LATTICE-FREE SIMPLEXES IN DIMENSION 4

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We use a numerical approach to discover lattice free simplexes in dimension 4 with width at least 3. We follow the methodologies of Mori, Morrison, and Morrison [3] and use a theoretical result proven by Barille, Bernardi, Borisov, and Kantor [1] to conjecture a complete list of empty-lattice simplexes in dimension 4. Similar work was done by Haase and Ziegler, however, using a different approach we were able to both produce more evidence for the conjecture and provide an explicit list of distinct empty-lattice simplexes in dimension 4.

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1.0 INTRODUCTION

Recently, convex lattice-free polytopes have received much attention. Here we summarize results in dimensions less than 4 and enumerate what we believe to be a complete list of empty lattice simplexes of exceptional width. We support this claim primarily with computational evidence. For the sake of completeness, we begin with some basic definitions and state, with proof, some important results about cones and polyhedra.

1.1 CONES AND POLYHEDRA

We begin by stating some basic definitions and fundamental results relating cones, polyhedra and polytopes.

Definition 1. *Given an inner product space V and a vector $c \in V$ then the hyperplane, P_c, is the orthogonal complement of $< c >$, i.e. $P_c = \{v \mid (v, c) = 0\}$.*

As a linear subspace, a hyperplane must pass through the origin. This is not the case for an affine hyperplane,

Definition 2. *Given an inner product space V, a vector $c \in V$, and a scalar a, we define an affine hyperplane, $P_{c,a}$, by $P_{c,a} = \{v \mid (v, c) = a\}$.*

Definition 3. *Given a vector space V, $l \in V^{\star}$, and a scalar c, we define*

1. *a linear half-space to be $\{v \in V \mid l(v) \leq 0\}$*
2. *an affine half-space to be $\{v \in V \mid l(v) \leq c\}$.*

Definition 4.

1. *A nonempty set C of points in Euclidean space is said to be a convex cone if $\alpha x + \beta y \in C$ for all $x, y \in C$ and $\alpha, \beta \geq 0$.*
2. *A convex cone is said to polyhedral if C is a finite intersection of linear half-spaces.*
3. *A cone is said to be generated by the vectors $\{x_1, x_2, \ldots, x_m\}$ if it has the form $\{\alpha_1 x_1 + \ldots + \alpha_m x_m \mid \alpha_1, \ldots, \alpha_m \geq 0\}$. This is the smallest convex cone containing $\{x_1, x_2, \ldots, x_m\}$ and we denote this by cone $\{x_1, x_2, \ldots, x_m\}$.*

The definition for a cone being polyhedral is equivalent to the following: A cone C is polyhedral if there exists a matrix A such that $C = \{x \mid Ax \leq 0\}$, where $Ax \leq 0$ means $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq 0$ for all i such that $1 \leq i \leq n$.

Definition 5. *Let X be a set of vectors. Then we define the linear, affine, and convex hull as follows:*

1. *lin.hull*$(X) = \{\alpha_1 x_1 + \ldots + \alpha_n x_n \mid x_i \in X, \alpha_i \in \mathbb{R}\}$
2. *aff.hull*$(X) = \{\alpha_1 x_1 + \ldots + \alpha_n x_n \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_1 + \ldots + \alpha_n = 1\}$
3. *conv.hull*$(X) = \{\alpha_1 x_1 + \ldots + \alpha_n x_n \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_1 + \ldots + \alpha_n = 1, \alpha_i \geq 0\}$

For a set of vectors X, the convex hull of X is the smallest convex set containing X.

Definition 6. *A simplex Δ in n-dimensional space is the convex hull of $n+1$ points, called the vertices of the simplex.*

1.2 INTRODUCING POLYTOPES

Definition 7. *A convex integral polytope, or simply integral polytope, is the convex hull of finitely many points in $\mathbb{Z}^n$. More generally, given a lattice, L, a lattice-polytope is simply a polytope whose vertex set is contained in L.*

Theorem 1 (Fundamental Theorem of Linear Inequalities). *Let $x_1, x_2, \ldots, x_m, y$ be vectors in an n-dimensional Euclidean space. Then precisely one of the following occurs:*

1. *$\alpha_i \geq 0$ and $y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m$ or*
2. *there exists a hyperplane P_c, containing $t-1$ linearly independent elements from $\{x_1, x_2, \ldots, x_m\}$, such that $(c, y) < 0$ and $(c, x_i) \geq 0$ for all i, where $t = rank\,(x_1, x_2, \ldots, x_m, y)$.*

Proof. First, we show the exclusivity of 1 and 2. Indeed, if both 1 and 2 hold, then $y = \alpha_1 x_1 + \ldots + \alpha_m x_m$ with $\alpha_i \geq 0$. We then obtain the following contradiction,

$$0 > (c, b) = \left(c, \sum_{i=1}^{m} \alpha_i x_i\right) = \sum_{i=1}^{m} \alpha_i (c, x_i) \geq 0$$

We now claim that without loss of generality we may assume that $span\,\{x_1, x_2, \ldots x_m\} = \mathbb{R}^n$. If not, we have two cases. First, consider the case with $y \in span\,\{x_1, x_2, \ldots x_m\}$. If so, we may simply restrict all of our arguments to the subspace generated by $span\,\{x_1, x_2, \ldots x_m\}$. However, if $y \notin span\,\{x_1, x_2, \ldots x_m\}$ then 1 cannot happen. We must show that 2 occurs. With that goal in mind, let π be the orthogonal projection onto $span\,\{x_1, x_2, \ldots x_m\}$. Further, let $c = \pi(y) - y$. Then $c \in span\,\{x_1, x_2, \ldots x_m\}^{\perp}$ and thus $(c, a_i) = 0$ for all i. Finally, we observe that,

$$\begin{aligned}
(c, y) &= (c, y) - (c, \pi(y)) + (c, \pi(y)) \\
&= (c, y - \pi(y)) + (c, \pi(y)) \\
&= -\|c\|^2 \\
&< 0
\end{aligned}$$

which proves the claim.

We now proceed algorithmically. Let $D = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ be a linearly independent set chosen from $\{x_1, x_2, \ldots, x_m\}$.

1. Write $y = \alpha_{i_1} x_{i_1} + \ldots + \alpha_{i_n} x_{i_n}$. If $\alpha_{i_k} \geq 0$ for all $1 \leq k \leq n$ we have arrived in situation 1 and we are done.
2. Otherwise, choose the smallest $h \in \{i_1, i_2, \ldots, i_n\}$ such that $\alpha_h < 0$. Then we can choose c such that hyperplane $P_c = span\,(D \backslash \{x_h\})$.
3. Normalize c such that $(c, x_h) = 1$. Hence, $(c, y) = \alpha_h < 0$.

4. If $(c, x_i) \geq 0$ for all $1 \leq i \leq m$ we have come to situation 2 and are done. If not proceed to the next step.
5. Choose the smallest s such that $(c, x_s) < 0$. Then reassign $D := (D \backslash \{x_h\}) \cup \{x_s\}$ and proceed to step 1.

Thus it remains only to prove that this process terminates. Let D_k be D as it is in the kth iteration. Then, if the process does not terminate there exists k, l with $k < l$ such that $D_k = D_l$ as there are only finitely many arrangements of the x_i's. Let r denote the largest index of an element to be removed from D during the iterations $k, k+1, \ldots, l-1$. Call this iteration p. Further, as $D_k = D_l$, x_r must be added back into D at some iteration q such that $k \leq q < l$. This gives

$$D_p \cap \{x_r + 1, \ldots, x_m\} = D_q \cap \{x_r + 1, \ldots, x_m\}. \tag{1.1}$$

Let $D_p = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$, $y = \alpha_{i_1} x_{i_1} + \ldots + \alpha_{i_n} x_{i_n}$, and let c' be the "c" vector found in the second step of our algorithm during iteration q. We can then obtain the following contradiction,

$$0 > (c', y) = (c', \alpha_{i_1} x_{i_1} + \ldots + \alpha_{i_n} x_{i_n}) = \sum_{k=1}^{n} \alpha_{i_k} (c', x_{i_k}) > 0, \tag{1.2}$$

which we now justify. The first inequality is clear from the second step of our algorithm and the second inequality can be seen as follows. By the second step of our algorithm, r is the smallest index with $\alpha_r < 0$. Thus, if $i_k < r$ then $\alpha_{i_k} \geq 0$ and $(c', x_{i_k}) \geq 0$. Secondly, by the fourth step of our algorithm, if $i_k = r$ then $\alpha_{i_k} < 0$ and $(c', x_{i_k}) < 0$. Finally, by (1.1) and the second step of our algorithm, if $i_k > r$ then $(c', x_{i_k}) = 0$ and the inequality follows. $\square$

Geometrically, this means that either y is in the cone generated by the x_i or that there is a hyperplane P_c that y and the cone generated by the x_i. The following corollaries are geometric consequences of the fundamental theorem. Further, we make rigorous the connection between polyhedra and polytopes.

Corollary 1 (Farkas, Minkowski, Weyl). *A convex cone is finitely generated if and only if it is polyhedral.*

Proof. Let $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$. We will show that *cone* $\{x_1, x_2, \ldots, x_m\}$ is polyhedral. Without loss of generality, we may assume that $x_1, x_2, \ldots, x_m$ span $\mathbb{R}^m$. Now consider all the linear half-spaces H of the form $\{x \mid (x, c) \leq 0\}$ such that $x_1, \ldots, x_m \in H$ and P_c is spanned by $n-1$ linearly independent vectors selected from $x_1, \ldots x_m$. Then, by the Fundamental Theorem of Linear Inequalities, the cone generated by $x_1, \ldots, x_m$ is the intersection of these half-spaces. Finally as there are only finitely many such half-spaces, our cone must be polyhedral.

Conversely, let C be a polyhedral cone, say $C = \{x \mid Ax \leq 0\}$ with $A = (a_1 \mid \ldots \mid a_m)$. We now consider the cone generated by the columns of the matrix A. By the first implication of our proof, we know that *cone* $\{a_1, \ldots, a_m\}$ is polyhedral. So,

$$\textit{cone}\,\{a_1, \ldots, a_m\} = \{x \mid Bx \leq 0\} \tag{1.3}$$

with $B = (b_1 \mid \ldots \mid b_t)$. We now claim that $C = \textit{cone}\,\{b_1, \ldots, b_t\}$.

Indeed, as $(b_j, a_i) \leq 0$, $b_1, \ldots, b_t \in C$ and thus *cone* $\{b_1, \ldots b_t\} \subseteq C$. Conversely, assume there exists a $y \in C$ such that $y \notin \textit{cone}\,\{b_1, \ldots, b_t\}$. As *cone* $\{b_1, \ldots, b_t\}$ is polyhedral, there exists some vector w such that $(w, b_j) \leq 0$ for all $j = 1, \ldots, t$ and $(w, y) > 0$. So, by (1.3), $w \in \textit{cone}\,\{a_1, \ldots, a_m\}$, and hence $(w, x) \leq 0$ for all x in C. However, this contradicts that $y \in C$ and $(w, y) > 0$. Thus $C = \textit{cone}\,\{b_1, \ldots, b_t\}$ and is finitely generated. □

Corollary 2 (Motzkin). *A set P of vectors in Euclidean space is a polyhedron if and only if $P = Q + C$ for some polytope Q and some polyhedral cone C.*

Proof. Let $P = \{x | Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. Now define a polyhedral cone Ω as follows

$$\Omega = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in \mathbb{R}^n, \lambda \in \mathbb{R}_+, Ax - \lambda b \leq 0 \right\}. \tag{1.4}$$

By Corollary 1, Ω is finitely generated, say by $\left\{ \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix} \right\}$. Without loss of generality we may assume that λ_i is either 0 or 1. Now, we define $Q = \textit{conv.hull}\,\{x_i | \lambda_i = 1\}$

and $C = cone\,\{x_i \mid \lambda_i = 0\}$. Now one clearly observes that $x \in P$ if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \Omega$ if and only $\begin{pmatrix} x \\ 1 \end{pmatrix} \in cone \left\{ \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix} \right\}$. It follows from this that $P = Q + C$.

Conversely, we now assume that $P = Q + C$ for some polytope Q and polyhedral cone C. Now let $Q = conv.hull\,\{x_1, x_2, \ldots, x_m\}$ and $C = cone\,\{y_1, y_2, \ldots, y_t\}$. We now observe that,

$$x_0 \in P \Leftrightarrow \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \in cone \left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} y_t \\ 0 \end{pmatrix} \right\}. \tag{1.5}$$

Then, by Corollary 1, we see that the cone in equation 1.4 is equal to $\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid Ax + \lambda x \leq 0 \right\}$ for some matrix A and vector b. Hence $x_0 \in P$ if and only if $Ax_0 \leq -b$, and therefore P is a polyhedron.

□

Corollary 3 (Minkowski, Steinitz, Weyl)**.** *A set P is a polytope if and only if P is a bounded polyhedron.*

1.3 FACES AND FACETS

Let P be a nonempty polyhedron defined by $Ax \leq b$, c a nonzero vector, and $\delta = max\,\{(c, x) \mid Ax \leq b\}$.

Definition 8. *The affine hyperplane $P_{c,\delta} = \{x \mid (c, x) = \delta\}$ is called a supporting hyperplane of P.*

We now state several equivalent definitions.

Definition 9. *A subset F of P is said to be a face of P if and only if one of the following holds:*

1. *$F = P$ or if F is the intersection of P with a supporting hyperplane of P.*

2. *There exists a vector c for which F is the set of vectors attaining* $\max\{(c, x) \mid x \in P\}$ *provided this maximum is finite.*
3. *F is nonempty and $F = \{x \in P \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

It then follows from these definitions that:

1. P has only finitely many faces.
2. Each face is itself a nonempty polyhedron.
3. If F is a face of P and $F' \subseteq F$, then F' is a face of P if and only if F' is a face of F.

Definition 10. *A facet of P is a maximal face distinct from P. Here maximal means maximal with respect to inclusion.*

2.0 PRELIMINARIES

In this paper we search for lattice-free polytopes in dimension 4 of exceptional width. By exceptional width, we mean width at least 3. In dimension 3, the situation is far less trivial; however, it was still proven in [6] and [8] that all lattice-free polytopes in dimension 3 have width one. The case of dimension 4 is more interesting. It was proven in [1] that all but finitely many lattice-free polytopes in dimension 4 have width bounded above by two. We provide a list of these exceptional cases and provide numerical evidence for completeness. From now on all polytopes will be convex.

Definition 11. *Let L be a lattice in $\mathbb{R}^n$, a simplex, Δ, is said to be lattice-free with respect to L if it intersects L on the vertices of Δ and no other points. Empty-lattice simplex is frequently used interchangably with lattice-free.*

We are now in a position to introduce the concept of width of a polytope.

Definition 12. *The width of a polytope P is defined by*

$$width\,(P) = \min_{l \in (\mathbb{Z}^n)^\star \setminus \{0\}} \max_{x,y \in P} \langle l, x - y \rangle \,. \tag{2.1}$$

2.1 KNOWN RESULTS ON LATTICE-FREE POLYTOPES

The next two theorems provide bounds on the width of empty-lattice simplexes. It was proven in 1998 by Banaszczyk, Litvak, Pajor, and Szarek that

Theorem 2. *In dimension d, the maximal width of empty-lattice simplexes is less than or equal to $dM \log(d)$ for some constant M [9].*

Further, in 2001, Sebő and Bárány found a lower bound for the maximal width of empty lattice simplexes in a given dimension by constructing an explicit example,

Theorem 3. *If d is even, then the maximal width of d-dimensional empty lattice simplexes is bounded below by $d-1$. Further, for any $d \geq 1$, the maximal width of d-dimension empty lattice simplexes is at least $d-2$ [10].*

Before this result, we only had a non-constructive linear lower bound given by Kantor in 1999.

Theorem 4. *Given any $\alpha > 0$ with $\alpha < \frac{1}{e}$ and d large enough there exists a lattice-free simplex, σ, of dimension d with $\text{width}(\sigma) > \alpha d$ [5].*

2.2 SOME KNOWN RESULTS ON LATTICE-FREE POLYTOPES IN SMALL DIMENSION

In dimension two it can be trivially shown that all empty-lattice simplexes have width at most one. We have an analogous result in dimension 3 known as White's Theorem [6, 8].

Theorem 5 (White)**.** *Empty-lattice simplexes in dimension 3 have width at most one.*

Due to the strong connection between terminal quotient singularities and lattice-free simplexes, we will often need to refer to them while stating the known results in this area. However, we will not need to refer to them to obtain or state our results. As terminal quotient singularities have received a great deal of attention in algebraic geometry, they provided a lot of the original motivation for our topic. It turns out that $\mathbb{Q}$-factorial terminal quotient singularities can be put in one-to-one correspondence with the equivalence classes of lattice-free simplexes. This will be why we can later borrow the methods of Mori, Morrison, and Morrison. White's Theorem was also proven using Bernoulli functions in 1984 by Morrison and Stevens [6]. Because Morrison and Stevens found their motivation in algebraic geometry, they were actually studying terminal quotient singularities in dimension 3. This is why White's Theorem is sometimes called the Terminal Lemma.

Later, in 1988, Mori, Morrison, and Morrison, tried to replicate the work of Morrison and Stevens and produce a "Terminal Lemma" in dimension 4. They succeeded in producing a conjectural classification of 4 dimensional terminal quotient singularities using computer based calculations. It is these methods that we will use to produce our conjectures, using the increased computational power of modern computers to extend their search.

Normally, such as in the work of Haase and Ziegler[4], we would search through simplexes generated by the 4 standard basis vectors and some integer point in $\mathbb{Z}^4$ and find which ones intersect $\mathbb{Z}^4$ only on the vertices of our simplex. Instead, Mori, Morrison, and Morrison consider the lattice $\mathbb{Z}^4$ with a rational point α adjoined where α has the form $\frac{1}{p}(a, b, c, d)$ for p prime and $a, b, c, d \in \mathbb{Z}$. We then only need to consider the standard simplex and see if it intersects $\mathbb{Z}^4 + \alpha\mathbb{Z}$ only at it's vertex set. If this occurs, then the induced simplex is lattice-free. To recover the simplex from our enlarged lattice, we need the affine transformation that replaces the fourth coordinate by 1 minus the sum of the first three coordinates. Then our simplex is generated by the four standard basis vectors and the point $(-a, -b, -c, n + a + b + c + 1)$.

Mori, Morrison, and Morrison found it convenient to add a fifth coordinate e such that $a + b + c + d + e \equiv 0 \pmod{p}$. Further, if $Q = (a, b, c, d, e)$ is our quintuple, they define $M_Q = max\{|a|, |b|, |c|, |d|, |e|\}$. They were able to prove the following result

Theorem 6. *Let Q be a quintuple of integers summing to zero, and p be a prime number. Suppose that either*

1. $Q = (\alpha, -\alpha, \beta, \gamma, -\beta - \gamma)$ *with* $0 < |\alpha|, |\beta|, |\gamma| < \frac{p}{2}$, *and* $\beta + \gamma \neq 0$, *or*
2. $Q = (\alpha, -2\alpha, \beta, -2\beta, \alpha + \beta)$ *with* $0 < |\alpha|, |\beta| < \frac{p}{2}$, *and* $\alpha + \beta \neq 0$, *or*
3. *Q is one of the 29 quintuples listed in Table 1.9 in [3] and $p > M_Q$.*

Then Q is p-terminal [3].

We reproduce Table 1.9 below.

This theorem along with extensive computer calculations led them to the following conjecture.

Conjecture 1 (Four-Dimensional Terminal Lemma)**.** *Fix $p \geq 421$. Up to the actions of $(\mathbb{Z}/p\mathbb{Z})^\star$ and S^4, each isolated four-dimensional terminal $\mathbb{Z}/p\mathbb{Z}$-quotient singularity of index*

Table 1: Table 1.9 from Mori, Morrison, and Morrison

Stable Quintuple	Linear Relations	Stable Quintuple	Linear Relations
$(9, 1, -2, -3, -5)$	02100, 11002, 20122	$(9, 7, 1, -3, -14)$	02001, 20221
$(9, 2, -1, -4, -6)$	01200, 02010, 20212	$(15, 7, -3, -5, -14)$	02001, 20221
$(12, 3, -4, -5, -6)$	02001, 10002, 12220	$(8, 5, 3, -1, -15)$	02211, 20011
$(12, 2, -3, -4, -7)$	02010, 11002, 20212	$(10, 6, 1, -2, -15)$	00210, 22012
$(9, 4, -2, -3, -8)$	01200, 02001, 20221	$(12, 5, 2, -4, -15)$	00210, 22012
$(12, 1, -2, -3, -8)$	02100, 12021, 20122	$(9, 6, 4, -1, -18)$	02221, 20001
$(12, 3, -1, -6, -8)$	02010, 10020, 12202	$(9, 6, 5, -2, -18)$	02221, 20001
$(15, 4, -5, -6, -8)$	02001, 20221	$(12, 9, 1, -4, -18)$	02001, 20221
$(12, 2, -1, -4, -9)$	01200, 02010, 20212	$(10, 7, 4, -1, -20)$	02221, 20001
$(10, 6, -2, -5, -9)$	02120, 10020, 12202	$(10, 8, 3, -1, -20)$	02221, 20001
$(15, 1, -2, -5, -9)$	02100, 20122	$(10, 9, 4, -3, -20)$	02221, 20001
$(12, 5, -3, -4, -10)$	02001, 02210, 20221	$(12, 10, 1, -3, -20)$	02001, 20221
$(15, 2, -3, -4 - 10)$	02010, 20212	$(12, 8, 5, -1, -24)$	02221, 20001
$(6, 4, 3, -1, -12)$	02221, 20001	$(15, 10, 6, -1, -30)$	02221, 20001
$(7, 5, 3, -1, -14)$	02221, 20001		

p is associated with one of the p-terminal quintuples given in the above theorem.[3]

This, a result from Sankaran in 1999 [11], and a general result proved by Borisov in 1999 [2], allowed Barrile, Bernardi, Borisov, and Kantor to give an almost complete classification of empty-lattice simplexes in dimension 4. In particular, they proved the following two results

Theorem 7. *Every $\mathbb{Q}$-factorial toric terminal 4-dimensional singularity is a cyclic quotient [1].*

Theorem 8. *Up to a finite number of exceptions, every empty lattice simplex of dimension 4 has width one or two [1].*

What we will provide in this paper, is a complete conjectural classification of these finite exceptions.

Remark 1. *The maximal width of an empty-lattice simplex in dimension 4 is at least 4. Later, we will explicitly give an example of such a simplex.*

3.0 RESULTS

3.1 OUR APPROACH

First, we begin by following the same approach as Mori, Morrison, and Morrison. Let $\tilde{\Delta}$ be the standard closed simplex in $\mathbb{R}^4$, i.e.

$$\tilde{\Delta} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 0 \leq x_i, \forall i = 1, \ldots, 4, \sum_{i=1}^{4} x_i \leq 1 \right\}. \tag{3.1}$$

Further, we define $\Omega(\Delta)$ to be the vertex set of a simplex Δ. In particular $\Omega\left(\tilde{\Delta}\right) = \{0, e_1, e_2, e_3, e_4\}$, with e_i a standard basis vector of $\mathbb{R}^4$. We observe that $\mathbb{Z}^4 \cap \tilde{\Delta} = \Omega$. We are interested in determining when we can adjoin a point α of the form $\frac{1}{n}(a, b, c, d)$ to $\mathbb{Z}^4$ and still maintain $(\mathbb{Z}^4 + \alpha\mathbb{Z}) \cap \tilde{\Delta} = \Omega$. Notice that here, unlike in the Mori, Morrison, and Morrison paper, we are not restricting n to be prime. This will enlarge our search. We also have checked up to a much larger n (in particular 1600).

We used a computer to check all a, b, c, d, n such that $1 \leq a, b, c, d, n \leq 1600$. The code is included in the appendix. The code given is not optimized in any significant way and is written for readability. The actual code that was used was modified slightly for speed. This produced a very large list of lattice-free simplexes, which we trimmed down considerably by using the second block of code in the appendix to find bounds on the width. Finally, we removed all isomorphic simplexes and obtained the list of quintuples corresponding to all of the simplexes of width 3 in dimension 4. The list has the form (a, b, c, d, e, n) and appears in the appendix. There are precisely 199 distinct simplexes of width 3 up to unimodular equivalence.

3.2 FINDING LATTICE-FREE SIMPLEXES OF EXCEPTIONAL WIDTH

First, we fix n and let $a = b = c = d = 1$. Then we define $\langle m \rangle_n$ to mean the unique integer between 0 and $n-1$ congruent to m modulo n. Then, let $M_k = \langle ka \rangle_n + \langle kb \rangle_n + \langle kc \rangle_n + \langle kd \rangle_n$ for $1 \leq k \leq n-1$. If $M_k > n$ for all k then the simplex associated to (a, b, c, d, e) and n is lattice-free, where $e = -a - b - c - d \pmod{n}$. Next we repeat this process for all $1 \leq a, b, c, d \leq n$. After we check all cases, we increment n and repeat the algorithm. We checked for $n \leq 1600$.

Then, using the results of Mori, Morrison, and Morrison, we can eliminate quintuples that we know correspond to simplexes of width less than or equal to two. We may eliminate any quintuple such that the sum of two coordinates is zero modulo n, the sum of any two plus twice a third is zero modulo n, or if the sum of one coordinate plus twice another is zero modulo n. This reduces the data set significantly. We can further reduce our data set by finding bounds on the width.

To do this, we consider linear functionals on $\mathbb{Z}^4 + \frac{1}{n}(a, b, c, d)\mathbb{Z}$. We generate our functionals by picking small integer coefficients x_1, x_2, x_3, x_4 such that $ax_1 + bx_2 + cx_3 + dx_4 \equiv 0 \pmod{n}$. Then, simply using a brute force search we find maps that explicitly evaluate to two or less. We can then throw away these quintuples as their width must be less than or equal to two. Doing this is computationally cheap, and so we can actually remove all of the width two simplexes using this method. To find the quintuples corresponding to simplexes of width 4, we simply need to tweak the code slightly to remove all of the quintuples corresponding to width 3 simplexes.

We now have two lists: one consisting of quintuples corresponding to simplexes of width 3 and one with quintuples corresponding to simplexes of width 4. Now, one can observe that there will be clear symmetries in these lists. The first to observe are when the coordinates of one entry are simply a permutation of the coordinates of another entry. For example, (a, b, c, d, e) generates the same simplex as (d, e, c, b, a) as long as we are working with the same n. We can find a second set of symmetries by multiplying the entries in the quintuples by $-1 \pmod{n}$ and then removing permutations as in the first one. For example, $(1, 23, 47, 52, 57)$ and $(3, 8, 13, 37, 59)$ with $n = 60$ both give the same simplex. The final reduction comes from finding a coordinate that is coprime to n and then multiplying the entries in the quintuple by the inverse of the coordinate modulo n. Then, as before, we remove the quintuples which are permutations of another one. For example, both $(1, 24, 45, 57, 59)$ and $(1, 41, 43, 47, 54)$ induce isomorphic simplexes with $n = 62$. Notice that if we were to multiply every entry of the first quintuple by $59^{-1} \pmod{62}$, we obtain $(41, 54, 47, 43, 1)$ which is just a permutation of the second.

The reductions we performed above are justified by the following remark from the paper by Barrile, Bernardi, Borisov, and Kantor [1].

Remark 2. *Even though dropping one of the five coordinates from a given quintuple gives five different cyclic quotient singularities, the corresponding empty simplexes are the same. Indeed, this simplex can be described as sitting in the affine subspace of $\mathbb{R}^5$ of points with sum of coordinates 1, with vertices $e_1, \ldots, e_5$. The lattice is the restriction to this affine subspace of a lattice in $\mathbb{R}^5$ that is obtained from $\mathbb{Z}^5$ by adding multiples of $\frac{k}{n}(a_1, a_2, a_3, a_4, a_5)$, where $(a_1, a_2, a_3, a_4, a_5)$ is the given quintuple. When we drop a coordinate, we project to $\mathbb{R}^4$, sending one of the vertices of the simplex to $(0, 0, 0, 0)$. This projection is an isomorphism between the lattice described above and the lattice $\mathbb{Z}^4 + \frac{1}{n}(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4})\mathbb{Z}$ [1].*

Finally, it remains to show that the list contains only non-isomorphic simplexes. This will actually follow from the reductions we performed above. To be precise, we say that two simplexes are isomorphic if there exists an invertible affine map from one lattice onto the other which sends the vertices of the first simplex onto the vertices of the second. Such a map induces an isomorphism on the sublattices generated by the vertices of the simplexes. As in the above remark, these sublattices can be seen as the subset $\mathbb{Z}^5$ such that the sum of the coordinates sum to one. Further, there is a natural correspondence between these maps and the permutations of our five vertices. This is given by the group S_5. Finally, as the point $\frac{1}{n}(a, b, c, d, e)$ is a representative of a generator of $\mathbb{Z}_n$, if any two of the quintuples were to represent the same simplex, the two generators would be related through multiplication by an invertible element of $\mathbb{Z}_n$. However, in our process of reduction, we have removed all such possibilities.

3.3 SUMMARY OF RESULTS

We found only one simplex of width 4. It corresponds to the quintuple $(1, 36, 84, 87, 95)$ and $n = 101$. These results concur with those found by Haase and Ziegler in [4], though both our methods and the breadth of our search differ. We can formulate the following result:

Theorem 9. *For $n \leq 1600$, there are precisely 199 distinct empty-lattice simplexes of width 3 and 1 of width 4 in dimension 4.*

Then, as all of the listed quintuples have an n such that $41 \leq n \leq 179$, we make the following conjecture:

Conjecture 2. *In dimension 4, all empty-lattice simplexes of width 3 correspond to the list given in the appendix and there is precisely one empty-lattice simplex of width 4 which corresponds to the quintuple $(1, 36, 84, 87, 95)$ and $n = 101$.*

Further, it should be noted that the quintuples in the appendix are all listed with 1 as the first entry. This is done to make clear the fact that they are distinct.

4.0 APPENDIX

4.1 CODE TO FIND LATTICE-FREE SIMPLEXES

```
import java.io.*;
public class SearchForSimplices
{

    public static void main(String[] args) throws IOException
    {
        OutputStreamWriter out = null;
        try
        {
            out = new OutputStreamWriter(new FileOutputStream("LatticeList.txt"));
        }
        finally
        {

        }
        int Ma, Mb, Mc, Md, M;
        boolean trigger;
        for (int n=2; n<=1600; n++)
        {
            for (int a=1; a<=n; a++)
            {
                //
                //We only need to consider cases where a divides n
                //
                if (n%a!=0)
                {
                    break;
                }
                for (int b=0; b<n; b++)
                {
                    for (int c=0; c<n; c++)
                    {
                        for (int d=0; d<n; d++)
                        {
                            trigger=true;
                            for (int k=1; k<n; k++)
                            {
                                //
                                //Here we are checking to see if the simplex
                                //is lattice-free
                                //
                                Ma = (k*a)%n;
                                Mb = (k*b)%n;
                                Mc = (k*c)%n;
```

```
                                    Md = (k*d)%n;
                                    M=Ma+Mb+Mc+Md;
                                    if (M<=n)
                                    {
                                        trigger = false;
                                        break;
                                    }
                                }
                                if (trigger)
                                {
                                    if (check(a,b,c,d,n))
                                    {
                                        out.write(Integer.toString(a)+ " " + Integer.↩
                                            toString(b) + " " + Integer.toString(c) + " " ↩
                                            + Integer.toString(d) + " " + Integer.toString↩
                                            (n) +"\n");
                                    }

                                }
                            }
                        }
                    }
                }
            }

        }

        //
        //This method checks to see if our lattice-free simplex falls into one
        //of the cases that we already know have width less than or equal to 2
        //
        static boolean check(int a, int b, int c, int d, int n)
        {
            int e=-a-b-c-d;
            //
            // check to see if the sum of one coordinate plus twice
            // another is zero mod n
            // there are 20 cases here
            //
            if ((a+2*b)%n==0 || (a+2*c)%n==0 \textbar\textbar (a+2*d)%n==0 || (a+2*e)%n↩
                ==0){return false;}
            if ((b+2*a)%n==0 || (b+2*c)%n==0 || (b+2*d)%n==0 || (b+2*e)%n==0){return false↩
                ;}
            if ((c+2*b)%n==0 || (c+2*a)%n==0 || (c+2*d)%n==0 || (c+2*e)%n==0){return false↩
                ;}
            if ((d+2*b)%n==0 || (d+2*c)%n==0 || (d+2*a)%n==0 || (d+2*e)%n==0){return false↩
                ;}
            if ((e+2*b)%n==0 || (e+2*c)%n==0 || (e+2*a)%n==0 || (e+2*d)%n==0){return false↩
                ;}

            //
            //check to see if the sum of any two coordinate plus twice a third is zero mod↩
                n
            // there are 30 cases here
            //
            if ((c+d+2*e)%n==0 || (c+2*d+e)%n==0 || (2*c+d+e)%n==0){return false;}
            if ((b+c+2*d)%n==0 || (b+2*c+d)%n==0 || (b+c+2*e)%n==0 || (b+2*c+e)%n==0 || (b↩
                +d+2*e)%n==0 || (b+e+2*d)%n==0){return false;}
            if ((2*b+c+d)%n==0 || (2*b+c+e)%n==0 || (2*b+e+d)%n==0){return false;}
            if ((a+b+2*e)%n==0 || (a+b+2*d)%n==0 || (a+b+2*c)%n==0){return false;}
            if ((a+d+2*e)%n==0 || (a+2*d+e)%n==0 || (a+c+2*e)%n==0 || (a+c+2*d)%n==0 || (a↩
                +2*c+d)%n==0 || (a+e+2*c)%n==0){return false;}
            if ((a+2*b+e)%n==0 || (a+2*b+d)%n==0 || (a+2*b+c)%n==0){return false;}
            if ((2*a+d+e)%n==0 || (2*a+c+e)%n==0 || (2*a+c+d)%n==0){return false;}
            if ((2*a+b+e)%n==0 || (2*a+b+d)%n==0 || (2*a+b+c)%n==0){return false;}

            //
            // check to see if the sum of any two coordinates is zero mod n
```

```
            // there are 10 cases to check here
            //
            if ((a+b)%n==0 || (a+c)%n==0 || (a+d)%n==0 || (a+e)%n==0){return false;}
            if ((b+c)%n==0 || (b+d)%n==0 || (b+e)%n==0){return false;}
            if ((d+c)%n==0 || (e+c)%n==0){return false;}
            if ((d+e)%n==0){return false;}

            return true;
        }

    }
```

4.2 CODE TO CHECK THE WIDTH OF LATTICE-FREE SIMPLEXES

```
  #!/usr/bin/python
import sys
import string

f=open("LatticeList.txt")
potentialFile=open("Potential.txt",'w')
count=0
temp=" "
for line in f:
    numbers=string.split(line)
    trigger=1
    a=int(numbers[0])
    b=int(numbers[1])
    c=int(numbers[2])
    d=int(numbers[3])
    n=int(numbers[4])
    for a1 in range(-3, 3):
        if (trigger==1):
            for a2 in range(-3, 3):
                if (trigger==1):
                    for a3 in range(-3, 3):
                        if (trigger==1):
                            for a4 in range(-15,15):
                                if (a1==a2==a3==a4==0):
                                    break
                                if (trigger==1):
                                    hold=a1*a+a2*b+a3*c+a4*d
                                    if (hold%n==0):
                                        width=(max(0,a1,a2,a3,a4)-min(0,a1,a2,a3,a4))
                                        if (width<=2):
                                            trigger=0
    if (trigger==1):
        temp=str(a) + " " + str(b) + " "  + str(c) + " " + str(d) + " " + str(n)+ "\n"
        potentialFile.write(temp)
```

4.3 SIMPLEXES OF WIDTH THREE

Table 2: Simplexes of Width Three

$(1, 10, 16, 18, 37, 41)$	$(1, 5, 17, 28, 35, 43)$
$(1, 6, 13, 32, 34, 43)$	$(1, 7, 12, 27, 41, 44)$
$(1, 7, 16, 25, 39, 44)$	$(1, 4, 18, 31, 40, 47)$
$(1, 11, 15, 28, 41, 48)$	$(1, 19, 39, 41, 44, 48)$
$(1, 11, 15, 28, 41, 48)$	$(1, 20, 35, 43, 45, 48)$
$(1, 6, 19, 32, 40, 49)$	$(1, 8, 13, 30, 46, 49)$
$(1, 7, 12, 33, 47, 50)$	$(1, 16, 21, 19, 43, 50)$
$(1, 7, 23, 33, 38, 51)$	$(1, 8, 19, 29, 45, 51)$
$(1, 12, 23, 31, 37, 52)$	$(1, 4, 21, 34, 46, 53)$
$(1, 4, 25, 34, 42, 53)$	$(1, 6, 21, 35, 43, 53)$
$(1, 8, 13, 34, 50, 53)$	$(1, 8, 14, 33, 50, 53)$
$(1, 10, 22, 24, 49, 53)$	$(1, 17, 21, 29, 40, 54)$
$(1, 33, 35, 43, 50, 54)$	$(1, 4, 18, 39, 48, 55)$
$(1, 10, 18, 39, 42, 55)$	$(1, 11, 23, 37, 40, 56)$
$(1, 9, 16, 35, 53, 57)$	$(1, 12, 21, 27, 55, 58)$
$(1, 19, 21, 24, 51, 58)$	$(1, 19, 21, 32, 43, 58)$
$(1, 20, 45, 53, 55, 58)$	$(1, 4, 22, 39, 52, 59)$
$(1, 8, 19, 44, 46, 59)$	$(1, 9, 19, 33, 56, 59)$
$(1, 9, 19, 42, 47, 59)$	$(1, 9, 21, 32, 55, 59)$
$(1, 11, 19, 42, 45, 59)$	$(1, 11, 26, 39, 41, 59)$
$(1, 13, 21, 27, 56, 59)$	$(1, 16, 22, 24, 55, 59)$
$(1, 18, 25, 27, 47, 59)$	$(1, 23, 47, 52, 57, 60)$
$(1, 39, 44, 47, 49, 60)$	$(1, 7, 12, 44, 58, 61)$
$(1, 9, 15, 39, 58, 61)$	$(1, 9, 16, 38, 58, 61)$

Continued on next page

Table 2 – continued from previous page

$(1, 9, 20, 34, 58, 61)$	$(1, 10, 26, 28, 57, 61)$
$(1, 10, 26, 32, 53, 61)$	$(1, 12, 25, 40, 44, 61)$
$(1, 6, 19, 45, 53, 62)$	$(1, 13, 20, 37, 53, 62)$
$(1, 15, 19, 36, 53, 62)$	$(1, 24, 45, 57, 59, 62)$
$(1, 37, 39, 51, 58, 62)$	$(1, 12, 17, 41, 55, 63)$
$(1, 17, 25, 30, 53, 63)$	$(1, 9, 15, 42, 61, 64)$
$(1, 15, 25, 28, 59, 64)$	$(1, 21, 23, 26, 57, 64)$
$(1, 21, 50, 59, 61, 64)$	$(1, 8, 17, 42, 62, 65)$
$(1, 8, 18, 41, 62, 65)$	$(1, 9, 15, 43, 62, 65)$
$(1, 9, 16, 42, 62, 65)$	$(1, 6, 19, 50, 58, 67)$
$(1, 7, 12, 50, 64, 67)$	$(1, 11, 20, 38, 64, 67)$
$(1, 12, 19, 50, 52, 67)$	$(1, 12, 20, 38, 63, 67)$
$(1, 14, 16, 40, 63, 67)$	$(1, 9, 15, 46, 65, 68)$
$(1, 16, 27, 45, 47, 68)$	$(1, 27, 55, 59, 62, 68)$
$(1, 8, 29, 45, 55, 69)$	$(1, 9, 20, 43, 65, 69)$
$(1, 6, 28, 47, 60, 71)$	$(1, 8, 28, 47, 58, 71)$
$(1, 10, 21, 52, 58, 71)$	$(1, 10, 31, 33, 67, 71)$
$(1, 10, 32, 46, 53, 71)$	$(1, 14, 20, 45, 62, 71)$
$(1, 14, 29, 47, 51, 71)$	$(1, 14, 30, 47, 50, 71)$
$(1, 14, 31, 37, 59, 71)$	$(1, 9, 15, 51, 70, 73)$
$(1, 9, 19, 47, 70, 73)$	$(1, 9, 20, 46, 70, 73)$
$(1, 16, 26, 34, 69, 73)$	$(1, 24, 28, 30, 63, 73)$
$(1, 13, 18, 45, 71, 74)$	$(1, 17, 23, 44, 63, 74)$
$(1, 21, 24, 35, 67, 74)$	$(1, 30, 51, 69, 71, 74)$
$(1, 49, 51, 54, 67, 74)$	$(1, 18, 49, 29, 51, 74)$
$(1, 12, 32, 34, 71, 75)$	$(1, 14, 29, 34, 72, 75)$
$(1, 16, 28, 34, 71, 75)$	$(1, 22, 24, 32, 71, 75)$
$(1, 15, 18, 49, 69, 76)$	$(1, 23, 65, 67, 72, 76)$

Continued on next page

Table 2 – continued from previous page

$(1, 30, 63, 65, 69, 76)$	$(1, 9, 19, 51, 74, 77)$
$(1, 9, 23, 47, 74, 77)$	$(1, 9, 24, 46, 74, 77)$
$(1, 10, 21, 48, 74, 77)$	$(1, 16, 18, 46, 73, 77)$
$(1, 6, 26, 56, 69, 79)$	$(1, 6, 37, 52, 62, 79)$
$(1, 6, 38, 51, 62, 79)$	$(1, 9, 15, 57, 76, 79)$
$(1, 9, 29, 43, 76, 79)$	$(1, 11, 24, 46, 76, 79)$
$(1, 11, 24, 59, 63, 79)$	$(1, 11, 26, 59, 61, 79)$
$(1, 11, 35, 48, 63, 79)$	$(1, 12, 26, 45, 74, 79)$
$(1, 24, 38, 45, 50, 79)$	$(1, 26, 29, 43, 59, 79)$
$(1, 12, 20, 52, 77, 81)$	$(1, 16, 35, 37, 75, 82)$
$(1, 49, 52, 69, 75, 82)$	$(1, 9, 15, 61, 80, 83)$
$(1, 9, 29, 48, 79, 83)$	$(1, 10, 33, 55, 67, 83)$
$(1, 11, 31, 45, 78, 83)$	$(1, 13, 29, 44, 79, 83)$
$(1, 15, 18, 55, 77, 83)$	$(1, 15, 31, 39, 80, 83)$
$(1, 15, 35, 39, 76, 83)$	$(1, 18, 22, 48, 77, 83)$
$(1, 29, 64, 76, 79, 83)$	$(1, 20, 71, 79, 81, 84)$
$(1, 39, 65, 71, 76, 84)$	$(1, 51, 53, 67, 80, 84)$
$(1, 12, 14, 62, 81, 85)$	$(1, 12, 21, 54, 82, 85)$
$(1, 12, 22, 53, 82, 85)$	$(1, 13, 24, 50, 82, 85)$
$(1, 12, 32, 46, 83, 87)$	$(1, 18, 20, 52, 83, 87)$
$(1, 8, 34, 59, 76, 89)$	$(1, 14, 38, 40, 85, 89)$
$(1, 20, 35, 38, 84, 89)$	$(1, 9, 15, 69, 88, 91)$
$(1, 12, 32, 50, 87, 91)$	$(1, 20, 81, 89, 91, 94)$
$(1, 57, 59, 75, 90, 94)$	$(1, 9, 15, 73, 92, 95)$
$(1, 9, 37, 51, 92, 95)$	$(1, 15, 28, 54, 92, 95)$
$(1, 21, 33, 44, 91, 95)$	$(1, 27, 36, 43, 83, 95)$
$(1, 33, 75, 84, 92, 95)$	$(1, 12, 25, 62, 94, 97)$
$(1, 12, 26, 61, 94, 97)$	$(1, 16, 29, 55, 93, 97)$

Continued on next page

Table 2 – continued from previous page

$(1,18,32,69,74,97)$	$(1,20,22,58,93,97)$
$(1,24,26,57,86,97)$	$(1,13,27,63,98,101)$
$(1,13,30,63,95,101)$	$(1,18,31,57,95,101)$
$(1,36,84,87,95,101)$	$(1,12,14,80,99,103)$
$(1,16,44,46,99,103)$	$(1,28,30,48,99,103)$
$(1,28,85,96,99,103)$	$(1,34,36,44,91,103)$
$(1,34,36,47,88,103)$	$(1,41,84,90,93,103)$
$(1,15,25,71,102,107)$	$(1,16,34,60,103,107)$
$(1,17,37,56,103,107)$	$(1,20,94,102,104,107)$
$(1,22,29,64,98,107)$	$(1,24,93,99,104,107)$
$(1,32,34,44,103,107)$	$(1,44,77,96,103,107)$
$(1,9,15,87,106,109)$	$(1,12,49,51,105,109)$
$(1,15,28,68,106,109)$	$(1,17,32,62,106,109)$
$(1,9,15,91,110,113)$	$(1,9,46,60,110,113)$
$(1,22,27,72,116,119)$	$(1,25,27,71,114,119)$
$(1,15,32,76,118,121)$	$(1,16,33,78,122,125)$
$(1,19,36,74,124,127)$	$(1,21,40,78,134,137)$
$(1,26,46,99,106,139)$	$(1,19,39,93,146,149)$
$(1,21,35,99,142,149)$	$(1,25,48,98,166,169)$
$(1,27,52,102,176,179)$	

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