ELLIPTIC EQUATIONS IN GRAPHS VIA STOCHASTIC GAMES

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Consider a connected finite graph E with set of vertices $\mathfrak{X}$. Choose a nonempty subset $Y \subset \mathfrak{X}$, not equal to the whole $\mathfrak{X}$, and call it the boundary $Y = \partial\mathfrak{X}$. We are given a real valued function $F : Y \to \mathbb{R}$. Our objective is to find function u on $\mathfrak{X}$, such that $u = F$ on Y and u satisfies the following equation for all $x \in \mathfrak{X} \setminus Y$

$$u(x) = \alpha \max_{y \in S(x)} u(y) + \beta \min_{y \in S(x)} u(y) + \gamma \left(\frac{\sum_{y \in S(x)} u(y)}{\#(S(x))} \right), \tag{1}$$

where $\alpha, \beta,$ and γ are some predetermined non-negative constants such that $\alpha + \beta + \gamma = 1$, for $x \in \mathfrak{X}$, $S(x)$ is the set of vertices connected to x by an edge, and $\#(S(x))$ denotes the cardinality of $S(x)$. We prove existence and uniqueness of the solution of the above Dirichlet problem and study qualitative properties of the solutions.

Keywords: p-harmonic function, infinity harmonic function, p-harmonious function, stochastic games, unique continuation.

TABLE OF CONTENTS

LIST OF FIGURES

PREFACE

I thank my family for continuing support during my studies and research. I also would like to thank my advisor for all the time and efforts he invested in me. In addition, I am grateful to my committee members for their insightful questions and comments.

1.0 INTRODUCTION

The main object of our interest is the p-Laplacian. The standard p-Laplacian for $1 < p < \infty$ is the Euler-Lagrange equation of the functional

$$\frac{1}{p}\int |\nabla u(x)|^p \, dx.$$

This is a solution to Euler-Lagrange equation.

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.1}$$

The following are two well-known problems in the theory of p-harmonic functions in $\mathbb{R}^n$.

Problem 1. Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is p-harmonic in $B_2(0)$ and $u \equiv 0$ in $B_1(0)$. Does it imply that $u \equiv 0$ in $B_2(0)$?

Problem 2. Assume that u and v are p-harmonic in $B_1(0)$, $u \le v$, and $u(0) = v(0)$. Does it follow that $u = v$ in $B_1(0)$?

The answers to both problems are known in $\mathbb{R}^2$ for all p, $1 < p < \infty$ [19] and in $\mathbb{R}^n$ for $p = 2$. The case of $n \neq 2$ remains open. **In this dissertation we consider these problems for p-harmonious functions on graphs.** These are discretization of p-harmonic functions suggested by random tug-of-war games [26], [20]. The answer to the problem 1 is false even for $p = 2$ in the case of graphs (see example in section 3.6). On the contrary the answer to problem 2 is true (see Theorem 6 and [20]).

We begin by considering the following ternary tree. We set the last row of the vertices $\{e_{000}, ..., e_{022}\}$ to be the boundary of the tree. We define a function F on this boundary by

$$F(e_{000}) = 2, F(e_{022}) = -1, \quad \text{and}$$

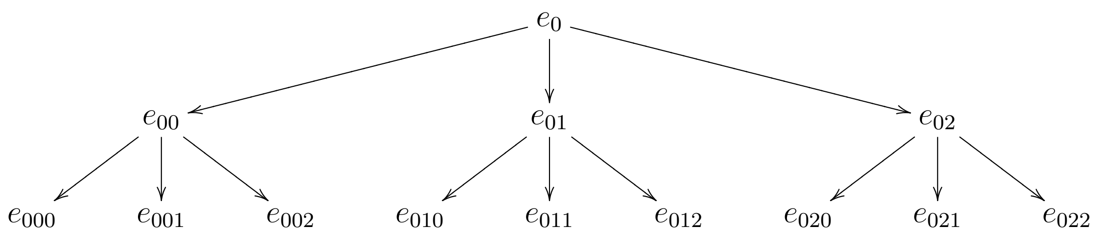

Figure 1: Ternary tree

$$F(e_{001}) = F(e_{002}) = F(e_{010}) = F(e_{011}) = F(e_{012}) = F(e_{020}) = F(e_{021}) = 0.$$

Let us play a very simple game with only one player on this tree. At the beginning of the game a token is placed at one of the vertices. The token moves to one of the three succeeding vertices with equal probabilities. Once the boundary is reached, the player receives a payoff equal $F(e_i)$ dollars, where e_i denotes the vertex to which the token has arrived. Let $u(e_j)$ be the amount the player can expect to earn, if the game starts at a vertex e_j. For example, if game start at e_{00}, then player can expect to receive 2/3 dollar. If game starts at e_{02}, then player can expect to pay 1/3 dollar. Finally, if game starts at e_0, then player can expect to get 1/9 dollar. Observe that function u satisfies the mean-value property:

$$u(e_j) = \frac{1}{3}u(e_{j_0}) + \frac{1}{3}u(e_{j_1}) + \frac{1}{3}u(e_{j_2})$$

where $\{e_{j_0}, e_{j_1}, e_{j_2}\}$ are the immediate successors of the vertex e_j.

Since the function u satisfies mean-value property, we can expect it to be a *solution* of some equation. In particular, we shall see that u solves the discrete Laplace equation in the tree.

To relate our example to the partial differential equations on $\mathbb{R}^n$ we introduce the following definitions.

Definition 1. *The* **gradient** *of a function u defined on the tree at vertex e_i is*

$$\nabla u(e_i) = (u(e_{i0}) - u(e_i), u(e_{i1}) - u(e_i), u(e_{i2}) - u(e_i))$$

where e_{i0}, e_{i1}, e_{i2} are succesors of e_i

Definition 2. *The averaging operator or* **divergence** *of a vector* $X = (x, y, z) \in \mathbb{R}^3$ *is*

$$\text{div}(X) = x + y + z$$

Definition 3. *The function* u *defined on the tree above is harmonic, if it satisfies Laplace equation*

$$\text{div}(\nabla u) = 0$$

It is now clear that a function is harmonic if and only if it satisfies the mean value property

$$u(e_i) = \frac{u(e_{i0}) + u(e_{i1}) + u(e_{i2})}{3}.$$

This setup has been described in detail for example in Kaufman, Llorente, and Wu in [13].

Next, let us consider another game with two players on the same tree. At the beginning of the game a token is placed at some vertex e_j and players toss a fair coin to decide who gets to move the token. If the outcome of the toss is heads, then player I moves the token to a succeeding vertex of his choice. In the case of tails, player II gets to move the token.We require the player always to move the token down the tree, since the tree is a directed graph. We also do not allow player to remain in the same position whenever it is his turn to move the token. The game stops once the token reaches some boundary vertex e_i. At this moment player I receives from player II $F(e_i)$ dollars. The goal of each player is to maximize the payoff. Let function $u(e_j)$ be the expected payoff to player I, when the game starts at a vertex e_j. In this simple situation we can directly compute

$$u(e_{00}) = 1, u(e_{01}) = 0, u(e_{02}) = -1/2, u(e_0) = 1/4$$

Observe that this function u satisfies a variant a mean-value property:

$$u(e_j) = \frac{1}{2} \max_{y \in \{e_{j_1}, e_{j_2}, e_{j_3},\}} u(y) + \frac{1}{2} \min_{y \in \{e_{j_1}, e_{j_2}, e_{j_3},\}} u(y)$$

This is the mean-value property of the infinity Laplacian studied by Le Gruyer [8] and by Peres, Schramm, Sheffield, and Wilson in [26].

The goal of this dissertation is to extend the above ideas to include the study of ∞-Laplacian in infinite trees and p-Laplacian in graphs following the work of Manfredi, Parviainen, and Rossi [20] in $\mathbb{R}^n$.

1.1 THE DIRICHLET PROBLEM ON GRAPHS

Consider a graph E with a finite set of vertices $\mathfrak{X}$. Choose a nonempty subset of $\mathfrak{X}$ and call it the boundary Y. Let $F : Y \to \mathbb{R}$ be a given function. We consider the following problem. Find a function u on $\mathfrak{X}$, such that $u = F$ on Y and u satisfies the following equation in $\mathfrak{X}$

$$u(x) = \alpha \max_{y \in S(x)} u(y) + \beta \min_{y \in S(x)} u(y) + \gamma \left(\frac{\sum_{y \in S(x)} u(y)}{\#(S(x))} \right) \tag{1.2}$$

where $\alpha, \beta,$ and γ are predetermined non-negative constants such that $\alpha + \beta + \gamma = 1$, $x \in \mathfrak{X}$, $S(x)$ is the set of vertices connected to x by an edge, and $\#(S(x))$ is the cardinality of $S(x)$. For a function $u : \mathfrak{X} \to \mathbb{R}$ we use

$$⨍_{S(x)} u = \frac{\sum_{y \in S(x)} u(y)}{\#(S(x))}$$

We can also state this Dirichlet problem in more traditional notation. We will need the following definitions.

Definition 4. *The Laplace operator on the graph is given by*

$$\Delta u(x) = \operatorname{div}(\nabla u) = ⨍_{S(x)} u - u(x)$$

Definition 5. *The infinity Laplacian on the graph is given by*

$$\Delta_\infty u(x) = \frac{1}{2}(\max_{S(x)} u + \min_{S(x)} u) - u(x)$$

Definition 6. *For $X = (x, y, z) \in \mathbb{R}^3$ we define the analog of the maximal directional derivative*

$$\langle X \rangle_\infty = \max\{x, y, z\}.$$

With the above notation in mind we can write (1.2) as

$$(\alpha - \beta) \langle \nabla u \rangle_\infty + 2\beta \Delta_\infty u + \gamma \Delta u = 0. \tag{1.3}$$

We call equation (1.3) the biased the p-Laplacian. Observe that when $\alpha = \beta$ we obtain the regular p-Laplacian, when $\alpha = \beta = 0$ we obtain the 2-Laplacian, and when $\alpha \neq \beta$ and $\gamma = 0$ we obtain the biased ∞-Laplacian. In this dissertation we study existence, uniqueness and a qualitative properties of the solutions to this equation.

We start by answering the question of existence of the solution of Dirichlet problem 1.2 in Theorem 1. In Theorem 3 we answer the question of uniqueness through comparison principle (Theorem 2) employing a martingale approach as the main tool. Theorem 4 emphasizes the connection between the optimal strategies and the solution of (1.2). A particular case of existence and uniqueness proof is presented in Theorem 5. Theorem 6 is the strong comparison principle. The case of a ternary directed tree includes the extension of some ideas obtained for finite graphs. In particular, the measure induced by the game on the boundary of a ternary directed tree is discussed in Theorem 7. Theorem 8 answers the question of existence and uniqueness of the solution of Dirichlet problem on the ternary directed tree. The example in the section 3.6 clarifies the problem of unique continuation.

2.0 GAME SETUP AND DEFINITIONS

We consider the following game on a connected graph E with vertex set $\mathfrak{X}$. The set $\mathfrak{X}$ is finite unless stated otherwise. We equip $\mathfrak{X}$ with the σ-algebra $\mathcal{F}$ of all subsets of $\mathfrak{X}$. For an arbitrary vertex x we define $S(x)$ the collection of vertices, which are connected to the vertex x by a single edge. In case $\mathfrak{X}$ is infinite, we require that $\mathfrak{X}$ is at least locally finite; i.e. the cardinality of $S(x)$ is finite. At the beginning of the game a token is placed at some point $x_0 \in \mathfrak{X}$. Then we toss a three-sided virtual coin. The side of a coin labeled 1 comes out with probability α and in this case player I chooses where to move the token among all vertices in $S(x)$. The side of a coin labeled 2 comes out with probability β and in this case player II chooses where to move the token among all vertices in $S(x)$. Finally, the side of a coin labeled 3 comes out with probability γ and in this case we choose the next point randomly (uniformly) among all vertices in $S(x)$. This setup has been described in [26] and in [24] and is known as "biased tug-of-war with noise". The game stops once we hit the boundary set Y. The set Y is simply predetermined non-emplty set of vertices at which game terminates. In the game literature set Y is the called set of absorbing states. Let $F : Y \to \mathbb{R}$ be the payoff function defined on Y. If game ends at some vertex $y \in Y$, then player I receives from player II the sum of $F(y)$ dollars.

Let us define the value of the game for player I. Firstly, we formalize the notion of a pure strategy. We define a strategy S_I for player I as a collection of maps $\{\sigma_I^k\}_{k\in\mathbb{N}}$, such that for each k

$$\sigma_I^k : \mathfrak{X}^k \to \mathfrak{X}$$

$$\sigma_I^k(x_0, ..., x_{k-1}) = x_k,$$

$$\text{where} \quad \mathfrak{X}^k = \underbrace{\mathfrak{X} \times \mathfrak{X} \times ... \times \mathfrak{X}}_{k\,\text{times}}$$

Hence, σ_I^k tells player I where to move given $(x_0, ..., x_{k-1})$ - the history of the game up to the step k. We call a strategy *stationary* if it depends only on the current position of the token.

Given two strategies for player I and II the transition probabilties for $k \geq 1$ are given by

$$\pi_k(x_0, ..., x_{k-1}; y) = \alpha\delta_{\sigma_I^k(x_0,...,x_{k-1})}(y) + \beta\delta_{\sigma_{II}^k(x_0,...,x_{k-1})}(y) + \gamma U_{S(x_{k-1})}(y),$$

where we have set

$$U_{S(x_{k-1})} \text{ is a uniform distribution on } S(x_{k-1}) \text{ and } \pi_0(y) = \delta_{x_0}(y).$$

We equip $\mathfrak{X}^k$ with product σ-algebra $\mathcal{F}^k$

$$\mathcal{F}^k = \underbrace{\mathcal{F} \otimes \mathcal{F} \otimes ... \otimes \mathcal{F}}_{k \text{ times}}$$

and then we define a probability measure on $(\mathfrak{X}^k, \mathcal{F}^k)$ in the following way:

$$\mu_0 = \pi_0 = \delta_{x_0},$$

$$\mu_k(A^k \times A) = \int_{A^k} \pi_k(x_0, ..., x_{k-1}; A) d\mu_{k-1},$$

where $A^{k-1} \times A$ is a rectangle in $(\mathfrak{X}^k, \mathcal{F}^k)$. The space of infinite sequences with elements from $\mathfrak{X}$ is $\mathfrak{X}^\infty$. Let $X_k : \mathfrak{X}^\infty \to \mathfrak{X}$ be the coordinate process defined by

$$X_k(h) = x_k, \text{ for } h = (x_0, x_1, x_2, x_3, ...) \in \mathfrak{X}^\infty.$$

We equip $\mathfrak{X}^\infty$ with product σ-algebra $\mathcal{F}^\infty$. For precise definition of $\mathcal{F}^\infty$ see [11].

The family of $\{\mu_k\}_{k\geq 0}$ satisfies the conditions of Kolmogorov extension theorem [29], therefore, we can conclude that there exists a unique measure $\mathbb{P}^{x_0}$ on $(\mathfrak{X}^\infty, \mathcal{F}^\infty)$ with the following property:

$$\mathbb{P}^{x_0}(B_k \times \mathfrak{X} \times \mathfrak{X} \times \times \mathfrak{X}...) = \mu_k(B_k), \text{ for } B_k \in \mathcal{F}^k \tag{2.1}$$

and

$$\mathbb{P}^{x_0}[X_k \in A | X_0 = x_0, X_1 = x_1, ..., X_{k-1} = x_{k-1}] = \pi_k(x_0, ..., x_{k-1}; A). \tag{2.2}$$

We are now ready to define the value of the game for player I. The boundary hitting time is

$$\tau = \inf_k \{X_k \in Y\}.$$

Consider strategies S_I and S_{II} for player I and player II respectively. We define

$$F_-^x(S_I, S_{II}) = \begin{cases} \mathbb{E}^x_{S_I,S_{II}}[F(X_\tau)] & \text{if } \mathbb{P}^x_{S_I,S_{II}}(\tau < \infty) = 1 \\ -\infty & \text{otherwise} \end{cases} \tag{2.3}$$

$$F_+^x(S_I, S_{II}) = \begin{cases} \mathbb{E}^x_{S_I,S_{II}}[F(X_\tau)] & \text{if } \mathbb{P}^x_{S_I,S_{II}}(\tau < \infty) = 1 \\ +\infty & \text{otherwise} \end{cases} \tag{2.4}$$

The value of the game for player I is

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{F}^x_-(S_I, S_{II})$$

and the value of the game for player II is

$$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{F}^x_+(S_I, S_{II})$$

These definitions penalize players severely for not being able to force the game to end. Whenever player I has a strategy to finish the game almost surely, then we simplify notation by setting

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}}[F(X_\tau)].$$

Similarly, for player II we set

$$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} E^x_{S_I,S_{II}}[F(X_\tau)].$$

The following lemma states rigorously whether player I has a strategy to finish the game almost surely:

Lemma 1. *If $\mathfrak{X}$ is a finite set, then player I (player II) has strategies to finish the game almost surely.*

Proof. When $\gamma = 0$, this result was already proven by Peres, Schramm, Sheffield, and Wilson in [26] in Theorem 2.2. When $\gamma \neq 0$, the statement follows from the fact that random walk on a finite graph is recurrent. □

We always have $u_I(x) \leq u_{II}(x)$. Whenever $u_I(x) = u_{II}(x)$ for all $x \in \mathfrak{X}$ we say that game has a value.

2.1 GAME THEORY FORMULATION

Following the notation of Maitra and Sudderth [17] we describe our game in the standard term used in Game Theory.

Our state space $\mathbb{S}$ is $\mathfrak{X}$. Consider the *gambling house* at the point $x_{k-1} \in \mathfrak{X}$ given by

$$\Gamma(x_{k-1}) = \left\{ \alpha\delta_{y_1} + \beta\delta_{y_2} + \gamma U_{S(x_{k-1})} : y_1, y_2 \in S(x_{k-1}) \right\} \tag{2.5}$$

This is a collection of probabilities in $S_{x_{k-1}}$. We call this gambling house a p-Laplacian gambling house. In the game described above we only consider pure strategies, i.e. given the history $(x_0, x_1, ...x_n)$ a player chooses next point x_{n+1} in $S(x_n)$ with probability one. However, we can generalize the notion of strategies to align it with standard for Game Theory terms.

Definition 7. *A strategy σ is a sequence $\sigma_0, \sigma_1, ...$ such that σ_0 is a probability on $\mathfrak{X}$ and, for $n = 1, 2, ...$, σ_n is a mapping, which assigns to each partial history $p = (x_1, ..., x_n)$ of length n a probability $\sigma(x_1, ..., x_n)$ on $\mathfrak{X}$*

We also would like to reproduce here the definition of conditional strategy from [17], since this notion is critical to the proof of mean-value property in general case.

Definition 8. *Suppose a player following the strategy σ has played for n periods and has experienced the partial history $p = (x_1, ..., x_n)$. The conditional strategy $\sigma[p]$ governs the continuation of play and is defined by setting*

$$\sigma[p]_0 = \sigma_n(x_1, ..., x_n)$$

and

$$\sigma[p]_m(y_1, ..., y_m) = \sigma_{m+n}(x_1, ...x_n, y_1, ..., y_m)$$

for all partial histories $(y_1, ..., y_2)$.

One can check that

$$\mathbb{P}_\sigma[X_{n+1} = y_1, ..., X_{n+m} = y_m | X_1 = x_1, ..., X_n = x_n] = \mathbb{P}_{\sigma[p]}[X_1 = y_1, ..., X_m = y_m],$$

whenever $\mathbb{P}_\sigma[X_1 = y_1, ..., X_m = y_m] > 0$.

3.0 MAIN RESULTS

3.1 EXISTENCE

Here is the first existence result for the equation (1.2).

Theorem 1. *(Dynamic Programing Principle = Mean Value Property)*

The value functions u_I and u_{II} satisfy the Dynamic Programing Principle (DPP) or Mean Value Property (MVP):

$$u_I(x) = \alpha \max_{y \in S(x)} u_I(y) + \beta \min_{y \in S(x)} u_I(y) + \gamma \fint_{S(x)} u_I(y)dy, \tag{3.1}$$

$$u_{II}(x) = \alpha \max_{y \in S(x)} u_{II}(y) + \beta \min_{y \in S(x)} u_{II}(y) + \gamma \fint_{S(x)} u_{II}(y)dy. \tag{3.2}$$

The above result is true in the general setting of discrete stochastic games (see Maitra and Sudderth, chapter 7 [17]). Here we provide a proof in the easier Markovian case. It turns out that optimal strategies are Markovian (see chapter 5 of [17]).

Proposition 1. *(The stationary case proof)*

In the game with stationary strategies value functions u_I and u_{II} satisfy the Dynamic Programing Principle (DPP) or Mean Value Property (MVP):

$$u_I(x) = \alpha \max_{y \in S(x)} u_I(y) + \beta \min_{y \in S(x)} u_I(y) + \gamma \fint_{S(x)} u_I(y)dy, \tag{3.3}$$

$$u_{II}(x) = \alpha \max_{y \in S(x)} u_{II}(y) + \beta \min_{y \in S(x)} u_{II}(y) + \gamma \fint_{S(x)} u_{II}(y)dy. \tag{3.4}$$

Proof. We will provide proof only for u_I; proof for u_{II} follows by symmetry. Take a set of vertices $\mathfrak{X}$, boundary Y and adjoin one vertex y^* to the boundary. Denote new boundary by $Y^* = Y \cup \{y^*\}$ and the new set of vertices by $\mathfrak{X}^* = \mathfrak{X} \setminus \{y^*\}$ and define

$$F^*(y) = \begin{cases} F(y) & \text{if} \quad y \in Y \\ u_I(y^*) & \text{if} \quad y = y^*. \end{cases} \tag{3.5}$$

Let $u_I(x)$ be the value of the game with $\mathfrak{X} \,\&\, Y$ and $u_I^*(x)$ be the value of the game with $\mathfrak{X}^* \,\&\, Y^*$ The goal is to show that

$$u_I^*(x) = u_I(x).$$

Once we prove the above, the main result follows by extending F to the set $S(x)$.

Remark 1. *The idea to extend F is used by [26] in the proof of Lemma 3.5.*

Hence, we have to show $u_I^*(x) = u_I(x)$. Since we consider only Markovian strategies we can think of them as mappings $S_I : \mathfrak{X} \to \mathfrak{X}$. For the game $\mathfrak{X}^* \,\&\, Y^*$ we define S_I^* as a restriction of S_I to $\mathfrak{X}^*$ Here are the steps in detail:

$$\begin{aligned} u_I^*(x) &= \sup_{S_I^*} \inf_{S_{II}^*} \left(E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \right) \\ &= \sup_{S_I^*} \inf_{S_{II}^*} \left(E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c} \right) \\ &= \sup_{S_I^*} \inf_{S_{II}^*} \left(E^x_{S_I^*, S_{II}^*} u_I(y^*) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c} \right) \\ &= \sup_{S_I^*} \inf_{S_{II}^*} \left(E^x_{S_I^*, S_{II}^*} \sup_{S_I} \inf_{S_{II}} E^{y^*}_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} \right. \\ &\left. + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c} \right) \\ &= \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \left(E^x_{S_I^*, S_{II}^*} \left(E^{y^*}_{S_I, S_{II}} F(X_\tau) \right) \chi_{\{X_{\tau^*} = y^*\}} \right. \\ &\left. + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c} \right). \end{aligned} \tag{3.6}$$

If we can show that

$$\begin{aligned} &\sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \left(E^x_{S_I^*, S_{II}^*} \left(E^{y^*}_{S_I, S_{II}} F(X_\tau) \right) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c} \right) \\ &\qquad = \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \left(E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}^c} \right), \end{aligned} \tag{3.7}$$

we can complete the proof in the following way:

$$\begin{aligned} u_I^*(x) &= \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \left(E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c} \right) \\ &= \sup_{S_I} \inf_{S_{II}} \sup_{S_I^*} \inf_{S_{II}^*} \left(E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c} \right) \\ &= \sup_{S_I} \inf_{S_{II}} \left(E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c} \right) \\ &= \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}} F(X_\tau) = u_I(x). \end{aligned} \tag{3.8}$$

Let us clarify (3.7). Actually, we have the following equalities

$$E^x_{S_I^*,S_{II}^*} E^{y^*}_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} = E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} \tag{3.9}$$

$$E^x_{S_I^*,S_{II}^*} F^*(X_{\tau^*})\chi_{\{X_{\tau^*}=y^*\}^c} = E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c} \tag{3.10}$$

Equation (3.9) could be thought as payoff computed for the trajectories that travel through a point y^*. Roughly speaking we first discount boundary points to the point y^* and then discount value at y^* back to x which is the same as to discount boundary points to x through trajectories that contain y^*, keeping in mind that S_i^* is just a restriction of S_i. Equation (3.10) is a payoff computed for the trajectories that avoid y^*, and, therefore, there is no difference between S_i^* and S_i, since S_i^* is just a restriction of S_i to $\mathfrak{X} \setminus \{y^*\}$. □

A function v that satisfies (3.1) property when $\alpha = \beta$ is called **p-harmonious** (see Manfredi, Parviainen, and Rossi [20]). The proof of Proposition 1 suggests the following corollary, which we present with the proof for the sake of completeness.

Corollary 1. *Consider a game on the graph E with finite set of vertices $\mathfrak{X}$ and with only stationary strategies. An optimal strategy for player I (player II) is to always move from vertex x to the maximum (minimum) of u_I (u_{II}) on $S(x)$.*

Proof. We will prove the result for player I and then by symmetry it follows for player II. Let us denote S_I^0 strategy of player I, where he always moves from vertex x to the maximum of u_I on $S(x)$. Then for two arbitrary strategies S_I and S_{II} we have to show that

$$E^x_{S_I^0,S_{II}} F(X_\tau) \geq E^x_{S_I,S_{II}} F(X_\tau) \tag{3.11}$$

The idea of the proof is the same as in the proof of the Proposition 1. We will initially extend the boundary by one vertex and then will extend the boundary up to $S(x)$, at which points all arguments become trivial. Fix a pair of arbitrary strategies S_I and S_{II}. Take a set of vertices $\mathfrak{X}$, boundary Y and adjoin one vertex y^* to the boundary. Denote new boundary by $Y^* = Y \cup \{y^*\}$ and the new set of vertices by $\mathfrak{X}^* = \mathfrak{X} \setminus \{y^*\}$ and define

$$F^*(y) = \begin{cases} F(y) & \text{if} \quad y \in Y \\ E^{y^*}_{S_I,S_{II}} F(X_\tau) & \text{if} \quad y = y^* \end{cases} \tag{3.12}$$

Accordingly, we define

$$\tau = \inf\{n : X_n \in Y\}$$

and

$$\tau^* = \inf\{n : X_n \in Y^*\}$$

Then we claim that the following is true:

Lemma 2. *For any pair of stationary strategies S_I and S_{II}*

$$E^x_{S_I,S_{II}} F(X_\tau) = E^x_{S_I,S_{II}} F(X_{\tau^*}). \tag{3.13}$$

Proof. (of Lemma) The left hand side of formula (3.13) is expected value for the game on $\mathfrak{X}$ and boundary Y and the right hand side of formula (3.13) is the expected value the game on $\mathfrak{X}^*$ and boundary Y^*

$$\begin{aligned} E^x_{S_I,S_{II}} F(X_{\tau^*}) &= E^x_{S_I,S_{II}} F(X_{\tau^*}) \chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_{\tau^*}) \chi_{\{X_{\tau^*}=y^*\}^c} \\ &= E^x_{S_I,S_{II}} E^{y^*}_{S_I,S_{II}} F(X_\tau) \chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_{\tau^*}) \chi_{\{X_{\tau^*}=y^*\}^c} \\ &= E^x_{S_I,S_{II}} F(X_\tau) \chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_{\tau^*}) \chi_{\{X_{\tau^*}=y^*\}^c} \\ &= E^x_{S_I,S_{II}} F(X_\tau) \chi_{\{X_{\tau^*}=y^*\}} + E^x_{S_I,S_{II}} F(X_\tau) \chi_{\{X_{\tau^*}=y^*\}^c} \\ &= E^x_{S_I,S_{II}} F(X_\tau) \end{aligned} \tag{3.14}$$

Here is a clarification:

$$\begin{aligned}
E^x_{S_I,S_{II}} E^{y^*}_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} &= E^{y^*}_{S_I,S_{II}} F(X_\tau) E^x_{S_I,S_{II}} \chi_{\{X_{\tau^*}=y^*\}} \\
&= E^{y^*}_{S_I,S_{II}} F(X_\tau) P^x_{S_I,S_{II}}(X_{\tau^*}=y^*) \\
&= \sum_{y\in Y} F(y) P^{y^*}_{S_I,S_{II}}(X_\tau = y) P^x_{S_I,S_{II}}(X_{\tau^*}=y^*) \\
&= \text{(by the Markov property)} \\
&= \sum_{y\in Y} F(y) P^x_{S_I,S_{II}}(\{X_\tau = y\}\cap\{X_{\tau^*}=y^*\}) \\
&= E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}}. \qquad (3.15)
\end{aligned}$$

The following is also true:

$$E^x_{S_I,S_{II}} F(X_{\tau^*})\chi_{\{X_{\tau^*}=y^*\}^c} = E^x_{S_I,S_{II}} F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c}, \qquad (3.16)$$

simply because on the set $\{X_{\tau^*}=y^*\}^c$ X_τ never hits the y^*. □

Applying the result of the lemma we can consider the regular game with set of vertices $\mathfrak{X}$ and boundary Y and the modified game where the boundary is extended all the way up to $S(x)$. We still denote such a game by *. Therefore, the following is true:

$$E^x_{S_I^0,S_{II}} F(X_\tau) = E^x_{S_I^0,S_{II}} F(X_{\tau^*}) \geq E^x_{S_I,S_{II}} F(X_{\tau^*}) = E^x_{S_I,S_{II}} F(X_\tau) \qquad (3.17)$$

□

Example. We would like to warn the reader that the Corollary 1 does not claim that tugging towards that maximum of F on the boundary would be an optimal strategy to the player I. Here is an counterexample. The boundary vertices are indicated by the numbers, which reflect the value of F at each vertex. We consider the game starting at vertex e_0 and require player II always pull towards the vertex labeled -1. For player one we choose S_I^a to be the strategy of always tugging towards vertex 3/2 and let S_I^b be the strategy of moving towards vertex 1. One can show that

$$E^{e_0}_{S_I^a,S_{II}} F(X_\tau) = -1\cdot 2/3 + 3/2\cdot 1/3 = -1/6 \qquad (3.18)$$

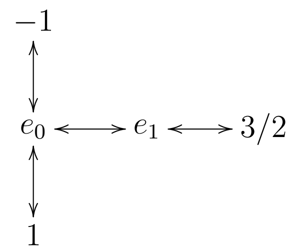

Figure 2: Counterexample - tugging towards the boundary

$$E^{e_0}_{S_I^b, S_{II}} F(X_\tau) = -1 \cdot 1/2 + 1 \cdot 1/2 = 0 \tag{3.19}$$

3.2 UNIQUENESS

Uniqueness will follow from the comparison principle below proven by using Doob's Optional Sampling Theorem.

Theorem 2. *(via Martingale) Let v be a solution of*

$$v(x) = \alpha \max_{y \in S(x)} v(y) + \beta \min_{y \in S(x)} v(y) + \gamma \fint_{S(x)} v(y)dy \tag{3.20}$$

on a graph E with a countable set of vertices $\mathfrak{X}$ and boundary Y. Assume

- $F(y) = u_I(y)$, *for all* $y \in Y$,
- $\inf_Y F > -\infty$,
- v *bounded from below, and*
- $v(y) \geq F(y)$, *for all* $y \in Y$

Then u_I is bounded from below on $\mathfrak{X}$ and

$$v(x) \geq u_I(x), \quad x \in \mathfrak{X}.$$

Proof. Note that we only need " $\leq$ " in equation (3.20). The theorem says that u_I is the smallest super-solution with given boundary value F.

We proceed as in Lemma 2.1 in [26]. Since the game ends almost surely,

$$u_I \geq \inf_Y F > -\infty$$

which proves that u_I is bounded from below. Now we have to show that

$$v(x) \geq \sup_{S_I} \inf_{S_{II}} F_-^x(S_I, S_{II}) = u_I(x)$$

If we fix an arbitrary strategy S_I, then we have to show that

$$v(x) \geq \inf_{S_{II}} F_-^x(S_I, S_{II}) \tag{3.21}$$

Consider a game that start at vertex x ($X_0 = x$). We have two cases

Case 1: If our fixed S_I cannot force the game to end a.s. (i.e. $\mathbb{P}^x_{S_I,S_{II}}(\tau < \infty) < 1$), then by the definition of F_-, $\inf_{S_{II}} F_-^x(S_I, S_{II}) = -\infty$ and the inequality (3.21) holds.

Case 2: Now assume that our fixed S_I forces the game to end despite all the efforts of the second player. Let player II choose a strategy of moving to $\min_{y \in S(x)} v(y)$ - denote such a strategy $\hat{S}_{II}$. If we prove that $v(X_k)$ is a supermartigale, then we can finish the proof in the following way:

$$\begin{aligned}
\inf_{S^{II}} \mathbb{E}^x_{S_I,S_{II}} F(X_\tau) &\leq \mathbb{E}^x_{S_I,\hat{S}_{II}} F(X_\tau) \\
&\leq \mathbb{E}^x_{S_I,\hat{S}_{II}} v(X_\tau) \\
&= \mathbb{E}^x_{S_I,\hat{S}_{II}} \liminf_k v(X_{\tau\wedge k}) \\
&\leq \liminf_k \mathbb{E}^x_{S_I,\hat{S}_{II}} v(X_{\tau\wedge k}) \leq \mathbb{E}^x_{S_I,S_{II}} v(X_0) \\
&= v(X_0) = v(x), \qquad (3.22)
\end{aligned}$$

where we have used Fatou lemma. The result follows after applying $\sup_{S_I}$ to both sides. Hence, we only need to prove that $v(X_k)$ is a supermartingale under the expectation $\mathbb{E}^x_{S_I,\hat{S}_{II}}$:

$$\begin{aligned}\mathbb{E}^x_{S_I,\hat{S}_{II}}[v(X_k)|X_0,...,X_{k-1}] &= \alpha v(X_k^I) + \beta v(X_k^{II}) \\ &+ \gamma \fint_{S(X_{k-1})} v(y)dy \\ &\leq \alpha \max_{y\in S(X_{k-1})} v(y) + \beta \min_{y\in S(X_{k-1})} v(y) \\ &+ \gamma \fint_{S(X_{k-1})} v(y)dy = v(X_{k-1}),\end{aligned} \tag{3.23}$$

where $v(X_k^I)$ indicates the choice of player I and $v(X_k^{II})$ indicates the choice of player II. Then $v(X_k^{II}) = \min_{y\in S(X_{k-1})} v(y)$ by choice of strategy for player II.

□

In case $\min_{y\in S(X_{k-1})} v(y)$ is not achieved (i.e. graph is not locally finite), we need to modify the above proof by making player II move within ϵ neighborhood of $\min_{y\in S(X_{k-1})} v(y)$. We can prove similar result for u_{II}. The next theorem is the extension of the result obtained in [21].

Theorem 3. *If graph E is finite and F is bounded below on Y, then $u_I = u_{II}$, so the game has a value.*

Proof. Clearly, finite E implies that F is bounded below. We included this redundant statement to suggest future possible extensions to an uncountable graph. We know that $u_I \leq u_{II}$ always holds, so we only need to show $u_I \geq u_{II}$. Assume F is bounded below. Similar to the proof of Lemma 2 we can show that u_I is a supermartingale bounded below by letting player I to choose an arbitrary strategy S_I and requiring player II always move to $\min_{y\in S(x)} u_I(y)$ from x - strategy S^o_{II}. For simplicity of the presentation we consider a case when $\min_{y\in S(x)} u_I(y)$ is achievable, for the general case we have to employ ϵ, like in Theorem

2. We start the game at x_0, so $X_0 = x_0$. Recall $u_{II}(x) = \inf_{S_{II}} \sup_{S_I} F_+(S_I, S_{II})$

$$\begin{aligned} u_{II}(x_0) &\leq \text{since E is finite} \leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S^o_{II}}[F(X_\tau)] = \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S^o_{II}}[u_I(X_\tau)] \\ &= \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S^o_{II}}[\liminf_k u_I(X_{\tau \wedge k})] \leq \sup_{S_I} \liminf_k \mathbb{E}^{x_0}_{S_I, S^o_{II}}[u_I(X_{\tau \wedge k})] \\ &\leq \sup_{S_I} \liminf_k \mathbb{E}^{x_0}_{S_I, S^o_{II}}[u_I(X_0)] = \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S^o_{II}}[u_I(X_0)] = u_I(x_0) \end{aligned} \tag{3.24}$$

□

3.3 CONNECTIONS AMONG GAMES, PARTIAL DIFFERENTIAL EQUATIONS AND DPP

This section summarizes some previous results as well as presents new prospectives on known issues.

Theorem 4. *Assume we are given a function u on the set of vertices $\mathfrak{X}$ and consider a strategy $\hat{S}_I$ ($\hat{S}_{II}$) where player I (player II) moves from vertex x to vertex z, where*

$$u(z) = \max_{y \in S(x)} u(y) \quad (u(z) = \min_{y \in S(x)} u(y))$$

Then the following are equivalent:

- *the process $u(X_n)$ is a martingale under the measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$,*
- *the function u is a solution of Dirichlet problem (1.2),*

In addition, $u(X_n)$ is a martingale under the measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$ implies that $\hat{S}_I$ and $\hat{S}_{II}$ are the optimal strategies.

Proof. Suppose that $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$. Fix an arbitrary point $x \in \mathfrak{X}$ and consider a game which starts at $x = X_0$, then

$$\begin{aligned} E^x_{\hat{S}_I, \hat{S}_{II}}[u(X_1)|X_0] &= \alpha u(X_1^I) + \beta u(X_1^{II}) + \gamma \fint_{S(X_0)} u(y) dy \\ &= \alpha \max_{y \in S(X_0)} u(y) + \beta \min_{y \in S(X_0)} + \gamma \fint_{S(X_0)} u(y) dy \\ &= u(X_0). \end{aligned} \tag{3.25}$$

Conversely, assume that u solves Dirichlet problem (1.2), then (3.25) implies that $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$.

Let us show final implication. By optimal strategy we mean

$$E^x_{\hat{S}_I,\hat{S}_{II}} F(X_\tau) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}} F(X_\tau). \tag{3.26}$$

The result relies on the fact that our game has a value and value of game function is the solution of the Dirichlet problem (1.2). Since $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$ we have

$$E^x_{\hat{S}_I,\hat{S}_{II}} F(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}} u(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}} u(X_0) = u(x). \tag{3.27}$$

By Uniqueness result (Theorem 2)

$$u(x) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}} F(X_\tau). \tag{3.28}$$

□

3.4 EXISTENCE AND UNIQUENESS WITHOUT GAMES

The idea of the following existence and uniqueness results is an adaptation of the arguments of Le Gruyer [8] that arose in the discussion with Adam Oberman.

Theorem 5. *Given a finite graph E with set of vertices $\mathfrak{X}$ and boundary Y, there exists the unique function u on $\mathfrak{X}$, which satisfies for every $x \in \mathfrak{X}$*

$$u(x) = \alpha \max_{S(x)} u + \beta \min_{S(x)} u + \gamma ⨍_{S(x)} u(y)dy \tag{3.29}$$

where $\gamma \neq 0$

Proof. We introduce the following non-linear average operator.

$$NA(u)(x) = \alpha \max_{S(x)} u + \beta \min_{S(x)} u + \gamma \fint_{S(x)} u(y)dy \tag{3.30}$$

First let us prove uniqueness by using a comparison principle. We prove that if u and v satisfy

$$u(x) = \alpha \max_{S(x)} u + \beta \min_{S(x)} u + \gamma \fint_{S(x)} u(y)dy, \tag{3.31}$$

$$v(x) = \alpha \max_{S(x)} v + \beta \min_{S(x)} v + \gamma \fint_{S(x)} v(y)dy \tag{3.32}$$

and $u \leq v$ on the boundary Y, then $u \leq v$ on the whole $\mathfrak{X}$. By the argument of contradiction assume that

$$M = \sup_{\mathfrak{X}}(u - v) > 0$$

Let

$$A = \{x \in \mathfrak{X} : u(x) - v(x) = M\}$$

By our assumptions $A \neq \emptyset$ and $A \cap Y = \emptyset$. Then we claim that exists $x_0 \in A$ and exists $y \in S(x_0)$ such that $y \notin A$. Otherwise, due to connectedness of our graph, if for all $x \in \mathfrak{X}$ and for all $y \in S(x)$ we have that $y \in A$, then we can easily conclude that $A \cap Y \neq \emptyset$. By definition of M we have

$$u(x_0) - v(x_0) \geq u(x) - v(x), \quad \forall x \in S(x_0) \tag{3.33}$$

$$v(x) - v(x_0) \geq u(x) - u(x_0), \quad \forall x \in S(x_0) \tag{3.34}$$

In particular

$$v(y) - v(x_0) > u(y) - u(x_0), \quad y \in S(x_0) \tag{3.35}$$

Now, we apply non-linear average operator to $v - v(x_0)$ and $u - u(x_0)$

$$0 = v(x_0) - v(x_0) = NA(v - v(x_0))(x_0) > NA(u - u(x_0))(x_0) = u(x_0) - u(x_0) = 0 \tag{3.36}$$

which gives us a contradiction. Observe that in the last step we used the fact that $\gamma \neq 0$. Now, let us prove existence. We consider a sequence

$$u_{n+1} = NA(u_n) \tag{3.37}$$

We claim that this sequence converges. We use the following result on convergence in metric spaces.

If every subsequence of a give sequence has a further subsequence converging to the same limit, then the whole sequence is convergent.

Choose a subsequence u_{n_k}, then due to compactness we know that there exist a convergent subsequence $u_{n_{k_l}}$. Similarly, choose a subsequence u_{n_m} different from u_{n_k}, and due to compactness conclude existence of a further convergent subsequence $u_{n_{m_r}}$. Since we are on the finite graph, limits of $u_{n_{k_l}}$ and $u_{n_{m_r}}$ satisfy equation (3.29), but due to uniqueness of the solution of (3.29) we obtain

$$\lim_{l\to\infty} u_{n_{k_l}} = \lim_{r\to\infty} u_{n_{m_r}} \tag{3.38}$$

Hence, $u = \lim_{n\to\infty} u_n$ is the solution. □

3.5 STRONG COMPARISON PRINCIPLE

Theorem 6. *Assume that u and v are solutions of equation (1.2) on $\mathfrak{X} \setminus Y$, $\gamma \neq 0$, $u \leq v$ on the boudary Y, and exists $x \in \mathfrak{X}$ such that $u(x) = v(x)$, then $u = v$ through the whole $\mathfrak{X}$.*

Proof. By Theorem 2 from the fact that $u \leq v$ on the boundary we know that $u \leq v$ on $\mathfrak{X}$. By definition of p-harmonious function we have

$$v(x) = \alpha \max_{y\in S(x)} v(y) + \beta \min_{y\in S(x)} v(y) + \gamma \fint_{S(x)} v(y)dy, \tag{3.39}$$

$$u(x) = \alpha \max_{y\in S(x)} u(y) + \beta \min_{y\in S(x)} u(y) + \gamma \fint_{S(x)} u(y)dy. \tag{3.40}$$

Since $u \geq v$ on $\mathfrak{X}$ we know that

$$\max_{y\in S(x)} v(y) \leq \max_{y\in S(x)} u(y),$$

$$\min_{y\in S(x)} v(y) \leq \min_{y\in S(x)} u(y),$$

$$\fint_{S(x)} v(y)dy \leq \fint_{S(x)} u(y)U(dy).$$

But since $u(x) = v(x)$, we actually have equalities

$$\max_{y\in S(x)} v(y) = \max_{y\in S(x)} u(y),$$

$$\min_{y\in S(x)} v(y) = \min_{y\in S(x)} u(y),$$

$$\fint_{S(x)} v(y)dy = \fint_{S(x)} u(y)dy.$$

From equality of average values and the fact that $u \geq v$ we conclude that $u = v$ on $S(x)$. Since our graph is connected, we immediately get the result. □

3.6 UNIQUE CONTINUATION

We can pose the following question. Let E be a finite graph with the vertex set $\mathfrak{X}$ and let $B_R(x)$ be the ball of radius R contained within this graph. Here we assign to every edge of the graph length one and let

$$d(x, y) = \inf_{x\sim y}\{|x \sim y|\},$$

where $x \sim y$ is the path connecting vertex x to the vertex y and $|x \sim y|$ is the number of edges in this path. Assume that u is a p-harmonious function on $\mathfrak{X}$ and $u = 0$ on $B_R(x)$. Does this mean that $u = 0$ on $\mathfrak{X}$? It seems like the answer to this question depends on the values of u on the boundary Y, as well as properties of the graph E itself. Here we can provide simple examples for particular graph, which shows that u does not have to be zero through the whole $\mathfrak{X}$.

164	-349	80	163	1	-164	1	163	96	-617	74
-349	-52	-19	28	1	-20	1	28	-38	-9	596
80	-19	-4	1	1	-2	1	1	-1	35	-217
163	28	1	0	0	0	0	0	1	-26	-26
1	1	1	0	0	0	0	0	-2	1	1
-164	-20	-2	0	0	0	0	0	1	7	52
1	1	1	0	0	0	0	0	1	1	1
163	28	1	0	0	0	0	0	-2	1	-53
80	-19	-4	1	1	-2	1	1	-1	-19	80
-349	-52	-19	28	1	-20	1	28	-19	2	-160
164	-349	80	163	1	-164	1	163	77	403	461

Figure 3: p=2, 8 neighbors

-31	21	-11	-5	1	3	1	-5	11	-21	23
21	-5	5	-3	-1	1	-1	3	-5	1	21
-11	5	0	1	-1	0	1	-1	0	5	-11
-5	-3	1	0	0	0	0	0	1	-3	5
3	-1	-1	0	0	0	0	0	-1	-1	3
1	1	0	0	0	0	0	0	0	1	1
3	-1	1	0	0	0	0	0	1	-1	3
-5	3	-1	0	0	0	0	0	-1	3	-5
11	-5	0	1	-1	0	1	-1	0	-5	11
-21	1	5	-3	-1	1	-1	3	-5	5	-21
23	21	-11	-5	1	3	1	-5	11	-21	31

Figure 4: p=infinity, 8 neighbors

The above example when $p = 2$ is rather interesting, since regular harmonic function is known to have the unique continuation property. One way to prove it is to exploit the fact that harmonic function is analytic and zeros of analytic functions are isolated. We would like to explore the discrepancy between the discrete example above and the continuous case. To make our study more general we would like to investigate a reformulation of the question of unique continuation for regular harmonic functions. In particular, we consider the following Dirichlet problem Let

$$\begin{cases} \Delta u = 0, \text{ in } B(0,1), \\ u = F, \text{ on } \partial B(0,1). \end{cases} \tag{3.41}$$

Assume that F is a continuous function and $u \equiv 0$ on $B(0,\epsilon)$, where $\epsilon << 1$. Then we know that

$$u(x) = \mathbb{E}^x F(X_\tau), \quad x \in B(0,\epsilon), \tag{3.42}$$

where X_t is the Brownian motion and $\tau = \inf\{t : X_t \in \partial B(0,1)\}$ [12]. We would like to show that $F = 0$ on $\partial B(0,\epsilon)$ a.e. This kind of problem is already known is statistics (see Lehmann and Romano [15]).

Definition 9. *A family $\mathcal{P}$ of probability distributions $\mathbb{P}$ is* **complete** *if*

$$E_{\mathbb{P}} F(X) = 0 \quad \textit{for all} \quad \mathbb{P} \in \mathcal{P} \tag{3.43}$$

implies

$$F(x) = 0, \ \textit{a.e. } \mathcal{P}. \tag{3.44}$$

Lehmann [15] established criterion for the completeness of the family of exponential probability distribution. In our case we have to work with the family of distributions with densities

$$\phi_x(y) = \frac{1-|x|^2}{|y-x|^2}, \quad y \in \partial B(0,1), \, x \in B(0,\epsilon), \tag{3.45}$$

(see Bass [3]).

The reason why we are looking for the new proof of unique continuation property is that due to stochastic game approach to the study of p-harmonic function, we might expect the following representation of p-harmonic function to hold for certain measure $\mathbb{P}^x$

$$u(x) = \mathbb{E}^x F(X_\tau). \tag{3.46}$$

With this representation the proof of unique continuation for the case $p = 2$ combined with the the notion of completeness of the family of measures could be used to prove the result for general p.

4.0 THE CASE OF A COUNTABLE DIRECTED GRAPH (TERNARY DIRECTED TREE)

All previous results were true for the case of a finite graph. In this chapter we will consider an extension to the case of a ternary directed tree. Our main goal would be as before finding functions satisfying certain mean value properties, as well as determining a measure on the boundary of a graph induced by our game. Let us first describe the graph in detail. As usual, we will denote our directed ternary tree by E and set of vertices by $\mathfrak{X}$. Here is a drawing of sample graph:

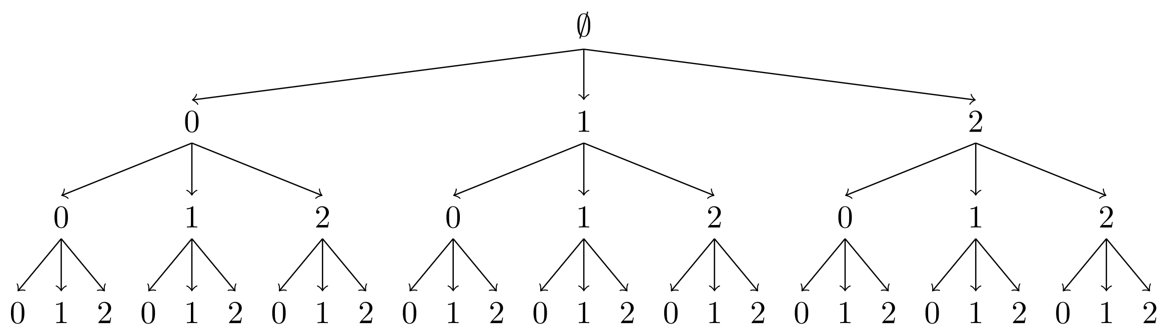

Figure 5: Directed tree - moving from top to bottom only

Every vertex is labeled either 0, 1 or 2. Each vertex has three successors. Therefore, on level 0 we only have one vertex labeled $\emptyset$, on level 1 we have 3 vertices, on level 2 we have 9 vertices, and on lever r we have 3^r vertices. We denote by $S(x)$ the three immediate successors of a vertex x. The boundary of our tree is different from what we had in the finite case.

Definition 10. *A* **branch** *of E is an infinite sequence of vertices, each followed by its immediate successor. The collection of all branches forms the* **boundary** *of the tree E and is denoted by Y.*

There is natural surjective mapping between set Y and the interval $[0,1]$. Every element of Y could be thought of as an ternary expansion of a number in the interval $[0,1]$. To be more precise we define the map for

$$v = (a_1, a_2, ..., a_k, ...) \in Y,$$

$$\psi(v) = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = x \in [0,1]. \tag{4.1}$$

Whenever $v_k = (a_1, a_2, ..., a_k)$ is only a finite sequence of vertices, we set

$$\psi(a_1, a_2, a_3, ..., a_k) = \psi(a_1, a_2, a_3, ..., a_k, 0, 0, 0, ...). \tag{4.2}$$

The function ψ is not a bijection simply because $1/3$ (in general all triadic rationals) does not have a unique ternary expansion

$$\psi(1\bar{0}) = \psi(0\bar{2}) = 1/3.$$

We also associate to a vertex v represented by a finite sequence $v_k = (a_1, a_2, a_3, ..., a_k)$ an interval I_v of length $\frac{1}{3^k}$ as follows

$$\theta : v \mapsto I_v = [\frac{\psi(v)}{3^k}, \frac{\psi(v)}{3^k} + \frac{1}{3^k}]. \tag{4.3}$$

With the above surjection in mind we consider a function $F : [0,1] \to \mathbb{R}$ and redefine it on the set Y by the following formula

$$F(v) = F(\psi(v)), \quad \text{for} \quad v \in Y, \tag{4.4}$$

where ψ is defined by (4.1).

Dirichlet problem: Given bounded function $F : [0,1] \to \mathbb{R}$ find a function u such that

$$u(v) = \alpha \max_{S(v)} u + \beta \min_{S(v)} u + \gamma \fint_{S(v)} u(w)dw \tag{4.5}$$

and takes boundary values F in the sense that

$$\lim_{n\to\infty} u(v_n) = u(v) = F(v), \quad \text{for all } v \in Y. \tag{4.6}$$

The value function of appropriate tug-of-war with noise game solves the problem as shown in [22]. Since game is different in the case of a ternary tree, let us provide some extra details.

4.1 GAME SETUP

We play the following game on this tree. We start the game by placing a token on some vertex and then toss a virtual coin with three sides. We require that the side of the coin labeled 0 comes out with probability $\alpha \geq 0$, the side of the coin labeled 1 comes out with probability $\gamma \geq 0$, and the side of the coin labeled 2 comes out with probability $\beta \geq 0$, where $\alpha + \gamma + \beta = 1$. If the outcome of the toss is 0, then player I chooses where to move the token among the three succeeding vertices $\{0, 1, 2\}$. If the outcome if the toss is 2, then player II chooses where to move the token among the three succeeding vertices $\{0, 1, 2\}$. If the outcome of the toss is 1, then we move the token randomly uniformly among the three succeeding vertices $\{0, 1, 2\}$. Formally, this game never stops and every run of the game generates an infinite sequence composed of 0, 1 and 2. Therefore, unlike the previous games we can not define a stopping time of hitting the boundary set Y, but we still can associate a payoff from player II to player I for $a \in Y$ by the equation (4.4). Hence, if game generates sequence $a \in Y$, then player I receives from player II $F(a)$ dollars. Now we need to define the expected pay off for an individual game and in order to do it we have to describe a measure on either set Y or on the interval $[0, 1]$, which is related to Y by (4.1). We accomplish this in the following section for particular choice of boundary function F in order to make the details of the construction more transperant. From now on it would be helpful to think of set Y in terms of interval $[0, 1]$ and vice versa.

4.1.1 CONSTRUCTION OF THE MEASURE WHEN F IS MONOTONE

When F is monotonically decreasing, the optimal strategy for player I is to always choose a node labeled 0 and the optimal strategy for player II is to choose a node labeled 2. As before the measure on the interval $[0,1]$ should depend on the starting point of the game and two strategies used by two players. Therefore, the measure that we about to describe should be properly denoted by

$$\mathbb{P}^v_{S_I,S_{II}},$$

where $v \in \mathfrak{X}$. For brevity we will write $\mathbb{P}^x_{S_I,S_{II}} = \mathbb{P}$. We will define the measure on the algebra of intervals in $[0,1]$ and then uniquely extend it to the $\sigma-$algebra generated by the collection of these intervals. By the ternary map (4.1) every infinite sequence in Y starting with 0 will end up in the interval $[0,1/3]$. Therefore, we set $\mathbb{P}([0,1/3]) = \alpha + 1/3\gamma$. Accordingly, every infinite sequence in Y starting with 1 will end up in the interval $[1/3,2/3]$ and we should set $\mathbb{P}([1/3,2/3]) = 1/3\gamma$. Finally, we conclude that $\mathbb{P}([2/3,1]) = \beta + 1/3\gamma$. The interval $[0,1/9]$ corresponds to the collection of sequences having first two coordinate equal 0. Therefore, $\mathbb{P}([0,1/9]) = (\alpha + 1/3\gamma)^2$. Continuing similarly we see that

$$\mathbb{P}([0,1/9]) = (\alpha + 1/3\gamma)^2, \qquad \mathbb{P}([1/9,2/9]) = (\alpha + 1/3\gamma)1/3\gamma,$$
$$\mathbb{P}([2/9,1/3]) = (\alpha + 1/3\gamma)(\beta + 1/3\gamma),$$

$$\mathbb{P}([1/3,4/9]) = 1/3\gamma(\alpha + 1/3\gamma), \qquad \mathbb{P}([4/9,5/9]) = (1/3\gamma)^2,$$
$$\mathbb{P}([5/9,2/3]) = 1/3\gamma(\beta + 1/3\gamma),$$

$$\mathbb{P}([2/3,7/9]) = (\beta + 1/3\gamma)(\alpha + 1/3\gamma), \quad \mathbb{P}([7/9,8/9]) = (\beta + 1/3\gamma)(1/3\gamma),$$
$$\mathbb{P}([8/9,1]) = (\beta + 1/3\gamma)^2.$$

Since every interval could be written as a union on intervals with triadic rational endpoints, we obtain a measure defined on the algebra of subintervals of $[0,1]$. Observe that when $\alpha < 1$, $\beta < 1$ and $\gamma < 1$ the measure defined above, in general, does not have atoms and, therefore, we do not have to worry about the endpoints of the intervals as well as set of triadic rationales, since both have measure zero.

We streamline the notation by setting

$$a = \alpha + \frac{1}{3}\gamma, \quad b = \frac{1}{3}\gamma, \quad c = \beta + \frac{1}{3}\gamma. \tag{4.7}$$

With this notation we choose node labeled 0 with probability a, node labeled 1 with probability b, and node labeled 2 with probability c. Hence, we can compute the probability of sequence $(a_1, a_2, a_3, ..., a_k)$ which is equal to the probability of the interval associated to $(a_1, a_2, a_3, ..., a_k)$ by (4.3).

Theorem 7. *Let $v = (a_1, a_2, a_3, ..., a_k)$ be a vertex at level k, then*

$$\mathbb{P}(I_v) = a^{n(0)} b^{n(1)} c^{n(2)}, \tag{4.8}$$

where $n(0)$ is the number of $\mathbf{0}$ in $(a_1, a_2, a_3, ..., a_k)$, $n(1)$ is the number of $\mathbf{1}$ in $(a_1, a_2, a_3, ..., a_k)$, and $n(2)$ is the number of $\mathbf{2}$ in $(a_1, a_2, a_3, ..., a_k)$.

Jose Llorente offered the following observation along with the idea of the proof.

Proposition 2. *Except for the case $\alpha = \beta = \gamma = \frac{1}{3}$, the probability $\mathbb{P}$ is singular with respect to Lebesgue measure.*

Proof. We will rely on the Proposition 1.1 from [16]. We will show that

$$\limsup_{n\to\infty} \frac{\mathbb{P}(I_n)}{|I_n|} = \infty, \quad \text{for } \mathbb{P}-a.e.\, x \in [0, 1], \tag{4.9}$$

where $\{I_n\}_{n\geq 1}$ is a decreasing sequence of intervals and $|\cdot|$ denotes Lebesgue measure. The above implies that measure $\mathbb{P}$ is singular with respect to Lebesgue measure. Theorem 7 shows that

$$\mathbb{P}(I_n) = \alpha^{n(0)} \beta^{n(1)} \gamma^{n(2)}, \tag{4.10}$$

where $n(0)$ is the number of $\mathbf{0}$ in the vertex associated to I_n, $n(1)$ is the number of $\mathbf{2}$ in the vertex associated to I_n and $n(2)$ is the number of $\mathbf{2}$ in the vertex associated to I_n. Observe that

$$|I_n| = 3^{-n}, \tag{4.11}$$

then

$$\frac{\mathbb{P}(I_n)}{|I_n|} = 3^n \alpha^{n(0)} \beta^{n(1)} \gamma^{n(2)},$$

$$\left(\frac{\mathbb{P}(I_n)}{|I_n|}\right)^{\frac{1}{n}} = 3\alpha^{\frac{n(0)}{n}}\beta^{\frac{n(1)}{n}}\gamma^{\frac{n(2)}{n}},$$

$$\lim_{n\to\infty}\left(\frac{\mathbb{P}(I_n)}{|I_n|}\right)^{\frac{1}{n}} = \lim_{n\to\infty} 3\alpha^{\frac{n(0)}{n}}\beta^{\frac{n(1)}{n}}\gamma^{\frac{n(2)}{n}} = 3\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}.$$

By considering $f(x,y) = x^x y^y (1-x-y)^{1-x-y}$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$ we conclude that

$$3\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma} > 1,$$

except when $\alpha = \beta = \gamma = \frac{1}{3}$. For some fixed α, β and γ there is $\eta > 0$ such that

$$\lim_{n\to\infty}\left(\frac{\mathbb{P}(I_n)}{|I_n|}\right)^{\frac{1}{n}} = 3\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma} \geq 1+\eta.$$

By definition of the limit for $\eta/2$ there is N, such that for all $n \geq N$

$$\left(\frac{\mathbb{P}(I_n)}{|I_n|}\right)^{\frac{1}{n}} \geq 1+\eta/2,$$

$$\frac{\mathbb{P}(I_n)}{|I_n|} \geq (1+\eta/2)^n$$

and

$$\limsup_{n\to\infty}\frac{\mathbb{P}(I_n)}{|I_n|} = \infty$$

□

4.2 EXISTENCE AND UNIQUENESS

The expected payoff of an individual game is

$$\int_{[0,1]} F(y)\mathbb{P}^v_{S_I,S_{II}}(dy) = \mathbb{E}^v_{S_I,S_{II}}[F].$$

We also define value of the game for the first player as

$$u_I(v) = \sup_{S_I} \inf_{S_{II}} \int_{[0,1]} F(y)\mathbb{P}^v_{S_I,S_{II}}(dy),$$

and value of the game for the second player as

$$u_{II}(v) = \inf_{S_{II}} \sup_{S_I} \int_{[0,1]} F(y)\mathbb{P}^v_{S_I,S_{II}}(dy).$$

The fact that the value function of a tug-of-war with noise game satisfies (4.5) is a restatement of the DPP [22]. Uniqueness follows by noting that the argument of Theorem 2 applies here as well.

Theorem 8. *The solution to Dirichlet problem (4.5) and (4.6) is unique and is given by the value of the game function*

$$u(v) = \sup_{S_I} \inf_{S_{II}} \int_{[0,1]} F(y)\mathbb{P}^v_{S_I,S_{II}}(dy) = \inf_{S_{II}} \sup_{S_I} \int_{[0,1]} F(y)\mathbb{P}^v_{S_I,S_{II}}(dy).$$

4.2.1 SPECIAL CASES

Special cases of the above game are interesting.

Case: $p = 2$ $(\gamma = 1, \alpha = 0, \beta = 0)$ In this case equation (4.5) is

$$u(x) = \fint_{S(x)} u(y)dy \tag{4.12}$$

and the solution is given by running random walk on the tree. The expression for the value of the game in this case simplifies to

$$u(v) = \int_{I_v} F(y)\mathbb{P}^v dy = \frac{1}{|I_v|}\int_{I_v} F(y)dy, \tag{4.13}$$

We remark that the last integral in (4.13) is with respect to Lebesgue measure.

Case: $p = \infty$ $(\gamma = 0,\ \alpha > 0,\ and\, \beta > 0)$ **and** F **is monotonically decreasing** This is a case of tug-of-war and the answer to Dirichlet problem is give by the value of the game function, which in this case has particularly simple form due to existence of optimal strategies.

$$u(x) = \sup_{S_I}\inf_{S_{II}}\int_{[0,1]} F(y)\mathbb{P}^x_{S_I,S_{II}}(dy) = \int_{[0,1]} F(y)\mathbb{P}^x_{S_I^*,S_{II}^*}(dy) \tag{4.14}$$

The optimal strategy S_I^* for player I is alway to pull to the left or to choose vertex labeled 0 and the optimal strategy S_{II}^* is to always pull to the right or to choose vertex labeled 2. If one tries to visualize the dynamics of this game, one can notice that the boundary set in this case is Cantor set. For the rigorous proof we can use the characterization of a Cantor set as set of real numbers whole triadic expansion does not contain digit 1. In addition, when $\alpha = \beta = 1/2$ the resulting measure on the unit interval $\mathbb{P}^x_{S_I^*,S_{II}^*}$ is the Cantor measure. This fact is remarkable enough to be stated as a theorem.

Theorem 9. *(Infinity Laplacian in ternary trees and the Cantor-like measure.)*
Let $F : [0,1] \to \mathbb{R}$ *bounded and monotonically decreasing. The solution to the Dirichlet problem*

$$\begin{cases} u(v) = \frac{1}{2}\max_{w\in S(v)} u(w) + \frac{1}{2}\min_{w\in S(v)} u(w), & v \in \mathfrak{X} \\ u(y) = F(y), & \textit{for all } y \in Y \end{cases} \tag{4.15}$$

is given by

$$u(v) = \int_{I_v} F(z)\mathfrak{C}^v(dz), \tag{4.16}$$

where $\mathfrak{C}^v$ *is the Cantor like measure.*

Proof. By the notation $\mathfrak{C}^v$ we mean the Cantor measure on the interval I_v with $\mathfrak{C}^v(I_v) = 1$. It is the measure corresponding to the distribution function given by the Cantor step function, which is constructed inductively by split I_v into three equal parts and setting the function to be constant on the middle part.

The only claim in the theorem that requires proof is

$$\mathbb{P}^v_{S_I^*, S_{II}^*} = \mathfrak{C}^v,$$

but it follows by checking that both measure coincide on the algebra of triadic intervals. By a triadic interval we mean an interval whose endpoints are triadic rational numbers. □

An interesting observation suggested by David Futer is the following one. In case F is not monotonically decreasing we can employ graph automorphism to describe the boundary measure $\mathbb{P}$ in more familiar terms. In particular, at each vertex of our tree we introduce a permutation σ such that

$$u(\sigma(v_k^i)) \leq u(\sigma(v_k^{i+1})), \quad 0 \leq i \leq 1.$$

Then we define

$$w(v_k^i) = u(\sigma(v_k^i)).$$

If we extend the notion of the permutation on the tree to the permutation of the interval $[0, 1]$ and define F^* to be the boundary function obtained from F through our permutation, then we can apply Theorem 9:

$$u(\sigma(v)) = w(v) = \int_{I_v} F^*(z)\mathfrak{C}^v(dz),$$

$$u(\sigma(v)) = w(v) = \int_{I_v} F^*(z)\mathfrak{C}^v(dz) = \int_{I_v} F(\sigma(z))\mathfrak{C}^v(dz).$$

If we change variables, we obtain

$$u(\sigma(v)) = \int_{I_{\sigma^{-1}(v)}} F(z)\mathfrak{C}^{\sigma^{-1}(v)} \circ \sigma^{-1}(dz).$$

Properties of the measure $\mathfrak{C}^{\sigma^{-1}(v)} \circ \sigma^{-1}$ present a definite research interest.

4.3 ITERATED FUNCTIONS SYSTEMS

The boundary set and the measure on this set that we described in this section could also be studied in terms of Iterated Functions Systems (IFS) (see [2]). We give here a definition IFS which suites our purpose.

Definition 11. *The Iterated Function Systems with probabilities is a collection of contractive mappings*

$$S_i : [0,1] \to [0,1], \quad i = 1, ..., n$$

with collection of weights

$$p_i, \quad \sum_{i=1}^{n} p_i = 1.$$

According to Theorem 2 [2], there exists a unique set B called *attractor* of the IFS with the property

$$B = \bigcup_{i=1}^{n} S_i(B). \tag{4.17}$$

The existence of set B follows from the contraction mapping principle. We can find set B by applying iteratively all mappings S_i to the interval $[0,1]$. Further we define Markov operator $M : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$, where $\mathcal{P}([0,1])$ is a set of probability measures on the interval $[0,1]$.

$$M(\nu) = \sum_{i=1}^{n} p_i \nu \circ S_i^{-1}. \tag{4.18}$$

In proper settings [2] this operator is a contraction and, therefore, exists unique probability measure with the property

$$M\mu = \mu. \tag{4.19}$$

This measure μ is know as invariant or self-similar. One of the central questions is the study of singularity or absolute continuity of measure μ with respect to Lebesgue measure. This question is already very interesting in case of only two contractive mappings as shown by the following example from [14].

Example.

$$S_1(x) = \rho x, \quad S_2(x) = 1 - \rho x, \quad 0 < \rho < 1.$$

The invariant measure $\mu(= \mu_\rho)$is

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}.$$

For $0 < \rho < \frac{1}{2}$ μ is a Cantor type measure and for $\frac{1}{2} < \rho < 1$ the measure is much more complicated. Due to Solomyak [27] we only know that μ is absolutely continuous with respect to Lebesgue measure. There is exist a remarkable connection to algebraic integers. In particular, when ρ^{-1} is Pisot-Vijayarahavan (P.V.) number, the invariant measure μ is singular. See [14] for this result and the definition of P.V. number. Let us demonstrate the connection of the IFS theory to our game.

Case: $p = 2$ $(\gamma = 1, \alpha = 0, \beta = 0)$ We define the IFS with probabilities

$$S_i : [0, 1] \to [0, 1]; \quad S_i(x) = \frac{1}{3}x + \frac{i}{3}, \quad \text{for } i = \{0, 1, 2\}, \tag{4.20}$$

$$p_0 = \alpha + \frac{1}{3}\gamma, \quad p_1 = \frac{1}{3}\gamma, \quad p_2 = \beta + \frac{1}{3}\gamma. \tag{4.21}$$

The attractor set B is the boundary of our ternary tree (4.1) and in our case it is simply the interval $[0, 1]$. The invariant measure μ is the measure on the boundary of our tree induced by the game i.e. $\mu = \mathbb{P}$. Since in our case is $\alpha = \beta = 0$ and $\gamma = 1$, μ is a Lebesgue measure supported on $[0, 1]$.

Case: $p = \infty$ $(\gamma = 0,\ \alpha > 0,\ and\, \beta > 0)$ **and F is monotonically decreasing** We define IFS with probabilities

$$S_i : [0, 1] \to [0, 1]; \quad S_i(x) = \frac{1}{3}x + \frac{i}{3}, \quad \text{for } i = \{0, 2\}, \tag{4.22}$$

$$p_0 = \alpha, \quad p_2 = \beta. \tag{4.23}$$

In this case the attractor set B is the Cantor set. When $\alpha = \beta = \frac{1}{2}$ we see that the invariant measure in this case is the Cantor measure.

Case: $2 < p < \infty \Leftrightarrow \gamma >, \alpha > 0, \mathit{and}\, \beta > 0$ **and** F **is monotonically decreasing** Similar to the previous cases we define the IFS with probabilities

$$S_i : [0,1] \to [0,1]; \quad S_i(x) = \frac{1}{3}x + \frac{i}{3}, \quad \text{for } i = \{0,1,2\}. \tag{4.24}$$

$$p_0 = \alpha + \frac{1}{3}\gamma, \quad p_1 = \frac{1}{3}\gamma, \quad p_2 = \beta + \frac{1}{3}\gamma. \tag{4.25}$$

The attractor set B is the interval $[0,1]$. The invariant measure in this case is singular with respect to Lebesgue measure, except when $p_0 = p_1 = p_2 = \frac{1}{3}$, as shown by Proposition 2.

5.0 CONNECTIONS TO $\mathbb{R}^N$

The connections between discrete results presented above and the partial differential equations on $\mathbb{R}^n$ are well illustrated by the following publications. -"Tug-of-war and infinity laplacian" Y. Peres, O. Schramm, S. Sheffield, and D. Wilson [26]; -"Tug-of-war with noise: a game theoretic view of the p-Laplacian" Y. Peres and S. Sheffield [25]; -"On absolutely minimizing lipschitz extensions and PDE $\Delta_\infty = 0$" E. Le. Gruyer [8]; -"On the definition and properties of p-Harmonious functions" Manfredi, M. Parviainen, and J.D. Rossi [21]. The following three concepts are important.

5.1 VISCOSITY SOLUTION

Viscosity solution of the ∞-Laplacian. One way to extent the results obtained in the previous sections is to approximate some subset of $\mathbb{R}^n$ by a sequence of graphs with decreasing length of the edges. Given the approximating sequence one can employ the notion of expansion in viscosity sense as studied by Manfredi, Parviainen, and Rossi in [20]. An alternative approach to connecting discrete results and the continuous case is to use the following notions: viscosity solution, AMLE and comparison with cones as outlined in [26]. For the sake of completeness we present here some details of the second approach. Consider the partial differential equation

$$\begin{cases} \Delta_\infty u = 0 & \text{on } U, \\ u = F & \text{on } \partial U. \end{cases} \tag{5.1}$$

where $\Delta_\infty u$ is give by

$$\Delta_\infty u = \left\langle D^2 u Du, Du \right\rangle .$$

The 1-homogenous version is

$$\Delta_\infty u = \frac{\langle D^2 u Du, Du\rangle}{\langle Du, Du\rangle}.$$

Definition 12. *Let $\mathcal{T}_x$ be the set of real valued functions ϕ such that*

- *$\phi \in \mathcal{C}^2$ in some neighborhood of x,*
- *$\Delta_\infty \phi$ is defined in some neighborhood of x in the following sense. Either $D\phi(x) \neq 0$ or $D\phi(x) = 0$ and the limit*

$$\Delta_\infty \phi(x) = \lim_{y\to x} 2\frac{\phi(y) - \phi(x)}{|y-x|^2}$$

exists.

Let Ω be a domain in $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be continuous. Set

$$\Delta^+_\infty u(x) = \inf\{\Delta_\infty \phi(x) : \phi \in \mathcal{T}_x \textit{and } x \textit{ is a local minimum of } \phi - u\}.$$

We say that u satisfies $\Delta^+_\infty u \geq g$ in a domain Ω, if for every $\phi \in \mathcal{C}^2$ s.t. $\phi - u$ has a local minimum at some $x \in \Omega$ satisfies $\Delta^+_\infty \phi(x) \geq g(x)$. In this case u is called **viscosity subsolution** *of $\Delta_\infty(u) = g$. Similarly, we set*

$$\Delta^-_\infty u(x) = \inf\{\Delta_\infty \phi(x) : \phi \in \mathcal{T}_x \textit{and } x \textit{ is a local maximum of } \phi - u\}$$

and call u a **viscosity supersolution** *of $\Delta_\infty(\cdot) = g$ if $\Delta^-_\infty u \leq g$ in Ω u is a* **viscosity solution** *of $\Delta_\infty(\cdot) = g$ if $\Delta^-_\infty u \leq g \leq \Delta^+_\infty u(x)$ in Ω (i.e., u is both a supersolution and a subsolution).*

5.2 ABSOLUTELY MINIMAL LIPSCHITZ EXTENSIONS

The second important concept is **Absolutely Minimal Lipschitz Extensions** (AMLE) introduced by G. Aronsson [1]. Consider the following problem. Given a metric space (X, d) set $Y \subset X$ and function F defined on ∂Y, find $\tilde{F}$ which is an extension of F to Y s.t. $Lip_{\partial Y} F = Lip_Y \tilde{F}$. Where

$$Lip_{\Omega} f = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in \Omega, \right\}.$$

The answer to this problem was found by McShane [18] and Whitney [28]

$$\tilde{F}(x) = \inf_{y \in Y} [F(y) + Lip_Y d(x,y)].$$

Now consider the problem of finding an extension $\tilde{F}$ such that

$$\text{for all} \quad open \quad U \subset X \setminus Y, \text{ we have that} \quad Lip_{\partial U} \tilde{F} = Lip_U \tilde{F}.$$

Such an $\tilde{F}$ is called an absolutely minimal Lipschitz extension. Function F is Absolutely Minimal (AM), if it is defined on a domain $\bar{U}$ and it is the AMLE of its restriction to ∂U.

5.3 COMPARISON WITH CONES

Definition 13. *Let $b, c \in \mathbb{R}$, then function $\phi(y) = b|y - z| + c$ is called the cone based at $z \in \mathbb{R}^n$.*

Definition 14. *Let $U \subset \mathbb{R}^n$ and $u \in C(U, \mathbb{R})$, then u satisfies* **comparison with cones** *from above on U, if for every open $W \subset \bar{W} \subset U$, for every $z \in \mathbb{R}^n \setminus W$ and for every cone ϕ based at z such that the inequality $u \leq \phi$ holds on ∂W, the same inequality is valid throughout W.*

Comparison with cones from below is defined similarly using the inequality $u \geq \phi$.

It turns out that all three concepts mentioned above (viscosity solution of infinity Laplacian, AMLE, and comparison with cones) are equivalent. Jensen [10] proved that viscosity solutions to $\Delta_\infty u = 0$ for domains in $\mathbb{R}^n$ satisfy comparison with cones (from above and below), and Crandall, Evans, and Gariepy [4] proved that

a function on $\mathbb{R}^n$ is absolutely minimal in a bounded domain U, if and only if it satisfies comparison with cones in U.

The authors of the first paper [26] discovered that a random turn zero-sum game "tug-of-war" provides insight into the question of existence and uniqueness of the solution of infinity Laplacian. Let us briefly describe the game. The game is played by 2 players on a graph E with vertex set $\mathfrak{X}$ and boundary set $Y \subset \mathfrak{X}$. At the beginning of the game a token is placed at some vertex x_0, then a fair coin is tossed and whichever player wins the coin toss moves the token to some vertex x_1. The vertex x_1 must be connected to x_0 by single edge. Then the coin is tossed again. The game stops when the token reaches any point at the boundary $y \in Y$. At the end of the game player I receives from player II the amount of $F(y)$ dollars.

The value of the game is the expected payoff that player I can get from player II, provided they both play their best.

The game also could be played on the length space with a step size less than some fixed ϵ. It seems like, it was Peres et al. observation that the value of the game satisfies comparison with cones that led them to the proof that value of the game function is the unique viscosity solution of infinity Laplacian.

The second paper by Peres et al. deals with the existence of the solution for p-Laplacian. tug-of-war with noise is the game suggested for this problem. Similar to "tug-of-war", there are 2 players. At the beginning of the game a token is placed at some point $x_0 \in U$, where U is a bounded subset of a length space $(\mathfrak{X}, d)$. Then a fair coin is tossed and whoever wins the coin toss moves a token to some point x_1, s.t. $x_1 \in B(x_0, \epsilon)$, after that a vector v_1 perpendicular to $x_1 - x_0$ is added. The magnitude of v is random, with a variance related to ϵ. Whenever, the game hits some point $y \in \partial U$, game stops and player I receives from player II $F(y)$ dollars.

6.0 FUTURE WORK

6.1 HARNACK'S INEQUALITY

Another direction of the research could be the proof of Harnack's inequality. If $\mathfrak{X}$ is the finite set of vertices and B_R is a ball of radius R such that $2B_R \subset \mathfrak{X}$, then we would like to check whether the following holds for some constant c

$$\sup_{B_R} u \leq c \inf_{B_R} u,$$

where u is non-negative solution of (1.2). One way to prove this inequality and get an estimate of c would be through the use of conditioning.

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