NUMERICAL ANALYSIS OF THE AERODYNAMIC NOISE PREDICTION IN DIRECT NUMERICAL SIMULATION AND LARGE EDDY SIMULATION

by

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This thesis presents the rigorous numerical analysis of the aerodynamic noise generation via Lighthill acoustic analogy, [36], which is a non-homogeneous wave equation describing the sound waves. Over more than five decades, the Lighthill analogy was extensively used as one of the major tools in engineering applications in acoustics. However, the first mathematical research of the Finite Element approximation for it is introduced here. Specifically, we focus on both Direct Numerical and Large Eddy Simulations. The more or less intuitive derivation of the Lighthill analogy is reviewed in section 1.3.

First, the semidiscrete and fully discrete Finite Element methods in DNS are presented and the effect of the computational error in the right-hand side of the wave equation is pointed out. The convergence of this error to zero is studied in the semidiscrete case. The computational results support obtained theoretical predictions.

Second, the numerical analysis, using the negative norms of the error, is presented in the semidiscrete case. The negative norms help obtain better convergence rate and require less regularity of the data than positive norms.

Third, the sound power is defined as a non-linear functional of acoustic variables and three independent ways of computing it in the semidiscrete case in DNS are presented. All of these methods are based on the Finite Element scheme presented earlier. The methods are compared from the point of view of computational cost, accuracy and simplicity. Again, the computational experiments are presented.

Finally, the concept of Large Eddy Simulation is introduced for aeroacoustic research via Lighthill analogy. Two subgrid scale models, these are van Cittert deconvolution and Bardina, are presented for the filtered acoustic analogy. The semidiscrete Finite Elemet Method is analyzed for both of them. We present the numerical experiments for this research as well.

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1.0 INTRODUCTION

Aeroacoustics is a large scientific field which studies the generation, prediction and control of noise generated by nonlinear interaction in turbulent flows. It is an area of great practical importance, inherently nonlinear and one in which the correct physical models are still under depate. This thesis has focused on the so called Lighthill acoustic analogy and has treated

- the rigorous numerical analysis of the continuous Galerkin-type Finite Element Method (FEM), both semidiscrete and fully discrete, for solving the Lighthill analogy,
- the error analysis in negative Sobolev norms for the numerical solutions of the Lighthill analogy,
- evaluating numerically the sound power of the noise and estimating the computational error for it,
- large eddy modeling and simulation for the Lighthill analogy and the corresponding numerical analysis.

Generally, the motivation for the noise research is dictated either by the need of noise reduction in technologies and providing quiet and healthy environment or by the need of using the noise itself as a part of the technological instrument. Prediction of the noise is an important problem in various engineering applications such as transport by trains and jet airplanes. For high velocities the aerodynamic noise tends to dominate other sources of noise, [63]. The engines of the next generation fighter jets are expected to produce more than 140 decibels of noise while 150 already damage internal organs, [45]. Home technology, such as coffee makers or climate systems can also generate level of aeroacoustic noise that, while not dangerous, is annoying. Another important application lie in ocean acoustics and submarine detection. Measuring characteristics of the sound emitted from a blood flow in a valve of a heart would help diagnose heart murmurs. Wind turbines and helicopter rotors also produce significant amount of noise that designers are constantly seeking to reduce.

As we see, these problems require

- reliable and applicable description of the noise if the information about the turbulent flow is provided,
- ways for controlling noise through control of the turbulent flow; in particular, ways for reducing the noise and at the same time maintaining the non-acoustical properties of the flow needed in certain applications.

The basic physical model for decribing the aerodynamic noise is very recent. It was proposed by Lighthill [36] in 1951. Given the turbulent flow's velocity $\mathbf{u}$ and density ρ, the Lighthill's model for the small acoustic pressure fluctuations $p^{'}$ is a wave equation with a nonlinear source term

$$\frac{1}{a_0^2}\frac{\partial^2 p^{'}}{\partial t^2} - \Delta p^{'} = \nabla \cdot (\nabla \cdot (\rho_0 \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho_0 \mathbf{f}), \tag{1.1}$$

with *deviatoric* stress tensor $\mathbb{S}$, the sound speed $a_0 = \sqrt{\frac{\partial p}{\partial \rho}}|_{\rho=\rho_0}$, the external body force $\mathbf{f}$ and the averaged density ρ_0. A rigorous mathematical derivation of the Lighthill model is given in Novotny and Layton [46]. The Lighthill model is the accepted approach to aeroacoustic noise prediction. Since it is not commonly studied in the mathematical literature, we review the model in section 1.3.

Definition 1. *Mach number of a flow is defined as $M = U/a_0$, where U is the characteristic velocity of the flow.*

For low Mach numbers the generated noise itself plays little role in changing the flow and thus the Lighthill model desribes a one-way process. Noise is generated by the flow whose motion is dependent solely on the known external forces. No feedback from the noise to the turbulent flow is considered, [36]. For small Mach numbers compressibility of the flow has negligible impact on the sound generation, see for example [63]. Therefore, noise can be predicted by solving the incompressible Navier-Stokes equations (NSE) for $\mathbf{u}$ and inserting the incompressible velocity and density ρ_0 into the right-hand side (RHS) of (1.1) and then solving (1.1) for the acoustic pressure q. The Navier-Stokes equations may be written as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho_0} \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

with the kinematic viscosity ν and the pressure p. In this incompressible case $\nabla \cdot \nabla \cdot \mathbb{S} = 0$, Lemma 4 of section 1.3, and noise is produced through the nonlinear term in (1.1) if $\nabla \cdot \mathbf{f} = 0$. More on computational practice with Lighthill analogy may be found in [13], [56] and [68].

Although the Lighthill analogy has been used as one of the main tools for computing the noise in lots of applications, not much significant mathematical support was provided for it. In this research the first rigorous analysis of a numerical method for computing the noise via Lighthill analogy is introduced.

The whole acoustic domain of our model equation (1.1) is divided in two parts. These are the turbulent region Ω_1 with the flow where the generation of sound occurs and the far field Ω_2 where the acoustic waves propagate. We suppose that Ω_1 is surrounded by Ω_2 and the whole domain is $\Omega = \Omega_1 \cup \Omega_2$, figure 1.

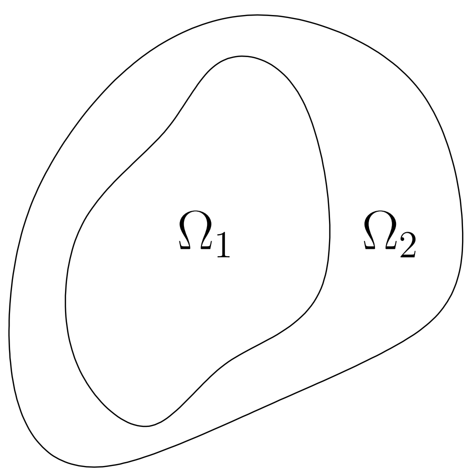

Figure 1: Turbulent flow region and surrounding acoustic region

Let $R(t,x) = \nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \rho\mathbf{f})$ inside Ω_1 and 0 around it in Ω_2. The Initial Boundary Value Problem is the following:

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = R(t,x) + \frac{1}{a_0^2}G(t,x)\ \forall(t,x)\in(0,T)\times\Omega, \tag{1.2}$$
$$q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x\in\Omega,$$
$$\nabla q\cdot\mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall(t,x)\in(0,T)\times\partial\Omega.$$

The functions $G(t,x)$ and $g(t,x)$ are arbitrary control functions that we add according to the problem's physics and goals. The case $g\equiv 0$ in (1.2) gives the non-reflecting boundary conditions of the first order. The computation of the incompressible NSE in Ω_1 is carried out on some mesh of size $h_1 < 1$, thus generating the numerical approximation R_{h_1} of the term R.

1.1 LITERATURE SURVEY

Lighthill analogy was first formulated in [36] in 1951. The results were based mostly on deep physical observations. It was shown that the noise generation is often dependent only on the term $\rho_0\nabla\cdot\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$. The strength of the noise is that which would be produced by a static distribution of acoustic quadrupoles. This follows from the solution of the wave equation obtained as an integral via Greens function. Also, the intensity of the noise was predicted to be proportional to the eighth power of the characteristic velocity of the flow, assumed the speed of sound is a constant. However, the direct computation of the integral in order to compute the fluctuation of pressure in (1.1) may be challenging, needless to say it is almost always impossible to evaluate analytically. Thus some efficient computational method is required for (1.1). Other acoustic analogies were presented in [38], [49], [28], [43] and rely on the Lighthill approach. [25] investigates the generation of sound by high Reynolds-number turbulent shear flows. According to this paper, Lighthill analogy explains prominent properties of this phenomenon very well, but some subtle features are better explained with other analogies.

[66] gives a good overview of modern computational methods for aeroacoustics and problems associated with them. Authors specified that science has entered the second so-called Golden Age of aeroacoustics, meaning the appearance of stricter noise regulations than they were during 1950s-70s and larger variety of problems, and, at the same time, more efficient and accurate methods for solving acoustic problems. [65] also gives a good critical review of common techniques for the computation of the aerodynamic sound.

Most of the work on the Lighthill analogy and aeroacoustics in general is related to computational and engineering aspects of the field. For example, [58] studies the prediction of the jet noise via Lighthill analogy and compares the results with experimental data. [13] and [56] present and validate the computational results for the fan noise predicted through Lighthill model using FEM. [57] provides wide results of aeroacoustic computations in the case of Direct Numerical Simulation (DNS), including acoustic spectrum. These results agree with those from [51], the latter were obtained analytically. [29] proposed the Linearized Euler Equations (LEE) for solving aeroacoustical problems. The presented numerical method uses finite volume scheme and the LEE are integrated in pseudo-time plane using a Runge-Kutta algorithm. The accuracy of the method is proven, as well as numerical examples for a few aeroacoustical model problems. In [1], a method that couples Finite Difference NSE solver in the turbulent region and the Discontinuous Galerkin (DG) LEE solver in the far field is presented from the point of view of computational performance, obtaining good results. [48] considers physical aspects of sound control in different technological areas.

RANS method is also used for the noise prediction, [9]. Large Eddy Simulation (LES) is another promising, quickly growing field in the Computational Fluid Dynamics and has been applied to the Computational Aeroacoustics (CAA) as well. The book [63] gives an excellent general overview of the common aeroacoustic problems and computational methods for them. Papers [67], [42] and [50] present the filtered Lighthill analogy with or without correcting subgrid scale tensor. The methods of parametrization of the subgrid scale tensor were addressed, for example, in [60], [8], [39]. The paper [59] is the one on which the LES chapter of this thesis is based on. It analyzes a simple case of noise generated by decaying isotropic turbulence and obtains numerical results that are compared with theoretical predictions.

The rigorous mathematical theory for the practical computational methods are to be the next research step. Complete mathematical theory would help understand which methods are to be used most efficiently and accurately in certain applications. It is also important to create a reliable mathematical basis for future CAA development.

1.2 THESIS CONTENTS

This thesis presents and studies both semidiscrete and the fully discrete Finite Element Methods for the Lighthill analogy in chapters 3 and 4 respectively. The analysis is based on Dupont [19] where the basic FEM scheme, both continuous and discrete in time, for the wave equation with RHS known exactly was analyzed. Our analysis differs by the presence of the computational error in the RHS of the wave equation in (1.2). We show in chapter 3 that the FEM formulation of the problem (1.2) has a stable solution for bounded time periods. The main convergence results are presented in Theorems 11 and 14. The right hand side of the error estimate has one bounding term that involves the error in the turbulent flow that generates the acoustic noise. The numerical experiments in chapter 5 support these theoretical results. We expand the analysis of chapter 3 through negative norms of the error, Theorem 18 of chapter 6.

The acoustic intensity or sound intensity is a nonlinear funtion of the acoustic variables, [37],

$$I = q \cdot \mathbf{v},$$

where $\mathbf{v}$ is the velocity of the fluid, and in the case of the far field it is a small velocity fluctuation about the zero state. The flux integral of the intensity along a surface S gives a sound power

$$A = \int_S I \cdot \mathbf{n} dS.$$

Is is often the fundamental quantity needed in a simulation. Chapter 7 studies three different ways on calculating the sound power on a given surface S and estimating the numerical error for it.

The first method uses the linearized continuity and momentum equations as a starting point in order to obtain exact formula for computing velocity $\mathbf{v}$ in the far field. Formally, the algorithm consists of three steps:

1. Compute q_h using FEM on Ω.
2. Compute the velocity by the formula
$$\mathbf{v}_h(t,\cdot) = -\frac{1}{\rho_0}\int_0^t \nabla q_h(\tau,\cdot)d\tau.$$
3. Compute the sound power $A_h = \int_S q_h \mathbf{v}_h \cdot \mathbf{n} dS$.

This is the cheapest method computationally, but theoretically is the least accurate. We prove an improvement in the rate of covergence when $S \subset \partial\Omega$.

The second approach considered in 7.2 is to obtain an upper bound for sound power. This bound is computed via the acoustic pressure q_h only. The numerical error in the bound is analyzed.

The last method is based on duality analysis. The convergence analysis techniques are only presented for the case $S \subset \partial\Omega$ and time-averaged sound power. This method breaks the problem in two computational subproblems, one is for finding q_h and the other is for finding $\mathbf{v}_{h_2}$ on the other independent grid of mesh size $h_2 < 1$ in Ω:

1. Compute q_h using FEM on Ω.
2. Compute the velocity $\mathbf{v}_{h_2}$ using FEM on Ω.
3. Compute the sound power $A_h = \int_S q_h \mathbf{v}_{h_2} \cdot \mathbf{n} dS$.

Although duality method gives the highest possible rate of convergence for the term containing q_h in the error, the scheme for $\mathbf{v}_{h_2}$ still requires more study since the rate of convergence it gives is 1 degree less than that for the term with q_h. From this point of view, we can only say that duality method is less preferable compared to the exact formula approach, since they both give the same rate of convergence in case $h = O(h_2)$ and computationally the duality method is much more expensive.

Chapter 7 also contains the numerical experiments of computing the sound power.

Finally, the numerical analysis of the Large Eddy Simulation applied to (1.2) will be presented in later sections. The filtering procedure of the Lighthill analogy (1.1) bears the necessity for presenting some subgrid scale model. Chapter 8 is an introduction to the LES in general and its application to the aeroacoustics. Section 8.1 presents the analysis for the zeroth order van Cittert deconvolution model. Section 8.2 studies the case of Bardina subgrid scale model. The final Section 8.3 contains the numerical results for an academic solution that uses the Bardina subgrid scale model.

1.3 DERIVATION OF THE LIGHTHILL ANALOGY

To understand Lighthill's contribution, we consider first the derivation of the far-field acoustic equation. We start with the compressible NSE for density ρ, velocity $\mathbf{u}$ and pressure p:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.3}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbb{S} + \rho \mathbf{f}. \tag{1.4}$$

In the far field the external forces $\mathbf{f}$ and the viscous stress tensor $\mathbb{S}$ are typically negligible. Additionally we have a relation $p = P(\rho, s)$ where s denotes the entropy. The wave equation is the result of linearization of the equations with respect to the rest state which is characterized by constants $\mathbf{u}_0 = 0$, p_0, ρ_0, $\mathbf{f} = 0$:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \rho = \rho_0 + r, p = p_0 + q. \tag{1.5}$$

Next differentiate the linearized continuity equation with respect to time and take the divergence of the linearized momentum equation. Subtraction of the results leads to the equation

$$\frac{\partial^2 r}{\partial t^2} - \Delta q = 0.$$

Using the relation between pressure and density gives the homogeneous wave equation in the form

$$\frac{1}{a_0^2} \frac{\partial^2 q}{\partial t^2} - \Delta q = 0. \tag{1.6}$$

The above wave equation only holds in the far field in which the sound propagates. Coupling equations for the turbulent region and the fluctuations requires some efficient physical model. Lighthill's approach has erased the gap between the turbulent region and the far field in (1.1).

The derivation of the Lighthill analogy is presented below. See, e.g., [36] for extensions, alternate derivation and complementary work. Rewrite (1.4) in the divergence form assuming (1.3):

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \nabla\cdot\mathbb{S} + \rho\mathbf{f}. \tag{1.7}$$

Differentiate (1.3) with respect to time and apply divergence operator to (1.7) :

$$\begin{aligned}&\frac{\partial^2\rho}{\partial t^2} + \frac{\partial}{\partial t}\nabla\cdot(\rho\mathbf{u}) = 0,\\ &\frac{\partial}{\partial t}\nabla\cdot(\rho\mathbf{u}) + \nabla\cdot\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \Delta p = \nabla\cdot\nabla\cdot\mathbb{S} + \nabla\cdot\rho\mathbf{f}.\end{aligned}$$

Subtraction of these two equations gives the following holding in Ω:

$$-\Delta p = \nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \nabla\cdot\mathbb{S} - \rho\mathbf{f}) - \frac{\partial^2\rho}{\partial t^2}. \tag{1.8}$$

Consider the far field where the perturbations of the pressure and density are defined with respect to the rest state. Then (1.8) is mathematically equivalent to

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta p = \nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \nabla\cdot\mathbb{S} - \rho\mathbf{f}) + \frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2} - \rho). \tag{1.9}$$

We choose a_0 to be the speed of sound in the medium at rest state. Equation (1.9) may already be called Lighthill's analogy. Now some considerations must be made. First, in the far field the last term $\frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2} - \rho) = \frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2} - r)$ is negligible because it is simply the LHS of the classical wave equation in the quiescent state (see [34] for details). Moreover, in this medium the first term on the RHS is also negligible because it consists of the nonlinear term and two linear terms that make no significant influence on the sound propagation in the far field. Therefore, in the far field equation (1.9) reduces to the wave equation (1.6) for the acoustic pressure. Lighthill's model extends equation (1.9) to the whole fluid including the turbulent region. Suppose that perturbations of the pressure and density are defined on the whole Ω and the last term on the RHS of (1.9) is negligible on Ω. These two suppositions together give a picture of the whole aerodynamical system as a field of wave propagation with the divergence term playing a role of a sound source.

Definition 2. $\mathbb{T} = \rho\mathbf{u}\otimes\mathbf{u} - \mathbb{S}$ *is called the Lighthill tensor.*

The Lighthill tensor is not negligble in the turbulent region and is negligible in laminar regions including the far field. The whole system is described by the following equation:

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = \nabla \cdot (\nabla \cdot \mathbb{T} - \rho \mathbf{f}). \tag{1.10}$$

This model of sound generated by turbulence allows breaking this problem in two subproblems. In the turbulent region we can use methods applicable for solving incompressible NSE and this will provide us with tensor $\mathbb{T}$. Knowing the RHS of the equation (1.10) we can solve the non-homogeneous hyperbolic problem for the whole domain. In the far field we set the RHS to zero.

In fact, for relatively small Mach numbers the compressibility of the flow has negligible impact on the sound generation (see, e.g., [63]). The fluctuations of the density $r = \rho - \rho_0$ in the RHS of (1.1) are the terms of high order and may be neglected. Thus we consider the coupled problem of (1.10) holding in Ω and

$$\begin{aligned} &\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla \cdot \mathbb{S} + \rho \mathbf{f}, \\ &\nabla \cdot \mathbf{u} = 0, \end{aligned} \tag{1.11}$$

holding in Ω_1. The boundary conditions for (1.11) depend on a certain application.

Lemma 3. *If* $\rho \equiv \rho_0$ *and* $\nabla \cdot \mathbf{u} = 0$*, then* $\nabla \cdot \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$.

Proof. Since ρ is constant,

$$\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \rho_0 (u_i u_j)_{,i} = \rho_0 (u_{i,i} u_j + u_i u_{j,i}),$$

where u_i denotes the i-th component of the vector $\mathbf{u}$ and repeating index means summation. Due to the incompressibility condition, the last expression equals

$$\rho_0 (u_{i,i} u_j + u_i u_{j,i}) = \rho_0 u_i u_{j,i} = \rho_0 \mathbf{u} \cdot \nabla \mathbf{u}.$$

Finally,

$$\nabla \cdot (\rho_0 \mathbf{u} \cdot \nabla \mathbf{u}) = \rho_0 \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \rho_0 (u_i u_{j,i})_{,j} = \rho_0 (u_{i,j} u_{j,i} + u_i u_{j,i,j}) = \rho_0 u_{i,j} u_{j,i}.$$

The last term is precisely $\rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$. ∎

Lemma 4. *If* $\nabla \cdot \mathbf{u} = 0$*, then* $\nabla \cdot \nabla \cdot \mathbb{S}(\mathbf{u}) = 0$*.*

Proof. Let $\mu > 0$ be the shear viscosity coefficient of the fluid. Since in incompressible flows

$$\nabla \cdot \mathbb{S}(\mathbf{u}) = \mu \Delta \mathbf{u},$$

then

$$\nabla \cdot \nabla \cdot \mathbb{S}(\mathbf{u}) = \mu \nabla \cdot \Delta \mathbf{u}.$$

Consider

$$\nabla \cdot \Delta \mathbf{u} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\Delta u_i) = \sum_{i=0}^{3} \frac{\partial^2}{\partial x_i^2}(\nabla \cdot \mathbf{u}) = 0.$$

□

The last two lemmas allow us to rewrite the RHS of the Lighthill analogy in the form

$$\rho_0 \cdot (\nabla \mathbf{u} : \nabla \mathbf{u}^t - \nabla \cdot \mathbf{f}).$$

1.4 THE EQUATION FOR THE FLUCTUATION OF VELOCITY

Since one of the methods for calculating the sound power uses some FEM scheme to compute the velocity fluctuation $\mathbf{v}$, defined by the first equality of (1.5), it is therefore important to derive an equation for $\mathbf{v}$. One way is to start from the first order system of two equations governing the fluctuations of pressure (or density)and velocity:

$$\frac{1}{\rho_0}\frac{\partial q}{\partial t} + a_0^2 \nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0}\nabla q = \frac{1}{\rho_0}\mathbf{F},$$

where $\mathbf{F}$ is zero in the far field and

$$\mathbf{F} = -\rho_0 \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \rho_0 \cdot \mathbf{f} + \nabla \cdot \mathbb{S}$$

in the turbulent region of the flow. A similar system was presented in [46], (1.2), and may be derived by simply linearizing the first two equations of (4) of [36].

Here we give our own derivation of the equation, rigorous up to some neccessary physical assumptions. Rewrite the exact mass and momentum conservation equations (1.3), (1.7) as

$$a_0^2 \frac{\partial \rho}{\partial t} + a_0^2 \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + a_0^2 \nabla \rho = a_0^2 \nabla \rho - \nabla p - \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \cdot \mathbb{S} + \rho \mathbf{f}.$$

Take the gradient of the first equation and differentiate the second with respect to time t and subtract to get

$$\frac{\partial^2(\rho \mathbf{u})}{\partial t^2} - a_0^2 \nabla(\nabla \cdot (\rho \mathbf{u})) = a_0^2 \nabla \frac{\partial \rho}{\partial t} - \nabla \frac{\partial p}{\partial t} - \frac{\partial}{\partial t} \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial}{\partial t} \nabla \cdot \mathbb{S} + \frac{\partial}{\partial t} \rho \mathbf{f},$$

or

$$\frac{\partial^2(\rho \mathbf{u})}{\partial t^2} - a_0^2 \nabla(\nabla \cdot (\rho \mathbf{u})) = a_0^2 \nabla \frac{\partial r}{\partial t} - \nabla \frac{\partial q}{\partial t} - \frac{\partial}{\partial t} \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial}{\partial t} \nabla \cdot \mathbb{S} + \frac{\partial}{\partial t} \rho \mathbf{f}.$$

Since it is assumed that Mach number M is small, the entropy production term $\nabla \frac{\partial}{\partial t}(a_0^2 r - q)$ is negligible even in the turbulent region, so it is dropped. Next, expand the left-hand side (LHS) of the last equation using (1.5). This gives

$$\frac{\partial^2(r \mathbf{v})}{\partial t^2} - a_0^2 \nabla(\nabla \cdot (r \mathbf{v})) + \rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} - \rho_0 a_0^2 \nabla(\nabla \cdot \mathbf{v}) = -\frac{\partial}{\partial t} \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial}{\partial t} \nabla \cdot \mathbb{S} + \frac{\partial}{\partial t} \rho \mathbf{f}.$$

Low Mach number allows to neglect compressibility and assume that the first two terms of the last equation are of high order compared to others and may be dropped. The following is the equation for the fluctuation of velocity $\mathbf{v}$:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - a_0^2 \nabla(\nabla \cdot \mathbf{v}) = \frac{1}{\rho_0} \frac{\partial}{\partial t}(-\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \cdot \mathbb{S} + \rho \mathbf{f}). \tag{1.12}$$

In the far field the RHS of (1.12) is reduced to zero. If $\frac{\partial \mathbf{f}}{\partial t}$ is negligible and the Reynolds number is high, then the RHS is simplified to

$$-\frac{\partial}{\partial t} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}).$$

2.0 NOTATION AND PRELIMINARIES

In this paper we assume that both Ω and Ω_1 are open bounded connected domains in $\mathbb{R}^n$, $n = 2, 3$, having smooth enough boundaries $\partial\Omega$ and $\partial\Omega_1$ respectively. $(\cdot, \cdot)$ and $\|\cdot\|$ without a subscript denote the $L^2(\Omega)$ or $L^2(\Omega_1)$ inner product and norm depending on which domain is considered at the moment. The norms $\|\cdot\|_{L^p(\Omega)}$ may be used for vector functions $\mathbf{u}$ with two or three components in a Banach space H. If $1 \leqslant p < \infty$, they should be understood as

$$\|\mathbf{u}\|_{L^p(\Omega)} = \left(\sum_{i=1}^{n} \|u_i\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

where u_i denotes i-th component of $\mathbf{u}$ and n is the number of components. The inner product should be understood as

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} (u_i, v_i).$$

$L^2(\partial\Omega)$ denotes the space of the real-valued square-integrable functions on the boundary $\partial\Omega$ of the domain Ω. The inner product in this space is denoted as $< \cdot, \cdot >$:

$$< u, v >= \int_{\partial\Omega} u \cdot v dS \text{ for } u, v \in L^2(\partial\Omega).$$

The norm induced by this inner product is denoted as $|\cdot|$:

$$|v| = \sqrt{< v, v >} \text{ for } v \in L^2(\partial\Omega).$$

For any integer $s \geqslant 0$ let $H^s(\Omega)$ denote a Sobolev space $W^{s,2}(\Omega)$ of real-valued functions on a domain Ω. The inner product and norm in the space $H^s(\Omega)$ are defined by

$$(u, v)_{H^s(\Omega)} = \sum_{|\alpha|=0}^{s} (\partial^\alpha u, \partial^\alpha v), \;\; \|u\|_{H^s(\Omega)} = \sqrt{(u, u)_{H^s(\Omega)}},$$

where α is a multiindex and $\partial^\alpha u$ denotes a weak partial derivative of the order $|\alpha|$ of the function u. Next, if B denotes a Banach space with norm $\|\cdot\|_B$ and $u : [0,T] \to B$ is Lebesgue measurable, then we define

$$\|u\|_{L^p(0,T;B)} = \left(\int_0^T \|u\|_B^p dt\right)^{\frac{1}{p}}, \|u\|_{L^\infty(0,T;B)} = esssup_{0\leqslant t\leqslant T}\|u(t,\cdot)\|_B,$$

and the space

$$L^p(0,T;B) = L^p(B) = \{u : [0,T] \to B | \|u\|_{L^p(0,T;B)} < \infty\} \text{ for } 1 \leqslant p \leqslant \infty.$$

Theorem 5. *Let $v \in H^1(\Omega)$. Then $v \in H^{\frac{1}{2}}(\partial\Omega)$ and the following inequality holds*

$$\|v\|_{L^2(\partial\Omega)} \leqslant C_{tr}\|v\|_1,$$

where C_{tr} is a constant that depends only on the geometry of the domain Ω.

2.1 FINITE ELEMENT SPACE

Let us build non-degenerate, edge-to-edge, shape regular mesh by introducing the partition $\Pi = \{T_1, T_2, ..., T_M\}$ of Ω into triangles. The characteristic size of the mesh $h < 1$ is defined by

$$h = max_{1\leqslant i\leqslant M} diam(T_i).$$

Following [11], define

$$M^m(\Omega) = \{u \in L^2(\Omega) \mid u|_T \in P_{m-1} \ \forall T \in \Pi\} \text{ and } M_0^m(\Omega) = M^m(\Omega) \cap C^0(\Omega),$$

where P_m is the space of polynomials of degree no more than m and $C^0(\Omega)$ is the space of continuous on Ω functions. Therefore, by M_0^m we mean the space of continuous piecewise polynomials of degree no more than $m-1$. The space $M_{00}^m(\Omega)$ consists of all functions from $M_0^m(\Omega)$ with zero trace on the boundary $\partial\Omega$.

From now on, C will denote a generic constant, not necessarilly the same in two places. As in [19], we suppose there exist a positive constant C and integer $k \geqslant 2$ such that the spaces $M_0^m(\Omega)$ have the property that for $0 \leqslant s \leqslant 1$ and $2 \leqslant m \leqslant k$, and $V \in H^m(\Omega)$

$$inf_{\chi \in M_0^m(\Omega)} \|V - \chi\|_{H^s(\Omega)} \leqslant Ch^{m-s} \|V\|_{H^m(\Omega)}.$$

Following [19], we define the H^1-projection $\hat{u} \in M_0^m(\Omega)$ for $u \in H^1(\Omega)$ by the formula:

$$a_0^2(\nabla u, \nabla u_h) + (u, u_h) = a_0^2(\nabla \hat{u}, \nabla u_h) + (\hat{u}, u_h) \;\forall u_h \in M_0^m(\Omega).$$

Below is the lemma that will be used in the proof of the main theorem about the error estimate.

Lemma 6. *(Dupont [19], Lemma 5) Let $u, \frac{\partial u}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L^2(H^k(\Omega))$ for some positive integer k, $m \geqslant k \geqslant 2$. Then for some positive constant C independent of h the error in the H^1-projection $\hat{u}$ satisfies*

$$\left\| \frac{\partial^r (u - \hat{u})}{\partial t^r} \right\|_{L^s(L^2(\Omega))} + \left\| \frac{\partial^r (u - \hat{u})}{\partial t^r} \right\|_{L^s(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant Ch^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively.

A mesh with above properties is called quasi-uniform, if there exist constants C_1 and C_2 independent of h, such that

$$C_1 \cdot diam(T_i) \leqslant diam(T_j) \leqslant C_2 \cdot diam(T_i)$$

for any distinct triangular elements T_i and T_j of the mesh.

If a mesh is quasi-uniform and functions v_h from the space $M_0^m(\Omega)$ built on this mesh satisfy the following regularity condition for non-negative integers l_1, l_2 and real numbers $p, q > 1$

$$v_h \in W^{l_1,p}(\Omega) \cap W^{l_2,q}(\Omega),$$

then the following inverse estimate holds (see [12]):

$$\|v_h\|_{W^{l_1,p}(\Omega)} \leqslant Ch^{l_2 - l_1 + min(0, \frac{n}{p} - \frac{n}{q})} \|v_h\|_{W^{l_2,q}(\Omega)}$$

for any $v_h \in M_0^m(\Omega)$ and some positive constant C independent on h.

For a given FEM space $M_0^m(\Omega)$, $m \geqslant 2$, consider the nodal basis consisting of functions ϕ_j. An arbitrary function $u \in H^m(\Omega)$ has a unique continuous representation on Ω and therefore we define a piecewise polynomial interpolant $I_h(u)$ for this function. If N_j denote the nodal points then

$$I_h(u) = \sum_j u(N_j)\phi_j.$$

In simulations of the incompressible NSE the FEM spaces for velocity X_h and pressure Q_h must satisfy the LBB-condition. It guarantees the stability of the approximate pressure. It is as follows:

$$inf_{q_h \in Q_h} sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|\nabla v_h\| \cdot \|q_h\|} \geqslant \beta_h > 0, \tag{2.1}$$

where β_h is bounded away from zero uniformly in h. More on the LBB-condition may be found in [35].

3.0 SEMIDISCRETE SCHEME

Definition 7. *Define*

$$Q(\mathbf{u},\mathbf{v}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{v}^t.$$

In fluid mechanics the term $\rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$ is called Q. The $Q > 0$ is used for eduction of persistent coherent vortices. It is interesting that, according to Lemmas 3 and 4 of section 1.3, this same quantity occurs in the RHS of (1.1) as the dominant sound source in its generation by turbulent flows.

Consider the following initial boundary-value problem

$$\begin{aligned}
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = a_0^2 (Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f}) + G(t,x)\ \forall (t,x) \in (0,T) \times \Omega_1, \qquad (3.1)\\
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = 0\ \forall (t,x) \in (0,T) \times \Omega / \Omega_1,\\
&q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x \in \Omega,\\
&\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall (t,x) \in (0,T) \times \partial\Omega,
\end{aligned}$$

where all functions on the RHS are known and $\mathbf{n}$ being the outward normal on the boundary $\partial\Omega$. The case $G \equiv 0$ refers to the turbulent flow being the only source of the sound. The question of proper boundary conditions depends on the physical problem. $g(t,x) \equiv 0$ gives the case of the first-order non-reflecting boundary conditions. Although more accurate non-reflecting boundary conditions are known, those in (3.1) with $g(t,x) \equiv 0$ are the first step in applications where the interest lies in the sound waves that propagate in infinite space without reflection. This allows a simulation to measure acoustic power of the waves generated solely by the turbulent flow. The non-zero boundary control function $g(t,x)$ may be used if we want to consider additional sources of sound on the boundary. Adding $g(t,x)$ to the right-hand side of the boundary condition has no effect on the error estimates.

In computations, $Q(\mathbf{u},\mathbf{u})$ is given approximately due to two reasons. First, Q consists of the solution of the incompressible NSE. Second, the solution of the incompressible NSE is found via computations and thus contains error which follows from inaccuracy of the scheme used. Let h_1 denote the mesh size of this scheme. The modeling error due to incompressibity is analyzed in [46]. The second one is of computational importance and is analyzed here.

The total error between the exact solution q of (3.1) and the approximate q_h will consist of the FEM error caused by computations and the perturbation of the RHS caused by replacing $Q(\mathbf{u},\mathbf{u}) - \rho_0\nabla\cdot\mathbf{f}$ with $Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f}$.

The variational formulation is as follows. Assume that

$$Q(\mathbf{u},\mathbf{u}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G \in L^2(0,T;L^2(\Omega_1)), q(0,\cdot) \in H^1(\Omega),$$

$$\frac{\partial q}{\partial t}(0,\cdot) \in L^2(\Omega), g \in L^2(0,T;L^2(\partial\Omega)).$$

Find $q \in L^2(0,T;H^1(\Omega))$ such that $\frac{\partial q}{\partial t} \in L^2(0,T;H^1(\Omega))$, $\frac{\partial^2 q}{\partial t^2} \in L^2(0,T;L^2(\Omega))$ and

$$\begin{aligned}\left(\frac{\partial^2 q}{\partial t^2}, v\right) &+ a_0^2\left(\nabla q, \nabla v\right) + a_0\left\langle\frac{\partial q}{\partial t}, v\right\rangle = \\ &= a_0^2\left(Q(\mathbf{u},\mathbf{u}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G, v\right)_{\Omega_1} + a_0^2 < g, v > \\ &\forall v \in H^1(\Omega), 0 < t < T,\end{aligned} \tag{3.2}$$

$$(q(0,\cdot),v) = (q_1(\cdot),v)\ \forall v \in H^1(\Omega), \tag{3.3}$$

$$\left(\frac{\partial q}{\partial t}(0,\cdot),v\right) = (q_2(\cdot),v)\ \forall v \in H^1(\Omega). \tag{3.4}$$

The condition that $Q(\mathbf{u},\mathbf{u}) \in L^2(0,T;L^2(\Omega_1))$ is satisfied if we impose the following regularity condition for $\mathbf{u}$:

$$\mathbf{u} \in L^4(0,T;W^{1,4}(\Omega_1)).$$

This fact easily follows from Holder's inequality.

The finite element approximation will be based on finite-dimensional spaces $M_0^m(\Omega) \subset H^1(\Omega)$ of continuous piecewise polynomials of degree no more than $m-1$, chapter 2. It is as follows. Assume that

$$Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \nabla \cdot \mathbf{f} + \frac{1}{a_0^2} G \in L^2(0,T; L^2(\Omega_1)),\ g \in L^2(0,T; L^2(\partial\Omega)).$$

Find such twice differentiable map $q_h : [0,T] \to M_0^m(\Omega)$ that

$$\begin{aligned} \left(\frac{\partial^2 q_h}{\partial t^2}, v_h \right) + a_0^2 \left(\nabla q_h, \nabla v_h \right) + a_0 \left\langle \frac{\partial q_h}{\partial t}, v_h \right\rangle = \\ = a_0^2 \left(Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \nabla \cdot \mathbf{f} + \frac{1}{a_0^2} G, v_h \right)_{\Omega_1} + a_0^2 < g, v_h > \end{aligned} \tag{3.5}$$

$$\forall v_h \in M_0^m(\Omega), 0 < t < T,$$

$$q_h(0,\cdot) \text{ approximates } q_1 \text{ well,}$$

$$\frac{\partial q_h}{\partial t}(0,\cdot) \text{ approximates } q_2 \text{ well.}$$

The regularity condition $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) \in L^2(0,T; L^2(\Omega_1))$ is handled by the following lemma.

Lemma 8. *Suppose the exact velocity* $\mathbf{u}$ *satisfies condition*

$$\mathbf{u} \in L^4(0,T; H^2(\Omega_1)) \cap L^4(0,T; W^{1,4}(\Omega_1))$$

and the mesh used for computing $\mathbf{u}_{h_1}$ *in* Ω_1 *is quasi-uniform. Finally, let* $\|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}$ *converge to zero no slower than* $O(h_1^{1+\frac{n}{4}})$*, where* $n = 2$ *or* 3 *is the dimension of the physical space. Then*

$$\mathbf{u}_{h_1} \in L^4(0,T; W^{1,4}(\Omega_1)),$$

and thus $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) \in L^2(0,T; L^2(\Omega_1))$.

Proof. Due to the triangle inequality, it is obvious that

$$\|\mathbf{u}_{h_1}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant \|\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} + \|\mathbf{u} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} + \|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))}.$$

Here I_{h_1} is a piecewise polynomial interpolant, chapter 2. The first term on the RHS is bounded due to the assumption of the lemma. The second term may be bounded as shown (see, for example, [12]):

$$\|\mathbf{u} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant C\|\nabla \mathbf{u}\|_{L^4(L^4(\Omega_1))}.$$

To bound the third term, we need to use the inverse estimate in the following manner:

$$\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant Ch_1^{-1-\frac{n}{4}}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(L^2(\Omega_1))}.$$

The final step is to use the triangle inequality

$$\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant Ch_1^{-1-\frac{n}{4}}\left(\|\mathbf{u} - I_{h_1}\mathbf{u}\|_{L^4(L^2(\Omega_1))} + \|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}\right).$$

The first term on the RHS may be bounded by $Ch^{1-\frac{n}{4}}\|\nabla\nabla\mathbf{u}\|_{L^4(L^2(\Omega_1))}$. The assumption on the speed of convergence of $\|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}$ finishes the proof. □

Theorem 9. *The solution q_h of (3.5) is stable and the following inequality holds:*

$$\left\|\frac{\partial q_h}{\partial t}\right\| + a_0\|\nabla q_h\| \leqslant C\left(\left\|Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|_{L^2(L^2(\Omega_1))} + \|g\|_{L^2(L^2(\partial\Omega))} + \right.$$
$$\left. + \left\|\frac{\partial q_h}{\partial t}(0,\cdot)\right\| + \|\nabla q_h(0,\cdot)\|\right)$$

with positive constant $C = C(t)$.

Proof. Set $v_h = \frac{\partial q_h}{\partial t}$. Then

$$\frac{1}{2}\frac{d}{dt}(\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2 \left\|\nabla q_h\right\|^2) + a_0 \left|\frac{\partial q_h}{\partial t}\right|^2 =$$
$$= a_0^2 \left(Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G, \frac{\partial q_h}{\partial t}\right)_{\Omega_1} + a_0^2 \left\langle g, \frac{\partial q_h}{\partial t}\right\rangle,$$

$$\frac{d}{dt}\left(\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2\|\nabla q_h\|^2\right) \leqslant a_0^4 \left\|Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|^2 + \left\|\frac{\partial q_h}{\partial t}\right\|^2 + \frac{a_0^3}{2}|g|^2,$$

$$\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2\|\nabla q_h\|^2 \leqslant \int_0^t \left(a_0^4 \left\|Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|^2 + \right.$$
$$\left. + \left\|\frac{\partial q_h}{\partial t}\right\|^2 + \frac{a_0^3}{2}|g|^2\right) d\tau + \left\|\frac{\partial q_h}{\partial t}(0,\cdot)\right\|^2 + a_0^2\|\nabla q_h(0,\cdot)\|^2.$$

Applying Gronwall's lemma to the inequality above finishes the proof. □

Remark 10. *The function $C(t)$ from the theorem may grow exponentially fast. This fact may be related to the phenomena of resonance which is common for hyperbolic problems.*

Consider the H^1-projection $\hat{q} \in L^2(0,T; M_0^m(\Omega))$ of the solution of (3.2)-(3.4) given by the formula

$$a_0^2(\nabla q, \nabla v_h) + (q, v_h) = a_0^2(\nabla \hat{q}, \nabla v_h) + (\hat{q}, v_h)\ \forall v_h \in M_0^m(\Omega). \tag{3.6}$$

Theorem 11. *Let the solution q of (3.2) satisfy the conditions: $q, \frac{\partial q}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 q}{\partial t^2} \in L^2(H^k(\Omega))$ for some positive integer k, $m \geqslant k \geqslant 2$. If the initial conditions are taken so that*

$$\|(q_h - \hat{q})(0,\cdot)\|_{H^1(\Omega)} + \left\|\frac{\partial}{\partial t}(q_h - \hat{q})(0,\cdot)\right\| \leqslant C_1 h^k$$

with some posititve constant C_1 independent of h, then the solution of (3.5) satisfies:

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(q - q_h)\right\|_{L^\infty(L^2(\Omega))} \leqslant$$
$$\leqslant C\left(h^k + \|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}\right)$$

with some constant $C > 0$ independent of h.

Proof. Denote $\psi = \hat{q} - q_h$, $\eta = q - \hat{q}$. The proof of the theorem technically resembles that of Theorem 2 in Dupont's work [19] except there is additional error term $Q - Q_{h_1}$ on the RHS of the error equation. The complete proof with all the details may be found in [40]. We will go straight to the inequality

$$\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2_{H^1(\Omega)} + \int_0^t \left|\sqrt{a}\cdot\frac{\partial \psi}{\partial t}\right|^2 \leqslant$$
$$C\left[\int_0^t \left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2_{H^1(\Omega)}\right) d\tau + \|\eta\|^2_{L^2(L^2(\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}\right.$$
$$\left. + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial \psi}{\partial t}(0,\cdot)\right\|^2 + \|\psi(0,\cdot)\|^2_{H^1(\Omega)} + \int_0^t \|Q_{h_1} - Q\|^2 d\tau\right]$$

with some positive constant C. Apply Gronwall's lemma to yield

$$\left\|\frac{\partial \psi}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))} + \|\psi\|^2_{L^\infty(H^1(\Omega))} + \left\|\sqrt{a}\cdot\frac{\partial \psi}{\partial t}\right\|^2_{L^2(L^2(\partial\Omega))} \leqslant$$
$$C\left[\left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + \|\eta\|^2_{L^2(L^2(\Omega))} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}\right.$$
$$\left. + \left\|\frac{\partial \psi}{\partial t}(0,\cdot)\right\|^2 + \|\psi(0,\cdot)\|^2_{H^1(\Omega)} + \int_0^t \|Q_{h_1} - Q\|^2 d\tau\right],$$

where $C = C(T)$ grows exponentially fast. Next we can use Lemma 6, i.e. for some constant C independent of h

$$\left\|\frac{\partial^r \eta}{\partial t^r}\right\|_{L^s(L^2(\Omega))} + \left\|\frac{\partial^r \eta}{\partial t^r}\right\|_{L^s(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant Ch^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively. If $q_h(0,\cdot)$, $\frac{\partial q_h}{\partial t}(0,\cdot)$ are taken so that $\|(q_h - \hat{q})(0,\cdot)\|_{H^1(\Omega)} + \left\|\frac{\partial}{\partial t}(q_h - \hat{q})(0,\cdot)\right\| \leqslant C_1 h^k$, where C_1 is independent of h, then there is a constant C independent of h such that

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(q - q_h)\right\|_{L^\infty(L^2(\Omega))} \leqslant C\left(h^k + \|Q_{h_1} - Q\|_{L^2(L^2(\Omega_1))}\right).$$

□

Now the estimate for $Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})$ must be found. Here we deal with another Finite Element scheme of the mesh size h_1 used for computing velocity field $\mathbf{u}_{h_1}$ of the turbulent flow in the inner domain Ω_1. Let X_{h_1} and Q_{h_1} denote the Finite Element spaces satisfying the LBB-condition (2.1).

Theorem 12. *Suppose the solution* $\mathbf{u}$ *of the incompressible NSE in* Ω_1 *satisfies the following regularity condition:*

$$\mathbf{u} \in L^\infty(0,T;H^2(\Omega_1)) \cap L^\infty(0,T;W^{1,4}(\Omega_1)) \cap L^2(0,T;W^{m,4}(\Omega_1))$$

for some integer $m \geqslant 2$. *In addition, assume that the mesh which is used for computing* $\mathbf{u}_{h_1}$, *is quasi-uniform. If the approximating space* $M_0^k(\Omega_1)$ *is used for computing velocity* $\mathbf{u}$ *with* $k \geqslant m$ *and the error* $\|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^\infty(L^2(\Omega_1))}$ *converges to zero no slower than* $O(h_1^{1+\frac{n}{4}})$, *where* $n = 2$ *or* 3 *is the dimension of the physical space, then there exists such positive constant* C *independent of* h_1 *that*

$$\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))} \leqslant$$
$$Ch_1^{-\frac{n}{4}} \cdot \left(\ h_1^{m-1}\|\partial^m \mathbf{u}\|_{L^2(L^4(\Omega_1))} + \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}\right).$$

Proof. It is easy to see that

$$Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) = \rho_0 \cdot (\nabla\mathbf{u} : \nabla\mathbf{u}^t - \nabla\mathbf{u}_{h_1} : \nabla\mathbf{u}_{h_1}^t) =$$
$$= \rho_0 \cdot (\nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t) + \rho_0 \cdot (\nabla(\mathbf{u} - \mathbf{u}_{h_1}) : \nabla\mathbf{u}_{h_1}^t).$$

Bound both terms separetely. For the L^2-norm of the first one obtain

$$\|\rho_0 \cdot (\nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t)\|^2 \leqslant C\int_\Omega |\nabla\mathbf{u}|^2 |\nabla(\mathbf{u} - \mathbf{u}_{h_1})|^2$$

for some suitable positive constant C. By Holder's inequality

$$C\int_{\Omega_1} |\nabla\mathbf{u}|^2 |\nabla(\mathbf{u} - \mathbf{u}_{h_1})|^2 \leqslant C\|\nabla\mathbf{u}\|^2_{L^4(\Omega_1)} \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|^2_{L^4(\Omega_1)}.$$

Consider the continuous piecewise polynomial interpolant $I_{h_1}(\mathbf{u})$ for $\mathbf{u}$. Obviously,

$$\nabla(\mathbf{u} - \mathbf{u}_{h_1}) = \nabla(\mathbf{u} - I_{h_1}(\mathbf{u})) + \nabla(I_{h_1}(\mathbf{u}) - \mathbf{u}_{h_1}).$$

Hence,

$$\|\rho_0 \cdot (\nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t)\| \leqslant$$
$$C\|\nabla\mathbf{u}\|_{L^4(\Omega_1)} \left(\|\nabla(\mathbf{u} - I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(I_{h_1}(\mathbf{u}) - \mathbf{u}_{h_1})\|_{L^4(\Omega_1)}\right).$$

In the same manner,

$$\|\rho_0 \cdot (\nabla(\mathbf{u}-\mathbf{u}_{h_1}) : \nabla \mathbf{u}_{h_1}^t)\| \leqslant$$
$$C\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \left(\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\|_{L^4(\Omega_1)}\right).$$

To bound the term $\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)}$ use the triangle inequality:

$$\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \leqslant \|\nabla(\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla \mathbf{u}\|_{L^4(\Omega_1)}.$$

Using the inverse estimate, we obtain

$$\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \leqslant Ch_1^{-1-\frac{n}{4}}\|\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u})\| + \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla \mathbf{u}\|_{L^4(\Omega_1)}.$$

The last two terms are bounded uniformly in time due to the regularity assumption of the theorem. For the first term on the RHS apply the triangle inequality as shown below:

$$Ch_1^{-1-\frac{n}{4}}\|\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u})\| \leqslant Ch_1^{-1-\frac{n}{4}}\left(\|\mathbf{u}-I_{h_1}(\mathbf{u})\| + \|\mathbf{u}_{h_1}-\mathbf{u}\|\right).$$

Both terms on the RHS are bounded uniformly in time, which follows from the assumption on the regularity and the speed of convergence. We obtain

$$\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\| \leqslant C(\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + C_1 h_1^{-\frac{n}{4}}\|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\|),$$

where $C = C(\mathbf{u})$ is a function of $\mathbf{u}$ independent of h_1. Further,

$$\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} \leqslant C_1 h_1^{m-1}\|\partial^m \mathbf{u}\|_{L^4(\Omega_1)}.$$

Next

$$\|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\| \leqslant \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\| + \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|,$$
$$\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\| \leqslant Ch_1^{m-1}\|\partial^m \mathbf{u}\| \leqslant C_1 h_1^{m-1}\|\partial^m \mathbf{u}\|_{L^4(\Omega_1)}.$$

So finally

$$\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\| \leqslant$$
$$C(h_1^{m-1-\frac{n}{4}}\|\partial^m \mathbf{u}\|_{L^4(\Omega_1)} + h_1^{-\frac{n}{4}}\|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|).$$

Since we are interested in $L^2(L^2)$-norm of $Q(\mathbf{u},\mathbf{u})-Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})$, we square and integrate the last inequality:

$$\int_0^t \|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|^2 d\tau \leqslant$$
$$Ch_1^{-\frac{n}{2}} \cdot \int_0^t (h_1^{2m-2}\|\partial^m \mathbf{u}\|^2_{L^4(\Omega_1)} + \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|^2) d\tau.$$

The statement of the theorem follows after extracting the square root of both sides of the last inequality. □

Remark 13. *The term $\|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$ may be bounded by Ch_1^p with some positive integer p, depending on which FEM space is used to solve the incompressible NSE in Ω_1. For example, for the space of MINI-element $p = 1$ and for Taylor-Hood element $p = 2$ (see [35] or [11] for details).*

4.0 FULLY DISCRETE SCHEME

Next, we construct a stable, second order in time, fully discrete scheme for the initial boundary-value problem (3.1). Denote $R^{'} = Q^{'} - \rho_0 \nabla \cdot \mathbf{f}$, where $Q^{'} = Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$.

Below we will follow Dupont's notations from [19]. Suppose the time step $\Delta t = T/N$ for some fixed positive integer N. If some function f is defined for time levels $i\Delta t$ with all integers i, $0 \leqslant i \leqslant N$, then denote by f_n the function f at the time level $t_n = n\Delta t$. Other notations are

$$f_{n+\frac{1}{2}} = \frac{1}{2}(f_{n+1} + f_n),\ f_{n,\frac{1}{4}} = \frac{1}{4}f_{n-1} + \frac{1}{2}f_n + \frac{1}{4}f_{n+1},$$

$$\partial_t f_{n+\frac{1}{2}} = \frac{f_{n+1} - f_n}{\Delta t},\ \partial_t^2 f_n = \frac{f_{n+1} - 2f_n + f_{n-1}}{(\Delta t)^2},\ \delta_t f_n = \frac{f_{n+1} - f_{n-1}}{2\Delta t}$$

and for any norm $\|\cdot\|_X$

$$\|f\|_{\check{L}^\infty(X)} = max_{0<n<N}\|f_n\|_X,\ \|f\|_{\hat{L}^\infty(X)} = max_{0\leqslant n<N}\|f_{n+\frac{1}{2}}\|_X.$$

We assume that the term $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$ is given either continuously or discretely in time. In the second case we additionaly impose that this term is defined for all the time levels t_n used for the wave equation. Consider the discrete scheme

$$\begin{aligned}(\partial_t^2 q_{h,n}, v_h) + a_0^2(\nabla q_{h,n,\frac{1}{4}}, \nabla v_h) + a_0 < \delta_t q_{h,n}, v_h >= \\ = a_0^2(R^{'}_{n,\frac{1}{4}} + \frac{1}{a_0^2}G_{n,\frac{1}{4}}, v_h) + a_0^2 < g_{n,\frac{1}{4}}, v_h >\end{aligned} \tag{4.1}$$

$$\forall v_h \in M_0^m(\Omega),\ \text{for } n = 1, ..., N-1,$$

$q_{h,0}$ and $q_{h,1}$ are the initial data.

Theorem 14. *Let q be the solution of (3.2) and $q, q_t \in L^\infty(H^k(\Omega))$ and $q_{tt} \in L^2(H^k(\Omega))$ for some integer k, $m \geqslant k \geqslant 2$. Also let $\frac{\partial^4 q}{\partial t^4} \in L^2(L^2(\Omega))$, $\frac{\partial^3 q}{\partial t^3} \in L^2(L^2(\partial\Omega))$. Finally, assume that the initial data satisfies conditions*

$$\|q_{h,0} - \hat{q}_0\|_{H^1(\Omega)} + \|q_{h,1} - \hat{q}_1\|_{H^1(\Omega)} + \left\| \frac{q_{h,1} - q_{h,0}}{\Delta t} - \frac{\hat{q}_1 - \hat{q}_0}{\Delta t} \right\| \leqslant Ch^k$$

with constant C independent of h. Then the solution q_h of (4.1) satisfies

$$\|\partial_t (q - q_h)\|_{\hat{L}^\infty(L^2(\Omega))} + \|q - q_h\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant C \left(h^k + \sum_{n=1}^{N-1} \Delta t \|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2 + (\Delta t)^2 \right).$$

with constant C independent of h.

Proof. The major part of the proof is following Dupont's work [19]. The exact solution q satisfies

$$(\partial_t^2 q_n, v_h) + a_0^2 (\nabla q_{n,\frac{1}{4}}, \nabla v_h) + a_0 < \delta_t q_n, v_h >= a_0^2 (R_{n,\frac{1}{4}} + \frac{1}{a_0^2}(G_{n,\frac{1}{4}} + r_n), v_h) +$$

$$+ a_0 < r'_n, v_h > + a_0^2 < g_{n,\frac{1}{4}}, v_h > .$$

Here r_n and r'_n are the approximation errors and

$$\|r_n\|^2 \leqslant C(\Delta t)^3 \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial^4 q}{\partial t^4} \right\|^2 d\tau \text{ and } |r'_n|^2 \leqslant C(\Delta t)^3 \int_{t_{n-1}}^{t_{n+1}} \left| \frac{\partial^3 q}{\partial t^3} \right|^2 d\tau.$$

Then set $\eta = \hat{q} - q$, $\psi = q_h - \hat{q}$. We leave all the details in [41]. The main idea is to use energy method and reduce the error equation to such an inequality that discrete Gronwall's lemma might be used. Notice that the error $Q - Q_{h_1}$ must be taken into account. Just as in the semidiscrete case, this term will appear on the RHS of the error inequality. The result will be

$$\|\partial_t \psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\psi\|_{\hat{L}^\infty(H^1(\Omega))} \leqslant$$

$$\leqslant C(\|\eta\|_{L^2(L^2(\Omega))} + \left\| \frac{\partial^2 \eta}{\partial t^2} \right\|_{L^2(L^2(\Omega))} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\| \frac{\partial^2 \eta}{\partial t^2} \right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} +$$

$$+ \|\partial_t \psi_{\frac{1}{2}}\| + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^1(\Omega)} + \sum_{n=1}^{N-1} \Delta t \|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2 + (\Delta t)^2).$$

Next use the triangle inequality

$$\|\partial_t e\|_{\hat{L}^\infty(L^2(\Omega))} + \|e\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant \|\partial_t \psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\partial_t \eta\|_{\hat{L}^\infty(L^2(\Omega))} + \|\eta\|_{\hat{L}^\infty(L^2(\Omega))}.$$

For the last two terms we have

$$\|\partial_t \eta\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant C\left(\left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + (\Delta t)^2 \left\|\frac{\partial^3 \eta}{\partial t^3}\right\|_{L^\infty(L^2(\Omega))}\right),$$

$$\|\eta\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant C\left(\|\eta\|_{L^\infty(L^2(\Omega))} + (\Delta t)^2 \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^\infty(L^2(\Omega))}\right).$$

Therefore, the final result will be

$$\|\partial_t e\|_{\hat{L}^\infty(L^2(\Omega))} + \|e\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C(\|\eta\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} +$$

$$+ \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + \|\partial_t \psi_{\frac{1}{2}}\| + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^1(\Omega)} + \sum_{n=1}^{N-1} \Delta t \|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2 + (\Delta t)^2).$$

Use Lemma 6 and obtain the theorem. □

Remark 15. *The term* $\sqrt{\sum_{n=1}^{N-1} \Delta t \|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2}$ *is a discrete analogue of the term* $\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))}$ *from Theorem 11.*

5.0 NUMERICAL EXPERIMENTS

In this chapter we present the results of some numerical experiments in two-dimensional case. Our main purpose is to check the rate of convergence for some exact smooth solution (not necessarily representing real physical phenomena) and compare the theoretical predictions with the experimental results. We focus on the case when the no-slip boundary condition is imposed for the NSE in the inner domain Ω_1. Physically this simulation may represent the turbulent flow in the center of the medium, which decays in space fast enough to vanish in the quiescent media. For example, this could be a large storm eddy that does not affect the air far from its epicentre but generates a noise.

Let Ω_1 and Ω be squares such that $\Omega_1 = [0,1]^2$ and $\Omega = [-0.25, 1.25]^2$, so Ω_1 is embedded into Ω symetrically, as shown on figure 2. The time-dependent incompressible flow is taking

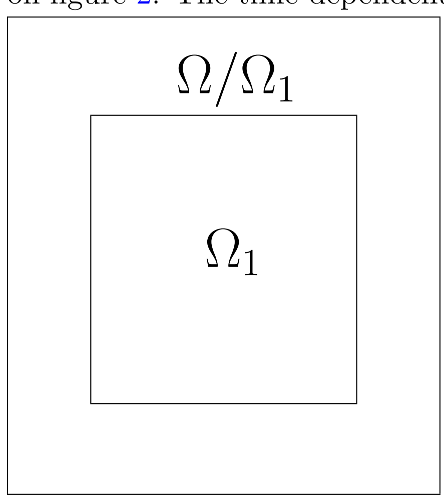

Figure 2: One domain inside the other

place inside Ω_1. The fluid's viscosity $\mu = 0.0172$ and density $\rho = 1.2047$ .The external forces

f are given explicitly by:

$$f_1(x,y,t) = -C\cdot(\mu/\rho)\cdot sin(\pi\cdot t)\cdot((x^2-2x^3+x^4)\cdot(-12+24y)+$$
$$+(2-12x+12x^2)\cdot(2y-6y^2+4y^3))+C^2\cdot(sin(\pi\cdot t))^2\cdot(x^4-2x^3+x^2)\cdot$$
$$\cdot(4x^3-6x^2+2x)\cdot((4y^3-6y^2+2y)^2-(y^4-2y^3+y^2)\cdot(12y^2-12y+2))+$$
$$+C\cdot(x^4-2x^3+x^2)\cdot(4y^3-6y^2+2y)\cdot\pi\cdot cos(\pi\cdot t)+(\nabla p)_1,$$

$$f_2(x,y,t) = -C\cdot(\mu/\rho)\cdot sin(\pi\cdot t)\cdot((-2x+6x^2-4x^3)\cdot(2-12y+12y^2)+$$
$$+(12-24x)\cdot(y^2-2y^3+y^4))+C^2\cdot(sin(\pi\cdot t))^2\cdot(y^4-2y^3+y^2)\cdot$$
$$\cdot(4y^3-6y^2+2y)\cdot((4x^3-6x^2+2x)^2-(x^4-2x^3+x^2)\cdot(12x^2-12x+2))-$$
$$-C\cdot(4x^3-6x^2+2x)\cdot(y^4-2y^3+y^2)\cdot\pi\cdot cos(\pi\cdot t)+(\nabla p)_2$$

with positive constant C and the fluid pressure p of our choice. Driven by this force **f**, the fluid has the following velocity:

$$u_1(x,y,t) = C\cdot(x^4-2x^3+x^2)\cdot(4y^3-6y^2+2y)\cdot sin(\pi\cdot t),$$
$$u_2(x,y,t) = -C\cdot(y^4-2y^3+y^2)\cdot(4x^3-6x^2+2x)\cdot sin(\pi\cdot t).$$

The pressure in this case is constant: $\nabla p = 0$. This incompressible flow gives a vortex with periodically changing direction. The velocity vector field for that flow looks like the one shown on figure 3. The no-slip boundary condition here is satisfied. The exact nonlinear term Q is given by

$$Q(x,y,t) = 2\cdot C^2\cdot(sin(\pi\cdot t))^2\cdot((4x^3-6x^2+2x)^2\cdot(4y^3-6y^2+2y)^2-$$
$$-(12x^2-12x+2)\cdot(y^4-2y^3+y^2)\cdot(x^4-2x^3+x^2)\cdot(12y^2-12y+2)).$$

Consider the following hyperbolic problem:

$$\frac{\partial^2 q}{\partial t^2} - a_0^2\Delta q = a_0^2 Q(\mathbf{u},\mathbf{u}) + G,\ \forall(x,t)\in\Omega\times(0,T) \tag{5.1}$$

with

$$\nabla q\cdot\mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g,\ \forall(x,t)\in\partial\Omega\times(0,T).$$

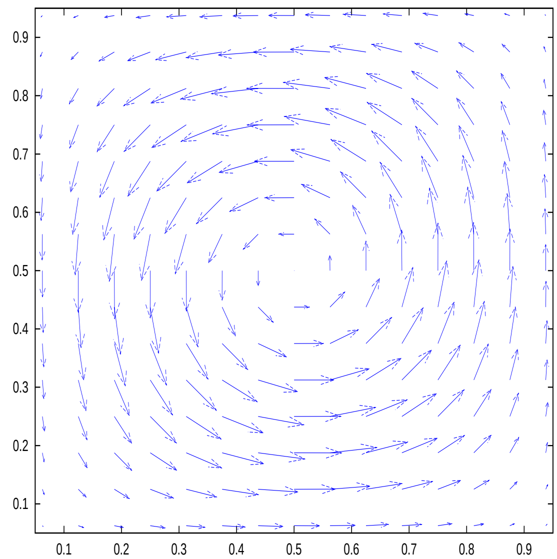

Figure 3: The flow for $C = 10$ and time level $T = 0.5$

We set $a_0 = 2$. As an exact solution we choose q to be the following:

$$q(x, y, t) = cos(\omega t + k(x + y) - k) + cos(\omega t - k(x + y) + k) + q_1(x, y, t),\ \forall (x, y, t) \in \Omega \times (0, T),$$

where $\omega = 2$, $k = \frac{\omega}{a_0 \sqrt{2}}$,

$$q_1(x, y, t) = \begin{cases} e^{4 - \frac{1}{\frac{1}{4} - (x - \frac{1}{2})^2 - (y - \frac{1}{2})^2}} \cdot (cos(\omega_1 t + k_1(x + y) - k_1) + \\ + cos(\omega_1 t - k_1(x + y) + k_1)),\ \text{if } (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{4}, \\ 0,\ \text{otherwise} \end{cases}$$

with $\omega_1 = 4$ and $k_1 = \frac{\omega_1}{a_0 \sqrt{2}}$. The plots of the acoustic pressure as a function of space are given for $t = 0$ and $t = 0.5$ on figures 4 and 5 respectively.

For our tests we take a uniform triangular mesh in Ω_1 of the size $N \times N$ with $N \geqslant 4$ even and $h_1 = 1/N$. Let the finite element space for the velocity field consist of piecewise linear functions, while for the pressure we use piecewise constants on the coarser mesh of size $2h_1$ (see figure 6). These spaces satisfy the LBB-condition, [24]. For the wave equation in Ω

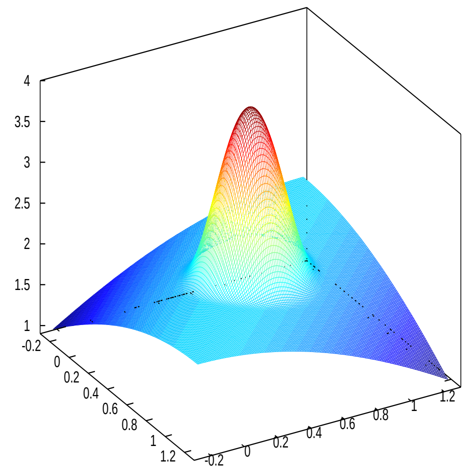

Figure 4: The graph of q at $t = 0$

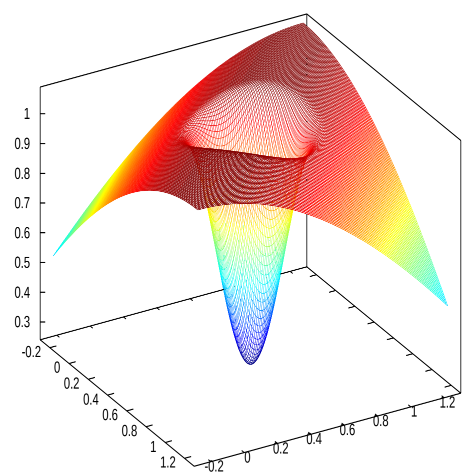

Figure 5: The graph of q at $t = 0.5$

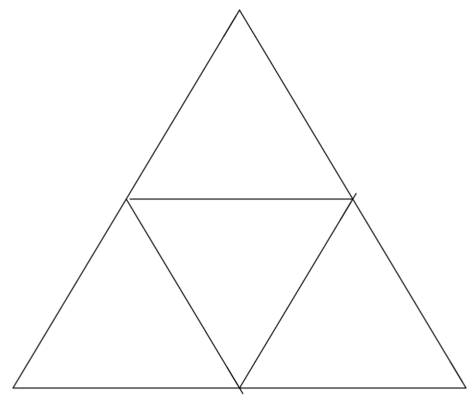

Figure 6: Four small triangles inside a big one

consider the trinagluar mesh of the same size $h = h_1$ and the space of the piecewise linears. Both grids for the NSE and the wave equation are the same in Ω_1. The example is shown on figure 7. For the simulation of the incompressible flow we choose Stabilized Extrapolated Backward Euler Method in time with parameter $\delta = 0.005$, [35]. This means that the values of the velocity $\mathbf{u}^h_{n+1}$ and pressure p^h_{n+1} at the time step $n+1$ can be found from their values at the previous step n via the relation:

$$\left(\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\Delta t_1}, \mathbf{v}^h\right) + \left(\frac{\mu}{\rho} + \delta\right)(\nabla \mathbf{u}^h_{n+1}, \nabla \mathbf{v}^h) + \frac{1}{2}(\mathbf{u}^h_n \cdot \nabla \mathbf{u}^h_{n+1}, \mathbf{v}^h)$$
$$- \frac{1}{2}(\mathbf{u}^h_n \cdot \nabla \mathbf{v}^h, \mathbf{u}^h_{n+1}) - (p^h_{n+1}, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}(t_{n+1}), \mathbf{v}^h) + \delta(\nabla \mathbf{u}^h_n, \nabla \mathbf{v}^h),\ \forall \mathbf{v}^h \in X^h$$

and

$$(\nabla \cdot \mathbf{u}^h_{n+1}, q^h) = 0,\ \forall q^h \in Q^h,$$

where X^h and Q^h denote the finite-dimensional spaces described earlier for velocity and pressure respectively.

The dimension of the space of piecewise linears built on the elements in Ω is equal to $d = (\frac{3}{2}N+1)^2$. If functions ϕ_i denote the basis functions in that space, then the solution q_h of the wave equation (3.5) can be written as a linear combination $q_h = \sum_i a_i\phi_i$ with coefficients

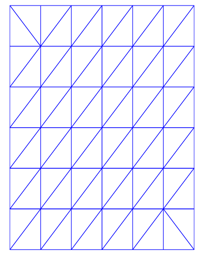

Figure 7: The grid for the whole Ω when $N = 4$

a_i. Let these coefficients organize a vector $\mathbf{q}_h = (a_1, a_2, ..., a_d)^t$. This vector satisfies a linear differential equation in the form

$$M\ddot{\mathbf{q}}_h + a_0 L\dot{\mathbf{q}}_h + a_0^2 S\mathbf{q}_h = \mathbf{f}_{RHS}$$

with the mass matrix M and the stiff matrix S and matrix L related to the boundary term in the LHS of (3.5). This equation may be rewritten as the first-order system of differential equations:

$$\begin{pmatrix} \dot{\mathbf{q}}_h \\ \dot{\mathbf{r}}_h \end{pmatrix} = \begin{pmatrix} 0 & I \\ -a_0^2 M^{-1} S & -a_0 M^{-1} L \end{pmatrix} \begin{pmatrix} \mathbf{q}_h \\ \mathbf{r}_h \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}\mathbf{f}_{RHS} \end{pmatrix}$$

Initial conditions $\mathbf{q}_h(0, \cdot)$ and $\mathbf{r}_h(0, \cdot)$ are found from the H^1-projections of the functions $q(0, \cdot)$ and $\frac{\partial q}{\partial t}(0, \cdot)$ via the formula (3.6).

For time integration we use the Trapezoidal Method with the time step $\Delta t = 0.025$, while for the Backward Euler Method above we use $\Delta t_1 = 0.0125 = \Delta t/2$. Every time step for the wave equation is done after two time steps for the NSE. We perform 20 steps for the wave equation until we reach the final time $T = 0.5$. This corresponds to the case, when the vortex in Ω_1 reaches its maximum velocity. Among the computed values of the error $\|q - q_h\|_{L^2(\Omega)}$ and $\|\frac{\partial}{\partial t}(q - q_h)\|_{L^2(\Omega)}$ at each time step we choose the greatest ones for both and add them. The result is the total error on the LHS of the inequality in Theorem 11. At the same time, we also compute the error $\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_h,\mathbf{u}_h)\|_{L^2(L^2(\Omega_1))}$.

Since the estimate from Theorem 12 is not sharp due to regularity assumptions, the actual rate of convergence for term Q may only be obtained experimentally. Suppose it is of order α. Then according to Theorem 11 the error for the acoustic pressure satisfies an inequality

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(q - q_h)\right\|_{L^\infty(L^2(\Omega))} \leqslant K_1 h^2 + K_2 h^\alpha \tag{5.2}$$

with some positive constants K_1 and K_2 independent of h. The actual rate of convergence for the acoustic pressure in this case is $\gamma = min(\alpha, 2)$.

The rate of convergence may be estimated by evaluating the ratios of the error related to the mesh of size $2h$ to the error related to the mesh of size h. Indeed, for the first and the second grids we have $error_1$ and $error_2$ respectively:

$$error_1 \sim K \cdot (2h)^\gamma, \; error_2 \sim K \cdot h^\gamma.$$

Division gives

$$\frac{error_1}{error_2} \sim 2^\gamma.$$

As we refine the mesh by halving h, i.e. doubling N, the above fraction approaches constant 2^γ. The tables below present the results of numerical simulations for different external force vectors $\mathbf{f}$.

Table 1: $C = 10$, $p(x, y, t) = x(1 - x)y(1 - y)$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	0.10927		0.9619	
8	0.0932	1.1720	0.3635	2.6463
16	0.0557	1.6736	0.1060	3.4292
32	0.02567	2.1704	0.0271	3.9056

According to Theorem 11, the rate of convergence for the solution of the wave equation in the absence of the error $Q - Q_{h_1}$ is expected to be quadratic, i.e. $k = 2$. This means that

Table 2: $C = 100$, $p(x, y, t) = const$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	8.8487		4.9921	
8	3.7567	2.3555	1.4432	3.4590
16	1.7709	2.1214	0.4119	3.5038
32	0.88578	1.9992	0.14264	2.8878

Table 3: $C = 100$, $p(x, y, t) = x(1 - x)y(1 - y)$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	9.1412		5.1537	
8	4.0375	2.2641	1.6182	3.1849
16	1.8930	2.1329	0.4889	3.3097
32	0.9338	2.0273	0.1626	3.0075

Table 4: $C = 100$, $p(x, y, t) = 4x(1 - x)y(1 - y)$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	10.3300		5.9369	
8	5.6410	1.8313	3.0508	1.9460
16	2.7856	2.0251	1.2187	2.5034
32	1.3353	2.0860	0.3774	3.2292

the ratio in this case must be reaching 4 as we refine our mesh. The experimental rate of convergence for the term Q appears to be linear. This fact follows from the third column of all tables where the ratio is approaching 2. So that means that the total rate of decrease of $L^\infty(L^2(\Omega))$-error for fluctuations of pressure must eventually reach 1 as we refine the mesh. This tendency of the rate to decrease may be seen in cases when the $L^2(L^2(\Omega_1))$-error for the term Q is large compared to the $L^\infty(L^2(\Omega))$-error for q and its time derivative. The example is presented below in the 5-th table. We can see that for $N = 32$ the ratio is dropping.

Table 5: $C = 300$, $p(x, y, t) = const$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	125.90		49.614	
8	42.593	2.9558	15.295	3.2437
16	19.714	2.1605	4.3770	3.4945
32	9.7621	2.0194	1.6236	2.6958

6.0 NEGATIVE NORM ANALYSIS

Let $R = Q - \rho_0 \nabla \cdot \mathbf{f}$ be given. Consider the problem

$$\begin{cases} q_{tt} - a_0^2 \Delta q = a_0^2 R + G, \text{ in } (0,T) \times \Omega, \\ \nabla q \cdot \mathbf{n} + \frac{1}{a_0} q_t = g, \text{ in } (0,T) \times \partial\Omega, \end{cases} \tag{6.1}$$

with initial conditions for $q(0,\cdot)$ and $q_t(0,\cdot)$. Here G and g are control functions. Introduce operators T and T_1 as shown below. Consider the elliptic problem

$$\begin{cases} -a_0^2 \Delta p + p = f, \text{ in } \Omega, \\ \nabla p \cdot \mathbf{n} = 0, \text{ in } \partial\Omega. \end{cases}$$

Definition 16. *$T : L^2(\Omega) \to H^1(\Omega)$ is a solution operator to this problem and is given by the formula $Tf = p$, for f being a given data.*

This operator is well-defined on the whole $L^2(\Omega)$, which follows from the Lax-Milgram theorem. Clearly, T is self-adjoint and positive definite.

For T_1 consider another elliptic problem

$$\begin{cases} -a_0^2 \Delta p + p = 0, \text{ in } \Omega, \\ \nabla p \cdot \mathbf{n} = g, \text{ in } \partial\Omega. \end{cases}$$

Definition 17. *$T_1 : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a solution operator to this problem and is given by the formula $T_1 g = p$.*

The existence of this operator again follows from the Lax-Milgram theorem. The trace operator is denoted $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.

Rewrite the given hyperbolic problem (6.1) in the form

$$\begin{cases} q_{tt} - a_0^2 \Delta q + q - q = a_0^2 R + G, \text{ in } (0,T) \times \Omega, \\ \nabla q \cdot \mathbf{n} = -\frac{1}{a_0} q_t + g, \text{ in } (0,T) \times \partial\Omega. \end{cases}$$

Now apply operator T to both sides of the wave equation and take into account the non-homogeneous boundary condition.

$$Tq_{tt} + q - Tq + \frac{1}{a_0} T_1(\gamma q_t - a_0 g) = T(a_0^2 R + G), \text{ in } (0,T) \times \Omega. \tag{6.2}$$

This is the main equation in the negative norm analysis to start from. Next define its semi-discrete analogue with operators T_h,$T_{1,h}$ and γ_h, see [62] for details.

$$T_h q_{h,tt} + q_h - T_h q_h + \frac{1}{a_0} T_{1,h}(\gamma_h q_{h,t} - a_0 g) = T_h(a_0^2 R_{h_1} + G), \text{ in } (0,T) \times \Omega. \tag{6.3}$$

The last term contains R_{h_1} which comes from the Direct Numerical Simulation (DNS) on the different grid of size h_1 in Ω_1.

Introduce the inner product and the norm

$$(u,v)_{-1} = (Tu, v), \ \|u\|_{-1} = \sqrt{(u,u)_{-1}},$$

and the semi-inner product and the semi-norm

$$(u,v)_{-1,h} = (T_h u, v), \ \|u\|_{-1,h} = \sqrt{(u,u)_{-1,h}},$$

defined on all functions $u, v \in L^2(\Omega)$. The error equation comes from subtracting the exact and discrete ones, i.e. if $e = q - q_h$ then

$$\begin{aligned} &T_h e_{tt} + e - T_h e + (T - T_h) q_{tt} - (T - T_h) q + \frac{1}{a_0}(T_1 \gamma - T_{1,h}\gamma_h) q_t - \\ &- (T_1 - T_{1,h}) g + \frac{1}{a_0} T_{1,h} \gamma_h e_t = (T - T_h)(a_0^2 R + G) + a_0^2 T_h (Q - Q_{h_1}). \end{aligned} \tag{6.4}$$

Multiply by e_t and integrate in space:

$$(T_h e_{tt}, e_t) + (e, e_t) = -\frac{1}{a_0}((T_1\gamma - T_{1,h}\gamma_h)q_t, e_t) + ((T_1 - T_{1,h})g, e_t) + (T_h e, e_t) - \\ -\frac{1}{a_0}(T_{1,h}\gamma_h e_t, e_t) + ((T - T_h)(a_0^2 R + G + q - q_{tt}), e_t) + a_0^2(T_h(Q - Q_{h_1}), e_t). \tag{6.5}$$

We moved the term $(T_{1,h}\gamma_h e_t, e_t)$ to the RHS because the operator $T_{1,h}\gamma_h$ is not positive definite and thus we cannot hide it in the LHS as a part of the global error. From this point, we work with the Neumann boundary condition that has no time derivative, since the mentioned term $(T_{1,h}\gamma_h e_t, e_t)$ is not of high order with respect to the others and it is not possible to increase the accuracy in this case. Therefore, we are now considering the problem

$$\begin{cases} q_{tt} - a_0^2 \Delta q = a_0^2 R + G, \text{ in } (0,T)\times\Omega, \\ \nabla q \cdot \mathbf{n} = g, \text{ in } (0,T)\times\partial\Omega, \end{cases} \tag{6.6}$$

and the equation (6.5) reduces to

$$(T_h e_{tt}, e_t) + (e, e_t) = (T_h e, e_t) + ((T - T_h)(a_0^2 R + G + q - q_{tt}), e_t) + \\ + a_0^2(T_h(Q - Q_{h_1}), e_t) + ((T_1 - T_{1,h})g, e_t). \tag{6.7}$$

Theorem 18. *Let the exact variational solution q of (6.6) satisfy conditions: $q, q_t \in L^\infty(H^k(\Omega))$, $q_{tt} \in L^2(H^k(\Omega))$ with integer k, $m \geqslant k \geqslant 2$. Also let the initial data satisfy conditions*

$$\|(q_h - \hat{q})(0,\cdot)\|_{H^1(\Omega)} + \left\| \frac{\partial}{\partial t}(q_h - \hat{q})(0,\cdot) \right\| \leqslant C_1 h^k$$

with the constant C_1 independent of h. Finally, let $a_0^2 R + G \in L^2(H^k(\Omega_1))$ and $g \in L^2(H^{\frac{1}{2}+k}(\partial\Omega))$. Then

$$\left\| \frac{\partial}{\partial t}(q - q_h) \right\|_{L^\infty(H^{-1}(\Omega))} + \|q - q_h\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$C(h^{k+1} + \frac{1}{h}\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} + h\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \\ + \left\| \frac{\partial}{\partial t}(q - q_h)(0,\cdot) \right\|_{-1} + \|(q - q_h)(0,\cdot)\|)$$

with constant C independent of h.

Proof. (6.7) is equivalent to

$$\frac{1}{2}\frac{d}{dt}\{\|e_t\|^2_{-1,h}+\|e\|^2\}=(T_he,e_t)+((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)+$$

$$+a_0^2(T_h(Q-Q_{h_1}),e_t)+((T_1-T_{1,h})g,e_t).$$

It is obvious that

$$(T_he,e_t)=(e,T_he_t).$$

Integration of (6.7) yields

$$\|e_t\|^2_{-1,h}+\|e\|^2\leqslant\int_0^t(\|e\|^2+\|T_he_t\|^2)+$$

$$+2\int_0^t|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|+2a_0^2\int_0^t|(T_h(Q-Q_{h_1}),e_t)|+$$

$$+2\int_0^t|((T_1-T_{1,h})g,e_t)|+\|e_t\|^2_{-1,h}(0)+\|e\|^2(0).$$

The term $\|T_he_t\|^2=\|e_t\|^2_{-2,h}\leqslant\|e_t\|^2_{-1,h}$.

Using Gronwall's lemma, we obtain

$$\|e_t\|^2_{-1,h}+\|e\|^2\leqslant$$

$$C(\int_0^t|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|+\int_0^t|(T_h(Q-Q_{h_1}),e_t)|+$$

$$+\int_0^t|((T_1-T_{1,h})g,e_t)|+\|e_t\|^2_{-1,h}(0)+\|e\|^2(0)).$$

Next,

$$|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|\leqslant\|(T-T_h)(a_0^2R+G+q-q_{tt})\|\cdot\|e_t\|\leqslant$$

$$\leqslant\frac{1}{4}h^{2s+2}\|a_0^2R+G+q-q_{tt}\|^2_{H^s(\Omega)}+h^2\|e_t\|^2$$

with integer $s\geqslant 0$, and

$$a_0^2|(T_h(Q-Q_{h_1}),e_t)|\leqslant C\left(\frac{1}{h^2}\|T_h(Q-Q_{h_1})\|^2+h^2\|e_t\|^2\right)=$$

$$=C\left(\frac{1}{h^2}\|Q-Q_{h_1}\|^2_{-2,h}+h^2\|e_t\|^2\right),$$

$$|((T_1-T_{1,h})g,e_t)|\leqslant\|(T_1-T_{1,h})g\|\cdot\|e_t\|\leqslant\frac{1}{4}h^{2s+2}\|g\|^2_{H^{\frac{1}{2}+s}(\partial\Omega)}+h^2\|e_t\|^2.$$

Thus

$$\|e_t\|^2_{L^\infty(H^{-1,h}(\Omega))} + \|e\|^2_{L^\infty(L^2(\Omega))} \leqslant$$

$$C(h^{2k+2}\|a_0^2R + G + q - q_{tt}\|^2_{L^2(H^k(\Omega))} + \frac{1}{h^2}\|(Q - Q_{h_1})\|^2_{L^2(H^{-2,h}(\Omega))} +$$

$$+h^{2k+2}\|g\|^2_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))} + h^2\|e_t\|^2_{L^2(L^2(\Omega))} + \|e_t\|^2_{-1,h}(0) + \|e\|^2(0)),$$

or

$$\|e_t\|_{L^\infty(H^{-1,h}(\Omega))} + \|e\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$C(h^{k+1}\|a_0^2R + G + q - q_{tt}\|_{L^2(H^k(\Omega))} + \frac{1}{h}\|(Q - Q_{h_1})\|_{L^2(H^{-2,h}(\Omega))} +$$

$$+h^{k+1}\|g\|_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))} + h\|e_t\|_{L^2(L^2(\Omega))} + \|e_t\|_{-1,h}(0) + \|e\|(0)).$$

According to V. Thomee's results, [62],

$$\|e_t\|_{-1} \leqslant C(\|e_t\|_{-1,h} + h\|e_t\|)$$

and therefore

$$\|e_t\|_{L^\infty(H^{-1}(\Omega))} + \|e\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$C(h^{k+1}\|a_0^2R + G + q - q_{tt}\|_{L^2(H^k(\Omega))} + \frac{1}{h}\|Q - Q_{h_1}\|_{L^2(H^{-2,h}(\Omega))} +$$

$$+h^{k+1}\|g\|_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))} + h\|e_t\|_{L^\infty(L^2(\Omega))} + \|e_t\|_{-1,h}(0) + \|e\|(0)).$$

For the initial data

$$\|e_t\|_{-1,h}(0) \leqslant C(\|e_t\|_{-1}(0) + h\|e_t\|(0)).$$

For the term $Q - Q_{h_1}$ we have

$$\frac{1}{h}\|(Q - Q_{h_1})\|_{-2,h} \leqslant C\left(\frac{1}{h}\|Q - Q_{h_1}\|_{-2} + h\|Q - Q_{h_1}\|\right).$$

The final result is, due to Theorem 11,

$$\|e_t\|_{L^\infty(H^{-1}(\Omega))} + \|e\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$C(h^{k+1} + \frac{1}{h}\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} + h\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|e_t\|_{-1}(0) + \|e\|(0)).$$

□

Theorem 19. *Suppose the exact solution* $\mathbf{u}$ *of the incompressible NSE satisfies condition*

$$\mathbf{u} \in L^\infty(H^1(\Omega_1))$$

and also has a continuous representation on Ω_1 *for almost all* $0 < t < T$*. Assume the mesh on* Ω_1 *used for the DNS of the incompressible NSE is quasi-uniform. Then the following estimate holds:*

$$\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} \leqslant C(\mathbf{u}) \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$$

with constant $C(\mathbf{u})$ *independent of* h_1*.*

Proof. The norm $\|\cdot\|_{-2}$ is equivalent to the norm

$$sup_{v \in H^2(\Omega)} \frac{(\cdot, v)}{\|v\|_2}.$$

Using this, we obtain

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot sup_{v \in H^2(\Omega)} \frac{(Q - Q_{h_1}, v)}{\|v\|_2}.$$

Since $Q - Q_{h_1}$ is zero outside the smaller domain Ω_1, it is obvious that

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot sup_{v \in H^2(\Omega_1)} \frac{(Q - Q_{h_1}, v)}{\|v\|_2}.$$

We know that

$$(Q - Q_{h_1}, v) \leqslant \rho_0 |(\nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t, v)| + \rho_0 |(\nabla(\mathbf{u} - \mathbf{u}_{h_1}) : \nabla \mathbf{u}_{h_1}^t, v)|.$$

For both terms use Holder's inequality. For example, for the first term we get

$$\rho_0 |(\nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t, v)| \leqslant C \|\nabla \mathbf{u}\|_{L^r(\Omega_1)} \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^p(\Omega_1)} \|v\|_{L^\infty(\Omega_1)},$$

where $\frac{1}{r} + \frac{1}{p} = 1$. Choose $p, r = 2$ and use Sobolev embedding $\|v\|_{L^\infty(\Omega_1)} \leqslant C\|v\|_2$. This gives

$$\|Q - Q_{h_1}\|_{-2} \leqslant C(\|\nabla \mathbf{u}\| + \|\nabla \mathbf{u}_{h_1}\|) \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|.$$

Next,

$$\|\nabla \mathbf{u}_{h_1}\| \leqslant \|\nabla \mathbf{u}\| + \|\nabla(\mathbf{u} - I_{h_1}\mathbf{u})\| + \|\nabla(\mathbf{u}_{h_1} - I_{h_1}\mathbf{u})\|,$$

where I_{h_1} is the piecewise polynomial interpolant, chapter 2. The first two terms on the RHS are bounded. For the last one we use the inverse estimate, [12],

$$\|\nabla(\mathbf{u}_{h_1} - I_{h_1}\mathbf{u})\| \leqslant Ch^{-1}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|.$$

Using triangle inequality, we obtain

$$h^{-1}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\| \leqslant h^{-1}\|\mathbf{u} - I_{h_1}\mathbf{u}\| + h^{-1}\|\mathbf{u} - \mathbf{u}_{h_1}\|.$$

These two terms are bounded for any continuous piecewise polynomial element satisfying LBB-condition, [35], and converging to the exact solution. Thus we showed that

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|$$

with some positive constant $C = C(\mathbf{u})$ depending on the solution $\mathbf{u}$. □

Remark 20. *If $h = O(h_1)$, then in order to have convergence for the total error in Theorem 18, it is necessary that $\|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$ converge superlinearly. This means we have to use high-order FEM scheme for the NSE. For example, Taylor-Hood element will be sufficient, [35].*

7.0 ESTIMATING THE ERROR IN ACOUSTIC POWER

Let surface S be Lipschitz continuous and belong to the far field. Consider the semidiscrete FEM scheme (3.5) for solving the problem (3.1). The acoustic power on S is given by

$$A(t) = \int_S q(t, \cdot)\mathbf{v}(t, \cdot) \cdot \mathbf{n} dS.$$

Its approximate analogue is defined as

$$A_h(t) = \int_S q_h(t, \cdot)\mathbf{v}_{h_2}(t, \cdot) \cdot \mathbf{n} dS.$$

Decompose the error in power in two terms:

$$A(t) - A_h(t) = \int_S (q - q_h)\mathbf{v} \cdot \mathbf{n} dS + \int_S q_h(\mathbf{v} - \mathbf{v}_{h_2}) \cdot \mathbf{n} dS. \tag{7.1}$$

Denote the terms on the RHS as $E_1(t)$ and $E_2(t)$ respectively. For computing q_h we use the semidiscrete FEM scheme.

Estimating the error in intesity depends on how the velocity $\mathbf{v}_{h_2}$ is computed.

7.1 METHOD 1: EXACT FORMULA

7.1.1 Statement of the algorithm

1. Compute q_h in Ω, using (3.5) with homogeneous initial conditions.
2. Compute the sound power directly as

$$A_h(t) = -\frac{1}{\rho_0} \int_S q_h(t, \cdot) \cdot \left(\int_0^t \nabla q_h(\tau, \cdot) d\tau \right) dS.$$

7.1.2 Analysis of the method

In order to find the exact formula for $\mathbf{v}$, consider the compressible linearized NSE in the far field:

$$\begin{cases} \frac{1}{a_0^2}\frac{\partial q}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0, \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla q = 0. \end{cases}$$

The second equation gives

$$\mathbf{v}(t,\cdot) = -\frac{1}{\rho_0}\int_0^t \nabla q(\tau,\cdot)d\tau + \mathbf{v}(0,\cdot).$$

Set $\mathbf{v}(0,\cdot) = 0$. This agrees with the homogeneous initial conditions for q. Thus define

$$\mathbf{v}_{h_2}(t,\cdot) = -\frac{1}{\rho_0}\int_0^t \nabla q_h(\tau,\cdot)d\tau + \mathbf{v}(0,\cdot). \tag{7.2}$$

The errors will be

$$E_1(t) = -\frac{1}{\rho_0}\int_S (q-q_h)(t,\cdot)\left(\int_0^t \nabla q(\tau,\cdot)\cdot \mathbf{n} d\tau\right) dS$$

and

$$E_2(t) = -\frac{1}{\rho_0}\int_S q_h(t,\cdot)\left(\int_0^t \nabla (q-q_h)(\tau,\cdot)\cdot \mathbf{n} d\tau\right) dS.$$

Using Fubini's theorem, write the first term in the form

$$E_1(t) = -\frac{1}{\rho_0}\int_0^t\int_S (q-q_h)(t,\cdot)\nabla q(\tau,\cdot)\cdot \mathbf{n} dS d\tau.$$

Next obtain the bound:

$$|E_1(t)| \leqslant C\int_0^t \left|\int_S (q-q_h)(t,\cdot)\nabla q(\tau,\cdot)\cdot \mathbf{n} dS\right| d\tau \leqslant$$

$$\leqslant C\|(q-q_h)(t,\cdot)\|_1 \cdot \int_0^t \|\nabla q(\tau,\cdot)\cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(S)} d\tau \leqslant$$

$$\leqslant C\|q-q_h\|_{L^\infty(0,T;H^1(\Omega))}\cdot \|\nabla q\cdot \mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}.$$

For the second term, again, using Fubini's theorem, we obtain

$$E_2(t) = -\frac{1}{\rho_0}\int_0^t\int_S q_h(t,\cdot)\nabla (q-q_h)(\tau,\cdot)\cdot \mathbf{n} dS d\tau.$$

Thus, in the same manner,

$$|E_2(t)| \leqslant C\|q_h(t,\cdot)\|_{H^1(\Omega)} \cdot \int_0^t \|\nabla(q-q_h)(\tau,\cdot)\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)} \leqslant$$

$$\leqslant C\|q_h\|_{L^\infty(0,T;H^1(\Omega))} \cdot \|\nabla(q-q_h)\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}.$$

For a regular enough function q, the rate of convergence in the term E_1 is no slower than that of $\|q-q_h\|_{L^\infty(0,T;H^1(\Omega))}$, which is $O(h^{k-1}+\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ for continuous piecewise polynomials of degree no more than $m-1$, $m \geqslant k \geqslant 2$. In the term E_2 the rate of convergence is defined by that of the term $\|\nabla(q-q_h)\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}$ which is $O(h^{k-\frac{3}{2}}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. Thus the rate of convergence for the total error may be estimated as $O(h^{k-\frac{3}{2}}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The conditions $\|\nabla q\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))} < \infty$ and $\|q_h\|_{L^\infty(0,T;H^1(\Omega))} < \infty$ will be guranteed by the regularity assumption $q \in L^\infty(H^k(\Omega))$ for $k \geqslant 2$ and the stability theorem for q_h (chapter 3) respectively.

The advantage of this approach is that the velocity and thus the sound power are computed quickly once q_h is known. The disadvantage is that we lose $\frac{3}{2}$ power of h compared to the L^2-norm of error in the fluctuation of pressure q. This is the least accurate method among those presented here.

In the particular case $S \subset \partial\Omega$ we can make an improvement. In the term $E_2(t)$, due to the boundary condition,

$$\nabla(q-q_h)\cdot\mathbf{n} = -\frac{1}{a_0}\left(\frac{\partial q}{\partial t} - \frac{\partial q_h}{\partial t}\right),$$

and so

$$\|\nabla(q-q_h)\cdot\mathbf{n}\|_{L^1(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant C\left\|\frac{\partial q}{\partial t} - \frac{\partial q_h}{\partial t}\right\|_{L^1(H^{\frac{1}{2}}(\Omega))} \leqslant$$

$$\leqslant C(h^{k-\frac{1}{2}} + h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}).$$

Then the total rate of convergence will be of order $O(h^{k-1}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$, which comes from the term $E_1(t)$. In the case $S \subset \partial\Omega$ there is a loss of only one power of h compared to the L^2-error in q.

7.2 METHOD 2: BOUNDING THE SOUND POWER

7.2.1 Statement of the algorithm

1. Compute q_h in Ω, using (3.5) with homogeneous initial conditions.
2. Complete the curve S to an arbitrary closed curve $\tilde{S}$ so that the new domain $\tilde{\Omega}$, bounded by that curve $\partial\tilde{\Omega} = \tilde{S}$, did not coincide with Ω_1.
3. Find the norms of the trace operator $\gamma_1 : H^1(\tilde{\Omega}) \to H^{\frac{1}{2}}(\tilde{S})$ and the normal trace operator $\gamma_2 : H_{div}(\tilde{\Omega}) \to H^{-\frac{1}{2}}(\tilde{S})$. Denote these norms as $C_1(\tilde{\Omega})$ and $C_2(\tilde{\Omega})$ respectively.
4. Compute
$$SQ_h(t) = \sqrt{\frac{2}{\rho_0^2}\|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4\rho_0^2}\left\|\frac{\partial q_h}{\partial t}\right\|^2_{L^2(\tilde{\Omega})}}.$$
5. Compute the bound for the sound power by the formula
$$P_h(t) = C_1(\tilde{\Omega})C_2(\tilde{\Omega})\|q_h(t,\cdot)\|_{H^1(\tilde{\Omega})} \cdot SQ_h(t).$$

7.2.2 Analysis of the method

Instead of finding the sound power exactly, we consider the question of finding a good upper bound. This may be used in applications where one needs to know whether the loudness surpasses a certain level. If Q is given exactly as a function of space and time, then using this method only has meaning if S is not a part of $\partial\Omega$ since otherwise it has absolutely no advantage compared to the first approach. We have

$$\int_S q\mathbf{v}\cdot\mathbf{n}dS \leqslant \|q\|_{H^{\frac{1}{2}}(S)} \cdot \|\mathbf{v}\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)} \leqslant C_1(\tilde{\Omega})\cdot C_2(\tilde{\Omega})\|q\|_{H^1(\tilde{\Omega})}\cdot\|\mathbf{v}\|_{H_{div}(\tilde{\Omega})}.$$

Here $\tilde{\Omega}$ is some domain of our choice that has S as a part of its boundary and that does not coincide with the turbulent region. Constant C_1 is the norm of the trace operator from $H^1(\tilde{\Omega})$ to $H^{\frac{1}{2}}(\partial\tilde{\Omega})$ and C_2 is a norm of the normal trace operator from $H_{div}(\tilde{\Omega})$ to $H^{-\frac{1}{2}}(\partial\tilde{\Omega})$. In fact, C_1 is constant C_{tr} from the trace theorem 5, chapter 2. Next,

$$\|\mathbf{v}\|_{H_{div}(\tilde{\Omega})} = \sqrt{\|\mathbf{v}\|^2_{L^2(\tilde{\Omega})} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\tilde{\Omega})}}.$$

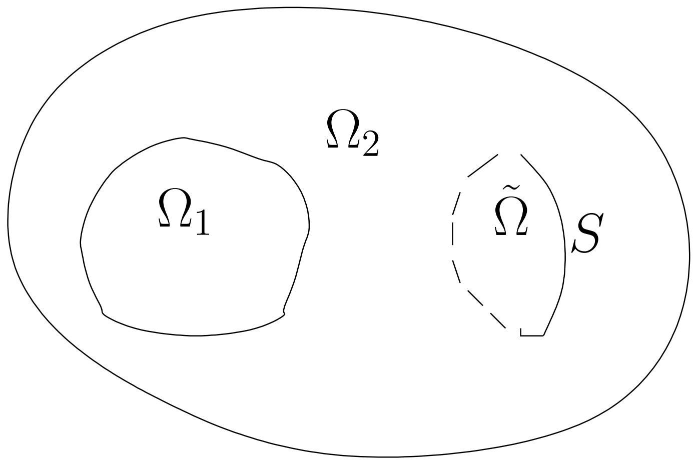

Figure 8: Domain $\tilde{\Omega}$

From the continuity equation of the linearized compressible NSE we have

$$\nabla \cdot \mathbf{v} = -\frac{1}{a_0^2 \rho_0} \frac{\partial q}{\partial t}$$

and also

$$\|\mathbf{v}(t,\cdot)\|_{L^2(\tilde{\Omega})} - \|\mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})} \leqslant \|\mathbf{v}(t,\cdot) - \mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})} = \left\| \int_0^t \frac{\partial \mathbf{v}}{\partial t} d\tau \right\|_{L^2(\tilde{\Omega})} =$$

$$= \frac{1}{\rho_0} \left\| \int_0^t \nabla q d\tau \right\|_{L^2(\tilde{\Omega})} \leqslant \frac{1}{\rho_0} \int_0^t \|\nabla q\|_{L^2(\tilde{\Omega})} d\tau.$$

Thus at time t

$$\|\mathbf{v}\|_{L^2(\tilde{\Omega})} \leqslant \frac{1}{\rho_0} \|\nabla q\|_{L^1(0,t;L^2(\tilde{\Omega}))} + \|\mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})}.$$

As in method 1, set $\mathbf{v}(0,\cdot) = 0$. That is why we obtain

$$\|\mathbf{v}\|_{H_{div}(\tilde{\Omega})} \leqslant \sqrt{\frac{2}{\rho_0^2} \|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}.$$

For simplicity, denote

$$SQ = \sqrt{\frac{2}{\rho_0^2} \|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}$$

and

$$SQ_h = \sqrt{\frac{2}{\rho_0^2}\|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde\Omega))} + \frac{1}{a_0^4\rho_0^2}\left\|\frac{\partial q_h}{\partial t}\right\|^2_{L^2(\tilde\Omega)} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde\Omega)}}.$$

Our bound will be

$$P = C_1(\tilde\Omega)C_2(\tilde\Omega)\|q\|_{H^1(\tilde\Omega)} \cdot SQ. \tag{7.3}$$

Introduce

$$P_h = C_1(\tilde\Omega)C_2(\tilde\Omega)\|q_h\|_{H^1(\tilde\Omega)} \cdot SQ_h. \tag{7.4}$$

The purpose now is to get the rate of convergence for the error $P - P_h$. If we require that

$$\mathbf{v}(0,\cdot) \in L^2(\Omega),$$

then both SQ and SQ_h will be bounded due to earlier regularity assumptions and stability theorem from chapter 3. Obviously,

$$P - P_h = C_1(\tilde\Omega)\cdot C_2(\tilde\Omega)\cdot(\|q\|_{H^1(\tilde\Omega)} - \|q_h\|_{H^1(\tilde\Omega)})\cdot SQ + \\ + C_1(\tilde\Omega)\cdot C_2(\tilde\Omega)\cdot\|q_h\|_{H^1(\tilde\Omega)}\cdot(SQ - SQ_h).$$

The first term of the error may be bounded by

$$C_1(\tilde\Omega)\cdot C_2(\tilde\Omega)\cdot\|q - q_h\|_{H^1(\tilde\Omega)}\cdot SQ,$$

and thus converges as $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The second term of the error may be bounded by

$$C_1(\tilde\Omega)\cdot C_2(\tilde\Omega)\cdot\|q_h\|_{H^1(\tilde\Omega)}\cdot\frac{|SQ^2 - SQ_h^2|}{SQ + SQ_h}.$$

Next,

$$|SQ^2 - SQ_h^2| =$$

$$= \frac{2}{\rho_0^2}\left(\|\nabla q\|^2_{L^1(0,t;L^2(\tilde\Omega))} - \|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde\Omega))}\right) + \frac{1}{a_0^4\rho_0^2}\left(\left\|\frac{\partial q}{\partial t}\right\|^2_{L^2(\tilde\Omega)} - \left\|\frac{\partial q_h}{\partial t}\right\|^2_{L^2(\tilde\Omega)}\right).$$

The first and the second terms in this expression converge as $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ and $O(h^k + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ respectively. Therefore, we conclude that the rate of convergence for the total error $P - P_h$ is of order $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The advantage of this approach is obvious: it gives more accurate approximation. A big disadvantage is that we compute the upper bound for the sound power instead of itself. This approach also suffers from the necessity for the user to know constants $C_1(\tilde\Omega)$ and $C_2(\tilde\Omega)$ whose behavior depends on the geometry of the domain $\tilde\Omega$ chosen.

7.3 METHOD 3: DUALITY

7.3.1 Statement of the algorithm

1. Compute q_h in Ω, using (3.5).
2. Compute $\mathbf{v}_{h_2}$ in Ω, using (7.11), which a FEM approximation for the variational problem (7.10).
3. Compute the sound power directly as

$$A_h(t) = \int_S q_h(t,\cdot)\mathbf{v}_{h_2}(t,\cdot)dS.$$

7.3.2 Analysis of the method

The error in the sound power cannot converge to zero faster than the error $q - q_h$ in the solution of the wave equation, i.e. the greatest rate of convergence may be of order $O(h^k)$. The way we may reach this rate is by using the duality approach. It also allows to reduce the regularity of the exact solution needed for reaching the desired rate of convergence. This advantage may be crucial if one works with turbulent irregular effects. In this case we work with time-averaged sound power

$$\bar{A} = \frac{1}{T}\int_0^T \int_S q\mathbf{v}\cdot\mathbf{n}dSd\tau$$

and the error

$$T(\bar{A} - \bar{A}_h) = \int_0^T \int_S (q - q_h)\mathbf{v}\cdot\mathbf{n}dSd\tau + \int_0^T \int_S q_h(\mathbf{v} - \mathbf{v}_{h_2})\cdot\mathbf{n}dSd\tau. \tag{7.5}$$

Also assume that $S \subset \partial\Omega$. Denote these error terms as $\bar{E}_1$ and $\bar{E}_2$ respectively.

Let us demonstrate duality approach by estimating the error term $\bar{E}_1$ first. First, write the variational formulation for the wave equation, using integration both in space and time. If v denotes a test function, then

$$\int_0^T \left(\frac{\partial^2 q}{\partial t^2}, v\right) + a_0^2 \int_0^T (\nabla q, \nabla v) + a_0 \int_0^T \left\langle \frac{\partial q}{\partial t}, v\right\rangle =$$
$$= a_0^2 \int_0^T (R + \frac{1}{a_0^2}G, v)_{\Omega_1} + a_0^2 \int_0^T < g, v > .$$

Integration by parts in time gives

$$\left(\frac{\partial q}{\partial t}(T),v(T)\right)-\left(\frac{\partial q}{\partial t}(0),v(0)\right)-\left(\frac{\partial v}{\partial t}(T),q(T)\right)+\left(\frac{\partial v}{\partial t}(0),q(0)\right)+\int_0^T\left(\frac{\partial^2 v}{\partial t^2},q\right)+$$

$$+a_0^2\int_0^T(\nabla q,\nabla v)+a_0<q(T),v(T)>-a_0<q(0),v(0)>-a_0\int_0^T\left\langle q,\frac{\partial v}{\partial t}\right\rangle=$$

$$=a_0^2\int_0^T(R+\frac{1}{a_0^2}G,v)_{\Omega_1}+a_0^2\int_0^T<g,v>.$$

The initial data is given:

$$q(0,\cdot)=q_1(\cdot),\frac{\partial q}{\partial t}(0,\cdot)=q_2(\cdot)$$

and thus

$$\left(\frac{\partial q}{\partial t}(T),v(T)\right)-\left(\frac{\partial v}{\partial t}(T),q(T)\right)+\int_0^T\left(\frac{\partial^2 v}{\partial t^2},q\right)+a_0^2\int_0^T(\nabla q,\nabla v)-a_0\int_0^T\left\langle q,\frac{\partial v}{\partial t}\right\rangle+$$

$$+a_0<q(T),v(T)>=a_0^2\int_0^T(R+\frac{1}{a_0^2}G,v)_{\Omega_1}+(q_2,v(0))-\left(\frac{\partial v}{\partial t}(0),q_1\right)+$$

$$+a_0<q_1,v(0)>+a_0^2<g,v>.$$

Consider function ψ by the formula

$$\psi(t,\mathbf{x})=\begin{cases}\mathbf{v}\cdot\mathbf{n},\ \text{if}\ \mathbf{x}\in S,\\ 0,\ \text{if}\ \mathbf{x}\in\partial\Omega/S.\end{cases}$$

The weak formulation for the dual problem with unknown function $\tilde{q}$ will be

$$\left(\frac{\partial v}{\partial t}(T),\tilde{q}(T)\right)-\left(\frac{\partial \tilde{q}}{\partial t}(T),v(T)\right)+\int_0^T\left(\frac{\partial^2 \tilde{q}}{\partial t^2},v\right)+a_0^2\int_0^T(\nabla \tilde{q},\nabla v)-$$

$$-a_0\int_0^T\left\langle v,\frac{\partial \tilde{q}}{\partial t}\right\rangle+a_0<\tilde{q}(T),v(T)>=a_0^2\int_0^T\langle\psi,v\rangle.$$

In order to get rid of the terms at final time T, we may reduce this formulation to the following point-wise problem:

$$\begin{cases}\tilde{q}_{tt}-a_0^2\Delta\tilde{q}=0,\ \text{on}\ \Omega\times(0,T),\\ \tilde{q}(T,\cdot)=0,\ \text{on}\ \Omega,\\ \tilde{q}_t(T,\cdot)=0,\ \text{on}\ \Omega,\\ \nabla\tilde{q}\cdot\mathbf{n}-\frac{1}{a_0}\tilde{q}_t=\psi,\ \text{on}\ \partial\Omega\times(0,T).\end{cases}\tag{7.6}$$

and so

$$\int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial t^2}, v\right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla v) - a_0 \int_0^T \left\langle v, \frac{\partial \tilde{q}}{\partial t}\right\rangle = a_0^2 \int_0^T \langle \psi, v\rangle \,.$$

Next we present the stability lemma which will be needed for the error analysis.

Lemma 21. *Let* $\mathbf{v} \in L^2(0, T; H^1(\Omega))$*. Then the variational solution of the dual problem is stable in the following sense:*

$$\left\|\frac{\partial \tilde{q}}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + a_0 \|\nabla \tilde{q}\|_{L^\infty(L^2(\Omega))} \leqslant a_0^{\frac{3}{2}} \|\mathbf{v}\cdot \mathbf{n}\|_{L^2(L^2(S))}.$$

Proof. The change of time variable $\tau = T - t$ will give us the problem

$$\int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial \tau^2}, v\right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla v) + a_0 \int_0^T \left\langle v, \frac{\partial \tilde{q}}{\partial \tau}\right\rangle = a_0^2 \int_0^T \langle \psi, v\rangle \,.$$

Set $v = \frac{\partial \tilde{q}}{\partial \tau}$.

$$\frac{1}{2}\left(\left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|^2_{\tau=T} - \left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|^2_{\tau=0}\right) + \frac{1}{2} a_0^2 (\|\nabla \tilde{q}\|^2_{\tau=T} - \|\nabla \tilde{q}\|^2_{\tau=0}) + a_0 \int_0^T \left|\frac{\partial \tilde{q}}{\partial \tau}\right|^2 =$$

$$= a_0^2 \int_0^T \left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau}\right\rangle.$$

Since we have homogeneous conditions at time $t = T$, or $\tau = 0$, we can simplify the equation:

$$\left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|^2_{\tau=T} + a_0^2 \|\nabla \tilde{q}\|^2_{\tau=T} + 2a_0 \int_0^T \left|\frac{\partial \tilde{q}}{\partial \tau}\right|^2 = 2a_0^2 \int_0^T \left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau}\right\rangle.$$

Bound the RHS using Young's inequality as shown below

$$\left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau}\right\rangle \leqslant \frac{a_0}{4} |\psi|^2 + \frac{1}{a_0} \left|\frac{\partial \tilde{q}}{\partial \tau}\right|^2.$$

This results in cancelling the boundary term with the time derivative:

$$\left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|^2_{\tau=T} + a_0^2 \|\nabla \tilde{q}\|^2_{\tau=T} \leqslant \frac{a_0^3}{2} \int_0^T |\psi|^2 = \frac{a_0^3}{2} \|\mathbf{v}\cdot \mathbf{n}\|^2_{L^2(0,T;L^2(S))}.$$

Extracting the square root out of both sides and using the fact that $|a| + |b| \leqslant \sqrt{2} \cdot \sqrt{a^2 + b^2}$, we obtain the formulation of the theorem. □

Remark 22. *The analogous stability result may be obtained for the FEM solution* $\tilde{q}_h$ *of the dual problem, if we use the same space* $M_0^m(\Omega)$ *of piecewise polynomials as for the original problem.*

Proceeding with the error analysis, set test function $v = q - q_h$ in the variational formulation of the dual problem:

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial t^2}, q - q_h \right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla (q - q_h)) - a_0 \int_0^T \left\langle q - q_h, \frac{\partial \tilde{q}}{\partial t} \right\rangle .$$

The LHS is exactly $a_0^2 \bar{E}_1$. Let us integrate by parts again:

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle =$$

$$= \int_0^T \left(\frac{\partial^2 (q - q_h)}{\partial t^2}, \tilde{q} \right) + \left(\tilde{q}(0), \frac{\partial (q - q_h)}{\partial t}(0) \right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla (q - q_h)) -$$
$$- \left(\frac{\partial \tilde{q}}{\partial t}(0), (q - q_h)(0) \right) + a_0 \langle (q - q_h)(0), \tilde{q}(0) \rangle + a_0 \int_0^T \left\langle \frac{\partial (q - q_h)}{\partial t}, \tilde{q} \right\rangle .$$

Let v_h be some arbitrary test function from the approximating space $M_0^m(\Omega)$. Then Galerkin orthogonality gives

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \int_0^T \left(\frac{\partial^2 (q - q_h)}{\partial t^2}, \tilde{q} - v_h \right) + \left(\tilde{q}(0), \frac{\partial (q - q_h)}{\partial t}(0) \right) -$$

$$- \left(\frac{\partial \tilde{q}}{\partial t}(0), (q - q_h)(0) \right) + a_0^2 \int_0^T (\nabla (\tilde{q} - v_h), \nabla (q - q_h)) + a_0 \langle (q - q_h)(0), \tilde{q}(0) \rangle +$$

$$+ a_0 \int_0^T \left\langle \frac{\partial (q - q_h)}{\partial t}, \tilde{q} - v_h \right\rangle + a_0^2 \int_0^T (Q - Q_{h_1}, v_h)_{\Omega_1} .$$

The next step will be the integration by parts of the second derivative term once:

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \left(\frac{\partial (q - q_h)}{\partial t}(T), \tilde{q}(T) - v_h(T) \right) - \left(\frac{\partial (q - q_h)}{\partial t}(0), \tilde{q}(0) - v_h(0) \right) -$$

$$- \int_0^T \left(\frac{\partial (q - q_h)}{\partial t}, \frac{\partial (\tilde{q} - v_h)}{\partial t} \right) + \left(\tilde{q}(0), \frac{\partial (q - q_h)}{\partial t}(0) \right) - \left(\frac{\partial \tilde{q}}{\partial t}(0), (q - q_h)(0) \right) +$$

$$+ a_0^2 \int_0^T (\nabla (\tilde{q} - v_h), \nabla (q - q_h)) + a_0 \langle (q - q_h)(0), \tilde{q}(0) \rangle + a_0 \int_0^T \left\langle \frac{\partial (q - q_h)}{\partial t}, \tilde{q} - v_h \right\rangle +$$

$$+ a_0^2 \int_0^T (Q - Q_{h_1}, v_h)_{\Omega_1} .$$

Let $v_h = \tilde{q}_h$ be the FEM solution for $\tilde{q}$. We assume that at time $t = T$ $\tilde{q}_h$ and $\frac{\partial \tilde{q}_h}{\partial t}$ are chosen to be the H^1-projections of the corresponding functions, just as in case of q_h and $\frac{\partial q_h}{\partial t}$ being

H^1-projections of exact functions at time $t=0$. This implies that the first term in the RHS is zero since $\tilde{q}(T)=0$ and H^1-projection of zero function is also zero. Finally, we have

$$a_0^2\int_0^T \langle \psi, q-q_h\rangle = \left(\frac{\partial(q-q_h)}{\partial t}(0), \tilde{q}_h(0)\right) - \int_0^T \left(\frac{\partial(q-q_h)}{\partial t}, \frac{\partial(\tilde{q}-\tilde{q}_h)}{\partial t}\right) - \tag{7.7}$$

$$-\left(\frac{\partial \tilde{q}}{\partial t}(0), (q-q_h)(0)\right) + a_0^2\int_0^T (\nabla(\tilde{q}-\tilde{q}_h), \nabla(q-q_h)) + a_0\left\langle (q-q_h)(0), \tilde{q}(0)\right\rangle +$$

$$+a_0\int_0^T \left\langle \frac{\partial(q-q_h)}{\partial t}, \tilde{q}-\tilde{q}_h\right\rangle + a_0^2\int_0^T (Q-Q_{h_1}, \tilde{q}_h)_{\Omega_1}.$$

Now we must bound optimally each term on the RHS.

Definition 23. *Let $r\in\mathbb{R}$ and $r>0$. Then $]r[$ denotes the smallest possible integer $s\in\mathbb{N}$ with a property $s\geqslant r$.*

Theorem 24. *Assume the initial data satisfies*

$$q(0,\cdot)\in H^k(\Omega), \frac{\partial q}{\partial t}(0,\cdot)\in H^k(\Omega),$$

where integer k satisfies $2\leqslant k\leqslant m$. Also let $q_h(0,\cdot)$, $\frac{\partial q_h}{\partial t}(0,\cdot)$ be H^1-projections of the initial data. If the exact solution q and the solution $\tilde{q}$ of the dual problem (7.6) satisfy regularity conditions

$$q, \tilde{q}\in L^\infty(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$

$$\frac{\partial q}{\partial t}, \frac{\partial \tilde{q}}{\partial t}\in L^\infty(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$

$$\frac{\partial^2 q}{\partial t^2}, \frac{\partial^2 \tilde{q}}{\partial t^2}\in L^2(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$

then

$$\bar{E}_1\leqslant C(h^k + h^{]\frac{k}{2}[-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))})$$

with some positive constant C independent of h.

Proof. Using stability lemma, for the first term of (7.7) we obtain

$$\left|\left(\frac{\partial(q-q_h)}{\partial t}(0),\tilde{q}_h(0)\right)\right| \leqslant \left\|\frac{\partial(q-q_h)}{\partial t}(0)\right\|_{H^{-1}(\Omega)} \cdot \|\tilde{q}_h(0)\|_{H^1(\Omega)} \leqslant Ch^{k+1}.$$

Next, in the same manner,

$$\left|\left(\frac{\partial\tilde{q}}{\partial t}(0),(q-q_h)(0)\right)\right| \leqslant \left\|\frac{\partial\tilde{q}}{\partial t}(0)\right\| \cdot \|(q-q_h)(0)\| \leqslant Ch^k,$$

$$a_0\left|\langle(q-q_h)(0),\tilde{q}(0)\rangle\right| \leqslant C\|(q-q_h)(0)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\tilde{q}(0)\|_{H^1(\Omega)} \leqslant Ch^k.$$

For the integral terms, using Theorem (11), we obtain:

$$\left|\int_0^T\left(\frac{\partial(q-q_h)}{\partial t},\frac{\partial(\tilde{q}-\tilde{q}_h)}{\partial t}\right)\right| \leqslant C\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{L^\infty(L^2(\Omega))} \cdot \left\|\frac{\partial(\tilde{q}-\tilde{q}_h)}{\partial t}\right\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C(h^{2]\frac{k}{2}[+2} + h^{]\frac{k}{2}[+1}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

$$a_0^2\left|\int_0^T(\nabla(\tilde{q}-\tilde{q}_h),\nabla(q-q_h))\right| \leqslant C\|\nabla(\tilde{q}-\tilde{q}_h)\|_{L^\infty(L^2(\Omega))} \cdot \|\nabla(q-q_h)\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C(h^{2]\frac{k}{2}[} + h^{]\frac{k}{2}[}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

$$a_0\left|\int_0^T\left\langle\frac{\partial(q-q_h)}{\partial t},\tilde{q}-\tilde{q}_h\right\rangle\right| \leqslant C\int_0^T\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\tilde{q}-\tilde{q}_h\|_{L^\infty(H^1(\Omega))} \leqslant$$

$$\leqslant Ch^{]\frac{k}{2}[}\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{L^1(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant C(h^{2]\frac{k}{2}[+\frac{1}{2}} + h^{]\frac{k}{2}[-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

and finally

$$\left|a_0^2\int_0^T(Q-Q_{h_1},\tilde{q}_h)_{\Omega_1}\right| \leqslant C\|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))} \cdot \|\tilde{q}_h\|_{L^\infty(H^1(\Omega))}.$$

Therefore, the total rate of convergence for $\bar{E}_1$ is given by

$$\bar{E}_1 \leqslant C(h^k + h^{]\frac{k}{2}[-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))}). \tag{7.8}$$

□

To obtain the bound for $\bar{E}_2$ it is necessary to formulate and solve a variational problem for $\mathbf{v}$. The equation for $\mathbf{v}$ will be as (1.12):

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - a_0^2 \nabla(\nabla\cdot\mathbf{v}) = \begin{cases} 0, \text{ in the far field } \Omega/\Omega_1, \\ \frac{\partial}{\partial t}(-\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \mathbf{f} + \nu\Delta\mathbf{u}) - \frac{1}{a_0^2\rho_0}\frac{\partial}{\partial t}\mathbf{G}_1, \text{ in } \Omega_1, \end{cases} \tag{7.9}$$

with initial conditions

$$\mathbf{v}(0,x) = \mathbf{v}_1(x), \frac{\partial \mathbf{v}}{\partial t}(0,x) = \mathbf{v}_2(x).$$

Here $\mathbf{G}_1$ is such control function that $\nabla\cdot\mathbf{G}_1 = G$. The boundary conditions are

$$\frac{\partial \mathbf{v}}{\partial t}\cdot\mathbf{n} + a_0\nabla\cdot\mathbf{v} = -\frac{1}{\rho_0}g, \text{ on } \partial\Omega\times(0,T).$$

The variational formulation for this problem will be as follows. Assume

$$\frac{\partial}{\partial t}(-\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \mathbf{f} + \nu\Delta\mathbf{u} - \frac{1}{a_0^2\rho_0}\mathbf{G}_1) \in L^2(0,T;L^2(\Omega_1)), \mathbf{v}(0,\cdot) \in H_{div}(\Omega),$$

$$\frac{\partial \mathbf{v}}{\partial t}(0,\cdot) \in L^2(\Omega), g \in L^2(0,T;L^2(\partial\Omega)).$$

Find $\mathbf{v} \in L^2(0,T;H_{div}(\Omega))$ such that $\frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T;H_{div}(\Omega))$ and $\frac{\partial^2 \mathbf{v}}{\partial t^2} \in$ $L^2(0,T;L^2(\Omega))$ and which satisfies

$$\left(\frac{\partial^2 \mathbf{v}}{\partial t^2}, \mathbf{w}\right) + a_0^2(\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{w}) + a_0\left\langle\frac{\partial \mathbf{v}}{\partial t}\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n}\right\rangle = \tag{7.10}$$

$$= \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}\mathbf{F}, \mathbf{w}\right) - \frac{a_0}{\rho_0} < g, \mathbf{w}\cdot\mathbf{n} >$$

$$\text{for } \forall\mathbf{w} \in H_{div}(\Omega), 0 < t < T,$$

$$(\mathbf{v}(0,\cdot), \mathbf{w}) = (\mathbf{v}_1(\cdot), \mathbf{w}) \ \forall\mathbf{w} \in H_{div}(\Omega),$$

$$\left(\frac{\partial \mathbf{v}}{\partial t}(0,\cdot), \mathbf{w}\right) = (\mathbf{v}_2(\cdot), \mathbf{w}) \ \forall\mathbf{w} \in H_{div}(\Omega).$$

Construct a space $M_0^{m_2}(\Omega)$ of vector continuous piecewise polynomials of degree no more than $m_2 - 1$, where $m_2 \geqslant 2$ is an integer. The mesh has a characteristic size $h_2 < 1$. The FEM semidiscrete formulation is as follows. Assume

$$\frac{\partial}{\partial t}(\mathbf{f} - \frac{1}{a_0^2\rho_0}\mathbf{G}_1) \in L^2(0,T;L^2(\Omega_1)), g \in L^2(0,T;L^2(\partial\Omega)).$$

Find a twice differentiable map $\mathbf{v}_{h_2} : [0, T] \to M_0^{m_2}(\Omega)$ such that

$$\left(\frac{\partial^2 \mathbf{v}_{h_2}}{\partial t^2}, \mathbf{w}_{h_2}\right) + a_0^2(\nabla \cdot \mathbf{v}_{h_2}, \nabla \cdot \mathbf{w}_{h_2}) + a_0 \left\langle \frac{\partial \mathbf{v}_{h_2}}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle = \tag{7.11}$$

$$= \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}\mathbf{F}_{h_1}, \mathbf{w}_{h_2}\right) - \frac{a_0}{\rho_0} < g, \mathbf{w}_{h_2} \cdot \mathbf{n} >$$

$$\text{for } \forall \mathbf{w}_{h_2} \in M_0^{m_2}(\Omega), 0 < t < T,$$

$$(\mathbf{v}_{h_2}(0, \cdot), \mathbf{w}_{h_2}) = (\mathbf{v}_1(\cdot), \mathbf{w}_{h_2}) \ \forall \mathbf{w}_{h_2} \in M_0^{m_2}(\Omega),$$

$$\left(\frac{\partial \mathbf{v}_{h_2}}{\partial t}(0, \cdot), \mathbf{w}_{h_2}\right) = (\mathbf{v}_2(\cdot), \mathbf{w}_{h_2}) \ \forall \mathbf{w}_{h_2} \in M_0^{m_2}(\Omega).$$

Definition 25. *Let vector function $\mathbf{u} \in H_{div}(\Omega)$. Then its H_{div}-projection $\hat{\mathbf{u}}$ is defined by the formula*

$$a_0^2(\nabla \cdot \hat{\mathbf{u}}, \nabla \cdot \mathbf{w}_{h_2}) + (\hat{\mathbf{u}}, \mathbf{w}_{h_2}) = a_0^2(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{w}_{h_2}) + (\mathbf{u}, \mathbf{w}_{h_2}), \ \forall \ \mathbf{w}_{h_2} \in M_0^{m_2}(\Omega).$$

Assume $\mathbf{u} \in H^l(\Omega)$, with $m_2 \geqslant l \geqslant 2$. Then

$$\|\mathbf{u} - \hat{\mathbf{u}}\|_1 \leqslant C h_2^{l-1} \cdot \|\mathbf{u}\|_l. \tag{7.12}$$

Theorem 26. *Let the solution $\mathbf{v}$ of (7.10) satisfy conditions: $\mathbf{v}, \frac{\partial \mathbf{v}}{\partial t} \in L^\infty(H^l(\Omega))$ and $\frac{\partial^2 \mathbf{v}}{\partial t^2} \in L^2(H^l(\Omega))$ for some positive integer l, $m_2 \geqslant l \geqslant 2$. Let the inital conditions be the H_{div}-projections of the corresponding initial functions:*

$$\mathbf{v}_{h_2}(0, \cdot) = \hat{\mathbf{v}}(0, \cdot), \ \frac{\partial \mathbf{v}_{h_2}}{\partial t}(0, \cdot) = \frac{\partial \hat{\mathbf{v}}}{\partial t}(0, \cdot).$$

Then the solution of (7.11) satisfies:

$$\|\mathbf{v} - \mathbf{v}_{h_2}\|_{L^\infty(H_{div}(\Omega))} + \left\|\frac{\partial}{\partial t}(\mathbf{v} - \mathbf{v}_{h_2})\right\|_{L^\infty(L^2(\Omega))} \leqslant C\left(h_2^{l-1} + \left\|\frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))}\right)$$

with some constant $C > 0$ independent of h_2.

Proof. The equation for the error has the form

$$\left(\frac{\partial^2 \mathbf{e}}{\partial t^2}, \mathbf{w}_{h_2}\right) + a_0^2(\nabla\cdot\mathbf{e}, \nabla\cdot\mathbf{w}_{h_2}) + a_0\left\langle \frac{\partial \mathbf{e}}{\partial t}\cdot\mathbf{n}, \mathbf{w}_{h_2}\cdot\mathbf{n}\right\rangle = \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1}), \mathbf{w}_{h_2}\right)_{\Omega_1}.$$

Decompose the error $\mathbf{e} = \mathbf{v} - \mathbf{v}_{h_2} = \mathbf{e}_1 + \mathbf{e}_2$, where $\mathbf{e}_1 = \mathbf{v} - \hat{\mathbf{v}}$ and $\mathbf{e}_2 = \hat{\mathbf{v}} - \mathbf{v}_{h_2}$. Notice that $\mathbf{e}_2 \in M_0^{m_2}(\Omega)$. It is obvious that

$$\left(\frac{\partial^2 \mathbf{e}_2}{\partial t^2}, \mathbf{w}_{h_2}\right) + a_0^2(\nabla\cdot\mathbf{e}_2, \nabla\cdot\mathbf{w}_{h_2}) + a_0\left\langle \frac{\partial \mathbf{e}_2}{\partial t}\cdot\mathbf{n}, \mathbf{w}_{h_2}\cdot\mathbf{n}\right\rangle = -a_0^2(\nabla\cdot\mathbf{e}_1, \nabla\cdot\mathbf{w}_{h_2}) -$$

$$-\left(\frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \mathbf{w}_{h_2}\right) + \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1}), \mathbf{w}_{h_2}\right)_{\Omega_1} - a_0\left\langle \frac{\partial \mathbf{e}_1}{\partial t}\cdot\mathbf{n}, \mathbf{w}_{h_2}\cdot\mathbf{n}\right\rangle.$$

Using the definition of the H_{div}-projection, we obtain

$$\left(\frac{\partial^2 \mathbf{e}_2}{\partial t^2}, \mathbf{w}_{h_2}\right) + a_0^2(\nabla\cdot\mathbf{e}_2, \nabla\cdot\mathbf{w}_{h_2}) + a_0\left\langle \frac{\partial \mathbf{e}_2}{\partial t}\cdot\mathbf{n}, \mathbf{w}_{h_2}\cdot\mathbf{n}\right\rangle = \left(\mathbf{e}_1 - \frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \mathbf{w}_{h_2}\right) +$$

$$+\frac{1}{\rho_0}\left(\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1}), \mathbf{w}_{h_2}\right)_{\Omega_1} - a_0\left\langle \frac{\partial \mathbf{e}_1}{\partial t}\cdot\mathbf{n}, \mathbf{w}_{h_2}\cdot\mathbf{n}\right\rangle.$$

Next we use energy method by setting $\mathbf{w}_{h_2} = \frac{\partial \mathbf{e}_2}{\partial t}$.

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + a_0^2\frac{1}{2}\frac{d}{dt}\left\|\nabla\cdot\mathbf{e}_2\right\|^2 + a_0\left|\frac{\partial \mathbf{e}_2}{\partial t}\cdot\mathbf{n}\right|^2 = \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1}), \frac{\partial \mathbf{e}_2}{\partial t}\right)_{\Omega_1} +$$

$$+\left(\mathbf{e}_1 - \frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \frac{\partial \mathbf{e}_2}{\partial t}\right) - a_0\left\langle \frac{\partial \mathbf{e}_1}{\partial t}\cdot\mathbf{n}, \frac{\partial \mathbf{e}_2}{\partial t}\cdot\mathbf{n}\right\rangle.$$

Using the fact that $(a,b) \leqslant \frac{1}{2\epsilon}\|a\|^2 + \frac{\epsilon}{2}\|b\|^2$ for any inner product $(\cdot,\cdot)$ and any $\epsilon > 0$, we can get

$$\frac{d}{dt}\left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + a_0^2\left\|\nabla\cdot\mathbf{e}_2\right\|^2\right) \leqslant \frac{1}{\rho_0}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{\Omega_1}^2 + \frac{1}{\rho_0}\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 +$$

$$2\|\mathbf{e}_1\|^2 + 2\left\|\frac{\partial^2 \mathbf{e}_1}{\partial t^2}\right\|^2 + \left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + \frac{a_0}{2}\left|\frac{\partial \mathbf{e}_1}{\partial t}\cdot\mathbf{n}\right|^2.$$

Add

$$\frac{d}{dt}\|\mathbf{e}_2\|^2 \leqslant \left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + \|\mathbf{e}_2\|^2\right)$$

to the previous inequality:

$$\frac{d}{dt}\left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + \|\mathbf{e}_2\|^2 + a_0^2\left\|\nabla\cdot\mathbf{e}_2\right\|^2\right) \leqslant \frac{1}{\rho_0}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{\Omega_1}^2 + \left(2 + \frac{1}{\rho_0}\right)\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 +$$

$$+\|\mathbf{e}_2\|^2 + 2\|\mathbf{e}_1\|^2 + 2\left\|\frac{\partial^2 \mathbf{e}_1}{\partial t^2}\right\|^2 + \frac{a_0}{2}\left|\frac{\partial \mathbf{e}_1}{\partial t}\cdot \mathbf{n}\right|^2.$$

Integrate assuming that initial data is approximated via H_{div}-projection.

$$\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + \|\mathbf{e}_2\|^2 + a_0^2\,\|\nabla\cdot\mathbf{e}_2\|^2 \leqslant \left(2+\frac{1}{\rho_0}\right)\int_0^t \left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2 + \|\mathbf{e}_2\|^2\right) d\tau +$$

$$+\frac{1}{\rho_0}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|^2_{L^2(L^2(\Omega_1))} + 2\|\mathbf{e}_1\|^2_{L^2(L^2(\Omega))} +$$

$$+2\left\|\frac{\partial^2 \mathbf{e}_1}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + \frac{a_0}{2}C_{tr}^2\left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|^2_{L^2(H^1(\Omega))},$$

where C_{tr} denotes the constant from the trace theorem. Applying Gronwall's lemma and extracting the square root of both sides yield

$$\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + \|\mathbf{e}_2\|_{L^\infty(H_{div}(\Omega))} \leqslant$$

$$C\left(\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))} + \|\mathbf{e}_1\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial^2 \mathbf{e}_1}{\partial t^2}\right\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|_{L^2(H^1(\Omega))}\right)$$

with some constant $C = C(T)$ growing exponentially fast. This implies, due to the triangle inequality, that

$$\left\|\frac{\partial \mathbf{e}}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + \|\mathbf{e}\|_{L^\infty(H_{div}(\Omega))} \leqslant$$

$$C\left(\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))} + \|\mathbf{e}_1\|_{L^\infty(H_{div}(\Omega))} + \left\|\frac{\partial^2 \mathbf{e}_1}{\partial t^2}\right\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|_{L^\infty(H^1(\Omega))}\right).$$

Using (7.12), we obtain the statement of the theorem. □

In order to estimate $\bar{E}_2$ it is necessary to formulate a corresponding dual problem. Similarly to the case with $\bar{E}_1$, the pointwise dual problem with unknown function $\tilde{\mathbf{v}}$ has the form

$$\begin{cases} \tilde{\mathbf{v}}_{tt} - a_0^2\nabla(\nabla\cdot\tilde{\mathbf{v}}) = 0, \text{ on } (0,T)\times\Omega, \\ \tilde{\mathbf{v}}(T,\cdot) = 0, \text{ on } \Omega, \\ \tilde{\mathbf{v}}_t(T,\cdot) = 0, \text{ on } \Omega, \\ \nabla\cdot\tilde{\mathbf{v}} - \frac{1}{a_0}\tilde{\mathbf{v}}_t\cdot\mathbf{n} = \xi, \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{7.13}$$

where

$$\xi(t,\mathbf{x}) = \begin{cases} q_h, & \text{if } \mathbf{x} \in S, \\ 0, & \text{if } \mathbf{x} \in \partial\Omega / S. \end{cases}$$

The equation in the weak form will be

$$\int_0^T \left(\frac{\partial^2 \tilde{\mathbf{v}}}{\partial t^2}, \mathbf{w}\right) + a_0^2 \int_0^T (\nabla \cdot \tilde{\mathbf{v}}, \nabla \cdot \mathbf{w}) - a_0 \int_0^T \left\langle \mathbf{w} \cdot \mathbf{n}, \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \mathbf{n} \right\rangle = a_0^2 \int_0^T \langle \xi, \mathbf{w} \cdot \mathbf{n} \rangle .$$

Next we present a stability lemma similar to Lemma 21. Its proof resembls that of Lemma 21.

Lemma 27. *The variational solution of the dual problem is stable and the following inequality holds:*

$$\left\| \frac{\partial \tilde{\mathbf{v}}}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} + a_0 \|\nabla \cdot \tilde{\mathbf{v}}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant a_0^{\frac{3}{2}} \|q_h\|_{L^2(0,T;L^2(S))}.$$

Remark 28. *The same stability result holds for the approximate solution* $\tilde{\mathbf{v}}_{h_2}$*.*

We will follow the same ideas as those used for obtaining the estimate for the error $\bar{E}_1$. Integrating the second derivative term by parts twice and then setting $\mathbf{w} = \mathbf{v} - \mathbf{v}_{h_2}$ lead to

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v} - \mathbf{v}_{h_2}) \cdot \mathbf{n} \rangle =$$

$$= \int_0^T \left(\frac{\partial^2 (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t^2}, \tilde{\mathbf{v}}\right) + a_0^2 \int_0^T (\nabla \cdot \tilde{\mathbf{v}}, \nabla \cdot (\mathbf{v} - \mathbf{v}_{h_2})) + a_0 \int_0^T \left\langle \tilde{\mathbf{v}} \cdot \mathbf{n}, \frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t} \cdot \mathbf{n} \right\rangle +$$
$$+ \left(\tilde{\mathbf{v}}(0), \frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t}(0)\right) - \left((\mathbf{v} - \mathbf{v}_{h_2})(0), \frac{\partial \tilde{\mathbf{v}}}{\partial t}(0)\right) + a_0 < \tilde{\mathbf{v}}(0) \cdot \mathbf{n}, (\mathbf{v} - \mathbf{v}_{h_2})(0) \cdot \mathbf{n} > .$$

Next use Galerkin orthogonality with a test function $\mathbf{w}_{h_2}$:

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v} - \mathbf{v}_{h_2}) \cdot \mathbf{n} \rangle =$$

$$= \int_0^T \left(\frac{\partial^2 (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t^2}, \tilde{\mathbf{v}} - \mathbf{w}_{h_2}\right) + a_0^2 \int_0^T (\nabla \cdot (\tilde{\mathbf{v}} - \mathbf{w}_{h_2}), \nabla \cdot (\mathbf{v} - \mathbf{v}_{h_2})) +$$
$$+ a_0 \int_0^T \left\langle (\tilde{\mathbf{v}} - \mathbf{w}_{h_2}) \cdot \mathbf{n}, \frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t} \cdot \mathbf{n} \right\rangle + \frac{1}{\rho_0} \int_0^T \left(\frac{\partial}{\partial t}(\mathbf{F}_{h_1} - \mathbf{F}), \mathbf{w}_{h_2}\right)_{\Omega_1} -$$
$$- \left((\mathbf{v} - \mathbf{v}_{h_2})(0), \frac{\partial \tilde{\mathbf{v}}}{\partial t}(0)\right) + a_0 < \tilde{\mathbf{v}}(0) \cdot \mathbf{n}, (\mathbf{v} - \mathbf{v}_{h_2})(0) \cdot \mathbf{n} > + \left(\tilde{\mathbf{v}}(0), \frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t}(0)\right).$$

Let $\mathbf{w}_{h_2} = \tilde{\mathbf{v}}_{h_2}$ be the FEM solution for $\tilde{\mathbf{v}}$ in the space $M_0^{m_2}(\Omega)$. We assume that at time $t = T$ the solution $\tilde{\mathbf{v}}_{h_2}$ is an H_{div}-projection of the exact solution $\tilde{\mathbf{v}}$, i.e. it is zero. The same goes for $\frac{\partial \tilde{\mathbf{v}}_{h_2}}{\partial t}(T,\cdot)$ since it is an H_{div}-projection of $\frac{\partial \tilde{\mathbf{v}}}{\partial t}(T,\cdot) = 0$. Then finally

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v} - \mathbf{v}_{h_2}) \cdot \mathbf{n} \rangle = \left(\frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t}(0), \tilde{\mathbf{v}}_{h_2}(0) \right) - \left((\mathbf{v} - \mathbf{v}_{h_2})(0), \frac{\partial \tilde{\mathbf{v}}}{\partial t}(0) \right) +$$

$$+ a_0 \langle \tilde{\mathbf{v}}(0) \cdot \mathbf{n}, (\mathbf{v} - \mathbf{v}_{h_2})(0) \cdot \mathbf{n} \rangle - \int_0^T \left(\frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t}, \frac{\partial (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_{h_2})}{\partial t} \right) +$$

$$+ a_0^2 \int_0^T (\nabla \cdot (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_{h_2}), \nabla \cdot (\mathbf{v} - \mathbf{v}_{h_2})) + a_0 \int_0^T \left\langle (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_{h_2}) \cdot \mathbf{n}, \frac{\partial (\mathbf{v} - \mathbf{v}_{h_2})}{\partial t} \cdot \mathbf{n} \right\rangle +$$

$$+ \frac{1}{\rho_0} ((\mathbf{F} - \mathbf{F}_{h_1})(0), \tilde{\mathbf{v}}_{h_2}(0))_{\Omega_1} + \frac{1}{\rho_0} \int_0^T \left(\mathbf{F} - \mathbf{F}_{h_1}, \frac{\partial \tilde{\mathbf{v}}_{h_2}}{\partial t} \right)_{\Omega_1}.$$

Now we have to estimate each term separately.

Theorem 29. *Assume the initial data satisfies*

$$\mathbf{v}(0,\cdot) \in H^l(\Omega), \frac{\partial \mathbf{v}}{\partial t}(0,\cdot) \in H^l(\Omega),$$

where integer l satisfies $2 \leqslant l \leqslant m_2$. Also let $\mathbf{v}_{h_2}(0,\cdot)$, $\frac{\partial \mathbf{v}_{h_2}}{\partial t}(0,\cdot)$ be H_{div}-projections of the initial data. If the exact solution $\mathbf{v}$ and the solution $\tilde{\mathbf{v}}$ of the dual problem (7.13) satisfy regularity conditions

$$\mathbf{v}, \tilde{\mathbf{v}} \in L^\infty(0,T; H^{]\frac{l}{2}[+1}(\Omega)),$$
$$\frac{\partial \mathbf{v}}{\partial t}, \frac{\partial \tilde{\mathbf{v}}}{\partial t} \in L^\infty(0,T; H^{]\frac{l}{2}[+1}(\Omega)),$$
$$\frac{\partial^2 \mathbf{v}}{\partial t^2}, \frac{\partial^2 \tilde{\mathbf{v}}}{\partial t^2} \in L^2(0,T; H^{]\frac{l}{2}[+1}(\Omega)),$$

then

$$\bar{E}_2 \leqslant C(h_2^{l-1} + h_2^{]\frac{l}{2}[-1} \left\| \frac{\partial}{\partial t}(\mathbf{F}_{h_1} - \mathbf{F}) \right\|_{L^2(L^2(\Omega_1))} + \|\mathbf{F} - \mathbf{F}_{h_1}\|_{L^1(L^2(\Omega_1))} + \|(\mathbf{F}_{h_1} - \mathbf{F})(0,\cdot)\|)$$

with some positive constant C independent of h_2.

Proof. For each term we have estimates

$$\left|\left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0), \tilde{\mathbf{v}}_{h_2}(0)\right)\right| \leqslant \left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0)\right\| \cdot \|\tilde{\mathbf{v}}_{h_2}(0)\| \leqslant Ch_2^l,$$

$$\left|\left((\mathbf{v}-\mathbf{v}_{h_2})(0), \frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right)\right| \leqslant \|(\mathbf{v}-\mathbf{v}_{h_2})(0)\| \cdot \left\|\frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right\| \leqslant Ch_2^l,$$

$$a_0\left|\langle\tilde{\mathbf{v}}(0)\cdot\mathbf{n}, (\mathbf{v}-\mathbf{v}_{h_2})(0)\cdot\mathbf{n}\rangle\right| \leqslant C\|\nabla\cdot\tilde{\mathbf{v}}(0)\| \cdot \|(\mathbf{v}-\mathbf{v}_{h_2})(0)\|_1 \leqslant Ch_2^{l-1},$$

$$\left|\int_0^T \left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}, \frac{\partial(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})}{\partial t}\right)\right| \leqslant$$

$$\leqslant C\left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\right\|_{L^\infty(L^2(\Omega))} \cdot \left\|\frac{\partial(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})}{\partial t}\right\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C\left(h_2^{2]\frac{l}{2}[} + h_2^{]\frac{l}{2}[}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))}\right),$$

$$a_0^2\left|\int_0^T (\nabla\cdot(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}), \nabla\cdot(\mathbf{v}-\mathbf{v}_{h_2}))\right| \leqslant C\|\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}\|_{L^\infty(H_{div}(\Omega))} \cdot \|\mathbf{v}-\mathbf{v}_{h_2}\|_{L^\infty(H_{div}(\Omega))} \leqslant$$

$$\leqslant C\left(h_2^{2]\frac{l}{2}[} + h_2^{]\frac{l}{2}[}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))}\right),$$

$$a_0\left|\int_0^T \left\langle(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})\cdot\mathbf{n}, \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\cdot\mathbf{n}\right\rangle\right| \leqslant$$

$$\leqslant C\,\|\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}\|_{L^\infty(H_{div}(\Omega))} \cdot \left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\right\|_{L^\infty(H_1(\Omega))} \leqslant$$

$$\leqslant C\left(h_2^{2]\frac{l}{2}[-1} + h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))}\right),$$

$$\left|\frac{1}{\rho_0}\left((\mathbf{F}-\mathbf{F}_{h_1})(0), \tilde{\mathbf{v}}_{h_2}(0)\right)_{\Omega_1}\right| \leqslant C\|(\mathbf{F}-\mathbf{F}_{h_1})(0)\| \cdot \|\tilde{\mathbf{v}}_{h_2}(0)\|.$$

Finally,

$$\left|\frac{1}{\rho_0}\int_0^T \left(\mathbf{F}-\mathbf{F}_{h_1}, \frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}\right)_{\Omega_1}\right| \leqslant C\|\mathbf{F}-\mathbf{F}_{h_1}\|_{L^1(L^2(\Omega_1))} \cdot \left\|\frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}\right\|_{L^\infty(L^2(\Omega))}.$$

The estimate for $\bar{E}_2$ will be

$$\bar{E}_2 \leqslant C(h_2^{l-1} + h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2(L^2(\Omega_1))} + \quad \|\mathbf{F}-\mathbf{F}_{h_1}\|_{L^1(L^2(\Omega_1))} +$$

$$+\|(\mathbf{F}-\mathbf{F}_{h_1})(0)\|).$$

□

Combining both estimates for $\overline{E}_1$ and $\overline{E}_2$, we obtain

$$|\bar{A} - \bar{A}_h| \leqslant$$

$$\leqslant C(h^k + h]^{\frac{k}{2}}[^{-\frac{1}{2}} \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|Q - Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))} +$$
$$+ h_2^{l-1} + h_2]^{\frac{l}{2}}[^{-1} \left\| \frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}) \right\|_{L^2(L^2(\Omega_1))} + \|\mathbf{F}_{h_1} - \mathbf{F}\|_{L^1(L^2(\Omega_1))} + \|(\mathbf{F}_{h_1} - \mathbf{F})(0)\|).$$

We see that in term $\overline{E}_1$ the rate of convergence is dictated by h^k whereas for the exact formula approach the convergence is of order h^{k-1}.

7.4 NUMERICAL EXPERIMENTS

In this section the results of computing the sound power in a two-dimensional simulation will be presented. More specifically, we will compute the line integrals of the the sound intesity multiplied by a fixed unit vector over certain straight line segments of the computational domain. We will test both the exact formula method, section 7.1, and the duality method requiring an additional scheme for the fluctuation of the velocity $\mathbf{v}$, section 7.3. Our main purpose is to obtain plots for the computed acoustic powers. The specific conditions of the experiment are presented below.

The domains Ω_1 and Ω are the circles of radiuses 0.33 and 1 respectively, and Ω_1 is embedded in Ω symetrically. A decaying flow, i.e. the one with no external forces, is taking place in Ω_1 and satisfies the no-slip boundary condition on $\partial\Omega_1$. Both the NSE and the wave equation are non-dimensionalized and presented in Ω_1 in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \Delta \mathbf{u} = 0,$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$M^2 \frac{\partial^2 q}{\partial t^2} - \Delta q = \nabla \mathbf{u} : \nabla \mathbf{u}^t,$$

and the last one is given with the boundary condition on $\partial\Omega$

$$M \frac{\partial q}{\partial t} + \nabla q \cdot \mathbf{n} = 0.$$

Here Reynolds number $Re = 16$ and Mach number $M = 0.075$. If $\mathbf{r}$ denotes the radius-vector of the point in space and r denotes its magnitude, then we present the following initial condition for the velocity:

$$\mathbf{u}_0 = 36.73 \cdot (0.33 - r) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{r}.$$

This gives a rotational flow similar to the one presented on picture 3.

For all of the tests we construct a uniform mesh in Ω_1 with $h_1 \approx 0.0207$. The Finite Element used is Taylor-Hood element, i.e. the piecewise quadratics for the velocity and piecewise linears for the pressure. This scheme satisfies the LBB-condition, [35], and is second order accurate.

As a time-stepping scheme, we use the Stabilized Extrapolated Backward Euler Method, same as in chapter 5, with parameter $\delta = 0.0075$. The simulation is carried out from $t = 0$ to $t = 0.6$ with a constant time step $\Delta t = 0.0025$. The examples of the computed velocity and pressure fields are shown below on figures 9 and 10:

For both methods of sections 7.1 and 7.3 we use the second order in time scheme presented in chapter 4 in order to compute the field of acoustic pressure q in Ω. The time step Δt is the same as for the NSE. For space discretization, piecewise quadratics are used on the uniform mesh of size $h \approx 0.028$. The initial conditions are set to be zero for both $q(0, \cdot)$ and $q_t(0, \cdot)$, i.e. $q_{h,0} = 0$ and $q_{h,1} = 0$. Three pictures of the pressure fluctuations are presented below for time levels $t = 0.2, 0.4, 0.6$.

For the evaluation of the time integral from (7.2), the Trapezoidal Method is used. For the duality argument, the scheme (7.11) is implemented with a second order time-stepping algorithm.

The initial conditions of (7.11) must be chosen carefully for they cannot be arbitrary once the initial conditions for q_h are given. The conditions for the fluctuation of velocity are to be found from

$$M^2 \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla q = \mathbf{F},$$

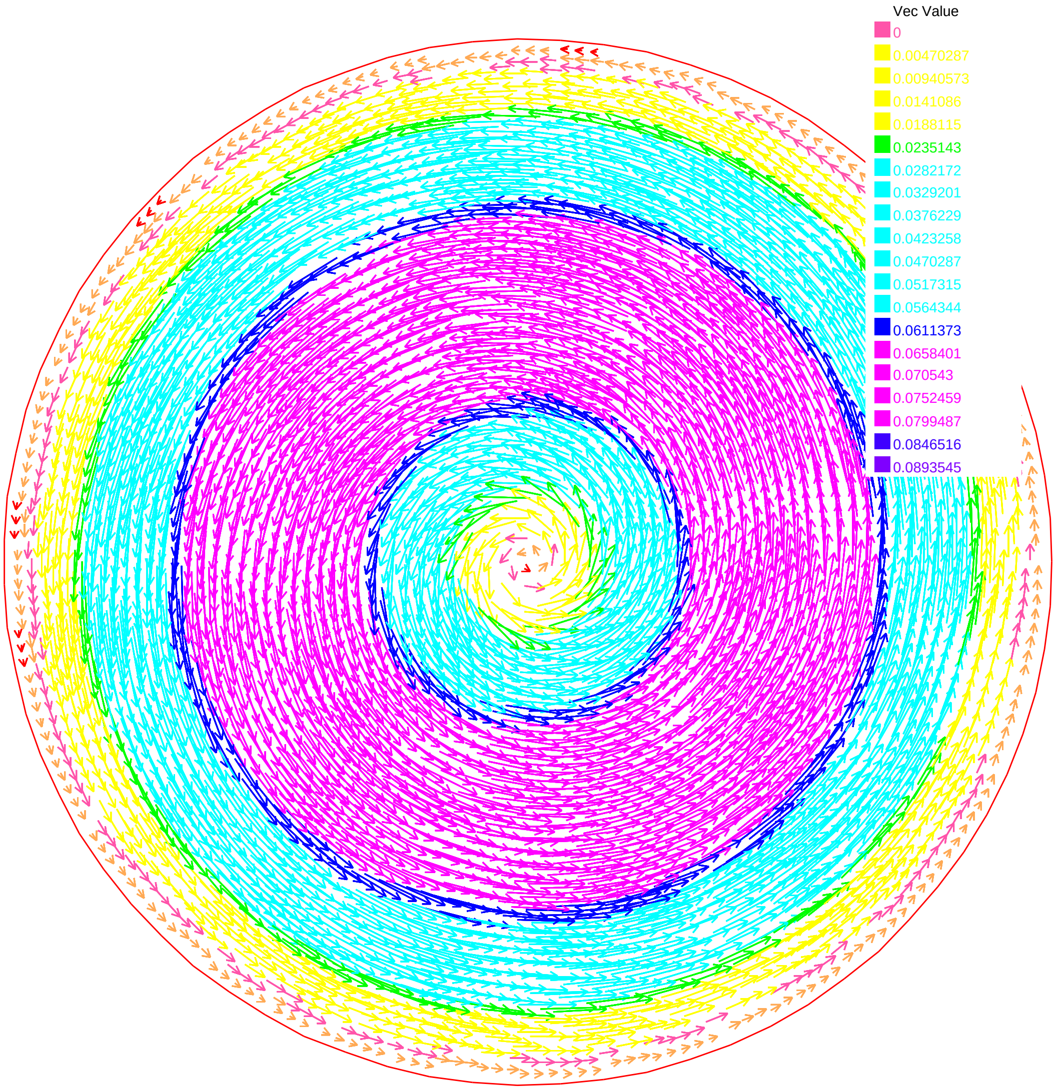

Figure 9: The velocity field in Ω_1 at time $t = 0.3$

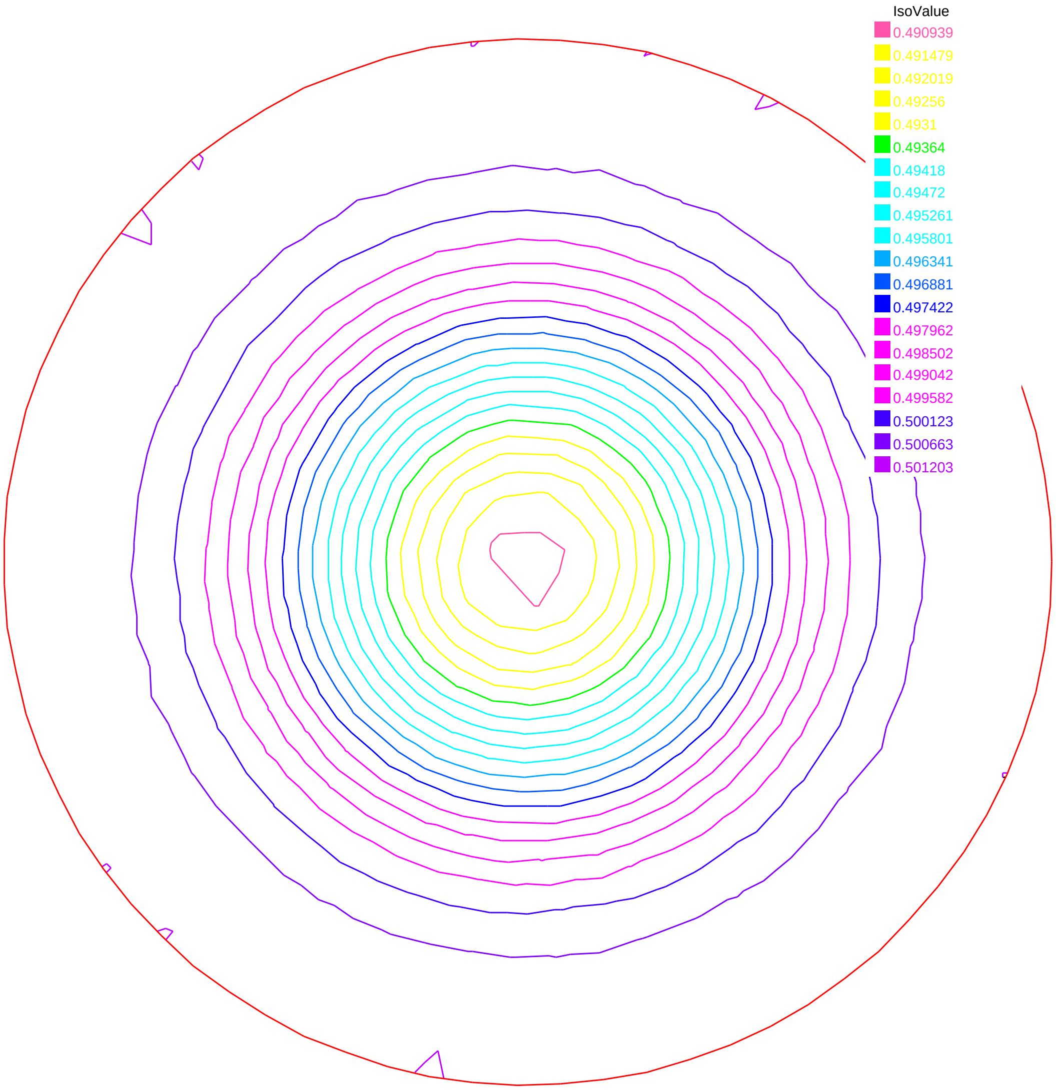

Figure 10: The pressure field in Ω_1 at time $t = 0.3$

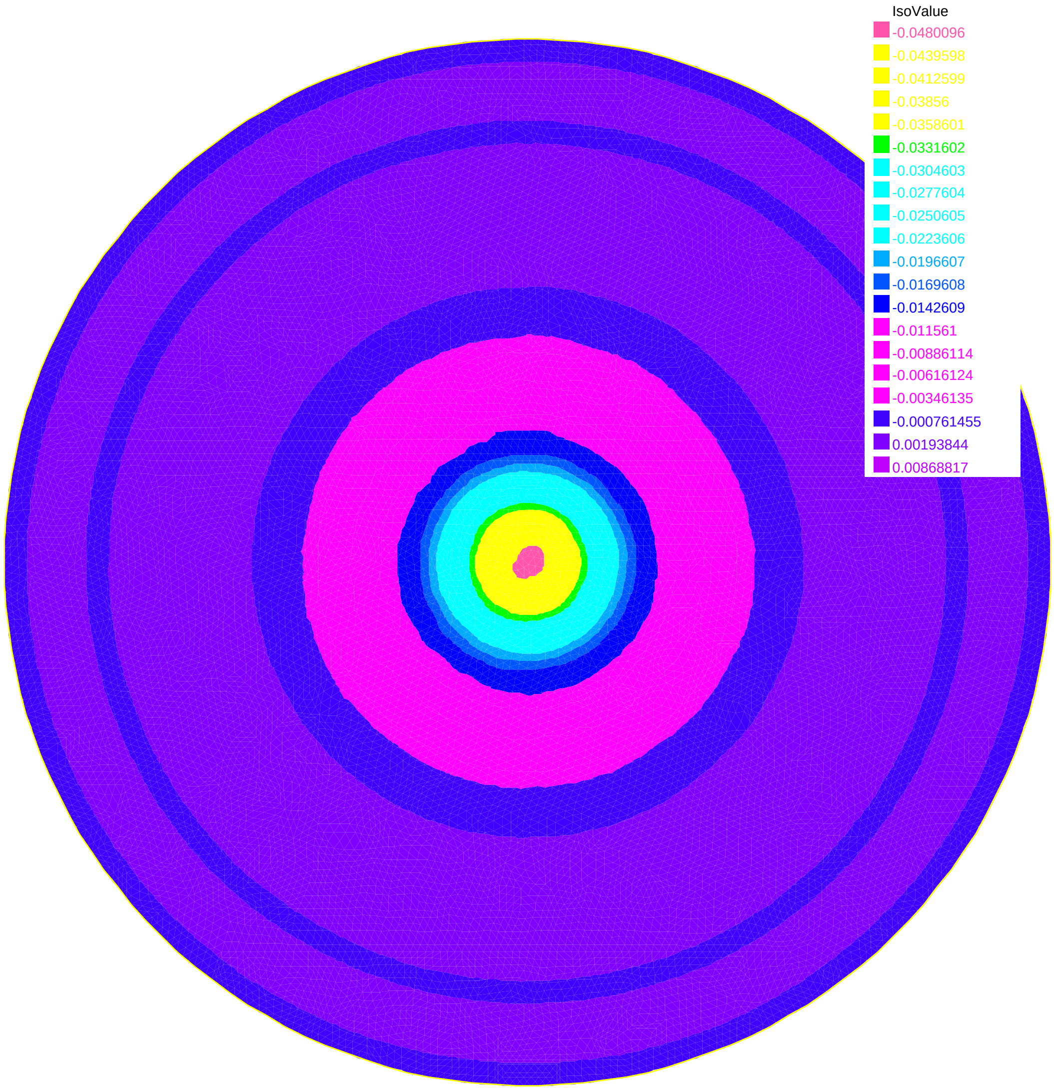

Figure 11: The acoustic pressure field in Ω at time $t = 0.2$

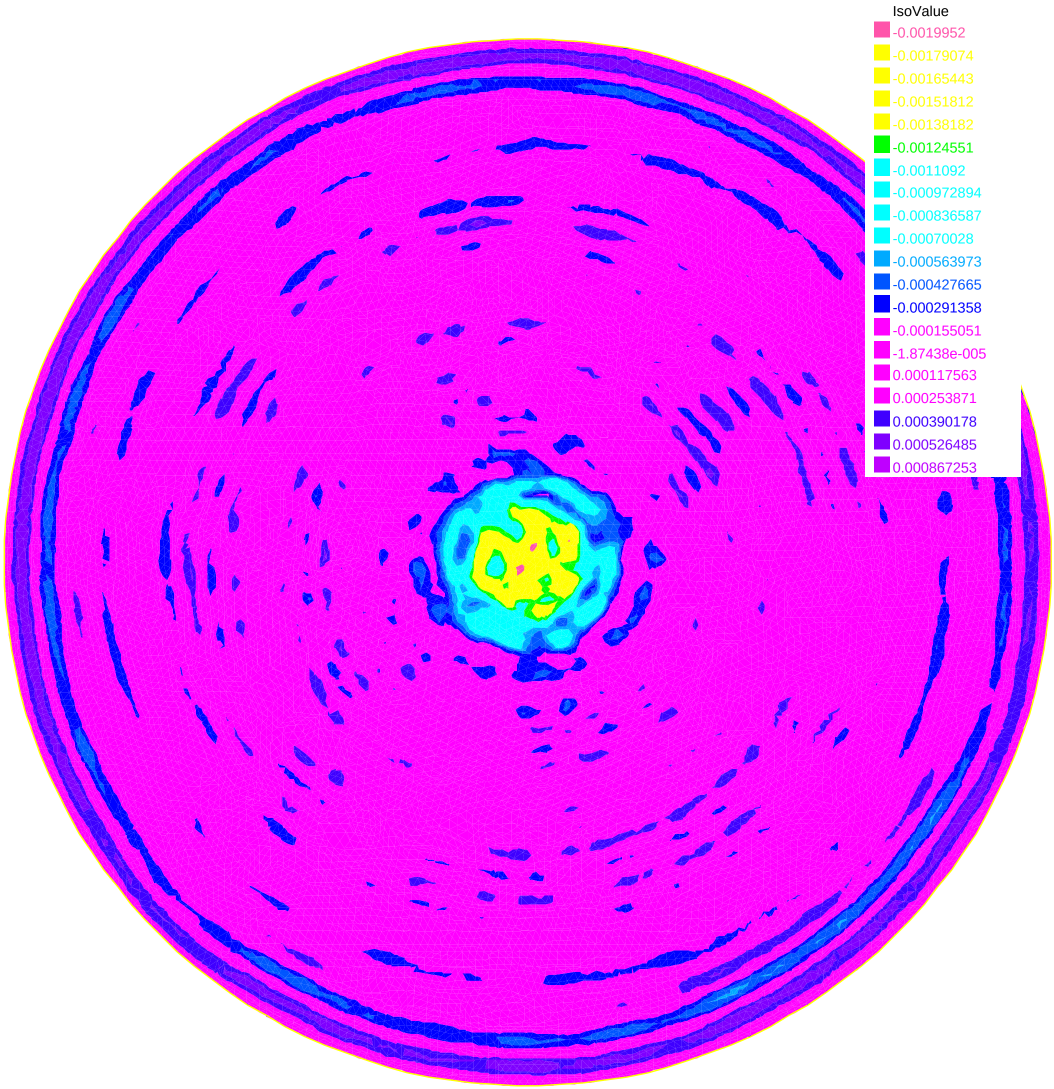

Figure 12: The acoustic pressure field in Ω at time $t = 0.4$

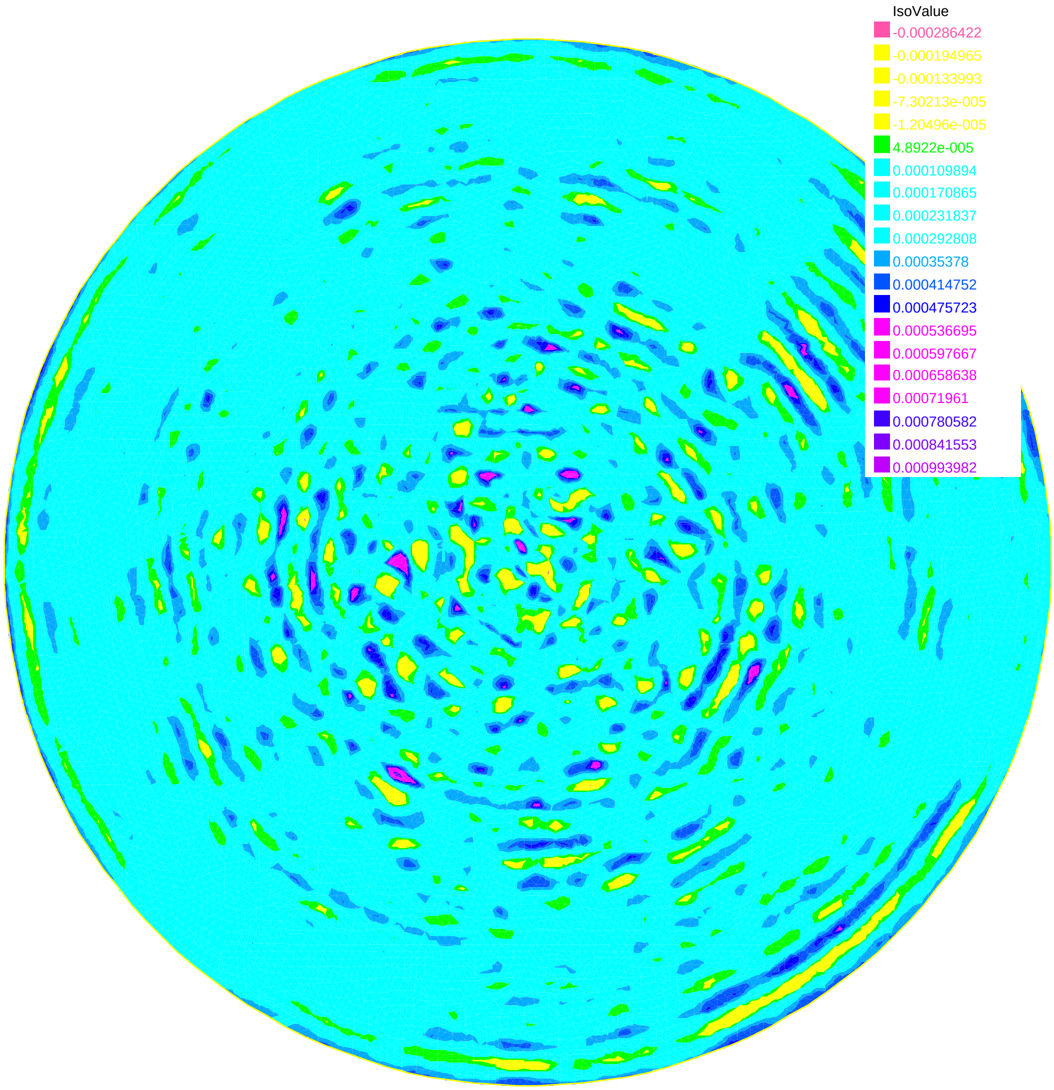

Figure 13: The acoustic pressure field in Ω at time $t = 0.6$

where $\mathbf{F}$ is zero in the far field Ω/Ω_1 and

$$\mathbf{F} = -\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{1}{Re}\Delta \mathbf{u} \equiv -\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{Re}\Delta \mathbf{u}$$

in Ω_1. $\mathbf{v}_{h,0}$ is set to 0. However, since $q_{h,0} = 0$, we have $\frac{\partial \mathbf{v}}{\partial t}(0, \cdot) = \mathbf{F}(0, \cdot)$. The following second-order approximating scheme is used for this condition:

$$\frac{\mathbf{v}_{h,1} - \mathbf{v}_{h,0}}{\Delta t} \equiv \frac{\mathbf{v}_{h,1}}{\Delta t} = \frac{\mathbf{F}_{h_1,0} + \mathbf{F}_{h_1,1}}{2},$$

i.e.

$$\mathbf{v}_{h,1} = 0.5 \cdot \Delta t \cdot (-\mathbf{u}_{h_1,0} \cdot \nabla \mathbf{u}_{h_1,0} + \frac{1}{Re}\Delta \mathbf{u}_{h_1,0} - \mathbf{u}_{h_1,1} \cdot \nabla \mathbf{u}_{h_1,1} + \frac{1}{Re}\Delta \mathbf{u}_{h_1,1}). \tag{7.14}$$

The sound power is computed for all time levels along two straight line segments, these are with end points $(0.34, -0.37)$, $(0.34, 0.37)$ and $(0.66, -0.37)$,$(0.66, 0.37)$. The normal vector $\mathbf{n} = (1, 0)^t$ for both cases. The first line is almost tangent to the boundary $\partial\Omega_1$ but is still in the far field. For the evaluation of the line integrals, we use the Midpoint Method, with 100 subintervals of length 0.0074 each. Additionally, the sound intensity is computed for all time levels at the point $(0.999, 1)$ in the same direction $(1, 0)^t$.

The graphs of the sound power and intensity are shown on pictures 14, 15 and 17 from $t = 0$ to 0.3. The blue and red curves present the cases of section 7.1 and 7.3 respectively. The results show that graphs obtained by two methods are hardly different from each other. We also present a zoomed version of the graphs for the case of the line segment at $x = 0.66$, 16. Note that the experiment was run both with and without the viscous term in the RHS of (7.9) and (7.14), and the differences in the graphs appeared to be too small for a human eye to see in the scales of the presented pictures. This proves that for this combination of Reynolds and Mach numbers the viscous term plays no significant role in affecting the acoustic intensity in the far field.

As we see on the graphs, the distance between zero and the elevation of the graph is the time required for the acoustic waves to travel from the turbulent region to the specified line or point in the far field. The following decay to zero is due to decay of the sound source in Ω_1.

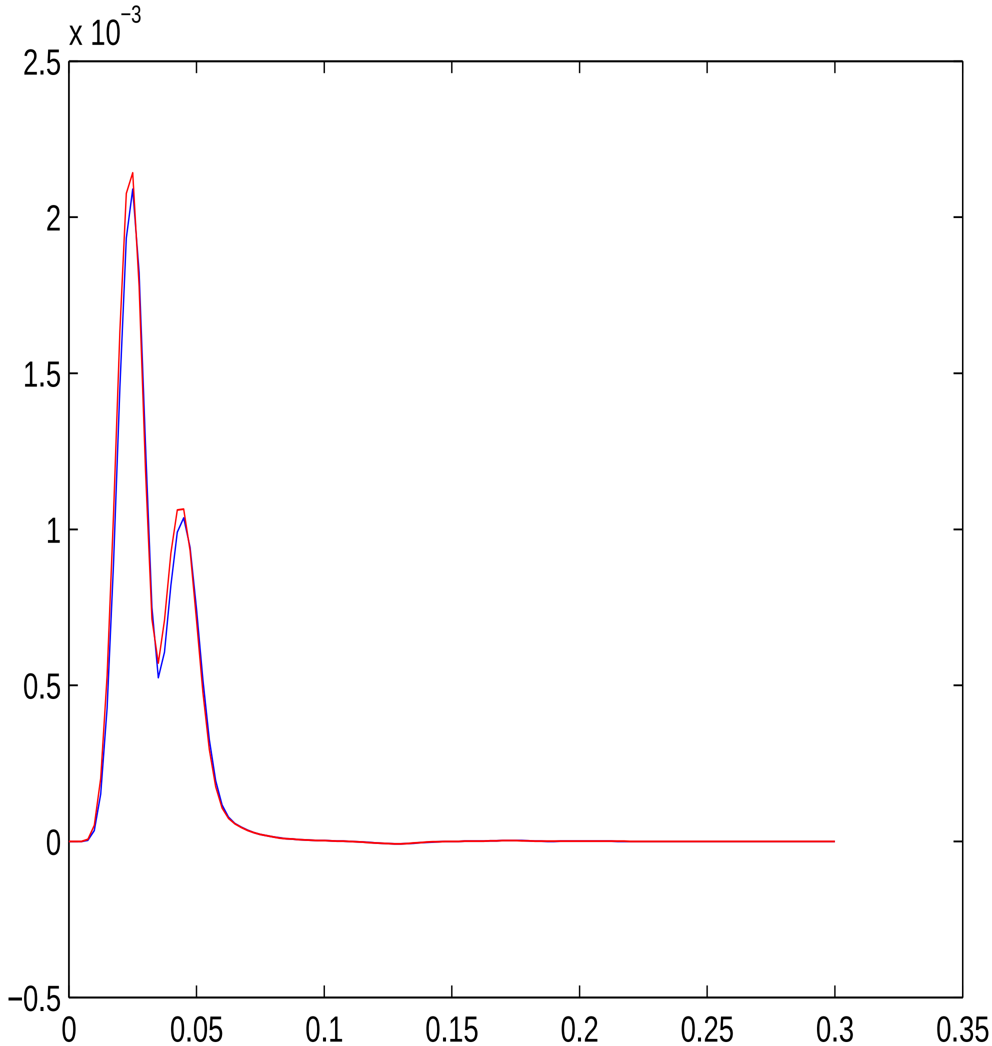

Figure 14: The sound power as a function of time along the line $(0.34, -0.37) - (0.34, 0.37)$

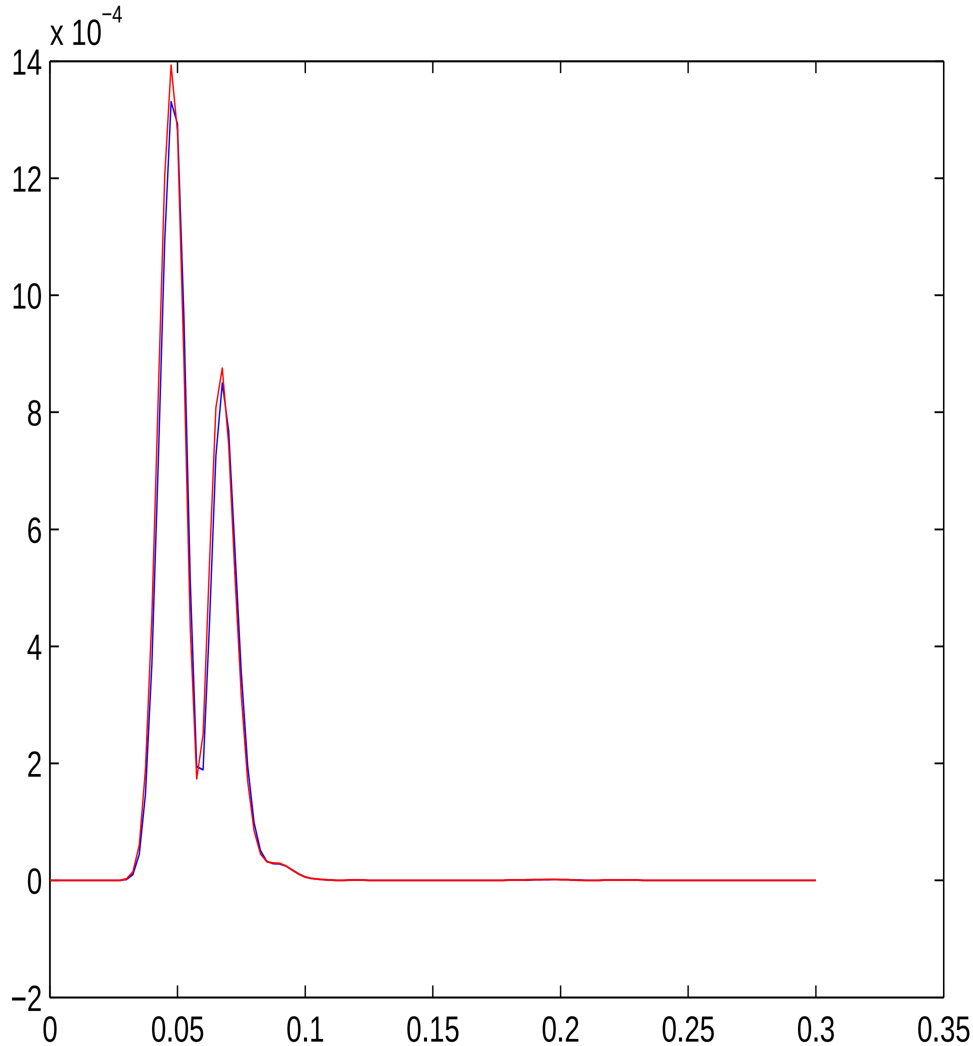

Figure 15: The sound power as a function of time along the line $(0.66, -0.37) - (0.66, 0.37)$

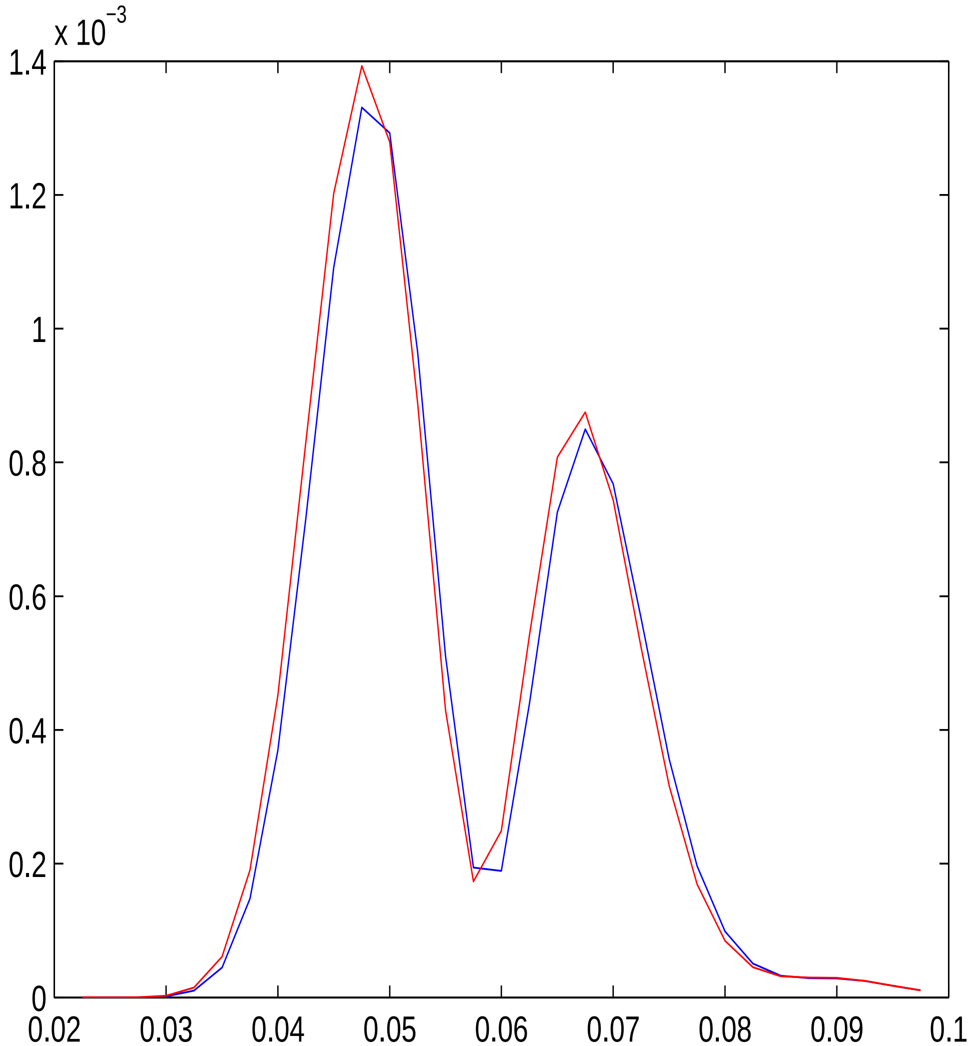

Figure 16: The sound power as a function of time along the line $(0.66, -0.37) - (0.66, 0.37)$

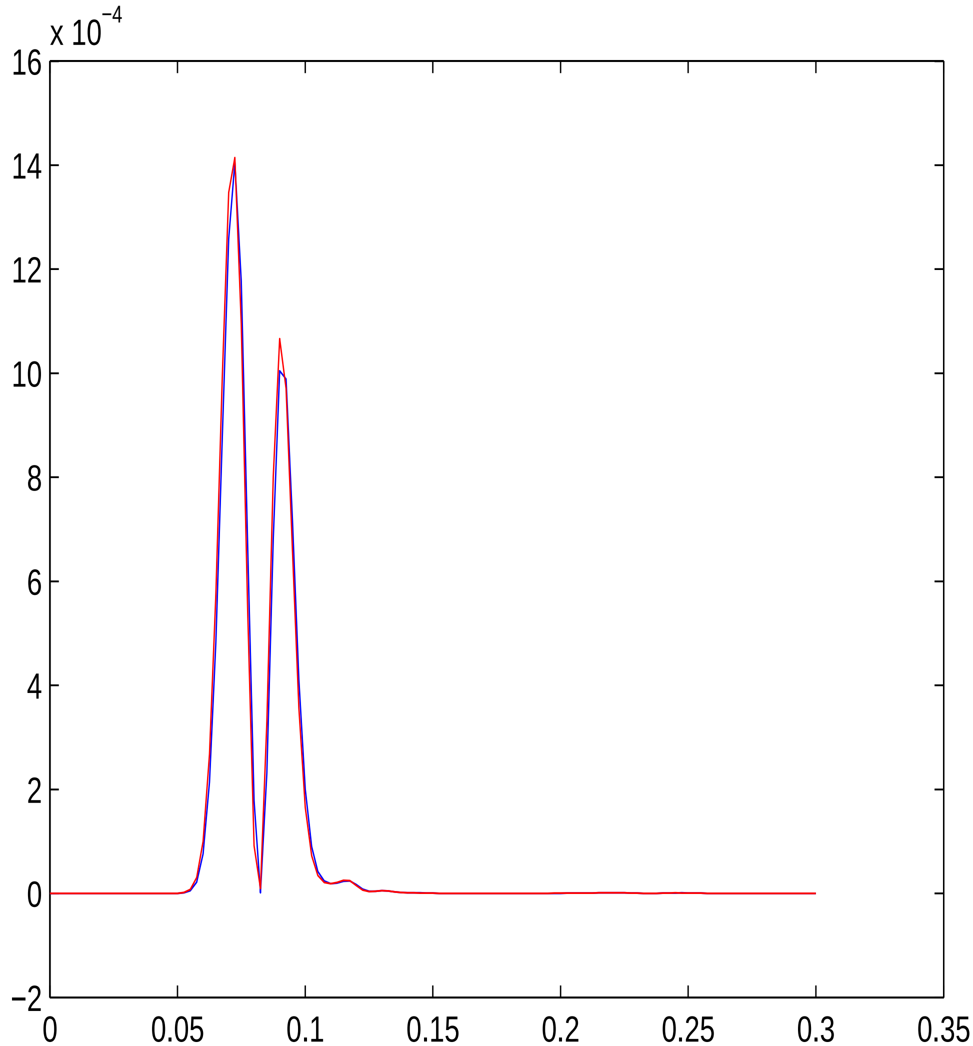

Figure 17: The intensity as a function of time at $(0.999, 0)$

The methods from 7.1 and 7.3 work well for a coarser mesh also. The picture 18 shows the results of the experiment for $h_1 \approx 0.052$ and $h \approx 0.0785$. The black and the blue graphs belong to the sound power computed via the method of 7.1 on the fine (the old one) and coarse mesh (the new one) respectively. The green and the red graphs are computed via the method of 7.3 on the same meshes. We see that the graphs of each pair are very close to each other.

Although this thesis has not covered a theory of fully discrete methods for computing a sound power, it should be evident that time steps must be taken small since the recent pictures show steep behavior of the graphs of the sound power. Indeed, the picture 19 below shows that for the fine mesh and for $\Delta t = 0.01$ oscillations start to occur. Moreover, the highest position of the graph happens at $t \approx 0.09$, whereas on 16 it is at $t \approx 0.05$.

Finally, we present a graph of the time averaged intensity on the boundary from $t = 0$ to 0.4, using the argument of section 7.3. The graph is shown on the picture 20. It is obvious that, as time continues increasing, the time averaged intensity drops to zero for this experiment.

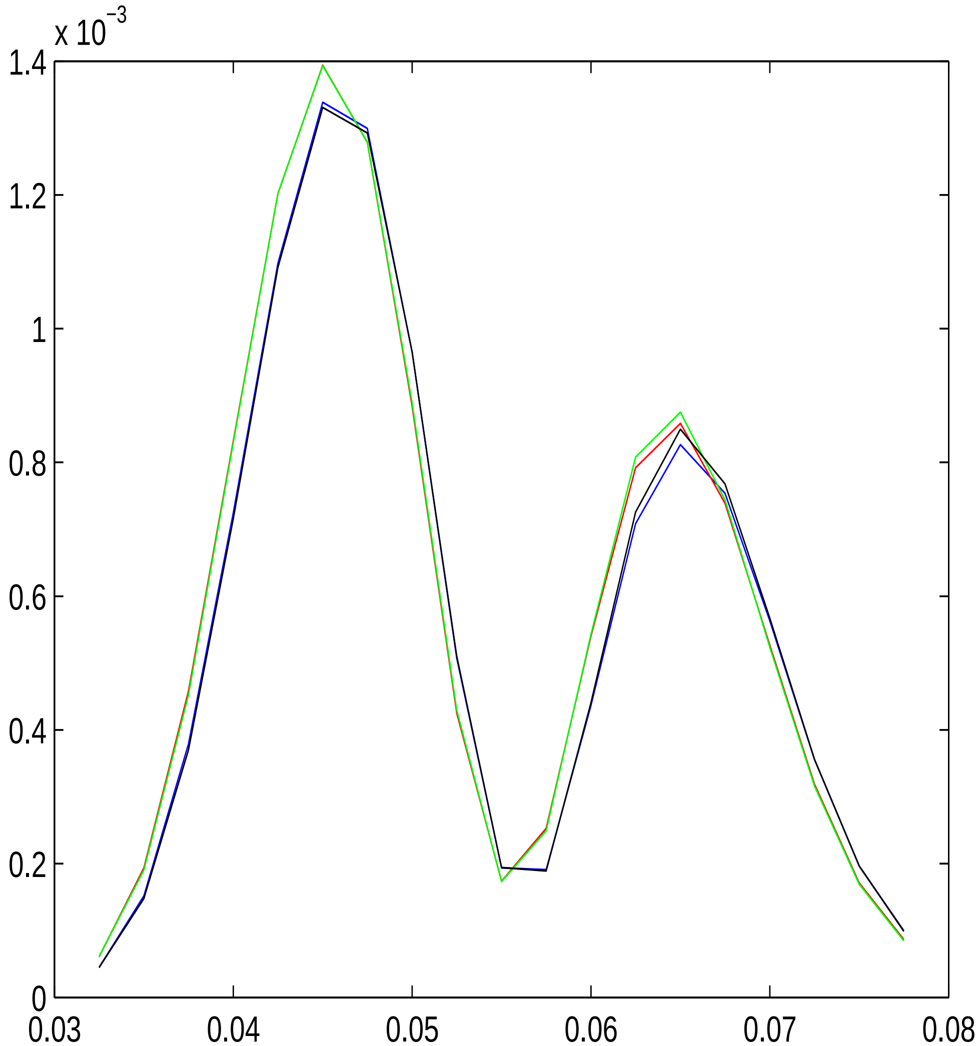

Figure 18: The sound power on the fine and coarse mesh along $(0.66, -0.37) - (0.66, 0.37)$

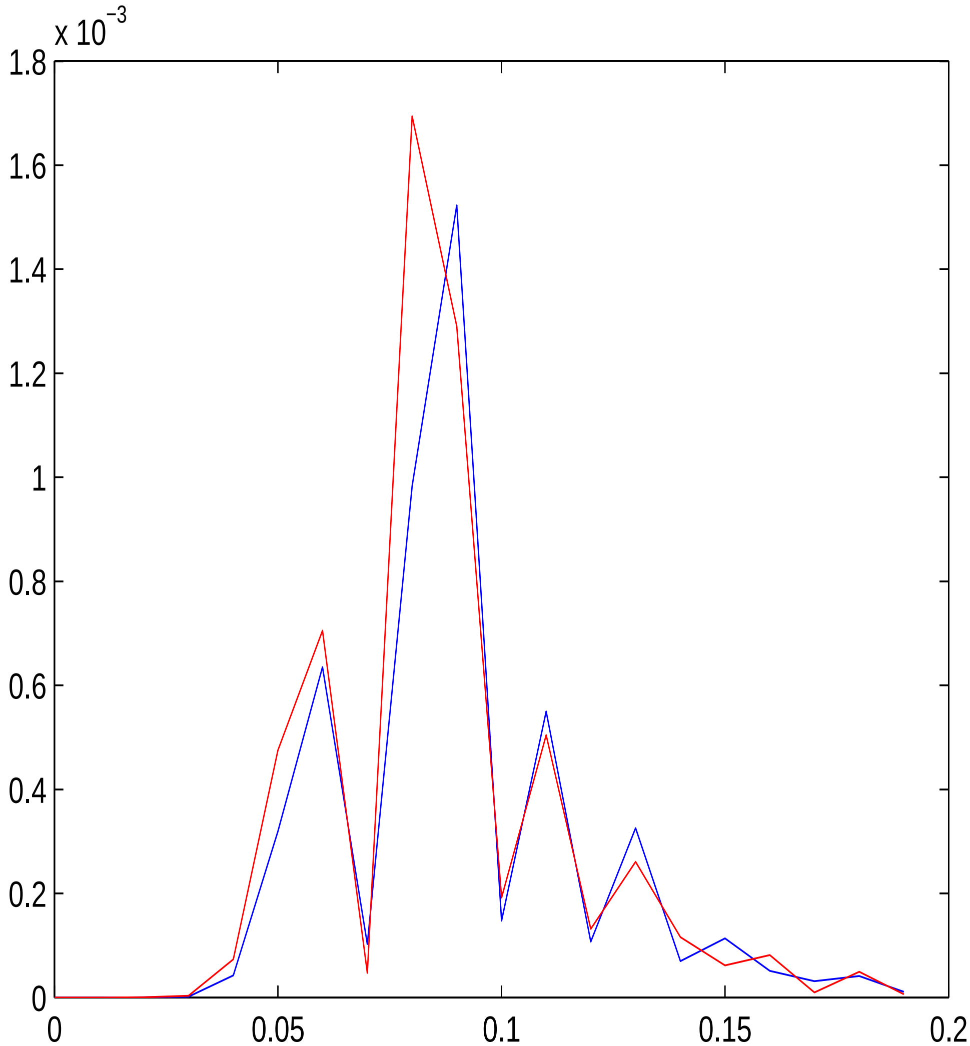

Figure 19: The sound power along $(0.66, -0.37) - (0.66, 0.37)$ for $\Delta t = 0.01$

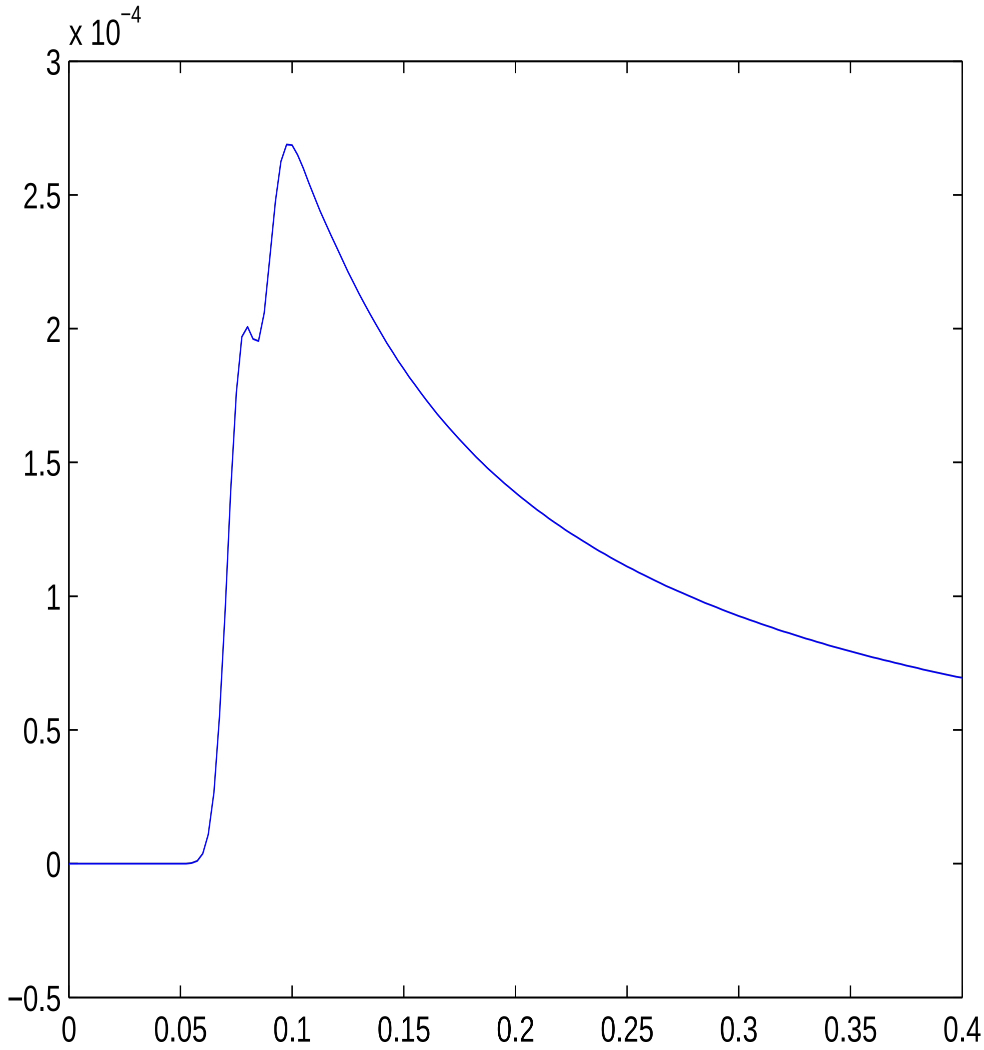

Figure 20: The time-averaged intensity as a function of time at $(0.999, 0)$

8.0 LARGE EDDY SIMULATION IN AEROACOUSTICS

Several computational strategies in noise generation have been developed during the last decades, among which the Large Eddy Simulation technique is recognized as being the most promising for unsteady simulation of turbulent flows in complex realistic systems. Since the full computation of all active scales which are present in a turbulent flow is far beyond the capability of available supercomputers due to the required memory and computational effort, LES introduced a scale-separation operator: scales smaller than the arbitrarily fixed cutoff length scale are removed from the computation, allowing for the use of much coarser computational grids and therefore tractable simulations for engineering purposes. Since governing equations of fluid mechanics are nonlinear ones and that turbulence is an intrisically nonlinear multiscale phenomenon, the effect of missing small scales on the large simulated scales must be taken into account. This is usually done adding a new term, referred to as a subgrid model, in the governing equations. The reader is referred to specialized books for a detailed introduction to LES [54, 55, 22].

While closing the Navier-Stokes equations for aerodynamics and combustion has received a large attention since the 1960s, definition of subgrid models for other physical mechanisms driven by turbulent fluctuations is still an almost open problem. This is true of noise generation by small scales in a turbulent flow, which has been addressed in very few papers only [59, 60]. In these works, a scale-similarity strategy is used to obtain a model for small-scale contribution, which can be interpreted as a low-order deconvolution method. This approach has been extensively and successfully used to close momentum equations for aerodynamic computations, and related mathematical analysis has been performed considering incompressible momentum equations [10, 32]. On the other hand, mathematical analysis of scale-similarity modeling for other physical mechanisms described by new governing equations, such as the Lighthill equation for turbulent noise production and acoustic wave propagation, has not been addressed up to now. The goal of the present paper is to provide the reader with such an analysis.

In this paper we will present the numerical analysis of the semidiscrete FEM for computing the noise generated by a turbulent flow in a field with no walls using the filtered Lighthill model. The starting point is the Lighthill analogy. While filtering the Lighthill analogy, we neglect the so called unresolved scales that satisfy the condition $l < \delta$, where δ is the cut-off length scale, typically corresponding to the allowed computational mesh size. For this purpose we will use the differential filter, [23], given by the condition

$$\begin{cases} \overline{\phi} - \delta^2 \Delta \overline{\phi} = \phi, \text{ on } \Omega_1, \\ \overline{\phi} = \phi, \text{ on } \partial\Omega_1, \end{cases} \tag{8.1}$$

where $\overline{\phi}$ means 'the filtered ϕ', and Ω_1 is the domain where both ϕ and $\overline{\phi}$ are defined. Assume that, as the flow approaches the boundary, its velocity, as well the external force are decaying to zero so that in a neighborhood of the boundary $\partial\Omega_1$ of a size δ the flow is already reduced to the rest state. In this case the condition $\overline{\mathbf{u}} = 0$ on $\partial\Omega_1$ (and the filtered velocity of higher orders) is physically justified. Moreover, two filters such as defined by

$$\begin{cases} \overline{\phi} - \delta^2 \Delta \overline{\phi} = \phi, \text{ on } \Omega_1, \\ \overline{\phi} = 0, \text{ on } \partial\Omega_1, \end{cases}$$

and

$$\begin{cases} \overline{\phi} - \delta^2 \Delta \overline{\phi} = \phi, \text{ on } \Omega, \\ \overline{\phi} = 0, \text{ on } \partial\Omega, \end{cases}$$

are equivalent for this class of functions ϕ, i.e. decaying as reaching the boundary $\partial\Omega_1$ and being equal to zero on Ω/Ω_1. This equivalence allows us to filter the Lighthill analogy whose RHS is defined on the whole Ω, including both the turbulent region Ω_1 and the far field. While filtering over Ω, we are implicitly assuming that $\mathbf{u}$ is defined on the whole Ω and is zero in the far field. In this part of the thesis, we use notation $p^{'}$ for the real acoustic pressure, since the letter q will be used as a filtered acoustic pressure. The filtering leads to

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = F_1,$$

with $q = \overline{p'}$, $\mathbf{F} = \overline{\mathbf{f}}$ and

$$F_1 = \begin{cases} \rho_0 \nabla \cdot \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \rho_0 \nabla \cdot \mathbf{F} \text{ on } \Omega_1, \\ 0 \text{ on } \Omega/\Omega_1. \end{cases}$$

The question of proper boundary conditions for the fluctuation of pressure q is a non-trivial one and, in addition, depends on an application. We specifically choose the non-reflecting boundary conditions of the first-order with a boundary control function g in the form

$$\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = 0 \text{ on } \partial\Omega.$$

Introducing the subrid scale tensor $\mathbb{R} = \overline{\mathbf{u} \otimes \mathbf{u}} - \overline{\mathbf{u}} \otimes \overline{\mathbf{u}}$, we can write the previous equation as

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = \rho_0 \nabla \cdot \nabla \cdot (\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) + \rho_0 \nabla \cdot \nabla \cdot \mathbb{R} - \rho_0 \nabla \cdot \mathbf{F}. \tag{8.2}$$

Note $\mathbb{R}$ is a symmetric tensor. More on the filtered Lighthill analogy may be found at [59]. The term $\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}$ is called the resolved Lighthill tensor. It is important to notice that the variable $\overline{\mathbf{u}}$ satisfies the exact filtered NSE. LES of the incompressible NSE requires that some subgrid scale model be introduced and used during the computations. The new function $\mathbf{v} \approx \overline{\mathbf{u}}$ satisfies that model, and the more accurate the model is, the closer $\mathbf{v}$ to the original $\overline{\mathbf{u}}$ is. This variable $\mathbf{v}$ will be used in the first term of the RHS of (8.2) instead of $\overline{\mathbf{u}}$, and since that point it is assumed that some model for the filetered NSE is already implemented. Note that here the notation $\mathbf{v}$ is not related to the fluctuation of velocity used in chapter 7.

Next, since the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ follows from the original NSE, using the idea of Lemma 3 from section 1.3, we further simplify the RHS of (8.2) so that we have

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = \rho_0 \nabla \mathbf{v} : \nabla \mathbf{v}^t + \rho_0 \nabla \cdot \nabla \cdot \mathbb{R} - \rho_0 \nabla \cdot \mathbf{F} \tag{8.3}$$

in Ω_1.

Since the subgrid scale tensor $\mathbb{R}$ contains the term $\overline{\mathbf{u} \otimes \mathbf{u}}$, some model for $\mathbb{R}$, containing only the filtered variables, is required. We will present and numerically analyze two models. For simplicity, we will preserve the same letter q in the wave equation, although we should always keep in mind that it is not exactly $\overline{p'}$ due to the inaccuracy of the used models. The models are the zeroth order van Cittert deconvolution model and the Bardina subgrid scale model and are given by the equations, respectively,

$$\mathbb{R} = \overline{\mathbf{v} \otimes \mathbf{v}} - \mathbf{v} \otimes \mathbf{v} \tag{8.4}$$

and

$$\mathbb{R} = \overline{\mathbf{v} \otimes \mathbf{v}} - \overline{\mathbf{v}} \otimes \overline{\mathbf{v}}. \tag{8.5}$$

Interestingly, for the filtered NSE and the filtered Lighthill analogy different subgrid scale models may be used. Although Bardina model is not stable in the LES of the incompressible NSE, it recovers fairly accurate results in terms of acoustic intensity, [59]. Using the definition of the differential filter (8.1), we can conclude that

$$\mathbb{R} - \delta^2 \Delta \mathbb{R} = \Delta(\mathbf{v} \otimes \mathbf{v})$$

for (8.4). Denoting $\mathbf{w} = \overline{\mathbf{v}}$, for (8.5) we get

$$\begin{cases} \mathbb{R} - \delta^2 \Delta \mathbb{R} = \delta^2 \Delta(\mathbf{w} \otimes \mathbf{w}) + \mathbf{v} \otimes \mathbf{v} - \mathbf{w} \otimes \mathbf{w}, \\ \mathbf{w} - \delta^2 \Delta \mathbf{w} = \mathbf{v}. \end{cases}$$

Coupling these results with (8.3) for each model respectively, we obtain the closed problem that can be studied numerically.

8.1 THE ZEROTH ORDER VAN CITTERT MODEL

In Ω_1 for any $0 < t < T$ we have

$$\begin{cases} \frac{1}{a_0^2}\frac{\partial q^2}{\partial t^2} - \Delta q = \rho_0 \nabla \mathbf{v} : \nabla \mathbf{v}^t + \rho_0 \nabla \cdot \nabla \cdot \mathbb{R} - \rho_0 \nabla \cdot \mathbf{F}, \\ \mathbb{R} - \delta^2 \Delta \mathbb{R} = \delta^2 \Delta(\mathbf{v} \otimes \mathbf{v}). \end{cases}$$

The boundary conditions are the following:

$$\mathbb{R} = 0 \ \forall (t, x) \in (0, T) \times \partial\Omega_1,$$

$$\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g \ \forall (t, x) \in (0, T) \times \partial\Omega.$$

The system above is decoupled with respect to variables $\mathbb{R}$ and q. This means that first we can solve the second equation for each time level t and then solve the first equation with $\mathbb{R}$ known. The grids for the first and the second equation have characteristic sizes $h < 1$ and $h_2 < 1$ respectively (we do not use index 1, since it is already used by the grid for solving the NSE; moreover, it has no relation to the mesh for computing $\mathbf{v}$ in chapter 7). In order to write the variational formulation, it is necessary to make regularity assumptions first. The RHS of the first equation has the double divergence of $\mathbb{R}$, so in order for the RHS to be of $L^2(0, T; L^2(\Omega_1))$, we can require that $\mathbb{R} \in L^2(0, T; H^2(\Omega_1) \cap H_0^1(\Omega_1))$. This regularity of $\mathbb{R}$ follows from the assumption that $\Delta(v_i v_j) \in L^2(0, T; L^2(\Omega_1))$ for any pair $i, j = \overline{1, n}$. This may be guaranteed by the condition $v_i v_j \in L^2(0, T; H^2(\Omega_1) \cap H_0^1(\Omega_1))$ for any pair $i, j = \overline{1, n}$. The last condition is satisfied if the following assumption is true about $\mathbf{v}$:

$$\mathbf{v} \in L^4(0, T; W^{2,4}(\Omega_1) \cap H_0^1(\Omega_1)). \tag{8.6}$$

This is the weakest assumption that can be required. Stronger assumptions for practical purposes could be

$$\mathbf{v} \in L^4(0, T; L^\infty(\Omega_1) \cap H_0^1(\Omega_1)), \nabla \mathbf{v} \in L^4(0, T; L^4(\Omega_1)), \nabla\nabla \mathbf{v} \in L^4(0, T; L^2(\Omega_1)).$$

The condition (8.6) is already sufficient for the term $\rho_0 \nabla \mathbf{v} : \nabla \mathbf{v}^t$ to be in $L^2(L^2(\Omega_1))$. Refer to chapter 3 for details. The variational formulation for the first equation is given by a formula

$$\left(\frac{\partial^2 q}{\partial t^2}, v \right) + a_0^2 \left(\nabla q, \nabla v \right) + a_0 \left\langle \frac{\partial q}{\partial t}, v \right\rangle = \\ = a_0^2 \rho_0 \left(\nabla \cdot \nabla \cdot \mathbb{R} + \nabla \mathbf{v} : \nabla \mathbf{v}^t - \nabla \cdot \mathbf{F}, v \right)_{\Omega_1} + a_0^2 < g, v > . \tag{8.7}$$

with v being a scalar test function on the whole domain Ω.

The variational formulation for the second equation is as follows. Let $\mathbf{v} \in L^4(0, T; W^{1,4}(\Omega_1) \cap H_0^1(\Omega_1))$. For all pairs $i, j = \overline{1, n}$ find $R_{ij} \in L^2(0, T; H_0^1(\Omega_1))$ such that

$$(R_{ij}, \hat{v}) + \delta^2 (\nabla R_{ij}, \nabla \hat{v}) = -\delta^2 (\nabla (v_i v_j), \nabla \hat{v}) \ \forall \hat{v} \in H_0^1(\Omega_1), 0 < t < T. \tag{8.8}$$

We can easily obtain the stability result by setting $\hat{v} = R_{ij}$. Then

$$\|R_{ij}\|^2 + \delta^2 \|\nabla R_{ij}\|^2 = -\delta^2 (\nabla (v_i v_j), \nabla R_{ij}) \leqslant \delta^2 \|\nabla (v_i v_j)\| \cdot \|\nabla R_{ij}\|.$$

There follows

$$\|R_{ij}\|_1 \leqslant max(1, \delta^2) \|\nabla (v_i v_j)\|.$$

Using the product differentiation rule and Holder's inequality we obtain

$$\|\nabla (v_i v_j)\| \leqslant \|v_i \nabla v_j\| + \|v_j \nabla v_i\| \leqslant C \|\mathbf{v}\|_{L^4(\Omega_1)} \cdot \|\nabla \mathbf{v}\|_{L^4(\Omega_1)} \leqslant C \|\mathbf{v}\|_{W^{1,4}(\Omega_1)}^2.$$

So the stability result is, after integration in time,

$$\|R_{ij}\|_{L^2(0,T;H_0^1(\Omega_1))} \leqslant C_1(\delta) \|\mathbf{v}\|_{L^4(0,T;W^{1,4}(\Omega_1))}^2,$$

where

$$lim_{\delta \to 0+} C_1(\delta) = C_0 > 0 \text{ and } lim_{\delta \to +\infty} C_1(\delta) = +\infty.$$

The Finite Element formulation will be as follows. For all $i, j = \overline{1, n}$ find such map $R_{ij,h_2} : [0, T] \to M_{00}^{m_2}(\Omega_1)$ that

$$(R_{ij,h_2}, \hat{v}_{h_2}) + \delta^2 (\nabla R_{ij,h_2}, \nabla \hat{v}_{h_2}) = -\delta^2 (\nabla (v_{i,h_1} v_{j,h_1}), \nabla \hat{v}_{h_2}) \tag{8.9}$$

$$\forall \hat{v}_{h_2} \in M_{00}^{m_2}(\Omega_1), 0 < t < T.$$

Theorem 30. *Let* $R_{ij} \in L^2(0,T; H^{m_2}(\Omega_1) \cap H_0^1(\Omega_1))$. *Then*

$$\|R_{ij} - R_{ij,h_2}\|_{L^2(0,T;H_0^1(\Omega_1))} \leqslant C(\delta)(h_2^{m_2-1} + \|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|_{L^2(0,T;L^2(\Omega_1))})$$

with some positive constant $C(\delta) = C_1 \cdot (1 + \sqrt{2} \cdot max(\delta^{-2}, \delta^2))$*, where* $C_1 > 0$ *and is independent of both* δ *and* h_2*.*

Proof. Denoting $e_{ij} = R_{ij} - R_{ij,h_2}$, subtract (8.8) and (8.9) to get the error equation

$$(e_{ij}, \hat{v}_{h_2}) + \delta^2(\nabla e_{ij}, \nabla \hat{v}_{h_2}) = -\delta^2(\nabla(v_i v_j - v_{i,h_1} v_{j,h_1}), \nabla \hat{v}_{h_2}).$$

Decompose the error as $e_{ij} = \eta_{ij} + \psi_{ij}$, where $\eta_{ij} = R_{ij} - \hat{R}_{ij}$ and $\psi_{ij} = \hat{R}_{ij} - R_{ij,h_2}$ and $\hat{R}_{ij} \in M_{00}^{m_2}(\Omega_1)$. Then we get

$$(\psi_{ij}, \hat{v}_{h_2}) + \delta^2(\nabla \psi_{ij}, \nabla \hat{v}_{h_2}) = -(\eta_{ij}, \hat{v}_{h_2}) - \delta^2(\nabla \eta_{ij}, \nabla \hat{v}_{h_2}) - \delta^2(\nabla(v_i v_j - v_{i,h_1} v_{j,h_1}), \nabla \hat{v}_{h_2}).$$

Since $\psi_{ij} \in M_{00}^{m_2}(\Omega_1)$ we can set $\hat{v}_{h_2} = \psi_{ij}$. This gives

$$\|\psi_{ij}\|^2 + \delta^2\|\nabla\psi_{ij}\|^2 \leqslant \|\eta_{ij}\| \cdot \|\psi_{ij}\| + \delta^2\|\nabla\eta_{ij}\| \cdot \|\nabla\psi_{ij}\| + \delta^2\|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\| \cdot \|\nabla\psi_{ij}\|.$$

Thus

$$\|\psi_{ij}\|_1 \leqslant \sqrt{2} \cdot max\left(\delta^{-2}, \delta^2\right)\left(\|\eta_{ij}\|_1 + \|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|\right),$$

so

$$\|e_{ij}\|_1 \leqslant \left(1 + \sqrt{2} \cdot max\left(\delta^{-2}, \delta^2\right)\right)(inf_{\hat{R} \in M_{00}^{m_2}(\Omega_1)}\|R_{ij} - \hat{R}\|_1 + \|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|).$$

Finally,

$$\|e_{ij}\|_1 \leqslant C(\delta)(h_2^{m_2-1} + \|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|).$$

Here $C(\delta)$ is such that

$$lim_{\delta \to 0+} C(\delta) = +\infty \text{ and } lim_{\delta \to +\infty} C(\delta) = +\infty.$$

□

Lemma 31. *Let the velocity* $\mathbf{v} \in L^4(0,T;W^{1,4}(\Omega_1) \cap H^2(\Omega_1) \cap H_0^1(\Omega_1))$*. Also let the mesh for* $M_{00}^{m_1}(\Omega_1)$*,* $m_1 \geqslant 2$*, used for computing* $\mathbf{v}_{h_1}$*, be quasi-uniform and* $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;H_0^1(\Omega_1))}$ *converge no slower than* $O(h_1^{\frac{n}{4}})$*. Then for any pair* $i,j = \overline{1,n}$

$$\|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|_{L^2(0,T;L^2(\Omega_1))} \leqslant C\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;W^{1,4}(\Omega_1))}.$$

Proof. Regroup terms in the subtraction in the way shown:

$$\nabla(v_i v_j - v_{i,h_1} v_{j,h_1}) = \nabla(v_i(v_j - v_{j,h_1})) + \nabla(v_{j,h_1}(v_i - v_{i,h_1})) =$$

$$= \nabla v_i \cdot (v_j - v_{j,h_1}) + v_i \cdot \nabla(v_j - v_{j,h_1}) + \nabla v_{j,h_1} \cdot (v_i - v_{i,h_1})) + v_{j,h_1} \cdot \nabla(v_i - v_{i,h_1}).$$

For all four terms the idea is to use Holder's inequality. For example, for the first one

$$\|\nabla v_i \cdot (v_j - v_{j,h_1})\| \leqslant C\|\nabla \mathbf{v}\|_{L^4(\Omega_1)} \cdot \|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(\Omega_1)}.$$

For all four we obtain, using the triangle inequality for a norm,

$$\|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\| \leqslant C(\|\mathbf{v}\|_{W^{1,4}(\Omega_1)} + \|\mathbf{v}_{h_1}\|_{W^{1,4}(\Omega_1)}) \cdot \|\mathbf{v} - \mathbf{v}_{h_1}\|_{W^{1,4}(\Omega_1)}.$$

What is left to show is boundness of $\|\mathbf{v}_{h_1}\|_{L^4(0,T;W^{1,4}(\Omega_1))}$. The way to show it is using the inverse inequality. Write

$$\mathbf{v}_{h_1} = \mathbf{v}_{h_1} + \mathbf{v} - \mathbf{v} + I_{h_1}(\mathbf{v}) - I_{h_1}(\mathbf{v}).$$

Then

$$\|\mathbf{v}_{h_1}\|_{L^4(0,T;W^{1,4}(\Omega_1))} \leqslant \|\mathbf{v}\|_{L^4(0,T;W^{1,4}(\Omega_1))} + \|\mathbf{v} - I_{h_1}(\mathbf{v})\|_{L^4(0,T;W^{1,4}(\Omega_1))} +$$

$$+ \|\mathbf{v}_{h_1} - I_{h_1}(\mathbf{v})\|_{L^4(0,T;W^{1,4}(\Omega_1))}.$$

The first two terms on the RHS are bounded. For the last one use the inverse estimate

$$\|\mathbf{v}_{h_1} - I_{h_1}(\mathbf{v})\|_{W^{1,4}(\Omega_1)} \leqslant h_1^{-\frac{n}{4}} \|\mathbf{v}_{h_1} - I_{h_1}(\mathbf{v})\|_1.$$

Decompose the last error as

$$\|\mathbf{v}_{h_1} - I_{h_1}(\mathbf{v})\|_1 \leqslant \|\mathbf{v}_{h_1} - \mathbf{v}\|_1 + \|\mathbf{v} - I_{h_1}(\mathbf{v})\|_1$$

The statement of the lemma follows immediately from convergence of these two terms. □

Remark 32. *Let us require in the lemma that* $\mathbf{v} \in L^4(0,T;W^{1,\infty}(\Omega_1) \cap H^{l_1}(\Omega_1) \cap H_0^1(\Omega_1))$ *with* $2 \leqslant l_1 \leqslant m_1$ *and convergence of* $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;H_0^1(\Omega_1))}$ *be no slower than* $O(h_1^{\frac{n}{2}})$*. Also let* $l_1 \geqslant 3$ *if* $n = 3$*. Then we may obtain in the same manner as before that*

$$\|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|_{L^2(0,T;L^2(\Omega_1))} \leqslant C\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;H_0^1(\Omega_1))}$$

for any pair $i, j = \overline{1,n}$*.*

Using the main convergence theorem 11, chapter 3, we can obtain

Theorem 33. *Let the solution* q *satisfy the conditions:* $q, \frac{\partial q}{\partial t} \in L^\infty(H^k(\Omega))$ *and* $\frac{\partial^2 q}{\partial t^2} \in L^2(H^k(\Omega))$ *for some positive integer* k*. Assume the approximating space of continuous piece-wise polynomials* $M_0^m(\Omega)$ *is used and* $m \geqslant k \geqslant 2$*. If the initial conditions are taken so that*

$$\|(q_h - \hat{q})(0,\cdot)\|_{H^1(\Omega)} + \left\|\frac{\partial}{\partial t}(q_h - \hat{q})(0,\cdot)\right\| \leqslant C_1 h^k$$

with some posititve constant C_1 *independent of* h*, where* $\hat{q}$ *means* H^1*-projection of* q*, then the solution* q_h *satisfies:*

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(q - q_h)\right\|_{L^\infty(L^2(\Omega))} \leqslant$$
$$\leqslant C\left(h^k + \|\nabla\cdot\nabla\cdot(\ \mathbb{R} - \mathbb{R}_{h_2})\|_{L^2(L^2(\Omega_1))} + \|\nabla\mathbf{v} : \nabla\mathbf{v}^t - \nabla\mathbf{v}_{h_1} : \nabla\mathbf{v}_{h_1}^t\|_{L^2(L^2(\Omega_1))}\right)$$

with some constant $C > 0$ *independent of* h*.*

Lemma 34. *Let* $\mathbf{v} \in L^4(0,T;W^{1,4}(\Omega_1) \cap H_0^1(\Omega_1))$ *and* $R_{ij} \in L^2(0,T;H^{m_2}(\Omega_1) \cap H_0^1(\Omega_1))$ *for any pair* $i, j = \overline{1,n}$*. Then, if the mesh for* $M_{00}^{m_2}(\Omega_1)$ *is quasi-uniform,*

$$\|\nabla\cdot\nabla\cdot(\mathbb{R} - \mathbb{R}_{h_2})\|_{L^2(L^2(\Omega_1))} \leqslant C(\delta)(h_2^{m_2-2} + h_2^{-1}\|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|_{L^2(L^2(\Omega_1))}).$$

Here $C(\delta)$ *is the same as in Theorem 30, up to some constant positive factor independent of both* δ *and* h_2*.*

Proof. Obviously, $\|\nabla \cdot \nabla \cdot (\mathbb{R} - \mathbb{R}_{h_2})\| \leqslant C \sum_{i,j=1}^{n} \|R_{ij} - R_{ij,h_2}\|_2$. Further, for each pair i, j

$$\|R_{ij} - R_{ij,h_2}\|_2 \leqslant \|R_{ij} - I_{h_2}(R_{ij})\|_2 + \|R_{ij,h_2} - I_{h_2}(R_{ij})\|_2,$$

where I_{h_2} is a piecewise polynomial interpolant into space $M_{00}^{m_2}(\Omega_1)$. Then

$$\|R_{ij} - I_{h_2}(R_{ij})\|_2 \leqslant C h_2^{m_2-2} \|R_{ij}\|_{m_2}.$$

The inverse estimate for the second term gives

$$\|R_{ij,h_2} - I_{h_2}(R_{ij})\|_2 \leqslant h_2^{-1} \|R_{ij,h_2} - I_{h_2}(R_{ij})\|_1,$$

then

$$\|R_{ij,h_2} - I_{h_2}(R_{ij})\|_1 \leqslant \|R_{ij} - I_{h_2}(R_{ij})\|_1 + \|R_{ij} - R_{ij,h_2}\|_1.$$

For the first term in the RHS,

$$\|R_{ij} - I_{h_2}(R_{ij})\|_1 \leqslant C h_2^{m_2-1} \|R_{ij}\|_{m_2}.$$

Using Theorem 30 in order to deal with the second term, we obtain

$$\|R_{ij} - R_{ij,h_2}\|_2 \leqslant C(\delta)(h_2^{m_2-2} + h_2^{-1} \|\nabla(v_i v_j - v_{i,h_1} v_{j,h_1})\|).$$

□

8.2 BARDINA MODEL

In Ω_1 for any $0 < t < T$ we have

$$\begin{cases} \frac{1}{a_0^2}\frac{\partial q^2}{\partial t^2} - \Delta q = \rho_0 \nabla \cdot \nabla \cdot \mathbb{R} + \rho_0 \nabla \mathbf{v} : \nabla \mathbf{v}^t - \rho_0 \nabla \cdot \mathbf{F}, \\ \mathbb{R} - \delta^2 \Delta \mathbb{R} = \delta^2 \Delta(\mathbf{w} \otimes \mathbf{w}) + \mathbf{v} \otimes \mathbf{v} - \mathbf{w} \otimes \mathbf{w}, \\ \mathbf{w} - \delta^2 \Delta \mathbf{w} = \mathbf{v}. \end{cases}$$

The boundary conditions are:

$$\mathbf{w} = 0 \ \forall (t, x) \in (0, T) \times \partial\Omega_1,$$

$$\mathbb{R} = 0 \ \forall (t, x) \in (0, T) \times \partial\Omega_1,$$

$$\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g \ \forall (t, x) \in (0, T) \times \partial\Omega.$$

As in the previous section, we need to make the regularity assumptions first. We are already assuming that $\mathbf{v} \in L^4(0, T; W^{1,4}(\Omega_1) \cap H_0^1(\Omega_1))$. First, we can solve the third equation for each time level t on the mesh of size $h_2 < 1$, then solve the second equation with $\mathbf{w}$ known on the mesh of size $h_3 < 1$ and finally solve the first equation for q on the mesh of size $h < 1$. The RHS of the first equation has the double divergence of $\mathbb{R}$, so in order for the RHS to be of $L^2(0, T; L^2(\Omega_1))$, we can require that $\mathbb{R} \in L^2(0, T; H^2(\Omega_1) \cap H_0^1(\Omega_1))$. This regularity of $\mathbb{R}$ follows from the assumption that $\mathbf{w} \in L^4(0, T; W^{2,4}(\Omega_1) \cap H_0^1(\Omega_1))$.

The variational formulation for the third equation starts with condition $\mathbf{v} \in L^4(0, T; H^{-1}(\Omega_1))$. For all $i = \overline{1, n}$ find $w_i \in L^4(0, T; H_0^1(\Omega_1))$ such that

$$(w_i, \hat{v}) + \delta^2(\nabla w_i, \nabla \hat{v}) = (v_i, \hat{v}) \ \forall \hat{v} \in H_0^1(\Omega_1), 0 < t < T. \tag{8.10}$$

We can easily obtain the stability result by setting $\hat{v} = w_i$. Then

$$\|w_i\|^2 + \delta^2 \|\nabla w_i\|^2 = (v_i, w_i) \leqslant \|v_i\|_{-1} \cdot \|w_i\|_1.$$

There follows

$$\|w_i\|_1 \leqslant max\left(1, \frac{1}{\delta^2}\right) \|v_i\|_{-1}.$$

So the stability result is

$$\|\mathbf{w}\|_{L^4(0,T;H_0^1(\Omega_1))} \leqslant C_1(\delta)\|\mathbf{v}\|_{L^4(0,T;H^{-1}(\Omega_1))},$$

where

$$lim_{\delta\to 0+}C_1(\delta) = +\infty \text{ and } lim_{\delta\to+\infty}C_1(\delta) = C_0 > 0.$$

The Finite Element formulation will be as follows. For all $i = \overline{1,n}$ find such map $w_{i,h_2} : [0,T] \to M_{00}^{m_2}(\Omega_1)$ that

$$(w_{i,h_2}, \hat{v}_{h_2}) + \delta^2(\nabla w_{i,h_2}, \nabla\hat{v}_{h_2}) = (v_{i,h_1}, \hat{v}_{h_2})\ \forall \hat{v}_{h_2} \in M_{00}^{m_2}(\Omega_1), 0 < t < T. \tag{8.11}$$

Theorem 35. *Assume* $\mathbf{v} \in L^4(0,T;H^{-1}(\Omega_1))$ *and* $\mathbf{w} \in L^4(0,T;H^{m_2}(\Omega_1)\cap H_0^1(\Omega_1))$. *Then*

$$\|\mathbf{w}-\mathbf{w}_{h_2}\|_{L^4(0,T;H_0^1(\Omega_1))} \leqslant C(\delta)(h_2^{m_2-1} + \|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;H^{-1}(\Omega_1))})$$

with the same constant $C(\delta)$ *as in Theorem 30, up to some constant positive factor independent of both* δ *and* h_2.

Proof. The technical details resemble those from the proof of Theorem 30 so we omit them here and get straight to inequality

$$\|w_i - w_{i,h_2}\|_1 \leqslant C(\delta)(inf_{\hat{w}\in M_{00}^{m_2}(\Omega_1)}\|w_i - \hat{w}\|_1 + \|v_i - v_{i,h_1}\|_{-1}),$$

and so

$$\|w_i - w_{i,h_2}\|_1 \leqslant C(\delta)(h_2^{m_2-1} + \|v_i - v_{i,h_1}\|_{-1}).$$

Here $C(\delta)$ is such that

$$lim_{\delta\to 0+}C(\delta) = +\infty \text{ and } lim_{\delta\to+\infty}C(\delta) = +\infty.$$

□

Lemma 36. *If* $n = 2$, *then*

$$\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;H^{-1}(\Omega_1))} \leqslant C(p)\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;L^p(\Omega_1))}\ \forall p,\ 1 < p < \infty.$$

If $n = 3$, *then*

$$\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;H^{-1}(\Omega_1))} \leqslant C\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;L^{\frac{6}{5}}(\Omega_1))}.$$

Proof. By definition,

$$\|\mathbf{v}-\mathbf{v}_{h_1}\|_{-1} = sup_{\hat{\mathbf{v}}\in H^1(\Omega_1)} \frac{(\mathbf{v}-\mathbf{v}_{h_1}:\hat{\mathbf{v}})}{\|\hat{\mathbf{v}}\|_1}.$$

The Holder's inequality and then the Sobolev embedding for $n=2$ give

$$(\mathbf{v}-\mathbf{v}_{h_1}:\hat{\mathbf{v}}) \leqslant \|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^p(\Omega_1)} \cdot \|\hat{\mathbf{v}}\|_{L^q(\Omega_1)} \leqslant C\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^p(\Omega_1)} \cdot \|\hat{\mathbf{v}}\|_1,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $1<q<\infty$ strictly. Constant C depends on q, and we can write this dependence in terms of p as $C=C(p)$. Then it's obvous that

$$\|\mathbf{v}-\mathbf{v}_{h_1}\|_{-1} \leqslant C(p)\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^p(\Omega_1)}$$

for any $1<p<\infty$. If $n=3$, then the Sobolev embedding gives

$$\|\hat{\mathbf{v}}\|_{L^q(\Omega_1)} \leqslant C(p)\|\hat{\mathbf{v}}\|_1$$

only for $q\leqslant 6$. Set $q=6$ and obtain the result of the lemma. □

The variational formulation for the second equation will be as follows. Let $\mathbf{v}\in L^4(0,T;L^4(\Omega_1))$ and $\mathbf{w}\in L^4(0,T;W^{1,4}(\Omega_1)\cap H_0^1(\Omega_1))$. For all $i,j=\overline{1,n}$ find $R_{ij}\in L^2(0,T;H_0^1(\Omega_1))$ such that

$$(R_{ij},\hat{v})+\delta^2(\nabla R_{ij},\nabla\hat{v}) = -\delta^2(\nabla(w_iw_j),\nabla\hat{v})+(v_iv_j-w_iw_j,\hat{v}) \tag{8.12}$$

$$\forall \hat{v}\in H_0^1(\Omega_1), 0<t<T.$$

The stability result follows from setting $\hat{v}=R_{ij}$.

$$\|R_{ij}\|^2+\delta^2\|\nabla R_{ij}\|^2 = -\delta^2(\nabla(w_iw_j),\nabla R_{ij})+(v_iv_j-w_iw_j,R_{ij}) \leqslant$$

$$\leqslant \delta^2\|\nabla(w_iw_j)\|\cdot\|\nabla R_{ij}\|+\|v_iv_j-w_iw_j\|_{-1}\cdot\|R_{ij}\|_1.$$

There follows

$$\|R_{ij}\|_1 \leqslant max\left(\delta^2,\frac{1}{\delta^2}\right)\left(\|\nabla(w_iw_j)\|+\|v_iv_j-w_iw_j\|_{-1}\right).$$

Holder's inequality and integration in time help obtain

$$\|R_{ij}\|_{L^2(0,T;H_0^1(\Omega_1))} \leqslant C(\delta)\left(\|\mathbf{w}\|^2_{L^4(0,T;W^{1,4}(\Omega_1))}+\|v_iv_j-w_iw_j\|_{L^2(0,T;H^{-1}(\Omega_1))}\right),$$

where

$$lim_{\delta\to 0+}C_1(\delta) = +\infty \text{ and } lim_{\delta\to +\infty}C_1(\delta) = +\infty.$$

The Finite Element formulation will be as follows. For all $i,j=\overline{1,n}$ find such map R_{ij,h_3} : $[0,T]\to M_{00}^{m_3}(\Omega_1)$ that

$$\begin{gathered}(R_{ij,h_3},\hat{v}_{h_3})+\delta^2(\nabla R_{ij,h_3},\nabla\hat{v}_{h_3}) = -\delta^2(\nabla(w_{i,h_2}w_{j,h_2}),\nabla\hat{v}_{h_3})+\\ +(v_{i,h_1}v_{j,h_1}-w_{i,h_2}w_{j,h_2},\hat{v}_{h_3})\end{gathered} \tag{8.13}$$

$$\forall\hat{v}_{h_3}\in M_{00}^{m_3}(\Omega_1), 0<t<T.$$

Theorem 37. *Assume* $\mathbf{v}\in L^4(0,T;L^4(\Omega_1))$*,* $\mathbf{w}\in L^4(0,T;W^{1,4}(\Omega_1)\cap H_0^1(\Omega_1))$ *and* $R_{ij}\in L^2(0,T;H^{m_3}(\Omega_1)\cap H_0^1(\Omega_1))$*. Then*

$$\|R_{ij}-R_{ij,h_3}\|_{L^2(0,T;H_0^1(\Omega_1))}\leqslant C(\delta)(h_3^{m_3-1}+\|v_iv_j-v_{i,h_1}v_{j,h_1}\|_{L^2(0,T;H^{-1}(\Omega_1))}+$$

$$+\|w_iw_j-w_{i,h_2}w_{j,h_2}\|_{L^2(0,T;H_0^1(\Omega_1))})$$

with the same positive constant $C(\delta)$ *as in Theorem 30, up to some constant positive factor independent of both* δ *and* h_3*.*

Proof. Similarly to the proof of Theorem 30, obtain

$$\|R_{ij}-R_{ij,h_3}\|_1\leqslant C(\delta)(inf_{\hat{R}\in M_{00}^{m_3}(\Omega_1)}\|R_{ij}-\hat{R}\|_1+\|v_iv_j-v_{i,h_1}v_{j,h_1}\|_{-1}+$$

$$+\|w_iw_j-w_{i,h_2}w_{j,h_2}\|_1),$$

or

$$\|R_{ij}-R_{ij,h_3}\|_1\leqslant C(\delta)(h_3^{m_3-1}+\|v_iv_j-v_{i,h_1}v_{j,h_1}\|_{-1}+\|w_iw_j-w_{i,h_2}w_{j,h_2}\|_1).$$

Here $C(\delta)$ is such that

$$lim_{\delta\to 0+}C(\delta) = +\infty \text{ and } lim_{\delta\to +\infty}C(\delta) = +\infty.$$

□

Similar to Lemma 31, we can obtain

Lemma 38. *Let* $\mathbf{w} \in L^4(0,T; W^{1,4}(\Omega_1) \cap H^2(\Omega_1) \cap H_0^1(\Omega_1))$. *Also let the mesh for* $M_{00}^{m_2}(\Omega_1)$, $m_2 \geqslant 2$, *used for computing* $\mathbf{w}_{h_2}$ *be quasi-uniform. If for a fixed* $p > 1$ *the error* $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^p(\Omega_1))}$ *converges no slower than* $O(h_2^{\frac{1}{2}})$ *for* $n = 2$ *or* $\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;L^{\frac{6}{5}}(\Omega_1))}$ *converges no slower than* $O(h_2^{\frac{3}{4}})$ *for* $n = 3$, *then for any pair* $i, j = \overline{1,n}$

$$\|w_i w_j - w_{i,h_2} w_{j,h_2}\|_{L^2(0,T;H_0^1(\Omega_1))} \leqslant C\|\mathbf{w} - \mathbf{w}_{h_2}\|_{L^4(0,T;W^{1,4}(\Omega_1))}$$

Proof. Since the trace of $w_i w_j - w_{i,h_2} w_{j,h_2}$ on $\partial\Omega_1$ is zero, the norms $\|\cdot\|_1$ and $\|\nabla \cdot\|$ are equivalent. Thus the proof of the lemma may be done for the error $\|\nabla(w_i w_j - w_{i,h_2} w_{j,h_2})\|$ instead of $\|w_i w_j - w_{i,h_2} w_{j,h_2}\|_1$.

The idea of the proof resembles that of Lemma 31. It is necessary to require that the rate of convergence of $\|\mathbf{w} - \mathbf{w}_{h_2}\|_{L^4(0,T;H_0^1(\Omega_1))}$ be no lower than $h_2^{\frac{n}{4}}$. Next use Theorem 35 that gives the estimate for $\|\mathbf{w} - \mathbf{w}_{h_2}\|_{L^4(0,T;H_0^1(\Omega_1))}$ in terms of $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;H^{-1}(\Omega_1))}$. This last error may be bounded, due to Lemma 36, by $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^p(\Omega_1))}$ with some $p > 1$ in a two-dimensional case and by $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^{\frac{6}{5}}(\Omega_1))}$ in a three-dimensional case. The statement of Lemma 38 then follows immediately. ◻

Remark 39. *Let us require in the lemma that* $\mathbf{w} \in L^4(0,T; W^{1,\infty}(\Omega_1) \cap H^{l_2}(\Omega_1) \cap H_0^1(\Omega_1))$ *with* $2 \leqslant l_2 \leqslant m_2$. *Let* $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^p(\Omega_1))}$ *converge no slower than* $O(h_2)$ *for some fixed* $p > 1$ *in a two-dimensional case and let* $\|\mathbf{v}-\mathbf{v}_{h_1}\|_{L^4(0,T;L^{\frac{6}{5}}(\Omega_1))}$ *converge no slower than* $O(h_2^{\frac{3}{2}})$ *in a three-dimensional case. Finally, let* $l_1 \geqslant 3$ *if* $n = 3$. *Then we may obtain in the same manner as before that*

$$\|w_i w_j - w_{i,h_2} w_{j,h_2}\|_{L^2(0,T;H_0^1(\Omega_1))} \leqslant C\|\mathbf{w} - \mathbf{w}_{h_2}\|_{L^4(0,T;H_0^1(\Omega_1))}$$

for any pair $i, j = \overline{1,n}$.

Lemma 40. *Let* $\mathbf{v} \in L^4(0,T; H^2(\Omega_1) \cap H_0^1(\Omega_1))$ *and the mesh used for computing* $\mathbf{v}_{h_1}$ *be quasi-uniform. In addition, let* $\mathbf{v} \in L^4(0,T; L^{\frac{2p}{2-p}}(\Omega_1))$ *with some* $1 < p < 2$ *for* $n = 2$, *and* $\mathbf{v} \in L^4(0,T; L^3(\Omega_1))$ *for* $n = 3$. *Then for any pair* $i, j = \overline{1,n}$

$$\|v_i v_j - v_{i,h_1} v_{j,h_1}\|_{L^2(0,T;H^{-1}(\Omega_1))} \leqslant C\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^2(\Omega_1))}$$

with positive constant $C = C(p)$ *in a two-dimensional case.*

Proof. Use triangle inequality

$$\|v_iv_j - v_{i,h_1}v_{j,h_1}\|_{-1} \leqslant \|v_i(v_j - v_{j,h_1})\|_{-1} + \|v_{j,h_1}(v_i - v_{j,h_1})\|_{-1}.$$

For each term the idea is to use Holder's inequality. The example will be shown for the first term. By definition,

$$\|v_i(v_j - v_{j,h_1})\|_{-1} = sup_{\hat{v}\in H^1(\Omega_1)} \frac{(v_i(v_j - v_{j,h_1}), \hat{v})}{\|\hat{v}\|_1}.$$

Then

$$(v_i(v_j - v_{j,h_1}), \hat{v}) \leqslant \|v_i\|_{L^{pr_1}(\Omega_1)} \cdot \|v_j - v_{j,h_1}\|_{L^{pr_2}(\Omega_1)} \cdot \|\hat{v}\|_{L^q(\Omega_1)}. \tag{8.14}$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$. We want q to be as large as possible. For $n = 2$ we require that $1 < q < \infty$. Set $pr_2 = 2$ and use the Sobolev embedding for the term $\hat{v}$ to obtain

$$(v_i(v_j - v_{j,h_1}), \hat{v}) \leqslant C\|\mathbf{v}\|_{L^{\frac{2p}{2-p}}(\Omega_1)} \cdot \|\mathbf{v} - \mathbf{v}_{h_1}\| \cdot \|\hat{v}\|_1.$$

In the end, the result will be

$$\|v_iv_j - v_{i,h_1}v_{j,h_1}\|_{-1} \leqslant C\left(\|\mathbf{v}\|_{L^{\frac{2p}{2-p}}(\Omega_1)} + \|\mathbf{v}_{h_1}\|_{L^{\frac{2p}{2-p}}(\Omega_1)}\right) \cdot \|\mathbf{v} - \mathbf{v}_{h_1}\|.$$

The way to show boundness of $\|\mathbf{v}_{h_1}\|_{L^{\frac{2p}{2-p}}(\Omega_1)}$ is similar to that in case of $\|\mathbf{v}_{h_1}\|_{W^{1,4}(\Omega_1)}$ from Lemma 31. Applying the inverse inequality consequently requires that $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^2(\Omega_1))}$ converge no slower than $O\left(h_1^{\frac{2(p-1)}{p}}\right)$, where p is fixed and $1 < p < 2$, and that is automatilcally guaranteed since the space $M_{00}^{m_1}(\Omega_1)$ with $m_1 \geqslant 2$ is used for computing the velocity field $\mathbf{v}_{h_1}$.

If $n = 3$, then proceeding with (8.14) we can only require that $q = 6$ in order to use the Sobolev embedding. Then $p = \frac{6}{5}$ and after setting $r_2 = \frac{5}{3}$ we end up with

$$\|v_iv_j - v_{i,h_1}v_{j,h_1}\|_{-1} \leqslant C\left(\|\mathbf{v}\|_{L^3(\Omega_1)} + \|\mathbf{v}_{h_1}\|_{L^3(\Omega_1)}\right) \cdot \|\mathbf{v} - \mathbf{v}_{h_1}\|.$$

Again, the boundness of $\|\mathbf{v}_{h_1}\|_{L^4(0,T;L^3(\Omega_1))}$ requires that $\|\mathbf{v} - \mathbf{v}_{h_1}\|_{L^4(0,T;L^2(\Omega_1))}$ converge no slower than $O\left(h_1^{\frac{1}{2}}\right)$, and that is satisfied. The lemma is proven. □

We present the following lemma without a proof, due to its resemblence to the proof of Lemma 34.

Lemma 41. *Let* $\mathbf{v} \in L^4(0,T;L^4(\Omega_1))$, $\mathbf{w} \in L^4(0,T;W^{1,4}(\Omega_1) \cap H_0^1(\Omega_1))$ *and* $R_{ij} \in L^2(0,T;H^{m_3}(\Omega_1) \cap H_0^1(\Omega_1))$ *for any pair* $i,j = \overline{1,n}$. *Then*

$$\|\nabla \cdot \nabla \cdot (\mathbb{R} - \mathbb{R}_{h_3})\|_{L^2(L^2(\Omega_1))} \leqslant$$

$$\leqslant C(\delta)(h_3^{m_3-2} + h_3^{-1}\|v_i v_j - v_{i,h_1} v_{j,h_1}\|_{L^2(H^{-1}(\Omega_1))} + h_3^{-1}\|w_i w_j - w_{i,h_2} w_{j,h_2}\|_{L^2(H^1(\Omega_1))})$$

with the same positive constant $C(\delta)$ *as in Theorem 30, up to some constant positive factor independent of both* δ *and* h_3.

8.3 NUMERICAL EXPERIMENT FOR TWO MODELS

The purpose of this section is to provide plots of a simulation using the filtered Lighthill analogy with both zeroth order van Cittert deconvolution model and Bardina subgrid scale model. The conditions of the experiment are the following.

The domains used are the same as in section 7.4. It is assumed there are no external forces acting on the flow in Ω_1. The filtered non-dimensionalized NSE have a form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \overline{p} - \frac{1}{Re}\Delta \mathbf{v} - \nabla \cdot (2\nu_T \mathbf{D}(\mathbf{v})) = 0, \tag{8.15}$$

$$\nabla \cdot \mathbf{v} = 0,$$

where the Reynolds number is taken as $Re = 16.2$, $\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ and ν_T is the turbulent viscosity coefficient. The mesh in Ω_1 is the same as it was in 7.3, i.e. it is uniform and $h_1 \approx 0.028$. Thus set the filter cutoff width $\delta = 0.028$. Let ν_T be a constant. Practically, this trivial model is not used in real applications, [35], since it reduces turbulent flows to a laminar one, but we are using it in this section due to simplicity for the purpose of acoustic simulation. Following [35], choose $\nu_T = \delta^2 = 0.000784$. The non-dimensionalized filtered wave equation is

$$M^2 \frac{\partial^2 q}{\partial t^2} - \Delta q = \nabla \mathbf{v} : \nabla \mathbf{v}^t + \nabla \cdot \nabla \cdot \mathbb{R}$$

and is given with the boundary condition on $\partial\Omega$

$$M\frac{\partial q}{\partial t} + \nabla q \cdot \mathbf{n} = 0.$$

The Mach number $M = 0.075$. As in 7.4, the initial condition for the filtered velocity is

$$\mathbf{v}_0 = 36.73 \cdot (0.33 - r) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{r}.$$

The Finite Element used for the filtered NSE is Taylor-Hood element. For time integration from $t = 0$ to $t = 0.6$ the same Stabilized Extrapolated Backward Euler is used as in 7.4, with the time step $\Delta t = 0.005$.

After each time step, the Poisson problem (8.9) is solved for van Cittert model or two Poisson problems (8.11) and (8.13) are solved consequently with obtained $\mathbf{v}_{h_1}$ as an input data. The pictures with plots of components of tensor $\mathbb{R}_{h_3}$ are presented below for both models. For the Bardina model, the plot of the twice filtered velocity $\mathbf{w}_{h_2}$ is presented.

For the wave equation we use the same second order in time scheme in Ω presented in chapter 4, starting with homogeneous initial conditions. Also, piecewise quadratics are used on the uniform mesh of size $h \approx 0.028$. Pictures of the filtered pressure fluctuation are presented for the final time level $t = 0.6$, for both models.

Additionally, pictures 30-32 show the subgrid scale tensor for van Cittert model with the same collection of values of isolines as for Bardina model from pictures 25-27. The empty spaces are those not included in the interval of values for the case of Bardina model.

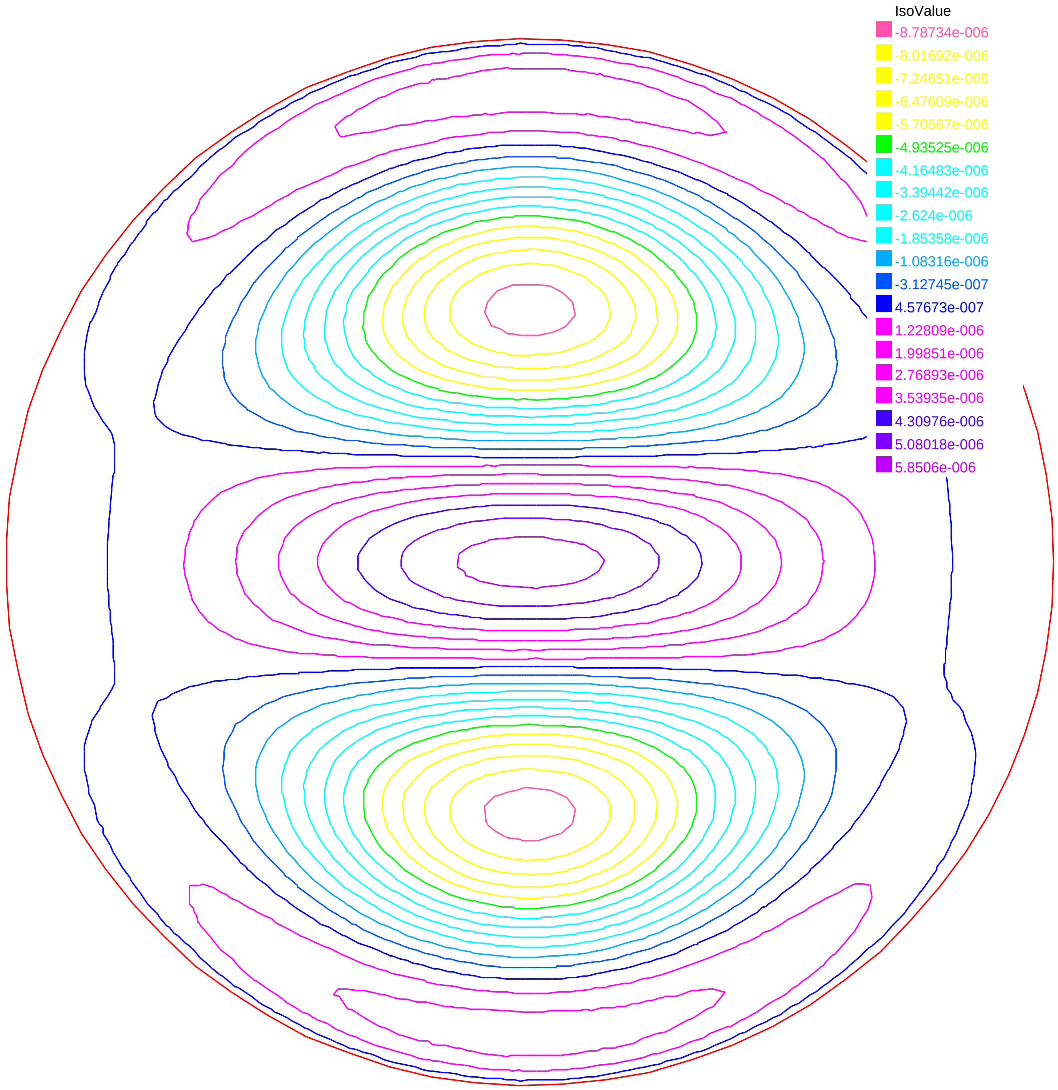

Figure 21: The subgrid scale tensor component R_{11} in Ω_1 at time $t = 0.6$, van Cittert

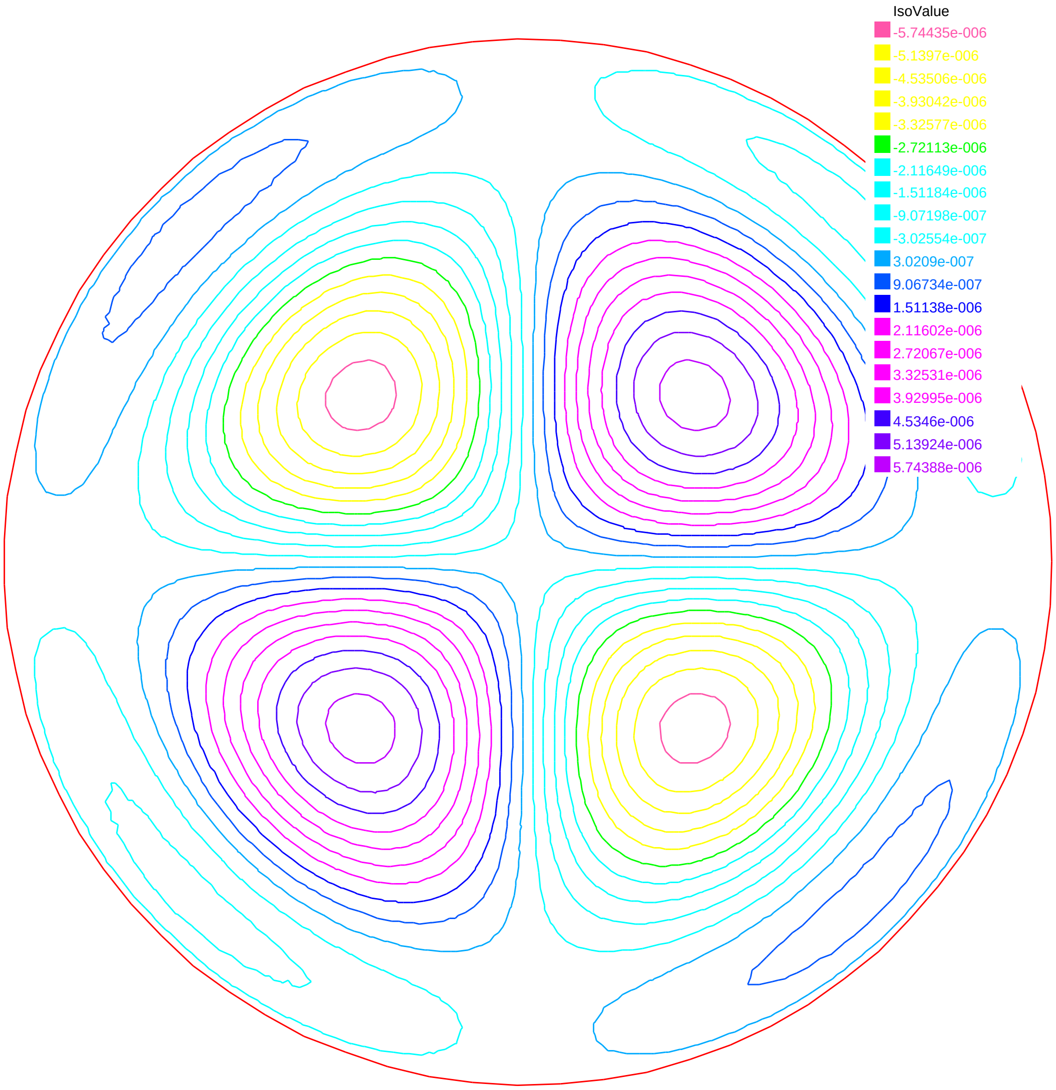

Figure 22: The subgrid scale tensor component $R_{12} = R_{21}$ in Ω_1 at time $t = 0.6$, van Cittert

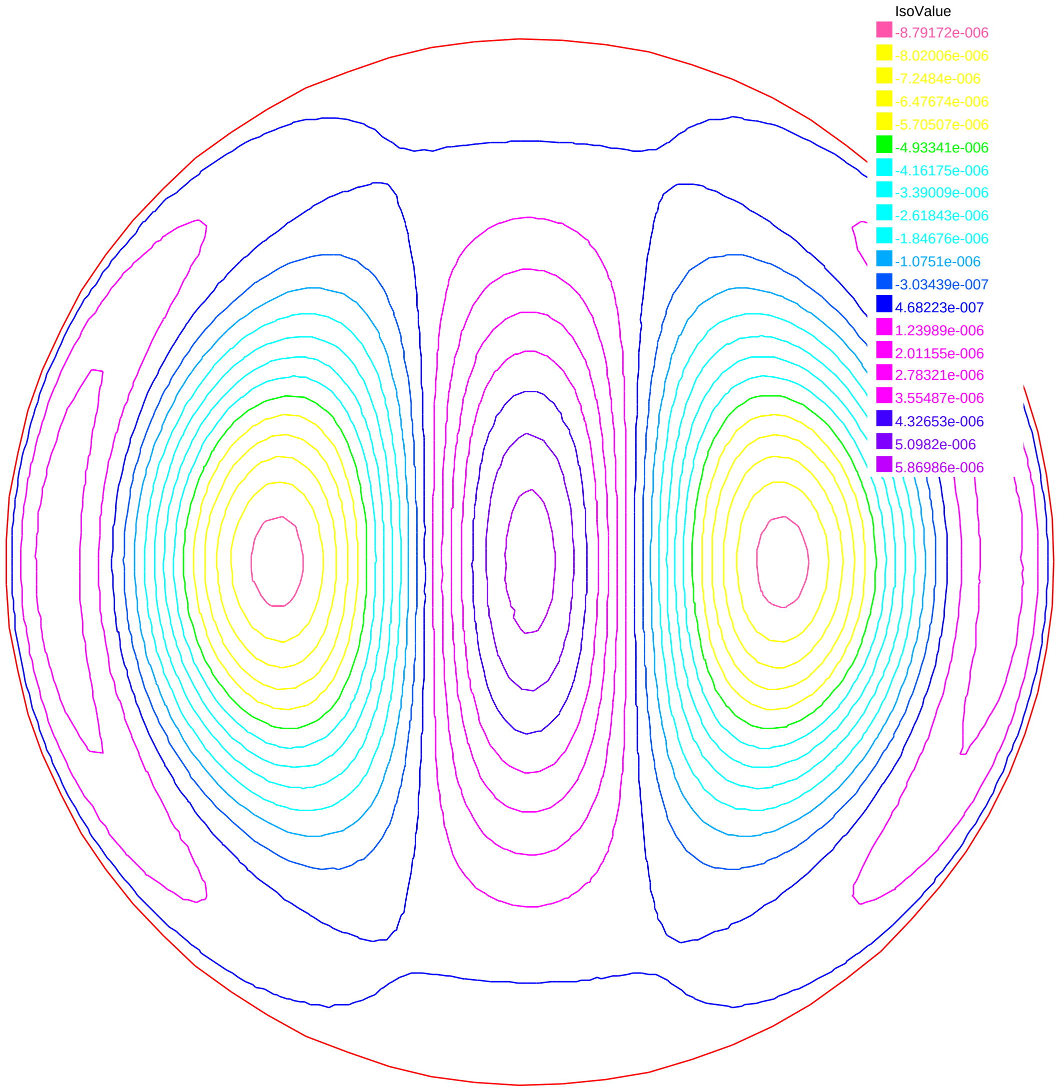

Figure 23: The subgrid scale tensor component R_{22} in Ω_1 at time $t = 0.6$, van Cittert

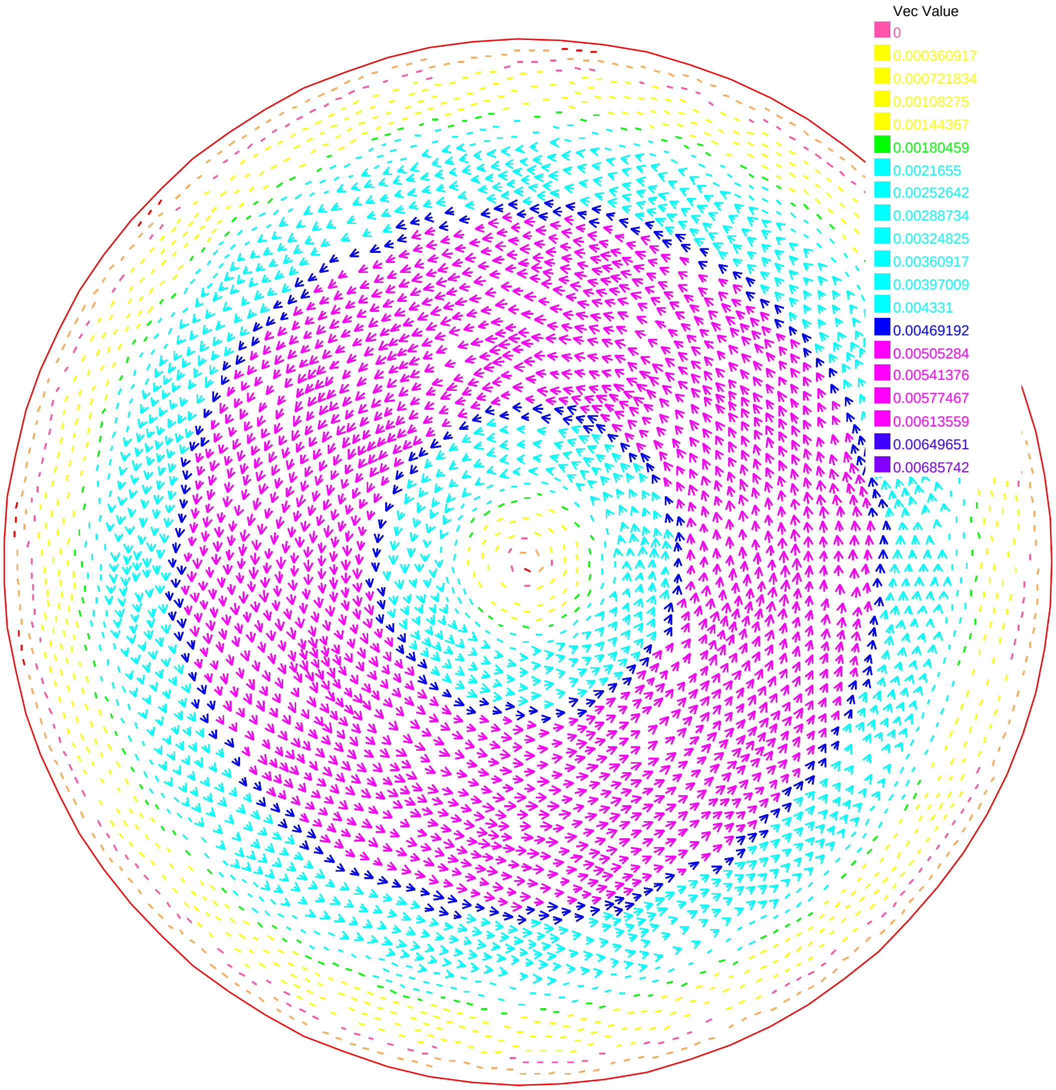

Figure 24: The twice filtered velocity $\mathbf{w}$ in Ω_1 at time $t = 0.6$

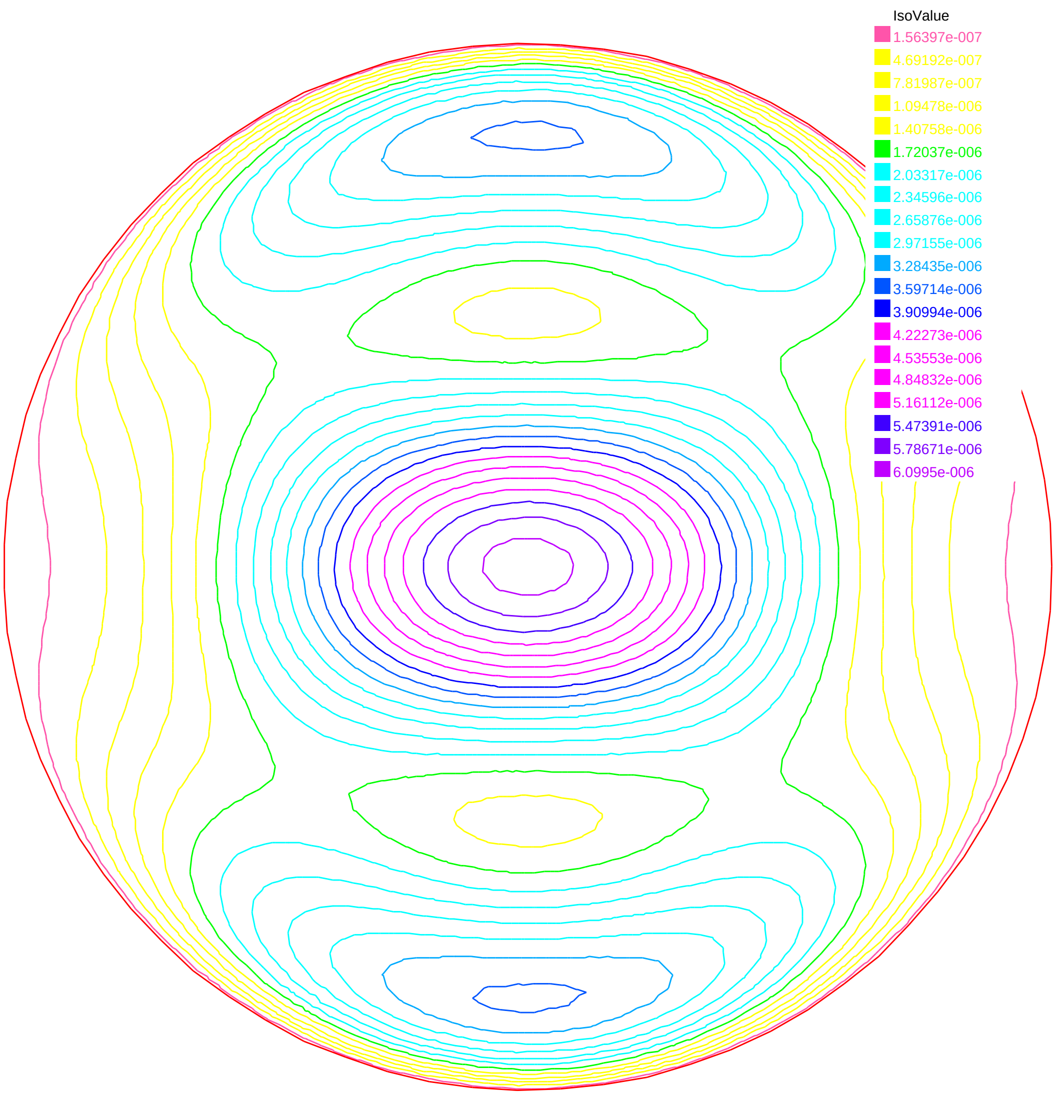

Figure 25: The subgrid scale tensor component R_{11} in Ω_1 at time $t = 0.6$, Bardina

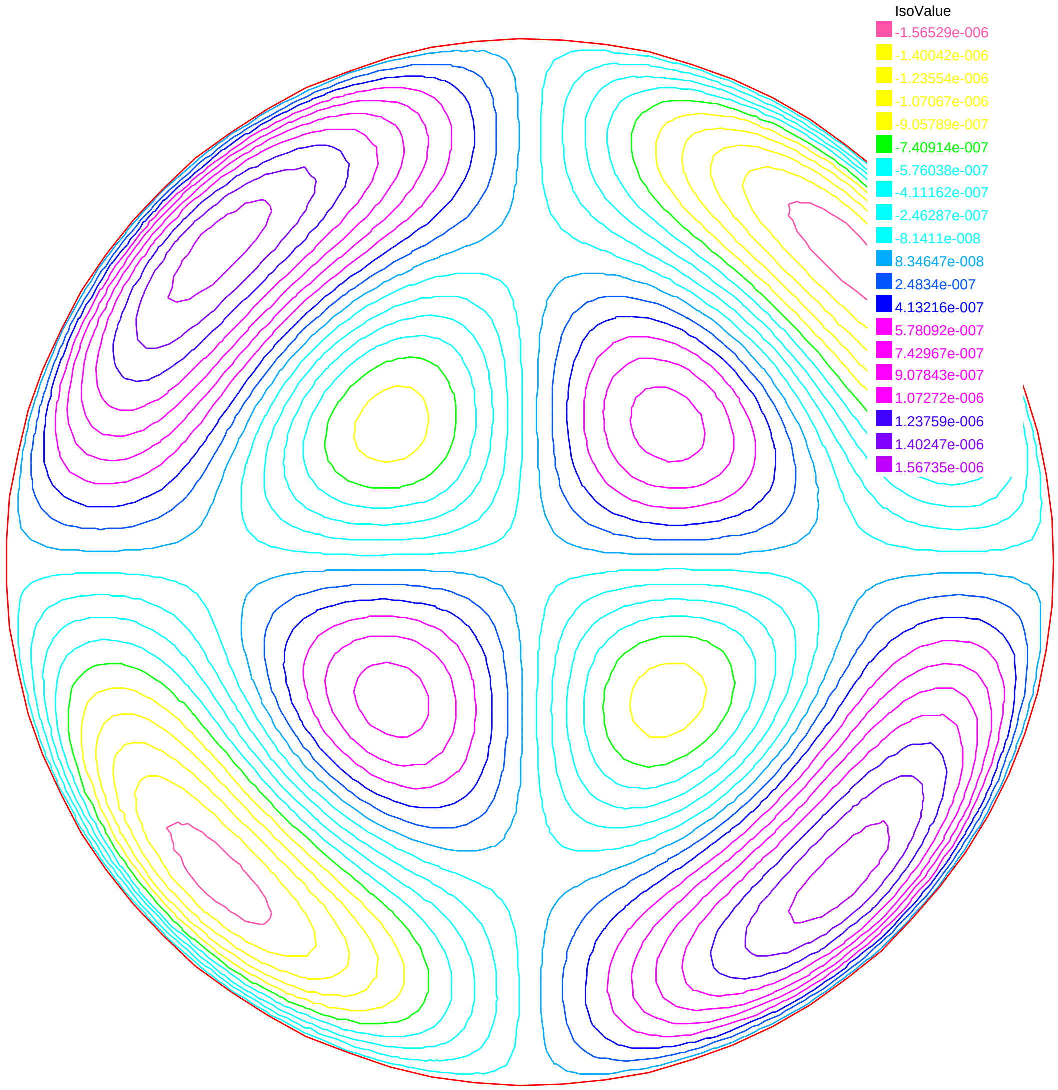

Figure 26: The subgrid scale tensor component $R_{12} = R_{21}$ in Ω_1 at time $t = 0.6$, Bardina

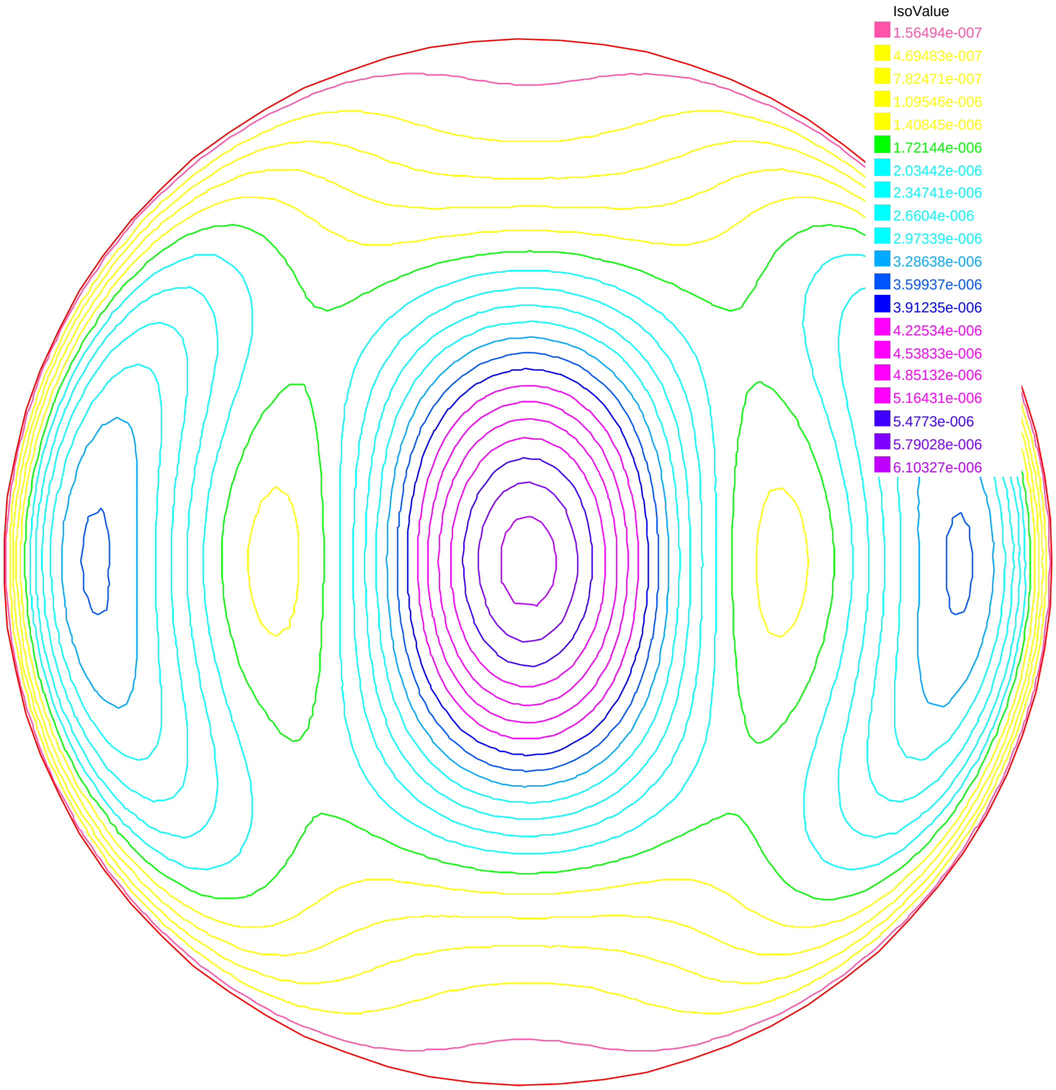

Figure 27: The subgrid scale tensor component R_{22} in Ω_1 at time $t = 0.6$, Bardina

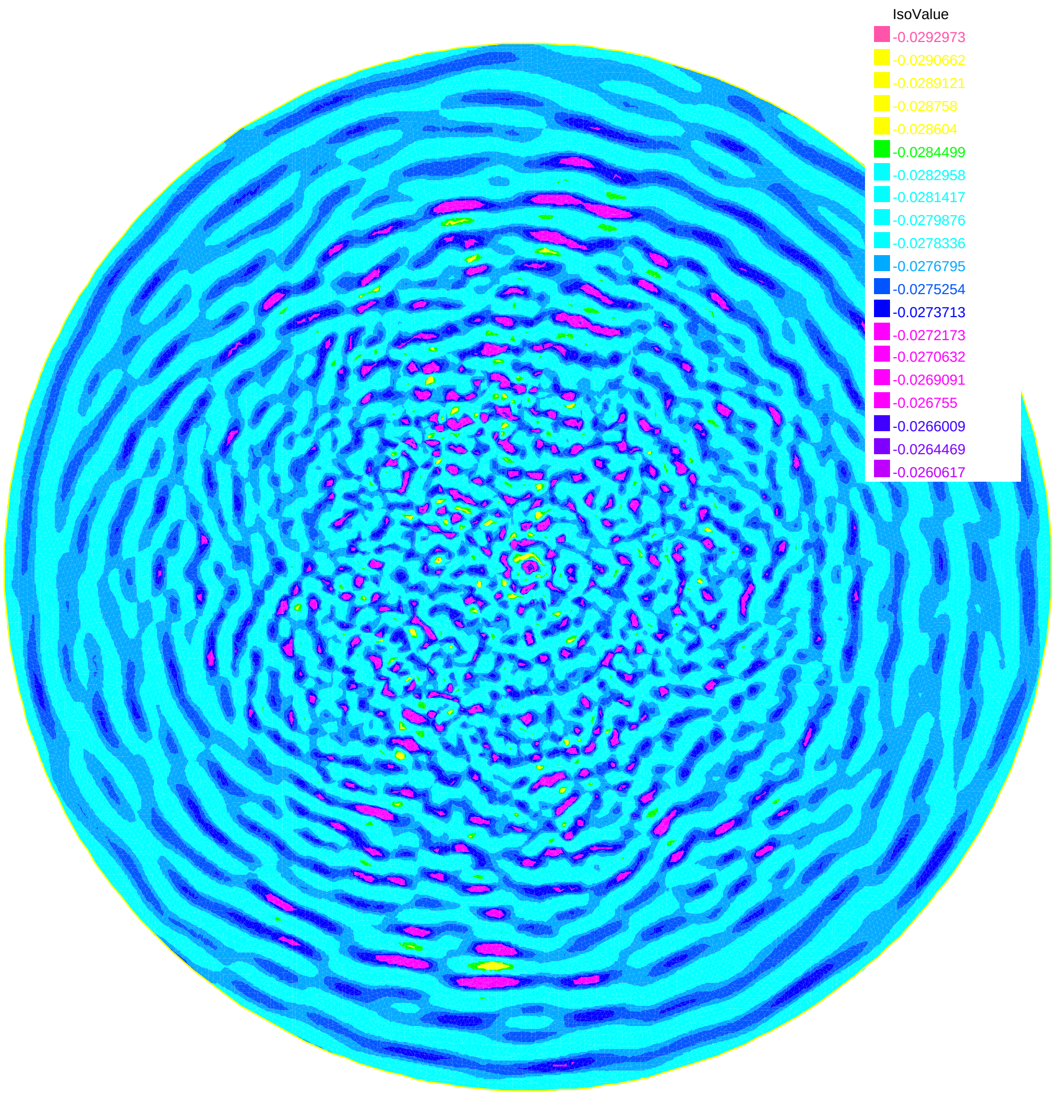

Figure 28: The filtered acoustic pressure field in Ω at time $t = 0.6$, van Cittert

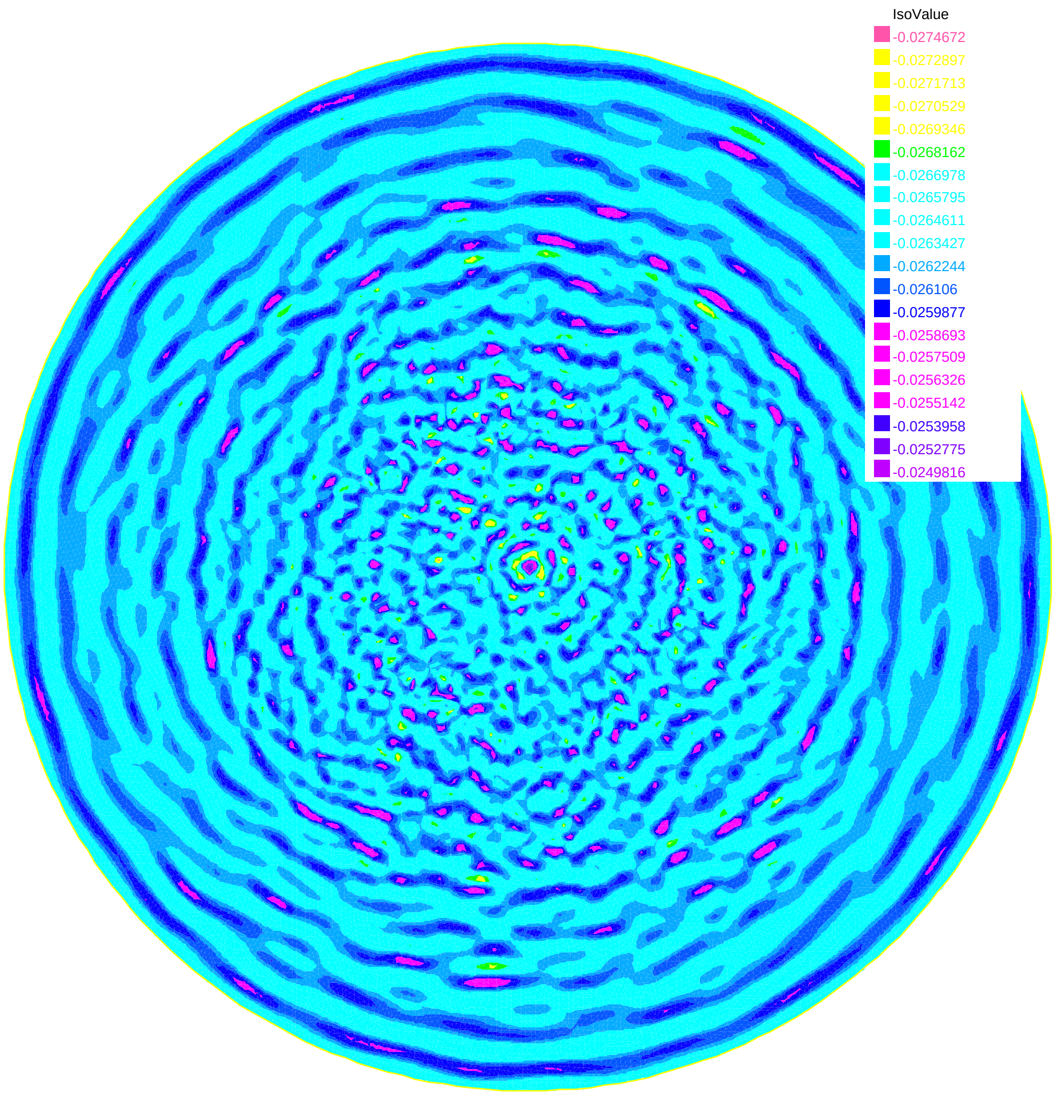

Figure 29: The filtered acoustic pressure field in Ω at time $t = 0.6$, Bardina

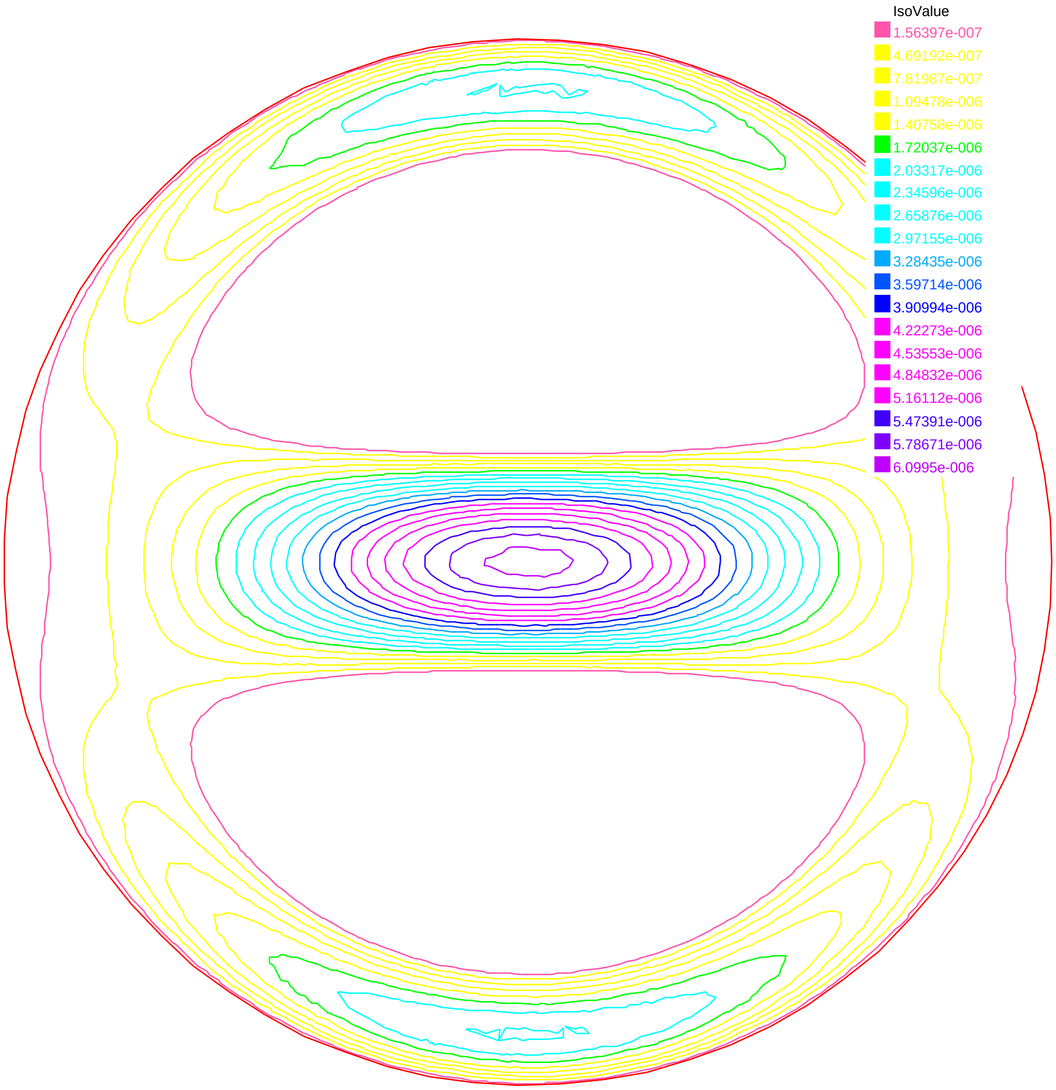

Figure 30: The subgrid scale tensor component R_{11} in Ω_1 at time $t = 0.6$, van Cittert

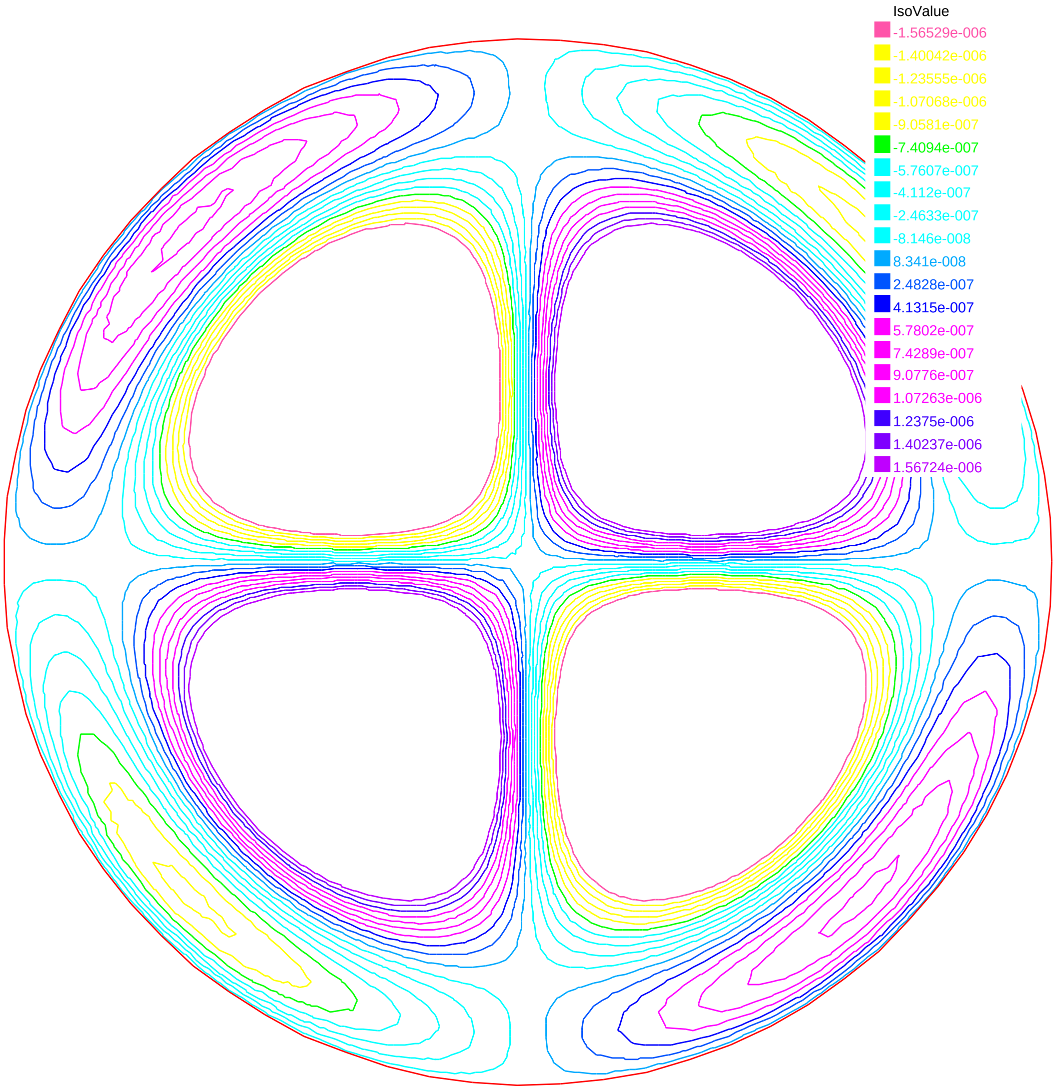

Figure 31: The subgrid scale tensor component R_{12} in Ω_1 at time $t = 0.6$, van Cittert

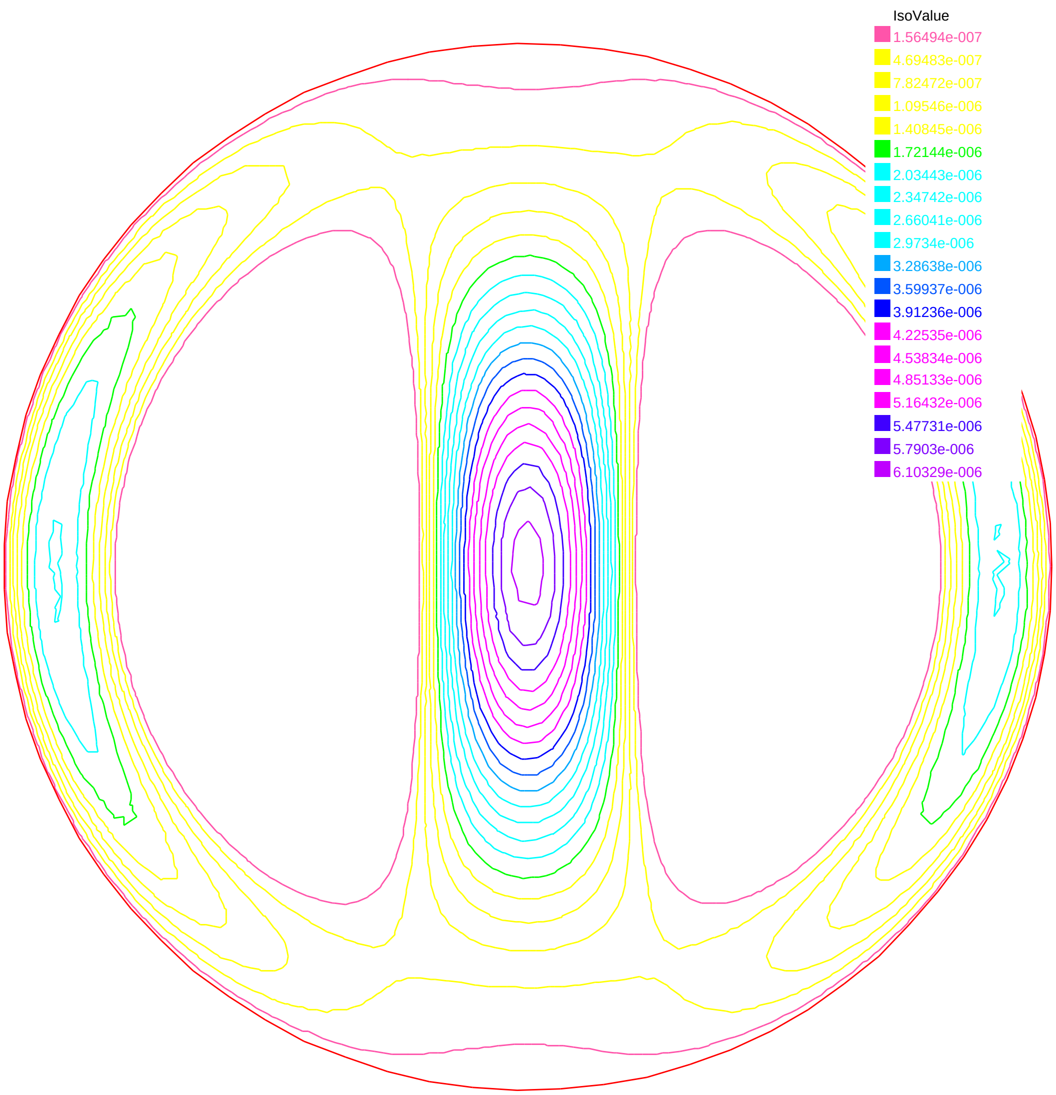

Figure 32: The subgrid scale tensor component R_{22} in Ω_1 at time $t = 0.6$, van Cittert

9.0 CONCLUSION AND FUTURE PROSPECTS

The main results of this thesis may be summarized as follows.

- The semidiscrete scheme for the Lighthill analogy was introduced and analyzed in case of the Direct Numeical Simulation. The stability and convergence were proven for the case of regular enough source term.
- Analogous result was shown for the fully discrete scheme.
- The numerical results agreed with the mentioned theoretical predictions.
- Analysis of the negative Sobolev norm of the solution error was performed for the semidiscrete scheme.
- The semidiscrete scheme for computing the fluctuation of velocity was presented and analyzed.
- Two methods for computation of the sound power were introduced and fully analyzed for the semidiscrete case. Both methods were supported by numerical expertiments.
- A method for bounding the sound power using only the fluctuation of pressure was given and analyzed.
- The Large Eddy Simulation for the Lighthill analogy was presented as two models, these are the zeroth order van Cittert deconvolution model and Bardina model. Both were analyzed and checked numerically.

The presented work gave the rigorous numerical analysis of noise generation in the most trivial case, that is turbulence driven by given forces in infinite space with no walls. Future work is to be directed toward solving more practical problems. These include the noise generated by jet planes, single wind turbines and complex wind farms and blood flows. In general, the boundary conditions for the fluctuations of pressure $p^{'}$, equation (1.1), are different for these problems and involve such phenomena as absorbtion and reflection. More on the engineering aspects of the wind turbine noise may be found at [64].

The obvious consequent problem is the noise control. Assuming the driving forces and the boundary conditions involve control functions, we need to present criterions for desired broadband noise levels. This problem requires results of the general control theory for the wave equation.

Also, the Large Eddy Simulation offers other subgrid scale models for the noise research. These could be higher order van Cittert deconvolution models or Lius subgrid scale model, [39], given by the equation

$$\mathbb{R} = 0.45 \cdot (\widetilde{\mathbf{v} \otimes \mathbf{v}} - \widetilde{\mathbf{v}} \otimes \widetilde{\mathbf{v}}),$$

where $\sim$ denotes a filter at scale 2δ. According to [59], Lius model recovers a little better results than Bardina model.

9.1 THE RESEARCH OF THE NOISE GENERATION IN THE NON-INERTIAL FRAMES

The research of the non-inertial effects in the noise generation, caused by motion of the frame of reference, represents a particular interest in the aerodynamic noise research. This research was aslo pioneered by Lighthill, [36]. This may be used, for example, in the research of the noise generated by wind turbines, where we may consider the rotating blades as a moving frame of reference. If the frame of reference is moving with non-zero acceleration, then the NSE in that frame have a form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho_0} \nabla p + 2[\zeta, \mathbf{u}] + [\zeta, [\zeta, \mathbf{r}]] + [\epsilon, \mathbf{r}] = \mathbf{g} - \mathbf{a},$$

where ζ is the angular velocity of the reference frame, ϵ is the angular acceleration and $\mathbf{r}$ is the radius-vector from the origin of the frame. $\mathbf{a}$ is the acceleration of the origin of the frame of reference and $\mathbf{g}$ is the gravity acceleration. The last three terms on the LHS are the Coriolis force, the centrifugal force and the Euler force respectively. The non-dimensionalization of this equation gives, beside the Reynolds number Re and the Froude number Fr, the Rossby number Ro, given by the formula

$$Ro = \frac{U}{2\zeta L},$$

where U is the characteristic velocity and L is the characteristic length of the flow domain. The general purpose is to answer the following question: if the velocity of the flow is being computed in the non-inertial frame, for what range of Ro can we neglect the fictitious forces when evaulating the sound power arising from the solution of the Lighthill analogy? In other words, if the sound powers for two cases, one containing the inertial forces and the other not, are not significantly different, then the velocity field may be sought from the equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho_0} \nabla p = \mathbf{g},$$

which is a great simplification of the previous model.

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