TOPOLOGICAL ALGEBRAIC STRUCTURE IN THE DENSITY TOPOLOGY AND ON SOUSLIN LINES

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This research investigates which topological algebraic structures can exist on two types of topological spaces: the real line $\mathbb{R}$ with the density topology; and any linearly ordered topological space (LOTS) satisfying the countable chain condition (CCC) that is not separable (i.e. any Souslin Line). Some surprising results are established in the density topology when considering the common group operations on $\mathbb{R}$. Indeed, this research shows that addition and multiplication are not topological group operations in this space. These theorems are then generalized to show that there are no topological group operations on $\mathbb{R}$ with the density topology. The case of cancellative topological semigroups, however, is left as an open question.

On the other hand, the conditions of existence of topological algebraic structures on Souslin lines is rather completely determined by this work. The main results in this space are that paratopological groups do not exist on any Souslin line, but cancellative topological semigroups do exist. The research on this space culminates with the construction of a cancellative topological semigroup on a Souslin line.

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In many science and engineering departments, professors and graduate students have a symbiotic relationship. The student receives invaluable training and mentoring while the professor receives a valued worker in the laboratory. Indeed, the student performs many tasks that the professor does not want to spend time on herself such as setting up equipment, writing computer code, collecting data, etc. There have been, in fact, cases in which a professor delayed a student's graduation so that she wouldn't lose the student's services. In pure mathematics, however, the relationship is much more of a one-way street. The professor gives a great deal to the student and receives little, if anything, in return. The situation is even more one-sided when the professor is Dr. Robert Heath. He is exceptionally generous, especially with his time and ideas. I owe him a great deal, and I thank him.

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1.0 INTRODUCTION

1.1 DEFINITIONS AND NOTATION

We begin with some, but not all, of the basic definitions of terms used throughout the dissertation. Many of these definitions will be familiar to mathematicians, especially topologists, but they are stated here for reference and completeness. Additional definitions will be provided in subsequent sections as they are needed.

In what follows, the nine basic axioms of Zermelo-Fraenkel Set Theory will be assumed as will the Axiom of Choice. This will be denoted **ZFC.** Familiarity with Lebesgue measure will also be assumed, and **measurable** will mean Lebesgue measurable as defined in [26]. The Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$ will be denoted m_1 and m_2, respectively.

Definition 1. *Given any set X a **topology** τ on X is defined to be a family of subsets $\tau = \{T_\alpha \subseteq X : \alpha < \kappa\}$ such that the following three conditions are satisfied. These subsets are said to be **open.***

1. Any union of elements of τ is in τ.
2. The intersection of any finite number of elements of τ is in τ.
3. X and $\emptyset$ are in τ.

Definition 2. *A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X.*

For brevity we will denote a topological space simply as X when the topology is clear from context or when a statement is independent of the topology.

Definition 3. *A subset $A \subseteq X$ of X is **closed** if its complement $X \backslash A$ is open.*

Definition 4. *A set is **clopen** if it is both closed and open. For example, the entire set, X, and the empty set $\emptyset$ are both clopen.*

Definition 5. *The **closure** of a set A, denoted $\overline{A}$, is the intersection of all closed sets containing it.*

Definition 6. *A map $f : X \longrightarrow Y$ between topological spaces is **continuous** if for each open subset U of Y the inverse image $f^{-1}(U)$ is open in X.*

Definition 7. *The **symmetric difference** between two sets A and B, denoted $A \triangle B$, is the set of all points in one and only one of the sets. i.e. $A \triangle B = \{x : x \in A, x \notin B\} \cup \{x : x \notin A, x \in B\} = (A \cup B) \setminus (A \cap B)$.*

Definition 8. *A topological space X is **Hausdorff** if, for any two points $x_1, x_2 \in X$, there exist disjoint open sets U_1 and U_2 containing x_1 and x_2, respectively.*

Definition 9. *A topological space X is **regular** if, for any point $x \in X$ and for any closed subset $C \subseteq X$ with $x \notin C$, the singleton $\{x\}$ is closed and there exists disjoint open sets U_1 and U_2 with $x \in U_1$ and $C \subseteq U_2$. The requirement that singletons be closed is sometimes omitted from the definition, and a topological space is said to have property T_3 if it is regular and singletons are closed. The distinction is irrelevant in this dissertation since all singletons are closed in the topologies being considered.*

Definition 10. *A topological space X is **completely regular** if, for any point $x \in X$ and for any closed subset $C \subseteq X$ with $x \notin C$, the singleton $\{x\}$ is closed and there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(C) = \{1\}$.*

Definition 11. *A topological space X is **normal** if, for each pair of disjoint closed sets $C \subseteq X$ and $D \subseteq X$, there exist disjoint open sets U_1 and U_2 with $C \subseteq U_1$ and $D \subseteq U_2$.*

Definition 12. *A map $f : X \longrightarrow Y$ is **injective** if for all $a, b \in X$, $a \neq b$ implies $f(a) \neq f(b)$. The map is **surjective** if every element of Y is the image of some element of X. The map is **bijective** if it is both injective and surjective.*

Definition 13. *A bijection $f : X \longrightarrow Y$ between topological spaces is a **homeomorphism** if both f and the inverse map $f^{-1} : Y \longrightarrow X$ are continuous.*

Definition 14. *A set X is **countable** if there is a bijective map from a subset of the natural numbers onto X.*

Definition 15. *An **open covering** of a topological space X is a collection of open subsets of X such that every point of X is contained in at least one of the subsets.*

Definition 16. *A topological space X is **compact** if every open covering has a finite sub-collection that covers X.*

Definition 17. *A G_δ **set** in a topological space X is a set equal to a countable intersection of open subsets of X.*

Definition 18. *Let X be a topological space, let A be a subset of X, and let $\mathcal{U}$ be a collection of subsets of X. The **star of** U **about** A, denoted $st(A,\mathcal{U})$, is the set $st(A,\mathcal{U}) = \cup\{U \in \mathcal{U} \colon A \cap U \neq \emptyset\}$. If A is a singleton (i.e. $A = \{x\}$), $st(\{x\},\mathcal{U})$ is denoted $st(x,\mathcal{U})$ for simplicity.*

Definition 19. *In a topological space (X, τ) a collection $\mathfrak{B}$ of elements of τ is a **base** of τ if, for all $T_\alpha \in \tau, T_\alpha$ is a union of elements of $\mathfrak{B}$.*

Definition 20. *A **local base** at a point $x \in X$ is a collection of open sets containing x such that any open set containing x has a subset in the collection.*

Definition 21. *A topological space X is **first countable** if at each $x \in X$ there is a countable local base.*

Definition 22. *A **Hamel Basis** is an algebraic basis for the vector space of real numbers over the field of rationals.*

Definition 23. *A binary relation, $\trianglelefteq$, on a set X is **reflexive** if for all a in X, $a \trianglelefteq a$.*

Definition 24. *A binary relation, $\trianglelefteq$, on a set X is **antisymmetric** if for all a and b in X, $a \trianglelefteq b$ and $b \trianglelefteq a$ implies $a = b$.*

Definition 25. *A binary relation, $\trianglelefteq$, on a set X is **transitive** if for all a, b, and c in X, $a \trianglelefteq b$ and $b \trianglelefteq c$ implies $a \trianglelefteq c$.*

Definition 26. *A binary relation, $\trianglelefteq$, on a set X is a **partial ordering** if it is reflexive, antisymmetric, and transitive. In general, there can exist a and b in X such that neither*

$a \trianglelefteq b$ nor $b \trianglelefteq a$.

Definition 27. *A binary relation $\trianglelefteq$ on a set X is **total** if for all a and b in X, $a \trianglelefteq b$ or $b \trianglelefteq a$.*

Definition 28. *A **total ordering,** or **linear ordering,** on a set X is a binary relation that is reflexive, antisymmetric, transitive, and total.*

Definition 29. *A linearly ordered topological space (**LOTS**) is a linearly ordered space for which open intervals are a base for the topology i.e. $\tau = \{(a, b) : a, b \in X\}$ where $(a, b) = \{x : a < x < b\}$.*

Definition 30. *A **well ordering** on a set X is a total ordering with the property that every non-empty subset of X has a least element.*

Definition 31. *A **tree** is a partially ordered set T that has a least element such that, for each $a \in T$, the set $\{b \in T : b \trianglelefteq a\}$ is well ordered.*

Definition 32. *A subset $A \subseteq X$ of a topological space X is a **dense subset** if $\overline{A} = X$.*

Definition 33. *A linearly ordered set $(X, \trianglelefteq)$ is **order dense** if for all $a, b \in X$ with $a < b$ there exists $c \in X$ such that $a < c < b$.*

Definition 34. *A subset $A \subseteq \mathbb{R}$ of the real numbers is a **c-dense subset** if for every non-empty open interval $U \subseteq \mathbb{R}$ the cardinality of $A \cap U$ is the cardinality of the continuum, $\mathfrak{c}$.*

Definition 35. *A subset A of a topological space X is **nowhere dense** if the complement of its closure, $X \setminus \overline{A}$, is a dense subset of X.*

Definition 36. *A topological space X satisfies the **Countable Chain Condition (CCC)** if every collection of pairwise disjoint open sets is countable.*

Definition 37. *A topological space X is **separable** if it has a countable dense subset.*

Definition 38. *A topological space X is **connected** if there does not exist a pair of disjoint nonempty open subsets of X whose union is X.*

Definition 39. *A **separation** of a topological space X is a pair of disjoint nonempty open subsets of X whose union is X. So, a space is connected if a separation does not exist.*

Definition 40. *A LOTS is **complete** if every nonempty subset with an upper bound has a least upper bound (supremum) and every nonempty subset with a lower bound has a greatest lower bound (infimum).*

Definition 41. *Two linearly ordered sets $(X, \trianglelefteq)$ and $(Y, \prec)$ are **isomorphic** if there exists a bijective map $f : X \longrightarrow Y$ such that for all $x_1, x_2 \in X$, $x_1 \trianglelefteq x_2$ if and only if $f(x_1) \prec f(x_2)$. The map, f, is called an **isomorphism**.*

Definition 42. *A **binary operation** $*$ is an operation on two variables in a nonempty set X (i.e. $* : X \times X \longrightarrow X$) that is defined for all pairs of elements in X and such that for all $a, b \in X$ the product $a * b$ is a unique element of X.*

Definition 43. *A **semigroup** is a pair $(X, *)$ where X is a nonempty set and $*$ is a binary operation such that the associative law holds (i.e. for all $x, y, z \in X$, $x * y \in X$ and $x * (y * z) = (x * y) * z$).*

Definition 44. *A **group** is a semigroup that contains an identity element e and an inverse element for every element of X (i.e. there exists $e \in X$ such that for every $x \in X$, $x * e = e * x = x$ and for all $x \in X$, there exists $x^{-1} \in X$ such that $x * x^{-1} = x^{-1} * x = e$).*

Definition 45. *For any Lebesgue measurable subset $E \subseteq \mathbb{R}$ and for any point $x \in \mathbb{R}$ the **density** of E at x is given by $D(E, x) = \lim\limits_{h \longrightarrow 0+} \frac{m_1(E \cap (x-h, x+h))}{2h}$.*

Definition 46. *A real number $x \in \mathbb{R}$ is a **density point** of a subset E of $\mathbb{R}$ if $D(E, x) = 1$, and x is a **dispersion point** of E if $D(E, x) = 0$. Equivalently, x is a dispersion point of E if and only if x is a density point of the complement of E. The set of all density points of E is denoted by $\Phi(E) = \{x \in \mathbb{R} : D(E, x) = 1\}$.*

Definition 47. *The set of natural numbers is denoted by $\boldsymbol{\omega}$ or $\boldsymbol{\omega}_0$ or $\mathbb{N}$. The set of rational numbers is denoted by $\mathbb{Q}$, and the set of real numbers is denoted by $\mathbb{R}$.*

Definition 48. *The first uncountable ordinal number will be denoted by $\boldsymbol{\omega}_1$.*

Definition 49. *A **Souslin line** is a totally ordered set L such that, in the order topology, L satisfies the CCC but is not separable.*

1.2 THE DENSITY TOPOLOGY

The density topology grew from the study of approximately continuous functions, which were defined by Denjoy [17] in 1916. A point $x \in \mathbb{R}$ is a **density point** of a measurable subset $E \subseteq \mathbb{R}$ if and only if the density $D(E, x) = \lim_{h \longrightarrow 0+} \frac{m_1(E \cap (x-h, x+h))}{2h} = 1$. A real valued function, $f(t)$, is said to be **approximately continuous** at a point $x \in \mathbb{R}$ if, for any $\epsilon > 0$, x is a density point of $\{t \in \mathbb{R} : f(t) \in (f(x) - \epsilon, f(x) + \epsilon)\}$. Many years later, Haupt and Pauc [18] showed that approximately continuous functions are those functions that are continuous when the range has the standard topology (given by unions of open intervals) and the domain has the topology $\tau_D = \{E \subseteq \mathbb{R} : E$ is measurable and for all $x \in E$, $D(E, x) = 1\}$, which they named the **density topology**. All of the standard open sets (unions of open intervals) are open in the density topology as are many sets that are not open in the standard topology. For example, the set of points that remain after all rational numbers are removed from any open interval is an open set in the density topology so that $(a, b) \setminus \mathbb{Q} \in \tau_D$. The density topology is, therefore, finer than the standard topology.

The study of approximately continuous functions and the density topology continued with the work of Goffman and Waterman [19], but the density topology soon began a life of its own. Today, the topological properties of $(\mathbb{R}, \tau_D)$ are well known. They are described by Tall [21] who first began to consider the density topology from a topological point of view. Consider the following known results:

Theorem 50 (Goffman and Waterman)**.** *$(\mathbb{R}, \tau_D)$ is connected. See [19].*

Theorem 51 (Goffman, Neugebauer, and Nishiura)**.** *$(\mathbb{R}, \tau_D)$ is completely regular, but it is not normal. See [20].*

Theorem 52 (Tall)**.** *$(\mathbb{R}, \tau_D)$ satisfies the CCC. See [21].*

Theorem 53 (Tall)**.** *$(\mathbb{R}, \tau_D)$ is neither separable nor first countable. See [21].*

Theorem 54 (Lebesgue Density Theorem)**.** *The measure of the symmetric difference between a set of positive measure and its set of density points is zero. In other words, for any measurable set $E \subset \mathbb{R}$, $m_1(E \triangle \Phi(E)) = 0$ where $\Phi(E)$ is the set of all density points of E. See [24] and [25].*

In higher dimensions there is more than one way to define both density and the density topology. For example, in $\mathbb{R}^2$ a point (x, y) is an **ordinary density point** of a measurable subset $E \subseteq \mathbb{R}^2$ if and only if the ordinary density

$$D_o(E,(x,y)) = \lim_{h \longrightarrow 0+} \frac{m_2(E \cap ((x-h, x+h) \times (y-h, y+h)))}{4h^2} = 1$$

and a **strong density point** if and only if the strong density

$$D_s(E,(x,y)) = \lim_{h \longrightarrow 0+,\ k \longrightarrow 0+} \frac{m_2(E \cap ((x-h, x+h) \times (y-k, y+k)))}{4hk} = 1$$

The **ordinary density topology** (denoted τ_o), then, consists of the set of measurable subsets of $\mathbb{R}^2$ such that every point in the subset is an ordinary density point, and the **strong density topology** (denoted τ_s) defined similarly with every point in the subset a strong density point. These definitions can, of course, be extended to higher dimensions. While the present work is mainly concerned with the one dimensional case, an interesting comparison will be made later between continuous functions of two variables with domain $(\mathbb{R}, \tau_D) \times (\mathbb{R}, \tau_D)$ and continuous funtions with domain $(\mathbb{R}^2, \tau_s)$ or $(\mathbb{R}^2, \tau_o)$.

1.3 SOUSLIN LINES

Any complete, separable, order dense LOTS with no first or last point is "the same" as the real numbers. That is to say, the space of real numbers, with their natural ordering $(\mathbb{R}, <)$, is isomorphic to any complete, separable, order dense, linearly ordered set with no first or last point. This is a well known fact. Indeed, for any other such LOTS $(S, \lhd)$ with a countable dense subset P there is an isomorphism from P to the rational numbers $\mathbb{Q}$. Thus, $(S, \lhd)$ is isomorphic to $(\mathbb{R}, <)$ by the uniqueness of the completion. In 1920, in the first volume of *Fundamenta Mathematicae [14],* Russian mathematician Mikhail Yakovlevich Souslin (1894-1919), whose name is also spelled Suslin in the literature, considered whether an isomorphism would still exist if "separable" was replaced by "CCC". Specifically, Souslin asked

> Un ensemble ordonné (linéairement) sans saut ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?

Paraphrasing, Souslin was asking whether the ordering of the real numbers is completely characterized by a LOTS that is complete, order dense, satisfies the CCC and does not have a first or last point. The hypothesis that separable could be replaced by CCC became known as Souslin's Hypothesis (SH). Any LOTS that is isomorphic to $(\mathbb{R}, <)$ must be separable since the isomorphism would preserve the countable dense set. Another way, then, of looking at Souslin's question is to ask whether a complete, order dense, LOTS without endpoints and satisfying the CCC must be separable. A complete order dense LOTS without endpoints can be constructed from any order dense LOTS by taking an open interval of the completion. Furthermore, if a LOTS satisfies the CCC and is not separable, it can be made to be order dense by defining an equivalence relation on any closed subinterval that is separable. Collapsing each of these subintervals to a point will make an order dense LOTS satisfying the CCC that is not separable. Therefore, the existence of a LOTS satisfying the CCC that is not separable would imply that SH is false. This is the definition of a Souslin line, and Souslin's Hypothesis is that there are no Souslin lines.

So, do Souslin lines exist? More than 45 years after Souslin first asked the question the surprising answer was found independently by Thomas Jech [15] and S. Tennenbaum [16]. The answer, however, wasn't a simple "yes" or "no". Jech and Tennenbaum proved that the existence of a Souslin line can be neither proven nor disproven using the standard axioms of set theory (ZFC). Additional axioms are needed, and a Souslin line can be proven to exist or not exist depending on which axioms are chosen. Indeed, Ronald Jensen proved the existence of Souslin lines assuming Gödel's axiom of contructibility (all sets are constructible). Gödel's axiom of contructibility implies $\diamondsuit$, and $\diamondsuit$ implies both the Continuum Hypothesis and the existence of Souslin lines. On the other hand, Martin's Axiom (MA) along with the negation of the Continuum Hypothesis ($\lceil$CH) implies that Souslin lines do not exist. In the following sections Souslin lines are assumed to exist.

1.4 TOPOLOGICAL GROUPS AND SEMIGROUPS

The following sections will be concerned with whether or not certain topological algebraic structures can exist on $(\mathbb{R}, \tau_D)$ and on Souslin lines. First, some definitions are required.

Definition 55. *A* ***topological group*** *is a triple* $(X, \tau, *)$ *such that* (X, τ) *is a topological space and* $(X, *)$ *is a group and in which both the binary operation* $* : (X, \tau) \times (X, \tau) \longrightarrow (X, \tau)$ *and the inverse function* $i(x) = x^{-1}$ *are continuous where the domain of* $*$ *is* $X \times X$*, and it has the product topology.*

Definition 56. *A* ***paratopological group*** *is a triple* $(X, \tau, *)$ *such that* (X, τ) *is a topological space and* $(X, *)$ *is a group and in which the binary operation* $* : (X, \tau) \times (X, \tau) \longrightarrow (X, \tau)$ *is continuous where the domain has the product topology. Note that the inverse function does not need to be continuous.*

Definition 57. *A* ***topological semigroup*** *is a triple* $(X, \tau, *)$ *such that* (X, τ) *is a topological space and* $(X, *)$ *is a semigroup and in which the binary operation* $* : (X, \tau) \times (X, \tau) \longrightarrow (X, \tau)$ *is continuous where the domain has the product topology.*

Definition 58. *A semigroup or topological semigroup is* ***cancellative*** *if for all* x, y, $z \in X$, $x * y = x * z$ *implies* $y = z$*, and* $y * x = z * x$ *also implies* $y = z$*.*

The existence of these structures on $(\mathbb{R}, \tau_D)$ and on Souslin lines will be analyzed. Obviously, groups and semigroups exist on $\mathbb{R}$ and other linearly ordered sets. The question, then, boils down to the one of continuity of the binary operation $*$ in the topology being studied. A study of the properties of binary operations is, therefore, a good place to start.

Let $*$ be a binary operation on a topological space, X, and let $a \in X$ be fixed. **Right translation** by a is a map $\phi_a : X \longrightarrow X$ defined for all $x \in X$ by $\phi_a(x) = x * a$, and **left translation** by a is a map ${}_a\phi : X \longrightarrow X$ defined for all $x \in X$ by ${}_a\phi(x) = a * x$. We will just say **translation** when a statement applies to both left and right translation.

Additionally, suppose the topological space has a total ordering denoted by $<$. Right translation by a will be called **order preserving** if for all $x, y \in X$ with $x < y$ translation gives $\phi_a(x) < \phi_a(y)$, and it will be called **order reversing** if $\phi_a(x) > \phi_a(y)$ for all $x < y$. If right translation is order preserving (reversing) for all $a \in X$ then right translation will be called order preserving (reversing) without any mention of the element a. The same holds, of course, for left translation, and the operation $*$ will be called order preserving (reversing) if both left and right translation are order preserving (reversing) for all $a \in X$.

We will use some well known results that follow directly from the definitions.

Theorem 59. *Let $*$ be a continuous binary operation on a topological space, X. For any $a \in X$, translation by a is continuous.*

Proof. Let $W \subset X$ be an arbitrary open set. We will show $\phi_a^{-1}(W)$ is open by showing for all $x \in \phi_a^{-1}(W)$ there exists an open set V with $x \in V$ such that $V \subset \phi_a^{-1}(W)$. Let $\phi\colon X \times X \longrightarrow X$ denote the map such that for all $x, y \in X$, $\phi(x, y) = x * y$. For any $x \in \phi_a^{-1}(W)$, $(x, a) \in \phi^{-1}(W)$. The continuity of ϕ implies $\phi^{-1}(W)$ is open. So, there exists open sets U and V with $a \in U$ and $x \in V$ such that $V \times U \subset \phi^{-1}(W)$. Now, for any $v \in V$, $(v, a) \in \phi^{-1}(W)$ so $v \in \phi_a^{-1}(W)$. Thus, $V \subset \phi_a^{-1}(W)$. Therefore, ϕ_a is continuous. A similar argument shows ${}_a\phi$ is continuous. $\square$

Although translation is easily shown to be a homeomorphism for topological groups and paratopological groups, the result that we'll be using is that translation maps open sets to open sets.

Theorem 60. *For any topological group or paratopological group, translation is an open map, i.e. translation maps open sets to open sets.*

Proof. For any open set $U \subset X$, $\phi_a(U) = \phi_{a^{-1}}^{-1}(U)$ is open since translation by a^{-1} is continuous. Therefore, right translation is an open map, and the same argument holds for left translation. $\square$

Additional useful properties of the binary operation depend on the topology. Let's start with the density topology on the real line.

2.0 TOPOLOGICAL GROUPS AND SEMIGROUPS ON $(\mathbb{R}, \tau_D)$

2.0.1 Addition

For any group or semigroup $(\mathbb{R}, *)$ on $\mathbb{R}$ to be a topological group or semigroup with respect to the density topology τ_D requires the operator to be a continuous map from $(\mathbb{R}, \tau_D) \times (\mathbb{R}, \tau_D)$ into $(\mathbb{R}, \tau_D)$. The existence of topological groups and semigroups in the density topology is mainly a question of continuity of binary operations on $(\mathbb{R}, \tau_D)$. The topology on the domain of the binary operation is, by definition, the product topology, which has not yet been studied much. Continuous transformations from $\mathbb{R}^2$ into $\mathbb{R}$ have been studied, however, in the case in which the domain has the "two dimensional" strong or ordinary density topology in lieu of the product topology, and the range has the density topology. Those continuous transformations are call **strongly density continuous** and **ordinary density continuous**, respectively. In [22] Ciesielski and Wilczynski show that the set of strongly density continuous functions is a proper subset of the set of ordinary density continuous functions. Furthermore, if a subset of $\mathbb{R}^2$ is open in the product topology then it is open in the strong density topology. So, it is easy to see that any continuous binary operation is strongly density continuous. Is the set of continuous binary operations equal to the set of strongly density continuous functions or is the containment proper?

In [22] Ciesielski and Wilczynski provide an important criterion for strong density continuous transformations. The criterion requires some definitions.

Definition 61. *For any open interval $U \subseteq \mathbb{R}$ the function $f : U \longrightarrow \mathbb{R}$ is **bi-Lipschitz** if there exists a constant $L \geq 1$ such that for every $a, b \in U$ $L^{-1} |a - b| \leq |f(a) - f(b)| \leq L |a - b|$ where $|\cdot|$ denotes the absolute value.*

Definition 62. *A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is **locally bi-Lipschitz** if for every $p \in \mathbb{R}^2$ there* *exists an open rectangle* $U = (a,b) \times (c,d)$ *with* $p \in U$ *and there exists a constant* $L \geq 1$ *such that for every* $x_o \in (a,b)$ *and* $y_o \in (c,d)$ *the coordinate functions* $g_{y_o}(x) = f(x, y_o)$ *and* $h_{x_o}(y) = f(x_o, y)$ *are bi-Lipschitz with constant* L.

In [22] Ciesielski and Wilczynski prove the following theorem.

Theorem 63 (Ciesielski and Wilczynski)**.** *If* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is locally bi-Lipschitz, then* f *is strongly density continuous.*

Now, consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by adding the components so that $f(x,y) = x + y$. In this case, both of the coordinate functions $g_{y_o}(x) = f(x, y_o) = x + y_o$ and $h_{x_o}(y) = f(x_o, y) = x_o + y$ are bi-Lipschitz with constant $L = 1$. The function f is locally bi-Lipschitz, and, by the theorem, f is strongly density continuous. Does this mean that addition is continuous as a binary operation? The answer turns out to be "no", and the explanation comes by way of a variation on a theorem of Hugo Steinhaus, which appeared in the first volume of *Fundamenta Mathematicae* (coincidently, the same volume as Souslin's problem). In [23] Steinhaus proves the following theorem.

Theorem 64 (Steinhaus)**.** *L'ensemble des distances* D *de deux ensembles* A *et* B *de mesures positives contient au moins un intervalle entier.*

Paraphrasing, Steinhaus proved that for any two sets A and B of positive measure, the set of distances between points contains an entire interval. In other words, if $C = \{|a - b| : a \in A,\ b \in B\}$ then C contains an interval. The variation on this theorem of Steinhaus that applies to the current question, is the following. (See, for example, Halmos [27], Chapter III, Section 16, Theorem B.)

Theorem 65. *Let* E *be a measurable subset of* $\mathbb{R}$ *with* $m_1(E) > 0$. *Then there exists* $\delta > 0$ *such that the interval*

$$(-\delta, \delta) \subseteq E - E := \{x - y : x, y \in E\}.$$

Theorem 66. *Addition is not a continuous binary operation on* $(\mathbb{R}, \tau_D)$, *and, therefore,* $(\mathbb{R}, \tau_D, +)$ *is not a topological group, nor even a topological semigroup.*

Proof. Suppose, in order to get a contradiction, that addition is a continuous binary operation. Fix an arbitrary irrational $a \in \mathbb{R}$, and note that $a + a$ is also irrational. Fix $\epsilon > 0$.

Define the open neighborhood V of $a+a$ in τ_D by

$$V := \{x \in \mathbb{R} : a + a - \epsilon < x < a + a + \epsilon \text{ and } x \notin \mathbb{Q}\}.$$

From above, $(x, y) \longrightarrow x + y$ is continuous at (a, a), from the τ_D product topology on $\mathbb{R}^2$ to the τ_D topology on $\mathbb{R}$. Thus, there exist τ_D open neighborhoods U_1 and U_2 of a such that

$$U_1 + U_2 := \{x + y : x \in U_1 \text{ and } y \in U_2\} \subseteq V.$$

Consider the open neighborhood U of a in τ_D given by $U := U_1 \cap U_2$. Clearly, $U + U \subseteq V$. Next, define $W := U - a = U + (-a)$. By our assumed continuity of addition, W is a τ_D open neighborhood of 0. Also, $U = a + W$. Now, from the definition of the topology τ_D, it is easy to check that $-W := \{-x : x \in W\}$ is another τ_D open neighborhood of 0. Consequently, $H := W \cap (-W)$ is a τ_D open neighborhood of 0 such that

$$-H = H \text{ and } (a + H) + (a + H) \subseteq V.$$

Hence,

$$\begin{aligned}(a + H) + (a + H) &= (a + H) + (a - H)\\ &= \{a + h_1 + a - h_2 : h_1, h_2 \in H\}\\ &= a + a + (H - H).\end{aligned}$$

Since $H \in \tau_D$, H is measurable and $m_1(H) > 0$. Thus, by the variation of Steinhaus' theorem, there exists an interval of the real numbers $(-\delta, \delta)$ with

$$a + a + (-\delta, \delta) \subseteq a + a + (H - H) = (a + H) + (a + H) \subseteq V.$$

However, $V \cap \mathbb{Q} = \emptyset$, and so we have reached a contradiction. Therefore, addition cannot be a continuous binary operation. $\square$

2.0.2 Multiplication

The set of continuous binary operations is, therefore, a proper subset of the set of strong density continuous functions, but the set is not empty. Continuous binary operations do exist in the form of constant functions. For example, for all $x, y \in \mathbb{R}$, let $x * y = 5$. This is a rather uninteresting continuous binary operation. Interesting continuous binary operations would at least be cancellative, but do any of these exist? What about multiplication? Is multiplication a continuous binary operation?

At first blush, multiplication might seem to follow easily by using logarithms and the result for addition. However, this requires the logarithm to be a density continuous function, and showing that functions are continuous in the density topology is frequently more difficult than expected. So, a multiplicative analog of Steinhaus's theorem results in an easier path. Consider only the positive real numbers denoted $\mathbb{R}^+$.

Theorem 67. *Any bounded subset $E \subset \mathbb{R}^+$ of positive measure contains 2 distinct points with a rational quotient.*

Proof. Suppose for contradiction there exists a bounded subset of positive measure such that no two distinct points of E have a rational quotient. For any $w \in \mathbb{R}^+$ define $w \times E = \{w \times x : x \in E\}$ where $\times$ is standard multiplication. Under the supposition the family of sets given by $\left(1 + \frac{1}{n}\right) \times E$ with $n \in \mathbb{N}$ is pairwise disjoint. To see this, suppose the sets $\left(1 + \frac{1}{h}\right) \times E$ and $\left(1 + \frac{1}{k}\right) \times E$ have a point α in common where $h, k \in \mathbb{N}$ and $h \neq k$. $\alpha \in \left(1 + \frac{1}{h}\right) \times E \Longrightarrow \left(\frac{h}{h+1}\right) \times \alpha \in E \Longrightarrow \beta = \left(1 + \frac{1}{k}\right) \times \left(\frac{h}{h+1}\right) \times \alpha \in \left(1 + \frac{1}{k}\right) \times E$. Thus, α and β are both in $\left(1 + \frac{1}{k}\right) \times E$ and $\frac{\beta}{\alpha} = \frac{h}{k} \times \frac{k+1}{h+1} \in \mathbb{Q}$. Let $a = \frac{k}{k+1} \times \alpha$ and $b = \frac{k}{k+1} \times \beta$. Now, a and b are distinct points in E and their quotient $\frac{b}{a} = \frac{\beta}{\alpha}$ is rational, which contradicts the supposition. So, the family of sets $\left(1 + \frac{1}{n}\right) \times E$ with $n \in \mathbb{N}$ is pairwise disjoint.

Let $V = \cup_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) \times E$.

$$
\begin{aligned}
m_1(V) &= \sum_{k=1}^{\infty} m_1\left(\left(1+\frac{1}{k}\right)\times E\right) \text{ since the sets are pairwise disjoint} \\
&= \sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)\times m_1(E) \\
&= m_1(E)\times\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right) \\
&= +\infty
\end{aligned}
$$

However, E bounded implies that there exists y such that $E \subset (0, y]$, and, therefore, $V \subset (0, 2 \times y]$. Thus, $m_1(V) \leq 2 \times y$, which is of course finite. This is a contradiction. Therefore, there exist 2 distinct points of E with a rational quotient. □

The preceeding theorem can be used to show that multiplication is also not a continuous binary operation. We thank Chris Lennard for showing us a proof that for any τ_D open neighborhood U of 1 contained in the interval $(1/a, a)$, U^{-1} is also a τ_D open neighborhood of 1. With his permission, we include this argument in our proof of Theorem 68.

Theorem 68. *Multiplication is not a continuous binary operation on* $(\mathbb{R}^+, \tau_D)$*, and, therefore,* $(\mathbb{R}^+, \tau_D, \times)$ *is not a topological group nor a topological semigroup.*

Proof. Suppose, in order to get a contradiction, that multiplication is a continuous binary operation. Fix $\epsilon \in (0, 1)$. Define the open neighborhood C of 1 in τ_D by

$$C := \{x \in \mathbb{R} : 1 - \epsilon < x < 1 + \epsilon \text{ and } x \notin \mathbb{Q}\} \cup \{1\}.$$

Note that 1 is the only rational element of C. From above, $(x, y) \longrightarrow x \times y$ is continuous at $(1, 1)$, from the τ_D product topology on $\mathbb{R}^2$ to the τ_D topology on $\mathbb{R}$. Thus, there exist τ_D open neighborhoods U_1 and U_2 of 1 such that

$$U_1 \times U_2 := \{x \times y : x \in U_1 \text{ and } y \in U_2\} \subseteq C.$$

Fix $a \in \mathbb{R}$ with $a > 1$. Then the interval $(1/a, a)$ is a τ_D open neighborhood of 1. Consider the open neighborhood U of 1 in τ_D given by $U := U_1 \cap U_2 \cap (1/a, a)$. Clearly, $U \times U \subseteq C$.

We claim that $U^{-1} := \{x^{-1} = 1/x : x \in U\}$ also belongs to τ_D. To see this, first note that the function $\phi : x \longrightarrow 1/x : (I, \tau_u) \longrightarrow (I, \tau_u)$ is a homeomorphism, where $I := [1/a, a]$ and τ_u is the usual topology on $\mathbb{R}$. Our function ϕ is also *absolutely continuous and monotone.* Now, $U \subseteq I$ and $U \in \tau_D$. In particular, U is Lebesgue-measurable. By (for example) [26], Chapter 3, Section 3, Proposition 15(v), page 63, there exists an F_σ set Y and a disjoint set of Lebesgue measure zero, Z, such that $U = Y \cup Z$. Thus,

$$U^{-1} = \phi(U) = \phi(Y) \cup \phi(Z).$$

Since ϕ is a homeomorphism, in the sense discussed above, it follows that $\phi(Y)$ is also an F_σ set. Moreover, by (for example) [26], Chapter 5, Section 4, Problem 18, page 111, $\phi(Z)$ has Lebesgue-measure zero. Hence, U^{-1} is a Lebesgue measurable subset of the interval $(1/a, a)$.

To prove that $U^{-1} \in \tau_D$, it remains to show that for all $z \in U^{-1}$, $D(U^{-1}, z) = 1$. Fix an arbitrary $z \in U^{-1}$. Then $x := 1/z \in U$. We wish to show that

$$\lim_{h \longrightarrow 0+} \frac{m_1\left(U^{-1} \cap (z-h, z+h)\right)}{2h} = 1.$$

Note that for all $u, v \in \mathbb{R}^+$ with $u < v$, for every Lebesgue measurable subset G of $[u, v]$,

$$\begin{aligned} m_1\left(G^{-1}\right) &= \int_{t \in G^{-1}} 1 dt = \int_{s \in G(t = s^{-1})} \left|-\tfrac{1}{s^2}\right| ds \\ &= \int_{s \in G} \tfrac{1}{s^2} ds. \end{aligned}$$

In particular, for all such G,

$$\frac{1}{v^2} m_1(G) \leq m_1\left(G^{-1}\right) \leq \frac{1}{u^2} m_1(G).$$

Next, fix an arbitrary $h > 0$ such that $z - h > 1/a$ and $z + h < a$.

$$U^{-1} \cap (z-h, z+h) = \left(U \cap \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)^{-1},$$

and so,

$$\begin{aligned} m_1\left(U^{-1} \cap (z-h, z+h)\right) &= m_1\left(\left(U \cap \left(\tfrac{1}{z+h}, \tfrac{1}{z-h}\right)\right)^{-1}\right) \\ &\geq (z-h)^2\, m_1\left(U \cap \left(\tfrac{1}{z+h}, \tfrac{1}{z-h}\right)\right). \end{aligned}$$

Recall that $x := 1/z \in U$ and $U \in \tau_D$. Thus, $D(U, x) = 1$. Hence, x is a *Lebesgue point* of the characteristic function of U, χ_U. (See, for example, Rudin [13], Section 7.6, page 138, for the definition of a *Lebesgue point*.) Moreover, the family of sets

$$\left(E_h := \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)_{h>0}$$

"shrinks to x nicely as $h \longrightarrow 0+$", in a natural variation on the definition of Rudin [13], Section 7.9, page 140. Indeed, for all $h > 0$ and small enough ($h < z/2$), we have

$$\left(x - \frac{h}{2z^2}, x + \frac{h}{2z^2}\right) \subseteq \left(\frac{1}{z+h}, \frac{1}{z-h}\right) \subseteq \left(x - \frac{2h}{z^2}, x + \frac{2h}{z^2}\right).$$

By a simple variation on the proof of Rudin [13], Section 7.9, Theorem 7.10, page 141, it follows that

$$\lim_{h \longrightarrow 0+} \frac{m_1\left(U \cap \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)}{m_1\left(\left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)} = \chi_U(x) = 1.$$

Therefore,

$$\begin{aligned} 1 \geq \frac{m_1\left(U^{-1} \cap (z-h, z+h)\right)}{2h} &\geq \frac{(z-h)^2 m_1\left(U \cap \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)}{2h} \\ &= \frac{(z-h)^2\left(\frac{1}{z-h} - \frac{1}{z+h}\right)}{2h} \frac{m_1\left(U \cap \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)}{m_1\left(\left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)} \\ &= \frac{(z-h)^2}{(z-h)(z+h)} \frac{m_1\left(U \cap \left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)}{m_1\left(\left(\frac{1}{z+h}, \frac{1}{z-h}\right)\right)} \\ &\longrightarrow (1)(1) = 1, \end{aligned}$$

as $h \longrightarrow 0+$.

At last we see that $D(U^{-1}, z) = 1$, for all $z \in U^{-1}$; and so, $U^{-1} \in \tau_D$.

Consequently, $H := U \cap U^{-1}$ is a τ_D neighborhood of 1 such that

$$H^{-1} = H \text{ and } H \times H \subseteq C.$$

Since $H \in \tau_D$, H is measurable and $m_1(H) > 0$; while H is a bounded subset of $\mathbb{R}^+$, by its construction. Thus, by Theorem 67, there exist $\alpha, \beta \in H$ with $\alpha \neq \beta$ such that $\alpha/\beta \in \mathbb{Q}$. Let $\gamma := 1/\beta \in H^{-1} = H$. Thus, $\alpha \times \gamma = \alpha/\beta \in \mathbb{Q}$ and $\alpha \times \gamma \in H \times H \subseteq C$. Further, $\alpha \times \gamma \neq 1$, which contradicts the fact that 1 is the only rational element of C. Therefore, multiplication is *not* a continuous binary operation on $(\mathbb{R}^+, \tau_D)$; and so $(\mathbb{R}^+, \tau_D, \times)$ is not a topological group, nor a topological semigroup. $\square$

2.0.3 Topological Groups and Semigroups

So, in $(\mathbb{R}, \tau_D)$ neither addition nor multiplication is a topological group operation. What, then, are the topological group operations on this space? The answer to this question requires a closer look at the properties of translation, which, as shown earlier, must be continuous for a continuous binary operation. The next result holds not only for topological group operations but also for cancellative topological semigroup operations. Some lemmas will first be required.

Lemma 69. *Let Y be a subspace of X, and define a **separation** of Y to be a pair of disjoint nonempty sets C and D whose union is Y, neither of which contains a limit point of the other. Y is **connected** if there exists no separation of Y.*

Proof. See [1]. □

Lemma 70. *For any continuous map $\phi : (\mathbb{R}, \tau_D) \longrightarrow (\mathbb{R}, \tau_D)$ that either preserves or reverses order, the inverse image of an open interval must be an open interval.*

Proof. The inverse image of any open interval must be open in τ_D since ϕ is continuous. Let U be any open interval and let $a, b \in \phi^{-1}(U)$. For any x such that $a < x < b$, $\phi(x) \in U$ since ϕ is order preserving or order reversing. Moreover, since $\phi^{-1}(U) \in \tau_D$, it cannot contain an end-point that is a real number. Therefore, $\phi^{-1}(U)$ is an open interval. □

Theorem 71. *Any cancellative topological semigroup operation on $(\mathbb{R}, \tau_D)$ is order preserving with respect to the usual ordering on $\mathbb{R}$.*

Proof. The proof will consist of 5 steps.

1. Fix $p \in \mathbb{R}$ and show that left translation by p is either order preserving or order reversing.

2.Fix $p \in \mathbb{R}$ and show that right translation by p is either order preserving or order reversing.

3. Show that left translation is either order preserving for all $x \in \mathbb{R}$, or it is order reversing for all $x \in \mathbb{R}$.

4. Show that left translation is order preserving.

5. Repeat the third and fourth steps for right translation.

Step 1 - Fix $p \in \mathbb{R}$ and let a, b, and c be arbitrary with $a < b < c$. Note that ${}_p\phi(x)$ is injective since the operation is cancellative (${}_p\phi(x) =_p \phi(y) \Longrightarrow p{*}x = p{*}y \Longrightarrow x = y$). Now, there are 6 possible orderings for ${}_p\phi(a)$, ${}_p\phi(b)$, and ${}_p\phi(c)$. First, suppose for contradiction that ${}_p\phi(a) <_p \phi(c) <_p \phi(b)$. The interval (a, b) doesn't contain c so ${}_p\phi(c)$ is not in the image of the interval ${}_p\phi(a, b)$ since ${}_p\phi(x)$ is injective. Thus, ${}_p\phi(a, b)$ is disconnected at ${}_p\phi(c)$. In other words, by Lemma 69, the sets $C =_p \phi(a, b) \cap (-\infty, {}_p\phi(c))$ and $D =_p \phi(a, b) \cap ({}_p\phi(c), \infty)$ form a separation of ${}_p\phi(a, b)$, and ${}_p\phi(a, b)$ is, therefore, not connected. It remains to check that $C \neq \emptyset$ and $D \neq \emptyset$. It is sufficient to show the former, because showing the latter is similar. The interval (a, ∞) is τ_D-open. Thus, $E := (-\infty, a]$ is τ_D-closed, with τ_D-interior equal to $(-\infty, a)$. Since a is not in this interior, a belongs to the τ_D-closure of the complement of E; i.e., a belongs to the τ_D-closure of (a, ∞). Thus, there exists a net $(\xi_k)_{k \in K}$ in (a, ∞) such that $\xi_k \underset{k}{\longrightarrow} a$ with respect to the τ_D topology. Now, ${}_p\phi$ is τ_D to τ_D continuous on $\mathbb{R}$. Therefore, ${}_p\phi(\xi_k) \underset{k}{\longrightarrow} {}_p\phi(a)$ with respect to the τ_D topology. But the τ_D topology is stronger than the standard topology on $\mathbb{R}$; and so, ${}_p\phi(\xi_k) \underset{k}{\longrightarrow} {}_p\phi(a)$ with respect to the standard topology. Thus, there exist $k_0 \in K$ such that ${}_p\phi(\xi_k) < {}_p\phi(c)$, for all $k \geq k_0$. Of course, we also have that $\xi_k \underset{k}{\longrightarrow} a$ with respect to the standard topology. Consequently, there exists $k_1 \geq k_0$ such that $\xi_{k_1} < b$. Hence, ${}_p\phi\left(\xi_{k_1}\right) \in C$. However, the interval (a, b) is connected since $(\mathbb{R}, \tau_D)$ is connected, and ${}_p\phi(a, b)$ is then the continuous image of a connected set, and it, therefore, must be connected. This is a contradiction. Hence, the ordering ${}_p\phi(a) <_p \phi(c) <_p \phi(b)$ is not possible. Similarly, the orderings ${}_p\phi(b) <_p \phi(a) <_p \phi(c)$, ${}_p\phi(b) <_p \phi(c) <_p \phi(a)$, and ${}_p\phi(c) <_p \phi(a) <_p \phi(b)$ all lead to a contradiction of connectedness. Therefore, it is straightforward to check that left translation by p either preserves order or reverses order; i.e., for all $x < y$ in $\mathbb{R}$, ${}_p\phi(x) <_p \phi(y)$; or for all $w < z$ in $\mathbb{R}$, ${}_p\phi(w) >_p \phi(z)$.

Step 2 - Replace left translation in Step 1 with right translation.

Step 3 - Suppose for contradiction that there exist elements p and q such that left translation by p is order preserving and left translation by q is order reversing. Without loss of generality, suppose $p < q$ and let

$r = \inf \{x \in \mathbb{R} : p < x$ and left translation by x is order reversing$\}$. Since $*$ is cancellative,

$r * p \neq r * q$, and since $\mathbb{R}$ is Hausdorff in the standard topology, there exists disjoint open intervals U_p and U_q such that $r * p \in U_p$ and $r * q \in U_q$. Thus, either every element of U_p is less than every element of U_q or vice versa.

Now, $r \in \left(\phi_p\right)^{-1}(U_p) \in \tau_D$ and $r \in \left(\phi_q\right)^{-1}(U_q) \in \tau_D$. Let $V = \left(\phi_p\right)^{-1}(U_p) \cap \left(\phi_q\right)^{-1}(U_q)$. V is open since it is the intersection of two open sets, and V is not empty since $r \in V$. Furthermore, both $\left(\phi_p\right)^{-1}(U_p)$ and $\left(\phi_q\right)^{-1}(U_q)$ are open intervals by Lemma 70 since U_p and U_q are open intervals and ϕ_p and ϕ_q either preserve or reverse order. Thus, V is an open interval. By definition of r, there exists $a, b \in V$ such that ${}_a\phi$ is order preserving and ${}_b\phi$ is order reversing. Thus $a * p < a * q$ and $b * p > b * q$. However, $a * p$ and $b * p$ are elements of U_p, and $a * q$ and $b * q$ are elements of U_q. This contradicts the fact that either every element of U_p is less than every element of U_q or vice versa. A similar argument holds for $p > q$. Therefore, left translation is either order preserving for all $x \in \mathbb{R}$, or it is order reversing for all $x \in \mathbb{R}$.

Step 4 - Suppose for contradiction that left translation is order reversing for all $x \in \mathbb{R}$. So, for any $a < b$, left translation by p gives $p * a > p * b$. Translating by p a second time gives $p * p * a < p * p * b$. Thus, left translation by $p * p$ is not order reversing. This contradicts the supposition.

Step 5 - Replace left translation in Steps 3 and 4 with right translation. $\square$

So, let's put this theorem to work right away to prove the following result for any paratopological group or topological group. For any element, a, the inverse will be denoted a^{-1}.

Corollary 72. *For any topological group or paratopological group on $(\mathbb{R}, \tau_D)$, $a < b \implies b^{-1} < a^{-1}$.*

Proof. Let e denote the group identity. From Theorem 71, translation is order preserving. So, $a < b \implies a * a^{-1} < b * a^{-1} \implies e < b * a^{-1} \implies b^{-1} * e < b^{-1} * b * a^{-1} \implies b^{-1} < a^{-1}$. $\square$

Theorem 71 is a key to proving the general result for topological groups, but some other results will first be needed.

Lemma 73. *Any set of positive measure contains a nonempty subset that is open in the density topology.*

Proof. Let $E \subset \mathbb{R}$ be any measurable set with $m_1(E) > 0$, and let $A = \{x \in E : x \notin \Phi(E)\}$. By the Lebesgue Density Theorem, $m_1(A) = 0$. Thus, $m_1(E \setminus A) > 0$ and all points of $E \setminus A$ are density points. Therefore, $E \setminus A$ is nonempty and open in the density topology. $\square$

Theorem 74. *For any paratopological group operation on $(\mathbb{R}, \tau_D)$, translation maps a set of positive measure to a set of positive measure.*

Proof. Let $E \subset \mathbb{R}$ be any measurable set with $m_1(E) > 0$, and let $q \in \mathbb{R}$ be arbitrary. By Lemma 73, there exists nonempty $B \subseteq E$ such that $B \in \tau_D$. Theorem 60 then implies $q * B$ is open in the density topology, and it is nonempty since B is nonempty. Thus, $m_1(q * E) \geq m_1(q * B) > 0$. Similarly, $m_1(E * q) > 0$. $\square$

Lemma 75. *For any paratopological group operation on $(\mathbb{R}, \tau_D)$, both left and right translation are continuous in the standard topology.*

Proof. For any $p \in \mathbb{R}$, left translation is injective since the operation is cancellative, and left translation is surjective since for any $q \in \mathbb{R}$ there exists $(p^{-1} * q)$ such that ${}_p\phi(p^{-1} * q) = q$. So, $({}_p\phi)^{-1}(x) =_{p^{-1}} \phi(x) = p^{-1} * x$. Let (a, b) be an arbitrary open interval. By Theorem 71, for any $x \in ({}_p\phi)^{-1}((a, b))$, $p^{-1} * a < x < p^{-1} * b$ so $({}_p\phi)^{-1}((a, b)) \subset (p^{-1} * a, p^{-1} * b)$. Conversely, for any $x \in (p^{-1} * a, p^{-1} * b)$ Theorem 71 implies $a < p * x < b$ and $x \in ({}_p\phi)^{-1}((a, b))$. So, $(p^{-1} * a, p^{-1} * b) \subset ({}_p\phi)^{-1}((a, b))$. Therefore, $({}_p\phi)^{-1}((a, b)) = (p^{-1} * a, p^{-1} * b)$ is an open interval in the standard topology. Thus, for any open interval in the standard topology, the inverse image is also an open interval in the standard topology. A similar argument shows that right translation is also continuous in the standard topology. $\square$

Lemma 76. *Let $*$ be a paratopological group operation on $(\mathbb{R}, \tau_D)$ and let $\{q_k\}_{k \in \mathbb{N}}$ be the sequence given by $q_k = e + \frac{1}{k}$ where e is the identity element and $k \in \mathbb{N}$. For any point $x \in \mathbb{R}$, the sequence of points $\{q_k * x\}_{k \in \mathbb{N}}$ converges in the standard topology to x; and the sequence $\left\{q_k^{-1} * x\right\}_{k \in \mathbb{N}}$ also converges in the standard topology to x.*

Proof. Let U be an arbitrary interval containing x. By Lemma 75, ϕ_x is continuous in the standard topology so $(\phi_x)^{-1}(U)$ is an open set, B. Furthermore, $e \in B$ since $e * x = x \in U$. Now, $\{q_k\}_{k\in\mathbb{N}}$ converges to e, and this implies that there exists a natural number $M \in \mathbb{N}$ such that $q_j \in B$ for all $j > M$. Thus, $q_j * x \in U$ for all $j > M$. Since U is an arbitrary interval containing x, $\{q_k * x\}_{k\in\mathbb{N}}$ converges to x in the standard topology. A similar argument shows that $\left\{q_k^{-1} * x\right\}_{k\in\mathbb{N}}$ also converges to x in the standard topology. □

Finally, the following theorem will be needed. The proof presented below for τ_D open sets V that are not open in the standard topology was shown to us by Chris Lennard. We include it with his permission.

Theorem 77. *Let $*$ be a paratopological group operation on $(\mathbb{R}, \tau_D)$ with identity e. For any $V \in \tau_D$ there exists $M \in \mathbb{N}$ such that each term of the sequence $\left\{m_1\left(\left(e + \frac{1}{k}\right) * V\right)\right\}_{k\in\mathbb{N}}$ is greater than a constant $\delta > 0$ for all $k > M$.*

Proof. Let τ_u denote the usual or standard topology on $\mathbb{R}$. Recall from, for example, Rudin [28], pages 37 and 38, that a function $f : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_u)$ is called *lower semicontinuous* if $f^{-1}((\alpha, \infty)) \in \tau_u$ for all $\alpha \in \mathbb{R}$. Moreover, as noted by Rudin [28], an example of such a function is $f := \chi_U$, the *characteristic function* of U, where U is any τ_u-open subset of $\mathbb{R}$.

It is also straightforward to check that a function $f : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_u)$ is *lower semicontinuous* if and only if for all $x \in \mathbb{R}$, for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $x_n \underset{n}{\longrightarrow} x$ and $x_n \leq x$ for all $n \in \mathbb{N}$, it follows that $f(x_n) \underset{n}{\longrightarrow} f(x)$.

Now, fix an arbitrary τ_D-open set V. We have that $m_1(V) > 0$. Without loss of generality, we may assume $V \subset (a, b)$, for some $a, b \in \mathbb{R}$ with $a < b$. Let $q_k := e + 1/k$, for all $k \in \mathbb{N}$. By Theorem 60, each $q_k * V$ is a τ_D-open set, and in particular, it is a measurable set.

CASE 1. The set V is open in the standard topology. We will use *Lebesgue's Dominated Convergence Theorem* to show that

$$m_1(q_k * V) \underset{k}{\longrightarrow} m_1(V).$$

Indeed, for each $k \in \mathbb{N}$, we define the measurable function $f_k := \chi_{q_k * V}$. Also, let $f := \chi_V$. Now, fix an arbitrary $x \in \mathbb{R}$. By Lemma 76, $q_k^{-1} * x \underset{k}{\longrightarrow} x$ in the standard topology, τ_u.

Fix an arbitrary $k \in \mathbb{N}$. Since $e < q_k$, we see that $q_k^{-1} < e^{-1} = e$, by Corollary 72. Thus, $q_k^{-1} * x < e * x = x$, by Theorem 71. Hence,

$$f_k(x) := \chi_{q_k * V}(x) = \chi_V\left(q_k^{-1} * x\right) \underset{k}{\longrightarrow} \chi_V(x) = f(x),$$

because $f := \chi_V : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_u)$ is *lower semicontinuous.* Using Theorem 71 again, we see that for all $k \in \mathbb{N}$, $q_k * V \subset (a, q_1 * b)$; and so for all $x \in \mathbb{R}$,

$$|f_k(x)| = \chi_{q_k * V}(x) \leq \chi_{(a, q_1 * b)} =: h(x).$$

Clearly, h is a Lebesgue-integrable function. Therefore, by *Lebesgue's Dominated Convergence Theorem,*

$$m_1(q_k * V) = \int_{\mathbb{R}} \chi_{q_k * V}(x)\, dm_1(x) = \int_{\mathbb{R}} f_k(x)\, dm_1(x) \underset{k}{\longrightarrow} \int_{\mathbb{R}} f(x)\, dm_1(x) = m_1(V).$$

Since $m_1(V) > 0$, the conclusion of our theorem follows in this case. Note, however, that we have reached a stronger conclusion when V is τ_u-open.

CASE 2. Assume that $V \subset (a, b)$ is an arbitrary τ_D-open set. So, V is Lebesgue-measurable and $0 < m_1(V) < \infty$. Fix $\epsilon > 0$. By (for example) Royden [26], Chapter 3, Section 3, Proposition 15(ii) and (iii), page 63, there exists a τ_u-open set U and a τ_u-closed set F such that

$$F \subset V \subset U \text{ and } m_1(U \backslash F) < \epsilon.$$

Also, we may assume that $U \subset (a, b)$. Fix an arbitrary $k \in \mathbb{N}$. Then

$$\begin{aligned} m_1(q_k * V) &= m_1(q_k * (V \cap U)) + m_1(q_k * (V \backslash U)) \\ &\geq m_1(q_k * (V \cap U)) \\ &= m_1(q_k * U) - m_1(q_k * U \backslash V) \\ &\geq m_1(q_k * U) - m_1(q_k * U \backslash F) \\ &\underset{k}{\longrightarrow} m_1(U) - m_1(U \backslash F) > m_1(V) - \epsilon, \end{aligned}$$

by CASE 1; because both U and $U \backslash F$ are τ_u-open subsets of (a, b).

Choose $\epsilon = m_1(V)/2$, and let $\delta := m_1(V) - m_1(V)/2 = m_1(V)/2 > 0$. Then there exists $M \in \mathbb{N}$ such that for all $k > M$,

$$m_1(q_k * V) > \delta.$$

□

The previous lemmas and theorems lead to the main result of this section.

Theorem 78. *There are no topological groups on* $(\mathbb{R}, \tau_D)$.

Proof. Suppose for contradiction that there is a topological group with a continuous group operation $*$. Let e be the group identity, and denote the inverse of any element x by x^{-1}. Furthermore, for any set $A \subset \mathbb{R}$ let $A^{-1} = \{x^{-1} : x \in A\}$. Now, let
$Q\prime = \left\{q_i^{-1} * q_j : i, j \in \mathbb{N}\right\}$. Note that $e \in Q\prime$, and $Q\prime$ is countable.

Fix $\delta > 0$. Consider the open set $U = \{x \in \mathbb{R} : e - \delta < x < e + \delta,\ x \notin Q\prime\} \cup \{e\}$. Note that e is the only element of $Q\prime$ contained in U. Since $e \in U$ and $*$ is continuous, there exists open sets $\widetilde{V}$ containing e and $\widetilde{W}$ containing e such that $\widetilde{V} * \widetilde{W} \subset U$. The intersection of the open sets $\widetilde{V} \cap \widetilde{W}$ is a nonempty open set. Denote it by $V\prime\prime = \widetilde{V} \cap \widetilde{W}$. So, $V\prime\prime * V\prime\prime \subset U$. Fix an interval (a, b) containing e. Let $V\prime = V\prime\prime \cap (a, b)$. Again, $V\prime$ is a nonempty open set since $V\prime\prime$ and (a, b) are both open sets containing e. Finally, let $V = V\prime \cap (V\prime)^{-1}$. Note that $(V\prime)^{-1}$ is an open set due to the continuity of the inverse operation, and it contains e since $e^{-1} = e$. Therefore, V is an open set containing e, $V = V^{-1}$, $V * V \subset U$, and $m_1(V)$ is finite.

Let $q_k = e + \frac{1}{k}$. By Theorem 77, there exists $M \in \mathbb{N}$ such that each term of the sequence $\{m_1(q_k * V)\}_{k \in \mathbb{N}}$ is greater than a constant $\delta > 0$ for all $k > M$. Thus, $\sum_{k=1}^{\infty} m_1(q_k * V) = +\infty$. However, for all $k \in \mathbb{N}$, $q_k * V$ is contained in the interval $(a, (e + 1) * b)$ which has finite measure. So, the family of sets $\{q_k * V\}$ cannot be pairwise disjoint. Therefore, there exists i, j with $i \neq j$ such that $(q_i * V) \cap (q_j * V) \neq \emptyset$. So, there exists $x, y \in V$ such that $q_i * x = q_j * y$ and $x = q_i^{-1} * q_j * y$. Now, $V = V^{-1}$ implies $y^{-1} \in V$ and $x * y^{-1} = q_i^{-1} * q_j$ with $x * y^{-1} \in U$. Now, $q_i^{-1} * q_j \neq e$ since $i \neq j$, but $q_i^{-1} * q_j \in Q\prime$. This contradicts the definition of U. Therefore, $*$ is not a continuous group operation. $\square$

3.0 TOPOLOGICAL GROUPS AND SEMIGROUPS ON SOUSLIN LINES

3.0.4 Topological Groups

With regard to Souslin lines, the first question is whether or not a topological group can exist on a Souslin line. This question, though, is relatively easy to answer using well known theorems. Consider the following such theorem.

Theorem 79. *Any linearly ordered topological space (LOTS) satisfying the CCC is first countable.*

Proof. Let L denote the LOTS and fix any arbitrary point $p \in L$ with the intent of showing that p has a countable neighborhood base. Assuming the Axiom of Choice, choose any point $x_1 < p$ as the first point of a sequence. Secondly, choose x_2 such that $x_1 < x_2 < p$. Thirdly, choose x_3 such that $x_1 < x_2 < x_3 < p$. Continue picking points greater than the previous point but always less than p until there are no points less than p and greater than all of the previous points in the (possibly uncountable) generalized sequence. This portion of the proof is more or less the same as the standard proof of the Well-ordering Theorem, and it can be made rigorous by means of a choice function and the union of extensions of sequences given by the choice function. When there are no more points less than p and greater than the previous point, either the last point picked is an immediate predecessor to p or the generalized sequence is increasing and converging to p from the left. In the first case in which p has an immediate predecessor, the predecessor by itself is a finite sequence coverging to p from the left. In the second case of the generalized sequence, if for any ordinal k, x_k is an immediate predecessor to x_{k+1} then omit x_{k+1} from the sequence. Each open interval (x_k, x_{k+1}) is then non-empty. Thus, since L satisfies the CCC, the increasing sequence converging to p must be countable. Now, repeat the above process choosing points greater than p to obtain a countable (or finite) sequence converging to p from the right. The left sequence and the right sequence then form a countable neighborhood base for p given by the family of open intervals (x_l, x_r) in which $x_l < p$ and $x_r > p$. Since p is arbitrary, L is first countable. $\square$

Thus, any Souslin line is first countable. The topological group question is then answered by a theorem arrived at by both Kakutani [7] and Birkhoff [8] independently.

Theorem 80 (Birkhoff and Kakutani)**.** *Let G be a topological group. G is metrizable if and only if G is first countable.*

Proof of the theorem is provided in the two references, and it is not repeated here. We see, then, that any LOTS satisfying the CCC and supporting a topological group is metrizable, and any metrizable topological space satisfying the CCC is also separable. Since Souslin lines are not separable, we conclude that no Souslin line can support a topological group.

3.0.5 Paratopological Groups

So, while a Souslin line can support a trivial topological semigroup, it cannot support a topological group. How much algebraic structure can a Souslin line support? First, can a Souslin line support a paratopological group? In order to answer this question, we'll need to use results from Ceder[9] and Lutzer [11], both of which appear in [10] and require some definitions.

Definition 81. *In a topological space X, the* ***diagonal*** *Δ is the subset of $X \times X$ given by $\Delta = \{(x, x) : x \in X\}$.*

Definition 82. *A topological space X has a* $\mathbf{G}_\delta-$**diagonal** *if its diagonal Δ is a G_δ set.*

Theorem 83 (Ceder)**.** *A topological space X has a $G_\delta - diagonal$ if and only if there exists a sequence $\mathcal{G}_n$ of open covers of X such that for each $x, y \in X$ with $x \neq y$ there exists $n \in \omega$ with $y \notin st\left(x, \mathcal{G}_n\right)$; i.e. for all $x \in X$, $\cap_n st\left(x, \mathcal{G}_n\right) = \{x\}$.*

Proof. Following a proof described in [10], suppose X has a $G_\delta - diagonal$. Let the diagonal $\Delta = \cap_n U_n$ with U_n open in $X \times X$. For each $x \in X$ and $n \in \omega$ let $g(n, x)$ be an open neighborhood of x such that $g(n,x) \times g(n,x) \subset U_n$. Let $\mathcal{G}_n = \{g(n,x) : x \in X\}$. We claim that for all $x \in X$ and for all $y \in X$ with $x \neq y$ there exists $n \in \omega$ with $y \notin st(x, \mathcal{G}_n)$. Suppose for contradiction $\{x, y\} \subset \cap_n st(x, \mathcal{G}_n)$ with $x \neq y$. For each n choose $z_n \in X$ such that $\{x, y\} \subset g(n, z_n)$. Then the point (x, y) of $X \times X$ is in $g(n, z_n) \times g(n, z_n) \subset U_n$. So, $(x, y) \in \cap_n U_n$, and this is a contradiction.

Conversely, suppose there exists a sequence of open covers $\mathcal{G}_n$ such that for each $x, y \in X$ with $x \neq y$ there exists $n \in \omega$ with $y \notin st(x, \mathcal{G}_n)$. Let $U_n = \cup\{G \times G : G \in \mathcal{G}_n\}$. Then $\Delta \subset \cap_n U_n$. Now, $(x, y) \in \cap_n U_n$ implies for each n there exists $G_n \in \mathcal{G}_n$ with $(x, y) \in G_n \times G_n$. Thus, $y \in \cap_n st(x, \mathcal{G}_n) = \{x\}$. So, $y = x$. Hence, $\Delta = \cap_n U_n$. □

Theorem 84 (Lutzer)**.** *Every linearly ordered space with a $G_\delta - diagonal$ is metrizable.*

Proof of Theorem 84 is provided in [10].

Ceder's theorem has been used extensively as a technique to show that a space has a $G_\delta - diagonal$. We will do the same for a first countable paratopological group. The result will then allow us to use Lutzer's theorem to show that a paratopological group cannot exist on a Souslin line.

Theorem 85. *Every first countable paratopological group has a $G_\delta - diagonal$.*

Proof. Let X be a first countable topological space with a paratopological group operation $*$. Let e be the group identity, and let $\{U_n\}$ be a decreasing neighborhood base at e. In other words, $U_1 \supset U_2 \supset U_3 \supset \ldots$. By Theorem 60, for all $x \in X$ and for any $n \in \omega$, $x * U_n$ is an open set. This implies that for any fixed n, $\{x * U_n : x \in X\}$ is an open cover for X since $e \in U_n$ for each n. Thus, $\mathcal{G}_n = \{x * U_n : x \in X\}$ is a sequence of open covers . Fix $p \in X$ with p arbitrary and suppose there exists $q \in X$ such that $q \in st(p, \mathcal{G}_n)$ for all $n \in \omega$. Then for each n there exists x_n such that $p, q \in x_n * U_n$. So, there exists $p_n, q_n \in U_n$ such that $p = x_n * p_n$ and $q = x_n * q_n$ and $\{p_n\}, \{q_n\}$ are both sequences converging to e since $\{U_n\}$ is a decreasing neighborhood base.

Now, $x_n^{-1} = p_n * p^{-1} = q_n * q^{-1}$ where $\{p_n * p^{-1}\}$ is a sequence converging to p^{-1} and $\{q_n * q^{-1}\}$ is a sequence converging to q^{-1}. To see this, let W be an arbitrary open set containing p^{-1} and let ϕ: $X \times X \longrightarrow X$ denote the map produced by the binary operation so that for all $x, y \in X$, $\phi(x, y) = x * y$. Thus, $(e, p^{-1}) \in \varphi^{-1}(W)$, and since φ is continuous there exists open sets A, B with $e \in A$ and $p^{-1} \in B$ such that $A \times B \subset \varphi^{-1}(W)$. Since $\{p_n\}$ converges to $e \in A$, there exists a natural number $m \in \omega$ such that $p_n \in A$ for all $n > m$, and this implies $p_n * p^{-1} \in W$ for all $n > m$. Finally, conclude $\{p_n * p^{-1}\}$ converges to p^{-1} since W can be any open set containing p^{-1}. Similarly, $\{q_n * q^{-1}\}$ is a sequence converging to q^{-1}.

Now, $\{p_n * p^{-1}\}$ and $\{q_n * q^{-1}\}$ must obviously converge to the same point since $p_n * p^{-1} = q_n * q^{-1}$ for each n. Hence, $p^{-1} = q^{-1}$ and $p = q$. Thus, $\mathcal{G}_n$ is a sequence of open covers satisfying the requirements of Theorem 83. Therefore, X has a $G_\delta - diagonal$. □

The three previous theorems combine to give the following.

Theorem 86. *Every linearly ordered topological space (LOTS) satisfying the CCC and for which there exists a paratopological group operation is separable.*

Proof. Such a space has a $G_\delta - diagonal$ by Theorem 79 and Theorem 85. The space is then metrizable by Theorem 84, and since any metrizable topological space satisfying the CCC is separable, the space is separable. □

This leads to the main result for paratopological groups and Souslin Lines.

Corollary 87. *There are no paratopological groups on any Souslin line.*

Proof. The proof is immediate from the theorem. □

3.0.6 Cancellative Topological Semigroups

So what about cancellative topological semigroups? Can a Souslin line support a cancellative topological semigroup? Let's start by trying to determine for a LOTS those properties of a topological semigroup operation that will actually make it a paratopological group. Consider the following.

Proposition 88. *A topological semigroup on a LOTS for which translation is both order preserving (or reversing) and surjective is a paratopological group.*

Proof. Suppose that X is a LOTS and $*$ is a topological semigroup operation for which translation is both order preserving (reversing) and surjective. The order preserving (reversing) property requires the operation to be cancellative, and a cancellative operation implies translation is injective. Indeed, let a be arbitrary and suppose $a * b = a * c$. This implies $b = c$ since translation by a is order preserving (reversing). Similarly, $b * a = c * a \implies b = c$. Translation is, therefore, a bijection. So, for any $a \in X$, since left translation by a is surjective, there exists some element $e \in X$ that is mapped to a by a. In other words $a * e = a$. The order preserving (reversing) property gives us $a*e = a \implies a*e*a = a*a \implies e*a = a$. Now, for any $x \in X$, $a * e * x = a * x \implies e * x = x$. Similarly, $x * e * a = x * a \implies x * e = x$. Therefore, e is an identity. Again, since translation by a is surjective, there exists some element that translation by a maps to the element e. So, there exists $b \in X$ such that $a * b = e$. This implies $b * a * b = b * e = b = e * b$ giving $b * a = e$. Thus, b is an inverse of a. Since a is arbitrary, an inverse exists for every element. We conclude that the topological semigroup is, in fact, a paratopological group. □

This leads us to the following result for Souslin lines.

Corollary 89. *For a topological semigroup on a Souslin Line, translation cannot be both order preserving (reversing) and surjective.*

Proof. The proof is immediate from the Proposition and Corollary 87. □

There are, then, some clues about how we might find a cancellative topological semigroup on a Souslin line, if one exists. From Corollary 89, we see that if a cancellative topological semigroup exists on a Souslin line, then the semigroup operation cannot be both order preserving (reversing) and surjective. Furthermore, in [12] Feng and Heath showed that any connected LOTS with a cancellative topological semigroup is metrizable. Therefore, a cancellative topological semigroup cannot exist on a connected Souslin line. These theorems provide a foundation upon which we can begin to see how a cancellative topological semigroup might exist on a Souslin line, and we will construct one below. Ironically, we will start with a connected Souslin line and discard points so that we end with another Souslin line that is not connected. We will then define a binary operation that is not surjective. First, however, we'll need some lemmas, a definition, and a theorem by Mary Ellen Rudin.

Lemma 90. *A complete order dense LOTS is connected.*

Proof. Let L be any complete order dense LOTS and suppose for contradiction that there exists disjoint open subsets U and V such that $U \cup V = L$. Let $(a, b) \subset V$ be any open interval in V. There exists a subset of U that is either bounded above by (a, b) or bounded below by (a, b). Without loss of generality, let a be an upper bound for a nonempty subset of U. Denote the set by $\widetilde{U} = \{x \in U : x \leq a\}$ and note that both $\widetilde{U}$ and $U/\widetilde{U}$ are open since there is an interval between them and U is open. Since L is complete, there exists a point $p = \sup \widetilde{U} \in L$, and since $\widetilde{U}$ is open, there exists a point c such that $(c, p) \subset \widetilde{U}$ and $(c, p) \neq \emptyset$. Let $\widetilde{V} = \{x \in V : x \geq p\}$ and note that both $\widetilde{V}$ and $V/\widetilde{V}$ are open. Now, the point p must be an element of either $\widetilde{U}$ or $\widetilde{V}$. Suppose $p \in \widetilde{U}$. Since $\widetilde{U}$ is open, there exists an open interval $(r, s) \subset \widetilde{U}$ containing p, but p is the least upper bound of $\widetilde{U}$. So, $(p, s) = \emptyset$, and this contradicts the fact that L is order dense. On the other hand, suppose $p \in \widetilde{V}$. Again since $\widetilde{V}$ is open, there exists an open interval $(r, s) \subset \widetilde{V}$ containing p, but by definition of $\widetilde{V}$, it contains no element less than p. So, $(r, p) = \emptyset$, and this contradicts the fact that L is order dense. Therefore, L must be connected. □

Definition 91. *A Souslin line is* ***hereditarily Souslin*** *if every open interval is itself a Souslin line.*

In [13], Mary Ellen Rudin proved the existence of a connected hereditarily Souslin line that we'll need.

Theorem 92 (Mary Ellen Rudin)**.** *If a Souslin line exists, then a Souslin line exists that is connected and hereditarily Souslin.*

Proof. Suppose a Souslin line S exists. First, make S complete by adding all suprema and infima. Since S is not separable and it satisfies the CCC, S contains at most countably many maximal separable nontrivial closed subintervals. Define two points to be equivalent if they are both in the same maximal separable closed subinterval. Collapse each of these equivalence classes to a point and let S' denote the resulting space. For any two points in S' there are uncountably many points between them. S' is, then, order dense, and it is connected by Lemma 90. Furthermore, S' does not contain any separable subintervals. So, S' is hereditarily Souslin. □

Lemma 93. *Let L be a Souslin line that is hereditarily Souslin and let $U \subset L$ be an arbitrary open interval. Let U be divided into infinitely many (necessarily countably many) abutting subintervals and let each subinterval be divided the same way. Continue indefinitely until all nested sequences of subintervals fail to properly contain a subinterval in their intersection. For any $\alpha < \omega_1$ there exists a nested sequence of subintervals of length α such that the intersection contains an open interval. Furthermore, none of the nested sequences of subintervals reaches a length of ω_1.*

Proof. Suppose for contradiction that there exists $\alpha < \omega_1$ such that all of the nested sequences of subintervals have length less than α. Let A be the set of all endpoints of the subintervals. Since A is countable, A cannnot be dense in the Souslin line U. So, $U \backslash \overline{A}$ is a nonempty open set, and it must be uncountable since otherwise $U \backslash \overline{A} \cup A$ would be a countable dense subset. Thus, there exists an uncountable open interval $(a, b) \subseteq U \backslash \overline{A}$. By transfinite induction a nested sequence of subsets of length α can be found as follows. Let $V_0 = U$ and note $(a, b) \subset V_0$. Now for successor ordinals, let $\beta < \alpha$ be an arbitrary ordinal with $(a, b) \subset V_\beta$. Clearly, since $A \cap (a, b) = \emptyset$ and since (a, b) is uncountable, there exists $V_{\beta+1}$ such that $(a, b) \subset V_{\beta+1} \subset V_\beta$. Finally, for any limit ordinal $\gamma < \alpha$, $(a, b) \subset V_\lambda$ for all $\lambda < \gamma$ implies $(a, b) \subset \cap_{\lambda<\gamma} V_\lambda$, which implies there exists $V_\gamma \supseteq (a, b)$. However, (a, b) must be a proper subset because, if $V_\gamma = (a, b)$ then there exists $V_{\gamma+1} \subset (a, b)$ (since (a, b) is uncountable), which contradicts $A \cap (a, b) = \emptyset$. Thus, there exists $V_\gamma \supset (a, b)$. Therefore, there exists a nested sequence of subintervals of length α, and this contradicts the supposition. Finally, to show that none of the nested sequences of subintervals reach a length of ω_1, suppose for contradiction there is a nested sequence of subintervals of length ω_1 denoted V_γ with $\gamma \leq \omega_1$. Since each V_γ is divided into infinitely many intervals, a subinterval $W_{\gamma+1} \subset V_\gamma$ can be chosen such that $W_{\gamma+1} \cap V_{\gamma+1} = \emptyset$. Therefore, there exists an uncountable collection of nonempty disjoint open sets $\mathcal{W} = \{W_\gamma : \gamma \leq \omega_1\}$, and this constradicts the fact that the Souslin line must satisfy the CCC. □

We will now construct a cancellative topological semigroup on a Souslin line.

Theorem 94. *There exists a cancellative topological semigroup on a Souslin line.*

Proof. Assume a Souslin line exists. By Theorem 92 there exists a Souslin line that is connected and hereditarily Souslin. So, let L' be a connected Souslin line that is hereditarily Souslin. Furthermore, assume L' has no first or last point since otherwise L' could be replaced by one of its open subintervals. A semigroup operation will be defined on a subset $L \subset L'$ that is itself a Souslin line. This operation will be a composition of continuous maps $\phi : L \longrightarrow \mathbb{R}^{\omega_1}$, $\psi : \phi(L) \times \phi(L) \longrightarrow \phi(L)$, and $\phi^{-1} : \phi(L) \longrightarrow L$ where $\mathbb{R}^{\omega_1}$ is an uncountable product space whose elements will be denoted with uncountably many components such as $a \in \mathbb{R}^{\omega_1}$, $a = \langle a_1, a_2, a_3, a_4, a_5, \ldots \rangle$. For any $a, b \in \mathbb{R}^{\omega_1}$, define ψ by $\psi(a,b) = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, \ldots \rangle$ where $a_i + b_i$ denotes the standard addition of real numbers. Note that ψ is associative. The semigroup operation is then defined as follows. For every $x, y \in L$, $x * y = \phi^{-1}\left(\psi\left(\phi\left(x\right), \phi\left(y\right)\right)\right)$. Of course, ϕ and L still need to be defined.

First, some subsets of $\mathbb{R}$ will be needed. Let H be a Hamel basis for $\mathbb{R}$ that is a c-dense subset of $\mathbb{R}$. Let $\widetilde{W}$ and T be disjoint countable subsets of H that are dense subsets of $\mathbb{R}$. Let $W = span\ \widetilde{W}$ over $\mathbb{N}$. Let $\widetilde{S} = span\left(\widetilde{W} \cup T\right)$ over $\mathbb{N}$ and let $S = \widetilde{S} \backslash W$. So, W and S are disjoint countable subsets of $\mathbb{R}$ such that

$$\begin{array}{cc} (i) & \forall a, b \in W,\ a + b \in W \\ (ii) & \forall a, b \in S,\ a + b \in S \\ (iii) & \forall a \in W \text{ and } \forall b \in S,\ a + b \in S \end{array}$$

In L' choose a collection of pairwise disjoint abutting open intervals with no first or last interval as follows. Choose both a strictly increasing sequence and strictly decreasing sequence. Such sequences exist since L'has no first or last point. Let each element of the sequences be both a left endpoint of an open interval and a right endpoint of an open interval. The result is infinitely many disjoint abutting open intervals with no first or last interval. Repeat the process in each of these open intervals and continue a countable infinity of times. Now let D_1 be the set of endpoints of all of these intervals. Assuming the Axiom of Choice, D_1 is countable. D_1 is not dense in L' since L' is not separable. So, $L' \backslash \overline{D_1} \neq \emptyset$, and, furthermore, $L' \backslash \overline{D_1}$ is open. Since L' is hereditarily Souslin and the above argument applies to each of the intial disjoint open intervals in L', the collection of maximal disjoint open intervals in $L'\backslash\overline{D_1}$ must be infinite. Since L' satisfies the CCC, however, the collection must be countable. Denote the collection by

$$\mathcal{V}_1 = \left\{V_{1i} = (c_{1i}, d_{1i}) \subset L'\backslash\overline{D_1} : \text{for all } i \neq j, V_{1i} \cap V_{1j} = \emptyset \text{ and } V_{1i} \cup V_{1j} \text{ is not an interval}\right\}.$$

Define a map $f_1 : D_1 \longrightarrow W$ such that f_1 is an order preserving bijection. In other words, define f_1 to be a surjective map such that for all $a, b \in D_1$, $a < b$ implies $f_1(a) < f_1(b)$. Now, define another map $g_1 : \cup_{i\in\omega} V_{1i} \longrightarrow S$ such that for each i, g_1 is constant on V_{1i} and preserves the order of the sets in $\mathcal{V}_1$. In other words, for all $i \in \omega$ and for all $a, b \in V_{1i}$, $g_1(a) = g_1(b)$, and for all $a \in V_{1i}$, $b \in V_{1j}$ with $i < j$, $g_1(a) < g_1(b)$. Furthermore, since W and S are dense subsets of $\mathbb{R}$, f_1 and g_1 can be defined such that for all $a \in D_1$ and $b \in \cup_{i\in\omega} V_{1i}$, $a < b \implies f_1(a) < g_1(b)$ and $a > b \implies f_1(a) > g_1(b)$. The cancellative topological semigroup operation $*$ will be defined on a subset $L \subset L'$, which is itself a Souslin line, and $*$ will be defined in terms of an injective map $\phi : L \longrightarrow \phi(L) \subset \mathbb{R}^{\omega_1}$ where $\phi(L)$ has the lexicographic order topology. Although L has not yet been defined, the definition of ϕ will commence by defining the first component of $\phi(x)$ to be $f_1(x)$ if $x \in D_1$ and $g_1(x)$ if $x \in \cup_{i\in\omega} V_{1i}$.

Now since L' is hereditarily Souslin, each $V_{1i} \in \mathcal{V}_1$ is itself a Souslin line. Repeat the preceeding process on each V_{1i} defining D_{2i}, $\mathcal{V}_{2i}$, V_{2ij}, f_{2i}, and g_{2i}. Let the second component of $\phi(x)$ be $f_{2i}(x)$ if $x \in D_{2i}$ and $g_{2i}(x)$ if $x \in \cup_{j\in\omega} V_{2ij}$. For each i discard the set $V_{1i}\backslash(D_{2i} \cup (\cup_{j\in\omega} V_{2ij}))$. Again, each $V_{2ij} \in \mathcal{V}_{2i}$ is itself a Souslin line, and the process can be repeated on each V_{2ij} defining D_{3ij}, $\mathcal{V}_{3ij}$, V_{3ijk}, f_{3ij}, and g_{3ij}. Let the third component of $\phi(x)$ be $f_{3ij}(x)$ if $x \in D_{3ij}$ and $g_{3ij}(x)$ if $x \in \cup_{k\in\omega} V_{3ijk}$. For each j discard the set $V_{2ij}\backslash(D_{3ijk} \cup (\cup_{k\in\omega} V_{3ijk}))$. By Lemma 93 this process can be continued to define $D_{\alpha i_1 i_2 i_3 \cdots i_n}$, $\mathcal{V}_{\alpha i_1 i_2 i_3 \cdots i_n}$, $f_{\alpha i_1 i_2 i_3 \cdots i_n}$, and $g_{\alpha i_1 i_2 i_3 \cdots i_n}$ for all successor ordinals $\alpha < \omega_1$ and $i_1, i_2, i_3 \ldots i_n \in \omega$. Furthermore, for all limit ordinals $\alpha < \omega_1$, the process can be repeated on the open intervals that exist by Lemma 93 to define the component for the successor ordinal $\alpha + 1$. Now, let the α component be equal to the $\alpha + 1$ component, and redefine the $\alpha + 1$ component to be the next successor. In effect, then, the limit ordinals are skipped. Let $L = \cup_\alpha \cup_{i_1} \cup_{i_2} \cup_{i_3} \cdots D_{\alpha i_1 i_2 i_3 \cdots}$. Note that L is not connected since points of L' are discarded. Indeed, for any $V_{1i} = (c_{1i}, d_{1i}) \in \mathcal{V}_1$, $c_{1i} \notin L$ so c_{1i} is a point of disconnection. L will later be shown to be a Souslin line itself, and the topological semigroup operation will be defined on it. Let L have the order topology. This is an important point since the order topology on L is quite different from the subspace or relative topology.

Complete the defininition of $\phi : L \longrightarrow \mathbb{R}^{\omega_1}$ as follows. For any $x \in L$ there exists $\beta < \omega_1$ such that $x \in D_{\beta i_1 i_2 i_3 \cdots i_n}$. Let the λth component of $\phi(x)$ be

$$(\phi(x))_\lambda = \begin{cases} g_{\lambda i_1 i_2 i_3 \cdots i_n}(x) & \lambda < \beta \\ f_{\lambda i_1 i_2 i_3 \cdots i_n}(x) & \lambda = \beta \\ 0 & \lambda > \beta \end{cases}$$

So, to clarify, consider the following 4 examples.

$$x \in D_1 \implies \phi(x) = \langle f_1(x), 0, 0, 0, 0, \cdots \rangle$$
$$x \in D_{2,5} \implies \phi(x) = \langle g_1(x), f_{2,5}(x), 0, 0, 0, 0, \cdots \rangle$$
$$x \in D_{3,7,2} \implies \phi(x) = \langle g_1(x), g_{2,7}(x), f_{3,7,2}(x), 0, 0, 0, 0, \cdots \rangle$$
$$x \in D_{4,93,1001,17} \implies \phi(x) = \langle g_1(x), g_{2,93}(x), g_{3,93,1001}(x), f_{4,93,1001,17}(x), 0, 0, \cdots \rangle$$

Note that ϕ is an injective map. Indeed, for any x in arbitrary $D_{\beta ijk\cdots}$ suppose there exists y such that $\phi(x) = \phi(y)$. Then $g_1(x) = g_1(y) \implies x, y \in V_{1i}$ and $g_{2i}(x) = g_{2i}(y) \implies x, y \in V_{2ij}$ and so on until the βth component. Since $x, y \in V_{\beta ijklm\cdots}$ and $f_{\beta ijklm\cdots}$ is injective, $x = y$. Thus, ϕ is injective, and the inverse map $\phi^{-1} : \phi(L) \longrightarrow L$ exists.

Let $\phi(L)$ have the lexicographic order topology (dictionary order topology) in which $a < b$ if and only if $a_\lambda < b_\lambda$ where the λth component is the first component for which the points differ, and recall that L has the order topology. From the definitions of the maps $f_{\alpha i_1 i_2 i_3 \cdots i_n}$ and $g_{\alpha i_1 i_2 i_3 \cdots i_n}$, $\phi : L \longrightarrow \phi(L)$ is order preserving, and, therefore, ϕ is an order preserving bijection between linearly ordered topological spaces. Any order preserving bijection between two LOTS is obviously a homeomorphism. Therefore, $\phi : L \longrightarrow \phi(L)$ is a homeomorphism.

Define the semigroup operation $* : L \times L \longrightarrow L$ by $x * y = \phi^{-1}(\psi(\phi(x), \phi(y)))$ for all $x, y \in L$, and note the following:

1. $*$ is commutative since ψ is commutative. So, for any $a \in L$ left and right translation by a are the same.

2. For any $x \in D_{\beta\ldots}$ and $y \in D_{\gamma\ldots}$ with $\beta < \gamma$ and for any $\lambda < \beta$, $x_\lambda + y_\lambda \in S$ since $x_\lambda, y_\lambda \in S$ and S is closed under addition.
3. For any $x \in D_{\beta\ldots}$ and $y \in D_{\gamma\ldots}$ with $\beta < \gamma$ and for any λ such that $\beta \leq \lambda < \gamma$, $x_\lambda + y_\lambda \in S$ since $x_\lambda \in W$, $y_\lambda \in S$ and for all $a_i \in W$ and for all $b_i \in S$, $a_i + b_i \in S$.
4. For any $x \in D_{\beta\ldots}$ and $y \in D_{\gamma\ldots}$ with $\beta < \gamma$ and for any $\lambda \geq \gamma$, $x_\lambda + y_\lambda = y_\lambda \in W$ since $x_\lambda = 0$.
5. For any $x, y \in D_{\beta\ldots}$, the βth component x_β, y_β is in W and $x_\beta + y_\beta \in W$ since W is closed under addition.

The above 5 points imply that for any $x \in D_{\beta\ldots}$ and $y \in D_{\gamma\ldots}$ with $\beta \leq \gamma$, $x * y$ is in $D_{\gamma\ldots}$. L is, therefore, closed under the operation $*$. Furthermore, note that for any $a \in D_{\beta\ldots}$, translation by a is not surjective since for all $x \in L$, $a * x \in D_{\gamma\ldots}$ with $\gamma \geq \beta$. In fact, translation by a does not even map intervals onto intervals.

In order to complete the proof, $*$ must be shown to be cancellative, associative, and continuous, and L must be shown to satisfy the CCC and not to be separable. First, to show $*$ is cancellative suppose $x * y = z * y$ for x, y, z arbitrary elements of L. This gives the following:

$$\phi(x * y) = \phi(z * y) \text{ since } \phi \text{ is injective}$$
$$\Longrightarrow \psi(\phi(x), \phi(y)) = \psi(\phi(z), \phi(y)) \text{ since } \phi(a * b) = \phi\left(\phi^{-1}(\psi(\phi(a), \phi(b)))\right)$$
$$\Longrightarrow \phi(x) = \phi(z) \text{ since addition of real numbers is cancellative}$$
$$\Longrightarrow x = z \text{ since } \phi \text{ is injective.}$$

The same argument shows that $y * x = y * z \Longrightarrow x = z$. Therefore, $*$ is cancellative.

Secondly, for associativity:

$$\begin{aligned}
(x * y) * z &= \phi^{-1}(\psi(\phi(x), \phi(y))) * z \\
&= \phi^{-1}(\psi(\psi(\phi(x), \phi(y)), \phi(z))) \\
&= \phi^{-1}(\psi(\phi(x), \psi(\phi(y), \phi(z)))) \text{ since } \psi \text{ is associative} \\
&= x * \phi^{-1}(\psi(\phi(y), \phi(z))) \\
&= x * (y * z).
\end{aligned}$$

Therefore, $*$ is associative.

Thirdly, $*$ is a composition of the maps ϕ, ψ, and ϕ^{-1}, and $*$ is continuous if each of them is continuous. ϕ and ϕ^{-1} are already known to be continuous since ϕ has been shown to be a homeomorphism. So, $\psi : \phi(L)\times\phi(L) \longrightarrow \phi(L)$ is the only map still needing to be addressed. Let $(a, b) \subseteq\ \phi(L)$ be an arbitrary non-empty open interval where $a = \langle a_1, a_2, a_3, a_4, a_5, \ldots\rangle$ and $b = \langle b_1, b_2, b_3, b_4, b_5, \ldots\rangle$ and $\phi(L)$ has the lexicographic order topology. $a < b$ implies a and b differ in at least one component. Let λ be the first such component. So, $a_\lambda < b_\lambda$. Let (p, q) be an arbitrary point in the inverse image $(p, q) \in \psi^{-1}(a, b)$. There are two meanings for $(,)$, and the appropriate meaning should be clear from context. (p, q) is a point in the product space $\phi(L) \times \phi(L)$ and (a, b) is an open interval in $\phi(L)$. Now, there are three cases to consider for the λth component of $\psi(p, q)$.

1. Suppose $a_\lambda < (\psi(p, q))_\lambda < b_\lambda$. By the continuity of addition on $\mathbb{R}$ (in the standard topology) there exists an open set $U \subset \mathbb{R}^2$ such that for any $(x, y) \in U$, $x + y \in (a_\lambda, b_\lambda)$. Thus, there exists an open set $V \subset \phi(L)\times\phi(L)$ containing (p, q) such that $V \subset \psi^{-1}(a, b)$.
2. Suppose $a_\lambda = (\psi(p, q))_\lambda < b_\lambda$. By the continuity of addition on $\mathbb{R}$ there exists an open set $U \subset \mathbb{R}^2$ such that for any $(x, y) \in U$, $x + y > a_\mu$ for some $\mu > \lambda$. Thus, there exists an open set $V \subset \phi(L) \times \phi(L)$ containing (p, q) such that $V \subset \psi^{-1}(a, b)$.
3. Suppose $a_\lambda < (\psi(p, q))_\lambda = b_\lambda$. By the continuity of addition on $\mathbb{R}$ there exists an open set $U \subset \mathbb{R}^2$ such that for any $(x, y) \in U$, $x + y < b_\mu$ for some $\mu > \lambda$. Thus, there exists an open set $V \subset \phi(L) \times \phi(L)$ containing (p, q) such that $V \subset \psi^{-1}(a, b)$.

So, for any point $(p, q) \in \psi^{-1}(a, b)$ there exists an open set $V \subset \psi^{-1}(a, b)$ containing (p, q). Thus, $\psi^{-1}(a, b)$ is an open set. Since (a, b) is an arbitrary open set, ψ is continuous. $*$ is, then, continuous since it's a composition of continuous functions. Therefore, $*$ is a cancellative topological semigroup operation on L.

Now, consider $L \subset L'$. L is totally ordered since L' is totally ordered, and it satisfies the CCC. Furthermore, for each $\alpha < \omega_1$ the endpoints of each interval $V_{\alpha i_1 i_2 i_3 \cdots i_n} \in \mathcal{V}_{\alpha i_1 i_2 i_3 \cdots i_n}$ are not in L since they are neither elements of $D_{\alpha i_1 i_2 i_3 \cdots i_n}$ nor $V_{\alpha i_1 i_2 i_3 \cdots i_n}$. The points in $V_{\alpha i_1 i_2 i_3 \cdots i_n}$, then, cannot be in the closure of $D_{\alpha i_1 i_2 i_3 \cdots i_n}$. So, $\overline{D}_{\alpha\cdots} \cap L = D_{\alpha\cdots}$. Thus, any dense subset must include at least one element of $D_{\alpha\cdots}$ for all $\alpha < \omega_1$. Any dense subset, then, must be uncountable. Therefore, L is not separable. Since L is a LOTS that satisfies the CCC but is not separable, it is a Souslin line. $\square$

4.0 OPEN QUESTIONS AND FUTURE RESEARCH

4.0.7 Density Topology

Theorem 78 states that there are no topological groups on $(\mathbb{R}, \tau_D)$. However, neither the paratopological group case nor the cancellative topological semigroup case is discussed. Perhaps the proof of Theorem 78 can be modified to address the paratopological group case. A review of the proof indicates that the continuity of the inverse operation is only employed in the construction of the open set V that contains the identity and is equal to its inverse V^{-1}. However, Corollary 72, which shows $e < a < b \Longrightarrow b^{-1} < a^{-1} < e$, suggests that this should be possible in the density topology without using the continuity of the inverse operation. For every point x in V that is greater than e, the inverse x^{-1} should be a density point of V provided V is symmetric in the sense of the interval (a^{-1}, a) containing e. This leads to the following conjecture.

Conjecture 95. *For any paratopological group on $(\mathbb{R}, \tau_D)$, the inverse operation is continuous, and, therefore, the paratopological group is actually a topological group.*

Proof of the conjecture would immediately lead to a corollary stating that there are no paratopological groups on $(\mathbb{R}, \tau_D)$.

On the other hand, the proof of Theorem 78 relies heavily on the existence of both an identity element and inverses of elements. Therefore, proof that cancellative topological semigroups do not exist on $(\mathbb{R}, \tau_D)$ would require a drastically different approach. Nevertheless, the fact that addition is not a continuous binary operation in the density topology, suggests that there might not be any continuous cancellative binary operations on $(\mathbb{R}, \tau_D)$.

4.0.8 Souslin

In addition to the Souslin line, there is also a tree named after Mikhail Yakovlevich Souslin. In order to define a Souslin tree, however, we must first define some other properties. First, recall the definition of a tree.

Definition 96. *A **tree** is a partially ordered set T that has a least element such that, for each $a \in T$, the set $\{b \in T : b \trianglelefteq a\}$ is well ordered.*

Trees have a number of different properties such as height, number and length of branches, and antichains. These terms are defined as follows.

Definition 97. *Every well ordered set is known to be isomorphic to a unique ordinal number called the **order type** of the set.*

Definition 98. *For any element a of a tree T, the order type of the well ordered set $\{b \in T : b \trianglelefteq a\}$ is called the **height** of a, and it is denoted $h(a)$.*

Definition 99. *The **αth level** of a tree T is the set $T_\alpha = \{x \in T : h(x) = \alpha\}$, and the **height** of the tree $h(T)$ is the least ordinal for which the level is empty. In other words, $h(T) = \min\{\text{ordinal } \alpha : T_\alpha = \emptyset\}$.*

Definition 100. *A **branch** of a tree T is a maximal linearly ordered subset of T. Since T has a least element, every branch is well ordered, and the order type of a branch is called its **length.** If the length of a branch equals the height of the tree, the branch is called **cofinal**.*

Definition 101. *An **antichain** in a tree T is a subset $A \subseteq T$ such that any two distinct elements $x, y \in A$ are incomparable.*

We are now able to define a Souslin tree.

Definition 102. *A **Souslin tree** is a tree of height ω_1 such that all antichains are countable and there are no cofinal branches.*

A well known theorem asserts that Souslin trees exist if and only if Souslin lines exist. Indeed, this equivalence is the reason for the name Souslin tree. We are, then, led to the following interesting question. Since there exists a cancellative topological semigroup on a

Souslin line (Theorem 94), does there exist a cancellative topological semigroup on a Souslin tree?

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