that the scheme can be regarded as providing a possible-world semantics for quantified modal logic.

Lewis is concerned with a translation between the following two languages:

- A quantified modal language, as we reviewed in Subsection IV.1.1. In addition to primitive predicates, individual variables, classical sentential operators, it has the modal operators $\Box$ and $\Diamond$; it has no other operators, or no function or constant symbols. We also assume it has no

[7] These are not in Lewis [24].

propositional variables; in other words, it has no 0-ary primitive predicates.[8] Let us call this language $\mathcal{L}_{\mathrm{M}}$.

- An extensional—or classical first-order—language of counterpart theory. It has all the vocabulary $\mathcal{L}_{\mathrm{M}}$ has (including, crucially, all the primitive predicates of $\mathcal{L}_{\mathrm{M}}$) except $\Box$ and $\Diamond$. Moreover, it has the special predicates $W$, $I$, $C$, and $A$ and the defined term @, as we reviewed in Subsection V.1.1. Let us call this language $\mathcal{L}_{\mathrm{C}}$.

Given any sentence $\varphi$ of $\mathcal{L}_{\mathrm{M}}$, the scheme Lewis proposes translates it into $\mathcal{L}_{\mathrm{C}}$. He denotes by $\varphi^{@}$ the translation of $\varphi$, that is, the sentence of $\mathcal{L}_{\mathrm{C}}$ that translates $\varphi$; its heuristic meaning is, as Lewis says, "$\varphi$ holds in the actual world".[9] In other words, $\varphi^{@}$ is an $\mathcal{L}_{\mathrm{C}}$ sentence expressing that the $\mathcal{L}_{\mathrm{M}}$ sentence $\varphi$ is actually true. To put it illustratively, a speaker of $\mathcal{L}_{\mathrm{C}}$ endorses that the $\mathcal{L}_{\mathrm{M}}$ sentence $\ulcorner\varphi\urcorner$ is actually true if and only if she endorses $\varphi^{@}$. Then Lewis proposes his translation scheme to define what $\varphi^{@}$ looks like.

Lewis's translation scheme is given by a family of maps $-^{z}$, for each individual term $z$ of $\mathcal{L}_{\mathrm{C}}$ (including @ in particular), from $\mathcal{L}_{\mathrm{M}}$ sentences to $\mathcal{L}_{\mathrm{C}}$ sentences.[10] These maps $-^{z}$ generalize the case of $\varphi^{@}$ in the sense that, given an $\mathcal{L}_{\mathrm{M}}$ sentence $\varphi$, $\varphi^{z}$ is an $\mathcal{L}_{\mathrm{C}}$ sentence expressing that the $\mathcal{L}_{\mathrm{M}}$ sentence $\varphi$ is true at world $z$.[11] To put it illustratively again, the speaker of $\mathcal{L}_{\mathrm{C}}$ endorses that the $\mathcal{L}_{\mathrm{M}}$ sentence $\ulcorner\varphi\urcorner$ is true at world $z$ if and only if she endorses $\varphi^{z}$.

To lay out this heuristic, illustrative idea rigorously, let us fix—in place of the speaker of $\mathcal{L}_{\mathrm{C}}$—an $\mathcal{L}_{\mathrm{C}}$ structure $\mathfrak{M}$ that models counterpart theory; that is,

- $\mathfrak{M} = (|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}}, @^{\mathfrak{M}}, F^{\mathfrak{M}})$ has a set $|\mathfrak{M}|$ as a domain of quantification;

- $(|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}}, @^{\mathfrak{M}})$ is a model of counterpart theory as in Definition 57; and moreover,

- $\mathfrak{M}$ also interprets every other primitive $n$-ary predicate $F$ of $\mathcal{L}_{\mathrm{C}}$, with $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^{n}$.

[8] We assume this because a proper treatment of propositional variables involves the modification of the semantics we make in Subsection V.2.3; after that we can dispose of this assumption.

[9] [24], 118.

[10] The language Lewis uses to formulate these maps may be $\mathcal{L}_{\mathrm{C}}$, but not necessarily. It has to have, of course, the identity predicate "=" applicable to pairs of $\mathcal{L}_{\mathrm{C}}$ sentences. It also has to be able to mention all the vocabulary of $\mathcal{L}_{\mathrm{M}}$, as well as the vocabulary of $\mathcal{L}_{\mathrm{C}}$ that is needed to express the translation.

[11] It is, however, not assumed that $z$ stands for a world. Indeed, it may not make sense to say "$z$, as in $\varphi^{z}$, stands for a world" in Lewis's or our use language, unless we fix, in particular, a particular model of counterpart theory.

Then the heuristic meaning of $\varphi^z$ is formulated, as a first approximation, as

$$\mathfrak{M} \vDash \varphi^z \iff \mathfrak{M} \vDash \text{“the } \mathcal{L}_\mathrm{M} \text{ sentence } \ulcorner\varphi\urcorner \text{ is true at world } z\text{”}.$$

It must be noted that $\varphi^z$ may contain free variables, say $\bar{x}$, including possibly $z$. Hence we need to modify our first approximation with an assignment $\alpha : \mathrm{var}(\mathcal{L}_\mathrm{C}) \to |\mathfrak{M}|$ of individuals to variables, as follows:

$$\text{(V.1)} \qquad \mathfrak{M} \vDash_\alpha \varphi^z \iff \mathfrak{M} \vDash_\alpha \text{“the } \mathcal{L}_\mathrm{M} \text{ sentence } \ulcorner\varphi\urcorner \text{ is true of fv}(\varphi) \text{ at world } z\text{”}.$$

The notation $\mathfrak{M} \vDash_\alpha \varphi^z$, as read as above, should be helpful in extracting Lewis's semantic idea out of his translation scheme, as we are going to do.

Lewis provides a recursive definition for maps $-^z$ with a base clause for atomic sentences and inductive clauses for compound sentences. Keeping the heuristic meaning of $\varphi^z$ in mind should make it seem natural that Lewis adopts the following clauses for classical operators:

$$\text{(V.2)} \qquad (\neg\varphi)^z = \neg\varphi^z,$$

$$\text{(V.3)} \qquad (\varphi \wedge \psi)^z = \varphi^z \wedge \psi^z,$$

$$\text{(V.4)} \qquad (\varphi \vee \psi)^z = \varphi^z \vee \psi^z,$$

$$\text{(V.5)} \qquad (\varphi \to \psi)^z = \varphi^z \to \psi^z,$$

$$\text{(V.6)} \qquad (\forall x \,.\, \varphi)^z = \forall x \,(Ixz \to \varphi^z),$$

$$\text{(V.7)} \qquad (\exists x \,.\, \varphi)^z = \exists x \,(Ixz \wedge \varphi^z).$$

To read these clauses, we note, for example, that "¬" to the right of "=" in (V.2) is the negation in $\mathcal{L}_\mathrm{C}$, while "¬" to the left is in $\mathcal{L}_\mathrm{M}$; hence (V.2) entails the first equivalence below (trivially, as the two $\mathcal{L}_\mathrm{C}$ sentences are just identical), while (III.2) entails the second:

$$\mathfrak{M} \vDash_\alpha (\neg\varphi)^z \overset{\text{(V.2)}}{\iff} \mathfrak{M} \vDash_\alpha \neg\varphi^z \overset{\text{(III.2)}}{\iff} \mathfrak{M} \nvDash_\alpha \varphi^z.$$

That is, we have

$$\text{(V.8)} \qquad \mathfrak{M} \vDash_\alpha (\neg\varphi)^z \iff \mathfrak{M} \nvDash_\alpha \varphi^z.$$

This means in terms of (V.1) that, in a given model $\mathfrak{M}$ of counterpart theory and with respect to the fixed interpretation $\alpha$ of variables, in particular $\bar{x}$, $z$ in terms of $\alpha(\bar{x})$, $\alpha(z)$, "the $\mathcal{L}_{\mathrm{M}}$ sentence $\ulcorner\neg\varphi\urcorner$ is true of $\bar{x}$ at $z$" holds iff "the $\mathcal{L}_{\mathrm{M}}$ sentence $\ulcorner\varphi\urcorner$ is true of $\bar{x}$ at $z$" does *not* hold. Or, put much less rigorously,

- $\neg\varphi$ is true of $\alpha(\bar{x})$ at $\alpha(z)$, iff
- $\varphi$ is *not* true of $\alpha(\bar{x})$ at $\alpha(z)$.

This heuristic reading of (V.2) should motivate us to adopt (V.2).

Similarly, (V.3)–(V.5) entail, and are motivated by, the following (V.9)–(V.11), respectively:

(V.9) $$\mathfrak{M} \vDash_{\alpha} (\varphi \wedge \psi)^{z} \iff \mathfrak{M} \vDash_{\alpha} \varphi^{z} \text{ and } \mathfrak{M} \vDash_{\alpha} \psi^{z},$$

(V.10) $$\mathfrak{M} \vDash_{\alpha} (\varphi \vee \psi)^{z} \iff \mathfrak{M} \vDash_{\alpha} \varphi^{z} \text{ or } \mathfrak{M} \vDash_{\alpha} \psi^{z},$$

(V.11) $$\mathfrak{M} \vDash_{\alpha} (\varphi \to \psi)^{z} \iff \mathfrak{M} \nvDash_{\alpha} \varphi^{z} \text{ or } \mathfrak{M} \vDash_{\alpha} \psi^{z}.$$

The clauses (V.6) and (V.7) entail (V.12) and (V.13) below; since their derivation is more involved than that of (V.8) above, we will lay out the derivation of (V.12) at the end of this subsection.

(V.12) $$\mathfrak{M} \vDash_{\alpha} (\forall y\,.\,\varphi)^{z} \iff \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^{z} \text{ for every } a \in |\mathfrak{M}| \text{ such that } (a, \alpha(z)) \in I^{\mathfrak{M}},$$

(V.13) $$\mathfrak{M} \vDash_{\alpha} (\exists y\,.\,\varphi)^{z} \iff \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^{z} \text{ for some } a \in |\mathfrak{M}| \text{ such that } (a, \alpha(z)) \in I^{\mathfrak{M}}.$$

Heuristically (and not rigorously), (V.12) for example means that, in a given model $\mathfrak{M}$ of counterpart theory, when $\bar{x}$ are the free variables occurring in $\forall y\,.\,\varphi$,

- $\forall y\,.\,\varphi$ is true of $\alpha(\bar{x})$ at $\alpha(z)$, iff
- $\varphi$ is true, at $\alpha(z)$, of $\alpha(\bar{x})$ and *every* $a$ (in place of $y$) that lives in $\alpha(z)$.

Lewis adopts the following clauses for modal operators:

(V.14) $$(\Box\varphi)^{z} = \forall z' \,\forall y_1 \,\cdots\, \forall y_n \\ ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \to [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'})),$$

(V.15) $$(\Diamond\varphi)^{z} = \exists z' \,\exists y_1 \,\cdots\, \exists y_n \\ ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \wedge [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'})).$$[12]

Here it is assumed that $\bar{x}$ are all distinct. We also assume that $\bar{y}$ and $z'$ are all distinct, new variables. It is important to note that (V.14) and (V.15) also require the assumption that the replaced variables $\bar{x}$ are all and only the free variables in $\varphi$.[13] Hence, with this assumption stated explicitly, (V.14) and (V.15) really are

(V.14) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$(\Box\varphi)^z = \forall z' \, \forall y_1 \cdots \forall y_n \; ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \rightarrow [y_n/x_n] \cdots [y_1/x_1](\varphi^{z'})),$$

(V.15) If $\bar{x}$ are all and only the free variables in $\varphi$, then

$$(\Diamond\varphi)^z = \exists z' \, \exists y_1 \cdots \exists y_n \; ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \wedge [y_n/x_n] \cdots [y_1/x_1](\varphi^{z'})).$$

To see why this assumption is required, suppose $x'$ does not actually occur in $\varphi$. Then, without the assumption, (V.14) yields both

$$(\Box\varphi)^z = \forall z' \, \forall y_1 \cdots \forall y_n \; ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \rightarrow [y_n/x_n] \cdots [y_1/x_1](\varphi^{z'})),$$

$$(\Box\varphi)^z = \forall z' \, \forall y_1 \cdots \forall y_n \, \forall y' \; ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n \wedge Iy'z' \wedge Cx'y') \rightarrow [y'/x'][y_n/x_n] \cdots [y_1/x_1](\varphi^{z'}));$$

[12] Lewis formulates (V.14) and (V.15), or T2i and T2j in his labels, with a different notation than ours. That is, "If $\phi$ is any $n$-place sentence and $\alpha_1 \ldots \alpha_n$ are any $n$ different variable, then $\phi\alpha_1 \cdots \alpha_n$ is the sentence obtained by substituting $\alpha_1$ uniformly for the alphabetically first free variable n $\phi$, $\alpha_2$ for the second, and so on" (footnote 11 of [24], 117). If it is assumed here that $\phi\alpha_1 \ldots \alpha_n$ does not make sense unless *exactly* $n$ variables occur freely in $\phi$ (see footnote 13 of this chapter for a reason why it should be assumed), then Lewis needs the minor modification of substituting $(\phi\gamma_1 \cdots \gamma_n)^{\beta_1}$ for $\phi^{\beta_1}\gamma_1 \cdots \gamma_n$ in his formulations of

T2i $(\Box\phi\alpha_1 \cdots \alpha_n)^\beta = \forall\beta_1 \, \forall\gamma_1 \cdots \forall\gamma_n \, (W\beta_1 \;\&\; I\gamma_1\beta_1 \;\&\; C\gamma_1\alpha_1 \;\&\; \cdots \;\&\; I\gamma_n\beta_1 \;\&\; C\gamma_n\alpha_n \;.\!\supset \phi^{\beta_1}\gamma_1 \cdots \gamma_n)$,

T2j $(\Diamond\phi\alpha_1 \cdots \alpha_n)^\beta = \exists\beta_1 \, \exists\gamma_1 \cdots \exists\gamma_n \, (W\beta_1 \;\&\; I\gamma_1\beta_1 \;\&\; C\gamma_1\alpha_1 \;\&\; \cdots \;\&\; I\gamma_n\beta_1 \;\&\; C\gamma_n\alpha_n \;\&\; \phi^{\beta_1}\gamma_1 \cdots \gamma_n)$

([24], 118). It is because it may be the case that $\phi$—take, for example, $\forall x\, Fxy$ for $\phi$—contains exactly $n$ free variables (so that $\phi\alpha_1 \cdots \alpha_n$ and $\phi\gamma_1 \cdots \gamma_n$ make sense under the assumption in question) but not $\beta_1$, whereas $\beta_1$ in addition to all the free variables of $\phi$ occurs freely in $\phi^{\beta_1}$, which means, under the assumption, that $\phi^{\beta_1}\gamma_1 \cdots \gamma_n$ does not make sense.

[13] Lewis does not quite explicitly state this assumption, but it follows from his notation $\phi\alpha_1 \cdots \alpha_n$, if the notation assumes that exactly $n$ variables occur freely in the sentence $\phi$ (see footnote 12).

but these equations are inconsistent because, while their left-hand sides are identical, the right-hand sides are not even equivalent. In this way, the well-definedness of maps $-^z$ requires the assumption in question.[14]

The clauses (V.14) and (V.15) entail and are motivated by:

(V.16) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M} \vDash_\alpha (\Box\varphi)^z \iff \mathfrak{M} \vDash_{[a_n/x_n]\cdots[a_1/x_1][w/z']\alpha} \varphi^{z'} \text{ for every } w \in W^{\mathfrak{M}} \text{ and } \bar{a} \in |\mathfrak{M}|^n$$
$$\text{such that each } i \text{ has } (a_i, w) \in I^{\mathfrak{M}} \text{ and } (\alpha(x_i), a_i) \in C^{\mathfrak{M}},$$

(V.17) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M} \vDash_\alpha (\Diamond\varphi)^z \iff \mathfrak{M} \vDash_{[a_n/x_n]\cdots[a_1/x_1][w/z']\alpha} \varphi^{z'} \text{ for some } w \in W^{\mathfrak{M}} \text{ and } \bar{a} \in |\mathfrak{M}|^n$$
$$\text{such that each } i \text{ has } (a_i, w) \in I^{\mathfrak{M}} \text{ and } (\alpha(x_i), a_i) \in C^{\mathfrak{M}}.$$

Again, we postpone the derivation of (V.16) until the end of this subsection. Let us read (V.16) and (V.17) heuristically (and not rigorously) in the way we read (V.8) and (V.12): that is, in a given model $\mathfrak{M}$ of counterpart theory,

- $\Box\varphi$ is true of $\alpha(\bar{x})$ at $\alpha(z)$, iff
- at *every* world $w$, $\varphi$ is true of *every* $\bar{a}$ such that, for each $i$,
  - $a_i$ lives in $w$, and
  - $a_i$ is a counterpart of $\alpha(x_i)$,

and

- $\Diamond\varphi$ is true of $\alpha(\bar{x})$ at $\alpha(z)$, iff
- at *some* world $w$, $\varphi$ is true of *some* $\bar{a}$ such that, for each $i$,
  - $a_i$ lives in $w$, and
  - $a_i$ is a counterpart of $\alpha(x_i)$,

[14] Strictly speaking, the well-definedness of $-^z$ requires more things, for example that each sentence has a privileged order of listing its free variables (Lewis uses the alphabetical order of variables). Such an order is required since the right-hand sides of

$$(\Box\varphi)^z = \forall z' \, \forall y_1 \, \forall y_2 \, ((Wz' \land Iy_1z' \land Cx_1y_1 \land Iy_2z' \land Cx_2y_2) \to [y_2/x_2][y_1/x_1](\varphi^{z'})),$$
$$(\Box\varphi)^z = \forall z' \, \forall y_1 \, \forall y_2 \, ((Wz' \land Iy_1z' \land Cx_2y_1 \land Iy_2z' \land Cx_1y_2) \to [y_2/x_1][y_1/x_2](\varphi^{z'}))$$

are distinct as sentences, although equivalent. We refrain, however, from being so strict because the difference between these sentences is not semantically significant, as long as they are equivalent.

Now that we have reviewed all the inductive clauses Lewis adopts for his recursive definition of translation maps $-^z$, let us review his base clause for atomic sentences, which is

$$\text{(V.18)} \qquad \varphi^z = \varphi \quad \text{for atomic } \varphi.$$

This implies that, for any $n$-ary primitive predicate $F$,

$$\mathfrak{M} \vDash_\alpha (F\bar{x})^z \overset{\text{(V.18)}}{\iff} \mathfrak{M} \vDash_\alpha F\bar{x} \overset{\text{(III.1)}}{\iff} \alpha(\bar{x}) \in F^{\mathfrak{M}};$$

that is,

$$\text{(V.19)} \qquad \mathfrak{M} \vDash_\alpha (F\bar{x})^z \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

Combined with (V.1), this can be read as:

$$\text{(V.20)} \qquad \mathfrak{M} \vDash_\alpha \text{“the } \mathcal{L}_{\mathrm{M}} \text{ sentence } \ulcorner F\bar{x} \urcorner \text{ is true of } \bar{x} \text{ at world } z\text{”} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}}.$$

Let us take an example, both to heuristically read this clause in concrete terms, and to clarify some worries it may cause. A typical example for an atomic sentence $F\bar{x}$ is a unary sentence "$x$ is a logician",[15] and Russell can be a typical example for an individual $\alpha(x)$; so, taking this example, and writing $w$ for a world $\alpha(z)$, we can read (V.20) heuristically (though not rigorously) as:

(i) Russell satisfies the atomic sentence "$x$ is a logician" of $\mathcal{L}_{\mathrm{M}}$ at $w$,[16] iff

(ii) Russell is a logician.

[15](V.18) does not seem to work well when $\varphi$ is a 0-ary primitive predicate, for example, "It rains a lot". It is certainly odd to apply (V.20) to this example and read it as saying

(i) the atomic sentence "It rains a lot" of $\mathcal{L}_{\mathrm{M}}$ is true at $w$, iff

(ii) it rains a lot.

This can be solved as follows. As $n$-ary predicates may be true of $n$-tuples, extend it by stipulating that 0-tuples are just worlds, so that 0-ary predicates may be true of worlds. Then we have (i) iff

(ii′) it is true of $w$ that it rains a lot,

which is no longer odd. This stipulation that 0-tuples are just worlds is not *ad hoc* but indeed natural.

[16]Recall that I use the phrases "$\varphi$ is true of $a$" and "$a$ satisfies $\varphi$" interchangeably.

One may well find the definiens (ii) puzzling, or even problematic. One may argue that, when read as a sentence used to provide possible-world semantics for $\mathcal{L}_{\mathrm{M}}$, (ii) does not seem to make sense unless it mentions a particular world. Putting this argument in Kripke's terms, Russell may be a logician in a world $w_0$, but may not be in another $w_1$; therefore Russell may have the properties of being-a-logician-at-$w_0$ and not-being-a-logician-at-$w_1$, but not the property of being-a-logician *simpliciter*. Nonetheless, even if one maintains that in possible-world semantics properties have to be relative to worlds, (ii) can still make sense, at least for individuals that live in some worlds or other such as Russell, because in counterpart theory each individual, including Russell, only lives in one world. That is, we can read (ii) as meaning that Russell is a logician in the unique world in which he lives, referring to the property of being-a-logician-at-the-world-in-which-one-lives.

One may, however, further worry that, granted (ii) makes sense, it gives rise to the following trouble for (i). That is, if (ii) holds and Russell is a logician *simpliciter* (or is a logician in the world in which he lives), then the "iff" implies that (i) holds—namely, Russell satisfies the $\mathcal{L}_{\mathrm{M}}$ sentence "$x$ is a logician" at $w$—not only when $w$ is the world in which Russell lives but indeed for every world $w$. This inference is, indeed, *correct*. And then it seems to follow (this inference is, indeed, *incorrect*) that, according to counterpart theory, if Russell is a logician then he is necessarily so and cannot be otherwise.

This second inference is incorrect due to Lewis's counterpart interpretation for $\Box$, namely (V.16), as follows. Suppose Russell lives in a world $w$ (and hence only in $w$). Also fix any world $w_0$. Then, by (V.16) (and the postulate P5 of counterpart theory), Russell is necessarily a logician—meaning that he satisfies the $\mathcal{L}_{\mathrm{M}}$ sentence "$\Box$($x$ is a logician)"—at $w_0$ if and only if,

- not only does he satisfy "$x$ is a logician" at $w$,
- but also, at every world $w' \neq w$, "$x$ is a logician" is true of all of his counterparts $a$ who live in $w'$.

Yet Russell is not included among such $a$, because he does not live in $w' \neq w$. Hence his satisfying "$x$ is a logician" at $w'$ is irrelevant for his satisfying "$\Box$($x$ is a logician)" at $w_0$.

There is indeed a more general sense in which the conclusion of the first inference—that if Russell is a logician *simpliciter* then he satisfies the $\mathcal{L}_{\mathrm{M}}$ sentence "$x$ is a logician" at every world, whether he lives in it or not—is not troublesome to Lewis. That is, his translation scheme guarantees that, as far as we are concerned with the logic of what sentences individuals satisfy at the worlds in which they live, there is no semantic significance to Russell's satisfying the $\mathcal{L}_\mathrm{M}$ sentence "$x$ is a logician"—or indeed any $\mathcal{L}_\mathrm{M}$ sentence—at any world other than the one in which he lives. This is because the semantics to which Lewis's semantic idea gives rise has autonomous domains of quantification, althouth, to state and prove this fact precisely and formally, we need to wait until we rewrite Lewis's translation scheme in semantic terms in Subsection V.2.[17]

It may be worth noting that the autonomy of domains of quantification in the semantics means that, not only is the conclusion of the first inference above unproblematic for Lewis, it is indeed semantically insignificant. To repeat the inference in question (with slight rephrasing), (V.20) implies that an atomic sentence of $\mathcal{L}_\mathrm{M}$ is either true at all worlds or true at no worlds; for example, either Russell is a logician and "$x$ is a logician" is true of him at all worlds, or he is not a logician and "$x$ is a logician" is true of him at no worlds. And, because of this technical import, (V.20) may well be more susceptible of some metaphysical interpretations than of others, as regards what, according to (V.20), are truthmakers for atomic sentences of $\mathcal{L}_\mathrm{M}$. Notwithstanding this conceptually significant question, how to interpret (V.20), or even whether to accept (V.20) at all, makes no difference to the logic of (DoQ-validity in) Lewis's semantics, due to the autonomy of domains of quantification in the semantics. We may agree with Lewis and accept (V.20) only for the case in which $\alpha(\bar{x})$ all live in $\alpha(z)$, so that, heuristically,

(i′) Russell satisfies the atomic sentence "$x$ is a logician" of $\mathcal{L}_\mathrm{M}$ at the world in which he lives, iff

(ii) Russell is a logician,

and, agreeing with Kripke, leave it possible that, even if "$x$ is a logician" is true of Russell at his world, it may not be at others. Yet, even under this condition, which is technically weaker than the one Lewis adopts, we end up with the same logic.

This completes our review of how, technically, Lewis's translation scheme works. His trans-

[17] One may be able to intuitively see this fact by reading the definientia in the inductive clauses of Lewis's definition of $-^z$. Observe that, in translating an $\mathcal{L}_\mathrm{M}$ sentence $\psi$ into $\mathcal{L}_\mathrm{C}$, whenever we come across an expression $\varphi^z$ (in our use language) for a subformula of $\psi^@$, and whenever a variable $x$ occurs freely in the $\mathcal{L}_\mathrm{C}$ sentence $\varphi^z$ (but not in $\psi^@$), $x$ is bound by a "bounded quantifier the domain of which is things in $z$", as in either $\forall x\,(Ixz \to (\cdots\varphi^z\cdots))$ or $\exists x\,(Ixz \wedge (\cdots\varphi^z\cdots))$ (or their equivalent). This means that it is irrelevant for the truth of $\psi^@$ whether the $\mathcal{L}_\mathrm{M}$ sentence $\varphi$ is true of (the referent of) $x$ at (the referent of) $z$ when (the referent of) $x$ does not live in (the referent of) $z$. Nevertheless, it is rather difficult to make a precise sense of this observation, because $\varphi^z$ as an $\mathcal{L}_\mathrm{C}$ sentence can be identical to an $\mathcal{L}_\mathrm{C}$ sentence $\varphi^{z'}$ for another variable $z' \neq z$. This is why our semantic rewrite will be more useful than Lewis's syntactic account in terms of $-^z$.

lation maps $-^z$ are defined recursively by (V.2)–(V.7), (V.14), (V.15), (V.18), the motivation for which is given by the heuristic readings we laid out, namely (V.8)–(V.13), (V.16), (V.17), (V.19). Let us close this subsection with deriving (V.12) and (V.16), as we promised (derivations for (V.13) and (V.17) are similar). (V.6) entails (V.12) as follows:

$$\begin{aligned}
\mathfrak{M} \vDash_\alpha (\forall y \,.\, \varphi)^z &\overset{\text{(V.6)}}{\iff} \mathfrak{M} \vDash_\alpha \forall y\, (Iyz \to \varphi^z) \\
&\overset{\text{(III.6)}}{\iff} \mathfrak{M} \vDash_{[a/y]\alpha} Iyz \to \varphi^z \text{ for every } a \in |\mathfrak{M}| \\
&\overset{\text{(III.5)}}{\iff} \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^z \text{ for every } a \in |\mathfrak{M}| \text{ such that } \mathfrak{M} \vDash_{[a/y]\alpha} Iyz \\
&\overset{\text{(III.1)}}{\iff} \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^z \text{ for every } a \in |\mathfrak{M}| \text{ such that } (a, \alpha(z)) \in I^{\mathfrak{M}}.
\end{aligned}$$

(V.14) entails (V.16) in a similar vein, though in a more complicated way. Suppose $\bar{x}$ are all and only the free variables actually occurring in $\varphi$; then, using (III.6), (III.5), and (III.1) in a manner similar to above, together with (III.3), we have

$$\begin{aligned}
&\mathfrak{M} \vDash_\alpha (\Box\varphi)^z \\
\overset{\text{(V.14)}}{\iff} \; &\mathfrak{M} \vDash_\alpha \forall z' \,\forall y_1 \,\cdots\, \forall y_n \\
&\qquad ((Wz' \wedge Iy_1z' \wedge Cx_1y_1 \wedge \cdots \wedge Iy_nz' \wedge Cx_ny_n) \to [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'})) \\
\iff \; &\mathfrak{M} \vDash_{[a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'}) \text{ for every } \bar{a} \in |\mathfrak{M}|^n \text{ and } w \in W^{\mathfrak{M}} \\
&\qquad \text{such that } (a_i, w) \in I^{\mathfrak{M}} \text{ and } (([a_n/y_n]\cdots[a_1/y_1][w/z']\alpha)(x_i), a_i) \in C^{\mathfrak{M}} \text{ for each } i.
\end{aligned}$$

Note that the application of (V.14) above assumes that $\bar{y}$, $z'$ are new variables, which implies

$$([a_n/y_n]\cdots[a_1/y_1][w/z']\alpha)(x_i) = \alpha(x_i)$$

for each $i$, and therefore, continuing the chain of equivalences above, we have

(V.21)

$$\begin{aligned}
&\mathfrak{M} \vDash_\alpha (\Box\varphi)^z \\
\iff \; &\mathfrak{M} \vDash_{[a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'}) \text{ for every } \bar{a} \in |\mathfrak{M}|^n \text{ and } w \in W^{\mathfrak{M}} \\
&\qquad \text{such that } (a_i, w) \in I^{\mathfrak{M}} \text{ and } (\alpha(x_i), a_i) \in C^{\mathfrak{M}} \text{ for each } i.
\end{aligned}$$

Here, note also that the SoS property (III.10) of classical first-order logic (Fact 6) implies

$$\begin{aligned} & \mathfrak{M} \vDash_{[a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'}) \\ \iff & \mathfrak{M} \vDash_{[a_n/x_n][a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} [y_{n-1}/x_{n-1}]\cdots[y_1/x_1](\varphi^{z'}) \\ \vdots & \\ \iff & \mathfrak{M} \vDash_{[a_1/x_1]\cdots[a_n/x_n][a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} \varphi^{z'} \end{aligned}$$

because $([a_n/y_n]\cdots[a_1/y_1][w/z']\alpha)(y_n) = a_n$ and so on. Moreover, because $\bar{x}$ are assumed to be all distinct, two assignments $[a_1/x_1]\cdots[a_n/x_n][a_n/y_n]\cdots[a_1/y_1][w/z']\alpha$ and $[a_n/x_n]\cdots[a_1/x_1][w/z']\alpha$ agree on all variables except $\bar{y}$, whereas $\varphi^{z'}$ contains none of $\bar{y}$; hence the local determination (III.9) of classical first-order logic (Fact 4) further implies

$$\mathfrak{M} \vDash_{[a_n/y_n]\cdots[a_1/y_1][w/z']\alpha} [y_n/x_n]\cdots[y_1/x_1](\varphi^{z'}) \iff \mathfrak{M} \vDash_{[a_n/x_n]\cdots[a_1/x_1][w/z']\alpha} \varphi^{z'}$$

Therefore, by substituting the right-hand side for the left-hand side of this in (V.21), we have (V.16).

## V.2 COUNTERPART-THEORETIC SEMANTICS

### V.2.1 Semantically Rewriting Lewis's Semantic Ideas

In Subsection V.1.2, we reviewed the translation scheme Lewis proposed for translating a quantified modal language into the extensional language of counterpart theory, and laid out Lewis's semantic ideas behind the scheme. In this subsection, we reformulate the semantic ideas in fully semantic terms, in order to extract, in a rigorous manner, a counterpart-theoretic semantics for quantified modal logic.

In our review of Lewis's scheme of translating a language $\mathcal{L}_\mathrm{M}$ of quantified modal logic into the extensional language $\mathcal{L}_\mathrm{C}$ of counterpart theory, we laid out (V.8)–(V.13), (V.16), (V.17), (V.19) to illustrate the semantic ideas behind the translation scheme. Recall that, to connect the $\mathcal{L}_\mathrm{C}$ sentence $\varphi^z$, Lewis's translation of the $\mathcal{L}_\mathrm{M}$ sentence $\varphi$, to semantic ideas, we used the heuristic clause

(V.1) $\mathfrak{M} \vDash_\alpha \varphi^z \iff \mathfrak{M} \vDash_\alpha$ "the $\mathcal{L}_\mathrm{M}$ sentence $\ulcorner\varphi\urcorner$ is true of $\bar{x}$ at world $z$".

In a sense, this provides an indirect semantics for $\mathcal{L}_{\mathrm{M}}$; we may say that (V.1) expresses a semantics, in our use language, for $\mathcal{L}_{\mathrm{C}}$ and, in particular, *for* the semantics a speaker of $\mathcal{L}_{\mathrm{C}}$ would have for $\mathcal{L}_{\mathrm{M}}$. Now we assimilate this middleperson, by merging the counterpart-theoretic vocabulary into our use language and replacing (V.1) with

$$\mathfrak{M}, w \vDash_\alpha \varphi \iff \text{in } \mathfrak{M}, \text{ the } \mathcal{L}_{\mathrm{M}} \text{ sentence } \varphi \text{ is true of } \alpha(\bar{x}) \text{ at world } w,$$

with $w$ in place of $\alpha(z)$. The replacement of one semantic relation by another,

$$\text{(V.22)} \qquad \mathfrak{M} \vDash_\alpha \varphi^z \quad \rightsquigarrow \quad \mathfrak{M}, w \vDash_\alpha \varphi$$

or, notationally speaking, bringing worlds from the right-hand side of $\vDash$ to the left-hand side, captures our key idea: We replace the classical semantics for (certain sentences of) the extensional, counterpart-theoretic language $\mathcal{L}_{\mathrm{C}}$ with a possible-world semantics for the quantified modal language $\mathcal{L}_{\mathrm{M}}$.

Needless to say, we need to define our new model $\mathfrak{M}$ rigorously to define the semantics rigorously. Recall that, in Subsection V.1.2, we used the following (classical) $\mathcal{L}_{\mathrm{C}}$ structure $\mathfrak{M}$ as our model:

- $\mathfrak{M} = (|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}}, @^{\mathfrak{M}}, F^{\mathfrak{M}})$ has a set $|\mathfrak{M}|$ as a domain of quantification;
- $(|\mathfrak{M}|, W^{\mathfrak{M}}, I^{\mathfrak{M}}, C^{\mathfrak{M}}, A^{\mathfrak{M}}, @^{\mathfrak{M}})$ is a model of counterpart theory as in Definition 57; and moreover,
- $\mathfrak{M}$ also interprets every other primitive $n$-ary predicate $F$ of $\mathcal{L}_{\mathrm{C}}$ with $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$.

For our new $\mathfrak{M}$, we use a model of a similar form, as (only) partly described as follows:

- $\mathfrak{M} = (|\mathfrak{M}|, W, I, C, A, @, F^{\mathfrak{M}})$ has a set $|\mathfrak{M}|$ as a domain of *individuals*—$|\mathfrak{M}|$ is not called a domain of *quantification*, since quantifiers are not interpreted relative to it, whereas it is, or at least can be, the range of assignments;
- $(|\mathfrak{M}|, W, I, C, A, @)$ is a counterpart structure as in Definition 56; and moreover,
- $\mathfrak{M}$ interprets every primitive $n$-ary predicate $F$ of $\mathcal{L}_{\mathrm{M}}$ with $F^{\mathfrak{M}}$ (though we delay characterization of its type).

We will call such a model a *counterpart-theoretic model* for $\mathcal{L}_M$. One difference is that, whereas $W, I, C, A, @$ were part of the vocabulary of $\mathcal{L}_C$ interpreted by $\mathfrak{M}$, they are now part of our model. Another is that, whereas we interpreted primitive predicates $F$ of $\mathcal{L}_C$ with $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$, we now interpret primitive predicates $F$ of $\mathcal{L}_M$ with some $F^{\mathfrak{M}}$; but we will not discuss what type the new $F^{\mathfrak{M}}$ should have until we discuss how we should rewrite the clause (V.19) for primitive predicates.

In defining the new semantics, we should also decide, regarding the notation $\mathfrak{M}, w \vDash_\alpha \varphi$, what range the assignment $\alpha$ should have. In the old notation $\mathfrak{M} \vDash_\alpha \varphi^z$ of Subsection V.1.2, $\alpha$ ranged over everything in $|\mathfrak{M}|$—whether it lives in a world or not—because what we were doing then was to interpret $\mathcal{L}_C$ using the classical semantics of first-order logic as reviewed in Subsection III.1.1. By contrast, Kripke, for one, lets $\alpha$ range over $D$, namely all the possible individuals that may or may not exist in a given world $w$. In this subsection, we tentatively settle for the largest possible range; that is, we let assignments be of the type $\mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$. Since $I \subseteq |\mathfrak{M}| \times |\mathfrak{M}|$, it is guaranteed that $D \subseteq |\mathfrak{M}|$, that is, that anything that lives in some world or another belongs to $|\mathfrak{M}|$.

Let us rewrite the semantic ideas (V.8)–(V.13), (V.16), (V.17), (V.19) into truth conditions in the new kind of structure. Applying the key idea (V.22) above, we rewrite

(V.8) $$\mathfrak{M} \vDash_\alpha (\neg\varphi)^z \iff \mathfrak{M} \nvDash_\alpha \varphi^z,$$

(V.9) $$\mathfrak{M} \vDash_\alpha (\varphi \wedge \psi)^z \iff \mathfrak{M} \vDash_\alpha \varphi^z \text{ and } \mathfrak{M} \vDash_\alpha \psi^z,$$

(V.10) $$\mathfrak{M} \vDash_\alpha (\varphi \vee \psi)^z \iff \mathfrak{M} \vDash_\alpha \varphi^z \text{ or } \mathfrak{M} \vDash_\alpha \psi^z,$$

(V.11) $$\mathfrak{M} \vDash_\alpha (\varphi \to \psi)^z \iff \mathfrak{M} \nvDash_\alpha \varphi^z \text{ or } \mathfrak{M} \vDash_\alpha \psi^z$$

to end up with

(IV.8) $$\mathfrak{M}, w \vDash_\alpha \neg\varphi \iff \mathfrak{M}, w \nvDash_\alpha \varphi,$$

(IV.9) $$\mathfrak{M}, w \vDash_\alpha \varphi \wedge \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ and } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.10) $$\mathfrak{M}, w \vDash_\alpha \varphi \vee \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.11) $$\mathfrak{M}, w \vDash_\alpha \varphi \to \psi \iff \mathfrak{M}, w \nvDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

that is, the same truth conditions as Kripke gives, as we reviewed in Section IV.1. Rewriting

(V.12) $$\mathfrak{M} \vDash_\alpha (\forall y \,.\, \varphi)^z \iff \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^z \text{ for every } a \in |\mathfrak{M}| \text{ such that } (a, \alpha(z)) \in I^{\mathfrak{M}},$$

(V.13) $$\mathfrak{M} \vDash_\alpha (\exists y \,.\, \varphi)^z \iff \mathfrak{M} \vDash_{[a/y]\alpha} \varphi^z \text{ for some } a \in |\mathfrak{M}| \text{ such that } (a, \alpha(z)) \in I^{\mathfrak{M}}$$

yields

$$\mathfrak{M}, w \vDash_\alpha \forall y . \varphi \iff \mathfrak{M}, w \vDash_{[a/y]\alpha} \varphi \text{ for every } a \text{ such that } Iaw,$$
$$\mathfrak{M}, w \vDash_\alpha \exists y . \varphi \iff \mathfrak{M}, w \vDash_{[a/y]\alpha} \varphi \text{ for some } a \text{ such that } Iaw,$$

which coincide with Kripke's (IV.14) and (IV.15) when we read $Iaw$ as $a \in D_w$—that is, when we read Lewis's "$a$ is in $w$" as Kripke's "$a$ exists in $w$". So let us set $a \in D_w$ iff $Iaw$ and write

$$D_w = \{ a \in |\mathfrak{M}| \mid Iaw \};$$

then Lewis's adoption of (IV.14) and (IV.15) means that $D_w$ serves as the domain of quantification for the world $w$ in Lewis's semantics, in the same way it does in Kripke's semantics. Thus, Lewis shares with Kripke the same semantic ideas, (IV.8)–(IV.11), (IV.14), (IV.15), regarding how to interpret the classical operators. Hence it is helpful to reintroduce

**Definition 58.** Fix a counterpart-theoretic model $\mathfrak{M} = (|\mathfrak{M}|, W, I, C, A, @, F^{\mathfrak{M}})$ for $\mathcal{L}_{\mathrm{M}}$. Then, for any $w \in W$, the set

$$D_w = \{ a \in |\mathfrak{M}| \mid Iaw \}$$

is called the *domain of quantification for* $w$. For any $w \in W$, an assignment $\alpha : \mathrm{var}(\mathcal{L}_{\mathrm{M}}) \to |\mathfrak{M}|$ is called a *DoQ-assignment for* $w \in W$, or a *$w$-DoQ-assignment*, if $\alpha : \mathrm{var}(\mathcal{L}_{\mathrm{M}}) \to D_w$. Also, a world-thing pair $(w, a) \in W \times |\mathfrak{M}|$, a world-tuple pair $(w, \bar{a}) \in W \times |\mathfrak{M}|^n$, or a world-assignment pair $(w, \alpha) \in W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L}_{\mathrm{M}})}$ is called a *DoQ-pair* if $a \in D_w$, if $\bar{a} \in {D_w}^n$, or if $\alpha : \mathrm{var}(\mathcal{L}_{\mathrm{M}}) \to D_w$, respectively.

Once we recall the discussion on 139, it is obvious that

$$\sum_{w \in W} D_w \subseteq W \times |\mathfrak{M}|, \qquad \sum_{w \in W} {D_w}^n \subseteq W \times |\mathfrak{M}|^n, \qquad \sum_{w \in W} {D_w}^{\mathrm{var}(\mathcal{L}_{\mathrm{M}})} \subseteq W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L}_{\mathrm{M}})}$$

are the sets of world-thing DoQ-pairs, of world-tuple DoQ-pairs, and of world-assignment DoQ-pairs, respectively.

In contrast to the agreement on the classical operators, Lewis's semantic ideas regarding how to interpret modal operators seem to differ from Kripke's; that is, while Lewis's

(V.16) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M} \vDash_\alpha (\Box\varphi)^z \iff \mathfrak{M} \vDash_{[a_n/x_n]\cdots[a_1/x_1][u/z']\alpha} \varphi^{z'} \text{ for every } u \in W^{\mathfrak{M}} \text{ and } \bar{a} \in |\mathfrak{M}|^n$$
$$\text{such that each } i \text{ has } (a_i, u) \in I^{\mathfrak{M}} \text{ and } (\alpha(x_i), a_i) \in C^{\mathfrak{M}},$$

(V.17) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M} \vDash_\alpha (\Diamond\varphi)^z \iff \mathfrak{M} \vDash_{[a_n/x_n]\cdots[a_1/x_1][u/z']\alpha} \varphi^{z'} \text{ for some } u \in W^{\mathfrak{M}} \text{ and } \bar{a} \in |\mathfrak{M}|^n$$
$$\text{such that each } i \text{ has } (a_i, u) \in I^{\mathfrak{M}} \text{ and } (\alpha(x_i), a_i) \in C^{\mathfrak{M}}.$$

can be rewritten, using also the $D_w$ notation as introduced above, as

(V.23) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_{[a_n/x_n]\cdots[a_1/x_1]\alpha} \varphi \text{ for every } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } C\alpha(x_i)a_i,$$

(V.24) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then

$$\mathfrak{M}, w \vDash_\alpha \Diamond\varphi \iff \mathfrak{M}, u \vDash_{[a_n/x_n]\cdots[a_1/x_1]\alpha} \varphi \text{ for some } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } C\alpha(x_i)a_i,$$

these appear to be different from Kripke's (IV.12) and (IV.13) or our revisions (IV.49) and (IV.50) of them. We can surely see that, in a special case where $\varphi$ is a closed sentence with no free variables, (V.23) and (V.24) boil down respectively to

$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi \text{ for every } u \in W,$$
$$\mathfrak{M}, w \vDash_\alpha \Diamond\varphi \iff \mathfrak{M}, u \vDash_\alpha \varphi \text{ for some } u \in W;$$

that is, Lewis's truth conditions for closed $\Box\varphi$ and $\Diamond\varphi$ are just Kripke's with the universal accessibility relation. Yet a general case of (V.23) and (V.24) involves the counterpart relation $C$, which make them appear quite different from Kripke's (IV.12) and (IV.13), or their modified versions (IV.49) and (IV.50). (Nonetheless, in a certain sense (V.23) and (V.24) subsumes (IV.49) and (IV.50) with significant modification).

Let us now discuss how to rewrite the clause (V.19) for primitive predicates, which is

$$\text{(V.19)} \qquad \mathfrak{M} \vDash_\alpha (F\bar{x})^z \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

While it is straightforward by (V.22) how to rewrite the left-hand side, we have (at least) two options for what type our new interpretation $F^{\mathfrak{M}}$ of $F$ should have.[18]

- One option is to simply carry over the type we used in Subsection V.1.2, namely $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$, so that we rewrite (V.19) with

  $$\text{(V.25)} \qquad \mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}}_{\mathrm{L}} \qquad \text{for an } n\text{-ary primitive predicate } F,$$

  where we write $F^{\mathfrak{M}}_{\mathrm{L}}$ (with "L" for Lewis) for this interpretation of $F$. In other words, we follow Lewis more faithfully by keeping the right-hand side of (V.19) unchanged.
- Another option (or a family of options) is Kripke's interpretation using, as we reviewed in Subsection IV.1.1, a set $F^{\mathfrak{M}}_{\mathrm{K}}$ (with "K" for Kripke) of pairs of worlds and (tuples of) things. Though we have a choice regarding what domain of "things" we should take, let us take the largest possible one, that is, $|\mathfrak{M}|$; so $F^{\mathfrak{M}}_{\mathrm{K}} \subseteq W \times |\mathfrak{M}|^n$. Then we rewrite (V.19) as

  $$\text{(IV.17)} \qquad \mathfrak{M}, w \vDash_\alpha F\bar{x} \iff (w, \alpha(\bar{x})) \in F^{\mathfrak{M}}_{\mathrm{K}} \qquad \text{for an } n\text{-ary primitive predicate } F,$$

  exactly keeping the form of Kripke's truth condition for atomic sentences.

To see the difference between the two options, take again the example "$x$ is a logician" for $F\bar{x}$. Then $F^{\mathfrak{M}}_{\mathrm{K}}$ consists of $\overrightarrow{F^{\mathfrak{M}}_{\mathrm{K}}}(w)$, for each world $w$, each of which stands for the property of being-a-logician-at-$w$. By contrast, $F^{\mathfrak{M}}_{\mathrm{L}}$ stands for the property of being-a-logician *simpliciter*—recall our discussion on p. 194 of how it makes sense to use the property of being-a-logician *simpliciter* in (V.19). Also recall the remark on pp. 195f. that Lewis gives a stronger constraint than Kripke does on how to interpret atomic sentences, namely that (given a fixed assignment) each atomic sentence is either true at all worlds or true at no worlds. To put this in terms of (V.25) and (IV.17), the former implies the following, while the latter does not, for any pair of worlds $w, w' \in W$:

$$\mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}}_{\mathrm{L}} \iff \mathfrak{M}, w' \vDash_\alpha F\bar{x}.$$

[18] Recall our tentative assumption that $\mathcal{L}_{\mathrm{M}}$ has no 0-ary primitive predicates.

In other words, the satisfaction relations obeying (V.25) with $F_{\mathrm{L}}^{\mathfrak{M}}$ are, technically speaking, exactly the satisfaction relations obeying (IV.17) along with the constraint that $F_{\mathrm{K}}^{\mathfrak{M}}$ is of the form $W \times F_{\mathrm{L}}^{\mathfrak{M}}$.

Nonetheless, as we remarked on pp. 195f. and we will show in Subsection V.2.2, this difference between (V.25) and (IV.17) will turn out to make no difference to the logic of (DoQ-validity in) the semantics. It is because the autonomy of domains of quantification implies that, when restricted to their semantically relevant parts, $F_{\mathrm{L}}^{\mathfrak{M}}$ and $F_{\mathrm{K}}^{\mathfrak{M}}$ are equivalent to each other. More precisely, while

$$F_{\mathrm{L}}^{\mathfrak{M}} \cap D^n \subseteq D^n, \qquad \text{where} \qquad D^n = \{\, \bar{a} \in |\mathfrak{M}|^n \mid \bar{a} \in {D_w}^n \text{ for some } w \in W \,\},$$

and

$$F_{\mathrm{K}}^{\mathfrak{M}} \cap \sum_{w \in W} {D_w}^n \subseteq \sum_{w \in W} {D_w}^n$$

are the only semantically significant parts respectively of $F_{\mathrm{L}}^{\mathfrak{M}}$ and of $F_{\mathrm{K}}^{\mathfrak{M}}$, subsets of $D^n$ correspond to subsets of $\sum_{w \in W} {D_w}^n$ along the bijection $(w, \bar{a}) \mapsto \bar{a}$ from $\sum_{w \in W} {D_w}^n$ to $D^n$ (which is bijective because each $\bar{a} \in D^n$ has exactly one $w$ in which all of $\bar{a}$ live). For the moment, we settle for a wider class of satisfaction relations with fewer constraint, that is, the semantics with $F_{\mathrm{K}}^{\mathfrak{M}}$ and (IV.17), since it will enable us to prove the autonomy in a stronger form than the semantics with $F_{\mathrm{L}}^{\mathfrak{M}}$ and (V.25) would; after proving the autonomy, we will reconsider $F_{\mathrm{L}}^{\mathfrak{M}}$ and (V.25).

Thus, for now, we rewrite (less faithfully) the clause (V.19) for primitive predicates as

(IV.17) $$\mathfrak{M}, w \vDash_{\alpha} F\bar{x} \iff (w, \alpha(\bar{x})) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

Moreover, we can now fully define our models:

**Definition 59.** Given a language $\mathcal{L}$ of quantified modal logic, we say a tuple

$$\mathfrak{M} = (|\mathfrak{M}|, W, I, C, A, @, F^{\mathfrak{M}})$$

is a *counterpart-theoretic model* for $\mathcal{L}$ if

- $(|\mathfrak{M}|, W, I, C, A, @)$ is a counterpart structure as in Definition 56; and moreover,
- $\mathfrak{M}$ is equipped with $F^{\mathfrak{M}} \subseteq W \times |\mathfrak{M}|^n$ for each $n$-ary primitive predicate $F$.

When $\mathfrak{M} = \mathfrak{M} = (|\mathfrak{M}|, W, I, C, A, @, F^{\mathfrak{M}})$ is a counterpart-theoretic model for $\mathcal{L}$, $W$ is called the *set of worlds* of $\mathfrak{M}$.

Let us close this subsection with a series of obvious definitions and results for the counterpart-theoretic semantics.

**Definition 60.** Given a language $\mathcal{L}$ of quantified modal logic, a *counterpart-theory-type satisfaction relation* for $\mathcal{L}$ is a pair $(\mathfrak{M}, \vDash)$ of a counterpart-theoretic model $\mathfrak{M}$ for $\mathcal{L}$ and any relation $(\mathfrak{M}, - \vDash_{-} -) \subseteq W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \times \mathrm{sent}(\mathcal{L})$, as in $\mathfrak{M}, w \vDash_{\alpha} \varphi$, where $W$ is the set of worlds of $\mathfrak{M}$. We say a counterpart-theory-type satisfaction relation for $\mathcal{L}$ is on $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$.

**Definition 61.** Given a quantified modal language $\mathcal{L}$, for each counterpart-theory-type satisfaction relation $(\mathfrak{M}, \vDash)$ for $\mathcal{L}$ with $W$ the set of worlds of $\mathfrak{M}$, we say

- a sentence $\varphi$ of $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, and write $\mathfrak{M} \vDash \varphi$, meaning that $\mathfrak{M}, w \vDash_{\alpha} \varphi$ for every $w \in W$ and assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$; and

- an inference $(\Gamma, \varphi)$ in $\mathcal{L}$ is *valid in* $(\mathfrak{M}, \vDash)$, meaning that if $\mathfrak{M} \vDash \psi$ for all $\psi \in \Gamma$ then $\mathfrak{M} \vDash \varphi$.

Given a class of counterpart-theory-type satisfaction relations for $\mathcal{L}$, we say a sentence or inference is *valid in* that class if it is valid in every member of that class.

**Definition 62.** A counterpart-theory-type satisfaction relation for a quantified modal language $\mathcal{L}$ is called *counterpart-theoretic*, and said to be simply a *counterpart-theoretic satisfaction relation* for $\mathcal{L}$, if it satisfies (IV.8)–(IV.11), (IV.14), (IV.15), (V.23), (V.24), and (IV.17):

(IV.8) $$\mathfrak{M}, w \vDash_\alpha \neg\varphi \iff \mathfrak{M}, w \nvDash_\alpha \varphi,$$

(IV.9) $$\mathfrak{M}, w \vDash_\alpha \varphi \wedge \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ and } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.10) $$\mathfrak{M}, w \vDash_\alpha \varphi \vee \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.11) $$\mathfrak{M}, w \vDash_\alpha \varphi \to \psi \iff \mathfrak{M}, w \nvDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.14) $$\mathfrak{M}, w \vDash_\alpha \forall x . \varphi \iff \mathfrak{M}, w \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in D_w,$$

(IV.15) $$\mathfrak{M}, w \vDash_\alpha \exists x . \varphi \iff \mathfrak{M}, w \vDash_{[a/x]\alpha} \varphi \text{ for some } a \in D_w,$$

(V.23) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then
$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_{[a_n/x_n]\cdots[a_1/x_1]\alpha} \varphi \text{ for every } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } C\alpha(x_i)a_i,$$

(V.24) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then
$$\mathfrak{M}, w \vDash_\alpha \Diamond\varphi \iff \mathfrak{M}, u \vDash_{[a_n/x_n]\cdots[a_1/x_1]\alpha} \varphi \text{ for some } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } C\alpha(x_i)a_i,$$

(IV.17) $$\mathfrak{M}, w \vDash_\alpha F\bar{x} \iff (w, \alpha(\bar{x})) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

Also, by the *counterpart-theoretic semantics* for $\mathcal{L}$, we mean the class of all counterpart-theoretic satisfaction relations for $\mathcal{L}$.

### V.2.2 Operational Form of Counterpart-Theoretic Semantics

In Subsection V.2.1, we laid out a satisfaction-relation formulation for Lewis's semantic ideas. In this subsection, we further rewrite it in an operational formulation and prove that the semantics has autonomous domains of quantifications.

First let us define the type of interpretations. It should be similar to the type of general Kripke-type, which we defined in Section IV.3 (Definition 51 on p. 167): the type can be similar to Kripke's because we chose (IV.17), the same type of condition as Kripke did, as the truth condition for atomic sentences; but the type has to be general rather than uniform because, under the truth conditions (V.23) and (V.24), $\Box$ and $\Diamond$ cannot be interpreted in uniformly. Then we define the notions of preservation of local determination and DoQ-restrictability in a similar manner to their definitions in Section IV.3 (Definition 52 on 170 and Definition 54 on 175); see Section IV.3 for the motivations behind these technical definitions.

**Definition 63.** Given a quantified modal language $\mathcal{L}$, a *counterpart-theory-type interpretation for* $\mathcal{L}$ is a pair of a counterpart-theoretic model $\mathfrak{M}$ for $\mathcal{L}$ and a map $[\![-]\!]$ that assigns,

- to each variable $x$, a map

$$[\![x]\!] : |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to |\mathfrak{M}|$$

  that satisfies

$$[\![x]\!] : \alpha \mapsto \alpha(x),$$

- to each sentence $\varphi$, a map

$$[\![\varphi]\!] : W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

  that satisfies

$$[\![Fx_1 \cdots x_n]\!] = F^{\mathfrak{M}} \circ (1_W \times \langle [\![x_1]\!], \ldots, [\![x_n]\!] \rangle),$$

- and, to each $n$-ary sentential operator $\otimes$ of $\mathcal{L}$, a family of maps

$$[\![\otimes]\!]^{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$$

  for all finite sets $\bar{x}$ of variables of $\mathcal{L}$, such that

$$[\![\otimes(\varphi_1, \ldots, \varphi_n)]\!] = [\![\otimes]\!]^{\bar{x}}([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!])$$

  for the set $\bar{x}$ of variables that occur freely in at least one of $\varphi_1, \ldots, \varphi_n$.

We say a counterpart-theory-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ interprets a sentential operator $\otimes$ of $\mathcal{L}$ *uniformly* if the family $[\![\otimes]\!]^{\bar{x}}$ is constant, that is, if there is a unique operation $f$ such that $[\![\otimes]\!]^{\bar{x}} = f$ for every $\bar{x}$; then we simply write $[\![\otimes]\!]$ for $[\![\otimes]\!]^{\bar{x}}$. We also say a counterpart-theory-type interpretation for $\mathcal{L}$ is *on* $\mathfrak{M}$ if its first coordinate is $\mathfrak{M}$, and is *over* a counterpart structure $\mathfrak{F}$ if it is on a counterpart-theoretic model over $\mathfrak{F}$.

**Definition 64.** Let $\mathcal{L}$ be a quantified modal language and let $\mathfrak{M}$ be a counterpart structure with a set $W$ of worlds. Then, for variables $\bar{y}$ of $\mathcal{L}$, we say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ for all finite sets $\bar{x}$ of variables *preserves local determination with the binding of* $\bar{y}$ if, for every finite set $\bar{x}$ of variables of $\mathcal{L}$, there is an operation $f^{\bar{x}}_{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}\setminus\bar{y}})$ such that, for every $B : W \times |\mathfrak{M}|^{\bar{x}} \to \mathbf{2}^n$,

$$f^{\bar{x}}_{\bar{x}}(B) \circ r_{\bar{x}\setminus\bar{y}} = f^{\bar{x}}(B \circ r_{\bar{x}}) : W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \to \mathbf{2}$$

(where $r_{\bar{x}} : (w, \alpha) \mapsto (w, \alpha{\restriction}\bar{x})$ and $r_{\bar{x}\setminus\bar{y}} : (w, \alpha) \mapsto (w, \alpha{\restriction}(\bar{x} \setminus \bar{y}))$), that is, that makes the following diagram commute.

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;- \circ r_{\bar{x}}\;} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})^n \\
{\scriptstyle f^{\bar{x}}}\downarrow & & \downarrow{\scriptstyle f^{\bar{x}}_{\bar{x}}} \\
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;- \circ r_{\bar{x}\setminus\bar{y}}\;]{} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}\setminus\bar{y}})
\end{array}$$

We also say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ (for a fixed $n$) preserves local determination *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and if the family preserves local determination with the binding of the variables that $\otimes$ binds. Moreover, we say a counterpart-theory-type interpretation for $\mathcal{L}$ preserves local determination if it interprets every sentential operator $\otimes$ of $\mathcal{L}$ with a family of operations that preserves local determination for $\otimes$.

**Definition 65.** Let $\mathcal{L}$ be a quantified modal language and $\mathfrak{M}$ be a counterpart structure with a set $W$ of worlds. Then, for any finite sets $\bar{x}$ and $\bar{y}$ of variables of $\mathcal{L}$, we say an operation

$$f : \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\bar{y}})^m$$

is *DoQ-restrictable* if it is restrictable to the sets $\sum_{w \in W} {D_w}^{\bar{x}}$ and $\sum_{w \in W} {D_w}^{\bar{y}}$ of DoQ-pairs, where $D_w$ is the domain of quantification for $w$; in other words, if there is $f_{\mathrm{DoQ}}$ that makes the diagram below commute.

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})^n & \xrightarrow{\;f\;} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{y}})^m \\
{\scriptstyle - \circ i}\downarrow & & \downarrow{\scriptstyle - \circ i} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right)^n & \xrightarrow[\;f_{\mathrm{DoQ}}\;]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{y}}\right)^m
\end{array}$$

**Definition 66.** Let $\mathcal{L}$ be a quantified modal language and $\mathfrak{M}$ be a counterpart structure with a set $W$ of worlds. Then we say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ is *DoQ-restrictable with the binding of variables $\bar{y}$* if

- the family $f^{\bar{x}}$ preserves local determination with the binding of $\bar{y}$, and, moreover,
- for each finite set $\bar{x}$ of variables, the operator $f^{\bar{x}}_{\bar{x}}$ that makes

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n & \xleftarrow{\;-\circ r_{\bar{x}}\;} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})^n \\
{\scriptstyle f^{\bar{x}}}\downarrow & \rotatebox{45}{$=$} & \downarrow{\scriptstyle f^{\bar{x}}_{\bar{x}}} \\
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;-\circ r_{\bar{x}\setminus\bar{y}}\;]{} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}\setminus\bar{y}})
\end{array}$$

  commute (which uniquely exists since the family $f^{\bar{x}}$ preserves local determination with the binding of $\bar{y}$) is DoQ-restrictable.

We also say a family of operations $f^{\bar{x}} : \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n \to \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})$ is DoQ-restrictable *for* a sentential operator $\otimes$ of $\mathcal{L}$ if $\otimes$ is $n$-ary and the family $f^{\bar{x}}$ is DoQ-restrictable with the binding of the variables that $\otimes$ binds. Moreover, we say a counterpart-theory-type interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ is *DoQ-restrictable* if it interprets each sentential operator $\otimes$ of $\mathcal{L}$ with a family of operators that is DoQ-restrictable for $\otimes$.

Now let us rewrite the satisfaction-relation formulation of Lewis's semantics into an operational form. Whereas (IV.17) for atomic sentences does not need rewriting (since it is part of Definition 63), it is obvious how to rewrite (IV.8)–(IV.11), (IV.14), (IV.15) for the classical first-order operators; that is, we simply extend the same conditions as we adopted for Kripke's semantics, since Lewis shares the same semantic ideas with Kripke regarding these operators. (V.23) and (V.24) for the modal operators can be rewritten with the help of the following notation. For any assignment $\alpha : \mathrm{var}(\mathcal{L}) \to |\mathfrak{M}|$ and any tuple $\bar{x}$ of variables of $\mathcal{L}$, let us write

$$C^{\bar{x}}(\alpha) = \{\, (w, [a_n/x_n] \cdots [a_1/x_1]\alpha) \in W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid a_i \in D_w \text{ and } C\alpha(x_i)a_i \text{ for each } i \leqslant n \,\};$$

that is, for every $(w, \beta) \in W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$,

$$(w, \beta) \in C^{\bar{x}}(\alpha) \iff \beta = [a_n/x_n] \cdots [a_1/x_1]\alpha \text{ for some } \bar{a} \in {D_w}^n \text{ such that } C\alpha(x_i)a_i \text{ for each } i \leqslant n.$$

So, in particular, for $\bar{x} = \varnothing$ and $n = 0$, we set

$$C^{\varnothing}(\alpha) = W \times \{\alpha\}.$$

Then it is easy to see that, using this notation, we can rewrite (V.23) and (V.24) as

$$(w, \alpha) \in [\![\Box]\!]^{\bar{x}}(A) \iff C^{\bar{x}}(\alpha) \subseteq A, \tag{V.26}$$

$$(w, \alpha) \in [\![\Diamond]\!]^{\bar{x}}(A) \iff C^{\bar{x}}(\alpha) \cap A \neq \varnothing. \tag{V.27}$$

So, to sum up, we enter:

**Definition 67.** Given a quantified modal language $\mathcal{L}$, a counterpart-theory-type interpretation for $\mathcal{L}$ is said to be *counterpart-theoretic*, and called simply a *counterpart-theoretic interpretation for* $\mathcal{L}$, if it interprets $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$ uniformly with the constant families of operations

(IV.34) $\quad [\![\neg]\!] = \neg_{\mathbf{2}} \circ -,$

(IV.35) $\quad [\![\wedge]\!] = \wedge_{\mathbf{2}} \circ -,$

(IV.36) $\quad [\![\vee]\!] = \vee_{\mathbf{2}} \circ -,$

(IV.37) $\quad [\![\rightarrow]\!] = \rightarrow_{\mathbf{2}} \circ -,$

(IV.40) $\quad [\![\forall x]\!] = \prod_{w \in W} [\![\forall x]\!]_w$, where $[\![\forall x]\!]_w(A) = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A$ for every $a \in D_w \,\}$,

(IV.41) $\quad [\![\exists x]\!] = \prod_{w \in W} [\![\exists x]\!]_w$, where $[\![\exists x]\!]_w(A) = \{\, \alpha \in |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})} \mid [a/x]\alpha \in A$ for some $a \in D_w \,\}$,

respectively, and if it interprets $\Box$ and $\Diamond$ with the families of operations $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ satisfying (V.26) and (V.27), respectively.

By our assumption that the classical and modal operators are the only sentential operators of a quantified modal language $\mathcal{L}$, counterpart-theoretic interpretations for $\mathcal{L}$ correspond one-to-one to counterpart-theoretic models for $\mathcal{L}$ and then to counterpart-theoretic satisfaction relations for $\mathcal{L}$. Hence the class of counterpart-theoretic interpretations for $\mathcal{L}$ can also be called the counterpart-theoretic semantics for $\mathcal{L}$. Then the counterpart-theoretic semantics preserves local determination and is DoQ-restrictable.

**Fact 54.** Given a quantified modal language $\mathcal{L}$, every counterpart-theoretic interpretation $(\mathfrak{M}, [\![-]\!])$ preserves local determination.

*Proof.* A proof that the families $[\![\neg]\!]^{\bar{x}}$, $[\![\wedge]\!]^{\bar{x}}$, $[\![\vee]\!]^{\bar{x}}$, $[\![\rightarrow]\!]^{\bar{x}}$, $[\![\forall x]\!]^{\bar{x}}$, $[\![\exists y]\!]^{\bar{x}}$ preserve local determination for $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$, respectively, is similar to the proof of Fact 45. A proof that the family $[\![\Box]\!]^{\bar{x}}$ preserves local determination for $\Box$ consists of fixing any $\bar{x}$ and showing that some $[\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}$ makes the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow{\;- \circ r_{\bar{x}}\;} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}}) \\
{\scriptstyle [\![\Box]\!]^{\bar{x}}}\big\downarrow & \leq & \big\downarrow{\scriptstyle [\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}} \\
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) & \xleftarrow[\;- \circ r_{\bar{x}}\;]{} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}})
\end{array}$$

To show it, let us write, for $\beta : \bar{x} \rightarrow |\mathfrak{M}|$,

$$C^{\bar{x}}_{\bar{x}}(\beta) = \{\, (w, \beta') \in W \times |\mathfrak{M}|^{\bar{x}} \mid \beta'(x_i) \in D_w \text{ and } C\beta(x_i)\beta'(x_i) \text{ for each } i \leqslant n \,\}$$

with $C^{\varnothing}_{\varnothing}(\varnothing) = W$, and define $[\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}$ so that, for every $(w, \beta) \in W \times |\mathfrak{M}|^{\bar{x}}$ and $B \subseteq W \times |\mathfrak{M}|^{\bar{x}}$,

(V.28) $$(w, \beta) \in [\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}(B) \iff C^{\bar{x}}_{\bar{x}}(\beta) \subseteq B.$$

Then the diagram above commutes because, for every $(w, \alpha) \in W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ and $B \subseteq W \times |\mathfrak{M}|^{\bar{x}}$,

$$\begin{aligned}
(w, \alpha) \in [\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}(B) \circ r_{\bar{x}} &\iff (w, \alpha \restriction \bar{x}) \in [\![\,\bar{x} \mid \Box\,]\!]^{\bar{x}}(B) \\
&\iff C^{\bar{x}}_{\bar{x}}(\alpha \restriction \bar{x}) \subseteq B \\
&\iff (u, \beta) \in B \text{ for every } (u, \beta) \in W \times |\mathfrak{M}|^{\bar{x}} \\
&\qquad\qquad \text{such that } \beta(x_i) \in D_u \text{ and } C\alpha(x_i)\beta(x_i) \text{ for each } i \leqslant n \\
&\iff (u, [a_n/x_n] \cdots [a_1/x_1]\alpha) \in B \circ r_{\bar{x}} \text{ for every } u \in W \\
&\qquad\qquad \text{and } \bar{a} \in D_u{}^n \text{such that } C\alpha(x_i)a_i \text{ for each } i \leqslant n \\
&\iff C^{\bar{x}}(\alpha) \subseteq B \circ r_{\bar{x}} \\
&\iff (w, \alpha) \in [\![\Box]\!]^{\bar{x}}(B \circ r_{\bar{x}}).
\end{aligned}$$

We can similarly prove that the family $[\![\Diamond]\!]^{\bar{x}}$ preserves local determination for $\Diamond$ by defining

(V.29) $$(w, \beta) \in [\![\,\bar{x} \mid \Diamond\,]\!]^{\bar{x}}(B) \iff C^{\bar{x}}_{\bar{x}}(\beta) \cap B \neq \varnothing$$

(and replacing "every" above with "some"). $\Box$

Our proof for DoQ-restrictability goes in a manner similar to our proof in Subsection IV.3.3 for the DoQ-restrictability of DoQ-autonomous Kripkean semantics.

**Fact 55.** Given a quantified modal language $\mathcal{L}$, a counterpart-theoretic interpretation $(\mathfrak{M}, [\![-]\!])$, and any finite set $\bar{x}$ of variables, the operations $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ satisfying (V.28) and $[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}$ satisfying (V.29), as in the proof for Fact 54 above, are DoQ-restrictable.

*Proof.* A proof that $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ is DoQ-restrictable amounts to showing that there is ${[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}$ making

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}}) & \xrightarrow{[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}} & \mathcal{P}(W \times |\mathfrak{M}|^{\bar{x}}) \\
{\scriptstyle - \circ i}\downarrow & \not= & \downarrow{\scriptstyle - \circ i} \\
\mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right) & \xrightarrow[{[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}]{} & \mathcal{P}\left(\sum_{w \in W} {D_w}^{\bar{x}}\right)
\end{array}$$

commute. So let us define ${[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}$ so that, for every $(u, \beta) \in \sum_{w \in W} {D_w}^{\bar{x}}$ and $B \subseteq \sum_{w \in W} {D_w}^{\bar{x}}$,

$$(u, \beta) \in {[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}(B) \iff C^{\bar{x}}_{\bar{x}}(\beta) \subseteq B. \tag{V.30}$$

Observe that, by definition, $C^{\bar{x}}_{\bar{x}}(\beta) \subseteq \sum_{w \in W} {D_w}^{\bar{x}}$ for any $\beta : \bar{x} \to |\mathfrak{M}|$, which implies the equivalence marked with $*$ below: For every $(u, \beta) \in \sum_{w \in W} {D_w}^{\bar{x}}$ and $B \subseteq W \times D^{\bar{x}}$, we have

$$\begin{aligned}
(u, \beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \cap \sum_{w \in W} {D_w}^{\bar{x}} &\iff (u, \beta) \in [\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \\
&\overset{(\mathrm{V.28})}{\iff} C^{\bar{x}}_{\bar{x}}(\beta) \subseteq B \\
&\overset{*}{\iff} C^{\bar{x}}_{\bar{x}}(\beta) \subseteq B \cap \sum_{w \in W} {D_w}^{\bar{x}} \\
&\overset{(\mathrm{V.30})}{\iff} (u, \beta) \in {[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}\left(B \cap \sum_{w \in W} {D_w}^{\bar{x}}\right);
\end{aligned}$$

thus $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}(B) \circ i = {[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}(B \circ i)$, making the diagram above commute. Hence $[\![\, \bar{x} \mid \Box \,]\!]^{\bar{x}}$ is DoQ-restrictable. Similarly, $[\![\Diamond]\!]^{\bar{x}}$ is DoQ-restrictable, with ${[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}$ such that

$$(u, \beta) \in {[\![\, \bar{x} \mid \Diamond \,]\!]^{\bar{x}}}_{\mathrm{DoQ}}(B) \iff C^{\bar{x}}_{\bar{x}}(\beta) \cap B \neq \varnothing \tag{V.31}$$

for every $(u, \beta) \in \sum_{w \in W} {D_w}^{\bar{x}}$ and $B \subseteq \sum_{w \in W} {D_w}^{\bar{x}}$. □

**Fact 56.** Given a quantified modal language $\mathcal{L}$, every counterpart-theoretic interpretation $(\mathfrak{M}, [\![-]\!])$ is DoQ-restrictable.

*Proof.* The families $[\![\Box]\!]^{\bar{x}}$ and $[\![\Diamond]\!]^{\bar{x}}$ are DoQ-restrictable for $\Box$ and $\Diamond$ by Fact 55. On the other hand, a proof that the families $[\![\neg]\!]^{\bar{x}}$, $[\![\wedge]\!]^{\bar{x}}$, $[\![\vee]\!]^{\bar{x}}$, $[\![\rightarrow]\!]^{\bar{x}}$, $[\![\forall x]\!]^{\bar{x}}$, $[\![\exists y]\!]^{\bar{x}}$ are DoQ-restrictable for $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall x$, $\exists x$, respectively, is similar to the proof of Corollary 8 (on 175). $\Box$

In this way, the counterpart-theoretic semantics is DoQ-restrictable. Before closing this subsection, let us observe that, not only is it restrictable to the domains of quantifications, the semantics is also restrictable to the domain of possible individuals in Kripke's sense, that is,

$$D = \bigcup_{w \in W} D_w \subseteq |\mathfrak{M}|.$$

The semantics is, indeed, restrictable to any set $E$ such that $D \subseteq E \subseteq |\mathfrak{M}|$. This follows from the following fact.

**Fact 57.** Let $\mathcal{L}$ be a given quantified modal language, $(\mathfrak{M}, [\![-]\!])$ be any counterpart-theoretic interpretation, and $E$ be any set such that $\bigcup_{w \in W} D_w \subseteq E \subseteq |\mathfrak{M}|$. Then, for every $n$-ary sentential operator $\otimes$ of $\mathcal{L}$ and any finite set $\bar{x}$ of variables of $\mathcal{L}$, there is an operation $[\![\otimes]\!]^{\bar{x}}{}_E$ such that

$$\begin{array}{ccc}
\mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})})^n & \xrightarrow{\ [\![\otimes]\!]^{\bar{x}}\ } & \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) \\
{\scriptstyle -\circ i}\downarrow & /\!\!/ & \downarrow{\scriptstyle -\circ i} \\
\mathcal{P}(W \times E^{\mathrm{var}(\mathcal{L})})^n & \xrightarrow[\ [\![\otimes]\!]^{\bar{x}}{}_E\ ]{} & \mathcal{P}(W \times E^{\mathrm{var}(\mathcal{L})})
\end{array}$$

commutes, where $i : W \times E^{\mathrm{var}(\mathcal{L})} \hookrightarrow W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ is the inclusion map.

*Proof.* When $[\![\otimes]\!]^{\bar{x}}$ is a postcomposition $f \circ -$ with $f : \mathbf{2}^n \rightarrow \mathbf{2}$, and in particular when $\otimes$ is $\neg$, $\wedge$, $\vee$, or $\rightarrow$, the same postcomposition $f \circ -$ yields such $[\![\otimes]\!]^{\bar{x}}{}_E$ as above. While $[\![\forall y]\!]^{\bar{x}} = \prod_{w \in W} [\![\forall y]\!]_w$, each $[\![\forall y]\!]_w$ is restrictable to $E$ by Fact 23 (on p. 110) since $D_w \subseteq E$; therefore $\prod_{w \in W} ([\![\forall y]\!]_w)_E$ serves as $[\![\forall y]\!]^{\bar{x}}{}_E$ for $\otimes = \forall y$. Similarly for $\otimes = \exists y$.

Let $\otimes = \Box$ and observe that, because $D_w \subseteq E$ for all $w \in W$, we have $C^{\bar{x}}(\alpha) \subseteq W \times E^{\mathrm{var}(\mathcal{L})}$ for any $\alpha : \mathrm{var}(\mathcal{L}) \to E$, which implies the equivalence marked with $*$ below: For every $A \subseteq W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}$ and $(w, \alpha) \in W \times E^{\mathrm{var}(\mathcal{L})}$,

$$\begin{aligned}(w,\alpha) \in [\![\Box]\!]^{\bar{x}}(A) \cap (W \times E^{\mathrm{var}(\mathcal{L})}) &\overset{\dagger}{\iff} (w,\alpha) \in [\![\Box]\!]^{\bar{x}}(A) \\ &\overset{(\mathrm{V.26})}{\iff} C^{\bar{x}}(\alpha) \subseteq A \\ &\overset{*}{\iff} C^{\bar{x}}(\alpha) \subseteq A \cap (W \times E^{\mathrm{var}(\mathcal{L})}) \\ &\overset{(\mathrm{V.26})}{\iff} (w,\alpha) \in [\![\Box]\!]^{\bar{x}}(A \cap (W \times E^{\mathrm{var}(\mathcal{L})})) \\ &\overset{\dagger}{\iff} (w,\alpha) \in [\![\Box]\!]^{\bar{x}}(A \cap (W \times E^{\mathrm{var}(\mathcal{L})})) \cap (W \times E^{\mathrm{var}(\mathcal{L})}),\end{aligned}$$

where the equivalences with † hold since $(w, \alpha) \in W \times E^{\mathrm{var}(\mathcal{L})}$. This means that, writing

$$\mathcal{P}(W \times E^{\mathrm{var}(\mathcal{L})}) \underset{i^* = - \circ i}{\overset{i_*}{\rightleftarrows}} \mathcal{P}(W \times |\mathfrak{M}|^{\mathrm{var}(\mathcal{L})}) ,$$

we have $i^* \circ [\![\Box]\!]^{\bar{x}} = i^* \circ [\![\Box]\!]^{\bar{x}} \circ i_* \circ i^*$, and so $i^* \circ [\![\Box]\!]^{\bar{x}} \circ i_*$ serves as $[\![\Box]\!]^{\bar{x}}{}_E$. Similarly for $\otimes = \Diamond$. $\Box$

**Corollary 9.** The counterpart-theoretic semantics for a given quantified language $\mathcal{L}$ is restrictable to the domain of possible individuals, that is, to the set

$$D = \bigcup_{w \in W} D_w$$

for a given counterpart-theoretic model $\mathfrak{M} = (|\mathfrak{M}|, W, I, C, A, @)$, with each $D_w$ being the domain of quantification for $w \in W$.

This result shows that, even though we have been taking $|\mathfrak{M}|$ as the range of assignments, we can safely restrict the range to the domain $D$ of possible individuals, as we did when dealing with Kripkean semantics in Chapter IV. Therefore, for the rest of this chapter, we take $D$ rather than $|\mathfrak{M}|$ as the range of assignments. (We omit the obvious series of redefinitions.)

### V.2.3 Bundle Formulation of Counterpart Theory

In this subsection, we introduce a series of notations that will be helpful in our analysis of agreement and disagreement between Kripke's and Lewis's semantic ideas.

It should be obvious from the exposition above in Subsection V.1.1 of the postulates P1–P8 of counterpart theory (for example, that the counterpart relation is a relation among things in worlds) that the notion of things in worlds plays a crucial conceptual role in counterpart theory. So let us introduce notation for the set of them. Recall the notation

$$D_w = \{ a \in |\mathfrak{M}| \mid Iaw \},$$

which we introduced in Subsection V.2, so that Lewis's truth conditions for quantifiers coincided with Kripke's; so, for both Lewis and Kripke, $D_w$ is the domain of quantification for the world $w \in W$. Recall also that every Kripke model assigns to each world $w \in W$ the set $D_w$ of individuals that exist in $w$, and moreover that Kripke defines $D$, the domain of all possible individuals, as the set of individuals that exist in some world or other, that is,

$$D = \bigcup_{w \in W} D_w.$$

This suggests that we should write $D$, in the framework of counterpart theory as well, for the set of things that live in some world or other; so we set

$$D = \bigcup_{w \in W} D_w = \{ a \in |\mathfrak{M}| \mid \exists w . Iaw \}.$$

The disjointness of ontology, which is expressed by the postulate

(P2) $$\forall x \forall y \forall z (Ixy \wedge Ixz \to y = z),$$

namely that everything lives in at most one world, means that $D$ is partitioned by the family of $D_w$ for all $w \in W$; so, using $\sum$ to signify disjoint union, we can write

$$D = \sum_{w \in W} D_w$$

to express the disjointness of ontology.

There is another notation useful in expressing the disjointness of ontology, namely P2, as well as other postulates. Note that the definition $D = \{ a \in |\mathfrak{M}| \mid \exists w . Iaw \}$ means that $D$ is the domain of the relation $I$. Then the postulate P2 above amounts to the statement that the relation $I$ is a function with domain $D$; let us refer to this function by $\pi$, so $\pi(a) = w$ means $Iaw$. On the other hand,

(P1) $$\forall a \, \forall w \, (Iaw \to Ww)$$

amounts to the statement that the range of the relation $I$ is contained in $W$. Therefore P1 and P2 together mean

$$\pi : D \to W,$$

which in turn entails both P1 and P2; thus $\pi : D \to W$ rewrites $I$ (with P1 and P2) in functional terms. We call $\pi$ a *residence map*, because it assigns to each $x \in D$ the world $\pi(x)$ in which $x$ lives. $D_w$ is then the set of *residents* of the world $w$. It is also helpful to note that, when $w \in W$, we can define $D_w$ as

$$D_w = \{ a \in D \mid Iaw \} = \pi^{-1}[\{w\}]$$

in terms of $D$ and $\pi$.

Given the rewrite of counterpart theory so far, let us carry on to observe how we can rewrite the other postulates in terms of $D$ and $\pi$. Recall that

(P3) $$\forall x \, \forall y \, (Cxy \to \exists w . Iyw),$$

(P4) $$\forall x \, \forall y \, (Cxy \to \exists w . Ixw)$$

together state that the counterpart relation $C$ is a relation among things in worlds; therefore they amount to the statement that $C$ is a relation on $D$, or, in notation, $C \subseteq D \times D$. It is also easy to see that $C \subseteq D \times D$ entails both P3 and P4.

In rewriting P5 and P6, it is useful to take the right transpose of the relation $C$. That is, because $C \subseteq D \times D$, we can take the map $\gamma : D \to \mathcal{P}D$ such that, when $a \in D$,

$$\gamma(a) = \{ b \in D \mid Cab \};$$

that is, $\gamma(a)$ is the set of counterparts of $a$. Note that $\gamma(a) \subseteq D$ implies P3, while P4 follows from our stipulation that the domain of $\gamma$ is $D$. Then recall that

(P5) $$\forall a\, \forall b\, \forall w\, (Iaw \land Ibw \land Cab \to a = b),$$

(P6) $$\forall a\, \forall w\, (Iaw \to Caa)$$

together mean that anything $a$, if it lives in any world, is the one and only counterpart of itself in its world $\pi(a)$. So, in terms of $\gamma$ and $\pi$, P5 and P6 mean that, for every $a \in D$, $\gamma(a)$ restricted to $\pi(a)$ consists of $a$ and only of $a$; or, in notation,

$$\gamma(a) \cap D_{\pi(a)} = \{a\}.$$

It is easy to see that this entails P5 and P6.

We turn now to the postulates on the notion of actuality, which are

(P7) $$\exists w\, (Ww \land \forall a\, (Iaw \equiv Aa)),$$

(P8) $$\exists a\, Aa.$$

Recall that P2, P7, and P8 imply that there uniquely exists a world @ such that $Aa$ iff $Ia@$, that is, such that anything is actual if and only if it lives in @. Assuming P2, therefore, P7 and P8 entail $@ \in W$ and $A = D_@ \neq \varnothing$, or, equivalently, that @ is in the range of $\pi$. On the other hand, P7 and P8 follow from $@ \in W$ and $A = D_@ \neq \varnothing$; so we regard these as our rewrite of P7 and P8.

To sum up, whereas Lewis's own formulation of counterpart theory is given in Definition 56, we can formulate it alternatively as follows.

**Definition 68.** We say $(W, \pi, \gamma, @)$ is a counterpart structure if

- $W$ is a set;
- $\pi$ is a function to $W$, that is, $\pi : D \to W$ for some set $D$;
- $\gamma$ is a map of the type $\gamma : D \to \mathcal{P}D$, that is, it assigns $\gamma(a) \subseteq D$ to each $a \in D$, while $\gamma(a)$ is defined only if $a \in D$;
- $\gamma(a) \cap \pi^{-1}[\{\pi(a)\}] = \{a\}$ for each $a \in D$;
- @ is in the range of $\pi$, that is, $@ \in W$ and $\pi^{-1}[\{@\}] \neq \varnothing$.

Though it seems obvious enough that the two formulations are equivalent, let us describe the equivalence more formally with the following notation, because it will be useful shortly in showing that the equivalence extends to the level of semantics.

**Notation 1.** Given a tuple $\mathfrak{M} = (X, W, I, C, A, @)$ of the type as in Definition 56—namely, $X$ is a set; $W, A \subseteq X$; $I, C \subseteq X \times X$; and $@ \in X$—that satisfies P2, we write

$$r(\mathfrak{M}) = (W, \pi_I, \overrightarrow{C}, @),$$

where $\pi_I$ is the relation $I$ regarded as a function (we can regard it so due to P2) and $\overrightarrow{C}$ is the right transpose of $C$. Also, given a tuple $\mathfrak{M} = (W, \pi, \gamma, @)$ of the type as in Definition 68—namely, $W$ is a set; $\pi : D \to W$ for some set $D$; $\gamma : D \to \mathcal{P}D$; and $@ \in W$—we write

$$s(\mathfrak{M}) = (W \cup D, W, I_\pi, \widetilde{\gamma}, \pi^{-1}[\{@\}], @),$$

where $I_\pi$ is the function $\pi$ regarded as a relation, and $\widetilde{\gamma}$ is the relation to which $\gamma$ gives rise by

$$(a, b) \in \widetilde{\gamma} \iff b \in \gamma(a).$$

In terms of this notation, the arguments we gave above up to Definition 68 amount to:

**Fact 58.** A tuple $\mathfrak{M} = (X, W, I, C, A, @)$ as in Notation 1 is a counterpart structure (in the sense of Definition 56) iff $r(\mathfrak{M})$ is a counterpart structure (in the sense of Definition 68). Also, a tuple $\mathfrak{M} = (W, \pi, \gamma, @)$ as in Notation 1 is a counterpart structure (in the sense of Definition 68) iff $s(\mathfrak{M})$ is a counterpart structure (in the sense of Definition 56).

Then the equivalence can be stated by the combination of Fact 58 and the following.

**Fact 59.** The operation $r \circ s$ restricted to the class of counterpart structures is the identity, in the sense that $r(s(\mathfrak{M})) = \mathfrak{M}$ for every counterpart structure $\mathfrak{M}$ in the sense of Definition 68. Moreover, for every counterpart structure $\mathfrak{M} = (X, W, I, C, A, @)$ in the sense of Definition 56, $s(r(\mathfrak{M}))$ is a counterpart structure (in the sense of Definition 56) of the form

$$s(r(\mathfrak{M})) = (W \cup D, W, I, C, A, @)$$

for $W \cup D \subseteq X$, where $D$ is the domain of the relation $I$.

Unlike $r \circ s$, the operation $s \circ r$ restricted to the counterpart structures is not strictly the identity, because, under the formulation of Definition 56, $X \setminus (W \cup D)$ may not be empty; for example, the set $X$ of things in a counterpart structure $\mathfrak{M}$ may contain a thing $x \notin W \cup D$ that neither is a world nor lives in a world, whereas $x$ is not in the set of things in $s \circ r(\mathfrak{M})$, namely $W \cup D$. Nevertheless, $\mathfrak{M}$ and $s(r(\mathfrak{M}))$ are essentially the same, as long as we focus on things in $W \cup D$ and ignore every $x \notin W \cup D$. So, the upshot of Fact 59 is that, whereas $r \circ s$ is strictly the identity, $s \circ r$ is essentially the identity, thereby making $r$ and $s$ essentially inverse to each other, as long as we focus on worlds and things in worlds. And this focus is justified by the fact that only worlds and things in worlds are semantically significant.

Let us then extend this equivalence to the level of semantics. It requires that we first formulate counterpart-theoretic semantics as based on Definition 68; but it is straightforward once we recall that, in Definition 59, we defined a counterpart-theoretic model for a given language $\mathcal{L}$ of quantified modal logic as a counterpart structure equipped with $F^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$ for each $n$-ary primitive predicate $F$ of $\mathcal{L}$,[19] and that the only semantically significant part of $F^{\mathfrak{M}}$ is its subset $F^{\mathfrak{M}} \cap D^n$ for

$$D^n = \{\, \bar{a} \in |\mathfrak{M}|^n \mid \bar{a} \in {D_w}^n \text{ for some } w \in W \,\},$$

where $D_w = \pi^{-1}[\{w\}]$. Hence we simply add $F^{\mathfrak{M}}$ to counterpart structures in the sense of Definition 68. It is helpful to note that the $D^n$ as above satisfies

$$D^n = \sum_{w \in W} {D_w}^n,$$

thereby extending $D = \sum_{w \in W} D_w$. Then we straightforwardly have:

**Definition 69.** Given a language $\mathcal{L}$ of quantified modal logic, we say a tuple

$$\mathfrak{M} = (W, \pi, \gamma, @, F^{\mathfrak{M}})$$

is a *counterpart-theoretic model* for $\mathcal{L}$ if

- $(W, \pi, \gamma, @)$ is a counterpart structure as in Definition 68; and moreover,
- $\mathfrak{M}$ is equipped with $F^{\mathfrak{M}} \subseteq \sum_{w \in W} {D_w}^n$ for each $n$-ary primitive predicate $F$ of $\mathcal{L}$, where $D_w = \pi^{-1}[\{w\}]$.

[19] Recall our assumption on p. 188 that $\mathcal{L}$ has no 0-ary primitive predicates.

**Definition 70.** Given a language $\mathcal{L}$ of quantified modal logic, a *counterpart-theoretic semantics* for $\mathcal{L}$ is a relation $(-,- \vDash_{-} -)$, as in $\mathfrak{M}, w \vDash_\alpha \varphi$, among

- a counterpart-theoretic model $\mathfrak{M} = (W, \pi, \gamma, @, F^{\mathfrak{M}})$ for $\mathcal{L}$,
- an element $w \in W$,
- an assignment $\alpha : \mathrm{var}(\mathcal{L}) \to D_w$, where $D_w = \pi^{-1}[\{w\}]$,[20] and
- a sentence $\varphi$ of $\mathcal{L}$

that satisfies

(IV.8) $$\mathfrak{M}, w \vDash_\alpha \neg\varphi \iff \mathfrak{M}, w \nvDash_\alpha \varphi,$$

(IV.9) $$\mathfrak{M}, w \vDash_\alpha \varphi \wedge \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ and } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.10) $$\mathfrak{M}, w \vDash_\alpha \varphi \vee \psi \iff \mathfrak{M}, w \vDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.11) $$\mathfrak{M}, w \vDash_\alpha \varphi \to \psi \iff \mathfrak{M}, w \nvDash_\alpha \varphi \text{ or } \mathfrak{M}, w \vDash_\alpha \psi,$$

(IV.14) $$\mathfrak{M}, w \vDash_\alpha \forall x\,.\,\varphi \iff \mathfrak{M}, w \vDash_{[a/x]\alpha} \varphi \text{ for every } a \in D_w,$$

(IV.15) $$\mathfrak{M}, w \vDash_\alpha \exists x\,.\,\varphi \iff \mathfrak{M}, w \vDash_{[a/x]\alpha} \varphi \text{ for some } a \in D_w,$$

(V.23) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then
$$\mathfrak{M}, w \vDash_\alpha \Box\varphi \iff \mathfrak{M}, u \vDash_{[\bar{a}/\bar{x}]\alpha} \varphi \text{ for every } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } a_i \in \gamma(\alpha(x_i)),$$

(V.24) If $\bar{x}$ are all and only the free variables actually occurring in $\varphi$, then
$$\mathfrak{M}, w \vDash_\alpha \Diamond\varphi \iff \mathfrak{M}, u \vDash_{[\bar{a}/\bar{x}]\alpha} \varphi \text{ for some } u \in W \text{ and } \bar{a} \in {D_u}^n$$
$$\text{such that each } i \text{ has } a_i \in \gamma(\alpha(x_i)),$$

$$\mathfrak{M}, w \vDash_\alpha F\bar{x} \iff \alpha(\bar{x}) \in F^{\mathfrak{M}} \qquad \text{for an } n\text{-ary primitive predicate } F.$$

[20] Note that this clause depends on the $w$ mentioned in the previous clause.

# VI.0 GENERALIZED TOPOLOGICAL SEMANTICS FOR FIRST-ORDER MODAL LOGIC

## VI.1 TOPOLOGICAL SEMANTICS FOR FIRST-ORDER MODAL LOGIC

### VI.1.1 Upshots from the Previous Chapters

In Chapters III through V, we observed some conditions that a semantics of first-order modal logic needs to satisfy in order to guarantee certain, both philosophically and mathematically desirable properties of a logic. In this subsection we give a technical summary of these logical properties and the semantic conditions they require.

In Chapter III, we extended the standard semantics for a classical first-order language to obtain a classical semantics for a non-classical first-order language. Then we observed in Chapter IV that, in a possible-world semantics equipped with a domain of possible individuals, each world $w \in W$ constitutes a classical interpretation, in our extended sense, for a first-order modal language. Yet we also observed that such a semantics may fail, as Kripke's did, to equip modal logic with all the rules and axioms of classical first-order logic; instead, Kripke ended up with free logic.

To analyze this failure, some notions introduced in Chapter III turned out helpful: We distinguished two notions of domains, a domain of individuals (the set of referents of free variables) and a domain of quantification (the range of quantifiers), and introduced the notion of the autonomy of the latter. Then, in Chapter IV, we observed that a possible-world semantics with a domain of possible individuals needs to have autonomous domains of individuals, in order to provide semantics for a simple union of classical first-order logic and modal logic. The autonomy, in the case of a possible-world semantics with domains, is roughly characterized as follows: The semantics has autonomous domains of quantification if, for each sentential operator $\otimes$ of the given language, an operation

$$[\![\otimes]\!] : [\![\, \bar{x}, \bar{y} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \otimes\varphi \,]\!]$$

that interprets $\otimes$ is (or is restrictable to, in the sense we rigorously defined in Chapters III and V) an operation of the type

$$\mathcal{P}(\sum_{w \in W} D_w{}^{n+m}) \to \mathcal{P}(\sum_{w \in W} D_w{}^{n}),$$

where we write $D_w$ for the domain of quantification—the range of quantifiers—of a world $w \in W$; hence the disjoint union $\sum_{w \in W} D_w$ is the domain of pairs of a world and an individual that exists in that world.

In Chapters IV and V, we discussed the free-variable sensitivity (and insensitivity) of interpretations of $\Box$. According to the semantic idea that we reviewed in these chapters and that is shared by David Lewis, the truth condition for a sentence $\Box\varphi$ is sensitive to the set of variables that occurs freely in $\varphi$. Even though this idea makes domains of quantification autonomous, one disadvantage of it is that, in general, it even fails to validate the rule

E $$\frac{\varphi \vdash \psi \qquad \psi \vdash \varphi}{\Box\varphi \vdash \Box\psi}$$

of **E**, the so-called minimal modal logic. This is because, under a free-variable-sensitive interpretation, $\Box$ is interpreted non-uniformly, that is, with different operations

$$[\![\, \bar{x} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \Box\varphi \,]\!],$$

$$[\![\, \bar{x} \mid \psi \,]\!] \mapsto [\![\, \bar{x} \mid \Box\psi \,]\!]$$

when $\varphi$ and $\psi$ have different sets of free variables, and therefore $[\![\, \bar{x} \mid \varphi \,]\!] = [\![\, \bar{x} \mid \psi \,]\!]$ fails to entail $[\![\, \bar{x} \mid \Box\varphi \,]\!] = [\![\, \bar{x} \mid \Box\psi \,]\!]$. A free-variable-sensitive interpretation of $\Box$ gives rise to rules restricted by a condition on free variables; for instance, Lewis's truth condition for $\Box$ makes valid a version of the rule M with a condition on variables, namely

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \text{ (every variable that occurs freely in } \varphi \text{ occurs freely in } \psi\text{)},$$

but not full M without the restriction. To restore full M, we need to interpret $\Box$ uniformly, that is, in a free-variable-insensitive manner. In other words, for each $n$ we use a single operation

$$[\![\Box]\!]_n : [\![\, \bar{x} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \Box\varphi \,]\!]$$

to interpret $\Box$ regardless of what subset of $\bar{x}$ occurs freely in $\varphi$,[1] so that

$$\begin{array}{ccc}
[\![\, \bar{x} \mid \varphi \,]\!] & \xrightarrow{\;[\![\Box]\!]_n\;} & [\![\, \bar{x} \mid \Box\varphi \,]\!] \\
{\scriptstyle {p_n}^{-1}} \downarrow & /\!/ & \downarrow {\scriptstyle {p_n}^{-1}} \\
[\![\, \bar{x}, y \mid \varphi \,]\!] & \xrightarrow[\;[\![\Box]\!]_{n+1}\;]{} & [\![\, \bar{x}, y \mid \Box\varphi \,]\!]
\end{array}$$

commutes.

We also saw in Chapter V how the ontology of Lewis's counterpart theory helps (at least technically) to model transworld identity of possible individuals in a more general way than Kripke's treatment does. As we showed, when we write $D$ for a domain of possible individuals (that exist in some world or other), Lewis's ontology can be characterized by a "residence map" $\pi : D \to W$, with which we read $\pi(a) = w$ as "$w$ is the unique world in which the individual $a$ lives" (so, when we write—as we did before—$D_w$ for the domain of quantification for the world $w$, $D$ amounts to $\sum_{w \in W} D_w$). Lewis moreover introduces a relation $C$ on $D$, with which we read $Cab$ as "$b$ is a counterpart (in $\pi(b)$) of $a$" and write $\overrightarrow{C}(a) = \{\, b \in D \mid Cab \,\}$ for the set of counterparts of $a$; then he gives the transworld identity of $a$ in terms of the counterparts of $a$, by setting

$$a \in [\![\, x \mid \Box\varphi \,]\!] \iff \overrightarrow{C}(a) \subseteq [\![\, x \mid \varphi \,]\!]$$

and more generally

$$\bar{a} \in [\![\, \bar{x} \mid \Box\varphi \,]\!] \iff \overrightarrow{C^n}(\bar{a}) \subseteq [\![\, \bar{x} \mid \varphi \,]\!].$$

This enables us to provide a countermodel for

$$x \neq y \vdash \Box(x \neq y),$$

which is not provable from the simple union of classical first-order logic and (propositional) modal logic.

[1]Recall that the notation $[\![\, \bar{x} \mid \varphi \,]\!]$ makes no sense if any variable other than $\bar{x}$ occurs freely in $\varphi$.

To interpret equality, however, there is one condition that semantics needs to satisfy in order to validate the rules and axioms on equality. Note that, writing $\Delta$ for the diagonal map $\Delta : a \mapsto (a, a)$, the validity of the rules and axioms on equality requires that

$$[\![ x \mid \varphi(x, x) ]\!] = \Delta^{-1}[[\![ x, y \mid \varphi(x, y) ]\!]]$$

hold for any sentence $\varphi$, even if it contains $\Box$. Therefore, to give rise to a union of modal logic and classical, fully first-order logic with the rules and axioms on equality, the semantics is required to make

$$\begin{array}{ccc} [\![ x, y \mid \varphi(x, y) ]\!] & \longmapsto & [\![ x, y \mid \Box\varphi(x, y) ]\!] \\ \Delta^{-1} \downarrow & /\!/ & \downarrow \Delta^{-1} \\ [\![ x \mid \varphi(x, x) ]\!] & \longmapsto & [\![ x \mid \Box\varphi(x, x) ]\!] \end{array}$$

commute.

To sum up, we have the following four conditions that a semantics for first-order modal logic should satisfy.

(i) An interpretation is defined on a structure $\pi : D \to W$ so that, for each $w \in W$, $\pi^{-1}[\{w\}]$ serves as the domain of quantification for the world $w$, with which we interpret classical first-order logic.

(ii) $\Box$ is interpreted by a general notion of counterparts defined on $D$.

(iii) $\Box$ is interpreted uniformly by

$$[\![\Box]\!]_n : [\![ \bar{x} \mid \varphi ]\!] \mapsto [\![ \bar{x} \mid \Box\varphi ]\!]$$

such that the following commutes.

$$\begin{array}{ccc} [\![ \bar{x} \mid \varphi ]\!] & \overset{[\![\Box]\!]_n}{\longmapsto} & [\![ \bar{x} \mid \Box\varphi ]\!] \\ {p_n}^{-1} \downarrow & /\!/ & \downarrow {p_n}^{-1} \\ [\![ \bar{x}, y \mid \varphi ]\!] & \underset{[\![\Box]\!]_{n+1}}{\longmapsto} & [\![ \bar{x}, y \mid \Box\varphi ]\!] \end{array}$$

(iv) The following commutes.

$$\begin{array}{ccc} [\![\, x, y \mid \varphi(x,y) \,]\!] & \longmapsto & [\![\, x, y \mid \Box\varphi(x,y) \,]\!] \\ \Delta^{-1} \downarrow & \not= & \downarrow \Delta^{-1} \\ [\![\, x \mid \varphi(x,x) \,]\!] & \longmapsto & [\![\, x \mid \Box\varphi(x,x) \,]\!] \end{array}$$

In terms of these, we can compare the two semantics for quantified modal logic reviewed in Chapters IV and V, one by Kripke and the other according to Lewis's semantic idea, as follows. On the one hand, Kripke has (iii) and (iv) at the cost of (ii), which keeps his treatment of equality from being general enough; also, he does not have (i), thereby failing to unify modal logic with classical first-order logic. On the other hand, Lewis surely has (i) and (ii); but he lacks (iii), thereby ending up with too restricted a modal logic, and lacks (iv), thereby failing to validate the rules and axioms on equality.[2]

Moreover, we have another desideratum; namely, as we argued in Subsection II.1.1,

(v) We should generalize the relational notion of accessibility to the topological, neighborhood notion.

For the rest of this section, we lay out a semantics for first-order modal logic that satisfies all these five desiderata.

### VI.1.2 Classical Semantics in a Category of Sets over a Set

For our purpose it is helpful to first prepare an underlying, classical semantics on the underlying, set structure of $\pi : D \to X$ as in the desideratum (i) of Subsection VI.1.1. In this subsection, we briefly lay out a semantics of classical first-order logic in $\mathbf{Sets}/X$, the category of sets sliced over a fixed set $X$, for a non-classical first-order language. We do this by categorically rewriting, and then "bundling up" over $X$, of the classical semantics we laid out in Chapter III.[3]

Let us fix any set $X$. Then, by a *set over* $X$, we mean any pair $(D, \pi)$ of a set $D$ and a map $\pi$ of the type $\pi : D \to X$; or we may simply mean a map $\pi$ with codomain $X$, because, once $\pi$ is given,

[2] From the viewpoint of the sheaf semantics we will lay out, we can put the comparison as follows. Kripke attains the sheaf properties (iii) and (iv) by taking a constant sheaf; but a constant sheaf makes transworld identity not general enough to serve the purpose of (ii). Also, he takes domains of quantification different from fibers of the constant sheaf, thereby failing (i). By contrast, Lewis liberalizes his semantics too much to attain the sheaf properties (iii) and (iv).

[3] We will not lay out basic definitions in category theory; see [3].

its domain $D$ is determined. When we take a pair $(D, \pi)$ as a set over $X$, we say $\pi$ is its *projection*. Moreover, given two sets $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, by a *map from* $(D, \pi_D)$ *to* $(E, \pi_E)$ *over* $X$, we mean any map $f : D \to E$ such that $\pi_E \circ f = \pi_D$, that is, that makes

$$\begin{array}{ccc} D & \xrightarrow{\ f\ } & E \\ & {\scriptstyle \pi_D}\searrow \ \ \  = \ \ \ \swarrow{\scriptstyle \pi_E} & \\ & X & \end{array}$$

commute. These kinds of structures form a category $\mathbf{Sets}/X$, the category $\mathbf{Sets}$ of sets sliced over $X$. That is,

- the objects of $\mathbf{Sets}/X$ are the sets over $X$; and
- the arrows of $\mathbf{Sets}/X$ from $(D, \pi_D)$ to $(E, \pi_E)$ are the maps from $(D, \pi_D)$ to $(E, \pi_E)$ over $X$.

Among many properties $\mathbf{Sets}/X$ has, it is important to our purpose that it has finite products. The 0-ary product is just $(X, 1_X)$. And, given sets $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, their (binary) product is just the pullback $D \times_X E$ of them in $\mathbf{Sets}$. Take the pullback

$$\begin{array}{ccc} D \times_X E & \xrightarrow{\ p_E\ } & E \\ {\scriptstyle p_D}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_E} \\ D & \xrightarrow[\ \pi_D\ ]{} & X \end{array}$$

in $\mathbf{Sets}$, paired with the map $\pi_D \circ p_D = \pi_E \circ p_E$; let us write $\pi_{D \times_X E}$ for it. Then $p_D$ and $p_E$ are maps over $X$ by definition. It is moreover immediate that $(D \times_X E, \pi_{D \times_X E})$ with projections $p_D$ and $p_E$ is the product in $\mathbf{Sets}/X$ of $(D, \pi_D)$ and $(E, \pi_E)$.

It is also important that, as we explained in Subsection I.3.1, sets over $X$, products over $X$, and maps over $X$ can be obtained by first taking sets, products, and maps over each $w \in X$ and then "bundling" them up over all $w \in X$.[4] To recall more precisely what this means, any set $(D, \pi)$ over $X$ can be written as the disjoint union of fibers $D_w = \pi^{-1}[\{w\}]$, that is, as

$$D = \sum_{w \in X} D_w.$$

[4]We can state this most precisely with the fact (that is familiar to category theorists) that $\mathbf{Sets}/X$ is categorically equivalent to the category $\mathbf{Sets}^X$ of sets and maps indexed by $w \in X$.

A map $f : D \to E$ is a map from $(D, \pi_D)$ to $(E, \pi_E)$ over $X$ iff it is of the form

$$f = \sum_{w \in X} (f_w : D_w \to E_w).$$

And, given any collection of sets $(D_1, \pi_1), \ldots, (D_n, \pi_n)$ over $X$, their product in **Sets**$/X$ is the *fibered product over X*,

$$\begin{aligned} D_1 \times_X \cdots \times_X D_n &= \sum_{w \in W} ((D_1)_w \times \cdots \times (D_n)_w) \\ &= \{ (a_1, \ldots, a_n) \in D_1 \times \cdots \times D_n \mid \pi_1(a_1) = \cdots = \pi_n(a_n) \}, \end{aligned}$$

paired with the map

$$\pi_1 \times_X \cdots \times_X \pi_n : D_1 \times_X \cdots \times_X D_n \to X :: (a_1, \ldots, a_n) \mapsto \pi_1(a_1) = \cdots = \pi_n(a_n),$$

together with a family of projections $p_i$ such that, for each $i$,

$$p_i : D_1 \times_X \cdots \times_X D_n \to D_i :: (a_1, \ldots, a_n) \mapsto a_i.$$

We should note that, given any set $(D, \pi)$ over $X$, the "diagonal map"

$$\Delta : D \to D \times_X D :: a \mapsto (a, a)$$

is a map over $X$, from $(D, \pi)$ to its two-fold product $(D \times_X D, \pi_{D \times_X D})$ in **Sets**$/X$.

Taking advantage of these structures over a fixed set $X$, we can define semantic structures and semantics as follows. First, we enter:

**Definition 71.** Given a first-order (perhaps non-classical) language $\mathcal{L}$, we say that a tuple $\mathfrak{M}$ is an *$\mathcal{L}$ structure in* **Sets**$/X$ if it consists of the following:

- An object $(D, \pi)$ of **Sets**$/X$—that is, a set over $X$—the projection $\pi$ of which is surjective.[5]
- For each $n$-ary primitive predicate $R$ of $\mathcal{L}$, a subobject $R^{\mathfrak{M}}$ of $(D^n, \pi^n)$, the $n$-fold product of $(D, \pi)$ in **Sets**$/X$—that is, a subset $R^{\mathfrak{M}} \subseteq D^n$ that is naturally paired with the projection $\pi \restriction R^{\mathfrak{M}} : R^{\mathfrak{M}} \to X$;
- in particular, $=^{\mathfrak{M}} = \Delta[D]$ if $\mathcal{L}$ has the equality predicate $=$.

[5] We need to require $\pi$ to be surjective in order to have classical first-order logic, as opposed to free logic, sound.

- For each $n$-ary function symbol $f$ of $\mathcal{L}$, a map $f^{\mathfrak{M}}$ of **Sets**$/X$ from $(D^n, \pi^n)$ to $(D, \pi)$—that is, a map $f^{\mathfrak{M}} : D^n \to D$ over $X$;
- in particular, with $n = 0$, that is, for each (individual) constant $c$ of $\mathcal{L}$, a map $c^{\mathfrak{M}}$ of **Sets**$/X$ from $(D^0, \pi^0) = (X, 1_X)$ to $(D, \pi)$—that is, a map $c^{\mathfrak{M}} : X \to D$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

Then we extend interpretations of primitive predicates and terms to all the sentences, using the same operations of sets and maps (in **Sets**$/X$) as we use in classical semantics in **Sets**.

**Definition 72.** Given a first-order (perhaps non-classical) language $\mathcal{L}$, by a *classical interpretation for* $\mathcal{L}$ *in* **Sets**$/X$, we mean a pair $(\mathfrak{M}, [\![-]\!])$ of an $\mathcal{L}$ structure $\mathfrak{M}$ in **Sets**$/X$ and and a map $[\![-]\!]$ (of the suitable type) that satisfies:

(VI.1) $\quad [\![\, \bar{x} \mid R\bar{x} \,]\!] = R^{\mathfrak{M}} \qquad$ for $n$-ary primitive predicate $R$;

(VI.2) $\quad [\![\, \bar{x} \mid \top \,]\!] = D^n;$

(VI.3) $\quad [\![\, \bar{x} \mid \neg\varphi \,]\!] = D^n \setminus [\![\, \bar{x} \mid \varphi \,]\!] \qquad$ (that is, $[\![\neg]\!] = D^n \setminus -$);

(VI.4) $\quad [\![\, \bar{x} \mid \varphi \wedge \psi \,]\!] = [\![\, \bar{x} \mid \varphi \,]\!] \cap [\![\, \bar{x} \mid \psi \,]\!] \qquad$ (that is, $[\![\wedge]\!] = \cap$);

(VI.5) $\quad [\![\, \bar{x} \mid \varphi \vee \psi \,]\!] = [\![\, \bar{x} \mid \varphi \,]\!] \cup [\![\, \bar{x} \mid \psi \,]\!] \qquad$ (that is, $[\![\vee]\!] = \cup$);

(VI.6) $\quad [\![\, \bar{x} \mid \varphi \to \psi \,]\!] = [\![\, \bar{x} \mid \varphi \,]\!] \to [\![\, \bar{x} \mid \psi \,]\!] \qquad$ (that is, $[\![\to]\!] = \to$);

(VI.7) $\quad [\![\, \bar{x} \mid \forall y \,.\, \varphi \,]\!] = \forall_p([\![\, \bar{x}, y \mid \varphi \,]\!]) \qquad$ (that is, $[\![\forall y]\!] = \forall_p$);

(VI.8) $\quad [\![\, \bar{x} \mid \exists y \,.\, \varphi \,]\!] = \exists_p([\![\, \bar{x}, y \mid \varphi \,]\!]) \qquad$ (that is, $[\![\exists y]\!] = \exists_p$);

(VI.9) $\quad [\![\, \bar{x}, y \mid \varphi \,]\!] = p^{-1}[[\![\, \bar{x} \mid \varphi \,]\!]] \qquad$ if $y$ is not free in $\varphi$;

(VI.10) $\quad [\![\, \bar{x}, \bar{y} \mid [t/z]\varphi \,]\!] = (1_{D^n} \times [\![\, \bar{y} \mid t \,]\!])^{-1}[[\![\, \bar{x}, z \mid \varphi \,]\!]];$

(VI.11) $\quad [\![\, \bar{x}, y \mid [y/z]\varphi \,]\!] = (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \varphi \,]\!]].$

We say that a binary sequent $\varphi \vdash \psi$ in $\mathcal{L}$ is *valid* in an interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$, and also that $(\mathfrak{M}, [\![-]\!])$ *validates* $\varphi \vdash \psi$, if

$$[\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$$

is the case (whenever it makes sense). We say that an inference is valid in $(\mathfrak{M}, [\![-]\!])$, and that the latter validates the former, if the inference preserves validity in $(\mathfrak{M}, [\![-]\!])$. By *classical semantics for* $\mathcal{L}$ *in* **Sets**$/X$, we mean the class of classical interpretations for $\mathcal{L}$ in **Sets**$/X$.

Note that, if $\mathcal{L}$ is classical (that is, if $\mathcal{L}$ only has classical sentential operators), (VI.1)–(VI.9) uniquely determine a classical interpretation on $\mathfrak{M}$ and moreover entail (VI.10) and (VI.11). On the other hand, as we argued in Subsection III.1.3, if $\mathcal{L}$ has non-classical sentential operators then we need to require (VI.10) and (VI.11), for classical first-order logic in $\mathcal{L}$ to be sound with respect to the semantics.

Let us close this subsection by revisiting an intuitive idea we mentioned in Subsection I.3.1. Recall that each classical interpretation $(M, [\![-]\!])$ in **Sets** (as we reviewed in Section I.2) is just a set $|M|$ equipped with interpretations of predicates, terms, and sentences. And note that, when we restrict an interpretation $(\mathfrak{M}, [\![-]\!])$ in **Sets**$/X$ on $(D, \pi)$ to fibers $D_w$, each

$$(D_w, R^{\mathfrak{M}}{}_w, f^{\mathfrak{M}}{}_w, c^{\mathfrak{M}}{}_w, [\![-]\!]_w)$$

is a classical interpretation in **Sets**. In other words, we can obtain an interpretation $(\mathfrak{M}, [\![-]\!])$ in **Sets**$/X$ on the set $(\sum_{w \in X} D_w, \pi)$ over $X$ by bundling up over $X$ a collection of classical interpretations $(D_w, [\![-]\!]_w)$ in **Sets**. Thus, we can obtain classical semantics in **Sets**$/X$ by bundling up classical semantics in **Sets**. This idea is not only conceptually illuminating, but also crucial in the completeness proof in Section VI.3. The only fact we need to check in order to make sure that this intuitive idea works formally is that all the operations and relations that interpret classical first-order logic commute with $\sum_{w \in X}$ (we omit the proof):

**Fact 60.** Given a first-order (perhaps non-classical) language $\mathcal{L}$ and any $\mathcal{L}$ structure $\mathfrak{M}$ in **Sets**$/X$, a map $[\![-]\!]$ (of the suitable type) satisfies (VI.1)–(VI.11), respectively, iff for each $w \in X$ the restriction $[\![-]\!]_w$ of $[\![-]\!]$ to $w \in X$ satisfies (VI.1)–(VI.11), respectively. Also,

$$[\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!] \iff [\![\, \bar{x} \mid \varphi \,]\!]_w \subseteq [\![\, \bar{x} \mid \psi \,]\!]_w \text{ for all } w \in X.$$

By Fact 60, the soundness of classical first-order logic with respect to classical semantics in **Sets** immediately entails

**Theorem 4.** *For any set X, classical first-order logic is sound with respect to classical semantics in* **Sets**$/X$.

### VI.1.3 Topological Spaces over a Space

In Subsection VI.1.2, we laid out how to interpret classical first-order logic in the category **Sets**$/X$ of sets over a fixed set $X$, regarding any surjection $\pi : D \to X$ as the underlying structure of $\pi$ as required in the desideratum (i) of Subsection VI.1.1. The goal of this subsection and the next, then, is to lay out structures needed to achieve the other desiderata. In this subsection, we add topological structures to the structures in **Sets**$/X$, so that (ii) and (v) are satisfied. Then, so that (iii) and (iv) hold, we will restrict our attention to the structures called sheaves in Subsection VI.1.4. In this subsection and the next, we omit proofs for facts we give, because they follow, as special, topological cases, from more general proofs we will give in Subsection VI.2.4.

Let us first recall the basic definitions of topology. Given any set $X$, any family $\mathcal{O}X \subseteq \mathcal{P}X$ of subsets of $X$ is called a *topology* on $X$ if it is

- closed under arbitrary joins, including the empty one (that is, $\varnothing$), and
- closed under finite meets, including the empty one (that is, $X$).

A pair of a set and any topology on it is called a *topological space*. We will write $X$ for a topological space, and $|X|$ for the underlying set of $X$ when we want it explicit that $|X|$ is without a topological structure. Given a topological space $X$, any $A \subseteq X$ is said to be *open* (in the space $X$) if $A \in \mathcal{O}X$, and *closed* (in $X$) if $X \setminus A \in \mathcal{O}X$.

Given any topological spaces $X$ and $Y$, we say that a map $f : Y \to X$ is *continuous* if $f^{-1}[U] \in \mathcal{O}Y$ for every $U \in \mathcal{O}X$, that is, if $f$ pulls open sets of $X$ back to open sets of $Y$. Continuous maps are clearly composable; hence we have the category **Top** of topological spaces and continuous maps. An isomoprhism in **Top** is called a *homeomorphism*; that is, $f : Y \to X$ is a homeomorphism if it is a continuous bijection with a continuous inverse, or, equivalently, if $X$ and $Y$ share the same topological structure, with points renamed by $f$.

Now, instead of slicing **Sets** with a set $|X|$ as we did in Subsection VI.1.2 to obtain the category **Sets**$/|X|$, let us fix a topological space $X$ and slice **Top** with $X$. Then we have the category **Top**$/X$ of *topological spaces over* $X$. Its objects are spaces over $X$, that is, pairs $(D, \pi)$ of a space $D$ and a continuous map $\pi : D \to X$, called the projection of $(D, \pi)$. Its arrows from a space $(D, \pi_D)$ over $X$

to another $(E, \pi_E)$ are continuous maps $f : D \to E$ over $X$, that is, continuous maps $f$ that make

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ {\scriptstyle \pi_D}\searrow & = & \swarrow{\scriptstyle \pi_E} \\ & X & \end{array}$$

commute. We add topological structures in this way to $\mathbf{Sets}/|X|$ and obtain $\mathbf{Top}/X$.

Recall that, in Subsection VI.1.2, we used $n$-fold products in $\mathbf{Sets}/|X|$ to interpret $n$-ary predicates, terms and sentences. To extend such an interpretation by adding topological structures, we should see in a detailed manner how $\mathbf{Top}/X$ has finite products. It is helpful to use the notion of a basis. Given any set $|D|$, a family $\mathcal{B} \subseteq \mathcal{P}(|D|)$ of subsets of $|D|$ is called a *basis for a topology on X* if

- $\mathcal{B}$ is closed under binary meets,[6] and
- for every $a \in |D|$, there is $B \in \mathcal{B}$ such that $a \in B$.

Given any basis $\mathcal{B}$ for a topology on $|D|$, the family $\mathcal{O}D \subseteq \mathcal{P}(|X|)$ defined by

$$A \in \mathcal{O}D \iff A = \bigcup_{i \in I} B_i \text{ for some collection } \{ B_i \in \mathcal{B} \mid i \in I \}$$

is a topology on $|D|$; we call such $|D|$ the topology generated by $\mathcal{B}$.

Then finite products in $\mathbf{Top}$ can be defined in the following manner. Given topological spaces $D_1, \ldots, D_n$, their product $D_1 \times \cdots \times D_n$ in $\mathbf{Top}$ is a pair of the cartesian product $|D_1| \times \cdots \times |D_n|$ and the topology $\mathcal{O}(D_1 \times \cdots \times D_n)$, called the *product topology*, that is generated by the basis

$$\mathcal{B} = \{ U_1 \times \cdots \times U_n \mid U_1 \in \mathcal{O}D_1, \ldots, U_n \in \mathcal{O}D_n \},$$

together with the projections

$$p_i : D_1 \times \cdots \times D_n \to D_i :: (a_1, \ldots, a_n) \mapsto a_i.$$

Note that, clearly, each projection $p_i$ is continuous.

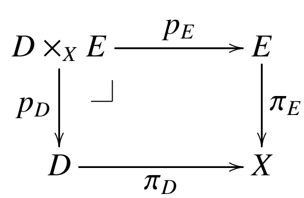
Using product topologies, we can explicitly define pullbacks in $\mathbf{Top}$. Given spaces $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, note that their fibered product $|D| \times_{|X|} |E|$ over $|X|$ in $\mathbf{Sets}$ is a subset of the cartesian

[6] This can be weakened to the condition that, for every $a \in |D|$, if $a \in B_0, B_1$ then $a \in B_2 \subseteq B_0 \cap B_1$ for some $B_2 \in \mathcal{B}$; but the definition above suffices for our purpose.

product $|D| \times |E|$; hence we write $\mathcal{O}(D \times_X E)$ for the subspace topology on $|D| \times_{|X|} |E|$ of the product topology $\mathcal{O}(D \times E)$, that is,

$$A \in \mathcal{O}(D \times_X E) \iff A = U \cap (|D| \times_{|X|} |E|) \text{ for some } U \in \mathcal{O}(D \times E).$$

Then the pair $D \times_X E = (|D| \times_{|X|} |E|, \mathcal{O}(D \times_X E))$ is the pullback in **Top** of $(D, \pi_D)$ and $(E, \pi_E)$ over $X$.




Now we can simply define finite products in **Top**$/X$ by saying that the 0-ary product in **Top**$/X$ is just $(X, 1_X)$, whereas, given spaces $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, their (binary) product in **Top**$/X$ is just the pullback $D \times_X E$ of them in **Top**. Or, more explicitly, given spaces $(D_1, \pi_1), \ldots, (D_n, \pi_n)$ over $X$, their product in **Top**$/X$ is the fibered product $|D_1| \times_{|X|} \cdots \times_{|X|} |D_n|$ over $|X|$ paired with the topology $\mathcal{O}(D_1 \times_X \cdots \times_X D_n)$ generated by the basis

$$\mathcal{B} = \{ U_1 \times_{|X|} \cdots \times_{|X|} U_n \mid U_1 \in \mathcal{O}D_1, \ldots, U_n \in \mathcal{O}D_n \},$$

together with the projections

$$p_i : D_1 \times_X \cdots \times_X D_n \to D_i :: (a_1, \ldots, a_n) \mapsto a_i.$$

It is crucial that each projection $p_i$ is continuous.

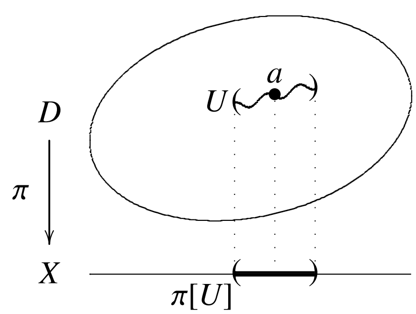
We have thus added topological structures to the structures in **Sets**$/X$ and obtained the category **Top**$/X$. These structures enable us to achieve the desiderata (ii) and (v) of Subsection VI.1.1 by interpreting $\Box$ with suitable topologies. To achieve (iii) and (iv), however, we need to use the notion of sheaves over a space $X$, which we introduce in Subsection VI.1.4.

### VI.1.4 Sheaves over a Topological Space

In Subsection VI.1.3, we introduced the category **Top**/$X$ of topological spaces over a fixed space $X$. Even though the structures in **Top**/$X$ give us the desiderata (ii) and (v) of Subsection VI.1.1, they are still too general to give us (iii) and (iv). In this subsection, to achieve (iii) and (iv), we consider a subcategory of **Top**/$X$, namely the category **LH**/$X$ of sheaves over $X$. (**LH** is for "local homeomorphisms", as we will explain.)

Recall that objects of the category **Top**/$X$ for a fixed space $X$ are spaces over $X$, or, equivalently, continuous maps with codomain $X$. Let us define a certain subclass of such objects, as follows.

**Definition 73.** Given topological spaces $X$ and $D$, a continuous map $\pi : D \to X$ is called a *local homeomorphism* if every $a \in D$ has some $U \in \mathcal{O}D$ such that $a \in U$, $\pi[U] \in \mathcal{O}D$, and the restriction $\pi \restriction U : U \to \pi[U]$ of $\pi$ to $U$ is a homeomorphism.



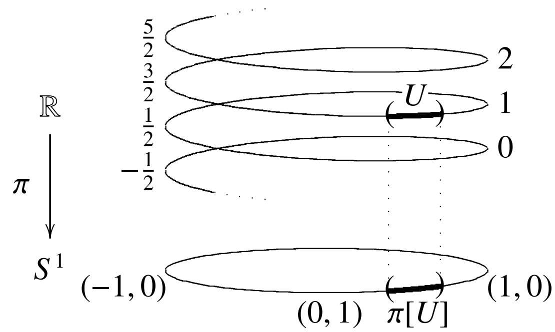

When this is the case, we say that the pair $(D, \pi)$ is a *sheaf over the space* $X$, and also that $X$, $D$, and $\pi$ are respectively the *base space*, *total space*, and *projection* of the sheaf.[7]

It is easy to check that local homeomorphisms are composable; so we have a category **LH** of topological spaces and local homeomorphisms. An example of a local homeomorphism is the map $\pi : \mathbb{R} \to S^1$ such that

$$\pi(a) = e^{i2\pi a} = (\cos 2\pi a, \sin 2\pi a),$$

which is clearly a local homeomorphism from $\mathbb{R}$ (with its usual topology) to the circle $S^1$ (with the subspace topology in $\mathbb{R}^2$); that is, $(\mathbb{R}, \pi)$ is a sheaf over $S^1$.

[7]The notion of a sheaf is usually defined as a certain functor, in which case the version used here is called an étale space. The functorial notion is equivalent (in the category-theoretical sense) to the version here.




Since sheaves over a space $X$ are spaces over $X$, we define the category of sheaves over $X$ as the full subcategory of $\mathbf{Top}/X$ whose objects are sheaves over $X$. In other words, we set the arrows in that category, called *maps of sheaves over* $X$, from a sheaf $(D, \pi_D)$ over $X$ to another $(E, \pi_E)$ to be just the continuous maps from $D$ to $E$ over $X$. Indeed, due to the following fact, this category is just the slice category $\mathbf{LH}/X$ over $X$ of the category $\mathbf{LH}$ of local homeomorphisms.

**Fact 61.** For any topological space $X$, maps of sheaves over $X$ are local homeomorphisms.

This fact turns out crucial for the purpose of providing semantics for first-order modal logic.[8] A few more facts are also crucial for logic. We say that a continuous map $f : D \to E$ is *open* if $f[U] \in \mathcal{O}E$ for every $U \in \mathcal{O}D$, that is, if (the direct-image operation under) $f$ maps every open set to an open set. Then we have:

**Fact 62.** Given any topological spaces $X$ and $D$, any continuous map $\pi : D \to X$ is a local homeomorphism, that is, $(D, \pi)$ is a sheaf over $X$, if and only if both $\pi$ and the diagonal map $\Delta : D \to D^2$ are open.

Moreover,

**Fact 63.** In $\mathbf{Top}$, open maps pull back local homeomorphisms to local homeomorphisms. That is, in the pullback diagram in $\mathbf{Top}$ below, if $\pi_D$ is open continuous and $\pi_E$ is a local homeomorphism, then $p_D$ is a local homeomorphism as well.

$$\begin{array}{ccc} D \times_X E & \xrightarrow{p_E} & E \\ {\scriptstyle p_D}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_E} \\ D & \xrightarrow[\pi_D]{} & X \end{array}$$

[8] Facts 61 and 62 are Exercise II.10 of [27].

**Corollary 10.** **LH**$/X$ has the same products as **Top**$/X$ does.

These structures of sheaves help us to achieve our desiderata (iii) and (iv) of Subsection VI.1.1; to see how they help, it is helpful to summarize the facts above as follows. Let us first recall that any topology is associated with a topological *interior operation*; that is, given any space $X$, there is an operation $\mathbf{int}_X : \mathcal{P}(|X|) \to \mathcal{P}(|X|)$ such that, for every $A \subseteq X$,

$$\mathbf{int}_X(A) = \bigcup_{U \in \mathcal{O}X, U \subseteq A} U = \{ x \in X \mid x \in U \subseteq A \text{ for some } U \in \mathcal{O}X \};$$

in other words, $\mathbf{int}_X(A)$ is the largest open set of $X$ contained in $A$. Then let us make the following observation.[9]

**Observation 4.** Given spaces $X$ and $Y$, a map $f : Y \to X$ is continuous iff

$$f^{-1}[\mathbf{int}_X(A)] \subseteq \mathbf{int}_Y(f^{-1}[A])$$

for every $A \subseteq X$, and, moreover, $f$ is open iff we further have

$$\mathbf{int}_Y(f^{-1}[A]) \subseteq f^{-1}[\mathbf{int}_X(A)]$$

for every $A \subseteq X$.

Thus we can characterize open continuous maps by the commutation of its inverse-image operation with interior operations. Therefore the facts above can be summarized by saying both

- that the following Fact 64 holds, due to the "only if" part of Fact 62, along with Fact 61 and Corollary 10; and
- in order for Fact 64 to hold, we *need* sheaves, due to the "if" part of Fact 62.

**Fact 64.** In **LH**$/X$ for any space $X$, the following diagram commutes for any map $f : D \to E$ of sheaves over $X$ (including the projection of a sheaf over $X$).

$$\begin{array}{ccc} \mathcal{P}(|E|) & \xrightarrow{\mathbf{int}_E} & \mathcal{P}(|E|) \\ {\scriptstyle f^{-1}}\downarrow & = & \downarrow{\scriptstyle f^{-1}} \\ \mathcal{P}(|D|) & \xrightarrow[\mathbf{int}_D]{} & \mathcal{P}(|D|) \end{array}$$

And this is what guarantees to us our desiderata (iii) and (iv), when we interpret $\Box$ with the interior operations **int** of suitable spaces, as we will in Subsection VI.1.5.

[9]In Section VI.2, we will not prove a more general fact that entails Observation 4, because we will define continuous maps and open maps between general neighborhood frames by simply extending Observation 4. We nonetheless omit a proof here for Observation 4, since it can be checked easily.

### VI.1.5 Topological-Sheaf Semantics for First-Order Modal Logic

In Subsections VI.1.3 and VI.1.4, we showed how to obtain the category $\mathbf{LH}/X$ of sheaves over a given topological space $X$ by adding topological structures to $\mathbf{Sets}/|X|$. In this subsection, we extend this insight to the level of semantics; that is, we give a topological semantics on the topological structures of $\mathbf{LH}/X$, by adding a topological interpretation of $\Box$ to classical semantics in $\mathbf{Sets}/|X|$, which we reviewed in Subsection VI.1.2.

Let us recall from Subsection VI.1.2 that, given a first-order modal language $\mathcal{L}$ and any set $|X|$, an $\mathcal{L}$ structure in $\mathbf{Sets}/|X|$ is defined to be a tuple $\mathfrak{M} = (\pi, R_i{}^{\mathfrak{M}}, f_j{}^{\mathfrak{M}}, c_k{}^{\mathfrak{M}})_{i\in I, j\in J, k\in K}$ that consists of

- a surjection $\pi : |D| \twoheadrightarrow |X|$ with some domain $|D|$;
- for each $n$-ary primitive predicate $R$, a subset $R^{\mathfrak{M}} \subseteq |D|^n$ of the $n$-fold product of $|D|$ over $|X|$;
- for each $n$-ary function symbol $f$, a map $f^{\mathfrak{M}} : |D|^n \to |D|$ over $|X|$;
- for each constant $c$, a map $c^{\mathfrak{M}} : |D|^0 \to |D|$ over $|X|$, that is, a map $c^{\mathfrak{M}} : |X| \to |D|$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

Now, rather than just any surjection $\pi$, we take a surjective local homeomorphism to further interpret modal operators. Then, to interpret a primitive predicate, we may take any arbitrary subset (of the type above). By contrast, to interpret function symbols and constants, we need to take maps of sheaves over $X$ rather than just any maps over $|X|$; in short, we assume $f^{\mathfrak{M}}$ and $c^{\mathfrak{M}}$ to be continuous. So, we enter:

**Definition 74.** Given a first-order modal language $\mathcal{L}$, by a *topological-sheaf model* for $\mathcal{L}$ over a given space $X$ we mean an $\mathcal{L}$ structure $\mathfrak{M} = (\pi, R_i{}^{\mathfrak{M}}, f_j{}^{\mathfrak{M}}, c_k{}^{\mathfrak{M}})_{i\in I, j\in J, k\in K}$ in $\mathbf{Sets}/|X|$ such that

- $\pi : D \to X$ is a local homeomorphism;
- $f^{\mathfrak{M}}$ is a map of sheaves over $X$ from $(D^n, \pi^n)$ to $(D, \pi)$, for each $n$-ary function symbol $f$ of $\mathcal{L}$;
- in particular, $c^{\mathfrak{M}}$ is a map of sheaves over $X$ from $(X, 1_X)$ to $(D, \pi)$, for each constant $c$ of $\mathcal{L}$.

On such a structure, we interpret the non-modal part of $\mathcal{L}$ as we did before in Subsection VI.1.2, and moreover $\Box$, $\Diamond$ with the interior operation of the corresponding space $D^n$. More precisely, recall that, given a first-order modal language $\mathcal{L}$ and a set $|X|$, a classical interpretation for $\mathcal{L}$ in $\mathbf{Sets}/|X|$ is a pair $(\mathfrak{M}, [\![-]\!])$ of an $\mathcal{L}$ structure $\mathfrak{M}$ in $\mathbf{Sets}/|X|$ and and a map $[\![-]\!]$ that interprets sentences classically by satisfying (VI.1)–(VI.11), in particular,

$$\text{(VI.9)} \qquad [\![\, \bar{x}, y \mid \varphi \,]\!] = p^{-1}[[\![\, \bar{x} \mid \varphi \,]\!]] \qquad \text{if } y \text{ is not free in } \varphi;$$

$$\text{(VI.10)} \qquad [\![\, \bar{x}, \bar{y} \mid [t/z]\varphi \,]\!] = (1_{D^n} \times [\![\, \bar{y} \mid t \,]\!])^{-1}[[\![\, \bar{x}, z \mid \varphi \,]\!]];$$

$$\text{(VI.11)} \qquad [\![\, \bar{x}, y \mid [y/z]\varphi \,]\!] = (1_{D^n} \times \Delta)^{-1}[[\![\, \bar{x}, y, z \mid \varphi \,]\!]].$$

Then we enter:

**Definition 75.** Given a first-order modal language $\mathcal{L}$, by a *topological-sheaf interpretation* for $\mathcal{L}$ over a given space $X$ we mean a classical interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ in $\mathbf{Sets}/|X|$ such that

- $\mathfrak{M}$ is a topological-sheaf model for $\mathcal{L}$ over $X$, and
- $[\![-]\!]$ satisfies

$$\text{(VI.12)} \qquad [\![\, \bar{x} \mid \Box\varphi \,]\!] = \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Box]\!] = \mathbf{int}_{D^n});$$

$$\text{(VI.13)} \qquad [\![\, \bar{x} \mid \Diamond\varphi \,]\!] = \mathbf{cl}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Diamond]\!] = \mathbf{cl}_{D^n}).$$

We call the class of such interpretations *topological-sheaf semantics* over the given space $X$; and by topological-sheaf semantics (*simpliciter*) we mean the class of topological-sheaf interpretations over some space or other.

We should note that the conditions (VI.12) and (VI.13) declare that we interpret $\Box$ and $\Diamond$ uniformly. Hence, by (VI.9), the well-definedness of the semantics requires that

$$\begin{array}{ccc}
[\![\, \bar{x} \mid \varphi \,]\!] & \xmapsto{\ \mathbf{int}_{D^n}\ } & [\![\, \bar{x} \mid \Box\varphi \,]\!] \\
{\scriptstyle p_n{}^{-1}} \downarrow & & \downarrow {\scriptstyle p_n{}^{-1}} \\
[\![\, \bar{x}, y \mid \varphi \,]\!] & \xmapsto[\ \mathbf{int}_{D^{n+1}}\ ]{} & [\![\, \bar{x}, y \mid \Box\varphi \,]\!]
\end{array}$$

should commute, but this is guaranteed by Fact 64. This is how we achieve the desideratum (iii) of Subsection VI.1.1 in our semantics. Similarly, Fact 64 guarantees the commutation required by (VI.10), achieving the desideratum (iv). Fact 64 also guarantees the commutation required by (VI.11).

We should note that topological-sheaf semantics over any given space $X$ is a subclass of classical semantics in $\mathbf{Sets}/|X|$. Therefore the soundness of classical first-order logic with respect to the latter (Theorem 4) immediately implies the same thing with respect to the former, and hence

**Theorem 5.** *Classical first-order logic is sound with respect to topological-sheaf semantics.*

Moreover, due to (VI.12), topological-sheaf semantics validates all the rules and axioms of modal logic **S4**. Therefore the logic in the following definition is sound with respect to topological-sheaf semantics.

**Definition 76.** First-order modal logic **FOS4** consists of the following two sorts of axioms and rules.

1. All axioms and rules of (classical) first-order logic.

2. The rules and axioms of propositional modal logic **S4**; that is, M, C, N, T, 4.

**Theorem 6.** **FOS4** *is sound with respect to topological-sheaf semantics.*

It is moreover complete, in the following strong form, which says any consistent theory extending **FOS4** has a "canonical" interpretation.

**Theorem 7** (Awodey-Kishida [5])**.** *For any consistent theory* $\mathbb{T}$ *of first-order modal logic extending* **FOS4***, there exists a topological-sheaf interpretation* $(\pi, [\![-]\!])$ *that validates all and only theorems of* $\mathbb{T}$.

I proved this theorem with Awodey in [5]. Indeed, we can also prove it as a correspondence result, as we will in Section VI.3, in a more general framework of semantics for first-order modal logic, which we will lay out in Section VI.2.

## VI.2 NEIGHBORHOOD SEMANTICS FOR FIRST-ORDER MODAL LOGIC

In Section VI.1, we laid out a semantics for first-order modal logic, taking advantage of topological structures and in particular sheaves over a space. In this section, we show that this semantics can be naturally extended by generalizing topological structures with more general structures of neighborhood frames.

### VI.2.1 Basic Definitions for Neighborhood Frames

As we saw in Section VI.1, any topology comes with interior and closure operations, and topological semantics uses them to interpret the modal operators $\Box$ and $\Diamond$. It gives rise to modal logic **S4**, since topological interior operations satisfy the corresponding rules and axioms. In this section, we consider a framework of more generalized "interior" operations, so that it gives rise to more general modal logics; in this subsection, we give a review of basic definitions in such a framework.

Let us first recall that any topological space $X$ comes with an interior operation $\mathbf{int}_X : \mathcal{P}(|X|) \to \mathcal{P}(|X|)$ such that, for every $A \subseteq X$,

$$\mathbf{int}_X(A) = \bigcup_{U \in \mathcal{O}X, U \subseteq A} U = \{\, x \in X \mid x \in U \subseteq A \text{ for some } U \in \mathcal{O}X \,\}.$$

We should also note that a topological space $X$ comes naturally with the notion of neighborhoods by the definition that, for every $x \in X$ and $A \subseteq X$,

$$A \text{ is a neighborhood of } x \iff x \in U \subseteq A \text{ for some } U \in \mathcal{O}X.$$

To sum these up, let us write $A \in \mathcal{N}_X(x)$ for "$A$ is a neighborhood (in $X$) of $x$", and we have

$$A \in \mathcal{N}_X(x) \iff x \in U \subseteq A \text{ for some } U \in \mathcal{O}X \iff x \in \mathbf{int}_X(A).$$

Our goal is to obtain a framework of interior operations without assuming rules or axioms assumed on them. Even though the notion of open sets may not make sense any more once we drop some rules and axioms, we can keep the equivalence between the left-most and right-most conditions, and obtain the following definition.

**Definition 77.** A *neighborhood frame* is a pair $X = (|X|, \mathcal{N}_X)$ that consists of

- a nonempty set $|X|$, called the *underlying set* of $X$, and
- an arbitrary map $\mathcal{N}_X : |X| \to \mathcal{P}\mathcal{P}(|X|)$, called a *neighborhood function* on $|X|$ (and of $X$).

Given a point $x \in X$, each $U \in \mathcal{N}_X(a)$ is called a *neighborhood* of $x$. Every neighborhood function $\mathcal{N}_X$ of $X$ is associated with an operation $\mathbf{int}_X : \mathcal{P}(|X|) \to \mathcal{P}(|X|)$, called the *interior operation* of $X$, such that, for every $A \subseteq X$ and $x \in X$,

$$A \in \mathcal{N}_X(x) \iff x \in \mathbf{int}_X(A).$$

Now that we have dropped the notion of open sets, we cannot use it to define continuity and openness of maps; nonetheless, the notion of interior operations is still with us, which is why we can use Observation 4 as a definition, as follows.

**Definition 78.** Given neighborhood frames $X$ and $Y$, a map $f : X \to Y$ is said to be *continuous* if

$$A \in \mathcal{N}_Y(f(x)) \implies f^{-1}[A] \in \mathcal{N}_X(x)$$

for every $A \subseteq Y$ and $x \in X$, and *open* if

$$f^{-1}[A] \in \mathcal{N}_X(x) \implies A \in \mathcal{N}_Y(f(x))$$

for every $A \subseteq Y$ and $x \in X$.[10] Or, equivalently, $f$ is continuous if

$$f^{-1}[\mathbf{int}_Y(A)] \subseteq \mathbf{int}_X(f^{-1}[A])$$

for every $A \subseteq Y$, and open if

$$\mathbf{int}_X(f^{-1}[A]) \subseteq f^{-1}[\mathbf{int}_Y(A)]$$

for every $A \subseteq Y$.

Both continuous maps and open maps are clearly composable. Therefore we have categories of neighborhood frames and such maps. In particular, we write **Nb** for the category of neighborhood frames and continuous maps.

Topological spaces and Kripke frames are familiar examples of neighborhood frames. Indeed, the category **Top** of topological spaces and continuous maps (in the usual sense) is a full subcategory of **Nb**, and so is the category of Kripke frames with certain maps. These subcategories can be characterized by certain subsets of the following properties, as we will show in Subsection VI.2.3.

**Definition 79.** Given any neighborhood frame $X = (|X|, \mathcal{N}_X)$, we say $X$ is

[10] The usual definition in topology of open maps may make it appear more natural to say $f$ is open if

$$A \in \mathcal{N}_X(x) \implies f[A] \in \mathcal{N}_Y(f(x)).$$

As we will show as Corollary 11, this definition agrees with Definition 78 if $X$ and $Y$ are monotone in the sense of Definition 79 below.

- *monotone*, or *M*, if

$$A \subseteq B \text{ and } A \in \mathcal{N}_X(x) \implies B \in \mathcal{N}_X(x);$$

- *closed under binary meets*, or *C*, if

$$A, B \in \mathcal{N}_X(x) \implies A \cap B \in \mathcal{N}_X(x);$$

- *normal*, or *N*, if for every $x \in X$

$$X \in \mathcal{N}_X(x);$$

- *reflexive*, or *T*, if

$$A \in \mathcal{N}_X(x) \implies x \in A;$$

- *closed under interior*, or *4*, if

$$A \in \mathcal{N}_X(x) \implies \mathbf{int}_X(A) \in \mathcal{N}_X(x);$$

- *nonempty* if $\mathcal{N}_X(x) \neq \varnothing$ for every $x \in X$;
- *consistent* if $\varnothing \notin \mathcal{N}_X(x)$ for every $x \in X$;
- *containing core* if for every $x \in X$ there is $C_x \in \mathcal{N}_X(x)$ such that

$$A \in \mathcal{N}_X(x) \implies C_x \subseteq A;$$

- *quasifiltered*, or *MC*, if it is M and C (that is, if monotone and closed under binary meets);
- *topological* if it is M, C, N, T, and 4;
- *Kripke* if it is monotone and containing core.

It is worth noting that some of these properties can also be defined in terms of interior operations.

**Remark 5.** A neighborhood frame $X$ is

(i) monotone iff its interior operation $\mathbf{int}_X$ is monotone, that is, if it satisfies

$$A \subseteq B \subseteq X \implies \mathbf{int}_X(A) \subseteq \mathbf{int}_X(B);$$

(ii) closed under binary meets iff $\mathbf{int}_X(A) \cap \mathbf{int}_X(B) \subseteq \mathbf{int}_X(A \cap B)$ for every $A, B \subseteq X$;

(iii) normal iff $\mathbf{int}_X(X) = X$;

(iv) reflexive iff $\mathbf{int}_X(A) \subseteq A$ for every $A \subseteq X$;

(v) closed under interior iff $\mathbf{int}_X(A) \subseteq \mathbf{int}_X(\mathbf{int}_X(A))$ for every $A \subseteq X$;

(vi) consistent iff $\mathbf{int}_X(\varnothing) = \varnothing$;

(vii) MC iff $\mathbf{int}_X(A) \cap \mathbf{int}_X(B) = \mathbf{int}_X(A \cap B)$ for every $A, B \subseteq X$.

Among these properties of neighborhood frames, M—being monotone—and C—being closed under binary meets—play the most significant roles in our generalization of topological-sheaf semantics. Let us enter

**Definition 80.** We introduce the following names for full subcategories of **Nb**.

- **MNb** for the category of monotone neighborhood frames;
- **CNb** for the category of neighborhood frames that are closed under binary meets; and
- **MCNb** for the category of MC neighborhood frames.

In Subsection VI.2.2, we study these subcategories as well as **Nb** regarding, in particular, finite products in them. For this purpose, it is helpful to observe that all the forgetful functors from them to **Sets** are both left and right adjoints. To see this, let us first introduce

**Definition 81.** Given any set $|X|$, by the *discrete* and *codiscrete neighborhood functions on* $|X|$ we mean $\mathcal{N}_{\mathcal{P}(|X|)}, \mathcal{N}_{\varnothing} : |X| \to \mathcal{P}\mathcal{P}(|X|)$, respectively, such that

$$\mathcal{N}_{\mathcal{P}(|X|)}(x) = \mathcal{P}(|X|),$$
$$\mathcal{N}_{\varnothing}(x) = \varnothing$$

for each $x \in |X|$. We also call $\mathbf{disc}(|X|) = (|X|, \mathcal{N}_{\mathcal{P}(|X|)})$ and $\mathbf{codisc}(|X|) = (|X|, \mathcal{N}_{\varnothing})$ respectively the *discrete* and *codiscrete neighborhood frames on* $|X|$.

Observe that discrete neighborhood frames are monotone and closed under binary meets, and so are codiscrete neighborhood frames. Moreover, observe

**Remark 6.** Any map $f : |X| \to |Y|$ is continuous from $\mathbf{disc}(|X|)$ to any neighborhood frame on $|Y|$, and continuous from any neighborhood frame on $|X|$ to $\mathbf{codisc}(|Y|)$.

When we write $\mathbb{C}$ for either **Nb**, **MNb**, **CNb**, or **MCNb**, these observations mean that **disc** and **codisc** with the identity on arrows, that is,

$$\mathbf{disc}(f : |X| \to |Y|) = f : \mathbf{disc}(|X|) \to \mathbf{disc}(|Y|),$$
$$\mathbf{codisc}(f : |X| \to |Y|) = f : \mathbf{codisc}(|X|) \to \mathbf{codisc}(|Y|),$$

are functors from **Sets** to $\mathbb{C}$. Remark 6 moreover implies $\mathbf{disc} \dashv \mathbf{U} \dashv \mathbf{codisc}$, where $\mathbf{U} : \mathbb{C} \to \mathbf{Sets}$ is the forgetful functor, that is,

$$\mathbf{U}X = |X|, \qquad \mathbf{U}(f : X \to Y) = f : |X| \to |Y|.$$

It follows from these adjunctions that, if $\mathbb{C} = \mathbf{Nb}, \mathbf{MNb}, \mathbf{CNb}, \mathbf{MCNb}$ has finite products, they are neighborhood frames on finite products in **Sets**, that is, (finite) cartesian products.

### VI.2.2 Products of Neighborhood Frames

Recall from Section VI.1 that topological-sheaf semantics interprets $n$-ary relations, terms, and sentences with $n$-fold products in $\mathbf{Top}/X$. This is why, to extend the semantics to sheaves over general neighborhood frames, we need finite products of neighborhood frames. In this subsection, we show that the categories of neighborhood frames we introduced in Subsection VI.2.1—**Nb**, **MNb**, **CNb**, and **MCNb**—all have arbitrary products, by describing them explicitly.

Let us first consider

**Definition 82.** Given any set $\{X_i\}_{i \in I}$ of neighborhood frames, their *subbasic product* is the neighborhood frame $X = (|X|, \mathcal{N}_X)$ consisting of

- the cartesian product $|X| = \prod_{i \in I} |X_i|$, along with the projections

$$p_i : |X| \to |X_i| :: x \mapsto x(i),$$[11]

- the neighborhood function $\mathcal{N}_X$ on $|X|$ such that

$$\mathcal{N}_X(x) = \bigcup_{i \in I} \{\, p_i^{-1}[U] \mid U \in \mathcal{N}_{X_i}(p_i(x)) \,\}.$$

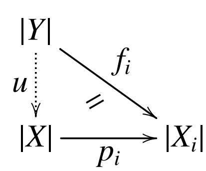
[11] Recall that the elements of the cartesian product $X$ are the maps $x$ of domain $I$ such that $x(i) \in |X_i|$ for all $i \in I$.

For the 0-ary case, we take the 0-ary cartesian product $\{*\}$ paired with $\mathcal{N}_\varnothing(*) = \varnothing$.

It is immediate from the definition that each projection $p_i$ is continuous from $X$ to $X_i$. Indeed, every neighborhood in $\mathcal{N}_X(x)$ is necessary for all $p_i$ to be continuous. That is, $\mathcal{N}_X$ is the "coarsest" neighborhood function on $|X|$ that has all $p_i$ continuous, in the sense that $\mathcal{N}_X(x) \subseteq \mathcal{N}(x)$ for every neighborhood function $\mathcal{N}$ on $|X|$ that has all $p_i$ continuous from $(|X|, \mathcal{N})$ to $X_i$. More generally,

**Fact 65.** Given any set $\{X_i\}_{i \in I}$ of neighborhood frames, their subbasic product $X$ together with the projections $p_i : X \to X_i$ is a product of $X_i$ in **Nb**.

*Proof.* For the case of $I = \varnothing$, $X = \mathbf{codisc}(\{*\})$ is clearly terminal in **Nb** by Remark 6. For $I \neq \varnothing$, let us write $X = (|X|, \mathcal{N}_X)$. Each projection $p_i$ is continuous by definition. Now fix any neighborhood frame $Y$ together with a continuous map $f_i : Y \to X_i$ for each $i \in I$. Then, since $|X|$ is a product of $|X_i|$ in **Sets**, there is a unique map $u : |Y| \to |X|$ such that, for each $i \in I$,




commutes. This $u$ is indeed continuous from $Y$ to $X$ because for each $y \in Y$ we have

$$\begin{aligned} A \in \mathcal{N}_X(u(y)) &\implies A = p_i^{-1}[U] \text{ for some } i \in I \text{ and } U \in \mathcal{N}_{X_i}(p_i(u(y))) \\ &\implies u^{-1}[A] = u^{-1}[p_i^{-1}[U]] = f_i^{-1}[U] \text{ for some } i \in I \text{ and } U \in \mathcal{N}_{X_i}(f_i(y)) \\ &\implies u^{-1}[A] \in \mathcal{N}_Y(y), \end{aligned}$$

where the first entailment is by the definition of $\mathcal{N}_X$, the second is by the commutation above, and the third is by the continuity of $f_i$. □

We should note, however, that subbasic products are not by themselves products in **MNb**, **CNb**, or **MCNb**, since in general they are neither monotone nor closed under binary meets. Nevertheless, they can still give rise to products in **MNb**, **CNb**, or **MCNb**, with the help of the following functors $\mathbf{M} : \mathbf{Nb} \to \mathbf{MNb}$ and $\mathbf{C} : \mathbf{Nb} \to \mathbf{CNb}$.

**Definition 83.** Given any neighborhood frame $X = (|X|, \mathcal{N}_X)$, we define $\widehat{X} = (|X|, \mathcal{N}_{\widehat{X}})$ by

$$A \in \mathcal{N}_{\widehat{X}}(x) \iff U \subseteq A \text{ for some } U \in \mathcal{N}_X(x),$$

and call it the *monotone neighborhood frame generated by a basis* $X$.

Clearly, any $\widehat{X}$ is monotone. This operation of generating a monotone neighborhood frame can be described in terms of interior operations as well:

**Remark 7.** When $\mathbf{int}_X$ and $\mathbf{int}_{\widehat{X}}$ are the interior operations of a neighborhood frame $(|X|, \mathcal{N}_X)$ and the monotone neighborhood frame $(|X|, \mathcal{N}_{\widehat{X}})$ generated by $(|X|, \mathcal{N}_X)$, for every $A \subseteq |X|$ we have

$$\mathbf{int}_{\widehat{X}}(A) = \bigcup_{U \subseteq A} \mathbf{int}_X(U).$$

*Proof.* If $f$ is continuous from $X$ to $Y$, then for every $x \in |X|$ we have

$$\begin{aligned} x \in \mathbf{int}_{\widehat{X}}(A) &\iff A \in \mathcal{N}_{\widehat{X}}(x) \\ &\iff U \subseteq A \text{ for some } U \in \mathcal{N}_X(x) \\ &\iff x \in \mathbf{int}_X(U) \text{ for some } U \subseteq A \\ &\iff x \in \bigcup_{U \subseteq A} \mathbf{int}_X(U) \end{aligned}$$

□

Indeed, this operation gives us a functor.

**Remark 8.** Given neighborhood frames $X$ and $Y$, a map $f : |X| \to |Y|$ is continuous from $\widehat{X}$ to $\widehat{Y}$ if it is continuous from $X$ to $Y$.

*Proof.* For every $A \subseteq Y$ and $x \in X$ we have

$$\begin{aligned} A \in \mathcal{N}_{\widehat{Y}}(f(x)) &\implies U \subseteq A \text{ for some } U \in \mathcal{N}_Y(f(x)) \\ &\implies f^{-1}[U] \subseteq f^{-1}[A] \text{ with } f^{-1}[U] \in \mathcal{N}_X(x) \\ &\implies f^{-1}[A] \in \mathcal{N}_{\widehat{X}}(x). \end{aligned}$$

□

Due to this fact, we can introduce

**Definition 84.** We write $\mathbf{M} : \mathbf{Nb} \to \mathbf{MNb}$ for the (faithful) functor such that

$$\mathbf{M}X = \widehat{X} \qquad \mathbf{M}(f : X \to Y) = f : \widehat{X} \to \widehat{Y}.$$

We can also introduce a functor $\mathbf{C}$ from $\mathbf{Nb}$ to $\mathbf{CNb}$ as follows.

**Definition 85.** Given any neighborhood frame $X = (|X|, \mathcal{N}_X)$, we define $\mathbf{C}X = (|X|, \mathcal{N}_{\mathbf{C}X})$ by

$$A \in \mathcal{N}_{\mathbf{C}X}(x) \iff A = U_0 \cap U_1 \text{ for some } U_0, U_1 \in \mathcal{N}_X(x).$$

**Remark 9.** Given neighborhood frames $X$ and $Y$, a map $f : |X| \to |Y|$ is continuous from $\mathbf{C}X$ to $\mathbf{C}Y$ if it is continuous from $X$ to $Y$.

*Proof.* For every $A \subseteq Y$ and $x \in X$ we have

$$\begin{aligned} A \in \mathcal{N}_{\mathbf{C}Y}(f(x)) &\implies A = U_0 \cap U_1 \text{ for some } U_0, U_1 \in \mathcal{N}_Y(f(x)) \\ &\implies f^{-1}[A] = f^{-1}[U_0] \cap f^{-1}[U_1] \text{ with } f^{-1}[U_0], f^{-1}[U_1] \in \mathcal{N}_X(x) \\ &\implies f^{-1}[A] \in \mathcal{N}_{\mathbf{C}X}(x). \end{aligned}$$ □

**Definition 86.** We write $\mathbf{C} : \mathbf{Nb} \to \mathbf{CNb}$ for the (faithful) functor such that $\mathbf{C}(f : X \to Y) = f : \mathbf{C}X \to \mathbf{C}Y$.

Obviously, $\widehat{X} = X$ if $X$ is already monotone, and $\mathbf{C}X = X$ if $X$ is already closed under binary meets; in other words,

**Remark 10.** $\mathbf{M} \circ \mathbf{i} = 1_{\mathbf{MNb}}$ for the inclusion functor $\mathbf{i} : \mathbf{MNb} \hookrightarrow \mathbf{Nb}$.

**Remark 11.** $\mathbf{C} \circ \mathbf{i} = 1_{\mathbf{CNb}}$ for the inclusion functor $\mathbf{i} : \mathbf{CNb} \hookrightarrow \mathbf{Nb}$.

Observe moreover that $\mathcal{N}_X(x) \subseteq \mathcal{N}_{\widehat{X}}(x)$ and $\mathcal{N}_X(x) \subseteq \mathcal{N}_{\mathbf{C}X}(x)$ by definition, and that $\mathbf{int}_X(A) \subseteq \mathbf{int}_{\widehat{X}}(A)$ by Remark 7. These observations help to show that $\mathbf{M}$ and $\mathbf{C}$ are right adjoints.

**Fact 66.** $\mathbf{i} \dashv \mathbf{M}$ for the inclusion functor $\mathbf{i} : \mathbf{MNb} \hookrightarrow \mathbf{Nb}$.

*Proof.* Since $\mathbf{M}$ is identity on arrows, it is enough to check that, given neighborhood frames $X$ and $Y$ such that $X$ is monotone, a map $f : |X| \to |Y|$ is continuous from $X$ to $Y$ iff continuous from $X$ to $\widehat{Y}$. The "only if" follows from Remark 8 because $\widehat{X} = X$, whereas the "if" is immediate since $\mathcal{N}_Y(y) \subseteq \mathcal{N}_{\widehat{Y}}(y)$ for every $y \in |Y|$. □

Similarly,

**Fact 67.** $\mathbf{i} \dashv \mathbf{C}$ for the inclusion functor $\mathbf{i} : \mathbf{CNb} \hookrightarrow \mathbf{Nb}$.

Therefore, by Remark 10 and Fact 66, and by Remark 11 and Fact 67, products in **MNb** and in **CNb** can be defined simply by first taking products taken in **Nb** and then applying to it **M** and **C**, respectively. For instance, given objects $X$ and $Y$ of **MNb**, their product in **MNb** is given as

$$X \times_{\mathbf{MNb}} Y = \mathbf{M}(\mathbf{i}X \times_{\mathbf{Nb}} \mathbf{i}Y),$$

where the subscripts of $\times$ denote in which categories the products are taken. To moreover define finite products in **MCNb**, it is enough to observe that **M** preserves the property C.

**Remark 12.** If a neighborhood frame $X$ is C, then so is $\widehat{X}$.

*Proof.* If $X$ is closed under binary meets, then for every $x \in X$ we have

$$\begin{aligned} A, B \in \mathcal{N}_{\widehat{X}}(x) &\implies U \subseteq A \text{ and } V \subseteq B \text{ for some } U, V \in \mathcal{N}_X(x) \\ &\implies U \cap V \subseteq A \cap B \text{ with } U \cap V \in \mathcal{N}_X(x) \\ &\implies A \cap B \in \mathcal{N}_{\widehat{X}}(x), \end{aligned}$$

that is, $\widehat{X}$ is also closed under binary meets. □

It immediately follows that the composition of **M** after **C** gives a functor $\mathbf{MC} : \mathbf{Nb} \to \mathbf{MCNb}$. Then, since $\mathbf{MC} \circ \mathbf{i} = 1_{\mathbf{MCNb}}$ and $\mathbf{i} \dashv \mathbf{MC}$ for the inclusion $\mathbf{i} : \mathbf{MCNb} \hookrightarrow \mathbf{Nb}$, products in **MCNb** are given simply by applying **MC** to products taken in **Nb**.[12] To describe finite products in **MCNb** more explicitly, we have

**Remark 13.** Given any MC neighborhood frames $X_1, \ldots, X_n$, the neighborhood frame

$$X_1 \times \cdots \times X_n = (|X_1| \times \cdots \times |X_n|, \mathcal{N}_{X_1 \times \cdots \times X_n})$$

with the neighborhood function $\mathcal{N}_{X_1 \times \cdots \times X_n}$ as follows is a product in **MCNb** of $X_1, \ldots, X_n$. Let $\mathcal{N}_X$ be the neighborhood function of the subbasic product of $X_1, \ldots, X_n$, so that $A \in \mathcal{N}_{\mathbf{C}X}(x)$ iff

- $A = U_1 \times \cdots \times U_n$ for some $U_1 \subseteq |X_1|, \ldots, U_n \subseteq |X_n|$ such that
  - for each $i$, either $U_i \in \mathcal{N}_{X_i}(p_i(x))$ or $U_i = |X_i|$, but
  - $U_i \in \mathcal{N}_{X_i}(p_i(x))$ for at least one $i$.

[12]The composition of **C** after **M** would also work, because **C** preserves the property M.

where we write $p_i : |X_1| \times \cdots \times |X_n| \to |X_i|$ for the projections. Then we set $\mathcal{N}_{X_1\times\cdots\times X_n} = \mathcal{N}_{\widehat{\mathbf{C}X}}$; that is, $A \in \mathcal{N}_{X_1\times\cdots\times X_n}(x)$ iff

- $U_1 \times \cdots \times U_n \subseteq A$ for some $U_1 \subseteq |X_1|, \ldots, U_n \subseteq |X_n|$ such that
  - for each $i$, either $U_i \in \mathcal{N}_{X_i}(p_i(x))$ or $U_i = |X_i|$, but
  - $U_i \in \mathcal{N}_{X_i}(p_i(x))$ for at least one $i$.

It is worth noting that the definition of products in **MCNb** in terms of applying the functor **MC** to subbasic products coincide with the usual definition of product spaces in case $X_i$ are topological spaces.

Let us close this subsection by observing a few more facts on **M** that will be useful later.

**Remark 14.** If a neighborhood frame $X$ is T or 4, then $\widehat{X}$ is also T or 4, respectively.

*Proof.* If $X$ is reflexive, that is, if $\mathbf{int}_X(U) \subseteq U$ for each $U \subseteq |X|$, then

$$\mathbf{int}_{\widehat{X}}(A) = \bigcup_{U \subseteq A} \mathbf{int}_X(U) \subseteq \bigcup_{U \subseteq A} U = A$$

by Remark 7, and hence $\widehat{X}$ is reflexive as well. If $X$ is closed under interior, then for every $x \in X$ we have

$$\begin{aligned} A \in \mathcal{N}_{\widehat{X}}(x) &\implies U \subseteq A \text{ for some } U \in \mathcal{N}_X(x) \\ &\implies \mathbf{int}_X(U) \subseteq \mathbf{int}_{\widehat{X}}(U) \subseteq \mathbf{int}_{\widehat{X}}(A) \text{ with } \mathbf{int}_X(U) \in \mathcal{N}_X(x) \\ &\implies \mathbf{int}_{\widehat{X}}(A) \in \mathcal{N}_{\widehat{X}}(x), \end{aligned}$$

that is, $\widehat{X}$ is also closed under interior. □

Also, the following trivial fact implies that **M** preserves the property N, since any neighborhood frame is nonempty if it is N.

**Remark 15.** If a neighborhood frame $X$ is nonempty, then $\widehat{X}$ is N.

Although **M** preserves the continuity of maps (Remark 8), it does not in general preserve the openness of maps; nonetheless, we still have

**Remark 16.** Given neighborhood frames $X$ and $Y$ and a map $f : |X| \to |Y|$, let us say that $X$ is closed under $f^* \circ f_!$ if

$$U \in \mathcal{N}_X(x) \implies f^{-1}[f[U]] \in \mathcal{N}_X(x).$$

Then $f$ is open from $\widehat{X}$ to $\widehat{Y}$ if it is open from $X$ to $Y$ and $X$ is closed under $f^* \circ f_!$.

*Proof.* If $f$ is open from $X$ to $Y$ and $X$ is closed under $f^* \circ f_!$, then

$$U \in \mathcal{N}_X(x) \implies f^{-1}[f[U]] \in \mathcal{N}_X(x) \implies f[U] \in \mathcal{N}_Y(f(x)),$$

and therefore, for every $U \subseteq Y$ and $x \in X$, we have

$$\begin{aligned} f^{-1}[A] \in \mathcal{N}_{\widehat{X}}(x) &\implies U \subseteq f^{-1}[A] \text{ for some } U \in \mathcal{N}_X(x) \\ &\implies f[U] \subseteq A \text{ with } f[U] \in \mathcal{N}_Y(f(x)) \\ &\implies A \in \mathcal{N}_{\widehat{Y}}(f(x)). \end{aligned}$$ □

**Remark 17.** Given neighborhood frames $X$ and $Y$ such that $Y$ is monotone, a map $f : |X| \to |Y|$ is open from $\widehat{X}$ to $Y$ iff the following holds for every $x \in |X|$:

$$A \in \mathcal{N}_X(x) \implies f[A] \in \mathcal{N}_Y(f(x)). \tag{VI.14}$$

*Proof.* If $f$ is open from $\widehat{X}$ to $Y$, then $A \subseteq f^{-1}[f[A]]$ implies

$$A \in \mathcal{N}_X(x) \implies f^{-1}[f[A]] \in \mathcal{N}_{\widehat{X}}(x) \implies f[A] \in \mathcal{N}_Y(f(x)).$$

On the other hand, if (VI.14) is the case, then for every $x \in |X|$ and $A \subseteq |Y|$ we have

$$\begin{aligned} f^{-1}[A] \in \mathcal{N}_{\widehat{X}}(x) &\implies U \subseteq f^{-1}[A] \text{ for some } U \in \mathcal{N}_X(x) \\ &\implies f[U] \subseteq A \text{ with } f[U] \in \mathcal{N}_Y(f(x)) \\ &\implies A \in \mathcal{N}_Y(f(x)). \end{aligned}$$ □

**Corollary 11.** Given monotone neighborhood frames $X$ and $Y$, a map $f : |X| \to |Y|$ is open from $X$ to $Y$ iff (VI.14) holds for every $x \in |X|$:

### VI.2.3 Some Subcategories of Neighborhood Frames

In this subsection, we review the rather obvious fact that topological spaces and Kripke frames form full subcategories of **MCNb**.

Every topological space $(|X|, \mathcal{O}X)$ has the interior operation $\mathbf{int}_X$ such that

$$x \in \mathbf{int}_X(A) \iff x \in U \subseteq A \text{ for some } U \in \mathcal{O}X,$$

and therefore gives rise to a neighborhood frame $(|X|, \mathcal{N}_X)$ by simply setting

$$A \in \mathcal{N}_X(x) \iff x \in \mathbf{int}_X(A).$$

Due to Observation 4, the usual definitions of continuous maps and open maps between topological spaces coincide with those in Definition 78. It follows that the category **Top** of topological spaces and continuous maps is a full subcategory of **Nb**.

Indeed, the topological spaces are just the neighborhood frames that are topological in the sense of Definition 79. To see this, it is useful to make the following two observations.

**Remark 18.** If a neighborhood frame $X$ is monotone, reflexive, and closed under interior, then

$$\bigcup_{i\in I} \mathbf{int}_X(A_i) = \mathbf{int}_X\Big(\bigcup_{i\in I} \mathbf{int}_X(A_i)\Big)$$

for any collection $\{A_i\}_{i\in I}$ of subsets of $|X|$.

*Proof.* The "⊇" part holds simply by (iv) of Remark 5, whereas "⊆" holds as follows. □

$$\cfrac{\cfrac{\cfrac{\mathbf{int}_X(A_j) \subseteq \bigcup_{i\in I} \mathbf{int}_X(A_i) \text{ for each } j \in I}{\mathbf{int}_X(\mathbf{int}_X(A_j)) \subseteq \mathbf{int}_X(\bigcup_{i\in I} \mathbf{int}_X(A_i)) \text{ for each } j \in I}\text{ by (i)}}{\mathbf{int}_X(A_j) \subseteq \mathbf{int}_X(\bigcup_{i\in I} \mathbf{int}_X(A_i)) \text{ for each } j \in I}\text{ by (v)}}{\bigcup_{i\in I} \mathbf{int}_X(A_i) \subseteq \mathbf{int}_X(\bigcup_{i\in I} \mathbf{int}_X(A_i))}$$

**Remark 19.** If a neighborhood frame $X$ is monotone, reflexive, and closed under interior, then

$$x \in \mathbf{int}_X(A) \iff x \in \mathbf{int}_X(B) \subseteq A \text{ for some } B \subseteq |X|$$

for every $A \subseteq |X|$.

*Proof.* If $x \in \mathbf{int}_X(A)$ then $x \in \mathbf{int}_X(A) \subseteq A$ by (iv) of Remark 5, whereas if $x \in \mathbf{int}_X(B) \subseteq A$ for some $B \subseteq |X|$ then $x \in \mathbf{int}_X(B) \subseteq \mathbf{int}_X(\mathbf{int}_X(B)) \subseteq \mathbf{int}_X(A)$ by (v) and (i). □

Therefore, given any neighborhood frame $(|X|, \mathcal{N}_X)$ that is topological in the sense of satisfying (i)–(v) of Remark 5, the family

$$\mathcal{O}X = \{\, \mathbf{int}_X(A) \mid A \subseteq X \,\}$$

is a topology on $|X|$, since it is closed under finite meets by (ii) and (iii) and closed under arbitrary joins by Remark 18. Moreover, by Remark 19, $\mathcal{O}X$ has the same interior operation $\mathbf{int}_X$ as $\mathcal{N}_X$ does, that is,

$$x \in \mathbf{int}_X(A) \iff x \in U \subseteq A \text{ for some } U \in \mathcal{O}X.$$

Thus **Top** is just the full subcategory of **Nb** of topological neighborhood frames with continuous maps; it is worth noting moreover that **Top** is a full subcategory of **MCNb**, since every topological neighborhood frame is MC by definition.

Recall that a *Kripke frame* is a pair of a set $|X|$ and any binary relation $R$ on $|X|$. Kripke frames correspond one-to-one to neighborhood frames that are Kripke in the sense of Definition 79, in the following way. Any Kripke frame $(|X|, R)$ trivially gives rise to a neighborhood frame $(|X|, \mathcal{N}_R)$ that is containing core, by using $\overrightarrow{R}(x) = \{\, y \in |X| \mid Rxy \,\}$ as a "core" of $\mathcal{N}_R(x)$, that is,

$$\mathcal{N}_R(x) = \{\overrightarrow{R}(x)\}$$

for each $x \in |X|$. Then we generate a monotone neighborhood frame $(|X|, \mathcal{N}_{\widehat{R}})$ by using $(|X|, \mathcal{N}_R)$ as a basis; to write it explicitly,

$$A \in \mathcal{N}_{\widehat{R}}(x) \iff \overrightarrow{R}(x) \subseteq A.$$

This neighborhood frame $(|X|, \mathcal{N}_{\widehat{R}})$ is clearly Kripke. On the other hand, given any Kripke neighborhood frame $(|X|, \mathcal{N}_X)$, each $x \in |X|$ has a "core" $C_x \in \mathcal{N}_X(x)$ such that

$$A \in \mathcal{N}_X(x) \implies C_x \subseteq A,$$

and therefore it gives rise to a Kripke frame $(|X|, R_X)$ with a binary relation $R_X$ on $|X|$ such that

$$R_X xy \iff y \in C_x.$$

These operations $(|X|, R) \mapsto (|X|, \mathcal{N}_{\widehat{R}})$ and $(|X|, \mathcal{N}_X) \mapsto (|X|, R_X)$ clearly give a one-to-one correspondence between Kripke frames and Kripke neighborhood frames; so let us say that they are associated with each other along that correspondence. The correspondence extends to the level of semantics—that is, $(|X|, R)$ and $(|X|, \mathcal{N}_{\widehat{R}})$ interpret $\Box$ in the same way—because

$$x \in \mathbf{int}_{\widehat{R}}(A) \iff A \in \mathcal{N}_{\widehat{R}}(x) \iff \overrightarrow{R}(x) \subseteq A \iff y \in A \text{ for all } y \text{ such that } Rxy.$$

Continuity and openness of maps between Kripke neighborhood frames correspond to kinds of maps that are well known in the field of Kripke semantics.

**Definition 87.** Given any Kripke frames $(|X|, R_X)$ and $(|Y|, R_Y)$, any map $f : |X| \to |Y|$ is said to be *monotone* if it "preserves order", that is, if

$$R_X xy \implies R_Y f(x) f(y)$$

for every $x, y \in |X|$. Moreover, a monotone map $f : X \to Y$ is called a *p-morphism* if

(VI.15) $$R_Y f(x) y \implies R_X xz \text{ and } f(z) = y \text{ for some } z \in X$$

for every $x \in X$ and $y \in Y$.[13]

Even though Definition 87 is more familiar, monotone maps and $p$-morphisms can be defined by the following alternative version.

[13] The name "$p$-morphism" was originally short for "pseudo-epimorphism".

**Remark 20.** Given any Kripke frames $(|X|, R_X)$ and $(|Y|, R_Y)$, a map $f : |X| \to |Y|$ is monotone from $(|X|, R_X)$ to $(|Y|, R_Y)$ iff

$$\overrightarrow{R_X}(x) \subseteq f^{-1}[\overrightarrow{R_Y}(f(x))],$$

or equivalently iff

$$f[\overrightarrow{R_X}(x)] \subseteq \overrightarrow{R_Y}(f(x)),$$

for every $x \in |X|$, and satisfies (VI.15) for every $x \in X$ and $y \in Y$ iff

$$\overrightarrow{R_Y}(f(x)) \subseteq f[\overrightarrow{R_X}(x)]$$

for every $x \in |X|$. Hence $f$ is a $p$-morphism from $(|X|, R_X)$ to $(|Y|, R_Y)$ iff

$$f[\overrightarrow{R_X}(x)] = \overrightarrow{R_Y}(f(x))$$

for every $x \in |X|$.

These notions coincide with the neighborhood notions of continuity and openness of maps.

**Fact 68.** Given any Kripke frames $(|X|, R_X)$ and $(|Y|, R_Y)$, consider the Kripke neighborhood frames $(|X|, \mathcal{N}_{\widehat{R_X}})$ and $(|Y|, \mathcal{N}_{\widehat{R_Y}})$ associated with them, and any map $f : |X| \to |Y|$. Then

- $f$ is monotone from $(|X|, R_X)$ to $(|Y|, R_Y)$ iff continuous from $(|X|, \mathcal{N}_{\widehat{R_X}})$ to $(|Y|, \mathcal{N}_{\widehat{R_Y}})$; and
- $f$ satisfies (VI.15) for every $x \in X$ and $y \in Y$ iff $f$ is open from $(|X|, \mathcal{N}_{\widehat{R_X}})$ to $(|Y|, \mathcal{N}_{\widehat{R_Y}})$.

*Proof.* By Remark 20, we have

$$\begin{aligned}
f \text{ is monotone} &\iff \overrightarrow{R_X}(x) \subseteq f^{-1}[\overrightarrow{R_Y}(f(x))] \text{ for every } x \in |X| \\
&\iff f^{-1}[\overrightarrow{R_Y}(f(x))] \in \mathcal{N}_{\widehat{R_X}}(x) \text{ for every } x \in |X| \\
&\iff f \text{ is continuous from } (|X|, \mathcal{N}_{\widehat{R_X}}) \text{ to } (|Y|, \mathcal{N}_{R_Y}) \\
&\iff f \text{ is continuous from } (|X|, \mathcal{N}_{\widehat{R_X}}) \text{ to } (|Y|, \mathcal{N}_{\widehat{R_Y}}),
\end{aligned}$$

where the third equivalence is because $N_{R_Y}(f(x)) = \{\overrightarrow{R_Y}(f(x))\}$, and the last is by Fact 66. Also, by Remark 20, we have

$$\begin{aligned}
\text{(VI.15)} &\iff \overrightarrow{R_Y}(f(x)) \subseteq f[\overrightarrow{R_X}(x)] \text{ for every } x \in |X| \\
&\iff f[\overrightarrow{R_X}(x)] \in \mathcal{N}_{\widehat{R_Y}}(f(x)) \text{ for every } x \in |X| \\
&\iff f \text{ is open from } (|X|, \mathcal{N}_{\widehat{R_X}}) \text{ to } (|Y|, \mathcal{N}_{\widehat{R_Y}}),
\end{aligned}$$

where the last equivalence holds by Remark 17, because $\mathcal{N}_{R_X}(x) = \{\overrightarrow{R_X}(x)\}$. □

It follows that the category **Kr** of Kripke frames and monotone maps is the full subcategory of **Nb** of Kripke neighborhood frames. The following fact is trivial but worth noting; it follows from this that **Kr** is indeed a full subcategory of **MCNb**.

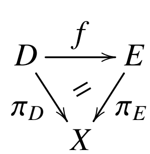
**Remark 21.** Any Kripke neighborhood frame is normal and closed under binary meets.

### VI.2.4 Neighborhood Frames over a Frame

The goal of this subsection is to give a general neighborhood version of Subsection VI.1.3, in which we reviewed the basic definitions of the category **Top** of topological spaces, sliced it over an arbitrary space $X$ to obtain **Top**$/X$, and explicitly described finite products in **Top** and in **Top**$/X$. Since we already introduced categories **Nb**, **MNb**, **CNb**, **MCNb** of neighborhood frames in Subsection VI.2.1 and described finite products in them in Subsection VI.2.2, in this subsection we explicitly describe finite products in the slice categories of these categories over a fixed neighborhood frame.

Slice categories are obtained in a manner straightforwardly extending what we did in Subsection VI.1.3. Let us pick one from **Nb**, **MNb**, **CNb**, **MCNb**, write $\mathbb{C}$ for it, and fix any object $X$ of $\mathbb{C}$. Then, by the definition of slice categories, the category $\mathbb{C}/X$ consists of the following.

- Its objects are *neighborhood frames over* $X$, that is, pairs $(D, \pi)$ of a neighborhood frame $D$ in $\mathbb{C}$ and a continuous map $\pi : D \to X$, called the projection of $(D, \pi)$.
- Its arrows from a neighborhood frame $(D, \pi_D)$ over $X$ to another $(E, \pi_E)$ are continuous maps $f : D \to E$ over $X$, that is, continuous maps $f$ that make the following commute.



We add neighborhood structures in this way to **Sets**$/|X|$ and obtain $\mathbb{C}/X$.

Recall that, in the topological case, pullbacks in **Top** give finite products in **Top**$/X$. Extending this to the neighborhood case, pullbacks in $\mathbb{C}$ give finite products in $\mathbb{C}/X$. Let us show that $\mathbb{C}$ has pullbacks, whether $\mathbb{C}$ is **Nb**, **MNb**, **CNb**, or **MCNb**; because $\mathbb{C}$ has products, it is enough to show that it has equalizers.

**Definition 88.** Given a neighborhood frame $X$ and any subset $|S| \subseteq |X|$, by the *subframe of* $X$ *on* $|S|$ we mean the neighborhood frame $(|S|, \mathcal{N}_S)$ that consists of $|S|$ and the neighborhood function on $|S|$ such that, for each $x \in |S|$,

$$\mathcal{N}_S(x) = \{\, i^{-1}[U] \mid U \in \mathcal{N}_X(i(x)) \,\},$$

where we write $i$ for the inclusion map $i : |S| \hookrightarrow |X|$, so that $i^{-1}[U] = U \cap |S|$.

By definition, the inclusion map $i$ is continuous from $S$ to $X$. Moreover, we should note

**Remark 22.** If a neighborhood frame $X$ is monotone, or closed under binary meets, then any subframe $S$ of it is also monotone, or closed under binary meets, respectively.

*Proof.* Fix any neighborhood frame $X$ and any subframe $S$ of it. If $X$ is monotone, then $A \subseteq B \subseteq |S|$ implies

$$\begin{aligned} A \in \mathcal{N}_S(x) &\implies A = U \cap |S| \text{ for some } U \in \mathcal{N}_X(x) \\ &\implies B = (B \cup (|X| \setminus |S|)) \cap |S| \text{ while } U \subseteq B \cup (|X| \setminus |S|) \text{ for } U \in \mathcal{N}_X(x) \\ &\implies B = V \cap |S| \text{ for some } V \in \mathcal{N}_X(x) \implies B \in \mathcal{N}_S(x). \end{aligned}$$

If $X$ is closed under binary meets, then

$$\begin{aligned} A, B \in \mathcal{N}_S(x) &\implies A = U \cap |S| \text{ and } B = V \cap |S| \text{ for some } U, V \in \mathcal{N}_X(x) \\ &\implies A \cap B = (U \cap V) \cap |S| \text{ with } U \cap V \in \mathcal{N}_X(x) \\ &\implies A \cap B \in \mathcal{N}_S(x). \end{aligned}$$

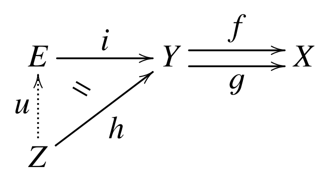
□

**Fact 69.** Given neighborhood frames $X$, $Y$ of $\mathbb{C}$ and any pair of continuous maps $f, g : Y \to X$, let $E = (|E|, \mathcal{N}_E)$ be the subframe of $Y$ on the set

$$|E| = \{\, y \in |Y| \mid f(y) = g(y) \,\} \subseteq |Y|.$$

Then $E$ together with the inclusion map $i : E \to Y$ is a coequalizer in $\mathbb{C}$ of $f$ and $g$.

*Proof.* First of all, $E$ is an object of $\mathbb{C}$ by Remark 22. $i$ is continuous by definition, and clearly $f \circ i = g \circ i$. Now fix any neighborhood frame $Z$ of $\mathbb{C}$ and a continuous map $h : Z \to Y$ such that $f \circ h = g \circ h$. Then, because $|E|$ together with $i$ is an equalizer of $f$ and $g$ in **Sets**, there is a unique map $u : |Z| \to |E|$ such that $i \circ u = h$, as in:

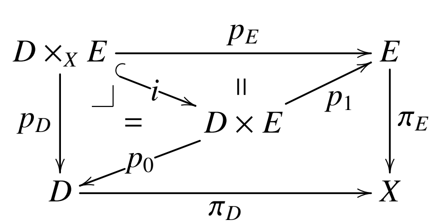


Therefore we only need to show that $u$ is continuous from $Z$ to $E$; but it is so because, for every $z \in |Z|$,

$$\begin{aligned} A \in \mathcal{N}_E(u(z)) &\implies A = i^{-1}[U] \text{ for some } U \in \mathcal{N}_Y(i(u(z))) \\ &\implies u^{-1}[A] = u^{-1}[i^{-1}[U]] = h^{-1}[U] \text{ for } U \in \mathcal{N}_Y(h(z)) \\ &\implies u^{-1}[A] \in \mathcal{N}_Z(z), \end{aligned}$$

where the last entailment is by the continuity of $z : Z \to Y$. □

Due to this fact, we can take pullbacks in $\mathbb{C}$ in exactly the same way as we do in **Top**. Given neighborhood frames $(D, \pi_D)$ and $(E, \pi_E)$ over $X$ in $\mathbb{C}$, we take their product $D \times E$ in $\mathbb{C}$ along with projections $p_0 : D \times E \to D$ and $p_1 : D \times E \to E$, and then the pullback $D \times_X E$ in $\mathbb{C}$ of $(D, \pi_D)$ and $(E, \pi_E)$ is just an equalizer of $\pi_D \circ p_0$ and $\pi_E \circ p_1$.



In other words, $D \times_X E$ is the subframe of the product $D \times E$ on the fibered product $|D| \times_{|X|} |E|$ over $|X|$. And we set $p_D = p_0 \circ i$ and $p_E = p_1 \circ i$ to be the projections from the pullback.

Then finite products in $\mathbb{C}/X$ is defined as follows. The 0-ary product in $\mathbb{C}/X$ is just $(X, 1_X)$. On the other hand, given objects $(D, \pi_D)$ and $(E, \pi_E)$ of $\mathbb{C}/X$, their (binary) product in $\mathbb{C}/X$ is just the pullback $D \times_X E$ of them in $\mathbb{C}$.

It will be useful later to observe that, in some nice cases, pullbacks in **MCNb** can be explicitly described as follows. Given a fibered product $|D| \times_{|X|} |E|$ and any subsets $A \subseteq |D|$ and $B \subseteq |E|$, let us write $A \times_{|X|} B$ for the fibered product of $A$ and $B$ over $|X|$, that is,

$$A \times_{|X|} B = \{ (a, b) \in |D| \times |E| \mid a \in A,\ b \in B, \text{ and } \pi_D(a) = \pi_E(b) \} = i^{-1}[A \times B]$$

for the projections $\pi_D : |D| \to |X|$ and $\pi_E : |E| \to |X|$ and the inclusion $i : |D| \times_{|X|} |E| \hookrightarrow |D| \times |E|$. Then we have

**Remark 23.** Given MC neighborhood frames $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, their pullback $D \times_X E$

$$\begin{array}{ccc} D \times_X E & \xrightarrow{p_E} & E \\ {\scriptstyle p_D}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_E} \\ D & \xrightarrow[\pi_D]{} & X \end{array}$$

in **MCNb** satisfies the "⇐" direction of

$$A \in \mathcal{N}_{D \times_X E}(a, b) \iff U_0 \times_{|X|} U_1 \subseteq A \text{ for some } U_0 \in \mathcal{N}_D(a) \text{ and } U_1 \in \mathcal{N}_E(b) \tag{VI.16}$$

for every $(a, b) \in |D| \times_{|X|} |E|$ and $A \subseteq |D| \times_{|X|} |E|$. Moreover, the "⇒" direction holds if $(D, \pi_D)$ and $(E, \pi_E)$ satisfy the following for every $(a, b) \in |D| \times_{|X|} |E|$:

$$\mathcal{N}_D(a) \neq \varnothing \iff \mathcal{N}_E(b) \neq \varnothing. \tag{VI.17}$$

*Proof.* For the "⇐" direction of (VI.16), observe that for every $(a, b) \in |D| \times_{|X|} |E|$ we have

$$\begin{aligned} U_0 \in \mathcal{N}_D(a) \text{ and } U_1 \in \mathcal{N}_E(b) &\implies U_0 \times U_1 \in \mathcal{N}_{D\times E}(a, b) \\ &\implies U_0 \times_{|X|} U_1 = i^{-1}[U_0 \times U_1] \in \mathcal{N}_{D\times_X E}(a, b) \end{aligned}$$

for the inclusion map $i : D \times_X E \hookrightarrow D \times E$. Hence "⇐" follows, because $D \times_X E$ is monotone by Remark 22.

For "⇒", assume (VI.17) and $A \in \mathcal{N}_{D\times_X E}(a, b)$. By the definition of $\mathcal{N}_{D\times_X E}$, we have $A = i^{-1}[U]$ for some $U \in \mathcal{N}_{D\times E}(a, b)$. This means, by Remark 13, that

(i) $U_0 \times U_1 \subseteq U$ for some $U_0 \in \mathcal{N}_D(a) \cup \{|D|\}$ and $U_1 \in \mathcal{N}_E(b) \cup \{|E|\}$,

(ii) but either $U_0 \in \mathcal{N}_D(a)$ or $U_1 \in \mathcal{N}_E(b)$.

(ii) implies by $(a, b) \in |D| \times_{|X|} |E|$ and (VI.17) that both $\mathcal{N}_D(a) \neq \varnothing$ and $\mathcal{N}_E(b) \neq \varnothing$, from which it follows that $\mathcal{N}_D(a) \cup \{|D|\} = \mathcal{N}_D(a)$ and $\mathcal{N}_E(b) \cup \{|E|\} = \mathcal{N}_E(b)$ because $D$ and $E$ are monotone. Thus (i) and (ii) boil down to

- $U_0 \times U_1 \subseteq U$ for some $U_0 \in \mathcal{N}_D(a)$ and $U_1 \in \mathcal{N}_E(b)$,

while we have $U_0 \times_{|X|} U_1 = i^{-1}[U_0 \times U_1] \subseteq i^{-1}[U] = A$. □

Let us note that the "⇒" direction of (VI.16) may fail if the nice condition (VI.17) does not hold; for instance, $U_0 \in \mathcal{N}_D(a)$ implies $U_0 \times_{|X|} |E| = {p_D}^{-1}[U_0] \in \mathcal{N}_{D\times_X E}(a, b)$ even if $\mathcal{N}_E(b) = \varnothing$, which can be the case when (VI.17) fails. On the other hand, there are many ways to guarantee (VI.17). For instance, it trivially holds if $D$ and $E$ are nonempty, and hence, in particular, if they are topological spaces. Moreover, it is significant for our purpose to note

**Remark 24.** Given MC neighborhood frames $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, (VI.17) of Remark 23 holds if $\pi_D$ and $\pi_E$ are both continuous and open.

*Proof.* Fix any $(a, b) \in |D| \times_{|X|} |E|$ and $A \in \mathcal{N}_D(a)$. Then $A \subseteq {\pi_D}^{-1}[\pi_D[A]]$ implies ${\pi_D}^{-1}[\pi_D[A]] \in \mathcal{N}_D(a)$ because $D$ is monotone. It follows since $\pi_D$ is open that $\pi_D[A] \in \mathcal{N}_X(\pi_D(a)) = \mathcal{N}_X(\pi_E(b))$, because $\pi_D(a) = \pi_E(b)$ for $(a, b) \in |D| \times_{|X|} |E|$. Therefore ${\pi_E}^{-1}[\pi_D[A]] \in \mathcal{N}_E(b)$ since $\pi_E$ is continuous. To sum up, we have the following for every $(a, b) \in |D| \times_{|X|} |E|$.

$$\mathcal{N}_D(a) \neq \varnothing \iff \mathcal{N}_X(\pi_D(a)) = \mathcal{N}_X(\pi_D(b)) \neq \varnothing \iff \mathcal{N}_E(b) \neq \varnothing. \qquad \square$$

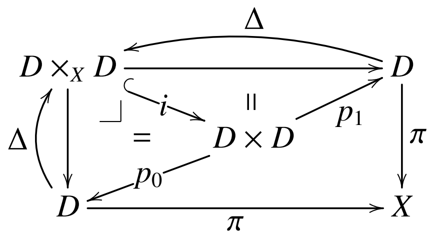
Let us close this subsection by observing

**Remark 25.** Given any neighborhood frame $(D, \pi)$ over $X$ of $\mathbb{C} = \mathbf{Nb}$, $\mathbf{MNb}$, $\mathbf{CNb}$ or $\mathbf{MCNb}$, its diagonal map

$$\Delta : D \to D \times_X D :: a \mapsto (a, a)$$

is continuous from $D$ to the product $D \times_X D$ in $\mathbb{C}/X$.

*Proof.* We first show $\Delta$ to be continuous from $D$ to the pullback $D \times_X D$ taken in $\mathbf{Nb}$.




Let us write $D \times D$ for the product in $\mathbf{Nb}$ of $D$, and $p_0$, $p_1$, $i$ for the projections and the inclusion as above (note that $p_0$, $p_1$ are projections from the product $D \times D$ rather than the pullback $D \times_X D$). Then observe that

$$p_0 \circ i \circ \Delta = p_1 \circ i \circ \Delta = 1_D,$$

since for every $a \in D$ we have $p_k \circ i \circ \Delta(a) = p_k \circ i(a, a) = p_k(a) = a$ for $k = 0, 1$. This observation implies the third line below: For each $a \in D$, we have

$$\begin{aligned}
A \in \mathcal{N}_{D \times_X D}(\Delta(a)) &\implies A = i^{-1}[U] \text{ for some } U \in \mathcal{N}_{D \times D}(i \circ \Delta(a)) \\
&\implies A = i^{-1}[U],\ U = p_k{}^{-1}[V] \text{ for some } i = 0, 1 \text{ and } V \in \mathcal{N}_D(p_k \circ i \circ \Delta(a)) \\
&\implies \Delta^{-1}[A] = \Delta^{-1}[i^{-1}[p_k{}^{-1}[V]]] = V \text{ for some } i = 0, 1 \text{ and } V \in \mathcal{N}_D(a) \\
&\implies \Delta^{-1}[A] \in \mathcal{N}_D(a).
\end{aligned}$$

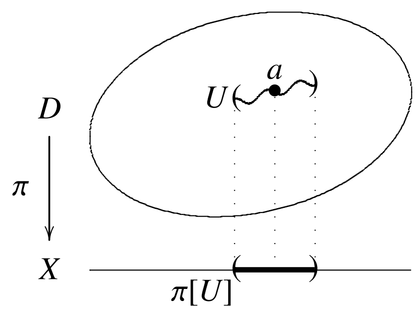
Thus $\Delta$ is continuous from $D$ to the pullback $D \times_X D$ in $\mathbf{Nb}$, establishing the case of $\mathbb{C} = \mathbf{Nb}$. This entails the other cases as follows. For $\mathbb{C} = \mathbf{MNb}$, $\Delta$ is continuous from $\mathbf{M}D = D$ to $\mathbf{M}(D \times_X D)$ by Remarks 8 and 10; similarly for $\mathbb{C} = \mathbf{CNb}$ and $\mathbf{MCNb}$. □

### VI.2.5 Sheaves over a Neighborhood Frame

**Definition 89.** Given neighborhood frames $X$ and $D$, a map $\pi : D \to X$ is called a *local isomorphism* if

(i) $\pi$ is continuous and open, and

(ii) every $a \in D$ with $\mathcal{N}_D(a) \neq \varnothing$ has some $U \in \mathcal{N}_D(a)$ such that $\pi{\restriction}U$ is injective.




When this is the case, we say that the pair $(D, \pi)$ is a *sheaf over the neighborhood frame* $X$, and also that $X$, $D$, and $\pi$ are respectively the *base frame*, *total frame*, and *projection* of the sheaf.

While identity maps are clearly local isomorphisms, local isomorphisms are composable under a certain condition, as follows.

**Fact 70.** Given local isomorphisms $f : D \to E$ and $g : E \to X$, their composition $g \circ f : D \to X$ is also a local isomorphism if $D$ is MC.

*Proof.* Fixing such $f, g$ as above, suppose $D$ is MC. Since open continuous maps are composable, we only need to show (ii) of Definition 89 for $g \circ f$; so let us fix any $a \in D$ with $\mathcal{N}_D(a) \neq \varnothing$. By (ii) for $f$, $f{\restriction}U$ is injective for some $U \in \mathcal{N}_D(a)$; this implies, because $U \subseteq f^{-1}[f[U]]$ and $D$ is M, that $f^{-1}[f[U]] \in \mathcal{N}_D(a)$. Hence $f[U] \in \mathcal{N}_E(f(a))$ because $f$ is open. Therefore, by (ii) for $g$, $g{\restriction}V$ is injective for some $V \in \mathcal{N}_E(f(a))$; this implies $f^{-1}[V] \in \mathcal{N}_D(a)$ since $f$ is continuous. Now, $U \cap f^{-1}[V] \in \mathcal{N}_D(a)$ since $D$ is C. Moreover $(g \circ f){\restriction}(U \cap f^{-1}[V])$ is injective. Thus $g \circ f$ is a local isomorphism. □

Due to this fact, we will restrict our attention to *MC* neighborhood frames and take the category **LI** of MC neighborhood frames and local isomorphisms, so that **LI** is a subcategory of **MCNb**.

Recall that, in Subsection VI.1.4, we listed some facts about local homeomorphisms—namely, Facts 61, 62, 63 and Corollary 10—that were essential for a sheaf semantics as we desired. For the rest of this subsection, we prove the local-isomorphism versions of those facts, as Theorems 10, 8, 9 and Corollary 13, respectively.

**Remark 26.** Given any neighborhood frame $X$ and a neighborhood frame $D$ closed under binary meets, any map $\pi : |D| \to |X|$ satisfies (ii) of Definition 89 if

(ii′) for each $a \in D$, for every $V \in \mathcal{N}_D(a)$ there is some $U \in \mathcal{N}_D(a)$ such that $U \subseteq A$ and $\pi \restriction U$ is injective.

**Theorem 8.** *Given any MC neighborhood frames $X$, $D$ and any open continuous map $\pi : D \to X$, (ii) of Definition 89 (or, equivalently, (ii′) of Remark 26) holds iff*

*(iii) the diagonal map $\Delta : D \to D \times_X D$ is open.*

*Proof.* Let us first note that, since $\pi$ is continuous and open, Remark 24 implies that Remark 23 applies to $\mathcal{N}_{D\times_X D}$. Now, assume (ii). Then (iii) follows, because we have the following.

$$\begin{aligned} \Delta^{-1}[A] \in \mathcal{N}_D(a) &\implies \text{ there is } U \in \mathcal{N}_D(a) \text{ such that } U \subseteq \Delta^{-1}[A] \text{ and } \pi\restriction U \text{ is injective} \\ &\implies U \times_{|X|} U \in \mathcal{N}_{D\times_X D}(\Delta(a)) \text{ with } U \times_{|X|} U \subseteq A \\ &\implies A \in \mathcal{N}_{D\times_X D}(\Delta(a)), \end{aligned}$$

where the first entailment is by (ii′), and the last is because $D$ is monotone; for the second entailment, $U \times_{|X|} U \in \mathcal{N}_{D\times_X D}(\Delta(a))$ is by Remark 23, and we can show $U \times_{|X|} U \subseteq A$ as follows. Fix any $(a, b) \in U \times_{|X|} U$; this means $a, b \in U$ and $\pi(a) = \pi(b)$, which together imply $a = b$ because $\pi\restriction U$ is injective. Therefore $a \in U \subseteq \Delta^{-1}[A]$ implies $(a, b) = \Delta(a) \in A$.

Assume (iii). To show (ii), suppose $\mathcal{N}_D(a) \neq \varnothing$; this implies $\Delta^{-1}[\Delta[D]] = D \in \mathcal{N}_D(a)$ since $D$ is monotone. Therefore $\Delta[D] \in \mathcal{N}_{D\times_X D}(\Delta(a))$ by (iii). Then, by Remark 23, $U_0 \times_{|X|} U_1 \subseteq \Delta[D]$ for some $U_0, U_1 \in \mathcal{N}_D(a)$. Writing $U = U_0 \cap U_1$, we have $U \in \mathcal{N}_D(a)$ since $D$ is closed under binary meets; moreover, $\pi\restriction U$ is injective, because $\pi(y) = \pi(z)$ for $y, z \in U$ means $(y, z) \in U \times_{|X|} U \subseteq \Delta[D]$ and hence $y = z$. Therefore (ii). □

**Lemma 5.** *Given MC neighborhood frames $(D, \pi_D)$ and $(E, \pi_E)$ over $X$, if $\pi_D$ and $\pi_E$ are open and continuous, then the projections $p_D$ and $p_E$ of the pullback $D \times_X E$ in* **MCNb***, as below, are open as well.*

$$\begin{array}{ccc} D \times_X E & \xrightarrow{\;p_E\;} & E \\ {\scriptstyle p_D}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi_E} \\ D & \xrightarrow[\;\pi_D\;]{} & X \end{array}$$

*Proof.* Suppose $\pi_D$ and $\pi_E$ are continuous and open; Remark 24 implies that Remark 23 applies to $\mathcal{N}_{D \times_X E}$. To show $p_D$ to be open, fix ${p_D}^{-1}[A] \in \mathcal{N}_{D \times_X E}(a, b)$. This means, by Remark 23, that

$$\frac{U_0 \times_{|X|} U_1 \subseteq {p_D}^{-1}[A]}{p_D[U_0 \times_{|X|} U_1] \subseteq A}$$

for some $U_0 \in \mathcal{N}_D(a)$ and $U_1 \in \mathcal{N}_E(b)$. Indeed, $p_D[U_0 \times_{|X|} U_1] = U_0 \cap {\pi_D}^{-1}[\pi_E[U_1]]$, because

$$\begin{aligned} a' \in p_D[U_0 \times_{|X|} U_1] &\iff (a', b') \in U_0 \times_{|X|} U_1 \text{ for some } b' \in E \\ &\iff a' \in U_0 \text{ and } \pi_D(a') = \pi_E(b') \text{ for some } b' \in U_1 \\ &\iff a' \in U_0 \text{ and } \pi_D(a') \in \pi_E[U_1] \\ &\iff a' \in U_0 \text{ and } a' \in {\pi_D}^{-1}[\pi_E[U_1]]. \end{aligned}$$

Now, since $E$ is monotone, $U_1 \subseteq {\pi_E}^{-1}[\pi_E[U_1]]$ and $U_1 \in \mathcal{N}_E(b)$ entails the first line below, and then the second and third follow because $\pi_E$ is open and $\pi_D$ continuous (we have $\pi_E(b) = \pi_D(a)$ on the second line by the assumption that $(a, b) \in |D| \times_{|X|} |E|$):

$$\begin{aligned} &{\pi_E}^{-1}[\pi_E[U_1]] \in \mathcal{N}_E(b), \\ &\qquad \pi_E[U_1] \in \mathcal{N}_X(\pi_E(b)) = \mathcal{N}_X(\pi_D(a)), \\ &{\pi_D}^{-1}[\pi_E[U_1]] \in \mathcal{N}_D(a). \end{aligned}$$

Therefore $U_0 \in \mathcal{N}_D(a)$ implies $p_D[U_0 \times_{|X|} U_1] = U_0 \cap {\pi_D}^{-1}[\pi_E[U_1]] \in \mathcal{N}_D(a)$ because $D$ is closed under binary meets. Hence $p_D[U_0 \times_{|X|} U_1] \subseteq A$ implies $A \in \mathcal{N}_D(a)$. Thus $p_D$ is open; the symmetric argument proves $p_E$ is open. □

**Lemma 6.** *In* **MCNb***, if $\pi : D \to X$ is a local isomorphism, then any continuous map $s : X \to D$ such that $\pi \circ s = 1_X$ is open.*

*Proof.* Fix any $x \in X$ and $s^{-1}[A] \in \mathcal{N}_X(x) = \mathcal{N}_X(\pi \circ s(x))$. Then $\pi^{-1}[s^{-1}[A]] \in \mathcal{N}_D(s(x))$ since $\pi$ is continuous. Therefore, by (ii′) of Remark 26, there is $U \in \mathcal{N}_D(s(x))$ such that $U \subseteq \pi^{-1}[s^{-1}[A]]$ and $\pi \restriction U$ is injective. Because $s$ and $\pi$ are continuous and because $D$ is closed under binary meets, $U \in \mathcal{N}_D(s(x))$ implies

$$\begin{aligned} s^{-1}[U] &\in \mathcal{N}_X(x) = \mathcal{N}_X(\pi \circ s(x)), \\ \pi^{-1}[s^{-1}[U]] &\in \mathcal{N}_D(s(x)), \\ U \cap \pi^{-1}[s^{-1}[U]] &\in \mathcal{N}_D(s(x)). \end{aligned}$$

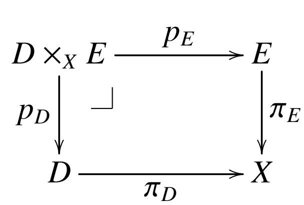
Now we claim $U \cap \pi^{-1}[s^{-1}[U]] \subseteq A$. Fix any $a \in U \cap \pi^{-1}[s^{-1}[U]]$. Since $U \subseteq \pi^{-1}[s^{-1}[A]]$, we have $a \in U, \pi^{-1}[s^{-1}[U]], \pi^{-1}[s^{-1}[A]]$. It follows that $s \circ \pi(a) \in U, A$. Note that $\pi \circ s = 1_X$ entails $\pi(a) = \pi \circ s \circ \pi(a)$; this implies, because $a, s \circ \pi(a) \in U$ and $\pi \restriction U$ injective, that $a = s \circ \pi(a) \in A$.

Therefore $U \cap \pi^{-1}[s^{-1}[U]] \in \mathcal{N}_D(s(x))$ implies $A \in \mathcal{N}_D(s(x))$ since $D$ is monotone. Thus $s$ is open. □

**Corollary 12.** In **MCNb**, if $\pi : D \to X$ is a local isomorphism, then any continuous map $s : X \to D$ such that $\pi \circ s = 1_X$ is a local isomorphism as well.

*Proof.* $s$ is continuous by assumption, and open by Lemma 6; that is, (i) of Definition 89 holds of $s$. To show (ii), note that $s$ is injective since $\pi \circ s = 1_X$; therefore (ii) holds of $s$ because $s \restriction U$ is trivially injective for any $U \in \mathcal{N}_X(x)$. □

**Theorem 9.** *In* **MCNb***, open maps pull back local isomorphisms to local isomorphisms. That is, in the pullback diagram in* **MCNb** *below, if $\pi_D$ is open continuous and $\pi_E$ is a local isomorphism, then $p_D$ is a local isomorphism as well.*



*Proof.* Suppose $\pi_D$ is open continuous and $\pi_E$ is a local isomorphism; then Remark 24 implies that Remark 23 applies to $\mathcal{N}_{D\times_X E}$. By Lemma 5, $p_D$ is open as well as continuous. Hence we only need to show (ii) of Definition 89 for $p_D$.

Fix any $(a,b) \in D \times_X E$ such that $\mathcal{N}_{D\times_X E}(a,b) \neq \varnothing$. Then Remark 23 implies $\mathcal{N}_E(b) \neq \varnothing$; hence (ii) for $\pi_E$ implies that there is $U \in \mathcal{N}_E(b)$ such that $\pi_E \restriction U$ is injective. Then ${p_E}^{-1}[U] \in \mathcal{N}_{D\times_X E}(a,b)$ because $p_E$ is continuous. We moreover claim $p_D \restriction ({p_E}^{-1}[U])$ is injective. Fix any $(a_0,b_0),(a_1,b_1) \in {p_E}^{-1}[U]$ such that $p_D(a_0,b_0) = p_D(a_1,b_1)$; this means $a_0 = a_1$, and also that

$$\pi_E(b_0) = \pi_E \circ p_E(a_0,b_0) = \pi_D \circ p_D(a_0,b_0) = \pi_D \circ p_D(a_1,b_1) = \pi_E \circ p_E(a_1,b_1) = \pi_E(b_1).$$

From this it follows that $b_0 = b_1$, because $(a_0,b_0),(a_1,b_1) \in {p_E}^{-1}[U]$ implies $b_0 = p_E(a_0,b_0) \in U$ and $b_1 = p_E(a_1,b_1) \in U$, while $\pi_E \restriction U$ is injective. In this way, any $(a_0,b_0),(a_1,b_1) \in {p_E}^{-1}[U]$ such that $p_D(a_0,b_0) = p_D(a_1,b_1)$ are identical; that is, $p_D \restriction ({p_E}^{-1}[U])$ is injective. Thus (ii) is true of $p_D$, making it a local isomorphism. □

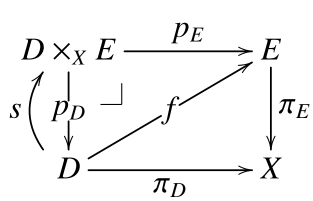
**Corollary 13.** **LI**/$X$ has the same products as **MCNb**/$X$ does.

**Theorem 10.** *Maps of sheaves over any given neighborhood frame $X$ are local isomorphisms. That is, if $\pi_D : D \to X$ and $\pi_E : E \to X$ are local isomorphisms, then any continuous $f : D \to E$ such that $\pi_E \circ f = \pi_D$ is a local isomorphism, too.*

*Proof.* Given such $\pi_D$, $\pi_E$, and $f$, take the pullback $D \times_X E$ in **MCNb** and define

$$s : |D| \to |D| \times_{|X|} |E| :: x \mapsto (x, f(x)),$$

as in the following diagram:




We claim $s$ is continuous. To show this, observe that, for every $U_0 \subseteq |D|$ and $U_1 \subseteq |E|$, we have

$$\begin{aligned} s^{-1}[U_0 \times_X U_1] &= \{\, a \in D \mid s(a) \in U_0 \times_X U_1 \,\} \\ &= \{\, a \in D \mid a \in U_0,\ f(a) \in U_1, \text{ and } \pi_D(a) = \pi_E(f(a)) \,\} \\ &= U_0 \cap f^{-1}[U_1], \end{aligned}$$

where the last equality is due to $\pi_E \circ f = \pi_D$. This implies the fifth entailment below, while the second entailment is by Remark 23: For every $a \in D$,

$$
\begin{aligned}
A \in \mathcal{N}_{D \times_X E}(s(a)) &\implies A \in \mathcal{N}_{D \times_X E}(a, f(a)) \\
&\implies U_0 \times_X U_1 \subseteq A \text{ for some } U_0 \in \mathcal{N}_D(a) \text{ and } U_1 \in \mathcal{N}_E(f(a)) \\
&\implies U_0 \times_X U_1 \subseteq A \text{ with } U_0, f^{-1}[U_1] \in \mathcal{N}_D(a) \\
&\implies U_0 \times_X U_1 \subseteq A \text{ with } U_0 \cap f^{-1}[U_1] \in \mathcal{N}_D(a) \\
&\implies U_0 \cap f^{-1}[U_1] = s^{-1}[U_0 \times_X U_1] \subseteq s^{-1}[A] \text{ with } U_0 \cap f^{-1}[U_1] \in \mathcal{N}_D(a) \\
&\implies s^{-1}[A] \in \mathcal{N}_D(a).
\end{aligned}
$$

Thus $s$ is continuous. Therefore $p_D \circ s = 1_D$ implies that $s$ is a local isomorphism by Corollary 12, while $p_E$ is a local isomorphism by Theorem 9, and hence $f = p_E \circ s$ is a local isomorphism as well by Fact 70. $\square$

By virtue of Theorems 8, 9, 10 and Corollary 13, we have a neighborhood version of Fact 64, the key fact that makes semantics work as we desire.

**Fact 71.** In $\mathbf{LI}/X$ for any neighborhood frame $X$, the following diagram commutes for any map $f : D \to E$ of sheaves over $X$ (including the projection of a sheaf over $X$).

$$
\begin{array}{ccc}
\mathcal{P}(|E|) & \xrightarrow{\ \mathbf{int}_E\ } & \mathcal{P}(|E|) \\
{\scriptstyle f^{-1}}\downarrow & \not= & \downarrow{\scriptstyle f^{-1}} \\
\mathcal{P}(|D|) & \xrightarrow[\ \mathbf{int}_D\ ]{} & \mathcal{P}(|D|)
\end{array}
$$

And we are finally ready for giving a sheaf semantics over neighborhood frames.

### VI.2.6 Neighborhood-Sheaf Semantics for First-Order Modal Logic

Now that we have laid out the category $\mathbf{LI}/X$ of sheaves over a neighborhood frame $X$ and made sure that it shares the same, essential property as $\mathbf{LH}/X$ has in order to give rise to a semantics for first-order modal logic, we can finally define neighborhood-sheaf semantics for first-order modal logic.

**Definition 90.** Given a first-order modal language $\mathcal{L}$, by a *neighborhood-sheaf model* for $\mathcal{L}$ over a given MC neighborhood frame $X$ we mean an $\mathcal{L}$ structure $\mathfrak{M} = (\pi, R_i^{\mathfrak{M}}, f_j^{\mathfrak{M}}, c_k^{\mathfrak{M}})_{i\in I, j\in J, k\in K}$ in $\mathbf{Sets}/|X|$ such that

- $\pi : D \to X$ is a local isomorphism in $\mathbf{LI}$;
- $f^{\mathfrak{M}}$ is a map of sheaves over $X$ from $(D^n, \pi^n)$ to $(D, \pi)$, for each $n$-ary function symbol $f$ of $\mathcal{L}$;
- in particular, $c^{\mathfrak{M}}$ is a map of sheaves over $X$ from $(X, 1_X)$ to $(D, \pi)$, for each constant $c$ of $\mathcal{L}$.

**Definition 91.** Given a first-order modal language $\mathcal{L}$, by a *neighborhood-sheaf interpretation* for $\mathcal{L}$ over a given MC neighborhood frame $X$ we mean a classical interpretation $(\mathfrak{M}, [\![-]\!])$ for $\mathcal{L}$ in $\mathbf{Sets}/|X|$ such that

- $\mathfrak{M}$ is a neighborhood-sheaf model for $\mathcal{L}$ over $X$, and
- $[\![-]\!]$ satisfies

(VI.12) $$[\![\, \bar{x} \mid \Box\varphi \,]\!] = \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Box]\!] = \mathbf{int}_{D^n});$$

(VI.13) $$[\![\, \bar{x} \mid \Diamond\varphi \,]\!] = \mathbf{cl}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]) \qquad (\text{that is, } [\![\Diamond]\!] = \mathbf{cl}_{D^n}).$$

We call the class of such interpretations *neighborhood-sheaf semantics* over the given neighborhood frame $X$; and then, by neighborhood-sheaf semantics (*simpliciter*), we mean the class of neighborhood-sheaf interpretations over some neighborhood frame or other.

Soundness obtains in exactly the similar way as it did before with **FOS4**. The only difference is that we have **MC** in place of **S4**, because it is the logic corresponding to MC neighborhood frames.

**Definition 92.** First-order modal logic **FOMC** consists of the following two sorts of axioms and rules.

1. All axioms and rules of (classical) first-order logic.

2. The rule and axiom of propositional modal logic **MC**; that is, M and C.

**Theorem 11.** **FOMC** *is sound with respect to topological-sheaf semantics.*

It is moreover complete, in the following strong form, which says any consistent theory extending FOMC has a "canonical" interpretation.

**Theorem 12.** *For any consistent theory* $\mathbb{T}$ *of first-order modal logic extending* **FOMC***, there exists a neighborhood-sheaf interpretation* $(\pi, [\![-]\!])$ *that validates all and only theorems of* $\mathbb{T}$*.*

We prove this theorem in Section VI.3, the final section of this dissertation.

## VI.3 COMPLETENESS

Let us say that a theory $\mathbb{T}$ is FOM if it satisfies all the rules and axioms of FOM, and FOMC if it satisfies all the rules and axioms of FOMC. In this section, we provide a proof for the completeness of the logic **FOMC**;[14] the completeness theorem we prove is of the following form.

**Theorem 13.** *For any consistent FOMC theory* $\mathbb{T}$ *in a first-order modal language* $\mathcal{L}$*, there exist a neighborhood frame* $X$*, a sheaf* $(D, \pi : D \to X)$ *over* $X$*, and a neighborhood-sheaf interpretation* $(\pi, [\![-]\!])$ *such that*

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$$

*for every pair of formulas* $\varphi$*,* $\psi$ *of* $\mathcal{L}$*.*

### VI.3.1 Sufficient Set of Models with All Names

We prove Theorem 13 by, given any consistent FOMC theory $\mathbb{T}$, constructing a neighborhood interpretation $(\pi, [\![-]\!])$ as needed in the statement of Theorem 13. In this subsection, we prepare the underlying set $X$ of the base frame $(X, \mathcal{N}_X)$ of the sheaf $(D, \pi : D \to X)$.

We achieve this preparation with two lemmas. One is the purification lemma (Lemma 1 on p. 79), which we already proved in Subsection III.1.3. And, using this lemma, we prove the other

[14] The proof is inspired by those of McKinsey and Tarski [30], Segerberg [39], and Butz and Moerdijk [8, 9, 31].

lemma, which we call the "lazy Henkinization" lemma. Let us first restate the purification lemma, in terms of $[\![-]\!]$ instead of $\vDash$.

**Lemma 1.** *For any first-order language $\mathcal{L}$ (that may have non-classical sentential operators, $\Box$ and $\Diamond$ for instance),*[15] *there exist*

- *a (purely) classical first-order language $\mathcal{L}^{\mathrm{pc}}$ (obtained by perhaps adding new primitive predicates to $\mathcal{L}$);*
- *a surjection $* : \mathrm{sent}(\mathcal{L}) \twoheadrightarrow \mathrm{sent}(\mathcal{L}^{\mathrm{pc}})$ such that*
  - *for every theory $\mathbb{T}$ in $\mathcal{L}$ that respects alpha-equivalence,*[16] *there is a theory $\mathbb{T}^{\mathrm{pc}}$ in $\mathcal{L}^{\mathrm{pc}}$ such that, for every pair of sentences $\varphi$, $\psi$ of $\mathcal{L}$,*

$$\mathbb{T}^{\mathrm{pc}} \text{ proves } \varphi^* \vdash \psi^* \iff \mathbb{T} \text{ proves } \varphi \vdash \psi;$$

- *a (class-sized) bijective operation $* : (M, [\![-]\!]) \mapsto (M^*, [\![-]\!]^*)$ from the class of classical interpretations for $\mathcal{L}$ to the class of those for $\mathcal{L}^{\mathrm{pc}}$ such that, for each classical interpretation $(M, [\![-]\!])$ for $\mathcal{L}$,*
  - *$M^*$ is an expansion of $\mathcal{L}$ structure $M$ to $\mathcal{L}^{\mathrm{pc}}$,*
  - *$(M^*, [\![-]\!]^*)$ is the unique classical interpretation for $\mathcal{L}^{\mathrm{pc}}$ on $M^*$, and, moreover,*
  - *for every sentence $\varphi$ of $\mathcal{L}$,*

$$[\![\, \bar{x} \mid \varphi^* \,]\!]^* = [\![\, \bar{x} \mid \varphi \,]\!].$$

This lemma can be used to prove the following.

**Lemma 7** (Lazy Henkinization lemma)**.** *Given a first-order language $\mathcal{L}$ and a consistent theory $\mathbb{T}$ in $\mathcal{L}$ that respects alpha-equivalence and has all the rules and axioms of classical first-order logic, there exist $\mathcal{L}^{\mathrm{Hen}}$, $\mathbb{T}^{\mathrm{Hen}}$ and $\mathfrak{M}$ such that*

[15] By a first-order language we mean a language with classical operators—by which we mean Boolean connectives and quantifiers—but perhaps with more operators. We say a first-order language is (purely) classical if it only has classical operators. Lemma 1 is trivial for classical $\mathcal{L}$, with $\mathcal{L}^{\mathrm{pc}} = \mathcal{L}$.

[16] Under the formulation of theories in terms of binary sequents, we say a theory $\mathbb{T}$ *respects alpha-equivalence* if

$$\varphi_0 \propto \varphi_1 \text{ and } \psi_0 \propto \psi_1 \implies \mathbb{T} \text{ proves } \varphi_0 \vdash \psi_0 \text{ iff it proves } \varphi_1 \vdash \psi_1,$$

where we write $\propto$ for alpha-equivalence.

*(i)* $\mathcal{L}^{\text{Hen}}$ *is an extension of* $\mathcal{L}$ *obtained by perhaps adding new constants, the set of which is written* $C = \{ c_i \mid i < \lambda \}$ *for some cardinal* $\lambda$ *(then* $\mathcal{L}^{\text{Hen}} = \mathcal{L} \cup C$ *and* $\mathcal{L} \cap C = \emptyset$*).*

*(ii)* $\mathbb{T}^{\text{Hen}}$ *is a theory in* $\mathcal{L}^{\text{Hen}}$ *extending* $\mathbb{T}$ *so that, for any sentences* $\varphi$, $\psi$ *of* $\mathcal{L}$*, any variables* $x_1, \ldots, x_n$ *of* $\mathcal{L}$*, and any (new) constants* $c_1, \ldots, c_n \in C$*,*

$$\mathbb{T}^{\text{Hen}} \textit{ proves } [\bar{c}/\bar{x}]\varphi \vdash [\bar{c}/\bar{x}]\psi \iff \mathbb{T} \textit{ proves } \varphi \vdash \psi,$$

*where we write* $[\bar{c}/\bar{x}]$ *for* $[c_n/x_n] \cdots [c_1/x_1]$.

*(iii)* $\mathfrak{M}$ *is a set of classical interpretations for* $\mathcal{L}^{\text{Hen}}$*, and moreover a sufficient set of models of* $\mathbb{T}^{\text{Hen}}$*, meaning, for any sentences* $\varphi$, $\psi$ *of* $\mathcal{L}^{\text{Hen}}$*,*

$$\mathbb{T}^{\text{Hen}} \textit{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!] \textit{ for every } (M, [\![-]\!]) \in \mathfrak{M}.$$

*(iv)* $\mathfrak{M}$ *is "named totally" by* $\mathcal{L}^{\text{Hen}}$*, in the sense that, for every* $a \in |M|$ *in every* $M \in \mathfrak{M}$*, there is a constant* $c$ *of* $\mathcal{L}^{\text{Hen}}$ *such that* $c^M = a$*.*

*Proof.* Suppose $\mathbb{T}$ is a consistent theory that respects alpha-equivalence and has all the rules and axioms of classical first-order logic. Then, by Lemma 1, there is a theory $\mathbb{T}^{\text{pc}}$ in $\mathcal{L}^{\text{pc}}$ such that, for every pair of sentences $\varphi$, $\psi$ of $\mathcal{L}$,

$$\mathbb{T}^{\text{pc}} \text{ proves } \varphi^* \vdash \psi^* \iff \mathbb{T} \text{ proves } \varphi \vdash \psi.$$

$\mathbb{T}^{\text{pc}}$ is a consistent classical first-order theory in the classical first-order language $\mathcal{L}^{\text{pc}}$, because $\mathbb{T}$ is a consistent theory that has all the rules and axioms of classical first-order logic. Therefore, by Gödel's completeness theorem for classical first-order logic (as generalized by Henkin for $\mathcal{L}^{\text{pc}}$ of any cardinality), there is a class $\mathbf{M} \neq \emptyset$ of classical interpretations for $\mathcal{L}^{\text{pc}}$ such that

$$\mathbb{T}^{\text{pc}} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!] \text{ for every } (M, [\![-]\!]) \in \mathbf{M},$$

which means $\mathbf{M}$ satisfies (iii) above (for $\mathbf{M}$, $\mathcal{L}^{\text{pc}}$, $\mathbb{T}^{\text{pc}}$ in place of $\mathfrak{M}$, $\mathcal{L}^{\text{Hen}}$, $\mathbb{T}^{\text{Hen}}$, respectively) if $\mathbf{M}$ is a set. While $\mathbf{M}$ may well be too large to be a set, the Löwenheim-Skolem theorem implies that there is a cardinal number $\lambda$ such that the set $\mathfrak{M}_0 = \{ (M, [\![-]\!]) \in \mathbf{M} \mid \|M\| \leqslant \lambda \} \subseteq \mathbf{M}$ satisfies

$$\mathbb{T}^{\text{pc}} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!] \text{ for every } (M, [\![-]\!]) \in \mathfrak{M}_0$$

for every pair of sentences $\varphi, \psi$ of $\mathcal{L}^{\mathrm{pc}}$. Then take the inverse-image $\mathfrak{M}_1$ of $\mathfrak{M}_0$ under the bijective operation $*$ as in Lemma 1; that is, $\mathfrak{M}_1$ is the set of classical interpretations for $\mathcal{L}$ such that

$$(M, [\![-]\!]) \in \mathfrak{M}_1 \iff (M^*, [\![-]\!]^*) \in \mathfrak{M}_0.$$

Note that $\|M\| \leqslant \lambda$ for every $(M, [\![-]\!]) \in \mathfrak{M}_1$, since $M^*$ is an expansion of $M$. Moreover, (iii) holds for $\mathfrak{M}_1$, $\mathcal{L}$, $\mathbb{T}$ in place of $\mathfrak{M}$, $\mathcal{L}^{\mathrm{Hen}}$, $\mathbb{T}^{\mathrm{Hen}}$ because, for every sentences $\varphi, \psi$ of $\mathcal{L}$,

$$\begin{aligned}
\mathbb{T} \text{ proves } \varphi \vdash \psi &\iff \mathbb{T}^* \text{ proves } \varphi^* \vdash \psi^* \\
&\iff [\![\, \bar{x} \mid \varphi^* \,]\!]^* \subseteq [\![\, \bar{x} \mid \psi^* \,]\!]^* \text{ for every } (M^*, [\![-]\!]^*) \in \mathfrak{M}_0 \\
&\iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!] \text{ for every } (M, [\![-]\!]) \in \mathfrak{M}_1
\end{aligned}$$

by Lemma 1.

Thus (iii) holds, along with (i) and (ii) trivially, for $\mathfrak{M}_1$, $\mathcal{L}$, $\mathbb{T}$ in place of $\mathfrak{M}$, $\mathcal{L}^{\mathrm{Hen}}$, $\mathbb{T}^{\mathrm{Hen}}$. Yet (iv) does not necessarily hold. To ensure (iv), we invoke a technique which may be called "lazy Henkinization", which is to take

$$\begin{aligned}
\mathcal{L}^{\mathrm{Hen}} &:= \mathcal{L} \cup C, \text{ adding new constants } C = \{\, c_i \mid i < \lambda \,\} \text{ for } \lambda \text{ as above, and} \\
\mathfrak{M} &:= \{\, (M_e, [\![-]\!]_e) \mid (M, [\![-]\!]) \in \mathfrak{M}_1 \text{ and } e : \lambda \twoheadrightarrow |M| \text{ is a surjection} \,\},
\end{aligned}$$

where we write $(M_e, [\![-]\!]_e)$ for the expansion of $(M, [\![-]\!])$ to $\mathcal{L}^{\mathrm{Hen}}$ with ${c_i}^{M_e} = e(i)$ for every $i < \lambda$.[17] Then obviously (i) and (iv) hold. Finally, define $\mathbb{T}^{\mathrm{Hen}}$ to be the theory of $\mathfrak{M}$; that is, we make (iii) hold by definition.

---

[17] The reader may wonder why we use "lazy Henkinization" rather than the usual method of adding Henkin constants to attain (iv). That method does not serve our purpose for the following reason. Suppose we add to $\mathcal{L}$ a constant $c_\varphi$ for each sentence $\varphi$ of $\mathcal{L}$, along with the corresponding Henkin axiom $\exists x . \varphi \vdash [c_\varphi / x]\varphi$ added to the extended theory $\mathbb{T}^{\mathrm{Hen}}$. Then, if (ii) holds, it implies that $\mathbb{T}^{\mathrm{Hen}}$ proves

$$\dfrac{\dfrac{\exists x . \varphi \vdash [c_\varphi / x]\varphi \qquad [c_\varphi / x]\varphi \vdash \exists x . \varphi}{\Box\exists x . \varphi \vdash \Box[c_\varphi / x]\varphi}\,\mathrm{E} \qquad [c_\varphi / x]\Box\varphi \vdash \exists x \Box\varphi}{\Box\exists x . \varphi \vdash \exists x \Box\varphi}\,(\Box[c_\varphi / x]\varphi = [c_\varphi / x]\Box\varphi)$$

Hence, for (ii) to be the case, $\Box\exists x . \varphi \vdash \exists x \Box\varphi$ must be provable in $\mathbb{T}$ as well, although it is not valid in neighborhood-sheaf semantics, as observed in Subsection I.3.6.

To show (ii), it suffices to show (VI.18) and (VI.19) below to be equivalent, due to the (iii) we saw for both $\mathfrak{M}$ and $\mathfrak{M}_1$; here we fix $\varphi$, $\psi$, $\bar{x}$, $\bar{c}$ as in the statement of (ii).

$$\llbracket \bar{y} \mid [\bar{c}/\bar{x}]\varphi \rrbracket_e \subseteq \llbracket \bar{y} \mid [\bar{c}/\bar{x}]\psi \rrbracket_e \text{ for every } (M_e, \llbracket - \rrbracket_e) \in \mathfrak{M}. \tag{VI.18}$$

$$\llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket \subseteq \llbracket \bar{x}, \bar{y} \mid \psi \rrbracket \text{ for every } (M, \llbracket - \rrbracket) \in \mathfrak{M}_1. \tag{VI.19}$$

We should note that, for every $(M, \llbracket - \rrbracket) \in \mathfrak{M}_1$ and $e : \lambda \twoheadrightarrow |\mathfrak{M}|$, we have $\llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket = \llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket_e$, and similarly for $\psi$, since $\varphi, \psi$ are sentences of $\mathcal{L}$ and since $(M_e, \llbracket - \rrbracket_e)$ is an expansion of $(M, \llbracket - \rrbracket)$. Now, if (VI.19), then for every $(M_e, \llbracket - \rrbracket_e) \in \mathfrak{M}$ we have

$$\begin{aligned} \bar{b} \in \llbracket \bar{y} \mid [\bar{c}/\bar{x}]\varphi \rrbracket_e &\implies (\bar{c}^{M_e}, \bar{b}) \in \llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket_e = \llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket \subseteq \llbracket \bar{x}, \bar{y} \mid \psi \rrbracket = \llbracket \bar{x}, \bar{y} \mid \psi \rrbracket_e \\ &\implies \bar{b} \in \llbracket \bar{y} \mid [\bar{c}/\bar{x}]\psi \rrbracket_e, \end{aligned}$$

and hence (VI.18). On the other hand, assume (VI.18) and fix any $(M, \llbracket - \rrbracket) \in \mathfrak{M}_0$. Note that, for every $\bar{a} \in |M|^n$, there is $e : \lambda \twoheadrightarrow |M|$ such that each $k$ ($1 \leqslant k \leqslant n$) has $e(k) = a_k$, that is, ${c_k}^{M_e} = a_k$; so, write $\bar{c}^{M_e} = \bar{a}$. Then, given any $\bar{a}$, $\bar{b}$, we can take such $e$ (for $\bar{a}$) to show

$$\begin{aligned} (\bar{a}, \bar{b}) \in \llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket &\implies (\bar{c}^{M_e}, \bar{b}) \in \llbracket \bar{x}, \bar{y} \mid \varphi \rrbracket_e \\ &\implies \bar{b} \in \llbracket \bar{y} \mid [\bar{c}/\bar{x}]\varphi \rrbracket_e \subseteq \llbracket \bar{y} \mid [\bar{c}/\bar{x}]\psi \rrbracket_e \\ &\implies (\bar{a}, \bar{b}) \in \llbracket \bar{x}, \bar{y} \mid \psi \rrbracket_e = \llbracket \bar{x}, \bar{y} \mid \psi \rrbracket. \end{aligned}$$

Thus (VI.19). □

We should note that any FOM or FOMC theory of first-order modal logic has all the rules and axioms of classical first-order logic; it moreover respects alpha-equivalence, due to classical first-order logic and the rule E of modal logic. That is, any consistent FOM or FOMC theory satisfies the condition for $\mathbb{T}$ as in Lemma 7. Let us also note the following.

**Remark 27.** For any (new) constants $c_1, \dots, c_n \in C$, if $\mathbb{T}$ has an axiom

$$\varphi \vdash \psi$$

in $\mathcal{L}$ then $\mathbb{T}^{\mathrm{Hen}}$ has the axiom

$$[\bar{c}/\bar{x}]\varphi \vdash [\bar{c}/\bar{x}]\psi,$$

and if $\mathbb{T}$ has a rule

$$\frac{\varphi_1 \vdash \psi_1 \qquad \cdots \qquad \varphi_n \vdash \psi_n}{\varphi \vdash \psi}$$

in $\mathcal{L}$ then $\mathbb{T}^{\mathrm{Hen}}$ has the rule

$$\frac{[\bar{c}/\bar{x}]\varphi_1 \vdash [\bar{c}/\bar{x}]\psi_1 \qquad \cdots \qquad [\bar{c}/\bar{x}]\varphi_n \vdash [\bar{c}/\bar{x}]\psi_n}{[\bar{c}/\bar{x}]\varphi \vdash [\bar{c}/\bar{x}]\psi}.$$

*Proof.* (ii) of Lemma 7 immediately implies the axiom part of Remark 27, and also implies the following when $\mathbb{T}$ has the first rule. □

$$\begin{array}{ccc}
\mathbb{T}^{\mathrm{Hen}} \text{ proves all } [\bar{c}/\bar{x}]\varphi_i \vdash [\bar{c}/\bar{x}]\psi_i & \overset{\text{(ii)}}{\Longleftrightarrow} & \mathbb{T} \text{ proves all } \varphi_i \vdash \psi_i \\
 & & \Downarrow \\
\mathbb{T}^{\mathrm{Hen}} \text{ proves } [\bar{c}/\bar{x}]\varphi \vdash [\bar{c}/\bar{x}]\psi & \underset{\text{(ii)}}{\Longleftrightarrow} & \mathbb{T} \text{ proves } \varphi \vdash \psi
\end{array}$$

This remark means that $\mathbb{T}^{\mathrm{Hen}}$ is FOM if $\mathbb{T}$ is, $\mathbb{T}^{\mathrm{Hen}}$ is FOMC if $\mathbb{T}$ is, and so on.

### VI.3.2 Frames of Models with Logical Topology

Given a first-order modal language $\mathcal{L}$ and a consistent FOMC theory $\mathbb{T}$ in $\mathcal{L}$, take $\mathcal{L}^{\mathrm{Hen}}$, $\mathbb{T}^{\mathrm{Hen}}$, and $\mathfrak{M}$ as given by Lemma 7. Then we use the set $\mathfrak{M}$ of classical interpretations for $\mathcal{L}^{\mathrm{Hen}}$ as a base frame and construct a sheaf over it by bundling up the domains of individuals of all $(M, [\![-]\!]) \in \mathfrak{M}$. Neighborhood functions for the base and total frames will be defined "logically", that is, by using the interpretations of $\Box$ sentences.

Let us make a small notational remark. From this subsection on, we write $|\mathfrak{M}|$ instead of $\mathfrak{M}$ for the set we constructed in Subsection VI.3.1, because it is just a set and without a neighborhood function; we will reserve $\mathfrak{M}$ for the neighborhood frame we will define on $|\mathfrak{M}|$. Also, for the sake of simplicity, we write $M$ for both an $\mathcal{L}$ structure and a classical interpretation for $\mathcal{L}$.[18]

Now we define a set $|\mathcal{D}|$ over the set $|\mathfrak{M}|$; we will equip $|\mathfrak{M}|$ and $|\mathcal{D}|$ with suitable neighborhood functions at the end of this subsection, so that the obtained neighborhood frame $\mathcal{D}$ with the projection forms a neighborhood-sheaf over the neighborhood frame $\mathfrak{M}$.

**Definition 93.** We define a set

$$|\mathcal{D}| := \sum_{M \in |\mathfrak{M}|} |M| = \{\, (M, a) \mid M \in |\mathfrak{M}| \text{ and } a \in |M| \,\}$$

over $|\mathfrak{M}|$, with the projection $\pi : |\mathcal{D}| \to |\mathfrak{M}| :: (M, a) \mapsto M$.

It should be clear that each $n$-fold product of $|\mathcal{D}|$ in **Sets**$/|\mathfrak{M}|$, that is, over $|\mathfrak{M}|$, can be written simply as a set of tuples of the form $(M, \bar{a})$ rather than $((M, a_1), \ldots, (M, a_n))$, so that

$$|\mathcal{D}|^n := \sum_{M \in |\mathfrak{M}|} |M|^n = \{\, (M, \bar{a}) \mid M \in |\mathfrak{M}| \text{ and } \bar{a} \in |M|^n \,\}$$

with the projection $\pi^n : |\mathcal{D}|^n \to |\mathfrak{M}| :: (M, \bar{a}) \mapsto M$.

Since the structure $M$ in each interpretation $(M, [\![-]\!]^M) \in |\mathfrak{M}|$ interprets the basic vocabulary of $\mathcal{L}^{\mathrm{Hen}}$ with $R^M$, $f^M$, $c^M$, we can let the entire $\pi : |\mathcal{D}| \to |\mathfrak{M}|$ interpret the same by bundling up all $R^M$, $f^M$, $c^M$.

[18] In our terminology (see Chapter III), an $\mathcal{L}$ structure $M$ consists of a domain $|M|$ and interpretations of primitive predicates and terms of $\mathcal{L}$. Then a map $[\![-]\!]$ extends these interpretations to interpret all sentences of $\mathcal{L}$, so that a pair $(M, [\![-]\!])$ is an interpretation for $\mathcal{L}$ on $M$. So, in the abusive notation we introduce here, $M$ refers to an interpretation when we write $M \in |\mathfrak{M}|$, whereas $M$ refers to a structure when we write $|M|$ for the domain of that structure or $c^M$ for the interpretation by that structure of a constant $c$, for instance. We should note that, when $\mathcal{L}$ is not classical, there may well be (and hence $|\mathfrak{M}|$ may well contain) several interpretations on the same structure $M$, which is why we cannot identify interpretations in $|\mathfrak{M}|$ with structures.

**Definition 94.** We write $\overline{\mathfrak{M}} = (\pi, R_i^{\overline{\mathfrak{M}}}, f_j^{\overline{\mathfrak{M}}}, c_k^{\overline{\mathfrak{M}}})_{i\in I, j\in J, k\in K}$ for the tuple that consists of $\pi$ and

- for each $n$-ary primitive predicate $R$ of $\mathcal{L}$,

$$R^{\overline{\mathfrak{M}}} = \sum_{M\in|\mathfrak{M}|} R^M = \{ (M, \bar{a}) \mid M \in |\mathfrak{M}|, \bar{a} \in R^M \} \subseteq |\mathcal{D}|^n,$$

- for each $n$-ary function symbol $f$ of $\mathcal{L}$,

$$f^{\overline{\mathfrak{M}}} = \sum_{M\in|\mathfrak{M}|} f^M : |\mathcal{D}|^n = \sum_{M\in|\mathfrak{M}|} |M|^n \longrightarrow \sum_{M\in|\mathfrak{M}|} |M| = |\mathcal{D}|$$
$$(M, \bar{a}) \longmapsto (M, f^M(\bar{a})),$$

- for each constant $c$ of $\mathcal{L}^{\text{Hen}}$,

$$c^{\overline{\mathfrak{M}}} = \sum_{M\in|\mathfrak{M}|} c^M : |\mathfrak{M}| = |\mathcal{D}|^0 \longrightarrow |\mathcal{D}|$$
$$M \longmapsto (M, c^M).$$

Indeed, the interpretations can be extended to all sentences and terms.

**Definition 95.** We define an interpretation $[\![-]\!]^{\overline{\mathfrak{M}}}$ by setting, for each sentence $\varphi$ of $\mathcal{L}^{\text{Hen}}$,

$$[\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} := \sum_{M\in|\mathfrak{M}|} [\![\, \bar{x} \mid \varphi \,]\!]^M = \{ (M, \bar{a}) \in |\mathcal{D}|^n \mid \bar{a} \in [\![\, \bar{x} \mid \varphi \,]\!]^M \} \subseteq |\mathcal{D}|^n,$$

and, for each term $t$ of $\mathcal{L}^{\text{Hen}}$ in the context of variables $\bar{x}$,

$$[\![\, \bar{x} \mid t \,]\!]^{\overline{\mathfrak{M}}} := \sum_{M\in|\mathfrak{M}|} [\![\, \bar{x} \mid t \,]\!]^M : |\mathcal{D}|^n = \sum_{M\in|\mathfrak{M}|} |M|^n \longrightarrow \sum_{M\in|\mathfrak{M}|} |M| = |\mathcal{D}|$$
$$(M, \bar{a}) \longmapsto (M, [\![\, \bar{x} \mid t \,]\!]^M(\bar{a})).$$

Our goal then is to show that, with appropriate neighborhood functions $\mathcal{N}_{\mathfrak{M}}$ on $|\mathfrak{M}|$ and $\mathcal{N}_{\mathcal{D}}$ on $|\mathcal{D}|$ in hand, $\overline{\mathfrak{M}}$ and $(\overline{\mathfrak{M}}, [\![-]\!]^{\overline{\mathfrak{M}}})$ form a neighborhood-sheaf model and interpretation as required in Theorem 13. Once we have $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$, this roughly consists of the following eight claims.

(a) $(|\mathfrak{M}|, \mathcal{N}_{\mathfrak{M}})$ and $(|\mathcal{D}|, \mathcal{N}_{\mathcal{D}})$ are MC frames (which we will prove as Claim 5).

(b) $\pi$ is a surjection (Claim 1).

(c) $\pi$ is a local isomorphism from $(|\mathcal{D}|, \mathcal{N}_{\mathcal{D}})$ to $(|\mathfrak{M}|, \mathcal{N}_{\mathfrak{M}})$ (Claim 9).

(d) $f^{\overline{\mathfrak{M}}}$ and $c^{\overline{\mathfrak{M}}}$ are continuous (Claim 10).

(e) $[\![-]\!]^{\overline{\mathfrak{M}}}$ extends $R^{\overline{\mathfrak{M}}}$, $f^{\overline{\mathfrak{M}}}$, $c^{\overline{\mathfrak{M}}}$ (Claim 2).

(f) $(\overline{\mathfrak{M}}, [\![-]\!]^{\overline{\mathfrak{M}}})$ interprets first-order operations of $\mathcal{L}^{\text{Hen}}$ with suitable operations (Claim 3).

(g) $(\overline{\mathfrak{M}}, [\![-]\!]^{\overline{\mathfrak{M}}})$ interprets $\Box$ with interior operations of suitable types (Claim 11).

(h) For every sentences $\varphi, \psi$ of $\mathcal{L}$, we have the following (Corollary 14):

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}.$$

(b) is immediate.

**Claim 1.** $\pi$ is a surjection.

*Proof.* Because $|M| \neq \varnothing$ for all $M \in |\mathfrak{M}|$. □

(e) and (f) are immediate from Fact 60 because we obtained $[\![-]\!]^{\overline{\mathfrak{M}}}$ by bundling up $[\![-]\!]^{M}$. Let us number (e) and (f) by entering:

**Claim 2.** $(\overline{\mathfrak{M}}, [\![-]\!]^{\overline{\mathfrak{M}}})$ interprets first-order operations of $\mathcal{L}^{\mathrm{Hen}}$ with suitable operations.

**Claim 3.** $(\overline{\mathfrak{M}}, [\![-]\!]^{\overline{\mathfrak{M}}})$ interprets $\Box$ with interior operations of suitable types.

(h) also follows from Fact 60.

**Claim 4.** For every sentences $\varphi, \psi$ of $\mathcal{L}^{\mathrm{Hen}}$,

$$\mathbb{T}^{\mathrm{Hen}} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}.$$

*Proof.* By (iii) of Lemma 7 and Fact 60 for $\mathcal{L}^{\mathrm{Hen}}$,

$$\begin{aligned} \mathbb{T}^{\mathrm{Hen}} \text{ proves } \varphi \vdash \psi &\overset{\text{(iii)}}{\iff} [\![\, \bar{x} \mid \varphi \,]\!]^{M} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{M} \text{ for all } M \in |\mathfrak{M}| \\ &\iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}. \end{aligned}$$ □

**Corollary 14.** For every sentences $\varphi, \psi$ of $\mathcal{L}$,

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}.$$

*Proof.* By (ii) of Lemma 7 as well as Claim 4, we have

$$\begin{aligned} \mathbb{T} \text{ proves } \varphi \vdash \psi &\overset{\text{(ii)}}{\iff} \mathbb{T}^{\mathrm{Hen}} \text{ proves } \varphi \vdash \psi \\ &\iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}. \end{aligned}$$ □

Let us close this subsection by defining neighborhood functions $\mathcal{N}_{\mathfrak{M}}$, $\mathcal{N}_{\mathcal{D}}$ on $|\mathfrak{M}|$, $|\mathcal{D}|$ and then showing (a)—that $(|\mathfrak{M}|, \mathcal{N}_{\mathfrak{M}})$ and $(|\mathcal{D}|, \mathcal{N}_{\mathcal{D}})$ are MC. We will prove (c), (d), and (g) in Subsection VI.3.4, and this will complete our proof for Theorem 13.

The key idea we use to define suitable neighborhood functions on $|\mathfrak{M}|$ and $|\mathcal{D}|$ is to define them "logically", in the sense of using $[\![-]\!]^{\overline{\mathfrak{M}}}$ to give what may be called the "topologies of necessity". Recall that, in a neighborhood-sheaf interpretation $[\![-]\!]$ on a given sheaf $\pi : D \to X$, we have

$$[\![\, \bar{x} \mid \varphi \,]\!] \in \mathcal{N}_{D^n}(\bar{a}) \iff \bar{a} \in [\![\, \bar{x} \mid \Box\varphi \,]\!]$$

for every $\bar{a} \in D^n$; in other words, $\mathcal{N}_{D^n}(\bar{a})$ serves as the set of ($n$-ary) properties that the $n$-tuple $\bar{a}$ necessarily satisfies. We use this insight (in the other direction) to define "logical bases" $\mathcal{N}^\dagger_0$, $\mathcal{N}^\dagger_1$ on $|\mathfrak{M}|$, $|\mathcal{D}|$, and generate $\mathcal{N}_{\mathfrak{M}}$, $\mathcal{N}_{\mathcal{D}}$, in the following way.

**Definition 96.** When $\mathbb{T}^{\text{Hen}}$ is FOE, let $\mathcal{N}^\dagger_n$ for each $n$ be the neighborhood function on $|\mathcal{D}|^n$ such that

$$\mathcal{N}^\dagger_n(M, \bar{a}) = \{\, [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \mid (M, \bar{a}) \in [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} \,\}$$

for every $(M, \bar{a}) \in |\mathcal{D}|$.

$\mathcal{N}^\dagger_n$ is not in general monotone. But we obtain monotone neighborhood functions $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$ by generating them on the bases of $\mathcal{N}^\dagger_0$ and $\mathcal{N}^\dagger_1$.

**Definition 97.** $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$ are the monotone neighborhood functions generated by the bases $\mathcal{N}^\dagger_0$ and $\mathcal{N}^\dagger_1$. Writing this definition explicitly, we set

$$U \in \mathcal{N}_{\mathfrak{M}}(M) \iff [\![\varphi]\!]^{\overline{\mathfrak{M}}} \subseteq U \text{ and } M \in [\![\Box\varphi]\!]^{\overline{\mathfrak{M}}} \text{ for some sentence } \varphi \text{ of } \mathcal{L}^{\text{Hen}},$$
$$U \in \mathcal{N}_{\mathcal{D}}(M, a) \iff [\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq U \text{ and } (M, a) \in [\![\, x \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} \text{ for some sentence } \varphi \text{ of } \mathcal{L}^{\text{Hen}}$$

for every $M \in |\mathfrak{M}|$ and $(M, a) \in |\mathcal{D}|$.

Note that the axiom E of $\mathbb{T}$ implies

$$\frac{[\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} = [\![\, \bar{x} \mid \psi \,]\!]^{\overline{\mathfrak{M}}}}{[\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} = [\![\, \bar{x} \mid \Box\psi \,]\!]^{\overline{\mathfrak{M}}}}$$

by Claims 3 and 4, and therefore

$$[\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \in \mathcal{N}_n^{\dagger}(M, \bar{a}) \iff (M, \bar{a}) \in [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}}$$

for every sentence $\varphi$ of $\mathcal{L}^{\mathrm{Hen}}$; in other words, we have the first half of the following.

**Remark 28.** For the interior operation $\mathbf{int}_n^{\dagger}$ associated with $\mathcal{N}_n^{\dagger}$,[19]

$$\mathbf{int}_n^{\dagger}([\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}) = [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}},$$
$$\mathbf{int}_n^{\dagger}(A) = \varnothing \qquad \text{if } A \subseteq |\mathcal{D}|^n \text{ is not of the form } [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}.$$

*Proof.* The second half holds since, if $A \subseteq |\mathcal{D}|^n$ is not of the form $[\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}$, then $A \in \mathcal{N}_n^{\dagger}(M, \bar{a})$ for no $(M, \bar{a}) \in |\mathcal{D}|^n$. □

$\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$ are monotone by definition. We moreover have:

**Claim 5.** $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$ are MC if $\mathbb{T}$ is FOMC.

**Remark 29.** $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$ are topological if $\mathbb{T}$ is FOS4.

These follow from the following fact, by Remark 12 and Remark 27.

**Fact 72.** Suppose $\mathbb{T}$ is an FOE theory. Then, for each $n$, $\mathcal{N}_n^{\dagger}$ is C, N, T, 4, respectively, if $\mathbb{T}^{\mathrm{Hen}}$ has the axiom C, rule N, axiom T, axiom 4, respectively.[20]

[19] We must not confuse this with the interior operation $\mathbf{int}_{\mathcal{D}^n}$ associated with the monotone neighborhood function $\mathcal{N}_{\mathcal{D}^n}$ generated by $\mathcal{N}_n^{\dagger}$. It is less trivial (and indeed false unless $\mathbb{T}$ is FOM) that $[\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} = \mathbf{int}_{\mathcal{D}^n}([\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}})$, which we will prove for a FOM $\mathbb{T}$ as Claim 11.

[20] On the other hand, as is easy to see, $\mathcal{N}_n^{\dagger}$ is not necessarily M even if $\mathbb{T}^{\mathrm{Hen}}$ has the rule M. This is why we need to generate the monotone $\mathcal{N}_{\mathfrak{M}}$ and $\mathcal{N}_{\mathcal{D}}$.

*Proof.* Suppose $\mathbb{T}^{\mathrm{Hen}}$ has the axiom C. Then we have the equalities marked with $*$ below by Remark 28, those with $\dagger$ by Claim 3, and the inclusion with ! by C and Claim 4:

$$\begin{aligned}
\mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}) \cap \mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \psi \rrbracket^{\overline{\mathfrak{M}}}) &\overset{*}{=} \llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}} \cap \llbracket \bar{x} \mid \Box\psi \rrbracket^{\overline{\mathfrak{M}}} \\
&\overset{\dagger}{=} \llbracket \bar{x} \mid \Box\varphi \wedge \Box\psi \rrbracket^{\overline{\mathfrak{M}}} \\
&\overset{!}{\subseteq} \llbracket \bar{x} \mid \Box(\varphi \wedge \psi) \rrbracket^{\overline{\mathfrak{M}}} \overset{*}{=} \mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \wedge \psi \rrbracket^{\overline{\mathfrak{M}}}) \\
&\overset{\dagger}{=} \mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}} \cap \llbracket \bar{x} \mid \psi \rrbracket^{\overline{\mathfrak{M}}}).
\end{aligned}$$

This means that $\mathcal{N}_n^\dagger$ is closed under binary intersection, because

$$\mathbf{int}_n^\dagger(A) \cap \mathbf{int}_n^\dagger(B) \subseteq \mathbf{int}_n^\dagger(A \cap B)$$

is trivially the case if either $A$ or $B$ is not of the form $\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}$ (which implies by Remark 28 that $\mathbf{int}_n^\dagger(A) \cap \mathbf{int}_n^\dagger(B) = \varnothing$). For the rest of this proof we use Claims 3, 4 and Remark 28 in a similar manner, but we omit the reference to it.

Suppose $\mathbb{T}^{\mathrm{Hen}}$ has the rule N. Then $\mathcal{N}_n^\dagger$ is normal because N implies

$$|\mathcal{D}|^n = \llbracket \bar{x} \mid \Box\top \rrbracket^{\overline{\mathfrak{M}}} = \mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \top \rrbracket^{\overline{\mathfrak{M}}}) = \mathbf{int}_n^\dagger(|\mathcal{D}|^n).$$

Suppose $\mathbb{T}^{\mathrm{Hen}}$ has the axiom T. Then T implies

$$\mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}) = \llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}} \subseteq \llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}},$$

whereas $\mathbf{int}_n^\dagger(A) = \varnothing \subseteq A$ for $A \subseteq |\mathcal{D}|^n$ that is not of the form $\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}$. Therefore $\mathcal{N}_n^\dagger$ is reflexive. Suppose $\mathbb{T}^{\mathrm{Hen}}$ has the axiom 4.

$$\mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}) = \llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}} \subseteq \llbracket \bar{x} \mid \Box\Box\varphi \rrbracket^{\overline{\mathfrak{M}}} = \mathbf{int}_n^\dagger(\mathbf{int}_n^\dagger(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}})),$$

whereas $\mathbf{int}_n^\dagger(A) = \varnothing \subseteq \mathbf{int}_n^\dagger(\mathbf{int}_n^\dagger(A))$ for $A \subseteq |\mathcal{D}|^n$ that is not of the form $\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}$. Thus $\mathcal{N}_n^\dagger$ is closed under interior. ◻

### VI.3.3 Products and Logical Topology

In Subsections VI.3.1 and VI.3.2, we constructed sets $|\mathfrak{M}|$, $|\mathcal{D}|$ of models and individuals with a projection $\pi$, and equipped $|\mathfrak{M}|$ and $|\mathcal{D}|$, and moreover products $|\mathcal{D}|^n$ in general, with an interpretation $[\![-]\!]$; then we defined neighborhood functions on $|\mathfrak{M}|$ and $|\mathcal{D}|$ to obtain neighborhood frames $\mathfrak{M}$ and $\mathcal{D}$. In this subsection we discuss neighborhood frames on $|\mathcal{D}|^n$.

Recall that we defined neighborhood frames $\mathfrak{M}$ and $\mathcal{D}$ "logically", with what may be called the "topologies of necessity" of $[\![-]\!]^{\overline{\mathfrak{M}}}$. We should then note that, on the $n$-fold product $|\mathcal{D}|^n$ of $|\mathcal{D}|$ in **Sets**$/|X|$, that is, over $|X|$, we can think of two neighborhood functions:

- one is the "logical topology" given by $[\![-]\!]^{\overline{\mathfrak{M}}}$;
- the other is the topology of the $n$-fold product of $\mathcal{D}$ in **MCNb**$/X$.

In neighborhood-sheaf semantics, $n$-ary sentences are interpreted by the $n$-fold product in **MCNb**$/X$ of a given sheaf; that is, with respect to the second neighborhood function above. Therefore it is helpful, for the sake of Theorem 13, to show that these two neighborhood functions coincide.

Let us introduce a notation to distinguish the two neighborhood functions, though we will show them identical immediately afterwards.

**Definition 98.** For each $n$, we write $\mathcal{N}_{\mathcal{D}^n}$ for the monotone neighborhood function generated by $\mathcal{N}_n^{\dagger}$, and $(\mathcal{N}_{\mathcal{D}})^n$ for the $n$-fold fibered product of $\mathcal{N}_{\mathcal{D}}$. Written explicitly, they are the neighborhood functions on $|\mathcal{D}|^n$ such that, for every $(M, \bar{a}) \in |\mathcal{D}|^n$,

$$U \in \mathcal{N}_{\mathcal{D}^n}(M, \bar{a}) \iff [\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq U \text{ and } (M, \bar{a}) \in [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} \text{ for some sentence } \varphi \text{ of } \mathcal{L}^{\mathrm{Hen}},$$

$$U \in (\mathcal{N}_{\mathcal{D}})^n(M, \bar{a}) \iff U_1 \times_X \cdots \times_X U_n \subseteq U \text{ for some } U_1 \in \mathcal{N}_{\mathcal{D}}(M, a_1), \ldots, U_n \in \mathcal{N}_{\mathcal{D}}(M, a_n).$$

And the following is the only fact we prove in this subsection.

**Fact 73.** When $\mathbb{T}$ is FOMC, $\mathcal{N}_{\mathcal{D}^n} = (\mathcal{N}_{\mathcal{D}})^n$ for each $n$.

*Proof.* Fix $(M, \bar{a}) \in |\mathcal{D}|^n$. To show that $(\mathcal{N}_{\mathcal{D}})^n(M, \bar{a}) \subseteq \mathcal{N}_{\mathcal{D}^n}(M, \bar{a})$, suppose $U \in (\mathcal{N}_{\mathcal{D}})^n(M, \bar{a})$. This means that $U_1 \times_X \cdots \times_X U_n \subseteq U$ for some $U_1 \in \mathcal{N}_{\mathcal{D}}(M, a_1)$, …, $U_n \in \mathcal{N}_{\mathcal{D}}(M, a_n)$. For each $i$, then, there is some sentence $\varphi_i$ of $\mathcal{L}^{\mathrm{Hen}}$ such that $[\![\, x_i \mid \varphi_i \,]\!] \subseteq U_i$ and $(M, a_i) \in [\![\, x_i \mid \Box\varphi_i \,]\!]$. Because $\mathbb{T}$ is FOMC, $\mathbb{T}^{\mathrm{Hen}}$ is also FOMC by Remark 27 and proves

$$\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \vdash \Box(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

Therefore $(M, \bar{a}) \in [\![\, x_1, \dots, x_n \mid \Box(\varphi_1 \wedge \cdots \wedge \varphi_n) \,]\!]$, whereas

$$[\![\, x_1, \dots, x_n \mid \varphi_1 \wedge \cdots \wedge \varphi_n \,]\!] = [\![\, x_1 \mid \varphi_1 \,]\!] \times_X \cdots \times_X [\![\, x_n \mid \varphi_n \,]\!] \subseteq U_1 \times_X \cdots \times_X U_n \subseteq U.$$

Thus $U \in \mathcal{N}_{\mathcal{D}^n}(M, \bar{a})$.

On the other hand, suppose $U \in \mathcal{N}_{\mathcal{D}^n}(M, \bar{a})$. This means that there is a sentence $\varphi$ of $\mathcal{L}^{\text{Hen}}$ such that $[\![\, \bar{x} \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq U$ and $(M, \bar{a}) \in [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}}$. Note that, by (iv) of Lemma 7, $\mathcal{L}^{\text{Hen}}$ has constants $c_1$, $\dots$, $c_n$ such that ${c_i}^M = a_i$ for each $i$. Given such $\bar{c}$, for each $i$ let us write

$$\varphi_i \qquad \text{for} \qquad [\bar{c}/\bar{x}]\varphi \wedge x_i = c_i$$

and we have $(M, a_i) \in [\![\, x_i \mid [\bar{c}/\bar{x}]\Box\varphi \,]\!] \cap [\![\, x_i \mid x_i = c_i \,]\!] \subseteq [\![\, x_i \mid \Box\varphi_i \,]\!]$, because FOMC $\mathbb{T}^{\text{Hen}}$ proves

$$[\bar{c}/\bar{x}]\Box\varphi \wedge x_i = c_i \vdash \Box([\bar{c}/\bar{x}]\varphi \wedge x_i = c_i);$$

hence $[\![\, x_i \mid \varphi_i \,]\!] \in \mathcal{N}_{\mathcal{D}}(M, a_i)$. Moreover, since FOMC $\mathbb{T}^{\text{Hen}}$ proves

$$[\bar{c}/\bar{x}]\varphi \wedge x_1 = c_1 \wedge \cdots \wedge x_n = c_n \vdash \varphi,$$

we have

$$[\![\, x_1 \mid \varphi_1 \,]\!] \times_X \cdots \times_X [\![\, x_n \mid \varphi_n \,]\!] = [\![\, \bar{x} \mid [\bar{c}/\bar{x}]\varphi \wedge x_1 = c_1 \wedge \cdots \wedge x_n = c_n \,]\!] \subseteq [\![\, \bar{x} \mid \varphi \,]\!] \subseteq U.$$

Thus $U \in (\mathcal{N}_{\mathcal{D}})^n(M, \bar{a})$. Therefore $\mathcal{N}_{\mathcal{D}^n}(M, \bar{a}) = (\mathcal{N}_{\mathcal{D}})^n(M, \bar{a})$ for every $(M, \bar{a}) \in |\mathcal{D}|^n$. $\Box$

### VI.3.4 Completing the Completeness Proof

In this subsection, we finally complete our proof of Theorem 13.

**Claim 6.** The projection $\pi : |\mathcal{D}| \to |\mathfrak{M}|$ is an open continuous map from $(|\mathcal{D}|, \mathcal{N}_1^\dagger)$ to $(|\mathfrak{M}|, \mathcal{N}_0^\dagger)$.

*Proof.* Claim 3 and Remark 28 imply

$$\pi^{-1}[\mathbf{int}_0^\dagger([\![\varphi]\!]^{\overline{\mathfrak{M}}})] = \pi^{-1}[[\![\Box\varphi]\!]^{\overline{\mathfrak{M}}}] = [\![\, x \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} = \mathbf{int}_1^\dagger([\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}) = \mathbf{int}_1^\dagger(\pi^{-1}[[\![\varphi]\!]^{\overline{\mathfrak{M}}}]).$$

On the other hand, fix any $A \subseteq |\mathfrak{M}|$ that is not of the form $[\![\varphi]\!]^{\overline{\mathfrak{M}}}$. Then, if $\pi^{-1}[A] \subseteq \mathcal{D}$ were of the form $[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}$, the surjectiveness of $\pi$ (Claim 1) and Claim 3 would imply

$$A = \pi[\pi^{-1}[A]] = \pi[[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}] = [\![\exists x\,.\,\varphi]\!];$$

thus $\pi^{-1}[A]$ is not of the form $[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}$, either. Therefore Remark 28 implies

$$\pi^{-1}[\mathbf{int}_0^\dagger(A)] = \pi^{-1}[\varnothing] = \varnothing = \mathbf{int}_1^\dagger(\pi^{-1}[A]).$$

Thus $\pi$ is continuous and open from $(|\mathcal{D}|, \mathcal{N}_1^\dagger)$ to $(|\mathfrak{M}|, \mathcal{N}_0^\dagger)$. □

**Claim 7.** If $\mathbb{T}$ is FOM, then $\mathcal{N}_1^\dagger$ is closed under $\pi^* \circ \pi_!$ for the projection $\pi : |\mathcal{D}| \to |\mathfrak{M}|$.

*Proof.* Suppose $\mathbb{T}$ is FOM. Then $\mathbb{T}^{\mathrm{Hen}}$ is FOM by Remark 27 and so proves

$$\frac{\varphi \vdash \exists x\,.\,\varphi}{\Box\varphi \vdash \Box\exists x\,.\,\varphi}$$

Therefore Remark 28 implies

$$\begin{aligned}[\![\, x \mid \varphi \,]\!] \in \mathcal{N}_1^\dagger(M, a) &\implies (M, a) \in [\![\, x \mid \Box\varphi \,]\!] \subseteq [\![\, x \mid \Box\exists x\,.\,\varphi \,]\!] \\ &\implies \pi^{-1}[\pi[[\![\, x \mid \varphi \,]\!]]] = [\![\, x \mid \exists x\,.\,\varphi \,]\!] \in \mathcal{N}_1^\dagger(M, a).\end{aligned}$$

□

**Claim 8.** The diagonal map $\Delta : |\mathcal{D}| \to |\mathcal{D}|^2$ is a continuous map from $(|\mathcal{D}|, \mathcal{N}_1^\dagger)$ to $(|\mathcal{D}|^2, \mathcal{N}_2^\dagger)$.

*Proof.* Claim 3 and Remark 28, together with $[x/y]\Box\varphi = \Box[x/y]\varphi$, imply

$$\begin{aligned}\Delta^{-1}[\mathbf{int}_2^{\dagger}([\![\, x, y \mid \varphi \,]\!]^{\overline{\mathfrak{M}}})] &= \Delta^{-1}[[\![\, x, y \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}}] \\ &= [\![\, x \mid [x/y]\Box\varphi \,]\!]^{\overline{\mathfrak{M}}} \\ &= [\![\, x \mid \Box[x/y]\varphi \,]\!]^{\overline{\mathfrak{M}}} \\ &= \mathbf{int}_1^{\dagger}([\![\, x \mid [x/y]\varphi \,]\!]^{\overline{\mathfrak{M}}}) \\ &= \mathbf{int}_1^{\dagger}(\Delta^{-1}[[\![\, x, y \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}]),\end{aligned}$$

whereas Remark 28 implies the following for any $A \subseteq |\mathcal{D}|^2$ that is not of the form $[\![\, x, y \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}$:

$$\Delta^{-1}[\mathbf{int}_2^{\dagger}(A)] = \Delta^{-1}[\varnothing] = \varnothing \subseteq \mathbf{int}_1^{\dagger}(\Delta^{-1}[A]).$$

Thus $\Delta$ is continuous from $(|\mathcal{D}|, \mathcal{N}_1^{\dagger})$ to $(|\mathcal{D}|^2, \mathcal{N}_2^{\dagger})$. □

**Claim 9.** If $\mathbb{T}$ is FOM, then the projection $\pi : \mathcal{D} \to \mathfrak{M}$ is a local isomorphism (with the diagonal map $\Delta : \mathcal{D} \to \mathcal{D}^2$).

*Proof.* By Theorem 8, it is enough to show that $\pi$ is continuous and open, and that $\Delta$ is open. To show $\pi$ continuous, it is enough by Remark 8 to show $\pi$ is continuous from $(|\mathcal{D}|, \mathcal{N}_1^{\dagger})$ to $(|\mathfrak{M}|, \mathcal{N}_0^{\dagger})$; but Claim 3 and Remark 28 imply

$$\pi^{-1}[\mathbf{int}_0^{\dagger}([\![\varphi]\!]^{\overline{\mathfrak{M}}})] = \pi^{-1}[[\![\Box\varphi]\!]^{\overline{\mathfrak{M}}}] = [\![\, x \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}} = \mathbf{int}_1^{\dagger}([\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}) = \mathbf{int}_1^{\dagger}(\pi^{-1}[[\![\varphi]\!]^{\overline{\mathfrak{M}}}]),$$

whereas Remark 28 implies

$$\pi^{-1}[\mathbf{int}_0^{\dagger}(A)] = \pi^{-1}[\varnothing] = \varnothing \subseteq \mathbf{int}_1^{\dagger}(\pi^{-1}[A])$$

for $A \subseteq \mathfrak{M}$ that is not of the form $[\![\varphi]\!]^{\overline{\mathfrak{M}}}$. Thus $\pi$ is continuous.

To show $\pi$ open, suppose $\pi^{-1}[U] \in \mathcal{N}^{\mathcal{D}}(M, a)$ for $U \subseteq \mathfrak{M}$; this means that $[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq \pi^{-1}[U]$ and $(M, a) \in [\![\, x \mid \Box\varphi \,]\!]^{\overline{\mathfrak{M}}}$ for a sentence $\varphi$ of $\mathcal{L}^{\mathrm{Hen}}$. The former entails

$$\frac{[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}} \subseteq \pi^{-1}[U]}{[\![\exists x\,.\,\varphi]\!]^{\overline{\mathfrak{M}}} = \pi[[\![\, x \mid \varphi \,]\!]^{\overline{\mathfrak{M}}}] \subseteq U}$$

by the adjunction $\pi_! \dashv \pi^*$, whereas the latter entails

$$\pi(M,a) \in \pi[[\![ x \mid \Box\varphi ]\!]^{\overline{\mathfrak{M}}}] = [\![\exists x \Box\varphi]\!]^{\overline{\mathfrak{M}}} \subseteq [\![\Box\exists x\,.\,\varphi]\!]^{\overline{\mathfrak{M}}}$$

since FOM $\mathbb{T}^{\text{Hen}}$ proves $\exists x \Box\varphi \vdash \Box\exists x\,.\,\varphi$; therefore $U \in \mathcal{N}_{\mathfrak{M}}(\pi(M,a))$. Thus $\pi$ is open.

To show $\Delta$ open, fixing any $(M,a) \in \mathcal{D}$, suppose $\Delta^{-1}[U] \in \mathcal{N}_{\mathcal{D}}(M,a)$ for $U \subseteq \mathcal{D}^2$; this means that there is a sentence $\varphi$ of $\mathcal{L}^{\text{Hen}}$ such that $[\![ x \mid \varphi ]\!]^{\overline{\mathfrak{M}}} \subseteq \Delta^{-1}[U]$ and $(M,a) \in [\![ x \mid \Box\varphi ]\!]^{\overline{\mathfrak{M}}}$. Then, by $\Delta_! \dashv \Delta^*$, the former implies

$$\frac{[\![ x \mid \varphi ]\!]^{\overline{\mathfrak{M}}} \subseteq \Delta^{-1}[U]}{[\![ x,y \mid \varphi \wedge x = y ]\!]^{\overline{\mathfrak{M}}} = \Delta[[\![ x \mid \varphi ]\!]^{\overline{\mathfrak{M}}}] \subseteq U}.$$

On the other hand, because FOE $\mathbb{T}^{\text{Hen}}$ proves

$$\cfrac{\cfrac{\varphi \vdash \varphi \wedge x = x \qquad \varphi \wedge x = x \vdash \varphi}{\Box\varphi \vdash \Box(\varphi \wedge x = x)}}{\Box\varphi \wedge x = y \vdash \Box(\varphi \wedge x = y)},$$

$(M,a) \in [\![ x \mid \Box\varphi ]\!]^{\overline{\mathfrak{M}}}$ entails

$$\Delta(M,a) \in \Delta[[\![ x \mid \Box\varphi ]\!]^{\overline{\mathfrak{M}}}] = [\![ x,y \mid \Box\varphi \wedge x = y ]\!]^{\overline{\mathfrak{M}}} \subseteq [\![ x,y \mid \Box(\varphi \wedge x = y) ]\!]^{\overline{\mathfrak{M}}}.$$

Therefore $U \in \mathcal{N}_{\mathcal{D}^2}(\Delta(M,a))$. Thus $\Delta : \mathcal{D} \to \mathcal{D}^2$ is open. □

Combining Claims 1, 5, and 9, we have

**Corollary 15.** If $\mathbb{T}$ is FOMC, then $(\mathcal{D}, \pi)$ is a surjective neighborhood sheaf over an MC neighborhood frame $\mathfrak{M}$.

We then show that the structure $\overline{\mathfrak{M}}$ defined on this sheaf is a neighborhood-sheaf model.

**Claim 10.** For any $n$-ary function symbol $f$ (or any constant $c$ as a 0-ary function symbol) of $\mathcal{L}^{\text{Hen}}$, $f^{\overline{\mathfrak{M}}} : \mathcal{D}^n \to \mathcal{D}$ is a continuous map over $\mathfrak{M}$.

*Proof.* $f^{\overline{\mathfrak{M}}}$ is over $\mathfrak{M}$ by definition. To show $f^{\overline{\mathfrak{M}}} : \mathcal{D}^n \to \mathcal{D}$ continuous, it is enough by Remark 8 to show it continuous from $(|\mathcal{D}|^n, \mathcal{N}_n^\dagger)$ to $(|\mathcal{D}|, \mathcal{N}_1^\dagger)$; but Claim 3 and Remark 28 imply

$$\begin{aligned}
(f^{\overline{\mathfrak{M}}})^{-1}[\mathbf{int}_1^\dagger(\llbracket z \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}})] &= (f^{\overline{\mathfrak{M}}})^{-1}[\llbracket z \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}}] \\
&= \llbracket \bar{y} \mid [f\bar{y}/z]\Box\varphi \rrbracket^{\overline{\mathfrak{M}}} \\
&\overset{!}{=} \llbracket \bar{y} \mid \Box[f\bar{y}/z]\varphi \rrbracket^{\overline{\mathfrak{M}}} \\
&= \mathbf{int}_m^\dagger(\llbracket \bar{y} \mid [f\bar{y}/z]\varphi \rrbracket^{\overline{\mathfrak{M}}}) \\
&= \mathbf{int}_m^\dagger((f^{\overline{\mathfrak{M}}})^{-1}[\llbracket z \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}])
\end{aligned}$$

(where the equality marked with ! is by $[f\bar{y}/z]\Box\varphi = \Box[f\bar{y}/z]\varphi$, due to the syntax of $\mathcal{L}^{\mathrm{Hen}}$), whereas Remark 28 implies

$$(f^{\overline{\mathfrak{M}}})^{-1}[\mathbf{int}_1^\dagger(A)] = (f^{\overline{\mathfrak{M}}})^{-1}[\varnothing] = \varnothing \subseteq \mathbf{int}_m^\dagger((f^{\overline{\mathfrak{M}}})^{-1}[A])$$

for $A \subseteq \mathfrak{M}$ that is not of the form $\llbracket\varphi\rrbracket^{\overline{\mathfrak{M}}}$. Thus $f^{\overline{\mathfrak{M}}}$ is continuous. □

Combined with Corollary 15, Claim 10 means:

**Corollary 16.** $\overline{\mathfrak{M}}$ is a neighborhood-sheaf model for $\mathcal{L}^{\mathrm{Hen}}$ and hence, when restricted to $\mathcal{L}$, is a neighborhood-sheaf model for $\mathcal{L}$.

As a last proof, we show that the interpretation $(\overline{\mathfrak{M}}, \llbracket - \rrbracket^{\overline{\mathfrak{M}}})$ defined over the neighborhood-sheaf model $\overline{\mathfrak{M}}$ is a neighborhood-sheaf interpretation.

**Claim 11.** If $\mathbb{T}$ is FOM, then $\llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}} = \mathbf{int}_{\mathcal{D}^n}(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}})$ for the interior operation $\mathbf{int}_{\mathcal{D}^n}$ associated with $\mathcal{N}_{\mathcal{D}^n}$.

*Proof.* By Claim 4, the rule M of $\mathbb{T}^{\mathrm{Hen}}$ means that

$$\frac{\llbracket \bar{x} \mid \psi \rrbracket^{\overline{\mathfrak{M}}} \subseteq \llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}}{\llbracket \bar{x} \mid \Box\psi \rrbracket^{\overline{\mathfrak{M}}} \subseteq \llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}}}.$$

This entails the last equality below, while Remark 7 entails the first since $\mathcal{D}^n$ is generated by $\mathcal{N}_n^\dagger$, and Remark 28 entails the second:

$$\mathbf{int}_{\mathcal{D}^n}(\llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}) = \bigcup_{U \subseteq \llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}} \mathbf{int}_{\mathcal{D}^n}(U) = \bigcup_{\llbracket \bar{x} \mid \psi \rrbracket^{\overline{\mathfrak{M}}} \subseteq \llbracket \bar{x} \mid \varphi \rrbracket^{\overline{\mathfrak{M}}}} \llbracket \bar{x} \mid \Box\psi \rrbracket^{\overline{\mathfrak{M}}} = \llbracket \bar{x} \mid \Box\varphi \rrbracket^{\overline{\mathfrak{M}}}.$$ □

The combination of Corollary 16 and Claims 2, 3, 11 means:

**Corollary 17.** $(\overline{\mathfrak{M}}, [\![-]\!])$ is a neighborhood-sheaf interpretation for $\mathcal{L}^{\mathrm{Hen}}$ and hence, when restricted to $\mathcal{L}$, is a neighborhood-sheaf interpretation for $\mathcal{L}$.

Finally, Corollary 14 means that $(\overline{\mathfrak{M}}, [\![-]\!])$ is a neighborhood-sheaf interpretation as required in Theorem 13. This completes our completeness proof for **FOMC** with respect to neighborhood-sheaf semantics. Moreover, the combination of Theorem 13 with Remark 29 proves Theorem 7, the completeness of **FOS4** with respect to topological-sheaf semantics.

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