Numerical Analysis of a Variational Multiscale Method for Turbulence

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NUMERICAL ANALYSIS OF A VARIATIONAL MULTISCALE METHOD FOR TURBULENCE

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This thesis is concerned with one of the most promising approaches to the numerical simulation of turbulent flows, the subgrid eddy viscosity models. We analyze both continuous and discontinuous finite element approximation of the new subgrid eddy viscosity model introduced in [43], [45], [44].

First, we present a new subgrid eddy viscosity model introduced in a variationally consistent manner and acting only on the small scales of the fluid flow. We give complete convergence of the method. We show convergence of the semi-discrete finite element approximation of the model and give error estimates of the velocity and pressure. In order to establish robustness of the method with respect to Reynolds number, we consider the Oseen problem. We present the error is uniformly bounded with respect to the Reynolds number.

Second, we establish the connection of the new eddy viscosity model with another stabilization technique, called Variational Multiscale Method (VMM) of Hughes et.al. [35]. We then show the advantages of the method over VMM. The new approach defines mean by elliptic projection and this definition leads to nonzero fluctuations across element interfaces.

Third, we provide a careful numerical assessment of a new VMM. We present how this model can be implemented in finite element procedures. We focus on herein error estimates of the model and comparison to classical approaches. We then establish that the numerical experiments support the theoretical expectations.

Finally, we present a discontinuous finite element approximation of subgrid eddy viscosity model.

We derive semi-discrete and fully discrete error estimations for the velocity.

To my parents: Hatice and Mustafa Kaya

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Introduction

0.1 Turbulence

In this thesis accurate and reliable solutions of turbulent flows are considered. Despite efforts of more than centuries, turbulence phenomenon is categorized as an unsolved problem. Turbulence is part of everyday's life. The majority of flows of industrial and technological applications are turbulent; natural flows are invariably so. There are many important and interesting physical phenomena which are connected with turbulent flows. Turbulence is observed in natural and engineering applications such as in weather prediction, air pollution, water pollution, aerodynamics and heat exchangers. In view of the importance of this subject, understanding turbulent flow is central to many important problems. So, it is natural that the study of turbulent flow has attracted wide-spread attention from scientists all over the world. However, progress has been limited and understanding turbulent flows remains a challenge. The main obstacles in turbulence are subjected to the features of turbulence, listed below:

1. Turbulence is diffusive.
2. Turbulence is not only chaotic motion and but also irregular motion.
3. Turbulence is rotational and three dimensional.
4. Turbulence is highly dissipative.
5. Turbulence is a continuum phenomenon. The smallest scales of turbulence are much larger than the molecular scales in the engineering application.
6. Turbulence is associated with high levels of vorticity fluctuations. Small scales are generated by the vortex stretching mechanism.

Description	Reynolds Number
Water droplet	6.4×10^{-1}
Blood flow	1.35×10^{2}
Car with 54 km/h	4×10^{6}
Small airplane	2×10^{7}
Planetary boundary layer	18×10^{12}
Geophysical flow	10^{20} and higher

Table 1. Some representative values of Reynolds numbers.

The equations of motion for an incompressible, viscous linear fluid are called *Navier Stokes Equations*. They were proposed in 1823 by the French Engineer C. M. L. H. Navier upon the basis of an oversimplified molecular model, [58]. Since turbulent flows are governed by the Navier-Stokes equations, in this thesis we consider the numerical solutions of the Navier-Stokes equations. These equations obey conservation of momentum and conservation of mass and their mathematical structure are best understood for incompressible fluids. Incompressible Navier-Stokes equations in the nondimensional form are given by the following equations:

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, p is the fluid pressure, $\mathbf{f}$ is an external force and ν is the kinematic viscosity, inversely proportional to Re, the Reynolds number. In Navier-Stokes equations the control parameter is the Reynolds number. As Re increases, the flow becomes more sensitive to perturbations, more complex in its structure and eventually turbulent. Table 1 presents some examples for 'real life' Reynolds number values. Nevertheless, turbulent flows exhibit a lack of robustness and predictability with respect to Reynolds number. Motivated by this question, the robustness of the model with respect to Reynolds numbers is discussed in this thesis.

Despite the introduction of the Navier-Stokes equations over a century ago, its predictions and consequences are still far from being completely understood today. One may think that since enormous computing power increases day by day, a straightforward procedure, called Direct Numerical Simulation (DNS) of Navier-Stokes equations, should solve the turbulence question. Unfortunately, having huge computing power is not enough to understand features of turbulence. One of the major problems in solving Navier-Stokes equations is the richness of scales. This broad range of scales changes with respect to Reynolds number and exceeds the limits of computing power available today. Because of this reason the use of a turbulence model becomes necessary. To do this, *questionably* derived models are typically used. The correct modeling of turbulence requires a thorough understanding of the physics of turbulent flows not yet attained. One of the objective of this work is to improve the numerical simulation of complex turbulent flows with a new discretization of (1). This new discretization is given with more fidelity to the physics of turbulent flows.

In addition to the richness of scales of Navier-Stokes equations, the question of turbulence becomes more complicated in the absence of a complete mathematical theory for the Navier-Stokes equations or any of the various turbulence models. The mathematical theory was founded by J. Leray [51] in 1934. Leray's theory describes the most abstract and complete mathematical aspects of the Navier-Stokes equations. Leray introduced the first description of turbulent solutions, namely weak solutions. The bold definition of Leray's leads to the fundamental question of uniqueness of weak solutions. Leray conjectured that turbulence is connected with the uniqueness in weak solutions of Navier-Stokes equations. In $2d$, the existence and the uniqueness of weak solutions are shown but in $3d$, uniqueness of weak solutions is still an open question Duchon and Robert [18], Galdi [22] (One Million Dollar Clay Prize problem). In this thesis, we discuss mathematical aspects of the new model for the Navier-Stokes equations.

0.2 Numerical Simulations of Turbulent Flows

The chaotic nature of turbulence gives rise to a statistical approach in comparing turbulent flows. For instance, it is impossible to find two turbulent flows such that the velocity at each point in one flow is equal to the velocity at the corresponding point in the other flow, at all times. Thus, we need a statistical approach to quantify turbulent flows. Statistical quantities in turbulent flows are the mean (e.g., average), the correlations between the different components of velocity and the correlation between the different velocities at the different points, etc.. The first statistical description is given by Kolmogorov [46] and will be considered in the following section.

0.2.1 Kolmogorov Energy Spectrum

An alternative description of multiscale turbulence was proposed by the meteorologist L. F. Richardson (1922):

> Big whirls have little whirls,
> That feed on their velocity,
> And little whirls have lesser whirls,
> and so on to viscosity.

The Russian mathematician A. N. Kolmogorov combined the idea of Richardson and the dimensional analysis and gave the first statistical description of the turbulent flows [46] in 1941. It is generally believed that the Kolmogorov cascade theory provides an approximate description of homogenous isotropic turbulence [9]. Kolmogorov quantified the statistics of the velocity fluctuations (small scales, unresolved scales) distribution and his ideas remain one of the corner stones of turbulent flows.

Kolmogorov's idea is that the velocity fluctuations in the inertial subrange are more affected by the interaction between small eddies in the turbulent field and less influenced by the large features of the flow. He also realized that the statistics of the small scales are independent from the large scales, the initial and boundary conditions, but depends only on the energy dissipation rate, ϵ and the kinematic viscosity, ν.

Using this physical argument Kolmogorov proposed his first hypothesis: *At sufficiently large Reynolds numbers, there will exist a range of high wave numbers in which the turbulence is in a state of statistical equilibrium influenced only by the parameters ϵ and ν. This state is universal*, [9]. His second hypothesis states: *At wave numbers which are much larger than of the large eddy scales and also much smaller than the scales of the dissipation range, which conditions can only be met at very high Reynolds numbers, there exists an inertial range in which the statistics of turbulence are determined solely by energy dissipation rate, ϵ*, [9].

From the first and second hypothesis, Kolmogorov derived an expression for the energy spectrum, denoted by $\mathbf{E}(k, t)$ where k is the wave number given by $k^2 = k_x^2 + k_y^2 + k_z^2$. Hence, energy

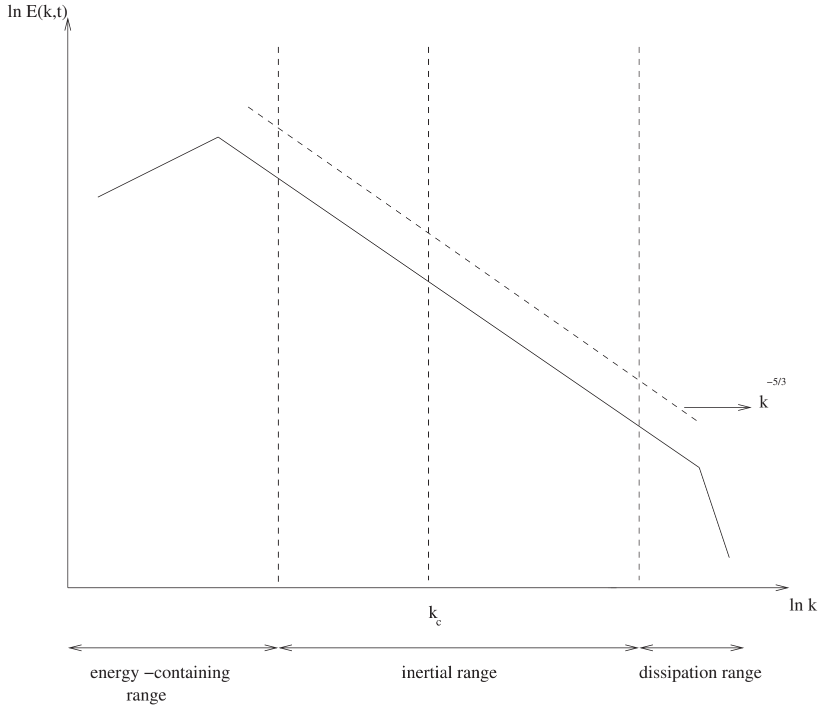

Figure 1. Kolmogorov's Energy Cascade.

spectrum in inertial range is given by

$$\mathbf{E}(k,t) = C\epsilon^{2/3}k^{-5/3}$$

where C is a nondimensional and universal constant, (Sreenivasan [67]) and these results are widely used in computational fluid mechanics to estimate the number of grid points to fully resolve turbulent flows.

The picture of turbulence that emerged with Kolmogorov was the following: turbulent energy is created at relatively large scales (in the *energy containing eddies*) and is transferred, through a process of eddies breaking into other eddies, to smaller and smaller scales till the dissipation effects, which increase at decreasing scales, predominate, and the energy is consumed at the smallest scales. This energy transferred from scale to scale progresses on the way through the inertial subrange, thus explaining the dependence of statistics of the subrange on the dissipation rate. This conception of energy transfer has been called the *energy cascade* and its graph is given by Figure 1. According to this theory, there is an energy loss around the cut off lengthscale k_c in inertial subrange. It is necessary to model this energy loss in such a way that the method/model must incorporate *energy drain* due to inertial effects near k_c of unresolved scales. This can be done with eddy viscosity models. In this thesis we consider one of these models, namely subgrid eddy viscosity models.

0.3 Techniques for Turbulent Simulations

In this section, we consider the following three approaches for modeling turbulent simulations:

1. Direct Numerical Simulations (DNS)
2. Reynolds Averaged Navier-Stokes (RANS)
3. Large eddy Simulations (LES).

These turbulence models are based on direct discretization, time averaged or space averaged quantities and have been used in engineering applications.

0.3.1 Direct Numerical Simulations (DNS)

In DNS, the Navier-Stokes equations are solved to determine the instantaneous flow field by using direct discretization. Since all length scales must be resolved, no modeling is required in this approach. This approach is the most desirable way of simulating a flow. Theoretically, the solutions obtained from DNS are as close to the solutions of (1). Unfortunately, the simulations of turbulent flow take way too much time and despite steady advances in computing power, attempts at the DNS of the Navier-Stokes equations have been limited to rather low Re. For instance, if one is interested in solving Navier-Stokes equations for $Re = 1000$ in a unit cube, according to Kolmogorov estimate, one needs a total of $Re^{-9/4} \sim 5.6$ million grid points. Hence although today's computers are much faster than they were decades ago, they still are not fast enough to simulate many fluid flows scenarios we are interested in. This prompts us to make use of alternatives to DNS, namely the use of turbulence models.

0.3.2 Reynolds Averaged Navier Stokes (RANS)

In this approach, suggested by Reynolds (1895), the velocity and pressure is decomposed into an ensemble mean flow and a fluctuating perturbation field. In other words, this approach requires to model all scales and to solve time-averaged equations. The decomposition of velocity and pressure leads to a set of differential equations for the mean flow quantities containing contributions from the time-varying, turbulent motion. An additional model is needed for the *Reynolds stresses* term to describe the effect of fluctuation on the mean. RANS has proven to be advantageous in providing data to industry. Thus RANS can be found in many commercial codes (Flowtech International AB, ARC3D). However, although RANS is able to satisfactorily model some turbulent flows, it sometimes unpredictably fails to model others. One of the problems is that there are terms in the equations to be solved whose values are unknown, for instance *Reynolds stresses*, which must be themselves modeled. Hence, modeling leads to new numerical equations which are different from original Navier-Stokes equations themselves. As a consequence, the RANS approach limits our ability, by providing only mean turbulence quantities. This means that it is desirable to find a better way to simulate turbulence.

0.3.3 Large Eddy Simulations (LES)

LES technique is a compromise between the computational efficiency of RANS and the high accuracy of DNS. According to LES procedure, the velocity and pressure can be written in two parts. These quantities are the sum of a mean component and fluctuations component as follows

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \quad p = \bar{p} + p', \tag{2}$$

in which the overbar denotes the mean quantity and the prime denotes the fluctuating part. $\bar{\mathbf{u}}$ and $\bar{p}$ are defined by filtering or mollification. In LES, the Navier Stokes equations are filtered in a way that the smallest spatial structure disappears from the solution. The goal is to compute the mean part accurately. After deciding how to define the mean flow structures of the flow, LES must construct closed equations for $\bar{\mathbf{u}}$. One simple approach is to convolve the Navier-Stokes equations with a filter function. By assuming that convolution commutes with differentiation, we obtain the space-averaged Navier-Stokes equations:

$$\begin{aligned} \bar{\mathbf{u}}_t + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) + \nabla \cdot \mathcal{R}(\mathbf{u}, \mathbf{u}) + \nabla \bar{p} - \nu \Delta \bar{\mathbf{u}} &= \bar{f}, \\ \nabla \cdot \bar{\mathbf{u}} &= 0, \end{aligned} \tag{3}$$

where $\mathcal{R}$, Reynolds stress tensor, is

$$\mathcal{R}(\mathbf{u},\mathbf{u}) = \overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}.$$

In general, since $\overline{\mathbf{u}\mathbf{u}} \neq \bar{\mathbf{u}}\bar{\mathbf{u}}$, the space-averaged Navier-Stokes equations are not closed. A main issue is to model Reynolds stress tensor to get a closed system. For the justification of LES models, we refer to [23], [39], [38], [64], [40]. The most common way of modeling Reynolds stress terms is to use eddy viscosity models considered in the following section. The reader can find the other modeling approaches in [50]. We emphasize that the major drawback of LES relative to RANS is that the computations are necessarily three-dimensional and time dependent (Piomelli and Chasnow [60]). This means that the computational cost is quite high. Also averaging Navier-Stokes equations leads to serious problems if the flow is given in a bounded domain, which is the most frequent case in applications. Already the first step of deriving equations for the large scales introduces an additional term, a so-called commutation error ([19]). This term is simply neglected in applications. However, the analysis in [19] shows that there are cases where this term does not vanish asymptotically. A second serious problem of the classical LES in bounded domains is the definition of appropriate boundary conditions for the large scales. This problem is unresolved. In applications, often physically motivated wall laws are used. A possible remedy of this dilemma is the definition of the large scales in a different way, namely by projection into appropriate spaces. This idea is the basis of variational multiscale methods (VMM), see Hughes et al. [35] Chapter 2.

As a result, LES is not the miracle cure to simulate most of the turbulent flows in engineering application.

Since all turbulence models have limitations, it is the author's belief that, careful consideration is needed in deciding what type of turbulence model will be used for what application.

0.4 Eddy Viscosity Models

One of the most widely used concepts in turbulence simulation for practical engineering applications is the eddy viscosity concept. These models are consistent with the idea of an energy cascade (Figure 1). Boussinesq [11] proposed that the turbulent stresses are proportional to the mean-velocity gradients. This concept can be generalized to model the Reynolds stress tensor as

$$\nabla \cdot R(\mathbf{u}, \mathbf{u}) \sim -\nabla \cdot (\nu_T \nabla^s \bar{\mathbf{u}}),$$

where $\nu_T \geq 0$ is called turbulent viscosity or eddy viscosity parameter and ∇^s is the symmetric part of deformation tensor defined by

$$\nabla^s \bar{\mathbf{u}} = \frac{\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T}{2}.$$

The extensions of this model with some extra terms can be found in Mohammadi and Pironneau [57]. Now, one needs to determine the eddy viscosity parameter, ν_T. The most commonly used eddy viscosity model is the Smagorinsky eddy viscosity model [65], in which the turbulent eddy viscosity parameter is given by

$$\nu_T = (C_s \delta)^2 \|\nabla^s \bar{\mathbf{u}}\|_F,$$

where C_s is the Smagorinsky constant, δ is the averaging radius and $\| \cdot \|_F$ is the Frobenius norm defined by for all $\mathbf{v} \in \mathbb{R}^N$

$$\|\mathbf{v}\|_F = \sqrt{\sum_{i,j=1}^{N} \mathbf{v}\mathbf{v}^T}.$$

Thus, with Boussniesq's assumption, we obtain the term $\nabla \cdot ((C_s \delta)^2 \|\nabla^s \bar{\mathbf{u}}\|_F \nabla^s \bar{\mathbf{u}})$. A complete mathematical theory for Navier-Stokes equations for the Smagorinsky model is constructed by Ladyzhenkaya [47]. In the work of Ladyzhenkaya, this extra term is considered as a correction term for flows with larger stresses. Also, one of the essential result in LES is given by Lilly [54]. Lilly estimated a universal value of Smagorinsky constant C_s as 0.17.

The justification of Smagorinsky eddy viscosity models is given by Zhang et. al. [70], [39], [53]. This concept is criticized because of the following facts:

- These eddy viscosity models require a priori knowledge of the flow to set C_s.
- The viscosity is applied over all wave numbers. This means that extra dissipation is added where it is needed (at the higher wave number) and also where it is not necessary (at the lower wave number). In addition, a viscosity model is too dissipative over the large flow structures.

- The eddy viscosity does not vanish for laminar flows.

This chapter will be concluded by pointing out that current eddy viscosity models are of limited usefulness on long time simulations because they are diffusive over the large flow structures. Limited accuracy in moderate time simulations is also observed, because the connection of eddy viscosity models to the physics of fluctuations is tenuous.

The objective of this thesis is to improve the numerical simulation of complex turbulent flows by formulating a new discretization of Navier-Stokes equations. This discretization which is introduced in Chapter 1 gives more realistic results to the physics of the turbulent flows.

0.5 Chapter Descriptions

The thesis is structured in four chapters. We analyze continuous finite element approximation in Chapter 1, Chapter 2, Chapter 3 and discontinuous finite element approximation of the model in Chapter 4.

Chapter 1 is dedicated to the presentation of the new eddy viscosity model resulting in a new discretization of Navier-Stokes equations. This idea was first introduced by [30], [49]. Unlike the traditional eddy viscosity model acting on all scales of the fluid flow, our new approach introduces viscosity only on small scales. We then investigate convergence of the continuous semi discrete finite element approximation of the corresponding model. First, a priori error estimates of velocity and pressure are proved in two successive sections. After giving the L^2 error estimate for velocity, the Reynolds number dependence of the new approach is reviewed for the Oseen problem.

Chapter 2 presents another stabilization technique called Variational Multiscale Method (VMM). We prove that our new model fits into the framework of VMM. The interest in this connection is that the new method allow fluctuations to be nonzero across meshlines.

Chapter 3 provides numerical assessment of the subgrid eddy viscosity model. First, we present how this model can be implemented in finite element procedures. Specifically, we perform two numerical experiments in $2d$. We present numerical results (including error tables, graphs as well as plots of the streamlines) corresponding to the model.

Chapter 4 addresses the discontinuous finite element approximation of the subgrid eddy viscosity model. We combine Discontinuous Galerkin (DG) and eddy viscosity techniques. This combination brings a new improvement over the continuous discretization. In order to investigate this, we first consider semi discretization of the DG approximation of the model. Finally, we analyze two fully discrete schemes based on: Backward Euler and Crank Nicholson.

Chapter 1

Finite Element Analysis of the New Subgrid Eddy Viscosity Models

1.1 The New Subgrid Eddy Viscosity Model

This chapter gives a numerical analysis of a special subgrid eddy viscosity method/model for the Navier-Stokes equations at higher Reynolds number. In this method, variationally consistent eddy viscosity is introduced acting only on the discrete fluctuations. This technique is inspired by earlier work of Guermond [30], Hughes [35] and Layton [49]. It can also be thought of as an extension to general domains and boundary conditions of the spectral vanishing viscosity idea of Maday and Tadmor [55], Chen, Du and Tadmor [14]. Specifically, this new method inserts eddy viscosity acting only on the smallest resolved mesh scales (Kaya [43]).

Consider the incompressible, viscous Navier-Stokes equations

$$\begin{aligned}
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} &= \mathbf{f} \quad \text{in } (0,T]\times\Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega,\\
\mathbf{u} &= \mathbf{0} \quad \text{in } [0,T]\times\partial\Omega,\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 \quad \text{in } \Omega,\\
\int_\Omega p\, d\mathbf{x} &= 0, \quad \text{in } (0,T].
\end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, $\mathbf{f}$ is the external force, ν is the kinematic viscosity, and $\Omega \subset \mathbb{R}^d (d = 2 \text{ or } d = 3)$ is a bounded, simply connected domain with polygonal boundary $\partial\Omega$.

The idea consists of stabilizing the discrete equations by adding eddy viscosity in a variationally consistent way. To motivate this approach, we consider a first variational formulation of (1.1) in the functional spaces

$$\begin{aligned}
\mathbf{X} &:= \{\mathbf{v} \in L^2(\Omega)^d : \nabla \mathbf{v} \in L^2(\Omega)^{d\times d} \text{ and } \mathbf{v} = 0 \text{ on } \partial\Omega\},\\
\mathbf{L} &:= \{\mathbb{S} \in L^2(\Omega)^{d\times d} : \mathbb{S}_{ij} = \mathbb{S}_{ji}\},\\
Q &:= L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q \; d\mathbf{x} = 0\}.
\end{aligned}$$

One nonstandard variational formulation of (1.1) is: find $\mathbf{u} : [0,T] \to \mathbf{X}$, $p : (0,T] \to Q$ and $\mathbb{G} : [0,T] \to \mathbf{L}$ satisfying

$$\begin{aligned}
(\mathbf{u}_t, \mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) &+ ((2\nu+\nu_T)\nabla^s\mathbf{u}, \nabla^s\mathbf{v}) \\
-(\nu_T\mathbb{G}, \nabla^s\mathbf{v}) &= (\mathbf{f},\mathbf{v}), \qquad (1.2)\\
(\mathbb{G} - \nabla^s\mathbf{u}, \mathbb{L}) &= 0, \qquad (1.3)
\end{aligned}$$

for all $(\mathbf{v}, q, \mathbb{L}) \in (\mathbf{X}, Q, \mathbf{L})$ where $(.,.)$ denotes the $L^2(\Omega)$ inner product, $b(\mathbf{u},\mathbf{u},\mathbf{v}) : \mathbf{X}\times\mathbf{X}\times\mathbf{X} :\to \mathbb{R}$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w}, \mathbf{v}) \qquad (1.4)$$

denotes the skew symmetrized trilinear form and $\nu_T > 0$ denotes the eddy viscosity parameter.

Next, we consider the finite element discretization of the Navier-Stokes equations based on (1.2), (1.3). We first construct a coarse finite element mesh $\Pi^H(\Omega)$ and a fine mesh $\Pi^h(\Omega)$, where $h << H$ typically. Conforming velocity-pressure finite element spaces are then constructed based upon $\Pi^h(\Omega)$ and $\Pi^H(\Omega)$. Let $\mathbf{X}^h \subset \mathbf{X}$, $\mathbf{X}^H \subset \mathbf{X}$, $Q^h \subset Q$ and $\mathbf{L}^H \subset \mathbf{L}$ be finite element spaces. Consider the approximation $(\mathbf{u}^h, p^h, \mathbb{G}^H)$ based on the variational formulation (1.2), (1.3) : find $\mathbf{u}^h \in \mathbf{X}^h, p^h \in Q^h, \mathbb{G}^H \in \mathbf{L}^H$ satisfying

$$(\mathbf{u}_t^h, \mathbf{v}^h) + b(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) - (p^h, \nabla\cdot\mathbf{v}^h) + (q^h, \nabla\cdot\mathbf{u}^h) + ((2\nu+\nu_T)\nabla^s\mathbf{u}^h, \nabla^s\mathbf{v}^h) - (\nu_T\mathbb{G}^H, \nabla^s\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h), \qquad (1.5)$$

for all $\mathbf{v}^h \in \mathbf{X}^h$ and $q^h \in Q^h$ where $\mathbb{G}^H \in \mathbf{L}^H \subset L^2(\Omega)$ is defined by

$$(\mathbb{G}^H - \nabla^s \mathbf{u}^h, \mathbb{L}^H) = 0, \tag{1.6}$$

for all $\mathbb{L}^H \in \mathbf{L}^H$.

Different choices of $\mathbf{L}^H$ give rise to different methods. Consider the following limit choices of $\mathbf{L}^H$:

- If $\mathbf{L}^H = \{0\}$: Then the equation (1.6) implies that $\mathbb{G}^H = 0$. Inserting this into (1.5) yields the usual artificial viscosity method.
- If $\mathbf{L}^H = \nabla^s \mathbf{X}^h$: Then, there exits a $\mathbf{v}^h \in \mathbf{X}^h$ such that $\mathbb{L}^H = \nabla^s \mathbf{v}^h$. The equation (1.6) implies we have $\mathbb{G}^H = \nabla^s \mathbf{u}^h$. Inserting $\mathbb{G}^H$ in (1.5) gives the Galerkin discretization of Navier-Stokes equations.

The effect of the extra term in (1.5) can be seen by using (1.6), to eliminate $\mathbb{G}^H$ from (1.5). Clearly, (1.6) implies that

$$\mathbb{G}^H = P_{L^H} \nabla^s \mathbf{u}^h \tag{1.7}$$

where $P_{L^H} : \mathbf{L} \to \mathbf{L}^H$ denotes the usual L^2 orthogonal projection. Insertion of this into (1.5) and simplification using properties of orthogonal projection yield:

$$\begin{aligned}(\mathbf{u}_t^h, \mathbf{v}^h) \;&+\; b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot \mathbf{u}^h) + (2\nu \nabla^s \mathbf{u}^h, \nabla^s \mathbf{v}^h) \\ &+\; (\nu_T (I - P_{L^H})(\nabla^s \mathbf{u}^h), (I - P_{L^H}) \nabla^s \mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h), \end{aligned} \tag{1.8}$$

for all $(\mathbf{v}^h, q^h) \in (\mathbf{X}^h, Q^h)$. The formulation (1.5), (1.6) thus introduces implicitly the extra stabilization term

$$(\nu_T (I - P_{L^H})(\nabla^s \mathbf{u}^h), (I - P_{L^H}) \nabla^s \mathbf{v}^h)). \tag{1.9}$$

This is an eddy viscosity term acting on the scales between H and h, i.e.,the small scales. The normal use of eddy viscosity (EV) models is in high Re/turbulent problems and thus we are specifically interested in the numerical analysis of (1.5), (1.6) for higher Re, (or smaller ν). As we mention in Section 0.4, standard EV models have often been noted to be too diffusive and over damp in the large flow structures (large eddies). Indeed, the physical interpretation of EV models is that they model energy lost around the cut off lengthscale due to inertial effects (eddy breakdown to below the meshwidth size). Standard EV, however, removes energy strongly from *all* resolved scales. Thus, the method (1.5), (1.6) is of special interest because (*i*) the eddy viscosity does not act on the large structures and (*ii*) it is introduced in a variationally consistent manner. This thesis will give a complete and systematic analysis of the finite element method (1.5), (1.6) for small ν.

The general idea of using two-grid discretizations to increase the *efficiency* of methods was pioneered by J. Xu (see, e.g., Marion and Xu [56]) and developed by Girault and Lions [25], [26], Layton [48]. This plus the physical ideas underlying eddy viscosity models and previous work [20] on stabilizations in viscoelasticity lead very naturally to the present method.

1.1.1 Preliminaries

Throughout this thesis, we use standard notation for Sobolev spaces (Adams [1]). For $s \geq 0$ and $r \geq 1$, the classical Sobolev space on a domain $E \subset \mathbb{R}^d (d = 2, 3)$ is

$$W^{k,r}(E) = \{v \in L^r(E) : \forall\, |m| \leq k,\, \partial^m v \in L^r(E)\},$$

where $\partial^m v$ are the partial derivatives of v of order $|m|$. The usual norm in $W^{k,r}(E)$ is denoted by $\|\cdot\|_{k,r,E}$ and the semi norm by $|\cdot|_{k,r,E}$. The L^2 inner-product is denoted by $(\cdot,\cdot)_E$ and only by $(\cdot,\cdot)$ if $E = \Omega$. For the Hilbert space $H^k(E) = W^{k,2}(E)$, the norm is denoted by $\|\cdot\|_{k,E}$. By $H^1_0(E)$ we shall understand the subspace of $H^1(E)$ of functions that vanish on ∂E. Since the case of scalar, vector or tensor functions will be clear from the context, we will not distinguish between these cases in the notation of $H^k(\Omega)$. The $H^1(\Omega)$ norm is defined by $\|\mathbf{v}\|_1 = \sqrt{\|\mathbf{v}\|^2 + \|\nabla \mathbf{v}\|^2}$. The norm of dual space $H^{-1}(\Omega) = (H^1(\Omega) \cap H^1_0(\Omega))^*$ is defined by

$$\|\boldsymbol{\phi}\|_{-1} = \sup_{\mathbf{v} \in (H^1(\Omega) \cap H^1_0(\Omega))^d} \frac{|(\boldsymbol{\phi}, \mathbf{v})_{L^2}|}{\|\mathbf{v}\|_1}.$$

Throughout the text, C denotes a generic constant which does not depend on ν, ν_T, h, H unless specified.

For any function $\boldsymbol{\phi}$ that depends on time t and space $\mathbf{x}$, denote

$$\boldsymbol{\phi}(t)(\mathbf{x}) = \boldsymbol{\phi}(t, \mathbf{x}), \quad \forall t \in [0, T], \forall \mathbf{x} \in \Omega.$$

If X denotes a functional space in the space variable with the norm $\| \cdot \|_X$ and if $\boldsymbol{\phi} = \boldsymbol{\phi}(t, \mathbf{x})$, then for $s > 0$:

$$\|\boldsymbol{\phi}\|_{L^s(0,T;X)} = [\int_0^T \|\boldsymbol{\phi}(t)\|_X^s dt]^{1/s}, \quad \|\boldsymbol{\phi}\|_{L^\infty(0,T;X)} = \max_{0 \le t \le T} \|\boldsymbol{\phi}(t)\|_X.$$

Recall that for a vector function $\boldsymbol{\phi}$, the tensor $\nabla \boldsymbol{\phi}$ is defined as $(\nabla \boldsymbol{\phi})_{i,j} = \frac{\partial \phi_i}{\partial x_j}$ and the tensor product of two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ is defined as $\boldsymbol{T} : \boldsymbol{S} = \sum_{i,j} T_{ij} S_{ij}$.

As usual, $\mathbf{V}$ denotes the space of divergence free functions

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0, \text{for all } q \in Q\}.$$

We assume that the velocity-pressure finite element spaces $\mathbf{X}^h \subset \mathbf{X}, Q^h \subset Q$ satisfy the discrete inf-sup or Babuska-Brezzi condition:

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|q^h\| \|\nabla \mathbf{v}^h\|} \ge \beta > 0. \tag{1.10}$$

where β is independent from h. This condition is now well understood and numerous example of attractive finite element spaces satisfying inf-sup condition exist, e.g., Gunzburger [31], Girault and Raviart [27]. Recall that under this condition, the space of discretely divergence free functions $\mathbf{V}^h$,

$$\mathbf{V}^h := \left\{\mathbf{v}^h \in \mathbf{X}^h : (\nabla \cdot \mathbf{v}^h, q^h) = 0, \text{for all } q^h \in Q^h\right\}, \tag{1.11}$$

is well-defined and the natural formulations of the discrete Navier Stokes problem in $(\mathbf{X}^h, Q^h)$ and $\mathbf{V}^h$ are equivalent. Further, if inf-sup condition holds and $\mathbf{u} \in \mathbf{V}$ then (see Girault and Raviart [27]),

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\| \le C(\beta) \inf_{\mathbf{v}^h \in \mathbf{X}^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\|.$$

Since (1.3) naturally occurs as a constraint associated with (1.2), well-posedness of the continuous reformulation depends upon another inf-sup condition, which we verify next.

Lemma 1.1.1. *(Inf-Sup Condition) The formulation (1.2), (1.3) satisfies following inf-sup condition:*

$$\inf_{\mathbb{L}\in\mathbf{L}} \sup_{(\mathbb{G},\mathbf{v})\in(\mathbf{L},\mathbf{X})} \frac{|\ (\nabla^s\mathbf{v}-\mathbb{G},\mathbb{L})\ |}{(\|\mathbb{G}\|^2+\|\nabla^s\mathbf{v}\|^2)^{\frac{1}{2}}\ \|\mathbb{L}\|} \geq 1. \tag{1.12}$$

Proof: Picking $\mathbf{v}=0$ and $\mathbb{G}=\mathbb{L}$ establishes the required inequality.

Corollary 1.1.1. *The continuous problem (1.2), (1.3) is equivalent to the usual variational formulation of the Navier-Stokes equations in* $(\mathbf{X}, Q)$.

We assume that the following approximation assumption, typical of piecewise polynomial velocity-pressure finite element spaces of degree $(k, k-1)$ holds : there is $k \geq 1$ such that for any $\mathbf{u} \in (H^{k+1}(\Omega))^d \cap \mathbf{X}$ and $p \in (H^k(\Omega) \cap Q)$:

$$\inf_{\mathbf{v}^h\in\mathbf{X}^h} \left\{\|\mathbf{u}-\mathbf{v}^h\| + h\|\nabla(\mathbf{u}-\mathbf{v}^h)\|\right\} \leq Ch^{k+1}\left\|\mathbf{u}\right\|_{k+1}, \tag{1.13}$$

$$\inf_{q^h\in Q^h} \|p-q^h\| \leq Ch^k \left\|p\right\|_k. \tag{1.14}$$

We often use the following inequalities.

Lemma 1.1.2. *(Young's inequality)*

$$ab \leq \frac{t}{p}a^p + \frac{t^{-q/p}}{q}b^q, \quad a,b,p,q,t \in \mathbb{R}, \quad \frac{1}{p}+\frac{1}{q}=1, \quad p,q \in (1,\infty), \quad t>0. \tag{1.15}$$

Lemma 1.1.3. *(Poincaré-Friedrichs) There is a* $C(\Omega) > 0$ *such that*

$$\|\mathbf{v}\|_{L^2} \leq C(\Omega)\|\nabla\mathbf{v}\|_{L^2}, \quad \forall\ \mathbf{v}\in\mathbf{X}. \tag{1.16}$$

Lemma 1.1.4. *(Korn's inequality) If* $\gamma(\mathbf{v})$ *is a semi-norm on* $L^2(\Omega)$ *which is a norm on the constants, then there is a* $C(\Omega) > 0$ *such that for*

$$\|\nabla\mathbf{v}\| \leq C(\Omega)[\gamma(\mathbf{v}) + \|\nabla^s\mathbf{v}\|],$$

for all $\mathbf{v}\in\mathbf{X}$.

As a consequence of Korn's inequality it follows that, taking $\gamma(\mathbf{v}) = \|\mathbf{v}\|$ then

$$\|\nabla \mathbf{v}\|_{L^2} \leq C \|\nabla^s \mathbf{v}\|_{L^2}, \tag{1.17}$$

for all $\mathbf{v} \in \mathbf{X}$. Thus, the norms $\|\nabla \mathbf{v}\|$ and $\|\nabla^s \mathbf{v}\|$ are equivalent.

Note that the skew symmetrized trilinear form $b(\mathbf{u}, \mathbf{u}, \mathbf{v})$ introduced in (1.4) has the following properties:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \text{ and } b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \text{ for all } \ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \tag{1.18}$$

Lemma 1.1.5. *Let $\Omega \subset \mathbb{R}^d$. Then,*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C \|\nabla^s \mathbf{u}\| \, \|\nabla^s \mathbf{v}\| \, \|\nabla^s \mathbf{w}\| \,, \tag{1.19}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$.

Lemma 1.1.6. *Let $\Omega \subset \mathbb{R}^3$,i.e.,$d = 3$. Then,*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C(\Omega) \sqrt{\|\mathbf{u}\| \, \|\nabla^s \mathbf{u}\|} \|\nabla^s \mathbf{v}\| \|\nabla^s \mathbf{w}\|,$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$.

Proof: By Lemma 2.1 p.12 of Temam [68]

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C(\Omega) \, \|\mathbf{u}\|_{1/2} \, \|\mathbf{v}\|_1 \, \|\mathbf{w}\|_1 \, .$$

The Poincaré-Friedrichs and Korn's inequality implies

$$\|\mathbf{v}\|_1 \leq C \|\nabla^s \mathbf{v}\|, \ \|\mathbf{w}\|_1 \leq C \|\nabla^s \mathbf{w}\|.$$

Lastly, an interpolation inequality implies

$$\|\mathbf{u}\|_{1/2} \leq C(\Omega) \, \|\mathbf{u}\|^{1/2} \, \|\mathbf{u}\|_1^{1/2} \leq C(\Omega) \, \|\mathbf{u}\|^{1/2} \, \|\nabla^s \mathbf{u}\|^{1/2}.$$

It will be important to introduce the notion of means/large scales and fluctuations/small scales that we use. We shall use over-bar and prime notation to denote large and small scales, respectively. If $\mathbf{X}^H \subset \mathbf{X}^h$, we can decompose $\mathbf{u}^h \in \mathbf{X}^h$ into discrete means and fine mesh fluctuations via

$$\mathbf{u}^h = \mathbf{u}^H + \mathbf{u}'_h, \tag{1.20}$$

for $\mathbf{u}^H \in \mathbf{X}^H$ and $\mathbf{u}' \in \mathbf{X}'_h$ where $\mathbf{X}^H$, $\mathbf{X}'_h$ denote the coarse mesh space and small scale space, respectively. The space for fluctuations $\mathbf{X}'_h$ and the decomposition (1.20) are determined by specifying how $\mathbf{u}^H \in \mathbf{X}^H$ is determined from $\mathbf{u}^h$.

Definition 1.1. *(Elliptic Projection) $P_E : \mathbf{X}^h \subset X \to \mathbf{X}^H$ is the projection operator satisfying*

$$(\nabla^s(\mathbf{u}^h - P_E\mathbf{u}^h), \nabla^s\mathbf{v}^H) = 0, \tag{1.21}$$

for all $\mathbf{v}^H \in \mathbf{X}^H$.

In Chapter 2 to present new method (1.5), (1.6) in a variational multiscale framework, we consider the following decomposition of deformation tensor. Let the spaces $\mathbf{L}^H \subset \mathbf{L}$, $\mathbf{X}^H \subset \mathbf{X}^h \subset \mathbf{X}$ as defined in Section 1. For the multiscale decomposition of the deformation tensor $\nabla^s\mathbf{u}^h$ set

$$\mathbb{D}(\mathbf{u}^h) = \nabla^s\mathbf{u}^h,$$

for all $\mathbf{v}^h \in \mathbf{X}^h$ and split

$$\mathbb{D}^h = \mathbb{D}^H + \mathbb{D}'_h$$

where $\mathbb{D}^H = P_{L^H}\mathbb{D}^h$ and $\mathbb{D}'_h = (I - P_{L^H})\mathbb{D}^h$.

Lemma 1.1.7. *Assume that $\mathbf{X}^H \subset \mathbf{X}^h$ and $\mathbf{L}^H = \nabla^s\mathbf{X}^H$. Then,*

$$(I - P_{L^H})\mathbb{D}(\mathbf{u}^h) = \mathbb{D}(I - P_E)\mathbf{u}^h \text{ and } P_{L^H}\mathbb{D}(\mathbf{u}^h) = \mathbb{D}(P_E\mathbf{u}^h),$$

where P_E is the elliptic projection operator into $\mathbf{X}^H$.

Proof: From $\mathbb{D}^H = P_{L^H}\mathbb{D}^h$, it follows that

$$(\mathbb{D}^h - \mathbb{D}^H, \mathbb{L}^H) = 0, \text{ for all } \mathbb{L}^H \in \mathbf{L}^H.$$

for all $\mathbb{L}^H \in \mathbf{L}^H$. Also, since $\mathbf{L}^H = \nabla^s \mathbf{X}^H$, $\mathbb{D}^H = \nabla^s \mathbf{w}^H$ for some $\mathbf{w}^H \in \mathbf{X}^H$. This yields:

$$(\nabla^s(\mathbf{u}^h - \mathbf{w}^H), \nabla^s \mathbf{v}^h) = 0, \tag{1.22}$$

for all $\mathbf{v}^h \in \mathbf{X}^H$. Then, consider the elliptic projector operator $P_E : \mathbf{X} \to \mathbf{X}^H$. From this map, define $P_E \mathbf{u}^h = \mathbf{w}^H$. Then, (1.22) can be written as:

$$(\nabla^s(\mathbf{u}^h - P_E \mathbf{u}^h), \nabla^s \mathbf{v}^h) = 0,$$

for all $\mathbf{v}^h \in \mathbf{X}^H$. This implies that $P_{L^H}\mathbb{D}(\mathbf{u}^h) = \nabla^s P_E \mathbf{u}^h$ and $(I - P_{L^H})\mathbb{D}(\mathbf{u}^h) = \nabla^s (I - P_E)\mathbf{u}^h$.

We shall assume that $\mathbf{L}^H = \nabla^s \mathbf{X}^H$, the condition of Lemma 1.1.7 holds and that the finite element spaces $\mathbf{X}^h$ and $\mathbf{X}^H$ satisfy an inverse-type inequality. Using Lemma 1.1.7, and applying the inverse estimate for $\mathbf{X}^H$, we have

$$\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\| = \|\mathbb{D}(P_E \boldsymbol{\phi}^h)\| \le C H^{-1} \|P_E \boldsymbol{\phi}^h\|_{L^2}, \tag{1.23}$$

We know want to find an estimate for the elliptic projection operator. By using the (1.21), for all $\boldsymbol{\phi}^h \in \mathbf{X}^h$ we can write

$$(\mathbb{D}(P_E \boldsymbol{\phi}^h), \mathbb{D}(\mathbf{v}^H)) = (\mathbb{D}(\boldsymbol{\phi}^h), \mathbb{D}(\mathbf{v}^H)), \tag{1.24}$$

for all $\mathbf{v}^H \in \mathbf{X}^H$. We claim that the following inequality holds for elliptic projection operator:

$$\|P_E \boldsymbol{\phi}^h\| \le C(\|\boldsymbol{\phi}^h\| + H\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|). \tag{1.25}$$

where $\boldsymbol{\phi}^H = P_E \boldsymbol{\phi}^h$. To show (1.25), we first define the following dual problem: given $\boldsymbol{\phi}^H \in L^2$, find $\boldsymbol{\psi} \in \mathbf{X}$

$$\begin{aligned} -\Delta \boldsymbol{\psi} &= \boldsymbol{\phi}^H \quad \text{in } \Omega \\ \boldsymbol{\psi} &= 0 \quad\ \ \text{on } \partial\Omega. \end{aligned} \tag{1.26}$$

Since the boundary $\partial\Omega$ is smooth enough, there exists a unique $\boldsymbol{\psi} \in \mathbf{X}$. We assume that the dual problem (1.26) is $H^2(\Omega)$ regular. This means that for any $\boldsymbol{\phi}^H \in L^2(\Omega)$ there exists a unique

$\boldsymbol{\psi} \in (\mathbf{X} \cap H^2(\Omega)^d)$ such that the following inequality holds

$$\|\boldsymbol{\psi}\|_2 \leq C\|\boldsymbol{\phi}^H\|. \tag{1.27}$$

Multiplying (1.26) with $\mathbf{v}^H \in \mathbf{X}^H$ and integrating over the domain yield

$$(-\Delta\boldsymbol{\psi}, \mathbf{v}^H) = (\boldsymbol{\phi}^H, \mathbf{v}^H).$$

Then, we set $\mathbf{v}^H = \boldsymbol{\phi}^H - \boldsymbol{\phi}^h$ in the last equation and applying Cauchy Schwarz and H^2 regularity. We obtain

$$\begin{aligned}
\|\boldsymbol{\phi}^H\|^2 - (\boldsymbol{\phi}^H, \boldsymbol{\phi}^h) &= (-\Delta\boldsymbol{\psi}, \boldsymbol{\phi}^H - \boldsymbol{\phi}^h) \\
\|\boldsymbol{\phi}^H\|^2 - (\boldsymbol{\phi}^H, \boldsymbol{\phi}^h) &\leq C\|\boldsymbol{\psi}\|_2\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\| \\
&\leq C\|\boldsymbol{\phi}^H\|\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|.
\end{aligned}$$

By applying Young's inequality, we get

$$\begin{aligned}
\|\boldsymbol{\phi}^H\|^2 &\leq (\boldsymbol{\phi}^H, \boldsymbol{\phi}^h) + C\|\boldsymbol{\phi}^H\|\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\| \\
&\leq \|\boldsymbol{\phi}^H\|\|\boldsymbol{\phi}^h\| + C\|\boldsymbol{\phi}^H\|\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\| \\
\|\boldsymbol{\phi}^H\|^2 &\leq C_1\|\boldsymbol{\phi}^h\|^2 + C_2\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|^2 \qquad (1.28)
\end{aligned}$$

where C_1, C_2 constant independent from H, h. Now, we need to estimate the term $\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|$ in (1.28). For this part we use the dual problem (1.26) as

$$\begin{aligned}
-\Delta\boldsymbol{\psi} &= \boldsymbol{\phi}^H - \boldsymbol{\phi}^h \quad \text{in } \Omega \\
\boldsymbol{\psi} &= 0 \qquad\qquad \text{on } \partial\Omega.
\end{aligned} \tag{1.29}$$

From H^2 regularity assumption, we now have

$$\|\boldsymbol{\psi}\|_2 \leq C\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|. \tag{1.30}$$

Then we use the definition of elliptic projection operator $(\mathbb{D}(\boldsymbol{\psi}^H), \mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)) = 0, \quad \forall \boldsymbol{\psi}^H \in \mathbf{X}^H$.

This gives

$$\begin{aligned}\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|^2 &= (-\Delta\boldsymbol{\psi}, \boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\\ &= (\nabla\boldsymbol{\psi}, \nabla(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h))\\ &= (\nabla(\boldsymbol{\psi} - \boldsymbol{\psi}^H), \nabla(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)).\end{aligned}$$

By using approximation results, Korn's inequality and H^2 regularity of (1.29), we get

$$\begin{aligned}\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|^2 &\leq CH\|\boldsymbol{\psi}\|_2\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|\\ &\leq CH\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\|\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|.\end{aligned}$$

Hence we get,

$$\|\boldsymbol{\phi}^H - \boldsymbol{\phi}^h\| \leq CH\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|.$$

Substituting this last result in (1.28) gives

$$\|P_E\boldsymbol{\phi}^h\| \leq C(\|\boldsymbol{\phi}^h\| + H\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|). \tag{1.31}$$

By using Lemma 1.1.7 and this final estimation in (1.23), we get

$$\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\| = \|\mathbb{D}(P_E\boldsymbol{\phi}^h)\| \leq CH^{-1}\|P_E\boldsymbol{\phi}^h\| \leq C(H^{-1}\|\boldsymbol{\phi}^h\| + \|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\|) \tag{1.32}$$

where $\|\mathbb{D}(\boldsymbol{\phi}^H - \boldsymbol{\phi}^h)\| = \|\mathbb{D}((I - P_E)\boldsymbol{\phi}^h)\| = \|(I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|$.

1.2 Error Estimations

1.2.1 Semi Discrete A Priori Error Estimation For Velocity

The first important question for the method (1.5), (1.6) is how the two scales and the eddy viscosity parameter h, H, ν_T should be coupled. The second important question for the method is the dependence of the error upon ν (i.e., Reynolds number). To answer the first question, this section considers the questions of stability, consistency and convergence of the method. We show convergence of the usual semi-discrete finite element approximation of the model as the mesh widths $h, H \to 0$ and give an estimate of the error. The error estimate reveals the correct coupling between h, H and ν_T. To answer the second question, we show the convergence analysis of the Oseen problem. Then, we show the error is uniform in Reynolds number.

The semi-discrete approximation is a map $\mathbf{u}^h : [0,T] \to \mathbf{V}^h$ satisfying

$$(\mathbf{u}^h_t, \mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + (2\nu \nabla^s \mathbf{u}^h, \nabla^s \mathbf{v}^h) + \nu_T((I - P_{L^H})\nabla^s \mathbf{u}^h, (I - P_{L^H})\nabla^s \mathbf{v}^h) = (f, \mathbf{v}^h), \tag{1.33}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ where $\mathbf{V}^h$ is the space of discretely divergence-free functions given by (1.11).

Proposition 1.2.1. *[Stability of method (1.5), (1.6)] The approximate solution of $\mathbf{u}^h$ of (1.5), (1.6), is stable. For any $t > 0$,*

$$\frac{1}{2}\|\mathbf{u}^h(t)\|^2 + \int_0^t (2\nu\|\nabla^s \mathbf{u}^h\|^2 + \nu_T\|(I - P_{L^H})\nabla^s \mathbf{u}^h\|^2)dt' \le \frac{1}{2}\|\mathbf{u}^h(0)\|^2 + \int_0^t (\mathbf{f}, \mathbf{u}^h)dt'.$$

In particular, $\sup_{0\le t\le T} \left\|\mathbf{u}^h\right\| \le C(f, \mathbf{u}_0)$.

Proof: Set $\mathbf{v}^h = \mathbf{u}^h$ in (1.33), use triangle inequality and the fact that $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{u}^h) = 0$:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^h\|^2 + 2\nu\|\nabla^s \mathbf{u}^h\|^2 + \nu_T\|(I - P_{L^H})\nabla^s \mathbf{u}^h\|^2 \le (\mathbf{f}, \mathbf{u}^h).$$

The result follows from integrating over $[0, t]$. Stability then easily follows by applying Cauchy-Schwarz inequality on the right hand side.

Corollary 1.2.1. *The solution $\mathbf{u}^h$ exist and is unique. If the discrete inf-sup condition (4.23) holds, then p^h exists and is unique.*

We note that by adding and subtracting terms, it is easy to see that the true solution $(\mathbf{u}, p)$ satisfies

$$\begin{aligned}(\mathbf{u}_t, \mathbf{v}^h) &+ b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - (p, \nabla \cdot \mathbf{v}^h) + (2\nu \nabla^s \mathbf{u}, \nabla^s \mathbf{v}^h) + (\nu_T(I - P_{L^H})\nabla^s \mathbf{u}, (I - P_{L^H})\nabla^s \mathbf{v}^h) \\ &= (\mathbf{f}, \mathbf{v}^h) + (\nu_T(I - P_{L^H})\nabla^s \mathbf{u}, (I - P_{L^H})\nabla^s \mathbf{v}^h),\end{aligned} \tag{1.34}$$

for all $\mathbf{v} \in \mathbf{V}^h$.

Theorem 1.1. *(Convergence) Suppose* $\nabla \mathbf{u} \in L^4(0,T;L^2(\Omega))$*. Then with* $C = C(\Omega)$*, there is a constant* $C^*(T) = \exp(C(1+\nu^{-3}) \|\nabla \mathbf{u}\|^4_{L^4(0,T;L^2(\Omega))})$ *such that*

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}^h\|^2_{L^\infty(0,T;L^2)} + C\nu\|\nabla^s(\mathbf{u}-\mathbf{u}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I-P_{L^H})\nabla^s(\mathbf{u}-\mathbf{u}^h)\|^2_{L^2(0,T;L^2)} \\
&\quad \leq \; C\|\mathbf{u}_0 - \mathbf{v}_0^h\|^2_{L^2(0,T;L^2)} + C^*(T) \inf_{\substack{\mathbf{v}^h \in \mathbf{X}^h \\ p^h \in Q^h}} \Big[(H^{-2}+\nu_T^{-1})\|(\mathbf{u}-\mathbf{v}^h)_t\|^2_{L^2(0,T,H^{-1})} \\
&\qquad +\nu\|\nabla^s(\mathbf{u}-\mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I-P_{L^H})\nabla^s(\mathbf{u}-\mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I-P_{L^H})\nabla^s\mathbf{u}\|^2_{L^2(0,T;L^2)} \\
&\qquad +(H^{-2}+\nu_T^{-1})\|\nabla^s(\mathbf{u}-\mathbf{v}^h)\|_{L^4(0,T;L^2)}\|\mathbf{u}-\mathbf{v}^h\|_{L^4(0,T;L^2)}(\|\nabla\mathbf{u}\|^2_{L^4(0,T;L^2)}+1) \\
&\qquad +\nu^{-1}\|p-q^h\|^2_{L^2(0,T;L^2)}\Big].
\end{aligned}$$

Proof: Let $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ and $\mathbf{v}^h \in \mathbf{V}^h$. Then, decompose the error into two parts: $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\phi}^h$, where $\boldsymbol{\eta} = \mathbf{u} - \mathbf{v}^h$ and $\boldsymbol{\phi}^h = \mathbf{u}^h - \mathbf{v}^h$. Let $\mathbf{v}^h \in \mathbf{V}^h$ denote an approximation of $\mathbf{u}$. Subtracting (1.33) from (1.34) yields,

$$\begin{aligned}
(\mathbf{e}_t, \mathbf{v}^h) \; &+ \; [b(\mathbf{u},\mathbf{u},\mathbf{v}^h) - b(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)] + (2\nu\nabla^s\mathbf{e}, \nabla^s\mathbf{v}^h) + (\nu_T(I-P_{L^H})\nabla^s\mathbf{e}, (I-P_{L^H})\nabla^s\mathbf{v}^h) \\
&= \; -(p, \nabla\cdot\mathbf{v}^h) + (\nu_T(I-P_{L^H})\nabla^s\mathbf{u}, (I-P_{L^H})\nabla^s\mathbf{v}^h). \qquad (1.35)
\end{aligned}$$

By using (1.53) and setting $\mathbf{v}^h = \boldsymbol{\phi}^h$, we obtain:

$$\begin{aligned}
&(\boldsymbol{\phi}^h_t, \boldsymbol{\phi}^h) + (2\nu\nabla^s\boldsymbol{\phi}^h, \nabla^s\boldsymbol{\phi}^h) + (\nu_T(I-P_{L^H})\nabla^s\boldsymbol{\phi}^h, (I-P_{L^H})\nabla^s\boldsymbol{\phi}^h) \\
&\quad = \; (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + [b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h) - b(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h)] + (2\nu\nabla^s\boldsymbol{\eta}, \nabla^s\boldsymbol{\phi}^h) \\
&\qquad +(\nu_T(I-P_{L^H})\nabla^s\boldsymbol{\eta}, (I-P_{L^H})\nabla^s\boldsymbol{\phi}^h) - (p-q^h, \nabla\cdot\boldsymbol{\phi}^h) \\
&\qquad +(\nu_T(I-P_{L^H})\nabla^s\mathbf{u}, (I-P_{L^H})\nabla^s\boldsymbol{\phi}^h). \qquad (1.36)
\end{aligned}$$

Note that, since $\boldsymbol{\phi}^h \in \mathbf{V}^h$, $(q^h, \nabla\cdot\boldsymbol{\phi}^h) = 0$, and we can write

$$(p, \nabla\cdot\boldsymbol{\phi}^h) = (p - q^h, \nabla\cdot\boldsymbol{\phi}^h),$$

for all $q^h \in Q^h$. We want to bound the terms on the right hand side of (1.36). Consider first the convection terms in (1.36). Adding and subtracting terms yields:

$$b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) = b(\mathbf{e}, \mathbf{u}, \boldsymbol{\phi}^h) + b(\mathbf{u}^h, \mathbf{e}, \boldsymbol{\phi}^h).$$

By writing $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\phi}^h$, and using skew-symmetry $b(\mathbf{u}^h, \boldsymbol{\phi}^h, \boldsymbol{\phi}^h) = 0$, this reduces to

$$b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) = b(\boldsymbol{\eta}, \mathbf{u}, \boldsymbol{\phi}^h) - b(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h) + b(\mathbf{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h). \tag{1.37}$$

We first bound the nonlinear terms on the right hand side of (1.37). After applying triangle inequality, the first nonlinear term is estimated by using the definition of nonlinear term, Cauchy Schwarz inequality, (1.32) and by using the following property of projection operator:

$$\begin{aligned}\|\nabla^s \boldsymbol{\phi}^h\| &\leq \|P_{L^H} \nabla^s \boldsymbol{\phi}^h\| + \|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\| \\ &\leq C(H^{-1}\|\boldsymbol{\phi}^h\| + \|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h)\|.\end{aligned} \tag{1.38}$$

Then we have

$$\begin{aligned} b(\boldsymbol{\eta}, \mathbf{u}, \boldsymbol{\phi}^h) &\leq C\|\nabla \mathbf{u}\|\|\boldsymbol{\eta}\|^{\frac{1}{2}}\|\nabla^s \boldsymbol{\eta}\|^{\frac{1}{2}}\|\nabla^s \boldsymbol{\phi}^h\| \\ &\leq C\|\nabla \mathbf{u}\|\|\boldsymbol{\eta}\|^{\frac{1}{2}}\|\nabla^s \boldsymbol{\eta}\|^{\frac{1}{2}}\left(H^{-1}\|\boldsymbol{\phi}^h\| + \|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|\right) \\ &\leq CH^{-2}\|\nabla \mathbf{u}\|^2\|\boldsymbol{\eta}\|\|\nabla \boldsymbol{\eta}\| + \|\boldsymbol{\phi}^h\|^2 + \frac{\nu_T}{4}\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2 + \nu_T^{-1}\|\nabla \mathbf{u}\|^2\|\boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\eta}\|. \end{aligned}$$

In order to bound the next nonlinear term, we use Lemma 1.1.6 and Young's inequality:

$$b(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h) \;\leq\; \|\nabla^s \boldsymbol{\phi}^h\|^{\frac{3}{2}}\|\boldsymbol{\phi}^h\|^{\frac{1}{2}}\|\nabla \mathbf{u}\| \leq \epsilon\|\nabla^s \boldsymbol{\phi}^h\|^2 + \frac{C}{\epsilon^3}\|\boldsymbol{\phi}^h\|^2\|\nabla \mathbf{u}\|^4.$$

The last nonlinear term on the right hand side of (1.37) is bounded by using Lemma 1.1.6, property of the projection operator, (1.38), (1.32) and Young's inequality.

$$\begin{aligned} b(\mathbf{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h) &\leq C\|\nabla \mathbf{u}^h\|^{\frac{1}{2}}\|\mathbf{u}^h\|^{\frac{1}{2}}\|\nabla^s \boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\phi}^h\| \\ &\leq (CH^{-2} + \nu_T^{-1})\|\nabla \mathbf{u}^h\|\|\mathbf{u}^h\|\|\nabla^s \boldsymbol{\eta}\| + \frac{1}{2}\|\boldsymbol{\phi}^h\|^2 + \frac{\nu_T}{8}\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2. \end{aligned}$$

The remaining terms in (1.36) are estimated by the Cauchy Schwarz, (1.38), (1.32), Young's, and Hölder inequalities as follows

$$
\begin{aligned}
\|(\boldsymbol{\eta}_t, \boldsymbol{\phi}^h)\| &\leq CH^{-2}\|\boldsymbol{\eta}_t\|_{-1}^2 + \|\boldsymbol{\phi}^h\|^2 + \frac{\nu_T}{4}\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\| \\
&\quad +4\nu_T^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2, \\
2\nu\|\nabla^s \boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\phi}^h\| &\leq 2\nu\|\nabla^s \boldsymbol{\eta}\|^2 + \frac{\nu}{2}\|\nabla^s \boldsymbol{\phi}^h\|^2, \\
\nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\eta}\|\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\| &\leq 4\nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\eta}\|^2 + \frac{\nu_T}{16}\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2, \\
\nu_T\|(I - P_{L^H})\nabla^s \mathbf{u}\|\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\| &\leq 4\nu_T\|(I - P_{L^H})\nabla^s \mathbf{u}\|^2 + \frac{\nu_T}{16}\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2, \\
\|(p - q^h, \nabla \cdot \boldsymbol{\phi}^h)\| &\leq \nu^{-1}\|p - q^h\|^2 + \frac{C\nu}{2}\|\nabla^s \boldsymbol{\phi}^h\|^2.
\end{aligned}
$$

Inserting these bounds into (1.36) and setting $\epsilon = \nu/2$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|^2 + \nu\|\nabla^s \boldsymbol{\phi}^h\|^2 + 7C\nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2 \\
\leq\ & C(H^{-2} + \nu_T^{-1})\|\boldsymbol{\eta}_t\|_{-1}^2 + C(H^{-2} + \nu_T^{-1})\|\nabla \mathbf{u}\|\|\boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\eta}\| \\
&+C(H^{-2} + \nu_T^{-1})\|\nabla^s \mathbf{u}^h\|^2\|\boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\eta}\| + C\nu\|\nabla^s \boldsymbol{\eta}\|^2 \\
&+C\nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\eta}\|^2 + C\nu_T\|(I - P_{L^H})\nabla \mathbf{u}\|^2 \\
&+\|\boldsymbol{\phi}^h\|^2 C(1 + \nu^{-3}\|\nabla \mathbf{u}\|^4) + \nu^{-1}\|p - q^h\|^2.
\end{aligned}
$$

Since by assumption $\nabla \mathbf{u} \in L^4(0, T; L^2(\Omega))$, Gronwall's inequality implies

$$
\begin{aligned}
&\max_{0 \leq t \leq T}\|\boldsymbol{\phi}^h\|^2 + C\int_0^T [2\nu\|\nabla^s \boldsymbol{\phi}^h\|^2 + C\nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\phi}^h\|^2]dt' \\
\leq\ & C^*(T)\|\boldsymbol{\phi}^h\|^2 + CC^*(T)\Big[\int_0^T (H^{-2} + \nu_T^{-1})\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu\|\nabla^s \boldsymbol{\eta}\|^2 + \nu_T\|(I - P_{L^H})\nabla^s \boldsymbol{\eta}\|^2 \\
&+\nu_T\|(I - P_{L^H})\nabla \mathbf{u}\|^2 + (H^{-2} + \nu_T^{-1})\|\boldsymbol{\eta}\|\|\nabla^s \boldsymbol{\eta}\|(\|\nabla \mathbf{u}\|^2 + \|\nabla \mathbf{u}^h\|^2)\Big]dt' \\
&+CC^*(T)\inf_{q^h \in Q^h}\int_0^T \nu^{-1}\|p - q^h\|^2 dt',
\end{aligned}
$$

where

$$C^*(T) = \exp(\int_0^T C(1+\nu^{-3})\|\nabla \mathbf{u}\|^4)dt'.$$

We can bound the remaining terms by using Cauchy-Schwarz inequality in $L^2(0,T)$ and Proposition 1.2.1. Thus,

$$\int_0^T \|\nabla \mathbf{u}\|^2 \|\boldsymbol{\eta}\| \|\nabla^s \boldsymbol{\eta}\| dt' \leq C \|\nabla \mathbf{u}\|^2_{L^4(0,T;L^2(\Omega))} \|\nabla^s \boldsymbol{\eta}\|_{L^4(0,T;L^2(\Omega))} \|\boldsymbol{\eta}\|_{L^4(0,T;L^2(\Omega))},$$

$$\int_0^T \|\nabla \mathbf{u}^h\|^2 \|\boldsymbol{\eta}\| \|\nabla^s \boldsymbol{\eta}\| dt' \leq C \|\nabla^s \boldsymbol{\eta}\|_{L^4(0,T;L^2(\Omega))} \|\boldsymbol{\eta}\|_{L^4(0,T;L^2(\Omega))}.$$

Applying the triangle inequality we have the infimum taken over only $\mathbf{V}^h$ instead of $\mathbf{X}^h$. Under the discrete inf-sup condition, it is known that if $\nabla \cdot \mathbf{u} = 0$ the infimum over $\mathbf{V}^h$ can be replaced by infimum taken over $\mathbf{X}^h$, (Girault and Raviart [27], Theorem 1.1, p.59). Thus the final result follows.

Remark 1.1. *For the error analysis, we assume regularity in time $\|\nabla u\| \in L^4(0,T)$ i.e. $\int_0^T \|\nabla u\|^4 dt < \infty$. With this assumption, it is known that the usual Leray [51] weak solution of the Navier-Stokes equations is unique, Ladyzhenskaya [47].*

The error estimates which are similar to Theorem 1.1 can be used as a guide to pick parameter scalings by balancing error terms in the case of smooth solutions. To illustrate this let us consider the Mini-element of Arnold, Brezzi and Fortin [3].

Corollary 1.2.2. *Suppose $\mathbf{u}, p$ are sufficiently smooth. Assume that $\Pi^h(\Omega)$ is obtained by refinement of $\Pi^H(\Omega)$ $(h << H)$. Let $\mathbf{X}^h, \mathbf{X}^H, \mathbf{L}^H := \nabla \mathbf{X}^H, Q^h$ denote a finite element space of linears plus bubble functions, piecewise linear on a coarser mesh width $H > h$, piecewise constants plus quadratics on coarser mesh, piecewise linear on a mesh width of h, respectively.*

With the choices

$$\nu_T \sim h, \;\; h \sim H^2,$$

the error in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is bounded by $C(\mathbf{u},\nu)h$.

Proof: By using approximation assumptions given Section 2,

$$\max_{0\le t\le T}\|\mathbf{u}-\mathbf{u}^h\|^2 + C(T)\int_0^T[\frac{\nu}{2}\|\nabla^s(\mathbf{u}-\mathbf{u}^h)\|^2+\frac{\nu_T}{4}\|(I-P_{L^H})\nabla^s\mathbf{u}\|^2]dt'$$
$$\le C^*(\mathbf{u},p)((H^{-2}+\nu_T^{-1})h^6+\nu h^2+\nu_T h^2+\nu_T H^2+H^{-2}h^3+\nu_T^{-1}h^3$$
$$+H^{-2}h^3+\nu^{-1}h^2).$$

We neglect higher order terms for balancing, i.e., $H^{-2}h^6, \nu_T^{-1}h^6, H^{-2}h^3$. It can be seen easily that with the natural choice of $\nu_T \sim h,\ H^2 \sim h$ the error is order h.

Remark 1.2. *If one considers Taylor-Hood element, with the choices*

$$\mathbf{X}^h := \left\{\mathbf{v}\in C^0(\bar\Omega) : \mathbf{v}\mid_\Delta \in P_2(\Delta),\ \textit{for all triangles}\ \Delta\ \textit{in}\,\Pi^h(\Omega)\right\},$$
$$\mathbf{X}^H := \left\{\mathbf{v}\in C^0(\bar\Omega) : \mathbf{v}\mid_\Delta \in P_2(\Delta),\ \textit{for all triangles}\,\Delta\ \textit{in}\,\Pi^H(\Omega)\right\},$$
$$\mathbf{L}^H := \left\{l^H\in L^2(\Omega)^d : l^H \mid_\Delta \in P_1(\Delta),\ \textit{for all triangles}\,\Delta \textit{in}\,\Pi^H(\Omega)\right\},$$
$$Q^h := \left\{\mathbf{v}\in C^0\bar\Omega : \mathbf{v}\mid_\Delta \in P_1(\Delta),\ \textit{for all triangles}\ \Delta\ \textit{in}\,\Pi^h(\Omega)\right\}.$$

$$\nu_T \sim h,\ \ h\sim H^2,$$

the error in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ *is bounded by* $C(\mathbf{u},\nu)h^2$.

1.2.2 Error Estimation For Pressure

In this section, we provide the L^2 error estimation for the pressure. We follow the pressure estimation of Girault and Raviart [27] for Navier-Stokes equations. The analysis of pressure requires the estimation of the derivative of velocity error. We first start with writing the error equation for the pressure and derive the estimation for the $\|(\mathbf{u}-\mathbf{u}^h)_t\|$.

It is easy to see that approximate solution $(\mathbf{u}^h,\mathbf{v}^h)$ satisfies the following equations:

$$(p^h,\nabla\cdot\mathbf{v}^h) = (\mathbf{u}_t^h,\mathbf{v}^h)+b(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)+(2\nu\nabla^s\mathbf{u}^h,\nabla^s\mathbf{v}^h)$$
$$+\nu_T((I-P_{L^H})\nabla^s\mathbf{u}^h,(I-P_{L^H})\nabla^s\mathbf{v}^h)+(\mathbf{f},\mathbf{v}^h), \qquad (1.39)$$

for all $\mathbf{v}^h\in\mathbf{X}^h$. Writing the equations (1.34) and (1.39) satisfied by p and p^h, respectively and setting $\sigma = p - p^h$ and $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$, we get the following equation for the pressure error:

$$
\begin{aligned}
(\sigma, \nabla \cdot \mathbf{v}^h) &= (\mathbf{e}_t, \mathbf{v}^h) + \nu(\nabla^s \mathbf{e}, \nabla^s \mathbf{v}^h) + \nu_T((I - P_{L^H})\nabla^s \mathbf{e}, (I - P_{L^H})\nabla^s \mathbf{v}^h) \\
&\quad + b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - \nu_T((I - P_{L^H})\nabla^s \mathbf{u}, (I - P_{L^H})\nabla^s \mathbf{v}^h). \quad (1.40)
\end{aligned}
$$

On the other hand, inf-sup condition (1.1.1) is equivalent to the following: for every $q^h \in Q^h$, there exists a nontrivial $\mathbf{v}^h \in \mathbf{X}^h$, such that

$$
(p^h - \tilde{p}, \nabla \cdot \mathbf{v}^h) \geq \beta \|\nabla \mathbf{v}^h\| \|p^h - \tilde{p}\|. \quad (1.41)
$$

In view of (1.41), we have

$$
\|\sigma\| \leq \|p - \tilde{p}\| + \beta^{-1} \frac{|(p^h - \tilde{p}, \nabla \cdot \mathbf{v}^h)|}{\|\nabla \mathbf{v}^h\|}, \quad (1.42)
$$

where $\tilde{p} \in Q^h$ approximation of p. Adding and subtracting $(\tilde{p}, \nabla \cdot \mathbf{v})$ into (1.40) gives

$$
\begin{gathered}
(p^h - \tilde{p}, \nabla \cdot \mathbf{v}^h) \leq C(\|p - \tilde{p}\| + \|\mathbf{e}_t\| + \nu \|\nabla^s \mathbf{e}\| \\
+ \nu_T \|(I - P_{L^H})\nabla^s \mathbf{e}\| + \nu_T \|(I - P_{L^H})\nabla^s \mathbf{u}\|) \|\nabla \mathbf{v}^h\| + |b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h)|.
\end{gathered}
$$

By using Lemma 1.1.5, the nonlinear terms can be rewritten as follows:

$$
\begin{aligned}
b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) &= -b(\mathbf{e}, \mathbf{e}, \mathbf{v}^h) + b(\mathbf{e}, \mathbf{u}, \mathbf{v}^h) + b(\mathbf{u}, \mathbf{e}, \mathbf{v}^h) \\
&\leq C(\|\nabla^s \mathbf{e}\| + \|\nabla \mathbf{u}\|) \|\nabla^s \mathbf{e}\| \|\nabla \mathbf{v}^h\|.
\end{aligned}
$$

Then (1.42) becomes

$$
\begin{aligned}
\|\sigma\| \leq C(&\|p - \tilde{p}\| + \|\mathbf{e}_t\| + \nu \|\nabla^s \mathbf{e}\| + \nu_T \|(I - P_{L^H})\nabla^s \mathbf{e}\| \\
&+ \nu_T \|(I - P_{L^H})\nabla^s \mathbf{u}\| + \|\nabla^s \mathbf{e}\|^2 + \|\nabla^s \mathbf{e}\| \|\nabla \mathbf{u}\|). \quad (1.43)
\end{aligned}
$$

To estimate the right hand side of (1.43), we need an estimation for $\|\mathbf{e}_t\|$. In order to get this estimation, let us define the following modified Stokes projection.

Definition 1.2. *(Modified Stokes Projection) Let* $(\mathbf{u}^S(t), p^S(t)) \in (\mathbf{X}^h, Q^h)$ *satisfy*

$$\nu(\nabla^s \mathbf{u}^S(t), \nabla^s \mathbf{v}^h) + \nu_T((I - P_{L^H})\nabla^s \mathbf{u}^S(t), (I - P_{L^H})\nabla^s \mathbf{v}^h) - (p^S, \nabla \cdot \mathbf{v}^h)$$
$$= \nu(\nabla^s \mathbf{u}(t), \nabla^s \mathbf{v}^h) + \nu_T((I - P_{L^H})\nabla^s \mathbf{u}, (I - P_{L^H})\nabla^s \mathbf{v}^h) - (p, \nabla \cdot \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \qquad (1.44)$$
$$(q^h, \nabla \cdot \mathbf{u}^S(t)) = 0, \quad \forall q^h \in Q^h. \qquad (1.45)$$

for all $(\mathbf{v}^h, q^h) \in (\mathbf{X}^h, Q^h)$.

We now derive estimates for the error $\mathbf{u}_t - \mathbf{u}_t^h$ and $p - p^h$. Let $(\mathbf{u}^S, p^S)$ satisfy (1.44) and (1.45) and define $\boldsymbol{\eta} = \mathbf{u} - \mathbf{u}^S$ and $\boldsymbol{\xi} = \mathbf{u}^h - \mathbf{u}^S$.

Theorem 1.2. *(Error Estimate for* $\|\mathbf{u}_t - \mathbf{u}_t^h\|$*) Assume that* $\mathbf{u} \in L^\infty(0, T; L^2)$ *and* $\nabla \mathbf{u} \in L^\infty(0, T; L^2)$*. Then there is a constant* $\tilde{C} = \exp(C(T)\nu^{-1})$ *such that the error* $\mathbf{u}_t - \mathbf{u}_t^h$ *satisfies for* $T \geq 0$

$$\|\mathbf{u}_t - \mathbf{u}_t^h\|^2_{L^2(0,T;L^2)} + 2\nu\|\nabla^s(\mathbf{u} - \mathbf{u}^h)\|^2_{L^2(0,T;L^2)} + \frac{\nu_T}{2}\|(I - P_{L^H})\nabla^s(\mathbf{u} - \mathbf{u}^h)\|^2_{L^2(0,T;L^2)}$$
$$\leq \tilde{C}\Big((\nu + \nu_T)\|\mathbf{u}_0 - \mathbf{u}_0^h\|^2_{L^2(0,T;L^2)} + \|(\mathbf{u} - \mathbf{v}^h)_t\|^2_{L^2(0,T;L^2)} + \|\nabla(\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)}$$
$$+ \nu_T^2 h^{-2} \|(I - P_{L^H})\nabla \mathbf{u}\|^2_{L^2(0,T;L^2)}\Big).$$

Proof: Adding and subtracting terms in (1.44)-(1.45) and setting $\mathbf{v}^h = \boldsymbol{\xi}_t$ result in

$$\begin{aligned}
&\|\boldsymbol{\xi}_t\|^2 + \nu\frac{d}{dt}\|\nabla^s\boldsymbol{\xi}\|^2 + \frac{\nu_T}{2}\|(I - P_{L^H})\nabla^s\boldsymbol{\xi}\|^2 \\
&\quad = (\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) - \nu_T((I - P_{L^H})\nabla^s\mathbf{u}, (I - P_{L^H})\nabla^s\boldsymbol{\xi}_t) + [b(\mathbf{u}, \mathbf{u}, \boldsymbol{\xi}_t) - b(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\xi}_t)] \\
&\quad = T_1 + T_2 + T_3. \qquad (1.46)
\end{aligned}$$

By using Cauchy Schwarz and Young's inequality, the first term is bounded by

$$T_1 \leq \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\|\boldsymbol{\eta}_t\|^2.$$

Also, an inverse inequality implies that the term T_2 can be estimated with

$$T_2 \leq \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\nu_T h^{-2}\|(I - P_{L^H})\nabla \mathbf{u}\|^2 .$$

Nonlinear terms can be rewritten as

$$\begin{aligned} b(\mathbf{u},\mathbf{u},\boldsymbol{\xi}_t) - b(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\xi}_t) &= b(\mathbf{u}-\mathbf{u}^S,\mathbf{u}^h,\boldsymbol{\xi}_t) - b(\boldsymbol{\xi},\mathbf{u}^h,\boldsymbol{\xi}_t) + b(\mathbf{u},\mathbf{u}-\mathbf{u}^h,\boldsymbol{\xi}_t) \\ &= b(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}_t) - b(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\xi}_t) + b(\boldsymbol{\xi},\mathbf{u},\boldsymbol{\xi}_t) - b(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\xi}_t) + b(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\xi}_t) - b(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\xi}_t) \\ &= T_{31} + T_{32} + T_{33} + T_{34} + T_{35} + T_{36}. \end{aligned} \tag{1.47}$$

We assume that $\boldsymbol{\xi} \in L^\infty((0,T)\times\Omega)$. We estimate each term in (1.47) separately. Using L^p bounds, we obtain:

$$\begin{aligned} T_{31} = (\boldsymbol{\xi}, \nabla^s\boldsymbol{\xi}, \boldsymbol{\xi}_t) &\leq \|\boldsymbol{\xi}\|_{L^\infty}\|\nabla^s\boldsymbol{\xi}\|\|\boldsymbol{\xi}_t\| \\ &= \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\|\nabla^s\boldsymbol{\xi}\|^2 . \end{aligned}$$

We use the integration by parts, inverse inequality, and L^p bounds to estimate T_{32}:

$$\begin{aligned} T_{32} = (\boldsymbol{\xi}, \nabla\boldsymbol{\eta}, \boldsymbol{\xi}_t) + \frac{1}{2}(\nabla\cdot\boldsymbol{\xi}, \boldsymbol{\xi}_t\cdot\boldsymbol{\eta}) &\leq \|\boldsymbol{\xi}\|_{L^\infty}\|\nabla^s\boldsymbol{\eta}\|\|\boldsymbol{\xi}_t\| + \frac{1}{2}\|\boldsymbol{\eta}\|_{L^4}\|\nabla^s\boldsymbol{\xi}\|\|\boldsymbol{\xi}_t\|_{L^4} \\ &\leq \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\|\nabla\boldsymbol{\eta}\|^2 + \frac{1}{2}h^{k+1/2}\|\boldsymbol{\eta}\|\|\nabla^s\boldsymbol{\xi}\| h^{-1/2}\|\boldsymbol{\xi}_t\| \\ &\leq \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\|\nabla^s\boldsymbol{\xi}\|^2 + C\|\nabla\boldsymbol{\eta}\|^2 . \end{aligned}$$

The next term T_{33} can be bounded by using integration by parts, Poincaré, Cauchy Schwarz and Young's inequality:

$$\begin{aligned} T_{33} &= (\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\xi}_t) + \frac{1}{2}(\nabla\cdot\boldsymbol{\xi}, \boldsymbol{\xi}_t\cdot\mathbf{u}) \\ &\leq \|\boldsymbol{\xi}\|\|\nabla\mathbf{u}\|_{L^\infty}\|\boldsymbol{\xi}_t\| + \frac{1}{2}\|\mathbf{u}\|_{L^\infty}\|\nabla^s\boldsymbol{\xi}\|\|\boldsymbol{\xi}_t\| \\ &\leq \frac{1}{16}\|\boldsymbol{\xi}_t\|^2 + C\|\nabla^s\boldsymbol{\xi}\|^2 . \end{aligned}$$

T_{34} is bounded like T_{32} :

$$
\begin{aligned}
T_{34} &= (\boldsymbol{\eta} \cdot \nabla \mathbf{u}^h, \boldsymbol{\xi}_t) + \frac{1}{2}(\nabla \cdot \boldsymbol{\eta}, \boldsymbol{\xi}_t \cdot \mathbf{u}^h) \\
&\le \|\boldsymbol{\eta}\|_{L^4} \|\nabla^s \mathbf{u}^h\|_{L^4} \|\boldsymbol{\xi}_t\| + \frac{1}{2}\|\mathbf{u}^h\|_{L^\infty} \|\nabla^s \boldsymbol{\eta}\| \|\boldsymbol{\xi}_t\| \\
&\le \frac{1}{16} \|\boldsymbol{\xi}_t\|^2 + C \|\nabla \boldsymbol{\eta}\|^2 .
\end{aligned}
$$

Note that the nonlinear term T_{35} and T_{36} are bounded exactly like T_{33}.

Combining all the bounds above and arranging terms, we obtain

$$
\begin{aligned}
&\frac{1}{2} \|\boldsymbol{\xi}_t\|^2 + \nu \frac{d}{dt} \|\nabla^s \boldsymbol{\xi}\|^2 + \frac{\nu_T}{2} \|(I - P_{L^H}) \nabla^s \boldsymbol{\xi}\|^2 \\
&\le C(\|\nabla \boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta}_t\|^2 + \|\nabla^s \boldsymbol{\eta}\|^2 + \nu_T^2 h^{-2} \|(I - P_{L^H}) \nabla \mathbf{u}\|^2).
\end{aligned}
$$

The application of Gronwall's inequality gives the required result.

Theorem 1.3. *(Pressure Estimation) Under the assumptions Theorem 1.1 and Theorem 1.2, the following estimate holds true*

$$
\begin{aligned}
\|p - p^h\|_{L^2(0,T;L^2)} \le C\|\mathbf{u}_0 - \mathbf{v}_0^h\|_{L^2(0,T;L^2)} &+ C^*(T) \inf_{\substack{\mathbf{v}^h \in \mathbf{X}^h \\ p^h \in Q^h}} \Big[(H^{-1} + \nu_T^{-1/2}) \|(\mathbf{u} - \mathbf{v}^h)_t\|_{L^2(0,T,H^{-1})} \\
&+ \nu^{1/2} \|\nabla^s (\mathbf{u} - \mathbf{v}^h)\|_{L^2(0,T;L^2)} + \nu_T \|(I - P_{L^H}) \nabla^s (\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} \\
&+ \nu_T^{1/2} \|(I - P_{L^H}) \nabla^s \mathbf{u}\|_{L^2(0,T;L^2)} + (H^{-1} + \nu_T^{-1/2}) \|\nabla^s (\mathbf{u} - \mathbf{v}^h)\|^{1/2}_{L^4(0,T;L^2)} \|\mathbf{u} - \mathbf{v}^h\|^{1/2}_{L^4(0,T;L^2)} \\
&\times (\|\nabla \mathbf{u}\|_{L^4(0,T;L^2)} + 1) + \nu^{-1/2} \|p - q^h\|_{L^2(0,T;L^2)} \Big] \\
&+ \tilde{C}((\nu + \nu_T)^{1/2} \|\mathbf{u}_0 - \mathbf{u}_0^h\|_{L^2(0,T;L^2)} + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(0,T;L^2)} + \|\nabla (\mathbf{u} - \mathbf{v}^h)\|_{L^2(0,T;L^2)} \\
&+ \nu_T h^{-1} \|(I - P_{L^H}) \nabla \mathbf{u}\|_{L^2(0,T;L^2)}) \\
C\|\mathbf{u}_0 - \mathbf{v}_0^h\|^2_{L^2(0,T;L^2)} &+ C^*(T) \inf_{\substack{\mathbf{v}^h \in \mathbf{X}^h \\ p^h \in Q^h}} \Big[(H^{-2} + \nu_T^{-1}) \|(\mathbf{u} - \mathbf{v}^h)_t\|^2_{L^2(0,T,H^{-1})} \\
&+ \nu \|\nabla^s (\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T \|(I - P_{L^H}) \nabla^s (\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} \\
&+ \nu_T \|(I - P_{L^H}) \nabla^s \mathbf{u}\|^2_{L^2(0,T;L^2)} + (H^{-2} + \nu_T^{-1}) \|\nabla^s (\mathbf{u} - \mathbf{v}^h)\|_{L^4(0,T;L^2)} \|\mathbf{u} - \mathbf{v}^h\|_{L^4(0,T;L^2)} \\
&\times (\|\nabla \mathbf{u}\|^2_{L^4(0,T;L^2)} + 1) + \nu^{-1} \|p - q^h\|^2_{L^2(0,T;L^2)} \Big],
\end{aligned}
$$

where $C^*(T) = \exp((7 + C\nu^{-3})\, \|\nabla \mathbf{u}\|^4_{L^4(0,T;L^2(\Omega))})$ *and* $\tilde{C} = \exp(CT\nu^{-1})$.

Proof: We want to bound (1.43). By using the Theorem 1.1 and Theorem 1.2 in (1.43). We obtain the required result.

1.3 Error Estimate For Velocity In L^2

We now give an error estimate in L^2 for the velocity by using the duality argument [33]. We first consider the linearized adjoint problem of the Navier-Stokes equations: given $\mathbf{g} \in L^2(\Omega)$, find $(\boldsymbol{\phi}, \boldsymbol{\chi}) \in (\mathbf{X}, Q)$ such that $\boldsymbol{\phi}(\tau) = 0$, and

$$\begin{aligned} -(\boldsymbol{\phi}_t, \mathbf{v}) + (2\nu \nabla^s \boldsymbol{\phi}, \nabla^s \mathbf{v}) + (\nu_T (I - P_{L^H}) \nabla^s \boldsymbol{\phi}, (I - P_{L^H}) \nabla^s \mathbf{v}) \\ + b(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}) + b(\mathbf{v}, \mathbf{u}, \boldsymbol{\phi}) + (\boldsymbol{\chi}, \nabla \cdot \mathbf{v}) - (q, \nabla \cdot \boldsymbol{\phi}) = (\mathbf{g}, \mathbf{v}). \end{aligned} \tag{1.48}$$

for all $(\mathbf{v}, q) \in (\mathbf{X}, Q)$. Since $(\mathbf{u}, p)$ is a nonsingular solution of (1) and the boundary $\partial\Omega$ is smooth enough, there exists a unique $(\boldsymbol{\phi}, \boldsymbol{\chi})$ to dual (1.48), [33] . We also assume that the linearized adjoint problem (1.48) is $H^2(\Omega)$ regular. This means that for any $\mathbf{g} \in L^2(\Omega)$ there exists a unique pair $(\boldsymbol{\phi}, \boldsymbol{\chi})$ in $(\mathbf{X} \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ such that the following inequality holds

$$\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\chi}\|_1 \le C\|\mathbf{g}\|. \tag{1.49}$$

The following error estimate is proved in [33]:

$$\|\boldsymbol{\phi}(0)\|_1^2 + \int_0^\tau \{\|\boldsymbol{\phi}\|_2^2 + \|\boldsymbol{\chi}\|_1\} dt \le C(\mathbf{u}_0, f) \int_0^\tau \|\mathbf{e}\|^2 \, dt. \tag{1.50}$$

Owing to (1.50), we now give the L^2 error estimate.

Theorem 1.4. *Assume that the assumptions of Theorem 1.1 and Theorem 1.3 hold and the solution of the dual problem (1.48) satisfies the stability estimates (1.49) and (1.50). Then, there exists a constant C, independent of* h, H, ν_T *such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;L^2(\Omega))} \le Ch^{k+1}(\|\mathbf{u}\|_{L^2(0,T;H^{k+1}(\Omega))} + \|p\|_{L^2(0,T;H^k(\Omega))}) \tag{1.51}$$

Proof: Choosing $\mathbf{v} = \mathbf{e} = \mathbf{u} - \mathbf{u}^h$, $q = p - p^h$ and $g = \mathbf{e}$ in (1.48) we have

$$\|\mathbf{e}\|^2 = -\frac{d}{dt}(\mathbf{e}, \boldsymbol{\phi}) + (\mathbf{e}_t, \boldsymbol{\phi}) + (2\nu\nabla^s\boldsymbol{\phi}, \nabla^s\mathbf{e})$$
$$+(\nu_T(I - P_{L^H})\nabla^s\boldsymbol{\phi}, (I - P_{L^H})\nabla^s\mathbf{e}) + b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}) + b(\mathbf{e}, \mathbf{u}, \boldsymbol{\phi})$$
$$+(\boldsymbol{\chi}, \nabla\mathbf{e}) - (p - p^h, \nabla\cdot\boldsymbol{\phi}). \tag{1.52}$$

Recall that we have the following error equation

$$(\mathbf{e}_t, \mathbf{v}^h) + [b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h)] + (2\nu\nabla^s\mathbf{e}, \nabla^s\mathbf{v}^h)$$
$$+(\nu_T(I - P_{L^H})\nabla^s\mathbf{e}, (I - P_{L^H})\nabla^s\mathbf{v}^h) - (p - p^h, \nabla\cdot\mathbf{v}^h)$$
$$+(q^h, \nabla\cdot(\mathbf{u} - \mathbf{u}^h) = (\nu_T(I - P_{L^H})\nabla^s\mathbf{u}, (I - P_{L^H})\nabla^s\mathbf{v}^h). \tag{1.53}$$

for all $(\mathbf{v}^h, q^h) \in \mathbf{X}^h \times Q^h$. Then, we choose $(\mathbf{v}^h, q^h) \approx (\tilde{\boldsymbol{\phi}}, \tilde{q})$ where $(\tilde{\boldsymbol{\phi}}, \tilde{q})$ are the best approximation of $(\boldsymbol{\phi}, \boldsymbol{\chi})$ in $(\mathbf{X}^h, Q^h)$. Note that we have the following estimation for these best approximations:

$$\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\|_1 \le Ch\|\boldsymbol{\phi}\|_2 \le Ch\left\|\mathbf{e}\right\|, \tag{1.54}$$
$$\|\boldsymbol{\chi} - \tilde{q}\|_1 \le Ch\|\boldsymbol{\chi}\|_1 \le Ch\left\|\mathbf{e}\right\|. \tag{1.55}$$

Subtracting (1.53) to right hand side of (1.52) results

$$\|\mathbf{e}\|^2 = -\frac{d}{dt}(\mathbf{e}, \boldsymbol{\phi}) + (\mathbf{e}_t, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}) + (2\nu\nabla^s(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \nabla^s\mathbf{e})$$
$$+(\nu_T(I - P_{L^H})\nabla^s(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), (I - P_{L^H})\nabla^s\mathbf{e}) + (\nu_T(I - P_{L^H})\nabla^s\mathbf{u}, (I - P_{L^H})\nabla^s\tilde{\boldsymbol{\phi}})$$
$$+b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}) + b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}) + b(\mathbf{e}, \mathbf{e}, \tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}) + b(\mathbf{e}, \mathbf{e}, \boldsymbol{\phi})$$
$$+(\boldsymbol{\chi} - \tilde{q}, \nabla\cdot\mathbf{e}) - (p - p^h, \nabla\cdot(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})).$$

We now integrate both sides $[0,\tau]$ and use $\boldsymbol{\phi}(\tau)=0$. This gives

$$\int_0^\tau \|\mathbf{e}\|^2\,dt \le (\boldsymbol{\phi}(0),\mathbf{e}(0)) + \int_0^\tau \Big\{(\mathbf{e}_t,\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}) + (2\nu\nabla^s(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}),\nabla^s\mathbf{e})$$
$$+(\nu_T(I-P_{L^H})\nabla^s(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}),(I-P_{L^H})\nabla^s\mathbf{e}) + (\nu_T(I-P_{L^H})\nabla^s\mathbf{u},(I-P_{L^H})\nabla^s\tilde{\boldsymbol{\phi}})$$
$$+b(\mathbf{u},\mathbf{e},\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}) + b(\mathbf{u},\mathbf{e},\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}) + b(\mathbf{e},\mathbf{e},\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}) + b(\mathbf{e},\mathbf{e},\boldsymbol{\phi})$$
$$(\boldsymbol{\chi}-\tilde{q},\nabla\cdot\mathbf{e}) - (p-p^h,\nabla\cdot(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}))\Big\}dt$$
$$= S_1+S_2+\cdots+S_{11}.$$

We first bound S_1, S_3 and S_{10} together. Using Cauchy Schwarz inequality and Young's inequality give

$$S_1+S_3+S_{10} \le \|\boldsymbol{\phi}(0)\|\,\|\mathbf{e}(0)\| + \int_0^\tau (\|2\nu\nabla^s(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}})\| + \|\boldsymbol{\chi}-\tilde{q}\|)\|\nabla^s\mathbf{e}\|dt.$$

Also, by using (1.54), (1.55) and (1.50)

$$S_1+S_3+S_{10} \le \varepsilon\,\|\boldsymbol{\phi}(0)\|^2 + \varepsilon\int_0^\tau (\|\boldsymbol{\phi}\|_2^2 + \|\boldsymbol{\chi}\|_1^2)dt$$
$$+\frac{1}{\varepsilon}\{\|\mathbf{e}(0)\|^2 + h^2\max\{C,\nu\}\int_0^\tau \|\nabla^s\mathbf{e}\|^2 dt\}.$$
$$\le \varepsilon C(\mathbf{u}_0,\mathbf{f})\int_0^\tau \|\mathbf{e}\|^2 dt + \frac{C(\nu)}{\varepsilon}\{\|\mathbf{e}(0)\|^2 + h^2\int_0^\tau \|\nabla^s\mathbf{e}\|^2 dt\}.$$

Using the same type of inequalities, approximation results and the stability bound (1.49) gives the following bound for S_2:

$$S_2 \le \int_0^\tau \|\mathbf{e}_t\|\,\|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|dt \le \frac{1}{\varepsilon}h^2\int_0^\tau \|\mathbf{e}_t\|^2\,dt + \varepsilon\int_0^\tau \|\mathbf{e}\|^2\,dt.$$

To bound the eddy viscosity term, we use $\|(I-P_{L^H})\| \le 1$ and (1.54). Hence, S_4 is bounded as

$$S_4 \le \int_0^\tau \nu_T\|(I-P_{L^H})\nabla^s(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}})\|\|(I-P_{L^H})\nabla^s\mathbf{e}\|dt$$
$$\le \varepsilon\int_0^\tau \|\mathbf{e}\|^2\,dt + \frac{1}{\varepsilon}\nu_T^2 h^2\int_0^\tau \|\nabla^s\mathbf{e}\|^2 dt.$$

Using approximation results, the consistency term S_5 can be estimated as follows

$$\begin{aligned}
S_5 &\leq \int_0^\tau \{\nu_T \, \|(I - P_{L^H})\nabla^s \mathbf{u}\| \, \|(I - P_{L^H})\nabla^s \tilde{\boldsymbol{\phi}}\|\} dt \\
&\leq \int_0^\tau \nu_T H^k |\mathbf{u}|_{k+1} H \|\nabla^s \tilde{\phi}\|_1 dt \leq \int_0^\tau C\nu_T H^{k+1} |\mathbf{u}|_{k+1} \|\tilde{\boldsymbol{\phi}}\|_2 \\
&\leq C \int_0^\tau \nu_T H^{k+1} |\mathbf{u}|_{k+1} (\|\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_2 + \|\boldsymbol{\phi}\|_2) dt \leq \int_0^\tau C\nu_T H^{k+1} |u|_{k+1} \|\phi\|_2 dt \\
&\leq \frac{1}{\varepsilon} \nu_T^2 H^{2k+2} \int_0^\tau \|\mathbf{u}\|_{k+1}^2 \, dt + \varepsilon \int_0^\tau \|\mathbf{e}\|^2 \, dt.
\end{aligned}$$

We then consider the nonlinear terms, S_6 and S_7, by using Lemma 1.1.6, (1.54) and Korn's inequality:

$$\begin{aligned}
S_6 + S_7 &\leq \int_0^\tau M \, \|\nabla \mathbf{u}\| \, \|\nabla \mathbf{e}\| \, \|\nabla(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})\| dt \\
&\leq \varepsilon \int_0^\tau \|\mathbf{e}\|^2 \, dt + \frac{C}{\varepsilon} h^2 \int_0^\tau \|\nabla \mathbf{u}\|^2 \, \|\nabla^s \mathbf{e}\|^2 \, dt.
\end{aligned}$$

The term S_8 can be estimated in the same way as above

$$S_8 \leq \int_0^\tau \|\nabla \mathbf{e}\|^2 \, \|\boldsymbol{\phi}\|_2 \, dt \leq \varepsilon \int_0^\tau \|\mathbf{e}\|^2 \, dt + \frac{1}{\varepsilon} h^2 \int_0^\tau \|\nabla^s \mathbf{e}\|^2 \, dt.$$

The estimate of the last nonlinear term S_9 follows from Korn's inequality and (1.49):

$$\begin{aligned}
S_9 &\leq \int_0^\tau \|\nabla \mathbf{e}\|^2 \, \|\nabla \boldsymbol{\phi}\| \, dt \leq C \int_0^\tau \|\nabla^s \mathbf{e}\|^2 \, \|\boldsymbol{\phi}\|_2 dt \\
&\leq C \int_0^\tau \|\nabla^s \mathbf{e}\|^2 \, \|\mathbf{e}\| dt \leq \varepsilon \int_0^\tau \|\mathbf{e}\|^2 \, dt + \frac{C}{\varepsilon} \int_0^\tau \|\nabla \mathbf{e}\|^4 \, dt.
\end{aligned}$$

Finally, S_{11} is bounded by using (1.54)

$$S_{11} \leq \int_0^\tau \|p - p^h\| \|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\| dt \leq \varepsilon \int_0^\tau \|\mathbf{e}\|^2 dt + \frac{1}{\varepsilon} h^2 \int_0^\tau \|p - p^h\|^2 dt.$$

Choosing ε sufficiently small and collecting all the estimates, we have

$$\int_0^\tau \|\mathbf{e}\|^2 \leq C\Big\{ \|\mathbf{e}(0)\|^2 + h^2 \int_0^\tau \|\nabla^s \mathbf{e}\|^2 \,(1+\nu_T^2 + \|\nabla \mathbf{u}\|^2)dt + h^2 \int_0^T \|\mathbf{e}_t\|^2 \, dt$$
$$+\nu_T^2 H^{2k+2} \int_0^\tau \|\mathbf{u}\|_{k+1}^2 \, dt + \int_0^\tau \|\nabla \mathbf{e}\|^4 \, dt + h^2 \int_0^\tau \|p - p^h\|^2 dt \Big\}.$$

Applying the Theorem 1.1 and Theorem 1.3 gives the required result.

In the next section, we investigate the Reynolds number dependence for the linearized Navier-Stokes equation called Oseen problem.

1.4 Reynolds Number Dependence for the Oseen Problem

Consider the solution of the time dependent linearized Navier-Stokes equations, i.e. the Oseen problem

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{in } [0,T]\times\partial\Omega, \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{1.56}$$

where $\mathbf{b}(x)$ is a smooth vector field with $\nabla\cdot\mathbf{b} = 0$ and $\mathbf{b} = 0$ on $\partial\Omega$. Simplifying the proof from the Navier-Stokes case, we show that the semi-discrete approximation of (1.56) is convergent. Indeed, the variational formulation of (1.56) is

$$(\mathbf{u}_t, \mathbf{v}) + (\nu\nabla\mathbf{u}, \nabla\mathbf{v}) + (\mathbf{b}\cdot\nabla\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) = (\mathbf{f}, \mathbf{v}). \tag{1.57}$$

for all $(\mathbf{v}, q) \in (\mathbf{X}, Q)$. We consider the new subgrid eddy viscosity method and the same finite element discretization as for the Navier-Stokes case. Thus, the semi discrete finite element approximation is a map $\mathbf{u}^h : [0,T] \to \mathbf{V}^h$ satisfying

$$(\mathbf{u}_t^h, \mathbf{v}^h) + (\nu\nabla\mathbf{u}^h, \nabla\mathbf{v}^h) + (\mathbf{b}\cdot\nabla\mathbf{u}^h, \mathbf{v}^h) + (\nu_T(I - P_{L^H})\nabla\mathbf{u}^h, (I - P_{L^H})\nabla\mathbf{v}^h) = (f, \mathbf{v}^h), \tag{1.58}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ where $\mathbf{V}^h$ is the space of discretely divergence-free functions. For error analysis, we need a second equation including $\mathbf{u}$. Multiply (1.56) by $\mathbf{v}^h \in \mathbf{V}^h$ and integrate over Ω. Rearrangements give for any $\mathbf{v}^h \in \mathbf{V}^h$

$$(\mathbf{u}_t, \mathbf{v}^h) + (\nu\nabla\mathbf{u}, \nabla\mathbf{v}^h) + (\mathbf{b}\cdot\nabla\mathbf{u}, \mathbf{v}) - (p - q^h, \nabla\cdot\mathbf{v}^h) \tag{1.59}$$

$$+(\nu_T(I - P_{L^H})\nabla\mathbf{u}, (I - P_{L^H})\nabla\mathbf{v}^h) = (\nu_T(I - P_{L^H})\nabla\mathbf{u}, (I - P_{L^H})\nabla\mathbf{v}^h) + (f, \mathbf{v}^h).$$

Theorem 1.5. *There is a constant $C^* = e^{C(t-T)}$ independent of ν such that*

$$\begin{aligned}
&\max_{0\le t\le T} \|\mathbf{u} - \mathbf{u}^h\|^2_{L^\infty(0,T;L^2)} + \nu\|\nabla(\mathbf{u} - \mathbf{u}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I - P_{L^H})\nabla(\mathbf{u} - \mathbf{u}^h)\|^2_{L^2(0,T;L^2)} \\
&\le \|\mathbf{u}_0 - \mathbf{v}^h_0\|^2_{L^2(0,T;L^2)} + C^* \inf_{\substack{\mathbf{v}^h\in\mathbf{X}^h\\ p^h\in Q^h}} [(H^{-2} + \nu_T^{-1})\|(\mathbf{u} - \mathbf{v}^h)_t\|^2_{L^2(0,T,H^{-1})} \\
&+ \nu\|\nabla(\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I - P_{L^H})\nabla(\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T\|(I - P_{L^H})\nabla\mathbf{u}\|^2_{L^2(0,T;L^2)} \\
&+ H^{-2}\|P_{L^H}\mathbf{b}(\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu_T^{-1}\|(I - P_{L^H})\mathbf{b}(\mathbf{u} - \mathbf{v}^h)\|^2_{L^2(0,T;L^2)} \\
&+ (H^{-2} + \nu_T^{-1})\|p - q^h\|^2_{L^2(0,T;L^2)}].
\end{aligned} \tag{1.60}$$

Proof: For the error analysis subtract (1.58) from (1.59) and set $e = \mathbf{u} - \mathbf{u}^h = \eta - \phi^h$ where $\eta = \mathbf{u} - \mathbf{v}^h$ and $\phi^h = \mathbf{u}^h - \mathbf{v}^h$. Following the same procedure in the proof of Theorem 1.1 gives the result.

Remark 1.3. *This result is important because it shows that the important the new method improves its robustness with respect to Reynolds number. The Oseen problem captures the convection terms contribution to the linearized problem but omits the reaction term. Thus, this error estimates shows that the new method ensures the uniformity in ν in the error contribution from the convection terms.*

To show the advantages of this new method we consider also the error in usual Galerkin formulation of Oseen problem ($\nu_T = 0$). The usual Galerkin variational formulation of Oseen problem is (1.57) and semi discrete approximation is

$$(\mathbf{u}^h_t, \mathbf{v}^h) - \nu(\nabla\mathbf{u}^h, \nabla\mathbf{v}^h) - (\mathbf{b}\cdot\nabla\mathbf{u}^h, \mathbf{v}^h) = (f, \mathbf{v}^h), \text{ for all } \mathbf{v}^h \in \mathbf{V}^h. \tag{1.61}$$

Then, if we subtract (1.61) from (1.57), we get the following theorem.

Theorem 1.6. *Let $\mathbf{u}^h$ be the usual Galerkin finite element approximation of the Oseen problem. Then, there is a constant C such that*

$$\max_{0\leq t\leq T} \|\mathbf{u}-\mathbf{u}^h\|^2_{L^\infty(0,T;L^2)} + \nu\|\nabla(\mathbf{u}-\mathbf{u}^h)\|^2_{L^2(0,T;L^2)} \leq C \inf_{\substack{\mathbf{v}^h\in\mathbf{X}^h \\ p^h\in Q^h}} \Big[\nu^{-1}\|(\mathbf{u}-\mathbf{v}^h)_t\|^2_{L^2(0,T,H^{-1})}$$

$$+\,\nu\|\nabla(\mathbf{u}-\mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu^{-1}\|\mathbf{b}(\mathbf{u}-\mathbf{v}^h)\|^2_{L^2(0,T;L^2)} + \nu^{-1}\|p-q^h\|^2_{L^2(0,T;L^2)}].$$

Remark 1.4. *If we compare Theorem 1.5 and Theorem 1.6, it is clear that the basic error estimate for the usual Galerkin formulation of Oseen problem depends badly on ν. In contrast, this new approach leads to an error estimate with error constants which are uniform in ν for the Oseen problem.*

Chapter 2

Connection with the Variational Multiscale Method

In this chapter, we give a slightly more general interpretation of Hughes's variational multiscale method (VMM) [35]. The VMM studied in this chapter is an extension of an approach for scalar convection diffusion equations which can be found in [44]. VMM approach has often noted fundamental conceptual restriction connected to the assumption that fluctuations cannot cross mesh lines. It seems this assumption does not reflect the behavior of small scales correctly. We show that a new consistently stabilized method of [43] fits into the framework of generalized variational multiscale methods as does another consistently stabilized method of [30]. In addition, this generalization removes the restriction of VMM. This chapter is organized as follows: Section 2.1 introduces the VMM method. Section 2.2 establishes the connection of VMM of Hughes and VMM of Chapter 1.

2.1 Introduction to Variational Multiscale Method (VMM)

The Variational Multiscale Method (VMM) is a finite element method for multiscale problem introduced by Hughes [35], which simultaneously discretizes coupled systems of both large and small scales. The usual semi discrete variational formulation of (1.1) is : find $\mathbf{u} : [0,T] \to \mathbf{X}$, $p : (0,T] \to Q$ satisfying

$$\begin{aligned} (\mathbf{u}_t, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) + (2\nu \nabla^s \mathbf{u}, \nabla^s \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \\ (q, \nabla \cdot \mathbf{u}) &= 0, \end{aligned} \tag{2.1}$$

for all $(\mathbf{v}, q) \in (\mathbf{X}, Q)$. The usual 1-scale semi-discrete finite element method for (2.1) arises by choosing appropriate finite dimensional subspaces $\mathbf{X}^h \subset \mathbf{X}$ and $Q^h \subset Q$ and determining $\mathbf{u}^h$ and p^h satisfying (2.1) restricted to $\mathbf{X}^h, Q^h$. In turbulent flows the effects of small scales velocity in $\mathbf{X} \setminus \mathbf{X}^h$ upon the large velocity $\mathbf{u}^h \in \mathbf{X}^h$ are widely thought to be critical for a realistic simulation.

Following Hughes [35], the velocity and pressure spaces are decomposed into means and fluctuations

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{X}' \ \text{ and } \ Q = \bar{Q} + Q' \tag{2.2}$$

where $\bar{\mathbf{X}} \subset \mathbf{X}$, $\bar{Q} \subset Q$ denote closed subspaces of large velocity and pressure scales. In this decomposition, $\bar{\mathbf{X}}$ is chosen as a standard finite element space, i.e. $\mathbf{X}^h$. Associated with $\bar{\mathbf{X}}$, $\bar{Q}$ is a projection operator: $\bar{P} : (\mathbf{X}, Q) \to (\bar{\mathbf{X}}, \bar{Q})$. The space of fluctuations $(\mathbf{X}', Q')$ is the complement of $\bar{\mathbf{X}}$, $\bar{Q}$ in $\mathbf{X}$, Q, respectively. Then $(\mathbf{X}', Q') := P'(\mathbf{X}, Q)$ where $P' = (I - \bar{P})$. By using (2.2), we write:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \ \text{ and } \ p = \bar{p} + p', \text{ where } (\bar{\mathbf{u}}, \bar{p}) = \bar{P}(\mathbf{u}, p), \ \ (\mathbf{u}', p') = P'(\mathbf{u}, p). \tag{2.3}$$

Inserting this decomposition into (2.1) and setting first $(\mathbf{v}, q) = (\bar{\mathbf{v}}, \bar{q})$ then $(\mathbf{v}, q) = (\mathbf{v}', q')$ gives the following coupled equations for $\bar{\mathbf{u}}$ and $\mathbf{u}'$:

$$(\bar{\mathbf{u}}_t + \mathbf{u}'_t, \bar{\mathbf{v}}) + b(\bar{\mathbf{u}} + \mathbf{u}', \bar{\mathbf{u}} + \mathbf{u}', \bar{\mathbf{v}}) + (2\nu\nabla^s(\bar{\mathbf{u}} + \mathbf{u}'), \nabla^s\bar{\mathbf{v}}) - (\bar{p} + p', \nabla \cdot \bar{\mathbf{v}}) = (\bar{f} + f', \bar{\mathbf{v}}), \tag{2.4}$$

$$(\bar{\mathbf{u}}_t + \mathbf{u}'_t, \mathbf{v}') + b(\bar{\mathbf{u}} + \mathbf{u}', \bar{\mathbf{u}} + \mathbf{u}', \mathbf{v}') + (2\nu\nabla^s(\bar{\mathbf{u}} + \mathbf{u}'), \nabla^s\mathbf{v}') - (\bar{p} + p', \nabla \cdot \mathbf{v}') = (\bar{f} + f', \mathbf{v}'), \tag{2.5}$$

for all $\bar{\mathbf{v}} \in \bar{\mathbf{X}}$ and for all $\mathbf{v}' \in \mathbf{X}'$. After rearranging terms, these coupled system yields

$$(\bar{\mathbf{u}}_t, \bar{\mathbf{v}}) + b(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \bar{\mathbf{v}}) - (2\nu\nabla^s\bar{\mathbf{u}}, \nabla^s\bar{\mathbf{v}}) - (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) - (\bar{f}, \bar{\mathbf{v}}) = (f', \bar{\mathbf{v}}) - a(\mathbf{u}', \bar{\mathbf{v}}) - (p', \nabla \cdot \bar{\mathbf{v}}), \tag{2.6}$$

where

$$a(\mathbf{u}', \bar{\mathbf{v}}) = (\mathbf{u}'_t, \bar{\mathbf{v}}) + b(\mathbf{u}', \mathbf{u}', \bar{\mathbf{v}}) + b(\bar{\mathbf{u}}, \mathbf{u}', \bar{\mathbf{v}}) + b(\mathbf{u}', \bar{\mathbf{u}}, \bar{\mathbf{v}}) + (2\nu\nabla^s\mathbf{u}', \nabla^s\bar{\mathbf{v}}),$$

and

$$(\mathbf{u}'_t, \mathbf{v}') + b(\mathbf{u}', \mathbf{u}', \mathbf{v}') - (2\nu\nabla^s\mathbf{u}', \nabla^s\mathbf{v}') - (p', \nabla \cdot \mathbf{v}') - (f', \mathbf{v}') = (\bar{f}, \mathbf{v}') - a(\bar{\mathbf{u}}, \mathbf{v}') - (\bar{p}, \nabla \cdot \mathbf{v}'), \tag{2.7}$$

where

$$a(\bar{\mathbf{u}}, \mathbf{v}') = (\bar{\mathbf{u}}_t, \mathbf{v}') + b(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}') + b(\mathbf{u}', \bar{\mathbf{u}}, \mathbf{v}') + b(\bar{\mathbf{u}}, \mathbf{u}', \mathbf{v}') + (2\nu\nabla^s\bar{\mathbf{u}}, \nabla^s\mathbf{v}').$$

A variational multiscale discretization from (2.6),(2.7) begins by selecting a finite dimensional subspace $\bar{\mathbf{X}} := \mathbf{X}^h$ for the approximate mean flow $\bar{\mathbf{u}} \in \bar{\mathbf{X}}, \bar{p} \in \bar{Q}$ and finite dimensional spaces

$\mathbf{X}'_h, Q'_h$ for the fluctuation. Note that ideally $\mathbf{X}'_h \subset \mathbf{X}'$ but usually $\mathbf{X}'_h \subsetneq \mathbf{X}'$.

Definition 2.1. *(Hughes et al., [35]) The VMM approximation to (2.1) is a pair* $(\bar{\mathbf{u}}, \bar{p}) : [0,T] \to (\bar{\mathbf{X}}, \bar{Q})$ *and* $(\mathbf{u}'_h, p'_h) : [0,T] \to (\mathbf{X}'_h, Q'_h)$ *satisfying* $\bar{\mathbf{u}}(0) = \bar{P}\mathbf{u}_0, \mathbf{u}'_h(0) = P'\mathbf{u}_0$ *and*

$$(\bar{\mathbf{u}}_t, \bar{\mathbf{v}}) + b(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \bar{\mathbf{v}}) + (2\nu\nabla^s\bar{\mathbf{u}}, \nabla^s\bar{\mathbf{v}}) - (\bar{p}, \nabla\cdot\bar{\mathbf{v}}) + (\bar{q}, \nabla\cdot\bar{\mathbf{u}}) - (\bar{f}, \bar{\mathbf{v}}) = (f', \bar{\mathbf{v}}) - a(\mathbf{u}'_h, \bar{\mathbf{v}}) + (p'_h, \nabla\cdot\bar{\mathbf{v}'_h}), \tag{2.8}$$

where

$$a(\mathbf{u}'_h, \bar{\mathbf{v}}) = (\mathbf{u}'_{h,t}, \bar{\mathbf{v}}) + b(\mathbf{u}'_h, \mathbf{u}'_h, \bar{\mathbf{v}}) + b(\bar{\mathbf{u}}, \mathbf{u}'_h, \bar{\mathbf{v}}) + b(\mathbf{u}'_h, \bar{\mathbf{u}}, \bar{\mathbf{v}}) + (2\nu\nabla^s\mathbf{u}'_h, \nabla^s\bar{\mathbf{v}}),$$

for all $\bar{\mathbf{v}} \in \bar{\mathbf{X}},\ \bar{q} \in \bar{Q}$ *and*

$$\begin{aligned}(\mathbf{u}'_{h,t}, \mathbf{v}'_h) &+ b(\mathbf{u}'_h, \mathbf{u}'_h, \mathbf{v}'_h) + (2\nu\nabla^s\mathbf{u}'_h, \nabla^s\mathbf{v}'_h) - (p', \nabla\cdot\mathbf{v}') + (q'_h, \nabla\cdot\mathbf{u}'_h)\\ &+ (\nu_T\nabla^s\mathbf{u}'_h, \nabla^s\mathbf{v}'_h) - (f', \mathbf{v}') = (\bar{f}, \mathbf{v}'_h) - a(\bar{\mathbf{u}}, \mathbf{v}'_h) + (\bar{p}, \nabla\cdot\mathbf{v}'_h),\end{aligned} \tag{2.9}$$

where

$$a(\bar{\mathbf{u}}, \mathbf{v}'_h) = (\bar{\mathbf{u}}_t, \mathbf{v}'_h) + b(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}'_h) + b(\mathbf{u}'_h, \bar{\mathbf{u}}, \mathbf{v}'_h) + b(\bar{\mathbf{u}}, \mathbf{u}'_h, \mathbf{v}'_h) + (2\nu\nabla^s\bar{\mathbf{u}}, \nabla^s\mathbf{v}'_h),$$

for all $\mathbf{v}'_h \in \mathbf{X}'_h,\ q'_h \in Q'_h$ *and where* ν_T *is the turbulent viscosity coefficient in a small scale artificial viscosity type stabilization term.*

Remark 2.1. *The work of Hughes (and co-workers) [35] has explored the choices*

- *the projection operator* $\bar{P}$ *as* L^2 *projection,*
- $(\bar{\mathbf{X}}, \bar{Q})$ = *standard velocity-pressure finite element spaces,* $(\mathbf{X}'_h, Q'_h)$ = *spaces of bubble functions vanishing on element edges,*
- ν_T = *a Smagorinsky eddy viscosity term.*

Remark 2.2. *Different choices of* $\mathbf{X}'_h$ *have been explored. In the case of periodic boundary conditions,* $\mathbf{X}'_h$ *is often constructed as the span of the next few exponentials [36]. For bounded domains,(motivated by Residual Free Bubble(RFB) [12] theory) the choices* $(\bar{\mathbf{X}}, \bar{Q})$ = *standard velocity-*

pressure finite element spaces, $(\mathbf{X}'_h, Q'_h)$ = *spaces of bubble functions vanishing on element edges have been explored. One important question which is actively being investigated is how rich the space* $\mathbf{X}'_h$ *must be.*

The VMM has been developed and connected with the other popular methods. The choice of bubble functions to model fluctuations imposes the following assumption: *the small scales exist only in the interior of element boundaries of the element domains.* This assumption leads to localizing calculations for the small scales in the sense that the problems are elementwise uncoupled. This choice of $(\mathbf{X}'_h, Q'_h)$, is made for the same reason as in RFB methods (Hughes [35]). Thus with VMMs, the small scales' equation can be solved element by element and inserted into large scales' equation. The introduced terms then closely represent the effect of the modelled small scales on the large scales.

Definition 2.2. *(Generalized VMM) A generalized VMM is determined by alternate choices of*

- *projection operator* $\bar{P}$ *defining large scales,*
- *the discrete spaces* $\bar{\mathbf{X}}$, $\bar{Q}$, $\mathbf{X}'_h$, Q'_h *where* $\mathbf{X}'_h$, Q'_h *are the complements of* $\bar{\mathbf{X}}$, $\bar{Q}$*, respectively,*
- *the small scale stabilization term*

$$(\nu_T \nabla^s \mathbf{u}'_h, \nabla^s \mathbf{v}'_h),$$

is added on the right hand side of (2.9)

2.2 Connection with the new approach and VMM

One contribution of this work is to show that (1.5), (1.6) is a suitably generalized VMM in the sense of Definition 2.2. To present this we use Lemma 1.1.7.

Remark 2.3. *Lemma 1.1.7 shows that the natural definition of means of* $\nabla^s \mathbf{u}$ *is by* L^2 *projection and means of* $\mathbf{u}$ *is by elliptic projection.*

Remark 2.4. *The VMM discretization of the incompressibility constraint is:*

$$\begin{aligned} (\nabla \cdot (\mathbf{u}^H + \mathbf{u}'_h), q^H) &= 0, \\ (\nabla \cdot (\mathbf{u}^H + \mathbf{u}'_h), q'_h) &= 0, \end{aligned} \tag{2.10}$$

for all $q^H \in Q^H$ and for all $q'_h \in Q'_h$. At this point an algorithmic choice must be made. If $\mathbf{u}^h = \mathbf{u}^H + \mathbf{u}'_h$ is to be discretely div-free with respect to Q^h then (2.10) is imposed (as stated) as a coupled system. In this case $\mathbf{u}^H$ may not be discretely div-free with respect to Q^H. This is the choice we make herein; it leads to the use of the elliptic projection operator P_E. If $\mathbf{u}^H$ is to be discretely div-free with respect to Q^H, then (2.10) uncouples and $\mathbf{u}'_h$ will not (in general) be discretely div-free with respect to Q'_h nor will $\mathbf{u}^h$ be discretely div-free with respect to Q^h. With this choice P_E should be replaced in our error analysis by the discrete Stokes projection. This issue does not arise in spectral discretization of periodic problem since spectral basis functions are chosen to be exactly div-free.

Theorem 2.1. *Assume that $\mathbf{X}^H \subset \mathbf{X}^h$ and $\mathbf{L}^H = \nabla \mathbf{X}^H$. Then (1.5),(1.6) is a generalized VMM wherein:*

- *the projector operator $P_E : \mathbf{X}^h \to \mathbf{X}^H$ a elliptic projection operator function. Thus, $\mathbf{u}^H = P_E \mathbf{u}^h$.*
- *$\mathbf{X}'_h$, is the complement of $\mathbf{X}^H$, in $\mathbf{X}^h$, respectively, given by $\mathbf{X}'_h = (I - P_E)\mathbf{X}^h$. Thus, $\mathbf{u}'_h = (I - P_E)\mathbf{u}^h$.*
- *the stabilization term is given by $(\nu_T \nabla^s \mathbf{u}'_h, \nabla^s \mathbf{v}'_h)$.*

Proof: From Lemma 1.1.7, the extra term in equation (1.9) can be written as

$$(\nu_T (I - P_{L^H})(\nabla^s \mathbf{u}^h), (I - P_{L^H})(\nabla^s \mathbf{v}^h)) = (\nu_T (\nabla^s \mathbf{u}'_h, \nabla^s \mathbf{v}'_h)).$$

Thus, the new method is equivalent to: find $(\mathbf{u}^h, p^h, g^H) \in (\mathbf{X}^h, Q^h, \mathbf{L}^H)$ satisfying

$$\begin{aligned}(\mathbf{u}^h_t, \mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot \mathbf{u}^h) + (2\nu \nabla^s \mathbf{u}^h, \nabla^s \mathbf{v}^h) \\ + (\nu_T \nabla^s \mathbf{u}'_h, \nabla^s \mathbf{v}'_h) = (f, \mathbf{v}^h),\end{aligned} \tag{2.11}$$

for all $\mathbf{v}^h \in \mathbf{X}^h$ and $q^h \in Q^h$. In equation (2.11), write $\mathbf{u}^h = \mathbf{u}^H + \mathbf{u}'_h$, and set alternately, $\mathbf{v}^h = \mathbf{v}^H$ and $\mathbf{v}^h = \mathbf{v}'_h$. Similarly, as in (2.8), (2.9), this gives the following coupled equations:

$$(\mathbf{u}_t^H, \mathbf{v}^H) + b(\mathbf{u}^H, \mathbf{u}^H, \mathbf{v}^H) + (2\nu\nabla^s\mathbf{u}^H, \nabla^s\mathbf{v}^H) - (p^H, \nabla\cdot\mathbf{v}^H)$$
$$+ (q^H, \nabla\cdot\mathbf{u}^H) - (f^H, \mathbf{v}^H) = (f', \mathbf{v}^H) - a_1(\mathbf{u}_h', \mathbf{v}^H), \text{ for all } \mathbf{v}^H \in \mathbf{X}^H,\ q^H \in Q^H, \tag{2.12}$$

where

$$a_1(\mathbf{u}_h', \mathbf{v}^H) = (\mathbf{u}_{h,t}', \mathbf{v}^H) + b(\mathbf{u}_h', \mathbf{u}_h', \mathbf{v}^H) + b(\mathbf{u}^H, \mathbf{u}_h', \mathbf{v}^H) + b(\mathbf{u}_h', \mathbf{u}^H, \mathbf{v}^H) - (p_h', \nabla\cdot\mathbf{v}^H) + (2\nu\nabla^s\mathbf{u}_h', \nabla^s\mathbf{v}^H),$$

and

$$(\mathbf{u}_{h,t}', \mathbf{v}_h') + b(\mathbf{u}_h', \mathbf{u}_h', \mathbf{v}_h') - (2\nu\nabla^s\mathbf{u}_h', \nabla^s\mathbf{v}_h') - (p', \nabla\cdot\mathbf{v}') + (q_h', \nabla\cdot\mathbf{u}_h')$$
$$+ (\nu_T\nabla^s\mathbf{u}_h', \nabla^s\mathbf{v}_h') - (f', \mathbf{v}') = (f^H, \mathbf{v}_h') - a_2(\mathbf{u}^H, \mathbf{v}_h'),\ \text{for all } \mathbf{v}_h' \in \mathbf{X}_h',\ q_h' \in Q_h', \tag{2.13}$$

where

$$a_2(\mathbf{u}^H, \mathbf{v}_h') = (\mathbf{u}_t^H, \mathbf{v}_h') + b(\mathbf{u}^H, \mathbf{u}^H, \mathbf{v}_h') + b(\mathbf{u}_h', \mathbf{u}^H, \mathbf{v}_h') + b(\mathbf{u}^H, \mathbf{u}_h', \mathbf{v}_h') - (p^H, \nabla\cdot\mathbf{v}_h') + (2\nu\nabla^s\mathbf{u}^H, \nabla^s\mathbf{v}_h').$$

As noted before, the Hughes' VMM uses the assumption that fluctuations vanish identically on the boundaries of the element. This seems to unlikely reflect the behavior of the small scales. The VMM studied in this thesis does not have this restriction. In the new VMM, (1.5), (1.6), fluctuations are allowed to be nonzero across boundaries. Indeed, let us choose the span of linear basis function with vertex in Π^H, for coarse mesh Π^H and the span of linear basis function with vertex in Π^h not in Π^H for the complement of $\mathbf{X}^H$ i.e. for the fluctuations. It is clear that with this new VMM's approach, fluctuations can be **nonzero across mesh edges**.

Chapter 3

Numerical Experiments

The objective of this chapter is to provide a careful numerical assessment of the model introduced in Chapter 1. In numerical tests, we consider the steady state Navier-Stokes equations.

We first describe the algorithm used for handling the nonlinearity and the subgrid eddy viscosity term. We then present two numerical examples: one with a known analytical solution that allows for a numerical study of the convergence rates; and one benchmark problem, driven cavity. In both cases, the mini-element spaces ($k = 1$) are used.

3.0.1 Algorithm

To solve the nonlinear system a Newton method is used. Given $(\mathbf{u}^{m-1}, p^{m-1})$, we find $(\mathbf{u}^m, p^m)$ satisfying

$$\begin{aligned}(2\nu\nabla^s\mathbf{u}^m, \nabla^s\mathbf{v}^h) + \frac{1}{2}b(\mathbf{u}^{m-1}, \mathbf{u}^m, \mathbf{v}^h) + \frac{1}{2}b(\mathbf{u}^m, \mathbf{u}^{m-1}, \mathbf{v}^h) - \frac{1}{2}b(\mathbf{u}^{m-1}, \mathbf{v}^h, \mathbf{u}^m)\\ -\frac{1}{2}b(\mathbf{u}^m, \mathbf{v}^h, \mathbf{u}^{m-1}) - (p^m, \nabla\cdot\mathbf{v}^h)\\ = (\mathbf{f}, \mathbf{v}^h) + \frac{1}{2}b(\mathbf{u}^{m-1}, \mathbf{u}^{m-1}, \mathbf{v}^h) - \frac{1}{2}b(\mathbf{u}^{m-1}, \mathbf{v}^h, \mathbf{u}^{m-1}) - (\nu_T(I - P_{L^H})\nabla^s\mathbf{u}^{m-1}, (I - P_{L^H})\nabla^s\mathbf{v}^h))\end{aligned} \tag{3.1}$$

$$(q^h, \nabla\cdot\mathbf{u}^m) = 0,$$

for all $(\mathbf{v}^h, q^h) \in (\mathbf{X}^h, Q^h)$. This algorithm leads to a linear system of the form $Ax = b$ with A nonsymmetric. To solve this linear system we use the iterative conjugate gradient squared method of [66]. The stopping criteria of this Newton method is based on the absolute residual.

We now show that the extra stabilization term $(\nu_T(I - P_{L^H})(\nabla^s\mathbf{u}^{m-1}), (I - P_{L^H})\nabla^s\mathbf{v}^h))$ requires a modification of the right-hand side of the linear system, that can be computed locally.

First, the extra stabilization term can be rewritten

$$(\nu_T(I - P_{L^H})\nabla^s\mathbf{u}^{m-1}, (I - P_{L^H})\nabla^s\mathbf{v}^h)) = \nu_T(\nabla^s\mathbf{u}^{m-1}, \nabla^s\mathbf{v}^h) - \nu_T(P_{L^H}\nabla^s\mathbf{u}^{m-1}, \nabla^s\mathbf{v}^h).$$

In this decomposition, adding the first term is straight-forward, as it is similar to the diffusive term $(2\nu\nabla^s\mathbf{u}^{m-1}, \mathbf{v}^h)$. The difficulty is to incorporate the second term, since it couples coarse and fine meshes. Denoting a basis of X^h by $\{\phi_j^h\}_{j=1}^{N^h}$, we want to compute $(P_{L^H}\nabla^s\mathbf{u}^{m-1}, \nabla^s\phi_j^h)$, for all j. Denoting a basis of $\mathbf{L}^H$ by $\{\psi_j^H\}_{j=1}^{N^H}$, we can write

$$P_{L^H}\nabla^s\mathbf{u}^{m-1} = \sum_{j=1}^{N^H}\beta_j\psi_j^H, \tag{3.2}$$

where the β_j's are unknown coefficients, uniquely defined. Thus, we have

$$(P_{L^H}\nabla^s\mathbf{u}^{m-1}, \nabla^s\phi_i^h)_i = (\sum_{j=1}^{N^H}\beta_j\psi_j^H, \nabla^s\phi_i^h)_i = R\begin{bmatrix}\beta_1\\ \beta_2\\ ...\\ \beta_{N^H}\end{bmatrix},$$

where R is the matrix that couples the fine and large scales: $R_{ij} = (\psi_j^H, \nabla^s\phi_i^h)$. To determine the unknown coefficients β_j's, it suffices to take the inner product to both sides of (3.2) with ψ_i^H:

$$\begin{bmatrix}\beta_1\\ \beta_2\\ ...\\ \beta_{N^H}\end{bmatrix} = S^{-1}(P_{L^H}\nabla^s\mathbf{u}^{m-1}, \psi_i^H) = S^{-1}(\nabla^s\mathbf{u}^{m-1}, \psi_i^H), \tag{3.3}$$

where $S = (\psi_j^H, \psi_i^H)$ is the mass matrix associated to $\mathbf{L}^H$. Thus, we have so far

$$(P_{L^H}\nabla^s\mathbf{u}^{m-1}, \nabla^s\phi_i^h)_i = RS^{-1}(\nabla^s\mathbf{u}^{m-1}, \psi_i^H)_i. \tag{3.4}$$

To conclude, we decompose $\nabla^s\mathbf{u}^{m-1}$ as

$$\nabla^s\mathbf{u}^{m-1} = \sum_{j=1}^{N^h}\alpha_j^{m-1}\nabla^s\phi_j^h,$$

and substitute this into (3.4):

$$(P_{L^H}\nabla^s\mathbf{u}^{m-1},\nabla^s\phi_i^h)_i = RS^{-1}R^T\begin{bmatrix}\alpha_1^{m-1}\\ \alpha_2^{m-1}\\ \cdots\\ \alpha_{N^h}^{m-1}\end{bmatrix}.$$

Since the α_j^{m-1}'s are known, it suffices to compute R and S. We note that if one chooses discontinuous piecewise polynomial basis functions for $\mathbf{L}^H$, the matrix S is the block diagonal and then computing $RS^{-1}R^T$ can be done locally on each element in the coarse mesh Π^H.

3.0.2 Convergence Rates

We consider the equation (1) on the domain $\Omega = [0,1]\times[0,1]$, with a body force obtained such that the true solution is given by $\mathbf{u} = (\mathbf{u}_1,\mathbf{u}_2)$,

$$\mathbf{u}_1 = 2x^2(x-1)^2y(y-1)(2y-1),\quad \mathbf{u}_2 = -y^2(y-1)^2 2x(x-1)(2x-1),$$
$$p = y.$$

The fluid viscosity is $\nu = 10^{-2}$, which gives a Reynolds number of the order 10^2. From Corollary 1.2.2 we choose $\nu_T = h$ and H such that $H^2 \leq h$. The theoretical analysis then predicts a convergence rate of $\mathcal{O}(h)$ for the velocity in the energy norm, $\mathcal{O}(h^2)$ for the velocity in the L^2 norm, and $\mathcal{O}(h)$ for the pressure. The domain is subdivided into triangles. First, the coarse mesh is chosen such that $H = 1/2$ and the fine mesh is a refinement of the coarse mesh, so that $h = 1/4$ (here, $h = H^2$). Other pairs of meshes are obtained by successive uniform refinements (see Figure 3.1 for the case $H = 1/8$ and $h = 1/16$). We choose for basis functions of L^H, discontinuous piecewise constants and two quadratics defined the reference elements. If $\mathcal{F}$ denote the affine mapping from the reference element to the physical element, we have:

$$\mathbf{L}^H = \{\mathbb{L} : \mathbb{L}|_E = \mathcal{F}\hat{\mathbb{L}},\ \forall\,\hat{\mathbb{L}}\in\hat{L}^H,\ \forall E\in\tau^H\},$$

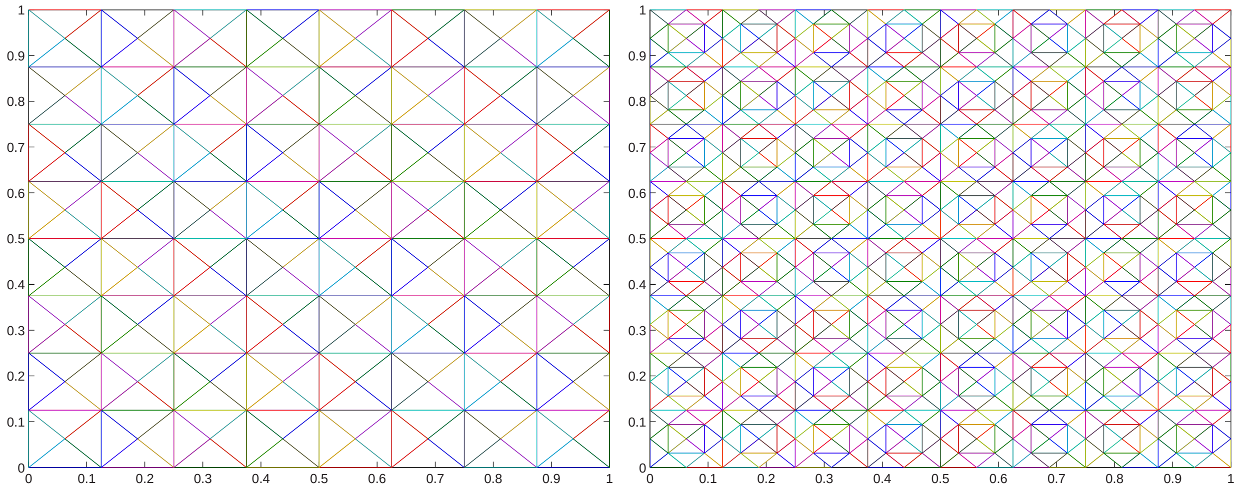

Figure 3.1. $H = 1/8$ with one refinement $h = 1/16$.

Table 3.1. Numerical errors and degrees of freedom..

meshes	N^h	L^2	Rate	H_0^1	Rate	L^2 pressure	Rate
H=1/2, h=1/4	218	0.0069		0.0509		4.3269e-04	
H=1/4, h=1/8	882	0.0017	2.0211	0.0241	1.0786	2.4448e-04	0.8236
H=1/8, h=1/16	3554	3.9446e-04	2.1076	0.0108	1.1580	9.6978e-05	1.3340
H=1/16, h=1/32	14274	8.1066e-05	2.2827	0.0046	1.2313	3.3879e-05	1.5173
H=1/32, h=1/64	57218	1.6313e-05	2.3131	0.0020	1.2016	1.1026e-05	1.6195

$$\hat{L}^H = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\partial b}{\partial x} & \frac{1}{2}\frac{\partial b}{\partial y} \\ \frac{1}{2}\frac{\partial b}{\partial y} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial b}{\partial x} \\ \frac{1}{2}\frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix} \right\},$$

where b denotes the bubble function defined as

$$b(x, y) = 27xy(1 - x - y).$$

Table 3.1 gives the errors and convergence rates for $\mathbf{u} - \mathbf{u}^h$ and $p - p^h$ in different norms. These numerical results demonstrate that the rates are optimal and better than theory predict. Figure 3.2 shows both computed solution and exact solution for the case $(H, h) = (1/8, 1/16)$.

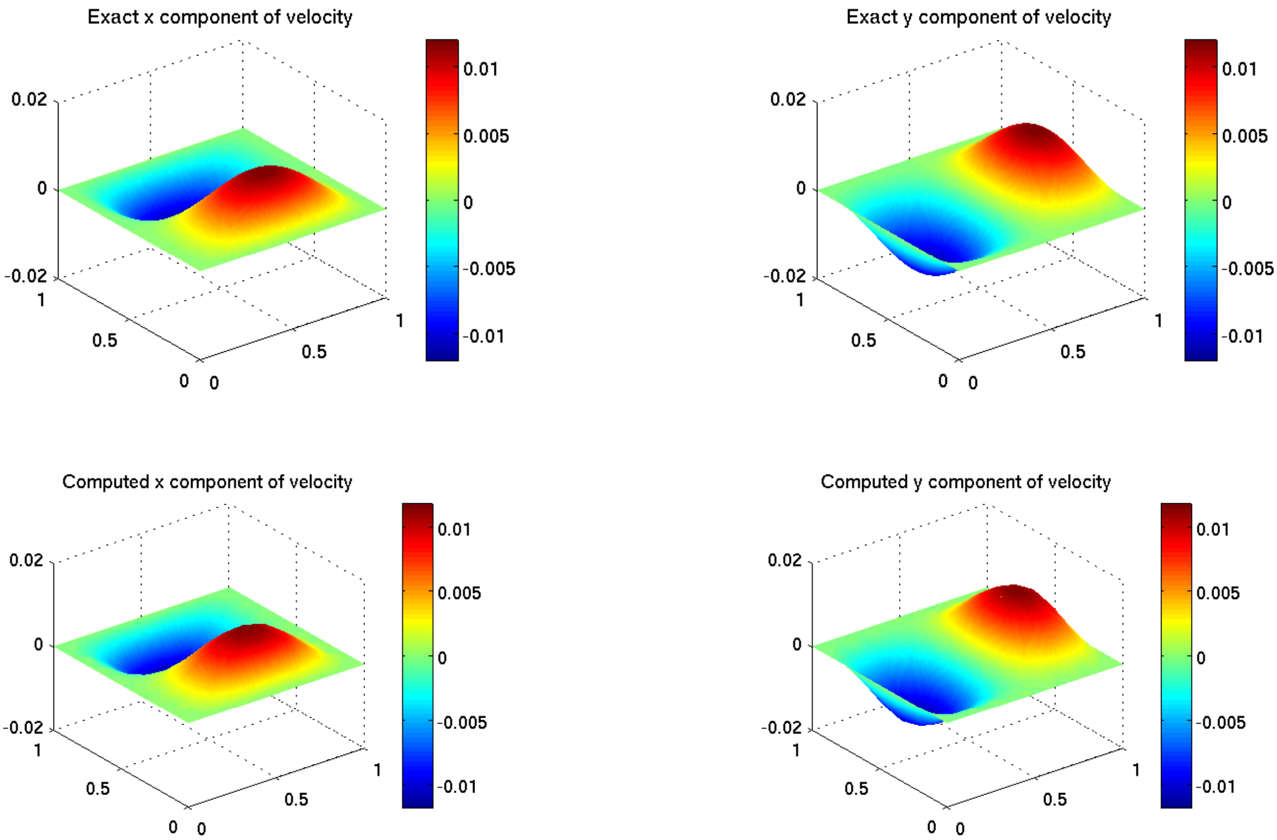

Figure 3.2. Comparison between the true solution and computed solution $(H, h) = (1/8, 1/16)$..

3.0.3 Driven Cavity Problem

Now, we are interested also in flows driven by interaction of a fluid with boundary. These flows have been widely used as test cases for validating incompressible fluid dynamic algorithms. Corner singularities for two dimensional fluid flows are very important, since most examples of physical interest have corners. Because of this reason, this section is devoted to driven cavity problem which as being favorable to the most of classical models. This problem is a classical test example used in Ghia, Ghia, Shin [24], J. E. Akin [2], and Cantekin, Westerink, and Luettich [13].

Driven cavity flow is enclosed in a square box, the boundary conditions $\mathbf{u} = (1, 0)$ for $0 < x < 1$, $y = 1$, a no slip condition at the other boundaries, and $\mathbf{f} = 0$ (Figure 3.3). The boundary conditions at the corners lead to a non-smooth solution. As we mentioned earlier, we use mini-element for the finite element discretization and this conforming pair of finite element spaces satisfy inf-sup condition, (1.12). We present the flow for different Reynolds number for fixed mesh where $H = 1/8$,

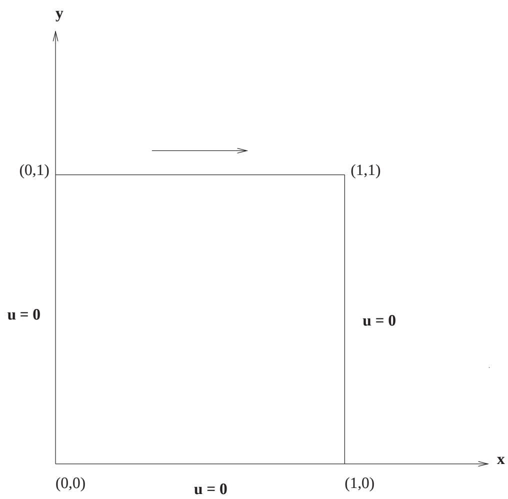

Figure 3.3. Driven cavity flow.

$h = 1/16$. The same basis functions $\hat{L}^H$ are chosen as Section 3.0.2.

The computational results for a set of different Reynolds numbers ($Re = 1, 100, 400, 2500$) are shown below. In these numerical tests, we observe the effect of Reynolds number on the flow pattern and the results are illustrated with Akin's velocity vectors [2]. For the low Reynolds number ($Re = 1$), the flow has only one vortex located above the center (Figure 3.4). When $Re = 100$, the flow pattern starts to form reverse circulation cells in lower corners (Figure 3.4). As we see in Figure 3.4, good agreement can be found between the case $Re = 1, 100$.

We also draw the x component of velocity along the vertical centerline and y component of velocity along the horizontal centerlines for $Re = 100$ and $Re = 400$. We compare the results obtained by Ghia, Ghia Shin's [24]. Ghia's algorithm is based on the time dependent streamfunction using the coupled implicit and multigrid methods. Their results are used as benchmark data as basis for comparison. Figure 3.5 and Figure 3.6 show that the results agree with Ghia's data using subgrid eddy viscosity method with Ghia's data.

As a next step we would like to compare numerical result with the small-small Smagorinsky model also known as artificial viscosity models. These models include adding only artificial viscosity as:

$$b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) + ((2\nu + \nu_T)\nabla^s \mathbf{u}, \nabla^s \mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

We compare the velocity streamline behavior for Navier-Stokes equation, subgrid eddy viscosity

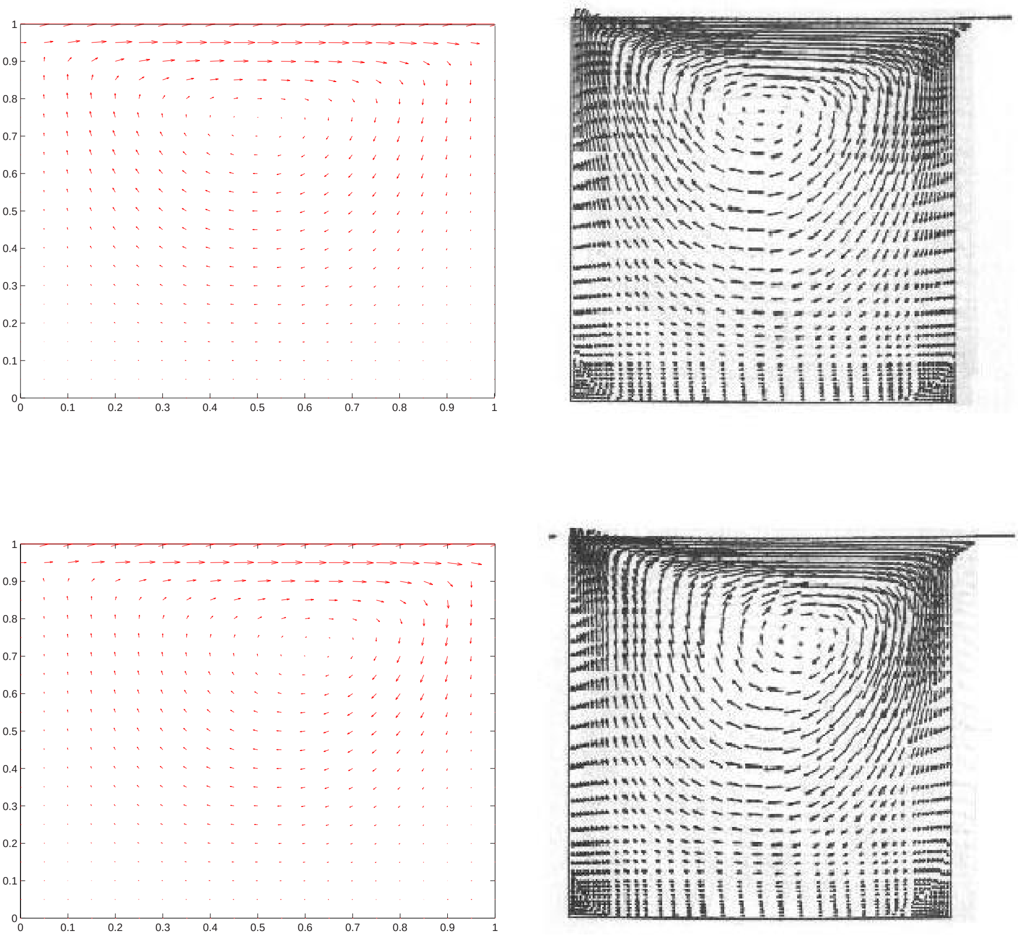

Figure 3.4. Velocity vectors for $Re = 1$ (above) and $Re = 100$ (below). Velocity vectors using subgrid eddy viscosity method (upper left, lower left) and Akin's velocity vectors a mesh of 40×40 elements (upper right, lower right), (J. Akin [2]).

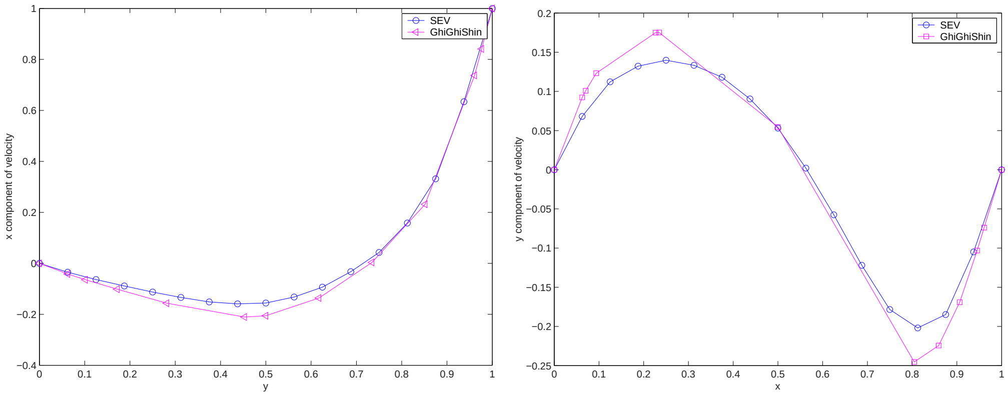

Figure 3.5. Vertical and horizontal midlines for $Re = 100$.

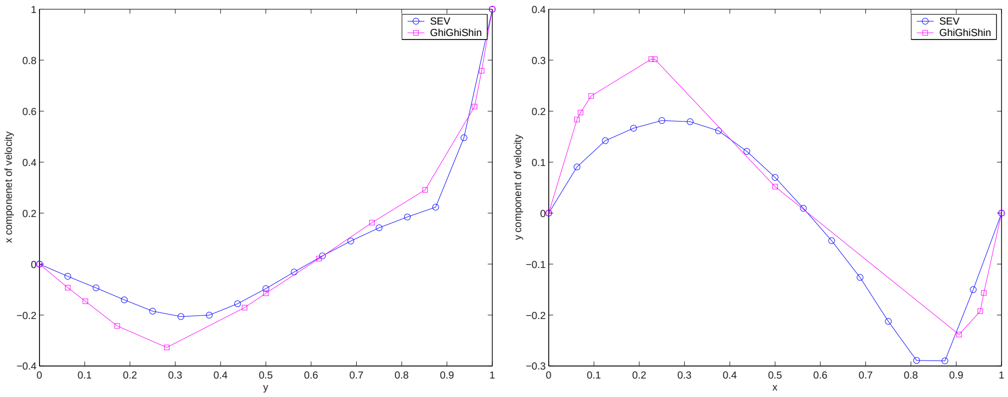

Figure 3.6. Vertical and horizontal midlines for $Re = 400$.

and artificial viscosity model for $Re = 2500$. As we mention in Introduction, one of the drawback of artificial viscosity model is that it introduces in general too much diffusion into the computed flow, i.e. the computed solution looks like a solution of a low Re. Figure 3.7 shows that the main eddy of artificial eddy viscosity model is too small. On the other hand, our numerical tests shows that the solution obtained with (1.5), (1.6) reproduce the main eddy of a high Re driven cavity flow much larger and centered more accurately than artificial viscosity model. With new subgrid eddy viscosity model, steady flow pattern becomes more complex with reverse circulation cells in both lower corners.

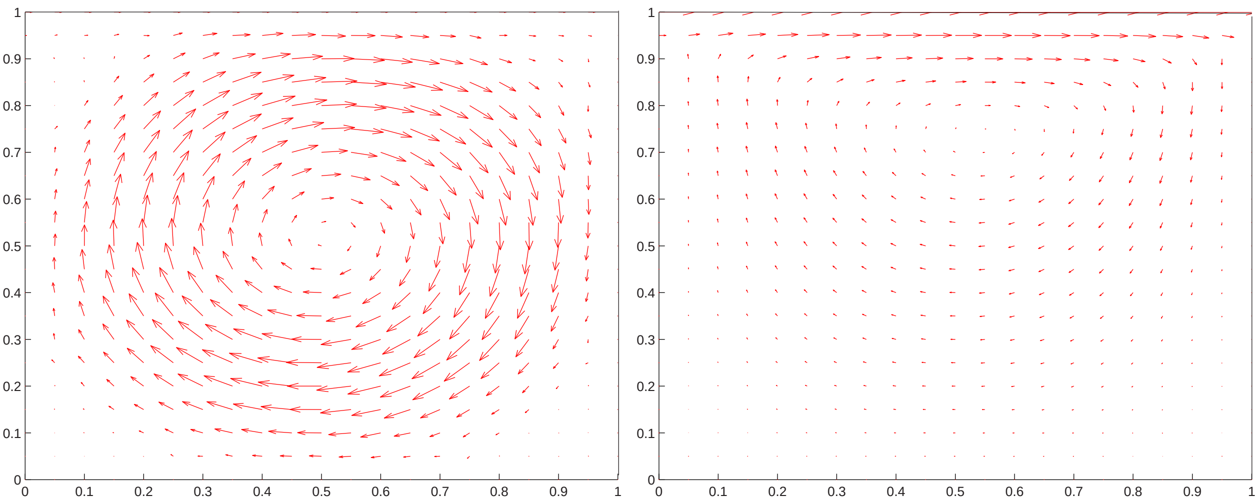

Figure 3.7. Velocity vectors for $Re = 2500$ for Subgrid Eddy Viscosity Model, Artificial Viscosity Model, from left to right $(H, h) = (1/8, 1/16)$.

Chapter 4

Discontinuous Approximations of Subgrid Eddy Viscosity Models

4.1 Introduction

The goal of this chapter is to analyze the subgrid eddy viscosity method combined with a different discretization. The discontinuous finite element approximation is used to discretize Navier-Stokes equations.

For any numerical method, the error equation arising from the Navier Stokes equations contains a convection-like term and a reaction (or stretching) term. Discontinuous Galerkin (DG) methods, first introduced in the work of Reed and Hill [61] and Lesaint and Raviart [52], are particularly efficient in controlling convective error terms. On the other hand, (generally nonlinear) eddy viscosity models are intended to give some control of the error's reaction like terms. Indeed, the exponential sensitivity of trajectories of the Navier Stokes equations (arising from reaction like term) is widely believed to be limited to the small scales. It is thus conjectured that by modeling their action on the large scales, the reaction like terms introducing exponential sensitivity will be contained.

DG methods have recently become more popular in the science and engineering community. They use piecewise polynomial functions with no continuity constraint across element interfaces. As a result, variational formulations must include jump terms across interfaces ([69]). The DG methods offers several advantages, including: (i) flexibility in the design of the meshes and in the construction of trial and test spaces, (ii) local conservation of mass, (iii) h-p adaptivity and (iv) higher order local approximations. DG methods have become widely used for solving computational fluid problems, especially diffusion and pure convection problems ([6, 59]). In addition, DG is applied for the elliptic problems [62, 63]. The reader should refer to Cockburn [15] for a historical review of DG methods. For the steady-state Navier-Stokes equations, a totally discontinuous finite element method is formulated in [28], while in [42], the velocity is approximated by discontinuous polynomials that are pointwise divergence-free, and the pressure by continuous polynomials.

Combining DG and eddy viscosity technique is clearly advantageous. While convective effects are accurately modelled by DG, the dispersive effects of small scales on the large scales are correctly taken into account with the eddy viscosity model. Besides, the fact that there is no constraint between the finite elements gives more freedom in choosing the appropriate the basis functions on the coarse and refined scales such as hierarchical basis functions for multiscale turbulent modeling. We consider in this chapter the combination of DG methods with a linear eddy viscosity model introduced in Chapter 1. We show that the errors are optimal with respect to the mesh size and depend on the Reynolds number in a reasonable fashion.

4.2 Notation and Preliminaries

We consider the stationary Navier-Stokes equations for incompressible flow as given 1.1. Let $\mathcal{K}_h = \{E_j, j = 1, ..., N_h\}$ denote a nondegenerate triangulation of the domain Ω. Let h denote the maximum diameter of the elements E_j in $\mathcal{K}_h$. We denote the edges of $\mathcal{K}_h$ by $\{e_1, e_2, ..., e_{P_h}, e_{P_h+1}, ..., e_{M_h}\}$, where $e_k \subset \Omega$ for $1 \le k \le P_h$ and $e_k \subset \partial\Omega$ for $P_{h+1} \le k \le M_h$. With each edge we associate a normal unit vector $\mathbf{n}_k$. For $k > P_h$, the unit vector $\mathbf{n}_k$ is taken to be outward normal to $\partial\Omega$. Let e_k be an edge shared by elements E_i and E_j with $\mathbf{n}_k$ exterior to E_i. We define the jump $[\phi]$ and average $\{\phi\}$ of a function ϕ by

$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}.$$

If e belongs to the boundary $\partial\Omega$, the jump and average of ϕ coincide with its trace on e.

We define the following *broken* norm for positive s:

$$|||\cdot|||_s = [\sum_{j=1}^{N_h} \|\cdot\|^2_{s,E_j}]^{1/2}.$$

From [68], if $\mathbf{f} \in L^2(0, T; (H^1_0)')$, there exists a solution $(\mathbf{u}, p)$ of (1.1) such that $\mathbf{u} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0)$. In addition, we will assume that $\mathbf{u}$ and p satisfy the following regularity properties:

- (R1) $\mathbf{u} \in \mathcal{C}^0(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$

- (R2) $\mathbf{u}_t \in L^2(0,T;H_0^1(\Omega))$,
- (R3) $\mathbf{u} \in L^\infty(0,T;W^{2,4/3}(\Omega)), \quad p \in L^\infty(0,T;W^{1,4/3}(\Omega))$.

The reader should refer to [10] for the justification of these regularity assumptions, except for the last one, that is needed here for the discontinuous Galerkin variational formulation. The following functional spaces are defined:

$$\mathbf{X} = \{\mathbf{v} \in (L^2(\Omega))^2 : \mathbf{v}|_{E_j} \in W^{2,4/3}(E_j), \quad \forall E_j \in \mathcal{K}_h\},$$
$$Q = \{q \in L_0^2(\Omega) : q|_{E_j} \in W^{1,4/3}(E_j), \quad \forall E_j \in \mathcal{K}_h\},$$

where $L_0^2(\Omega)$ is given by

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.$$

We associate to $(\mathbf{X}, Q)$ the following norms:

$$\|\mathbf{v}\|_X = (\|\!|\nabla \mathbf{v}|\!\|_0^2 + J(\mathbf{v},\mathbf{v}))^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{X}, \quad \|q\|_Q = \|q\|_{0,\Omega}, \quad \forall q \in Q,$$

where the jump term J is defined as

$$J(\mathbf{u},\mathbf{v}) = \sum_{k=1}^{M_h} \frac{\sigma}{|e|} \int_{e_k} [\mathbf{u}] \cdot [\mathbf{v}]. \tag{4.1}$$

In this jump term, $|e|$ denotes the measure of the edge e and σ is a constant parameter that will be specified later.

Recall the following property of norm $\|\cdot\|_X$ ([28]): for each real number $p \in [2,\infty)$ there exists a constant $C(p)$ such that

$$\|\mathbf{v}\|_{L^p(\Omega)} \leq C(p)\|\mathbf{v}\|_X, \quad \forall \mathbf{v} \in \mathbf{X}. \tag{4.2}$$

For any positive integer r, the finite-dimensional subspaces are

$$\mathbf{X}^h = \{\mathbf{v}^h \in \mathbf{X} : \mathbf{v}^h \in (\mathbb{P}_r(E_j))^2, \quad \forall E_j \in \mathcal{K}_h\},$$
$$Q^h = \{q^h \in Q : q^h \in \mathbb{P}_{r-1}(E_j), \quad \forall E_j \in \mathcal{K}_h\}.$$

We assume that for each integer $r \geq 1$, there exists an operator $R_h \in \mathcal{L}(H^1(\Omega); \mathbf{X}^h)$ such that

$$\|R_h(\mathbf{v}) - \mathbf{v}\|_X \leq Ch^r |\mathbf{v}|_{r+1,\Omega}, \quad \forall \mathbf{v} \in H^{r+1}(\Omega) \cap H^1_0(\Omega), \tag{4.3}$$

$$|\mathbf{v} - \boldsymbol{R}_h(\mathbf{v})|_{0,E_j} \leq Ch_{E_j}^{r+1} |\mathbf{v}|_{r+1,\Delta_{E_j}}, \quad \forall \mathbf{v} \in H^{r+1}(\Omega),\ 1 \leq j \leq N_h, \tag{4.4}$$

where Δ_{E_j} is a suitable macro element containing E_j. Note that for $r = 1, 2$ and 3, the existence of this interpolant follows from [17, 16, 21]. The bounds (4.3) and (4.4) are proved in [28] and in [29] respectively.

Also, for each integer $r \geq 1$, there is an operator $r_h \in \mathcal{L}(L^2_0(\Omega); Q_h)$ such that for any E_j in $\mathcal{K}_h$

$$\int_{E_j} z_h (r_h(q) - q) = 0, \quad \forall z_h \in \mathbb{P}_{r-1}(E_j), \quad \forall q \in L^2_0(\Omega), \tag{4.5}$$

$$\|q - r_h(q)\|_{m,E_j} \leq Ch_{E_j}^{r-m} |q|_{r,E_j}, \ \forall q \in H^r(\Omega) \cap L^2_0(\Omega), \quad m = 0, 1. \tag{4.6}$$

Finally, we recall some trace and inverse inequalities, that hold true on each element E in $\mathcal{K}_h$, with diameter h_E:

$$\|\mathbf{v}\|_{0,e} \leq C(h_E^{-1/2} \|\mathbf{v}\|_{0,E} + h_E^{1/2} \|\nabla \mathbf{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in \mathbf{X}, \tag{4.7}$$

$$\|\nabla \mathbf{v}\|_{0,e} \leq C(h_E^{-1/2} \|\nabla \mathbf{v}\|_{0,E} + h_E^{1/2} \|\nabla^2 \mathbf{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in \mathbf{X}, \tag{4.8}$$

$$\|\mathbf{v}\|_{L^4(e)} \leq Ch_E^{-3/4} (\|\mathbf{v}\|_{0,E} + h_E \|\nabla \mathbf{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in \mathbf{X}, \tag{4.9}$$

$$\|\mathbf{v}^h\|_{0,e} \leq Ch_E^{-1/2} \|\mathbf{v}^h\|_{0,E}, \quad \forall e \in \partial E, \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.10}$$

$$\|\nabla \mathbf{v}^h\|_{0,e} \leq Ch_E^{-1/2} \|\nabla \mathbf{v}^h\|_{0,E}, \quad \forall e \in \partial E, \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.11}$$

$$\|\nabla \mathbf{v}^h\|_{0,E} \leq Ch_E^{-1} \|\mathbf{v}^h\|_{0,E}, \quad \forall \mathbf{v}^h \in \mathbf{X}^h. \tag{4.12}$$

$$\|\mathbf{v}^h\|_{L^4(E)} \leq Ch_E^{-1/2} \|\mathbf{v}^h\|_{0,E}, \quad \forall \mathbf{v}^h \in \mathbf{X}^h. \tag{4.13}$$

4.3 Variational Formulation and Scheme

Let us first define the bilinear forms $a : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ and $c : \mathbf{X} \times Q \to \mathbb{R}$:

$$a(\mathbf{v}, \mathbf{w}) = \sum_{j=1}^{N_h} \int_{E_j} \nabla \mathbf{v} : \nabla \mathbf{w} - \sum_{k=1}^{M_h} \int_{e_k} (\{\nabla \mathbf{v}\}\mathbf{n}_k \cdot [\mathbf{w}] - \epsilon_0 \{\nabla \mathbf{w}\}\mathbf{n}_k \cdot [\mathbf{v}]), \tag{4.14}$$

$$c(\mathbf{v}, q) = -\sum_{j=1}^{N_h} \int_{E_j} q \nabla \cdot \mathbf{v} + \sum_{k=1}^{M_h} \int_{e_k} \{q\}[\mathbf{v}] \cdot \mathbf{n}_k, \tag{4.15}$$

where ϵ_0 takes the constant value 1 or -1. Throughout the chapter, we will assume the following hypothesis: if $\epsilon_0 = 1$, the jump parameter σ is chosen to be equal to 1; if $\epsilon_0 = -1$, the jump parameter σ is bounded below by $\sigma_0 > 0$ and σ_0 is sufficiently large. Based on this assumption, we can easily prove the following lemma.

Lemma 4.3.1. *There is a constant $\kappa > 0$ such that*

$$a(\mathbf{v}^h, \mathbf{v}^h) + J(\mathbf{v}^h, \mathbf{v}^h) \geq \kappa \|\mathbf{v}^h\|_X^2, \quad \forall \mathbf{v}^h \in \mathbf{X}^h. \tag{4.16}$$

In addition to these bilinear forms, we consider the following upwind discretization of the term $\mathbf{u} \cdot \nabla \mathbf{z}$ ([28]):

$$\begin{aligned} \tilde{b}(\mathbf{u}, \mathbf{z}, \boldsymbol{\theta}) = &\sum_{j=1}^{N_h} \Big(\int_{E_j} (\mathbf{u} \cdot \nabla \mathbf{z}) \cdot \boldsymbol{\theta} + \int_{\partial E_j^-} |\{\mathbf{u}\} \cdot \mathbf{n}_{E_j}| (\mathbf{z}^{\text{int}} - \mathbf{z}^{\text{ext}}) \cdot \boldsymbol{\theta}^{int} \Big) \\ &+ \frac{1}{2} \sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \mathbf{u}) \mathbf{z} \cdot \boldsymbol{\theta} - \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\mathbf{u}] \cdot \mathbf{n}_k \{\mathbf{z} \cdot \boldsymbol{\theta}\}, \end{aligned} \tag{4.17}$$

for all $\mathbf{u}, \mathbf{z}, \boldsymbol{\theta}$ in $\mathbf{X}$ and where on each element the inflow boundary is:

$$\partial E_j^- = \{\mathbf{x} \in \partial E_j : \{\mathbf{u}\} \cdot \mathbf{n}_{E_j} < 0\},$$

and the superscript int (resp ext) refers to the trace of the function on a side of E_j coming from the interior of E_j (resp. coming from the exterior of E_j on that side). Note that the form $\tilde{b}$ is not linear with respect to its first argument, but linear with respect to its second and third argument. To avoid any confusion, if necessary, in the analysis, we will explicitly write $\tilde{b}(\mathbf{u}, \mathbf{z}, \boldsymbol{\theta}) = \tilde{b}_{\mathbf{u}}(\mathbf{u}, \mathbf{z}, \boldsymbol{\theta})$

when the inflow boundaries ∂E_j^- are defined with respect to the velocity $\{\mathbf{u}\}$. We finally recall the positivity of $\tilde{b}$ proved in [28].

$$\tilde{b}(\mathbf{u},\mathbf{z},\mathbf{z}) \geq 0, \quad \forall \mathbf{u},\mathbf{z} \in \mathbf{X}. \tag{4.18}$$

Remark 4.1. *If $\mathbf{u},\mathbf{z},\boldsymbol{\theta} \in H_0^1(\Omega)$, then $b(\mathbf{u},\mathbf{z},\boldsymbol{\theta}) = \tilde{b}(\mathbf{u},\mathbf{z},\boldsymbol{\theta})$ in Chapter 1.*

With these forms, we consider a variational problem of (1.1): for all $t > 0$ find $\mathbf{u}(t) \in \mathbf{X}$ and $p(t) \in Q$ satisfying

$$\begin{aligned}
&(\mathbf{u}_t(t),\mathbf{v}) + \nu(a(\mathbf{u}(t),\mathbf{v}) + J(\mathbf{u}(t),\mathbf{v})) \\
&+\tilde{b}(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + c(\mathbf{v},p(t)) = (\mathbf{f}(t),\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X},
\end{aligned} \tag{4.19}$$

$$c(\mathbf{u}(t),q) = 0, \quad \forall q \in Q, \tag{4.20}$$

$$(\mathbf{u}(0),\mathbf{v}) = (\mathbf{u}_0,\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}. \tag{4.21}$$

We shall now show the equivalence of the strong and weak solutions.

Lemma 4.3.2. *Every strong solution of (1.1) is also a solution of (4.19)-(4.21). Conversely, if $\mathbf{u} \in L^\infty(0,T;H^2(\Omega))$ and $p \in L^2(0,T;H^1(\Omega))$ are a solution of (4.19)-(4.21) then $(\mathbf{u},p)$ satisfies (1.1).*

Proof: Fix $t > 0$. Let $(\mathbf{u},p)$ be the solution of (1.1). Since $\mathbf{u}(t) \in H_0^1(\Omega)$, by the trace theorem $[\mathbf{u}(t)] \cdot \mathbf{n}_k = 0$ on each edge. Also, $\nabla \cdot \mathbf{u}(t) = 0$, thus $\mathbf{u}$ satisfies (4.20). Multiplying the first Navier-Stokes equation (1.1) by $\mathbf{v} \in \mathbf{X}$, integrating over each element and summing over all elements yield

$$\begin{aligned}
\sum_{j=1}^{N_h} \int_{E_j} (\mathbf{u}_t \cdot \mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v}) - \nu \sum_{k=1}^{M_h} \int_{e_k} [\nabla \mathbf{u} \mathbf{n}_k \cdot \mathbf{v}] + \sum_{j=1}^{N_h} \int_{E_j} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} \\
- \sum_{j=1}^{N_h} \int_{E_j} p \nabla \cdot \mathbf{v} + \sum_{k=1}^{M_h} \int_{e_k} [p \mathbf{v} \cdot \mathbf{n}_k] = \int_\Omega \mathbf{f} \cdot \mathbf{v}.
\end{aligned}$$

The boundary terms are rewritten as:

$$\sum_{k=1}^{M_h} \int_{e_k} [\nabla \mathbf{u} \mathbf{n}_k . \mathbf{v}] = \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \mathbf{u}\} \mathbf{n}_k \cdot [\mathbf{v}] + \sum_{k=1}^{M_h} \int_{e_k} [\nabla \mathbf{u}] \mathbf{n}_k \cdot \{\mathbf{v}\}.$$

The first part of the lemma is then obtained because the jumps of $\mathbf{u}, \nabla \mathbf{u}\mathbf{n}_k$ and of p are zero almost everywhere.

Conversely, let $(\mathbf{u}, p)$ be a solution to (4.19)-(4.21). First, let E belong to $\mathcal{K}_h$ and choose $\mathbf{v} \in \mathcal{D}(E)^2$, extended by zero outside E. Then, $(\mathbf{u}, p)$ satisfy in the sense of distributions

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad E. \tag{4.22}$$

Next consider $\mathbf{v} \in \mathcal{C}^1(\bar{E})$ such that $\mathbf{v} = \mathbf{0}$ on ∂E, extended by zero outside E, $\nabla \mathbf{v} \cdot \mathbf{n} = 0$ on ∂E except on one side e_k. We multiply (4.22) by $\mathbf{v}$ and integrate by parts. We then obtain

$$\int_{e_k} \{\nabla \mathbf{v}\}\mathbf{n}_k \cdot [\mathbf{u}] = 0,$$

which implies that $[\mathbf{u}] = \mathbf{0}$ almost everywhere on e_k. If e_k belongs to the boundary $\partial\Omega$, this implies that $\mathbf{u}|_{e_k} = \mathbf{0}$. Thus, $\mathbf{u} \in H_0^1(\Omega)$. Finally, choose $\mathbf{v} \in \mathcal{C}^1(\bar{E})$, with $\mathbf{v} = \mathbf{0}$ on ∂E except on one side e_k, extended by zero outside. Integrating by parts (4.22), we have

$$\int_{e_k} (-\nu \nabla \mathbf{u}\mathbf{n}_E + p\mathbf{n}_E) \cdot \mathbf{v} = \int_{e_k} \{-\nu \nabla \mathbf{u}\mathbf{n}_E + p\mathbf{n}_E\} \cdot \mathbf{v}.$$

Since $\mathbf{v}$ is arbitrary, this means that the quantity $-\nu \nabla \mathbf{u}\mathbf{n}_k + p\mathbf{n}_k$ is continuous across e_k. Therefore, the equation (4.22) is satisfied over the entire domain Ω. The initial condition in (1.1) is straightforward.

We recall a discrete inf-sup condition and a property satisfied by R_h (see [28]).

Lemma 4.3.3. *There exists a positive constant β_0, independent of h such that*

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{c(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_X \|q^h\|_0} \geq \beta_0. \tag{4.23}$$

Furthermore, the operator R_h satisfies:

$$c(R_h(\mathbf{v}) - \mathbf{v}, q^h) = 0, \quad \forall q^h \in Q^h, \quad \forall \mathbf{v} \in H_0^1(\Omega). \tag{4.24}$$

In order to subtract the artificial diffusion introduced by the eddy viscosity on the coarse grid, we consider the same finite element discretization introduced in Chapter 1. We define the following bilinear $g : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$:

$$g(\mathbf{v}, \mathbf{w}) = \sum_{j=1}^{N_h} \int_{E_j} (I - P_{L^H})\nabla \mathbf{v} : (I - P_{L^H})\nabla \mathbf{w}, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{X}.$$

For all $t > 0$, we seek a discontinuous approximation $(\mathbf{u}^h(t), p^h(t)) \in \mathbf{X}^h \times Q^h$ such that

$$(\mathbf{u}_t^h(t), \mathbf{v}^h) + \nu(a(\mathbf{u}^h(t), \mathbf{v}^h) + J(\mathbf{u}^h(t), \mathbf{v}^h)) + \nu_T g(\mathbf{u}^h(t), \mathbf{v}^h)$$
$$+\tilde{b}(\mathbf{u}^h(t), \mathbf{u}^h(t), \mathbf{v}^h) + c(\mathbf{v}^h, p^h(t)) = (\mathbf{f}(t), \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.25}$$

$$c(\mathbf{u}^h(t), q^h) = 0, \quad \forall q^h \in Q^h, \tag{4.26}$$

$$(\mathbf{u}^h(0), \mathbf{v}^h) = (\mathbf{u}_0, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h. \tag{4.27}$$

Lemma 4.3.4. *There exists a unique solution to (4.25)-(4.27).*

Proof: The equations (4.25) and (4.26) reduce to the ordinary differential system

$$\frac{d\mathbf{u}^h}{dt} + \nu A\mathbf{u}^h + B\mathbf{u}^h + \nu_T G\mathbf{u}^h = \mathbf{F}.$$

By continuity, a solution exists. To prove uniqueness, we choose $\mathbf{v}^h = \mathbf{u}^h$ in (4.25), $q^h = p^h$ in (4.26); we apply the coercivity equation (4.16) and the generalized Cauchy-Schwarz

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^h\|_{0,\Omega}^2 + \nu\kappa\|\mathbf{u}^h\|_X^2 \leq \|\mathbf{f}\|_{L^{4/3}(\Omega)}\|\mathbf{u}^h\|_{L^4(\Omega)} \leq \frac{\nu\kappa}{2}\|\mathbf{u}^h\|_X^2 + \frac{C}{\nu\kappa}\|\mathbf{f}\|_{L^{4/3}(\Omega)}^2.$$

Integrating over $[0, t]$ yields:

$$\|\mathbf{u}^h(t)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu\kappa\|\mathbf{u}^h\|_{L^2(0,T;X)}^2 \leq \|\mathbf{u}^h(0)\|_0^2 + \frac{C}{\nu\kappa}\|\mathbf{f}\|_{L^2(0,T;L^{4/3}(\Omega))}^2.$$

Since $\mathbf{u}^h$ is bounded in $L^\infty(0, T; L^2(\Omega))$, it is unique [8]. The existence and uniqueness of p^h is obtained from the inf-sup condition stated above.

Remark 4.2. *From a continuum mechanics point of view, it might be advantageous to consider* *the symmetrized velocity tensor. In this case, the bilinear form a is replaced by*

$$a(\mathbf{v},\mathbf{w}) = \sum_{j=1}^{N_h}\int_{E_j}\nabla^s\mathbf{v} : \nabla^s\mathbf{w} - \sum_{k=1}^{M_h}\int_{e_k}(\{\nabla^s\mathbf{v}\}\boldsymbol{n}_k\cdot[\mathbf{w}] - \epsilon_0\{\nabla^s\mathbf{w}\}\boldsymbol{n}_k\cdot[\mathbf{v}]),$$

where $\nabla^s\mathbf{v} = 0.5(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$ *and the term relating the coarse and refined mesh is replaced by* $\sum_{j=1}^{N_h}\int_{E_j}(I-P_{L^H})\nabla^s\mathbf{u} : (I-P_{L^H})\nabla^s\mathbf{v}^h$*. It is easy to check that all the results proved in this chapter also hold true for the symmetrized tensor formulation.*

4.4 Semi-discrete A Priori Error Estimate

In this section, priori error estimates for the continuous in time problem, are derived. The estimates are optimal in the fine mesh size h. The effects of the coarse scale appear as higher order terms.

Theorem 4.1. *Let* $(\mathbf{u},p)$ *be the solution of (1.1) satisfying R1-R3. In addition, we assume that* $\mathbf{u}_t\in L^2(0,T;H^{r+1}(\Omega)), \mathbf{u}\in L^\infty(0,T;H^{r+1}(\Omega))$ *and* $p\in L^2(0,T;H^r(\Omega))$*. Then, the continuous in time solution* $\mathbf{u}_h$ *satisfies*

$$\|\mathbf{u}-\mathbf{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \kappa^{1/2}\nu^{1/2}\|\mathbf{u}-\mathbf{u}^h\|_{L^2(0,T;X)}$$
$$+\nu_T^{1/2}\|(I-P_{L^H})\nabla(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2(\Omega))} \le Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$
$$+\nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))} + |\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) + \nu_T^{1/2}H^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\mathbf{u}_0|_{r+1,\Omega},$$

where C *is a positive constant independent of* h, H, ν *and* ν_T*.*

Proof: We fix $t>0$ and for simplicity, we drop the argument in t. Defining $\mathbf{e}^h = \mathbf{u}-\mathbf{u}^h$ and subtracting (4.25), (4.26), (4.27) from (4.19), (4.20), (4.21) respectively yields

$$(\mathbf{e}_t^h,\mathbf{v}^h) + \nu a(\mathbf{e}^h,\mathbf{v}^h) + \nu J(\mathbf{e}^h,\mathbf{v}^h) + \nu_T g(\mathbf{e}^h,\mathbf{v}^h) + \tilde{b}(\mathbf{u},\mathbf{u},\mathbf{v}^h)$$
$$-\tilde{b}(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) = -c(\mathbf{v}^h,p-p^h) + \nu_T g(\mathbf{u},\mathbf{v}^h), \quad \forall\mathbf{v}^h\in\mathbf{X}_h, \quad \forall t>0, \tag{4.28}$$

$$c(\mathbf{e}^h,q^h) = 0, \quad \forall q^h\in Q^h, \quad \forall t>0, \tag{4.29}$$

$$(\mathbf{e}^h(0),\mathbf{v}^h) = 0, \quad \forall\mathbf{v}^h\in\mathbf{X}^h. \tag{4.30}$$

Decompose the error $\mathbf{e}^h = \boldsymbol{\eta} - \boldsymbol{\phi}^h$, where $\boldsymbol{\phi}^h = \mathbf{u}^h - R_h(\mathbf{u})$ and $\boldsymbol{\eta}$ is the interpolation error $\boldsymbol{\eta} = \mathbf{u} - R_h(\mathbf{u})$. Set $\mathbf{v}^h = \boldsymbol{\phi}^h$ in (4.28) and $q^h = r_h(p) - p_h$ in (4.29):

$$
\begin{aligned}
&(\boldsymbol{\phi}_t^h, \boldsymbol{\phi}^h) + \nu a(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + \nu J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + \nu_T g(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) \\
&+\tilde{b}_{\mathbf{u}^h}(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) - \tilde{b}_{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) = (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta}, \boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta}, \boldsymbol{\phi}^h) \\
&+\nu_T g(\boldsymbol{\eta}, \boldsymbol{\phi}^h) + c(\boldsymbol{\phi}^h, p - r_h(p)) - \nu_T g(\mathbf{u}, \boldsymbol{\phi}^h), \quad \forall t > 0. \qquad (4.31)
\end{aligned}
$$

We now bound the terms on the right hand-side of (4.31). The first three terms are rewritten as:

$$
\begin{aligned}
(\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta}, \boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta}, \boldsymbol{\phi}^h) &= (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + \nu \sum_{j=1}^{N_h} \int_{E_j} \nabla \boldsymbol{\eta} : \nabla \boldsymbol{\phi}^h \\
-\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\eta}\} \mathbf{n}_k \cdot [\boldsymbol{\phi}^h] &+ \nu \epsilon_0 \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\phi}^h\} \mathbf{n}_k \cdot [\boldsymbol{\eta}] + \nu J(\boldsymbol{\eta}, \boldsymbol{\phi}^h) \\
&= S_1 + \cdots + S_5.
\end{aligned}
$$

Using Cauchy-Schwarz and Young's inequalities and the approximation result (4.3), the first two terms are bounded as follows:

$$
\begin{aligned}
S_1 &\leq \|\boldsymbol{\eta}_t\|_{0,\Omega} \|\boldsymbol{\phi}^h\|_{0,\Omega} \leq \frac{1}{2} \|\boldsymbol{\phi}^h\|_{0,\Omega}^2 + C h^{2r+2} |\mathbf{u}_t|_{r+1,\Omega}^2, \\
S_2 &\leq 2\nu \sum_{j=1}^{N_h} \|\nabla \boldsymbol{\eta}\|_{0,E_j} \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \leq \frac{\kappa \nu}{8} \|\nabla \boldsymbol{\phi}^h\|_0^2 + C \nu h^{2r} |\mathbf{u}|_{r+1,\Omega}^2.
\end{aligned}
$$

To bound the third term, we insert the standard Lagrange interpolant of degree r, denoted by $L_h(\mathbf{u})$.

$$
\begin{aligned}
-\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\eta}\} \mathbf{n}_k \cdot [\boldsymbol{\phi}^h] &= -\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla (\mathbf{u} - L_h(\mathbf{u}))\} \mathbf{n}_k \cdot [\boldsymbol{\phi}^h] \\
&-\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla (L_h(\mathbf{u}) - R_h(\mathbf{u}))\} \mathbf{n}_k \cdot [\boldsymbol{\phi}^h].
\end{aligned}
$$

By using the inequalities (4.8) and (4.11), the definition of the jump (4.1), and the approximation results (4.3), the third term can be bounded by

$$S_3 \le \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

Then, from the trace inequalities (4.7), (4.11) and the approximation result (4.3), we have

$$\begin{aligned} S_4 &\le C\nu (\sum_{k=1}^{M_h} \frac{\sigma}{|e|} \|[\boldsymbol{\eta}]\|^2_{0,e_k})^{1/2} (\sum_{k=1}^{M_h} \frac{|e|}{\sigma} \|\{\nabla \boldsymbol{\phi}^h\}\|^2_{0,e_k})^{1/2} \\ &\le \frac{\kappa\nu}{8} |||\nabla \boldsymbol{\phi}^h|||^2_0 + C\nu h^{2r} |\mathbf{u}|^2_{r+1,\Omega}. \end{aligned}$$

The jump term is bounded by the approximation result (4.3) as follows:

$$S_5 \le \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu J(\boldsymbol{\eta}, \boldsymbol{\eta}) \le \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

The eddy viscosity term in the right-hand side of (4.31) is bounded by (4.3),

$$\nu_T g(\boldsymbol{\eta}, \boldsymbol{\phi}^h) \le \frac{\nu_T}{4} |||(I - P_{L^H})\nabla \boldsymbol{\phi}^h|||^2_0 + C\nu_T h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

Because of (4.5), the pressure term is reduced to

$$c(\boldsymbol{\phi}^h, p - r_h(p)) = \sum_{k=1}^{M_h} \int_{e_k} \{p - r_h(p)\}[\boldsymbol{\phi}^h] \cdot \mathbf{n}_k,$$

which is bounded by using Cauchy-Schwarz inequality, trace inequality (4.7) and approximation result (4.6)

$$\begin{aligned} c(\boldsymbol{\phi}^h, p - r_h(p)) &\le C(\|p - r_h(p)\|^2_0 + \sum_{j=1}^{N_h} h^2_{E_j} |p - r_h(p)|^2_{1,E_j})^{1/2} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h)^{1/2} \\ &\le \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\frac{h^{2r}}{\nu} |p|^2_{r,\Omega}. \end{aligned}$$

The last term on the right-hand side of (4.31), corresponding to the consistency error, is bounded using Cauchy-Schwarz inequality

$$\nu_T g(\mathbf{u}, \boldsymbol{\phi}^h) \leq \frac{\nu_T}{4} \|(I - P_{L^H})\nabla \boldsymbol{\phi}^h\|_0^2 + C\nu_T H^{2r} |\mathbf{u}|_{r+1,\Omega}^2.$$

Thus far, the terms in the right-hand side of (4.31) are bounded by

$$\frac{1}{2}\|\boldsymbol{\phi}^h\|_0^2 + Ch^{2r}|\mathbf{u}_t|_{r+1,\Omega}^2 + C(\nu + \nu_T)h^{2r}|\mathbf{u}|_{r+1,\Omega}^2 + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2$$
$$+C\nu_T H^{2r}|\mathbf{u}|_{r+1,\Omega}^2 + \frac{\kappa\nu}{4}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\|(I - P_{L^H})\nabla\boldsymbol{\phi}^h\|_0^2.$$

Consider now the nonlinear terms in (4.31). We first note that since $\mathbf{u}$ is continuous,

$$\tilde{b}_{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) = \tilde{b}_{\mathbf{u}^h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h).$$

Therefore, adding and subtracting the interpolant $R_h(\mathbf{u})$ yields:

$$\tilde{b}_{\mathbf{u}^h}(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) - \tilde{b}_{\mathbf{u}^h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) = \tilde{b}_{\mathbf{u}^h}(\mathbf{u}^h, \boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + \tilde{b}_{\mathbf{u}^h}(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h)$$
$$-\tilde{b}_{\mathbf{u}^h}(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h) - \tilde{b}_{\mathbf{u}^h}(\boldsymbol{\eta}, R_h(\mathbf{u}), \boldsymbol{\phi}^h) - \tilde{b}_{\mathbf{u}^h}(\mathbf{u}, \boldsymbol{\eta}, \boldsymbol{\phi}^h).$$

To simplify the writing, we drop the subscript $\mathbf{u}_h$ and write $\tilde{b}(\cdot,\cdot,\cdot)$ for $\tilde{b}_{\mathbf{u}_h}(\cdot,\cdot,\cdot)$. From the inequality (4.18), the first term is positive. We then bound the other terms. We first note, that we can rewrite the form $\tilde{b}$ as

$$\tilde{b}(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\phi}^h - \frac{1}{2} c(\boldsymbol{\phi}^h, \mathbf{u} \cdot \boldsymbol{\phi}^h). \tag{4.32}$$

The first term, using the L^p bound (4.2), is bounded by

$$\sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\phi}^h \leq \|\boldsymbol{\phi}^h\|_{L^4(\Omega)} \|\nabla \mathbf{u}\|_{L^4(\Omega)} \|\boldsymbol{\phi}^h\|_{L^2(\Omega)}$$
$$\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\mathbf{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be the piecewise constant vectors such that

$$\mathbf{c}_1|_{E_j} = \frac{1}{|E_j|}\int_{E_j} \mathbf{u}, \quad \mathbf{c}_2|_{E_j} = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{\phi}^h, \quad 1 \le j \le N_h.$$

We rewrite using (4.29) and (4.24):

$$c(\boldsymbol{\phi}^h, \mathbf{u}\cdot\boldsymbol{\phi}^h) = c(\boldsymbol{\phi}^h, \mathbf{u}\cdot\boldsymbol{\phi}^h - \mathbf{c}_1\cdot\mathbf{c}_2) = c(\boldsymbol{\phi}^h, (\mathbf{u}-\mathbf{c}_1)\cdot\boldsymbol{\phi}^h) + c(\boldsymbol{\phi}^h, \mathbf{c}_1\cdot(\boldsymbol{\phi}^h - \mathbf{c}_2)).$$

Then, expanding the first term

$$c(\boldsymbol{\phi}^h, (\mathbf{u}-\mathbf{c}_1)\cdot\boldsymbol{\phi}^h) = -\sum_{j=1}^{N_h}\int_E (\mathbf{u}-\mathbf{c}_1)\cdot\boldsymbol{\phi}^h \nabla\cdot\boldsymbol{\phi}^h$$
$$+\sum_{k=1}^{M_h}\int_{e_k}\{(\mathbf{u}-\mathbf{c}_1)\cdot\boldsymbol{\phi}^h\}[\boldsymbol{\phi}^h]\cdot\mathbf{n}_k = S_6 + S_7.$$

The first term is bounded, for $s > 2$, using the inverse inequality (4.12) and (4.2)

$$S_6 \le C\sum_{j=1}^{N_h}\|\mathbf{u}-\mathbf{c}_1\|_{L^s(E_j)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)}\|\nabla\boldsymbol{\phi}^h\|_{L^2(E_j)} \le C\|\boldsymbol{\phi}^h\|_{0,\Omega}|\mathbf{u}|_{W^{1,s}(\Omega)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(\Omega)}$$
$$\le C\|\boldsymbol{\phi}^h\|_{0,\Omega}|\mathbf{u}|_{W^{1,s}(\Omega)}\|\boldsymbol{\phi}^h\|_X \le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\mathbf{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}\|\boldsymbol{\phi}^h\|_0^2.$$

The bound for the second term is more technical. First, passing to the reference element $\hat{E}$, and using the trace inequality (4.10), we obtain

$$S_7 \le C\sum_{k=1}^{M_h}|e_k||E|^{-1/2}\|\boldsymbol{\phi}^h\|_{0,E}\|(\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{\hat{e}}$$
$$\le C\sum_{k=1}^{M_h}|e_k||E|^{-1/2}\|\boldsymbol{\phi}^h\|_{0,E}(\|(\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} + \|\hat{\nabla}((\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}}).$$

The L^2 term is bounded as, for $s > 2$,

$$\|(\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \le \|\hat{\mathbf{u}}-\hat{\mathbf{c}}_1\|_{L^s(\hat{E})}\|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})}$$
$$\le h|E|^{-1/s-(s-2)/(2s)}|\mathbf{u}|_{W^{1,s}(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \le C|\mathbf{u}|_{W^{1,s}(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}.$$

Note for the gradient term, we write

$$\|\hat{\nabla}((\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}} = \|(\hat{\nabla}\hat{\mathbf{u}}\cdot\hat{\boldsymbol{\phi}}^h + (\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\nabla\hat{\boldsymbol{\phi}}^h)\|.$$

Let us first bound

$$\|\hat{\nabla}\hat{\mathbf{u}}\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq \|\hat{\nabla}\hat{\mathbf{u}}\|_{L^s(\hat{E})}\|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})}$$
$$\leq Ch|E|^{-1/s}\|\nabla\mathbf{u}\|_{L^s(E)}|E|^{-(s-2)/2s}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \leq C\|\nabla\mathbf{u}\|_{L^s(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}.$$

Now the other term is

$$\|(\hat{\mathbf{u}}-\hat{\mathbf{c}}_1)\cdot\hat{\nabla}\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq \|\hat{\mathbf{u}}-\hat{\mathbf{c}}_1\|_{L^\infty(\hat{E})}\|\hat{\nabla}\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq Ch\|\mathbf{u}\|_{L^\infty(E)}\|\nabla\boldsymbol{\phi}^h\|_{0,E}.$$

Combining all the bounds above and using (4.2), we have

$$S_7 \leq C\sum_{j=1}^{N_h}\|\boldsymbol{\phi}^h\|_{0,E_j}[|\mathbf{u}|_{W^{1,s}(E_j)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)}$$
$$+\|\nabla\mathbf{u}\|_{L^s(E_j)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)} + h|\mathbf{u}|_{L^\infty(E_j)}\|\nabla\boldsymbol{\phi}^h\|_{L^2(E_j)}] \leq \frac{\kappa\nu}{32}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{\phi}^h\|_0^2.$$

Now,

$$c(\boldsymbol{\phi}^h, \mathbf{c}_1\cdot(\boldsymbol{\phi}^h-\mathbf{c}_2)) = -\sum_{j=1}^{N_h}\int_E \mathbf{c}_1\cdot(\boldsymbol{\phi}^h-\mathbf{c}_2)\nabla\cdot\boldsymbol{\phi}^h$$
$$+\sum_{k=1}^{M_h}\int_{e_k}\{\mathbf{c}_1\cdot(\boldsymbol{\phi}^h-\mathbf{c}_2)\}[\boldsymbol{\phi}^h]\cdot\mathbf{n}_k = S_8 + S_9.$$

The first term is bounded by (4.12)

$$S_8 \leq C\sum_{j=1}^{N_h}\|\mathbf{c}_1\|\|\boldsymbol{\phi}^h-\mathbf{c}_2\|_{0,E_j}h^{-1}\|\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\leq C\sum_{j=1}^{N_h}\|\mathbf{c}_1\|\|\nabla\boldsymbol{\phi}^h\|_{0,E_j}\|\boldsymbol{\phi}^h\|_{0,E_j} \leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)}^2\|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Similarly, the second term is bounded

$$S_9 \leq C \sum_{j=1}^{N_h} \|\mathbf{c}_1\| \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}_h\|_{0,E_j} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\phi}^h\|^2_{0,\Omega}.$$

Thus,

$$\tilde{b}(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h) \leq \frac{5\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{\phi}^h\|^2_{0,\Omega}.$$

Let us now bound $\tilde{b}(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h)$.

$$\tilde{b}(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \left(\int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h + \int_{\partial E_j^-} |\{\boldsymbol{\phi}^h\} \cdot \mathbf{n}_{E_j}| (\boldsymbol{\eta}^{\text{int}} - \boldsymbol{\eta}^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,int} \right)$$
$$-\frac{1}{2} c(\boldsymbol{\phi}^h, \boldsymbol{\eta} \cdot \boldsymbol{\phi}^h).$$

The first term is easily bounded:

$$\sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h \leq \sum_{j=1}^{N_h} \|\boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{L^4(E_j)} \|\nabla \boldsymbol{\eta}\|_{L^4(E_j)}$$
$$\leq \frac{\kappa\nu}{32} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\mathbf{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))} \|\boldsymbol{\phi}^h\|^2_{0,\Omega}.$$

The second term is bounded using inequalities (4.9), (4.12), (4.2) and (4.4)

$$\sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\phi}^h\} \cdot \mathbf{n}_{E_j}| (\boldsymbol{\eta}^{\text{int}} - \boldsymbol{\eta}^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,int}) \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\phi}^h\|_{L^4(\partial E_j)} \|\boldsymbol{\eta}^h\|_{L^4(\partial E_j)} \|\boldsymbol{\phi}^h\|_{L^2(\partial E_j)}$$
$$\leq C \sum_{j=1}^{N_h} h^{-3/2} h^{r+1} |\mathbf{u}|_{r+1,\Omega} \|\boldsymbol{\phi}^h\|^2_{0,\Omega} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + C \|\mathbf{u}\|^2_{L^\infty(0,T;H^{r+1}(\Omega))} \|\boldsymbol{\phi}^h\|^2_{0,\Omega}.$$

The last term in $\tilde{b}(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h)$ is bounded like the terms S_6, S_7, S_8 and S_9 of $\tilde{b}(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h)$. The remaining nonlinear terms are bounded in a similar fashion.

$$\tilde{b}_{\mathbf{u}^h}(\boldsymbol{\eta}, R_h(\mathbf{u}), \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\eta} \cdot \nabla R_h(\mathbf{u})) \cdot \boldsymbol{\phi}^h$$

$$+ \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\eta}\} \cdot \mathbf{n}_{E_j}| (R_h(\mathbf{u})^{\text{int}} - R_h(\mathbf{u})^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,\text{int}} + \frac{1}{2} \sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \boldsymbol{\eta}) R_h(\mathbf{u}) \cdot \boldsymbol{\phi}^h$$

$$-\frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\eta}] \cdot \mathbf{n}_k \{R_h(\mathbf{u}) \cdot \boldsymbol{\phi}^h\} = S_{10} + \cdots + S_{13}$$

Using the bound (4.2) and the approximation result (4.3), we have

$$S_{10} \leq \|\boldsymbol{\eta}\|_{L^2(\Omega)} \|\nabla R_h(\mathbf{u})\|_{L^4(\Omega)} \|\boldsymbol{\phi}^h\|_{L^4(\Omega)} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + C\|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

The inequalities (4.7), (4.10), (4.2) and the approximation result (4.3) yield

$$S_{11} \leq C \sum_{j=1}^{N_h} h_{E_j}^{-1/2} (\|\boldsymbol{\eta}\|_{0,E_j} + h_{E_j} \|\nabla \boldsymbol{\eta}\|_{0,E_j}) h_{E_j}^{-1/2} \|\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\leq C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

Similarly, we have

$$S_{12} \leq \sum_{j=1}^{N_h} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\phi}^h\|_{0,E_j} \|\nabla \cdot \boldsymbol{\eta}\|_{0,E_j} \leq C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\mathbf{u}|^2_{r+1,\Omega}.$$

Note that S_{13} is bounded exactly like S_{11}. The other nonlinear term is bounded using (4.3) and (4.10)

$$\tilde{b}_{\mathbf{u}^h}(\mathbf{u}, \boldsymbol{\eta}, \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \int_{E_j} (\mathbf{u} \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\mathbf{u}\} \cdot \mathbf{n}_{E_j}| (\boldsymbol{\eta}^{\text{int}} - \boldsymbol{\eta}^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,\text{int}}$$
$$\leq C \sum_{j=1}^{N_h} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)} \|\nabla \boldsymbol{\eta}\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{0,E_j} + C \sum_{j=1}^{N_h} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\eta}\|_{0,\partial E_j} \|\boldsymbol{\phi}^h\|_{0,\partial E_j}$$
$$\leq C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\mathbf{u}|^2_{r+1,\Omega}$$

Combining all bounds above and using (4.16), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_0^2 + \frac{\kappa\nu}{2}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\|(I - P_{L^H})\nabla\boldsymbol{\phi}^h\|_0^2 \leq C(\frac{1}{\nu}+1)\|\boldsymbol{\phi}^h\|_0^2$$
$$+Ch^{2r}(\nu + \frac{1}{\nu} + \nu_T)|\mathbf{u}|_{r+1,\Omega}^2 + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2 + Ch^{2r}|\mathbf{u}_t|_{r+1,\Omega}^2 + C\nu_T H^{2r}|\mathbf{u}|_{r+1,\Omega}^2.$$

Integrating over 0 and t, noting that $\|\boldsymbol{\phi}^h(0)\|_0$ is of the order h^r and using Gronwall's lemma, yields:

$$\|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|_{L^2(0,t;X)}^2 + \nu_T\|(I - P_{L^H})\nabla\boldsymbol{\phi}^h\|_{L^2(0,t;L^2(\Omega))}^2$$
$$\leq Ce^{C(1+\nu^{-1})}h^{2r}[(\nu+\nu^{-1}+\nu_T)|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + \nu^{-1}|p|_{L^2(0,T;H^r(\Omega))}^2$$
$$+|\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}^2 + \nu_T H^{2r}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2] + Ch^r|\mathbf{u}_0|_{r+1,\Omega}^2.$$

where the constant C is independent of ν, ν_T, h, H but depends on $\|\mathbf{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}$. The theorem is obtained using the approximation results (4.3), (4.4) and the following inequality:

$$\|\mathbf{u}(t) - \mathbf{u}^h(t)\|_0^2 + \kappa\nu\|\mathbf{u}(t) - \mathbf{u}^h(t)\|_{L^2(0,T;X)}^2 + \nu_T\|(I - P_{L^H})\nabla(\mathbf{u}(t) - \mathbf{u}^h(t))\|_{L^2(0,T;L^2(\Omega))}^2$$
$$\leq \|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|_{L^2(0,T;X)}^2 + \nu_T\|(I - P_{L^H})\nabla\boldsymbol{\phi}^h\|_{L^2(0,T;L^2(\Omega))}^2$$
$$+\|\boldsymbol{\eta}(t)\|_0^2 + \kappa\nu\|\boldsymbol{\eta}\|_{L^2(0,T;X)}^2 + \nu_T\|(I - P_{L^H})\nabla\boldsymbol{\eta}\|_{L^2(0,T;L^2(\Omega))}^2.$$

Remark 4.3. *One of the most important property of the Theorem 4.1 is that the new method improves its robustness with respect to the Reynolds number. In most cases, error estimations of Navier Stokes equations gives a Gronwall constant that depends on the Reynolds number as $1/\nu^3$. In contrast, this approach leads to a better error estimate with a Gronwall constant depending on $1/\nu$.*

Optimal convergence rates are obtained for Theorem 4.1 if ν_T and H are appropriately chosen.

Corollary 4.4.1. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$. If the relation $\beta \geq 2r(\alpha-1)/\alpha$ is satisfied, then the estimate becomes*

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} = \mathcal{O}(h^r).$$

For example, one may choose for a linear approximation the pair $(\nu_T, H) = (h, h^{1/2})$, for quadratic approximation $(\nu_T, H) = (h, h^{3/4})$ or $(\nu_T, H) = (h^2, h^{1/2})$, and for cubic approximation $(\nu_T, H) = (h, h^{5/6})$ or $(\nu_T, H) = (h^2, h^{2/3})$.

Remark 4.4. *By using the inf-sup condition (4.23) and the estimates for the* $\|(\mathbf{u}-\mathbf{u}^h)_t\|$, *a priori error estimate for the pressure can also be derived as in Chapter 1. Since the analysis is lengthy and is similar, we present this analysis in Appendix A.*

4.5 Fully discrete scheme

In this section, we formulate two fully discrete finite element schemes for the discontinuous eddy viscosity method. Let Δt denote the time step, let $M = T/\Delta t$ and let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the interval $(0, T)$. We denote the function ϕ evaluated at the time t_m by ϕ_m and the average of ϕ at two successive time levels by $\phi_{m+\frac{1}{2}} = \frac{1}{2}(\phi_m + \phi_{m+1})$.

Scheme 1: Given $\mathbf{u}_0^h$, find $(\mathbf{u}_m^h)_{m\geq 1}$ in $\mathbf{X}^h$ and $(p_m^h)_{m\geq 1}$ in Q^h such that

$$\frac{1}{\Delta t}(\mathbf{u}_{m+1}^h - \mathbf{u}_m^h, \mathbf{v}^h) + \nu(a(\mathbf{u}_{m+1}^h, \mathbf{v}^h) + J(\mathbf{u}_{m+1}^h, \mathbf{v}^h)) + \tilde{b}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \mathbf{v}^h)$$
$$+\nu_T g(\mathbf{u}_{m+1}^h, \mathbf{v}^h) + c(\mathbf{v}^h, p_{m+1}^h) = (\mathbf{f}_{m+1}, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.33}$$
$$c(\mathbf{u}_{m+1}^h, q^h) = 0, \quad \forall q^h \in Q^h. \tag{4.34}$$

Scheme 2: Given $\tilde{\boldsymbol{u}}_0^h, \tilde{\boldsymbol{u}}_1^h, \tilde{p}_1^h$, find $(\tilde{\boldsymbol{u}}_m^h)_{m\geq 2}$ in $\mathbf{X}^h$ and $(\tilde{p}_m^h)_{m\geq 2}$ in Q^h such that

$$\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}_{m+1}^h - \tilde{\boldsymbol{u}}_m^h, \mathbf{v}^h) + \nu(a(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \mathbf{v}^h) + J(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \mathbf{v}^h)) + \tilde{b}(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \mathbf{v}^h)$$
$$+\nu_T g(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \mathbf{v}^h) + c(\mathbf{v}^h, \tilde{p}_{m+\frac{1}{2}}^h) = (\mathbf{f}_{m+\frac{1}{2}}, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.35}$$
$$c(\tilde{\boldsymbol{u}}_{m+1}^h, q^h) = 0, \quad \forall q^h \in Q^h. \tag{4.36}$$

For both schemes, the initial velocity is defined to be the L^2 projection of $\mathbf{u}_0$. Scheme 1 is based on a backward Euler discretization. Scheme 2 is based on a Crank-Nicolson discretization, and requires the velocity and pressure at the first step. The approximations $\tilde{\boldsymbol{u}}_1^h$ and $\tilde{p}_1^h$ can be obtained by a first order scheme (see [5]). We will show that Scheme 1 is first order in time, and Scheme 2 second order in time. First, we prove the stability of the schemes.

Lemma 4.5.1. *The solution* $(\mathbf{u}_m^h)_m$ *of (4.33),(4.34) remains bounded in the following sense*

$$
\begin{gathered}
\|\mathbf{u}_m^h\|_{0,\Omega}^2 \leq K, \quad m = 0, \ldots, M, \\
\Delta t \sum_{m=0}^{M-1} \|\mathbf{u}_{m+1}^h\|_X^2 \leq \frac{K}{2\nu}, \quad \Delta t \sum_{m=0}^{M-1} |\!|\!|(I - P_{L^H})\nabla \mathbf{u}_{m+1}^h|\!|\!|_0^2 \leq \frac{K}{2\nu_T},
\end{gathered}
$$

where $K = \|\mathbf{u}_0\|_{0,\Omega}^2 + \|\mathbf{f}\|_{L^2([0,T]\times\Omega)}^2$.

The solution $(\tilde{\boldsymbol{u}}_m^h)_m$ *of (4.35),(4.36) remains bounded in the following sense*

$$
\begin{gathered}
\|\tilde{\boldsymbol{u}}_m^h\|_{0,\Omega}^2 \leq \tilde{K}, \quad m = 0, \ldots, M, \\
\Delta t \sum_{m=0}^{M-1} \|\tilde{\boldsymbol{u}}_{m+1}^h\|_X^2 \leq \frac{\tilde{K}}{2\nu}, \quad \Delta t \sum_{m=0}^{M-1} \|(I - P_{L^H})\nabla \tilde{\boldsymbol{u}}_{m+1}^h\|_{0,\Omega}^2 \leq \frac{\tilde{K}}{2\nu_T},
\end{gathered}
$$

where $\tilde{K} = \|\mathbf{u}_0\|_{0,\Omega}^2 + 2\|\mathbf{f}\|_{L^2([0,T]\times\Omega)}^2$.

Proof: Choose $\mathbf{v}^h = \mathbf{u}_{m+1}^h$ in (4.33) and $q^h = p_{m+1}^h$ in (4.34). We multiply by $2\Delta t$ and sum over m. Then, from the positivity of $\tilde{b}$, (4.16), we have

$$
\begin{aligned}
\|\mathbf{u}_m^h\|_{0,\Omega}^2 - \|\mathbf{u}_0^h\|_{0,\Omega}^2 + 2\kappa\nu\Delta t \sum_{j=0}^{m-1} \|\mathbf{u}_{j+1}^h\|_X^2 + 2\nu_T \Delta t \sum_{j=0}^{m-1} |\!|\!|(I - P_{L^H})\nabla \mathbf{u}_{j+1}^h|\!|\!|_0^2 \\
\leq \Delta t \sum_{j=0}^{m-1} \|\mathbf{f}_{j+1}\|_{0,\Omega}^2 + \Delta t \sum_{j=0}^{m-1} \|\mathbf{u}_{j+1}^h\|_{0,\Omega}^2.
\end{aligned}
$$

The result is obtained by using a discrete version of Gronwall's lemma [32] and the fact that $\|\mathbf{u}_0^h\|_{0,\Omega} \leq \|\mathbf{u}_0\|_{0,\Omega}$.

For Scheme 2, the proof is similar. Choose $\mathbf{v}^h = \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}$ in (4.35) and $q^h = \tilde{p}_{m+\frac{1}{2}}^h$ in (4.36). The rest of the proof follows as above.

Theorem 4.2. *Under the assumptions of Theorem 4.1 and if* $\mathbf{u}_t$ *and* $\mathbf{u}_{tt}$ *belong to* $L^\infty(0,T;L^2(\Omega))$, *there is a constant C independent of h, H, ν and ν_T such that*

$$\max_{m=0,\dots,M} \|\mathbf{u}_m - \mathbf{u}_m^h\|_{0,\Omega} + (\nu\kappa\Delta t \sum_{m=0}^{M-1} \|\mathbf{u}_{m+1} - \mathbf{u}_{m+1}^h\|_X^2)^{1/2}$$

$$+(\nu_T \Delta t \sum_{m=0}^{M} \|\!|(I - P_{L^H})(\nabla \mathbf{u}_{m+1} - \mathbf{u}_{m+1}^h)\|\!|_0^2)^{1/2} \leq Ch^r |\mathbf{u}_0|_{r+1,\Omega}$$

$$+Ce^{CT\nu^{-1}}[h^r(\nu + \nu^{-1} + \nu_T)^{1/2} |\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T^{1/2} H^r |\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$

$$+\nu^{-1/2}\Delta t(\|\mathbf{u}_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}) + h^r \nu^{-1/2} |p|_{L^2(0,T;H^r(\Omega))}]$$

Proof: As in the continuous case, we set $\mathbf{e}_m = \mathbf{u}_m - \mathbf{u}_m^h$. We subtract to (4.33) and (4.34) the equations (4.19) and (4.20) evaluated at time $t = t_{m+1}$.

$$(\mathbf{u}_t(t_{m+1}), \mathbf{v}^h) - \frac{1}{\Delta t}(\mathbf{u}_{m+1}^h - \mathbf{u}_m^h, \mathbf{v}^h) + \nu[a(\mathbf{e}_{m+1}, \mathbf{v}^h) + J(\mathbf{e}_{m+1}, \mathbf{v}^h)]$$

$$+\nu_T g(\mathbf{e}_{m+1}, \mathbf{v}^h) + \tilde{b}(\mathbf{u}_{m+1}, \mathbf{u}_{m+1}, \mathbf{v}^h) - \tilde{b}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \mathbf{v}^h)$$

$$+c(\mathbf{v}^h, p_{m+1} - p_{m+1}^h) = \nu_T g(\mathbf{u}_{m+1}, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.37}$$

$$c(\mathbf{e}_{m+1}, q^h) = 0, \quad \forall q^h \in Q^h. \tag{4.38}$$

Define $\boldsymbol{\phi}_m = \mathbf{u}_m^h - (R_h(\mathbf{u}))_m$, $\boldsymbol{\eta}_m = \mathbf{u}_m - (R_h(\mathbf{u}))_m$. Choose $\mathbf{v}^h = \boldsymbol{\phi}_{m+1}$ in (4.37) and $q^h = p_{m+1}^h$ in (4.38). Adding and subtracting the interpolant and using (4.16) yields the following error equation:

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T \|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0^2$$

$$+\tilde{b}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - \tilde{b}(\mathbf{u}_{m+1}, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})$$

$$\leq \|\frac{\partial \mathbf{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\mathbf{u}_{m+1} - \mathbf{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$

$$+\nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + \nu_T\|\!|(I - P_{L^H})\nabla\boldsymbol{\eta}_{m+1}\|\!|_0 \|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0$$

$$+\nu_T\|\!|(I - P_{L^H})\nabla\mathbf{u}_{m+1}\|\!|_0 \|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0.$$

We rewrite the nonlinear terms:

$$
\begin{aligned}
&\tilde{b}_{\mathbf{u}_m^h}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - \tilde{b}_{\mathbf{u}_{m+1}}(\mathbf{u}_{m+1}, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}) \\
&= \tilde{b}_{\mathbf{u}_m^h}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - \tilde{b}_{\mathbf{u}_m^h}(\mathbf{u}_{m+1}, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}).
\end{aligned}
$$

We now drop the subscript $\mathbf{u}_m^h$.

$$
\begin{aligned}
&\tilde{b}_{\mathbf{u}_m^h}(\mathbf{u}_m^h, \mathbf{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - \tilde{b}_{\mathbf{u}_m^h}(\mathbf{u}_{m+1}, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}) \\
&= \tilde{b}(\mathbf{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) - \tilde{b}(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + \tilde{b}(\boldsymbol{\phi}_m, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}) \\
&-\tilde{b}(\boldsymbol{\eta}_m, \mathbf{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) - \tilde{b}(\mathbf{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) - \tilde{b}(\mathbf{u}_{m+1} - \mathbf{u}_m, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1}).
\end{aligned}
$$

Thus, we rewrite the error equation as

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T\|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0^2 \\
&+\tilde{b}(\mathbf{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) \le |\tilde{b}(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + |\tilde{b}(\boldsymbol{\phi}_m, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1})| + |\tilde{b}(\boldsymbol{\eta}_m, \mathbf{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})| \\
&+|\tilde{b}(\mathbf{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + |\tilde{b}(\mathbf{u}_{m+1} - \mathbf{u}_m, \mathbf{u}_{m+1}, \boldsymbol{\phi}_{m+1})| + |c(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})| \\
&+\|\frac{\partial \mathbf{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\mathbf{u}_{m+1} - \mathbf{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} \\
&+\nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + \nu_T\|\!|(I - P_{L^H})\nabla\boldsymbol{\eta}_{m+1}\|\!|_0\|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0 \\
&+\nu_T\|\!|(I - P_{L^H})\nabla\mathbf{u}_{m+1}\|\!|_0\|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+1}\|\!|_0 \le |T_0| + \cdots + |T_{10}|.
\end{aligned}
$$

We want to bound the terms $T_0, T_2....T_{10}$. T_0 can be handled as in Theorem 4.1. Then, T_0 is bounded as

$$
T_0 \le \frac{\kappa\nu}{6}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}(\|\mathbf{u}\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2 + \|\mathbf{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2)\|\boldsymbol{\phi}_m\|_{0,\Omega}^2
$$

Also, the term T_1 is bounded exactly like the term (4.32) in the proof of Theorem 4.1. Here, the constant vectors are

$$
\mathbf{c}_1 = \frac{1}{|E_j|}\int_{E_j} \mathbf{u}_{m+1}, \quad \mathbf{c}_2 = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{\phi}_{m+1}.
$$

Then, T_1 can be rewritten as:

$$T_1 = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}_m \cdot \nabla \mathbf{u}_{m+1}) \cdot \boldsymbol{\phi}_{m+1} - \frac{1}{2} c(\boldsymbol{\phi}_m, (\mathbf{u}_{m+1} - \mathbf{c}_1) \cdot \boldsymbol{\phi}_{m+1})$$
$$-\frac{1}{2} c(\boldsymbol{\phi}_m, \mathbf{c}_1 \cdot (\boldsymbol{\phi}_{m+1} - \mathbf{c}_2)) \leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} \|\boldsymbol{\phi}_m\|_{0,\Omega}^2.$$

Expanding T_2, we obtain:

$$T_2 = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\eta}_m \cdot \nabla \mathbf{u}_{m+1}^I) \cdot \boldsymbol{\phi}_{m+1} + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\eta}_m\} \cdot \mathbf{n}_{E_j}| (\mathbf{u}_{m+1}^{I,\text{int}} - \mathbf{u}_{m+1}^{I,\text{ext}}) \cdot \boldsymbol{\phi}_{m+1}^{int}$$
$$+\frac{1}{2} \sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \boldsymbol{\eta}_m) \mathbf{u}_{m+1}^I \cdot \boldsymbol{\phi}_{m+1} - \frac{1}{2} \sum_{k=1}^{P_h} \int_{e_k} [\boldsymbol{\eta}_m] \cdot \mathbf{n}_k \{\mathbf{u}_{m+1}^I \cdot \boldsymbol{\phi}_{m+1}\}$$
$$= T_{21} + \cdots + T_{24}.$$

The bound for T_{21} is obtained using (4.2) and (4.4):

$$T_{21} \leq \|\boldsymbol{\eta}_m\|_{0,\Omega} \|\nabla \mathbf{u}_{m+1}^I\|_{L^4(\Omega)} \|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)}$$
$$\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} h^{2r} \|\mathbf{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2 |\mathbf{u}_m|_{r+1,\Omega}^2.$$

Similarly for the term T_{22}, the inequalities (4.3) and (4.10) give

$$T_{22} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\eta}_m\|_{L^2(\partial E_j)} \|\mathbf{u}_{m+1}^I\|_{L^\infty(\Omega)} \|\boldsymbol{\phi}_{m+1}\|_{L^2(\partial E_j)}$$
$$\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} h^{2r} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)}^2 |\mathbf{u}_m|_{r+1,\Omega}^2.$$

The estimate of T_{23} is obtained by using a bound on interpolant, Cauchy-Schwarz inequality, the approximation result (4.3), Young's inequality and L^p bound (4.2).

$$T_{23} \leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} h^{2r} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)}^2 |\mathbf{u}_m|_{r+1,\Omega}^2.$$

The term T_{24} is bounded exactly as for T_{22}. Because of the regularity of $\mathbf{u}$, the approximation result (4.3), we can bound T_3.

$$T_3 \le C\|\mathbf{u}_m\|_{L^\infty(\Omega)} h^r |\mathbf{u}_{m+1}|_{r+1,\Omega} \|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$
$$\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}\|\mathbf{u}\|^2_{L^\infty([0,T]\times\Omega)}|\mathbf{u}_m|^2_{r+1,\Omega}.$$

The term T_4 is bounded using the estimate (4.2).

$$T_4 \le \Delta t\|\mathbf{u}_t\|_{L^\infty(t_m,t_{m+1};L^2(\Omega))}\|\nabla\mathbf{u}_{m+1}\|_{L^4(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)}$$
$$\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2\|\mathbf{u}_t\|^2_{L^\infty(t_m,t_{m+1};L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}.$$

By property of the interpolant (4.24) and properties of $r_h(p)$ (4.5), (4.6), we now bound T_5.

$$T_5 = c(\boldsymbol{\phi}_{m+1}, p^h_{m+1} - (r_h(p))_{m+1}) - c(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1})$$
$$= -c(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1}) = \sum_{k=1}^{M_h}\int_{e_k}\{p_{m+1} - (r_h(p))_{m+1}\}[\boldsymbol{\phi}_{m+1}]\cdot\mathbf{n}_k$$
$$\le \sum_{k=1}^{M_h}\|[\boldsymbol{\phi}_{m+1}]\|_{0,e_k}|e_k|^{1/2-1/2}\|p_{m+1}\|_{0,e_k} \le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}|p_{m+1}|^2_{r,\Omega}.$$

From a Taylor expansion, we have

$$T_6 \le C\Delta t\|\boldsymbol{\phi}_{m+1}\|_X\|\mathbf{u}_{tt}(t^*)\|_{0,\Omega} \le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2\|\mathbf{u}_{Tm}\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

To bound T_7, we assume that $h \le \Delta t$ and we use (4.4) and (4.2).

$$T_7 \le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\frac{h^{2r+2}}{\Delta t^2}(|\mathbf{u}_{m+1}|^2_{r+1,\Omega} + |\mathbf{u}_m|^2_{r+1,\Omega})$$
$$\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}(|\mathbf{u}_{m+1}|^2_{r+1,\Omega} + |\mathbf{u}_m|^2_{r+1,\Omega}).$$

The terms T_8, T_9 and T_{10} are exactly bounded as in Theorem 4.1. Combining all the bounds of the terms $T_0, \ldots, T_{10}$, multiplying by $2\Delta t$ and summing over m, we obtain:

$$\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_0\|_{0,\Omega}^2 + \nu\kappa\Delta t \sum_{i=0}^{m} \|\boldsymbol{\phi}_{i+1}\|_X^2 + \nu_T \Delta t \sum_{i=0}^{m} \|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{i+1}\|\!|_0^2$$
$$\leq Ce^{CT\nu^{-1}}[h^{2r}(\nu + \nu^{-1} + \nu_T)|\mathbf{u}|^2_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T H^{2r}|\mathbf{u}|^2_{L^2(0,T;H^{r+1}(\Omega))}$$
$$+\nu^{-1}\Delta t^2(\|\mathbf{u}_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}) + h^{2r}\nu^{-1}|p|^2_{L^2(0,T;H^r(\Omega))}]$$

The final result is obtained by noting that $\|\boldsymbol{\phi}_0\|_{0,\Omega}$ is of order h^r and by using approximation results and a triangle inequality.

Theorem 4.3. *Assume that* $\mathbf{u}_{tt} \in L^\infty(0,T;(H^1(\Omega))^2)$, $p_{tt} \in L^\infty(0,T;H^1(\Omega))$, $\mathbf{u}_{ttt} \in L^\infty(0,T;(H^2(\Omega))^2)$ *and* $\mathbf{f}_{tt} \in L^\infty(0,T;(L^2(\Omega))^2)$. *Under the assumptions of Theorem 4.1, there is a constant* C *independent of* h, H, ν *and* ν_T *such that*

$$\max_{m=0,\ldots,M} \|\mathbf{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + (\nu\kappa\Delta t \sum_{m=0}^{M-1} \|\mathbf{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_X^2)^{1/2}$$
$$+(\nu_T\Delta t \sum_{m=0}^{M-1} \|\!|(I - P_{L^H})\nabla\mathbf{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|\!|_0^2)^{1/2} \leq Ce^{CT\nu^{-1}}[h^r\nu^{-1/2}\|p\|_{L^2(0,T;H^r(\Omega))}$$
$$+h^r(\nu + \nu^{-1} + \nu_T)^{1/2}\|\mathbf{u}\|_{L^2(0,T;H^{r+1}(\Omega))} + \Delta t^2\nu^{1/2}\|\mathbf{u}_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+\Delta t^2\nu^{-1/2}(\|\mathbf{u}_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|p_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+\|\mathbf{f}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}) + \nu_T^{1/2}H^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\mathbf{u}_0|_{r+1,\Omega}.$$

Proof: The proof is derived in a similar fashion as for the backward Euler scheme. Using the same notation, the error equation is obtained by subtracting the equation (4.19) evaluated at the time $t = t_{m+1/2}$ to the equation (4.35) and adding and subtracting the interpolant $(R_h(\mathbf{u}))_{m+1/2}$.

After some manipulation, we obtain

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + \nu_T\|\!|(I-P_{L^H})\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}\|\!|_0^2 \\
&+\tilde{b}(\tilde{\mathbf{u}}^h_{m+\frac{1}{2}}, \boldsymbol{\phi}^h_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) \le |\tilde{b}(\boldsymbol{\phi}_{m+\frac{1}{2}}, \boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |\tilde{b}(\boldsymbol{\phi}_{m+\frac{1}{2}}, \mathbf{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|\tilde{b}(\boldsymbol{\eta}_{m+\frac{1}{2}}, \mathbf{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |\tilde{b}(\mathbf{u}_{m+\frac{1}{2}}, \boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|\tilde{b}(\mathbf{u}_{m+\frac{1}{2}} - \mathbf{u}(t_{m+\frac{1}{2}}), \mathbf{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |\tilde{b}(\mathbf{u}(t_{m+\frac{1}{2}}), \mathbf{u}_{m+\frac{1}{2}} - \mathbf{u}(t_{m+\frac{1}{2}}), \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|c(\boldsymbol{\phi}_{m+\frac{1}{2}}, \tilde{p}^h_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))| + \|\mathbf{u}_t(t_{m+\frac{1}{2}}) - \frac{1}{\Delta t}(\mathbf{u}_{m+1} - \mathbf{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \\
&+\frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} + \|\mathbf{f}_{m+\frac{1}{2}} - \mathbf{f}(t_{m+\frac{1}{2}})\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \\
&+\nu|a(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+1}) + J(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+1})| \\
&+\nu_T\|\!|(I-P_{L^H})\nabla\boldsymbol{\eta}_{m+\frac{1}{2}}\|\!|_0\|\!|(I-P_{L^H})\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}\|\!|_0 \\
&+\nu_T\|\!|(I-P_{L^H})\nabla\mathbf{u}_{m+\frac{1}{2}}\|\!|_0\|\!|(I-P_{L^H})\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}\|\!|_0 \le A_0 + \cdots + A_{13}.
\end{aligned}
$$

The terms $A_0, A_1, A_2, A_3, A_8, A_{11}$ and A_{12} are bounded exactly like the terms T_0, T_1, T_2, T_3, T_7, T_9 and T_{10} respectively. From a Taylor expansion, we bound the terms A_4 and A_5:

$$
\begin{aligned}
A_4 + A_5 &= \sum_{j=1}^{N_h}\int_{E_j}(\mathbf{u}_{m+\frac{1}{2}} - \mathbf{u}(t_{m+\frac{1}{2}}))\cdot\nabla\mathbf{u}_{m+\frac{1}{2}})\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&+\sum_{j=1}^{N_h}\int_{E_j}\mathbf{u}(t_{m+\frac{1}{2}})\cdot\nabla(\mathbf{u}_{m+\frac{1}{2}} - \mathbf{u}(t_{m+\frac{1}{2}}))\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&= \frac{\Delta t^2}{8}\sum_{j=1}^{N_h}\int_{E_j}(\mathbf{u}_{tt}(t^*))\cdot\nabla\mathbf{u}_{m+\frac{1}{2}})\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} + \frac{\Delta t^2}{8}\sum_{j=1}^{N_h}\int_{E_j}\mathbf{u}(t_{m+\frac{1}{2}})\cdot\nabla(\mathbf{u}_{tt}(t^*))\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&\le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}\Delta t^4\|\mathbf{u}_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}.
\end{aligned}
$$

With (4.20), (4.24) and (4.36), the pressure term can be rewritten as:

$$
\begin{aligned}
A_6 &= c(\boldsymbol{\phi}_{m+\frac{1}{2}}, \tilde{p}^h_{m+\frac{1}{2}} - p_{m+\frac{1}{2}}) + c(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}})) \\
&= -c(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}) + c(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}})) \\
&= \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \mathbf{n}_k - \sum_{j=1}^{N_h} \int_{E_j} (p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))\nabla \cdot \boldsymbol{\phi}_{m+\frac{1}{2}} \\
&\quad + \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}})\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \mathbf{n}_k \\
&\leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1} h^{2r}(|p_{m+1}|_{r,\Omega}^2 + |p_m|_{r,\Omega}^2) + C\nu^{-1}\Delta t^4 \|p_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned}
$$

We now bound A_7, using a Taylor expansion,

$$
A_7 \leq C\Delta t^2 \|\mathbf{u}_{ttt}(t^*)\|_{0,\Omega} \|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}\Delta t^4 \|\mathbf{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}.
$$

Using also a Taylor expansion, we bound A_9:

$$
A_9 \leq C\nu^{-1}\Delta t^4 \|\mathbf{f}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\kappa\nu}{64} \|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2.
$$

Finally the last term A_{10} is handled as follows:

$$
\begin{aligned}
A_{10} &= \nu[a(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})] \\
&+\nu[a(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})] = A_{101} + A_{102}.
\end{aligned}
$$

The term A_{101} is bounded like T_8. The term A_{102} reduces to

$$
\begin{aligned}
A_{102} &= \nu \sum_{j=1}^{N_h} \int_{E_j} \nabla(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}_{m+\frac{1}{2}}) : \nabla \boldsymbol{\phi}_{m+\frac{1}{2}} \\
&-\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla(\mathbf{u}(t_{m+\frac{1}{2}}) - \mathbf{u}_{m+\frac{1}{2}})\mathbf{n}_k\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu\Delta t^4 \|\mathbf{u}_{tt}\|^2_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}
$$

Combining all the bounds above yield:

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \frac{\nu\kappa}{2}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + \frac{\nu_T}{2}|\!|\!|(I - P_{L^H})\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}|\!|\!|_0^2 \\
&\leq C\nu^{-1}(\|\boldsymbol{\phi}_m\|_{0,\Omega}^2 + \|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2) + Ch^{2r}(\nu + \nu^{-1} + \nu_T)(|\mathbf{u}_{m+1}|_{r+1,\Omega}^2 + |\mathbf{u}_m|_{r+1,\Omega}^2) \\
&\quad + Ch^{2r}\nu^{-1}(|p_{m+1}|_{r,\Omega}^2 + |p_m|_{r,\Omega}^2) + C\Delta t^4\nu\|\mathbf{u}_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}^2 \\
&\quad + C\Delta t^4\nu^{-1}(\|\mathbf{u}_{tt}\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|p_{tt}\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|\mathbf{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
&\quad + \|\mathbf{f}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2) + C\nu_T H^{2r}(|\mathbf{u}_{m+1}|_{r+1,\Omega}^2 + |\mathbf{u}_m|_{r+1,\Omega}^2).
\end{aligned}
$$

The end of the proof is similar to the one of Theorem 4.2.

Corollary 4.5.1. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$ where $\beta \geq 2r(\alpha-1)/\alpha$ (see Corollary 4.4.1), then the estimates in Theorem 4.2 and Theorem 4.3 are optimal.*

$$
\begin{aligned}
\max_{m=0,\dots,M}\|\mathbf{u}_m - \mathbf{u}_m^h\|_{0,\Omega} + (\Delta t\sum_{m=0}^{M-1}\|\mathbf{u}_{m+1} - \mathbf{u}_{m+1}^h\|_X^2)^{1/2} &= \mathcal{O}(h^r + \Delta t), \\
\max_{m=0,\dots,M}\|\mathbf{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + (\Delta t\sum_{m=0}^{M-1}\|\mathbf{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_X^2)^{1/2} &= \mathcal{O}(h^r + \Delta t^2).
\end{aligned}
$$

Chapter 5

Conclusions and Future Research

In this thesis we have performed and analyzed a subgrid eddy viscosity method for solving the time dependent Navier-Stokes equations. This method has the advantage that the diffusivity is introduced only on the small scales. By adding an eddy viscosity term on the coarse scale, the resulting method dissipates the energy of scales near the cutoff wave number. The method is robustness with respect to the Re. Specifically, the analysis for the Oseen problem establishes that the error is bounded uniformly in Re. Numerical test shows the new stabilization technique is robust and efficient in solving Navier-Stokes equations for a wide range of Reynolds numbers.

In addition, we have analyzed the stability and convergence of totally discontinuous schemes for solving the time-dependent Navier-Stokes equations. Both semi discrete approximation and fully discrete are constructed for velocity. In addition, semi discrete approximation of pressure is obtained. We have showed that these estimations are optimal.

This thesis suggests several directions for further research. One of the future directions is to extend the numerical simulations with time dependent Navier-Stokes equations and to do experiments for the computational benchmarks problems. Specifically, we should test the model for channel flow and flows over step problems.

Another research direction of this work is to reduce the dependence of the estimate on the Reynolds number. One way is to extend the analysis for some popular nonlinear eddy viscosity. We can try to use modified version of the approach in [41].

Also, we should perform a posteriori error estimation to improve the accuracy of the method while reducing computational cost. This can be done by investigation of explicit error estimates involving the computed numerical solution and the subgrid eddy viscosity constant. It would also be of interest to study adaptive subgrid eddy viscosity, in which eddy viscosity affects local regions of the flow.

Appendix A

Discontinuous L^2 Pressure Estimation

Theorem A.1. *Under the assumptions of Theorem 4.1, and if $a(.,.)$ is symmetric ($\epsilon_0 = -1$), the following estimate holds true*

$$\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;L^2(\Omega))} + \nu^{1/2}\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;X)} \leq Ce^{CT\nu^{-1}}[h^r|\mathbf{u}_0|_{r+1,\Omega}$$
$$+h^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))} + h^r|\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu_T H^r h^{-1}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}].$$

where C and is a positive constant independent of h, H, ν and ν_T. If $a(.,.)$ is nonsymmetric ($\epsilon_0 = 1$), the estimate is suboptimal, of order h^{r-1}.

Proof: We introduce the modified Stokes problem: for any $t > 0$, find $(\mathbf{u}^S(t), p^S(t)) \in \mathbf{X}^h \times Q^h$ such that

$$\nu(a(\mathbf{u}^S(t), \mathbf{v}^h) + J(\mathbf{u}^S(t), \mathbf{v}^h)) + \nu_T g(\mathbf{u}^S(t), \mathbf{v}^h) + c(\mathbf{v}^h, p^S(t))$$
$$= \nu(a(\mathbf{u}(t), \mathbf{v}^h) + J(\mathbf{u}(t), \mathbf{v}^h)) + \nu_T g(\mathbf{u}(t), \mathbf{v}^h) + c(\mathbf{v}^h, p(t)), \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{A.1}$$
$$c(\mathbf{u}^S(t), q^h) = 0, \quad \forall q^h \in Q^h. \tag{A.2}$$

For any $t > 0$, there exists a unique solution to (A.1), (A.2). Furthermore, it is easy to show that the solution satisfies the error estimate:

$$\kappa^{1/2}\nu^{1/2}\|\mathbf{u}(t) - \mathbf{u}^S(t)\|_X + \nu_T^{1/2}\|(I - P_H)\nabla(\mathbf{u} - \mathbf{u}^S)\|_{0,\Omega}$$
$$\leq h^r(\nu + \nu^{-1} + \nu_T)^{1/2}|\mathbf{u}|_{r+1,\Omega} + \nu^{-1/2}|p|_{r,\Omega} + |\mathbf{u}_t|_{r+1,\Omega}) + \nu_T^{1/2}H^r|\mathbf{u}|_{r+1,\Omega}, \quad \forall t > 0.$$

Define $\boldsymbol{\eta} = \mathbf{u} - \mathbf{u}^S$ and $\boldsymbol{\xi} = \mathbf{u}^h - \mathbf{u}^S$, and choose the test function $\mathbf{v}^h = \boldsymbol{\xi}_t$. The resulting error equation is:

$$
\begin{aligned}
&\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi}, \boldsymbol{\xi}_t) + \frac{\nu}{2}\frac{d}{dt}J(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi}, \boldsymbol{\xi}) \\
&= (\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) - \nu_T g(\mathbf{u}, \boldsymbol{\xi}_t) + \tilde{b}(\mathbf{u}, \mathbf{u}, \boldsymbol{\xi}_t) - \tilde{b}(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\xi}_t). \qquad \text{(A.3)}
\end{aligned}
$$

The first term in the right-hand side of (A.3) is easily bounded.

$$
(\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) \leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}_t|_{r+1,\Omega}^2.
$$

The consistency error term is bounded using the inverse inequality (4.12) and the projection operator.

$$
\begin{aligned}
\nu_T g(\mathbf{u}, \boldsymbol{\xi}_t) &\leq C\nu_T H^r |\mathbf{u}|_{r+1,\Omega} \|\nabla \boldsymbol{\xi}_t\|_{0,\Omega} \\
&\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\nu_T^2 H^{2r} h^{-2} |\mathbf{u}|_{r+1,\Omega}^2.
\end{aligned}
$$

Let us rewrite the nonlinear terms

$$
\begin{aligned}
&\tilde{b}(\mathbf{u}, \mathbf{u}, \boldsymbol{\xi}_t) - \tilde{b}(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\xi}_t) = \tilde{b}(\mathbf{u} - \mathbf{u}^S, \mathbf{u}^h, \boldsymbol{\xi}_t) - \tilde{b}(\boldsymbol{\xi}, \mathbf{u}^h, \boldsymbol{\xi}_t) + \tilde{b}(\mathbf{u}, \mathbf{u} - \mathbf{u}^h, \boldsymbol{\xi}_t) \\
&= \tilde{b}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}_t) - \tilde{b}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\xi}_t) + \tilde{b}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\xi}_t) - \tilde{b}(\boldsymbol{\eta}, \mathbf{u}^h, \boldsymbol{\xi}_t) + \tilde{b}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\xi}_t) - \tilde{b}(\mathbf{u}, \boldsymbol{\eta}, \boldsymbol{\xi}_t).
\end{aligned}
$$

In what follows, we assume that $\boldsymbol{\xi}$ belongs to $L^\infty((0,T) \times \Omega)$.

We now consider each of the nonlinear terms. Expanding $\tilde{b}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}_t)$ results

$$
\begin{aligned}
\tilde{b}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}_t) &= \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi} \cdot \boldsymbol{\xi}_t + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\xi}\} \cdot \mathbf{n}_{E_j}| (\boldsymbol{\xi}^{int} - \boldsymbol{\xi}^{ext}) \cdot \boldsymbol{\xi}_t^{int} \\
&+ \frac{1}{2}\sum_{j=1}^{N_h} (\nabla \cdot \boldsymbol{\xi}) \boldsymbol{\xi} \cdot \boldsymbol{\xi}_t - \frac{1}{2}\sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\xi}] \cdot \mathbf{n}_k \{\boldsymbol{\xi} \cdot \boldsymbol{\xi}_t\} \\
&= S_{14} + \cdots + S_{17}.
\end{aligned}
$$

The first term is bounded as

$$
\begin{aligned}
S_{14} &\le \|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega} \\
&\le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.
\end{aligned}
$$

From the definition of jump term, S_{15} is bounded

$$
\begin{aligned}
S_{15} &\le C\sum_{j=1}^{N_h}\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\boldsymbol{\xi}^{int}-\boldsymbol{\xi}^{ext}\|_{0,\partial E_j}\|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j} \\
&\le C\sum_{k=1}^{M_h}\|[\boldsymbol{\xi}]\|_{0,e_k}\left(\frac{\sigma}{|e_k|}\right)^{1/2-1/2}\|\boldsymbol{\xi}_t\|_{0,e_k} \\
&\le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.
\end{aligned}
$$

The bound for S_{16} and S_{17} is the same as S_{14}.

$$
\begin{aligned}
S_{16} &\le C\sum_{j=1}^{N_h}\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega} \\
&\le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.
\end{aligned}
$$

$$
\begin{aligned}
S_{17} &\le C\sum_{k=1}^{M_h}\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|[\boldsymbol{\xi}]\|_{0,e_k}\|\boldsymbol{\xi}_t\|_{0,e_k} \\
&\le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.
\end{aligned}
$$

We expand the term $\tilde{b}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\xi}_t)$

$$
\begin{aligned}
\tilde{b}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\xi}_t) &= \sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}\cdot\boldsymbol{\xi}_t + \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\xi}\}\cdot\mathbf{n}_{E_j}|(\boldsymbol{\eta}^{int}-\boldsymbol{\eta}^{ext})\cdot\boldsymbol{\xi}_t^{int} \\
&+\frac{1}{2}\sum_{j=1}^{N_h}(\nabla\cdot\boldsymbol{\xi})\boldsymbol{\eta}\cdot\boldsymbol{\xi}_t - \frac{1}{2}\sum_{k=1}^{M_h}\int_{e_k}[\boldsymbol{\xi}]\cdot\mathbf{n}_k\{\boldsymbol{\eta}\cdot\boldsymbol{\xi}_t\} \\
&= S_{18}+\cdots+S_{21}.
\end{aligned}
$$

By using approximation results and L^p bounds, S_{18} and S_{19} are bounded as

$$
\begin{aligned}
S_{18} &\leq \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{\eta}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega} \\
&\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\mathbf{u}|_{r+1,\Omega}^2,
\end{aligned}
$$

$$
\begin{aligned}
S_{19} &\leq C \sum_{j=1}^{N_h} \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\boldsymbol{\eta}^{int} - \boldsymbol{\eta}^{ext}\|_{0,\partial E_j} \|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j} \\
&\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\mathbf{u}|_{r+1,\Omega}^2.
\end{aligned}
$$

We use the inverse inequality, and L^p bounds to bound S_{20} and S_{21}

$$
\begin{aligned}
S_{20} &\leq C \sum_{j=1}^{N_h} \|\boldsymbol{\eta}\|_{L^4(\Omega)} \|\nabla \boldsymbol{\xi}\|_{L^2(\Omega)} \|\boldsymbol{\xi}_t\|_{L^4(\Omega)} \\
& C h^{r+1/2} |\mathbf{u}|_{r+1,\Omega} \|\nabla \boldsymbol{\xi}\|_{L^2(\Omega)} h^{-1/2} \|\boldsymbol{\xi}_t\|_{0,\Omega} \\
&\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C \|\boldsymbol{\xi}\|_X^2.
\end{aligned}
$$

$$
\begin{aligned}
S_{21} &\leq C \sum_{k=1}^{M_h} \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\boldsymbol{\eta}\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k} \\
&\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\mathbf{u}|_{r+1,\Omega}^2.
\end{aligned}
$$

The following nonlinear term $\tilde{b}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\xi}_t)$ can be expanded as

$$
\begin{aligned}
\tilde{b}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\xi}_t) &= \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\xi}_t + \frac{1}{2} \sum_{j=1}^{N_h} (\nabla \cdot \boldsymbol{\xi}) \mathbf{u} \cdot \boldsymbol{\xi}_t \\
&- \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\xi}] \cdot \mathbf{n}_k \{\mathbf{u} \cdot \boldsymbol{\xi}_t\} = S_{22} + \cdots + S_{24}.
\end{aligned}
$$

Similarly, S_{22} and S_{23} are bounded by using L^p bounds, jump term and approximation results.

$$\begin{aligned} S_{22} &\leq \|\boldsymbol{\xi}\|_{L^4(\Omega)}\|\nabla \mathbf{u}\|_{L^4(\Omega)}\|\boldsymbol{\xi}_t\|_{L^2(\Omega)} \\ &\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C|\mathbf{u}|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}\|\boldsymbol{\xi}\|_X^2. \end{aligned}$$

$$\begin{aligned} S_{23} &\leq C\sum_{j=1}^{N_h}\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega} \\ &\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2. \end{aligned}$$

$$\begin{aligned} S_{24} &\leq C\sum_{k=1}^{M_h}\|\mathbf{u}\|_{L^\infty(\Omega)}\|[\boldsymbol{\xi}]\|_{0,e_k}\|\boldsymbol{\xi}_t\|_{0,e_k}\left(\frac{\sigma}{|e_k|}\right)^{1/2-1/2} \\ &\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2. \end{aligned}$$

Again, we expand $\tilde{b}(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\xi}_t)$

$$\begin{aligned} \tilde{b}(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\xi}_t) &= \sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{\eta}\cdot\nabla\mathbf{u}^h\cdot\boldsymbol{\xi}_t + \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\eta}\}\cdot\mathbf{n}_{E_j}|(\mathbf{u}^{h,int}-\mathbf{u}^{h,ext})\cdot\boldsymbol{\xi}_t^{int} \\ &+\frac{1}{2}\sum_{j=1}^{N_h}(\nabla\cdot\boldsymbol{\eta})\mathbf{u}^h\cdot\boldsymbol{\xi}_t - \frac{1}{2}\sum_{k=1}^{M_h}\int_{e_k}[\boldsymbol{\eta}]\cdot\mathbf{n}_k\{\mathbf{u}^h\cdot\boldsymbol{\xi}_t\} \\ &= S_{25}+\cdots+S_{28}. \end{aligned}$$

By using L^p bounds and approximation results the terms are bounded as

$$\begin{aligned} S_{25} &\leq \|\boldsymbol{\eta}\|_{L^4(\Omega)}\|\nabla\mathbf{u}^h\|_{L^4(\Omega)}\|\boldsymbol{\xi}_t\|_{L^2(\Omega)} \\ &\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2, \end{aligned}$$

$$\begin{aligned} S_{26} &\leq C\sum_{j=1}^{N_h}\|\mathbf{u}^h\|_{L^\infty((0,T)\times\Omega)}\|\{\boldsymbol{\eta}\}\|_{0,\partial E_j}\|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j} \\ &\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2, \end{aligned}$$

$$S_{27} \leq C\sum_{j=1}^{N_h} \|\mathbf{u}^h\|_{L^\infty((0,T)\times\Omega)}\|\nabla\boldsymbol{\eta}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2.$$

$$S_{28} \leq C\sum_{k=1}^{M_h} \|\mathbf{u}^h\|_{L^\infty(\Omega)}\|[\boldsymbol{\eta}]\|_{0,e_k}\|\boldsymbol{\xi}_t\|_{0,e_k}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2.$$

Use L^p bounds to bound $\tilde{b}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\xi}_t)$

$$\tilde{b}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\xi}_t) = \sum_{j=1}^{N_h}\int_{E_j}\mathbf{u}\cdot\nabla\boldsymbol{\xi}\cdot\boldsymbol{\xi}_t + \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\mathbf{u}\}\cdot\mathbf{n}_{E_j}|(\boldsymbol{\xi}^{int}-\boldsymbol{\xi}^{ext})\cdot\boldsymbol{\xi}_t^{int}$$
$$= S_{29} + S_{30}.$$

$$S_{29} \leq \|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

$$S_{30} \leq C\sum_{j=1}^{N_h}\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\|\boldsymbol{\xi}^{int}-\boldsymbol{\xi}^{ext}\|_{0,\partial E_j}\|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}$$
$$\leq C\sum_{k=1}^{M_h}\|[\boldsymbol{\xi}]\|_{0,e_k}\|\boldsymbol{\xi}_t\|_{0,e_k}(\frac{\sigma}{|e_k|})^{1/2-1/2}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

Lastly, if we expand $\tilde{b}(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\xi}_t)$, we get

$$\tilde{b}(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\xi}_t) = \sum_{j=1}^{N_h}\int_{E_j}\mathbf{u}\cdot\nabla\boldsymbol{\eta}\cdot\boldsymbol{\xi}_t + \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\mathbf{u}\}\cdot\mathbf{n}_{E_j}|(\boldsymbol{\eta}^{int}-\boldsymbol{\eta}^{ext})\cdot\boldsymbol{\xi}_t^{int}$$
$$= S_{31} + S_{32}.$$

These terms are bounded as following:

$$
\begin{aligned}
S_{31} &\leq \|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\|\nabla\boldsymbol{\eta}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega}\\
&\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2,
\end{aligned}
$$

$$
\begin{aligned}
S_{32} &\leq C\sum_{j=1}^{N_h}\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\|\boldsymbol{\eta}^{int}-\boldsymbol{\eta}^{ext}\|_{0,\partial E_j}\|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}\\
&\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2.
\end{aligned}
$$

Collecting all the bounds with (A.3) gives:

$$
\begin{aligned}
&\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi},\boldsymbol{\xi}_t) + \frac{\nu}{2}\frac{d}{dt}J(\boldsymbol{\xi},\boldsymbol{\xi}) + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi},\boldsymbol{\xi})\\
&\leq \frac{1}{2}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2 + Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2\\
&+Ch^{2r}|\mathbf{u}_t|_{r+1,\Omega}^2 + C\nu_T^2H^{2r}h^{-2}|\mathbf{u}|_{r+1,\Omega}^2. \qquad \text{(A.4)}
\end{aligned}
$$

In the case where the bilinear form a is symmetric ($\epsilon_0=-1$), the inequality becomes

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \frac{\nu}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|_X^2 + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi},\boldsymbol{\xi}) \leq C\|\boldsymbol{\xi}\|_X^2\\
&+Ch^{2r}|\mathbf{u}|_{r+1,\Omega}^2 + Ch^{2r}|\mathbf{u}_t|_{r+1,\Omega}^2 + C\nu_T^2H^{2r}h^{-2}|\mathbf{u}|_{r+1,\Omega}^2. \qquad \text{(A.5)}
\end{aligned}
$$

Integrating between 0 and t, and using Gronwall's lemma yields:

$$
\begin{aligned}
&\|\boldsymbol{\xi}_t\|_{L^2(0,T;L^2(\Omega))}^2 + \nu\|\boldsymbol{\xi}\|_{L^\infty(0,T;X)}^2 + \nu_T\max_{0\leq t\leq T}g(\boldsymbol{\xi},\boldsymbol{\xi}) \leq Ce^{CT\nu^{-1}}[h^{2r}|\mathbf{u}_0|_{r+1,\Omega}^2\\
&+Ch^{2r}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + Ch^{2r}|\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}^2 + +C\nu_T^2H^{2r}h^{-2}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2].
\end{aligned}
$$

In the case where the bilinear form a is non-symmetric($\epsilon_0=1$), we rewrite (A.4) as

$$
a(\boldsymbol{\xi},\boldsymbol{\xi}_t) = \frac{1}{2}\frac{d}{dt}|||\nabla\boldsymbol{\xi}|||_0^2 - \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}\}\mathbf{n}_k\cdot[\boldsymbol{\xi}_t] + \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}_t\}\mathbf{n}_k\cdot[\boldsymbol{\xi}].
$$

The bound is then suboptimal: $\mathcal{O}(h^{r-1})$.

We now derive an error estimate for the pressure.

Theorem A.2. *Assume that $a(.,.)$ is symmetric ($\epsilon_0 = -1$) and $\nu \leq 1$. In addition, we assume that $\mathbf{u} \in L^2(0,T;H^{r+1})$, $\mathbf{u}_t \in L^2(0,T;H^{r+1})$ and $p \in L^2(0,T;H^r)$. Then, the solution p^h satisfies the following error estimate*

$$\|p^h - r_h(p)\|_{L^2(0,T;L^2(\Omega))} \leq Ce^{CT\nu^{-1}}[\nu h^r|\mathbf{u}_0|_{r+1,\Omega}$$

$$+\nu h^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu h^r|\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu\nu_T H^r h^{-1}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}]$$

$$+C\nu^{1/2}h^r|\mathbf{u}_0|_{r+1,\Omega} + C\nu h^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu h^r|p|_{L^2(0,T;H^r(\Omega))}$$

$$+C\nu_T H^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$

$$+Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}$$

$$+|\mathbf{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) + \nu_T^{1/2}H^r|\mathbf{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\mathbf{u}_0|_{r+1,\Omega}.$$

where C is independent of h, H, ν and ν_T. Again, if $\epsilon_0 = 1$, the estimate is suboptimal.

Proof: The error equation can be written for all $\mathbf{v}^h$ in $\mathbf{X}^h$:

$$-c(\mathbf{v}^h, p^h - r_h(p)) = (\mathbf{u}_t^h - \mathbf{u}_t, \mathbf{v}^h) + \nu a(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + \nu J(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h)$$

$$+\nu_T g(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + \tilde{b}(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - \tilde{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) + \nu_T g(\mathbf{u}, \mathbf{v}^h) - c(\mathbf{v}^h, p - r_h(p)).$$

From the inf-sup condition (4.23), there is $\mathbf{v}^h \in \mathbf{X}^h$ such that

$$c(\mathbf{v}^h, p^h - r_h(p)) = -\|p^h - r_h(p)\|_0^2, \quad \|\mathbf{v}^h\|_X \leq \frac{1}{\beta_0}\|p^h - r_h(p)\|_{0,\Omega}.$$

Thus, we have

$$\|p^h - r_h(p)\|_{0,\Omega}^2 = (\mathbf{u}_t^h - \mathbf{u}_t, \mathbf{v}^h) + \nu\sum_{j=1}^{N_h}\int_{E_j}\nabla(\mathbf{u}^h - \mathbf{u}) : \nabla\mathbf{v}^h$$

$$-\nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla(\mathbf{u}^h - \mathbf{u})\}\mathbf{n}_k \cdot [\mathbf{v}^h] + \nu\epsilon_0\sum_{k=1}^{M_h}\int_{e_k}\{\nabla\mathbf{v}^h\}\mathbf{n}_k \cdot [\mathbf{u}^h - \mathbf{u}] + \nu J(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h)$$

$$+\nu_T g(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + \tilde{b}(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - \tilde{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) + \nu_T g(\mathbf{u}, \mathbf{v}^h) - c(\mathbf{v}^h, p - r_h(p)).$$

All the terms above can be handled as in Theorem 4.1. The resulting inequality is

$$\|p^h - r_h(p)\|_{0,\Omega}^2 \leq C\nu^2\|\mathbf{u}_t^h - \mathbf{u}_t\|_{0,\Omega}^2 + C\nu^2\|\mathbf{u}^h - \mathbf{u}\|_X^2 + C\nu^2 h^{2r}|\mathbf{u}|_{r+1,\Omega}^2 + C\nu^2 h^{2r}|p|_{r,\Omega}^2$$
$$+C\nu_T^2 H^{2r}|\mathbf{u}|_{r+1,\Omega}^2 + C\nu_T^2 g(\mathbf{u}^h - \mathbf{u}, \mathbf{u}^h - \mathbf{u}) + C\|\mathbf{u}^h - \mathbf{u}\|_{0,\Omega}^2.$$

We now integrate between 0 and T, and use Theorem 4.1 and Theorem A.1 to conclude.

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