REVENUE MAXIMIZATION USING PRODUCT BUNDLING

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Submitted to the Graduate Faculty of

The Joseph M. Katz Graduate School of Business in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2009

UNIVERSITY OF PITTSBURGH

THE JOSEPH M. KATZ GRADUATE SCHOOL OF BUSINESS

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July 29, 2009

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Product bundling is a business strategy that packages (either physically or logically), prices and sells groups of two or more distinct products or services as a single economic entity. This practice exploits variations in the reservation prices and the valuations of a bundle vis-à-vis its constituents. Bundling is an effective instrument for price discrimination, and presents opportunities for enhancing revenue without increasing resource availability. However, optimal bundling strategies are generally difficult to derive due to constraints on resource availability, product valuation and pricing relationships, the consumer purchase process, and the rapid growth of the number of possible alternatives.

This dissertation investigates two different situations—vertically differentiated versus independently valued products—and develops two different approaches for revenue maximization opportunities using product bundling, when resource availability is limited. For the vertically differentiated market with two products, such as the TV market with prime time and non-prime time advertising, we derive optimal policies that dictate how the seller (that is, the broadcaster) can manage their limited advertising time inventories. We find that, unlike other markets, the revenue maximizing strategy may be to offer only the bundle, only the components, or various combinations of the bundle and the components. The optimality of these strategies critically depends on the availability of the two advertising time resources. We also show how the network should focus its programming quality improvement efforts, and investigate how the

"value of bundling," defined as the network's and the advertisers' benefit from bundling, changes as the resource availabilities change. We then propose and study a bundling model for the duopolistic situation, and extend the results from the monopolistic to the duopolistic case.

For the independently valued products, we develop stochastic mathematical programming models for pricing bundles of n components. Specializing this model for two components in a deterministic setting, we derive closed-form optimal product pricing policies when the demand functions are linear. Using the intuition garnered from these analytical results, we then investigate two procedures for solving large-scale problems: a greedy heuristic, and a decomposition method. We show the effectiveness of both methods through computational experiments.

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ACKNOWLEDGMENTS

This work could not have been completed without the constant, significant support of my family, my Pitt professors, and my friends. This is but a small token of appreciation for all the guidance that I have received during this period.

First and foremost, I would like to thank my advisor, Professor Prakash Mirchandani, for his mentorship, his supervision, his patience, and his willingness to help me at crazy hours. I am inspired by you, and I hope that I have learned from your work ethic, and your character. I am truly grateful for the opportunity to have worked with you.

My other committee members deserve a lot of thanks for this present work. I would like to thank Professor Esther Gal-Or for her insights and her willingness to help develop this thesis; her contribution was absolutely critical to the success of this dissertation. I am also grateful to Professor Venkatesh for his helpful suggestions about bundling, and to Dr. Brady Hunsaker for being a great teacher. Special thanks are in order for Professor Jerry May, who has been a great role-model for me, for his wit, humor, personality, and guidance during the first years of the program. I would also like to thank Dr. Srinivas Bollapragada for valuable discussions about the problem addressed in this dissertation, and for his help in starting the computational work.

None of this would have been possible without the unconditional support of my family. I would like to thank my wife, Alia, for her love, help, support, patience, understanding, and for constantly being next to me through the highs and the lows. My parents, Liana and Ion, deserve all the thanks in the world for shaping me into what I am today, and for believing in my abilities. This work is dedicated to all of you.

Finally, I would like to thank my friends for being there when I needed it. Scott and Katie, thank you for being wonderful role models and friends, and for encouraging me to earn a doctoral degree. Carrie, thank you for all your amazing help during these years. Radu, Jons, Tudor, Dana, Victor and Silvia, and the rest of the Romanian gang—thank you for your friendship. It's what keeps us sane.

1.0 INTRODUCTION

Across both service and manufacturing industries, a frequently used practice for increasing revenue and profitability combines *components* (that is, individual products) into packages of products. This strategy of selling packages, referred to as *bundling*, allows companies to satisfy customers who may not be interested in buying the individual products, or who derive greater consumer surplus from the packages than they do from the individual products. Thus, by offering product bundles along with the components, a company can increase its market size by appealing to a larger population. Bundling is also an effective instrument for price discrimination, and presents opportunities for enhancing revenue without increasing resource availability. Examples of bundling schemes include automotive option packages (for example, bundling a navigation system with a premium audio system), vacation packages (for example, bundling air, hotel and car rental reservations), software packages (for example, bundling word processing and spreadsheet software), and food product assortments (bundling different flavors of marinade sauces) and cosmetic products (for example, shampoo and conditioner, or mascara, eyeliner and eye shadow, etc).

According to Stremersch and Tellis (2002), bundling manifests itself on a product basis, where there is some degree of integration among the bundle components (e.g. the “quadruple play” packages offered by telecommunication companies, which integrate phone, Internet, television and mobile services), or on a price basis (e.g. season tickets for a sports team), where the absence of sufficient value-adding integration may require price-discounting. As these examples indicate, a seller offering multiple components faces the following alternative strategies (Adams & Yellen, 1976; Guiltinan, 1987): (i) *pure-components* strategy, in which the seller only offers the components as separate items; (ii) *pure-bundling* strategy, in which the seller only offers the bundle, but does not offer the components; and (iii) *mixed-bundling* strategy, in which the seller offers both the components and the bundle(s). Our work examines a mixed-bundling situation, and refers to both components and bundles as *products*. Given a distribution of customers' willingness to pay and the component availability, our objective is to determine the revenue maximizing pricing strategy. We examine two different situations—vertically differentiated versus independently valued products—and develop two different approaches for revenue maximization opportunities using product bundling. For the vertically differentiated market with two products, such as the television market with prime time and non-prime time advertising, we derive optimal policies that dictate how the seller (that is, the broadcaster) can manage their limited advertising time inventories, in a monopolistic as well as a duopolistic environment. For the independently valued products assumption we analytically derive optimal policies for the two components/one bundle scenario, and derive heuristics for pricing arbitrary number of products.

In the context of TV advertising, broadcasters use a multi-pronged strategy to capture revenues from the roughly $150 billion dollar advertising market in the US.[1] The market for selling television advertising time is split into two different parts: the *upfront* market, which accounts for about 60%-80% of airtime sold and takes place in May every year, and the *scatter*

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[1] 2007 TNS media intelligence report (http://www.tns-mi.com/news/03252008.htm). Of this amount, television advertising accounts for roughly $64 billion annually.

market which takes place during the remainder of the year. In the first stage of their strategy, broadcast networks make decisions about how much advertising time to sell in the upfront market and how much to keep for the scatter market. On their part, clients purchase advertising time in bulk, guided by their medium-term advertising strategy, during the upfront market, at prices that may eventually turn out to be higher or lower than the scatter market prices. The scatter market, on the other hand, allows advertisers to adopt a "wait-and-see" approach to verify the popularity of various network shows, and to tailor their decisions to match their short term advertising strategy. In the television advertising market, capacity constraints play a significant role in determining the broadcaster's optimal strategy. Particularly, the relative scarcity of the two resources, prime time and non-prime time, is the main driver of any sort of optimality analysis. The prime time resource availability constraint is far more likely to be binding than the non-prime time resource availability constraint. Prime time on television is usually the slot from 8:00 pm until 11:00 pm Monday to Saturday, and 7:00 pm to 11:00 pm on Sunday. Hence, the ratio of prime to non-prime time availability is about 1:8 (or 1:6 on Sunday).

1.1 REVENUE MANAGEMENT AND PRODUCT BUNDLING

Due to the peculiarities of the TV advertising market, our work diverges from the current revenue management stream in several ways. Traditionally, the revenue management literature has focused on the airlines, hotels, cruise, automotive rental markets, and railway markets. In the airlines, hotels, etc. situation, even though there is a strict ordering of the components, the bundle may not be preferred to the "prime" product. For example, in the airline setting, an individual may prefer a business seat to an economy seat on the same flight, but the individual is not likely to buy a bundle consisting of a both a business and an economy seat. Similarly, a traveler may prefer a suite in a hotel to an ordinary double room on the same day, but is unlikely to rent a bundle consisting of the suite and the double room. Likewise, an executive may prefer a full-size car to a compact during the same trip, but is unlikely to rent both the full-size car and the compact. This is quite different from a vertically differentiated market (e.g. the TV advertising market) where all advertisers prefer buying a bundle of prime time advertising and non-prime advertising to buying just prime time advertising, and prefer buying prime time advertising to buying just non-prime advertising.

The advertising market is different from other revenue management applications in another way. A "bundle" in the airlines industry could be two flight segments. Consider three demands: Pittsburgh to New York, New York to Boston, and Pittsburgh to Boston. Airline revenue management applications might consider a bundle of Pittsburgh to New York and New York to Boston to meet the Pittsburgh to Boston demand. But clearly a preference order does not exist in this case for the traveler: in fact the bundle may have lower utility than the Pittsburgh to New York product for someone wanting to travel to New York from Pittsburgh. Similar situations also occur in the hotel industry where a bundle may consist of a hotel room on Sunday and a hotel room on Monday. A business traveler, wanting to spend Sunday night with her family might have no interest in purchasing the bundle. So, in these cases, consuming the bundle may have lower utility than consuming the individual components.

The methodologies developed and the approaches adopted in the revenue management literature are also different from our work. While the revenue management literature is quite extensive, none of the revenue management papers adopt the "market segmentation through self selection" (that is, second degree price discrimination) approach as we do in this work. On the contrary, “fences” (Saturday night stay requirement to prevent a business traveler from using a leisure fare, or a student id requirement to prevent a regular patron from buying a discounted concert ticket) are typically constructed to prevent spillage. Moreover, in most revenue management papers, the prices are exogenously given rather than endogenously determined, as in our work. While there are papers that consider simultaneous pricing and inventory management decisions (Dana Jr & Petruzzi, 2001; Petruzzi & Dada, 1999; Raz & Porteus, 2006), multiple components and/or the availability of a bundle are not considered in these papers at all.

Finally, according to Bollapragada and Mallik (2008), Zhang (2006), and Araman and Popescu (2009), the current practice in the broadcasting industry is to use subjective methods for media planning decision, and only a few analytical models have been developed. Talluri and van Ryzin (2004), the most up-to-date, comprehensive reference on revenue management, describes some scheduling models for this application context. Araman and Popescu (2009) suggest that technical complexity argues for decomposing the general media planning problem into smaller, tractable problems, and consider allocating advertising time capacity between the upfront and scatter markets given the uncertainty of the audience. Bollapragada and Mallik (2008) focus on how to manage the “rating points” inventory for servicing the upfront market. Zhang (2006) develops a hierarchical approach for matching advertisers to shows and then constructing a broadcast schedule. Up to this point, none of these recent articles, as well as other recent television media related papers, consider the issue of bundling problem applied onto the TV advertising market.

1.2 OBJECTIVES OF THIS WORK

The current thesis aims to address two major issues. The main contribution of this research is to study the effectiveness of various bundling strategies on a market characterized by ordered preferences (i.e., the TV advertising market). Here, we show how the various strategies shift as a function of the advertising time inventory, and we also study the impact of the distribution of clients over the overall profitability of the TV network. Chapters 3 and 4 of the dissertations study this problem under different assumptions. In Chapter 3, we start with a monopolistic framework, and we assume that the buyers (in this case, the advertisers interested in purchasing advertising time) self select into different segments (non-purchasers, buyers of only one type of advertising product, or buyers of bundles containing both the "lower quality" and the "higher quality" product). In the TV context, the "lower quality" time is called *non-prime time* and the "higher quality" time is referred to as *prime time*. The availability of these two resources is limited. Traditionally, the bundling research has consistently found that mixed bundling strategies dominate other bundling strategies, because a bundle is able to capture extra revenues by reducing the heterogeneity of the consumers. Moreover, the issue of whether bundling benefits only one player (buyer or seller) involved in the transaction, or whether bundling is a win-win proposition when the resource availability is limited, is not addressed in the literature. Our work in Chapter 3 seeks to investigate these issues in the TV media context, and see whether conventional wisdom with respect to bundling holds, or whether the intrinsic characteristic of this market induces a different behavior. Additionally, we are also interested in finding out as to what are the incentives for improving programming quality in the context of selling bundles of prime and non-prime time. In Chapter 4, we address the same questions but this time in a duopolistic setting. Hence, Chapter 4 extends the results of Chapter 3. Overall, the contributions of the first two essays can be summarized in the following main discussion points:

The pure components strategy may dominate the mixed bundling strategy. Past bundling literature in a monopoly setting (Adams & Yellen, 1976; McAfee, McMillan, & Whinston, 1989; Stigler, 1963) has demonstrated that the mixed bundling strategy (weakly) dominates the pure components and the pure bundling strategies for independently valued components. We investigate the generality of this result and show that when all customers have a common preference ranking of the products, and the resource availability is unconstrained, then the pure bundling strategy is optimal. More importantly, we show that with constrained resource availability, the optimal strategy depends on the scarcity of the resources. In particular, we show that the pure components strategy may be the optimal strategy, dominating the mixed bundling strategy, when the resource availabilities are low. Thus, the clean, unambiguous structure of the optimality of the mixed bundling strategy breaks down when the preferences for the products are ordered and the resources are limited.

The skewness of the customer distribution is important in addition to its heterogeneity. Schmalensee (1984) points out that the reason for the dominance of the mixed bundling strategy stems from the fact that it allows greater price discrimination by reducing the *heterogeneity* of the customers. Our results show that the *skewness* of the distribution of customers, in addition to the heterogeneity, affects the benefits of mixed bundling. To our knowledge, previous research has not studied the impact of skewness on bundling.

Should programming quality be improved? *If so, which one*? One question that effective managers always ask is how they can do better: in this case, how can the broadcasters' profit be increased? Should we try to improve the ratings of the prime time or the non-prime time programming? We answer this question by concluding that, under fairly mild additional assumptions, it is always (that is, under all bundling strategies) better to improve the ratings of the resource that is more plentiful. Additionally, when the availabilities of the two resources are equal to each other, it is always better to increase the ratings of prime time programming, but the relative benefit from quality improvement of prime time programming depends on the overall resource scarcity.

Does bundling in our context improve consumer and social welfare? When decision makers evaluate a new strategy, they need to consider not just what the impact on their bottom line would be, but also how customers and society, in general, would be impacted. Are there situations where everyone (in this case, the broadcaster and the advertisers as a group) would be better off? We answer this question in the affirmative by computing the value of bundling for the broadcaster, the advertiser, and aggregate, in both monopolistic and duopolistic settings. The consumer and social welfare measures have been studied in other contexts, and sometimes a similar phenomenon has been observed. However, the relationship of the value of bundling to the scarcity of resources, and as a result, to the optimal bundling strategy, has not been addressed at all in the literature.

In a competitive environment with a strong (in terms of ratings) network and a weaker one, the strong network uses the non-prime time product as a deterrent. Previous literature on bundling in competitive environments has shown that the bundle can be used as a deterrent (Nalebuff, 2004). Our work suggests that in a vertically differentiated product market with a weak and a strong player, the strong firm uses the lower quality bundle component as a deterrent, because in the limit it can be sacrificed and given away for free (the marginal costs are zero in our model), in order to protect the bundle.

In the final essay, we depart from the vertical differentiation model and study the general mixed bundling problem. Here, we formulate the general mixed bundling problem under both stochastic and deterministic demands, and investigate the properties of the deterministic approach. We can summarize the contributions of this work along the following discussion points:

We investigate the connections between the optimal bundling strategies for vertically differentiated and independently valued products. We show analytically that the regions defined by the number of binding capacity constraints are similar in both situations. However, the dominant strategies are very different within each such region. (As we will discuss later, the assumptions underlying the two situations are quite different.)

We analyze effective solution methodologies for large-scale versions of the mixed bundling problem. We investigate two different approaches that are computationally efficient: a generic greedy heuristic for pricing arbitrary bundles of products with independent valuations, and a decomposition method. We investigate the theoretical performance of the heuristic, and observe that in practice it tends to perform well for moderate-sized problems. Then, we formulate and evaluate a decomposition method that is geared towards large-scale problem instances. We find that, in our limited computational experiments, on average more than 99.9% of the constraints of the optimization model are naturally satisfied by the solution, and therefore, it is possible to save valuable computational time by doing an implicit, rather than explicit enumeration of all the model constraints. Even more interesting, only a fraction of all possible bundles end up being offered in practice. With careful selection rules which we expand upon in Chapter 5, we can save a lot of computational time by identifying these candidates. Therefore, the overall theme of this last essay is the development of efficient algorithmic approaches for this hard combinatorial problem.

1.3 OVERVIEW OF CHAPTERS

Following this introductory chapter, the second chapter provides an overview of the current research stream, as it is applicable to revenue management and product bundling. This chapter builds an introductory foundation upon which this work can extend the current state of the art. Subsequent chapters will address relevant research literature in a more focused manner, in their corresponding introductory sections.

Chapter 3 examines a basic monopolistic setting in which the TV network seeks to maximize its revenues from sales of limited prime and non-prime advertising time, using different bundling strategies. We examine the impact of the relative scarcity of advertising time on the different strategies, quantify the effect of the shadow prices on the bundle composition, and look at the network relative incentive to improve the programming quality (and thus, its ratings). We show how changing the distribution of the customers affects our results. We also show how the value of bundling changes (the value of bundling is the net benefit for both the advertisers and the network) as the relative availability of the two advertising time resources changes.

Chapter 4 extends the monopolistic framework to a competitive duopolistic environment. We examine the impact of competition on the bundling strategies, and we also quantify the value of bundling. We present and interpret several structural properties of the value of bundling function, and show that there exist certain scenarios where bundling is a win-win proposition for all parties.

Chapter 5 approaches the bundling issue when the products are independently valued with known demand curves. We also provide a heuristic approach that generates near-optimal solution to the revenue maximizing mixed bundling problem, and conclude with a worst-case behavior analysis of its performance, along with several computational experiments. Finally, we introduce a decomposition-based framework that efficiently solves large scale instances of the mixed bundling problem, formulated as a convex optimization program.

Finally, Chapter 6 is a summary of the work developed in the thesis, with discussion on limitations and plans for improvement and extensions.

2.0 REVENUE MANAGEMENT AND PRODUCT BUNDLING

Businesses that sell perishable goods or services often have to manage a relatively fixed inventory of a product over a planning horizon. *Revenue management* (sometimes also referred to as demand management) is the active administration of all processes that could generate extra revenues from an existing inventory (or in some cases, capacity), by making better decisions with respect to pricing and/or allocation of a particular (or an entire line of) good(s) or service(s) that the company is offering. Organizations that use revenue management techniques often employ various techniques, such as priority rules for inventory allocation, customer segmentation, forecasting, and the dynamic adjustment of prices. Successful implementations of revenue management techniques have led to increased revenues and profits for many organizations across various industries, most notably airline, hotel, restaurant and car-rental businesses. Opportunities are now arising for the introduction of revenue management techniques into non-traditional areas, such as healthcare and the entertainment and advertising industries. Making decisions about the prices to charge and the availability of those products or services for each market segment over a period of time with the goal of increasing the expected profit pertains to revenue management. Thus, revenue management is sometimes referred to as "the art of maximizing the profit generated from managing a limited capacity of a product over a finite horizon, by selling each product to the right customer, at the right time, for the right price." (Talluri & Van Ryzin, 2004)

One of the major pillars supporting the revenue management foundation is the concept of *market segmentation* into multiple classes (e.g., leisure versus business travelers), where different types of products (e.g., seats on an airline with restricted or fully refundable fares) are targeted to each class. Another important operational driver is the idea that some resources are *perishable*. A resource is perishable if after a certain date becomes either unavailable or it ages at a significant cost. Seats on a flight or in a theater, rooms in a hotel, space on a cargo train, are a few examples of such perishable inventory, so the main focus of revenue is on the allocation of limited and perishable capacity to different demand classes (Elmaghraby & Keskinocak, 2003).

Revenue management, or yield management as it was initially called, started in the airline industry, back in the late 1970s, as a need for airline companies to cope with the increased competition when many fares became available, following the Airline Deregulation Act of 1978. Airlines had to manage the discounted fares that became part of their product offers, and the opportunities for revenue management techniques and models were acknowledged very fast. Their positive impact on revenue was attested by many companies. For example, American Airlines had a $1.4 billion in incremental revenue over the three year period between 1989-1992 (Smith, Leimkuhler, & Darrow, 1992). Recent successful applications of revenue management principles span industries beyond airlines. For example, Geraghty and Johnson (1997) report that successful revenue management saved the National car rental company from bankruptcy. In another study, Bollapragada et al. (2002) report significantly improved revenues after optimizing NBC's commercial scheduling systems, and Metters et al. (2008) report the successful application of revenue management-based segmentation at Harrah's Cherokee Casino. Other interesting applications of revenue management can be encountered in car rental businesses (Savin, Cohen, Gans, & Katalan, 2005), media advertising (Araman & Popescu, 2009;

Fridgeirsdottir & Roels, 2009), internet service providers (Nair, Bapna, & Brine, 2001), cargo shipping (L. H. Lee, Chew, & Slim, 2007; Pak & Dekker, 2004), and restaurants (Kimes, 1999). The most comprehensive survey articles that encapsulate the past literature and main results in revenue management are, chronologically, those of Weatherford and Bodily (1992), McGill and van Ryzin (1999), and Talluri and van Ryzin (2004).

Traditionally, revenue management research is broadly split along two dimensions. The *quantity based* revenue management is mainly concerned with capacity allocation decisions. In the airline case, for example, one of the tactical decisions is to determine the number of seats to make available to each fare class from a shared inventory and how many requests from each class to accept, in order to maximize total expected revenues, taking into account the probabilistic nature of future demand for a flight (Belobaba, 1989). In other words, given a booking request for a seat in an itinerary in a specific booking class, the fundamental revenue management decision is whether to accept or reject this booking, considering the past and future demands. In the hotel industry, the manager has to decide at the operational level, for example, whether or not to rent a room to a customer that requests it, considering the reservations already made, future reservation requests, and the potential walk-ins (customers that show up without a reservation). So it is not at all uncommon to deny an advanced booking (in either business) to price-sensitive customers for peak travel periods because it is anticipated that there will be enough demand from higher paying customers. The analysis of capacity (seat) allocation, (that is, controlling the mix of discount fares and early booking restrictions) and overbooking (selling more seats than available when cancellations and no-shows are allowed) are supported by a thorough understanding of customer behavior and the capability to accurately forecast future demand. The three most important airline and hotel revenue management interrelated aspects and areas of research are forecasting, seat allocation and overbooking. On the other hand, the *price based* revenue management is concerned with pricing decisions. These decisions can be different, depending on the industry. For example, in the airlines and the hotel industry, one form of control is that of bid prices, where the request for a seat (or a room) is accepted only if the price offered exceeds a threshold established by the seller (for example, in the airlines industry the most common way to compute a bid control for a flight is to sum the dual prices of all the capacity constraints associated with each leg of that particular flight). In the retail industry, an efficient form of price control is that of markdown pricing, where, at certain time intervals during the season, the prices for different items are permanently reduced. Finally, across various industries, an efficient technique is that of dynamic pricing, which refers to the adjustment of prices (either upwards or downwards) at various moments during the planning horizon.

Revenue management is attributable to bringing new ideas and models that changed the paradigm about doing business. In one form or another, revenue management applications and their consequences are felt more and more, be it when renting a hotel room or a car online or trying to find a deal in a superstore by buying a bundling of products. Revenue management is actively trying to reach new business settings and one of the current research focuses is finding ways to better incorporate customer behavior, lifetime customer value and competitive response into the revenue management decisions (Phillips, 2005).

2.1 BUNDLING STRATEGIES

Bundling is a prevalent business strategy. Most of the bundling papers are built on the early study of Stigler (1963), who concludes that bundling is profitable when the reservation prices of the components are negatively correlated. Later, Adams and Yellen (1976) show that the profitability of bundling can stem from its ability to sort customers into groups with different reservation price characteristics, thus extracting greater consumer surplus. They examine the three basic bundling strategies (pure components, pure bundling, and mixed bundling), compare these strategies in terms of seller profit and find that mixed bundling at least weakly (meaning that the revenues collected from a mixed bundling strategy are at least as high as those collected if some other strategy were followed) dominates pure bundling, since customers with negatively correlated reservation prices prefer individual products, while the others prefer the bundle. A related paper, (Dansby & Conrad, 1984) finds the same effect, as well as the study made by McAffee, McMillan and Whinston (1989). Bundling can also be used strategically, as an entry barrier, as Nalebuff (2004) shows in a recent paper.

Schmalensee, in two early papers (1982, 1984) relaxes the assumption that the reservation prices of the individual products are negatively correlated, and examines the case of a monopolist offering two products. He constructs a class of examples within which the profitability of bundling can be analyzed as a function of production costs, the mean and variance of the distribution of reservation prices for each product, and the correlation between the reservation prices of the two products. Schmalensee also demonstrates that mixed bundling combines the advantages of pure bundling and pure components strategies, because this policy enables the seller to reduce effective heterogeneity among those buyers with high reservation prices for both goods, while still selling at a high markup to those buyers willing to pay a high price for only one of the goods. An interesting consequence is that bundling can be profitable when demands are uncorrelated or even positively correlated.

Keeping it in the same two-product scenario, Venkatesh and Kamakura (2003) examine the relationship between the products—that is, whether they are complements, substitutes, or independent—and derive analytical solutions, based on the bivariate uniform distribution of consumers' reservation prices, for pricing either a pure bundle or the components separately, and do a numerical simulation for the mixed bundling scenario. Earlier, while examining a situation within the entertainment industry (pricing season tickets for an event), Venkatesh and Mahajan (1993) determined that mixed bundling can dominate both pure bundling and components strategies under certain conditions of the prices. They also derive analytical and numerical results for the profit maximizing equations when the probability density function that describes the customers' reservation prices follows a Weibull distribution.

Bakos and Brynjolfsson (1999) study the strategy of bundling a large number of information goods (that have zero marginal costs), such as those increasingly available on the Internet, and selling them for a fixed price. Interestingly, they find that bundling very large numbers of unrelated information goods and offering only the bundle (that is, a pure bundling strategy) can be surprisingly profitable, and can dominate the mixed bundling strategy. This research contrasts with the physical bundling scenario, where a negative product correlation seems to be the main driver of mixed bundling profitability. In a latter paper (2000), the same authors extend their research to a general competition model on the Internet via bundling. Keeping the same approach, Altinkemer (2001) examines the bundling strategy in the online environment of e-banking strategies.

Salinger (1995) focuses on the graphical analysis of bundling and deals with the two-product case, while assuming additive reservation prices. He explores the implications of the relationship between the bundle and aggregated components demand curves for the profitability and welfare effects of bundling, and finds that if it does not lower costs, bundling tends to be profitable when reservation values are negatively correlated and high relative to costs. If bundling lowers costs and costs are high relative to reservation values, positively correlated reservation values increase the incentive to bundle. On the other hand, Soman and Gourville (2001) illustrate how the bundling of services can hurt consumption, due to its nature of hiding costs from consumers (they argue that bundling increases the valuation complexity).

The internal valuation of bundles is a well-established marketing research area (Yadav, 1994; Yadav & Monroe, 1993). Chung and Rao (2003) examine the valuation of bundles comprised of heterogeneous products that could belong to several categories, and its implication on any optimal bundle pricing. For an ample study of factors that drive bundle purchase intentions, the handbook of Fuerderer, Herrmann and Wuebker (1999) provides a comprehensive treatment of the subject.

One notable shortcoming of most of these research papers is the relatively small number of optimal bundle prices derivations. One important study that examines this topic is the paper of Hanson and Martin (1990) which provides a practical method for calculating optimal bundle prices. The basis of the approach is to formulate the model as a mixed integer linear program using disjunctive programming. The authors also consider one of the most serious problems facing a product line manager addressing the bundling issue: the exponential growth in possible bundles which results from increasing the number of components considered. An algorithm for finding optimal solutions is given along with computational results. A different approach is undertaken by Bitran & Ferrer (2007), who provide a utility-maximizing analytical model for pricing bundles that will compete with other bundles on markets characterized by rapid technological innovation. Cready (1991) also develops profitability conditions for bundles that can be sold at a premium price.

According to Stremersch and Tellis (2002), a significant number of published bundling studies are fuzzy about some basic terms and principles and do not provide a comprehensive framework on the economic optimality of bundling. They provide a new synthesis of the field of bundling based on a critical review and extension of the marketing, economics and law literature, while clearly and consistently defining bundling terms and principles. They also propose a framework of twelve propositions that prescribe the optimal bundling strategy in various contexts, which incorporate all the important factors that influence bundling optimality.

2.2 FORECASTING

Forecasting is a critical component of any revenue management system, and in particular forecasting of sensible variables, such as demand and price sensitivities. There are studies (Polt, 1998) which suggest that a 20% reduction in the forecast error can yield a 1% increase in the revenues generated from the system. Moreover, in a very elegant paper, Cooper, Homem-de-Mello and Kleywegt (2006) show how incorrect assumptions about customer behavior result in lost sales, which trigger in return further incorrect capacity allocations in a downward spiral that could get out of control.

The survey paper of McGill and van Ryzin (1999) lists, in chronological order, most relevant forecasting research in the airline industry. They present historical results of models for both demand distributions and arrival processes, as well as issues related to uncensoring demand data and aggregate and disaggregate forecasting. In terms of demand distributions, the early work of Beckman and Bobkowski (1958) and Lyle (1970) offer evidence, after testing various distributions for the passengers arrivals, that the gamma distribution provides the most reasonable fit for the data. But later, various empirical studies, like in Belobaba (1987), have shown that the normal distribution, as a limiting distribution for both the binomial and Poisson distributions, is a good continuous approximation to aggregate airline demand distribution.

Regarding the customers' arrival distribution, various forms of Poisson processes have been proposed and used: homogeneous, nonhomogeneous and compound Poisson processes, in the research works of Lee and Hersh (1993), Gallego and van Ryzin (1994), Zhao and Zheng (2000), Bitran and Mondschein (1995) just to mention a few. For example, Weatherford et al. (1993) modeled the passengers arrivals as a nonhomogeneous Poisson process to investigate how to optimally implement decision rules for two fare classes, where the arrival rates are modeled with Beta functions and total demand using a Gamma distribution. They showed that that under certain characteristics of the arriving population, the simple static decision rule is a very good approximation to the optimal advanced static rule and can be applied as a heuristic to three or more classes.

Forecasting is one of the central issues in revenue management as its accuracy level has a great impact over the results of the revenue management systems. The regression technique, as a forecasting method, was showed to improve the efficiency of the revenue management systems (Boyd & Bilegan, 2003; Sa, 1987). Exponential smoothing and moving averages, as part of disaggregate forecasting systems, are also commonly implemented by airlines and hotels.

Even if these Poisson processes and smoothing approaches provide insights into future bookings in the same class, it is recognized, though, that these methods may fail to reflect the possible relations that may exist between various fare classes (diversion and possible sell-ups, for example). Weatherford (1999) and Weatherford et al. (2001) provide evidence that more sophisticated, disaggregated forecast methods are needed to improve the forecasting activity. One step in this direction is taken by Lan et. al. (2008) who derive booking policies for the airline network revenue management problem in the absence of information about the demand.

2.3 THE MEDIA ADVERTISING MARKET

While there has been extensive work in the marketing literature regarding the impact of advertising on sales and on consumers (see for instance Kanetkar, Weinberg and Weiss (1992) and Gal-Or et al. (2006)), the operational problem of air-time inventory management is relatively recent. From a scheduling perspective, the work done for NBC studios (Bollapragada, Bussieck, & Mallik, 2004; Bollapragada, et al., 2002) presents a coherent, deterministic optimization model for creating an advertising plan, while observing several scheduling constraints. In the same deterministic framework, Kimms and Muller-Bungart (2007) proposed a unified approach for the separate problems of matching advertisers to shows and scheduling commercials in different slots. In a related paper, Zhang (2006) tackles the same problem using a two-stage approach.

In contrast, new work is emerging that focuses on the inherent uncertainties of the problem. The major issue is that of audience (rating) uncertainty— this in respect drives the allocation decision between selling capacity during the upfront market, and selling the reminder on the scatter market. In this context, the work of Araman and Popescu (2009) deals with the issue of properly allocating and then adjusting inventory time during the upfront season in order to deal properly with the rating variability. They also mention the connection between the broader issue of capacity allocation under uncertainty in the media market, and the random yield production planning problem (Bollapragada & Morton, 1999), if no holding costs are assumed. Similarly, Bollapragada and Mallik (2008) derive a value-at-risk model for allocating rating points between upfront and scatter markets.

2.4 BUNDLING IN COMPETITIVE ENVIRONMENTS

In the recent past, researchers from the economics and marketing domains have investigated bundling related issues in a competitive environment. Matutes and Regibeau (1992) analyzed the interactions between two players engaged in a duopolistic competition, and showed that the optimal strategy is for companies to provide compatible products (such that consumers could theoretically form their own bundle by purchasing each component from a different firm), but to offer a discount if all components are purchased from the same firm. If the components are "incompatible" (i.e., components from different competitors cannot form a bundle), then they argue that the optimal strategy is pure bundling. In the context of market expansion, Kopalle et al. (1999) show that if the market has limited growth potential, the equilibrium strategy tends to be to offer pure components, in the limit, because there is less incentive to attract customers with discounts when the market is saturated. In a recent paper, Armstrong and Vickers (2009) show that bundling can harm customer welfare if customers are heterogeneous in their demand and there are costs associated with purchasing from one firm. If the heterogeneity is reduced, then bundling can increase customer welfare. Thanassoulis (2007) also looks at customer welfare in the context of mixed bundling and finds that if the buyers have brand-specific tastes, or incur firm-specific costs, then their welfare is reduced, but on the other hand it increases when the differentiation between components increases. Chen (1997) shows that bundling is an equilibrium strategy in a duopoly where at least one good that could be part of the bundle is produced under perfect competition, and that if both players in the duopoly commit to bundling, then they increase their profits, but the social welfare is reduced. This idea is confirmed by Gans and King (2006) who find that if competitors can negotiate bundling arrangements, consumers will end up consuming a sub-optimal bundling mix. Separately from the optimality of bundling question, Nalebuff (2004) shows that in a competitive model where a company has market power in two goods, it can protect its turf from potential entrants by packaging these goods into a bundle.

2.5 SUMMARY

As we have mentioned previously, bundling has received considerable attention in the economics and marketing literature. Most of the research conducted in this area studies the conditions under which bundling is profitable for the seller and/or the customer, with the general result being that the profitability of bundling depends on the distribution of reservation prices. We note that bundling studies in economics and marketing literature make an implicit assumption that there is an ample supply of products that could be acquired at a certain cost. In this thesis, however, we assume that there is a fixed amount of perishable inventory for each product to be sold over a finite horizon, and we study how individual and bundle products should be priced to maximize revenue from this limited inventory.

We should also note that while the existing research in marketing and economics studies the performance of different bundling strategies, the emphasis is not necessarily on explicitly optimizing the bundle and the individual product prices. In this thesis, our focus is on optimizing the bundle and individual prices when resources are scarce. We also seek to complement the extant revenue management revenue stream in the following way. In most revenue management papers, the prices are exogenously given rather than endogenously determined, as in our work. While there are papers that consider simultaneous pricing and inventory management decisions (Dana Jr & Petruzzi, 2001; Petruzzi & Dada, 1999; Raz & Porteus, 2006), multiple components and/or the availability of a bundle are not considered in these papers at all. Our contribution to this research stream is to show that bundling can be used as a successful capacity management strategy, where the resources managed are exactly of the type studied by the revenue management literature (fixed and perishable).

3.0 MIXED BUNDLING PRICING STRATEGIES FOR THE TV ADVERTISING MARKET

3.1 INTRODUCTION

Advertising accounts for about two thirds of the total revenue[2] for a typical television broadcast network. While the quality of the programming affects the ratings and thus the demand for television advertising, effective strategies for selling the advertising time are an important determinant of the broadcaster's revenue. Determining such strategies is particularly important because the broadcaster's available advertising time is limited either by competitive reasons (as in the US, where commercials account for roughly eight minutes for every 30 minute block of time) or by government regulations (as in the European Union,[3] where commercials are limited to at most 20 % of the total broadcast time). Moreover, the advertising time is a perishable resource; if it is not used for showing a revenue-generating commercial, the time and the corresponding potential revenue is lost forever.

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[2] Ad Revenue Down, CBS Posts Profit Drop of 52%. *The New York Times*, February 18, 2009. http://www.nytimes.com/2009/02/19/business/media/19cbs.html.

[3] http://www.europarl.europa.eu/sides/getDoc.do?language=NL&type=IM-PRESS&reference=20071112IPR12883

Broadcasters therefore use a multi-pronged strategy to capture revenues from the roughly $150 billion dollar advertising market in the US.[4] The market for selling television advertising time is split into two different parts: the *upfront* market, which accounts for about 60%-80% of airtime sold and takes place in May every year, and the *scatter* market which takes place during the remainder of the year. In the first stage of their strategy, broadcast networks make decisions about how much advertising time to sell in the upfront market and how much to keep for the scatter market. On their part, clients purchase advertising time in bulk, guided by their medium-term advertising strategy, during the upfront market (at prices that may eventually turn out to be higher or lower than the scatter market prices). The scatter market, on the other hand, allows advertisers to adopt a "wait-and-see" approach to verify the popularity of various network shows, and tailoring their decisions to match their short term advertising strategy.

Our work develops revenue maximizing strategies as they apply to broadcast networks making decisions during the scatter market period. The broadcaster makes available for sale limited amounts of advertising time during different categories of daily viewing times. Advertisers value these categories differently because television audience size varies by the time of the day. In particular, evening time, called *prime time*, traditionally attracts the most viewers, and as such is deemed more valuable by the advertisers, while the rest of the viewing time is referred to as *non-prime time*. A critical decision for the broadcaster is how to price these products (that is, the advertising time sold in the different categories) at levels that maximize revenue. Optimally aligning the prices with the advertiser's willingness to pay ensures that the network neither leaves "money on the table," nor uses the advertising resource inefficiently.

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[4] 2007 TNS media intelligence report (http://www.tns-mi.com/news/03252008.htm). Of this amount, television advertising accounts for roughly $64 billion annually.

Moreover, ad hoc pricing can lead to improper market segmentation: advertisers with a higher propensity to pay may end up buying a less expensive product. Likewise, some potential advertisers may be priced out of the market due to improper pricing, even though doing so may be unprofitable for the network. The broadcast network faces yet another decision which is based on an evaluation of the benefits of enhancing the programming quality. Improving quality requires effort (time and money), but can lead to higher ratings. However, the impact of better quality on the network's profitability may be different depending on whether it relates to prime or to non prime time programming. The question that broadcasters need to answer is the amount of effort they should apply to improve programming quality.

The complexity in the analysis for the situations described above gets amplified significantly if the network decides to use bundling—the strategy of combining several individual products for sale as a package (Stigler, 1963). In this regard, the broadcaster has several options available (Adams & Yellen, 1976): (i) *pure components* strategy, that is, offer for sale the different categories of advertising time as separate items only; (ii) *pure bundling* strategy, that is, offer for sale advertising time from the different categories only as a unified product; and (iii) *mixed bundling* strategy, that is, offer for sale both the bundle and the pure components. Mixed bundling offers an opportunity to the broadcaster to more precisely segment the market. However, as the number of components increases, the number of bundles that can be offered in a mixed-bundling strategy increases exponentially. As a consequence, the number of pricing relationships that need to hold also increases exponentially. Specifically, the broadcaster needs to ensure that the price of each bundle should be no more than the price of its component parts. Otherwise, the advertiser can simply buy the separate parts instead of the bundle (Schmalensee, 1984). If the number of bundles is exponential, so is the number of such pricing constraints. To keep the problem tractable, and since our intent is to draw out qualitative managerial insights to help the broadcaster make decisions regarding the available advertising time resources during the scatter market, we begin by assuming that the components each consist of one unit of prime and non-prime time respectively, and the bundle consists of one unit each of the two components. We later show that under some situations these earlier results apply with a simple recalibration of the units of measurement of the components. When the bundle composition can be chosen by the advertiser, one might consider potentially using an elegant approach proposed by Hitt and Chen (2005). This approach, *customized* bundling, allows buyers to themselves create for a fixed price idiosyncratic bundles of a specified cardinality from a larger set of available items. Wu et al. (2008) use nonlinear programming to further explore the properties of customized bundling. The customized bundling approach is not needed for the equal proportions television advertising case that we are considering; moreover, as we discuss later, we assume that the available resources are limited, and so the customized bundling model does not directly apply. Therefore, we focus on the seller (that is, the network broadcaster) creating and offering the bundle for sale.

In the television advertising case (as opposed to other bundling situations), the two components have a fundamental structural relationship. Since viewership during prime time hours exceeds the viewership during non-prime time hours, all advertisers prefer to advertise during prime time as compared to advertising during non-prime time hours. Therefore, the prime time product offered is more attractive than the non-prime time product. This natural ordering of the advertising products offered by the broadcaster implies that, given suitably low prices for the three products, all advertisers prefer the non-prime time product to no advertising, the prime time product to the non-prime time product, and the bundle to the prime time product. In the bundling context, this type of preference ordering between the components does not always exist. Indeed, the traditional bundling literature has focused on independently valued products (Adams & Yellen, 1976; Bakos & Brynjolfsson, 1999; McAfee, et al., 1989; Schmalensee, 1984) or assumed that the bundle consists of substitutable or complementary components (Venkatesh & Kamakura, 2003). Products are independently valued if the reservation price of the bundle is the sum of the reservation prices of the components. When the relationship is complementary, the reservation price of the bundle may exceed the sum of the reservation prices of its components (Guiltinan, 1987), and when the components are substitutable, the bundle's reservation price may (though not necessarily) be lower than the sum of the reservation prices of the parts. (Marketers may still offer the bundle to exploit market segmentation benefits, and because the variable cost of the bundle may be a subadditive function of the component variable costs.) Substitutable products may (as in the case of a slower versus a faster computer system) or may not (Coke versus Pepsi, or a slower versus a faster automobile) be amenable to a universally consistent ordering. Regardless, independent and complementary products clearly lack the natural ordering that we see for television advertising, where all advertisers prefer prime time advertising to non-prime time advertising.

This type of ordering in the advertisers' preferences also exists in some other commercially important practical situations. Radio or news magazine advertising are obvious examples. Additionally, in online advertising, advertisers prefer placing an advertisement on the front page of a website to placing it on a lower ranking page. Billboard advertising also exhibits this relationship. Here, placing a billboard advertisement featured along an interstate highway is preferred to placing the same advertisement on a secondary road, where the exposure to the advertisement may be more limited. While in this paper we use television advertising as a prototypical example, our model and results apply to other situations that exhibit the preference ordering. As we will see, this preference ordering in the products leads to some counter-intuitive and insightful results.

Another distinctive feature of our research concerns the total amounts of each type of advertising time available for sale. As is the case in practice, we assume that these amounts are limited, and investigate how the broadcast network's decisions change as the availabilities change. In contrast, previous bundling literature has not modeled resource availabilities.

This chapter is organized as follows. Section 3.2 discusses our modeling assumptions and develops a nonlinear pricing model for a bundling situation when the resources have limited availability. The output from this model is a set of optimal product prices that automatically segments the market, and correspondingly sets the fraction of the market that is covered by each product. Advertisers decide on the product they wish to purchase based on the prices they are offered and their willingness to pay—which in turn depends on the "efficiency" with which they can generate revenues from viewers of their advertisements. In Section 3.3, assuming that the distribution of the advertiser's efficiency parameter (which measures the effectiveness with which the advertiser translates viewers into revenue) is uniform, we analyze the properties of the optimal prices, and shadow prices. Interestingly, the tightness *and* the relative tightness of the advertising resources plays a pivotal role in not only affecting the product prices but also influencing whether or not to offer the bundle, and if the bundle is offered, the type of bundling strategy to adopt. When prime and non-prime time resource availability is unconstrained, the broadcaster offers only the bundle. On the other hand, the broadcaster offers the bundle in conjunction with some components only when there is "enough" prime and non-prime time advertising resource. We also analyze the shadow prices of advertising resources, and evaluate how the broadcast network should focus its quality improvement efforts to improve total revenue. Due to bundling, the shadow price of the prime time resource (non-prime time resource) can decrease or remain the same even when its availability is kept unchanged but the availability of only the non-prime time resource (prime time resource) is increased. Our analysis shows that when the relative availability of the two resources is comparable, it always makes more sense for the network to improve the ratings of the prime time product. This section also explores the value of bundling. Section 3.4 relaxes two of the assumptions in our original model. Using specific instances from the Beta family of distributions to model the density function of advertiser efficiencies, we show numerically that the general nature of our conclusions is quite robust. We also investigate how to implement, and the impact of, a generalization of the definition of the bundle to allow for an unequal mix in its constituent components. Section 3.5 concludes the chapter by identifying some future research directions.

3.2 THE GENERAL MIXED BUNDLING MODEL

A monopolist television broadcasting network, which we refer to as the *broadcaster*, considers offering for sale on the scatter market its available advertising time, that is, its advertising *inventory*. This inventory is of two types: prime time and non-prime time. The availability of both of these inventories, which we interchangeably refer to also as *resources*, is fixed, with q_P denoting the amount of advertising time available during prime time hours, and q_N denoting the amount of advertising time available during non-prime time hours. The broadcaster's objective is to maximize the total revenue it generates from selling its inventory. As in the information goods situation in Bakos and Brynjolfsson (1999), we can assume that the variable costs of both resources is zero for our situation, and so maximizing the revenue is equivalent to maximizing the contribution. In order to do so, the broadcaster sells three products corresponding to selling one unit of each of the two resources separately, and selling a bundle which consists of one unit of each resource.

The market consists of *advertisers* interested in purchasing advertising time from the broadcaster. In line with the bundling literature (Adams & Yellen, 1976; Schmalensee, 1984), we assume that the marginal utility of a second unit of a product is zero for all advertisers. Advertisers have a strict ordering of their preferences: They consider advertising during non-prime time to be more desirable than not advertising, prime time advertising to be more desirable than non-prime time advertising, and the bundle that combines both prime and non-prime time advertising to be the most desirable. This preference is a consequence of prime time ratings being higher than non-prime time ratings. We designate the ratings of the non-prime time, prime time, and the bundle options by α, β, and γ, respectively, where, $0 < \alpha < \beta < \gamma$. We also assume that the relationship between the ratings is "concave" in nature, that is, $\alpha + \beta \geq \gamma$. This assumption is reasonable because of diminishing returns seen in advertising settings: in this case, the same individual might see an advertisement shown during both prime and non-prime time periods, and so the rating of the bundle is less than the sum of the ratings of the prime and non-prime advertisements.

Advertisers differ in their willingness to pay for the three advertising products due to their varied ability to translate eyeballs into purchase decisions of viewers and the consequent profits. Advertisers who are more successful in generating higher profits have a greater willingness to pay for the more desirable products—which are also more expensive. We designate by the parameter t the intrinsic efficiency of an advertiser to generate profits out of advertisements, and assume that this efficiency is distributed on the unit interval according to some probability density function $f(t)$ and cumulative distribution function $F(t)$. The willingness to pay of an advertiser with efficiency t for an advertisement placed in time period i is thus equal to $t \times r_i$, where r_i is the rating of the i^{th} product, i equal to prime, non-prime or the bundle.

Given the above distribution of the efficiency parameter of advertisers and their willingness to pay function, an optimal strategy for the broadcaster segments the population of advertisers into at most four groups as described in Figure 1, with the thresholds T^*, T^{**}, and T^{***} demarcating the different market segments.[5] With this strategy, advertisers in the highest range of efficiency parameters (interval $[T^*, 1]$) choose to purchase the bundle. Those in the second highest range of efficiency parameters (interval $[T^{**}, T^*)$) choose to advertise during prime-time. Those in the third highest range (interval $[T^{***}, T^{**})$) choose the non-prime product, and those in the lowest range refrain from advertising altogether. An interval of zero length implies that it is not optimal for the broadcaster to offer the corresponding product. The values of the threshold parameters T^*, T^{**}, and T^{***} are determined to guarantee that the advertiser located at a given threshold level is indifferent between the two choices made by the advertisers in the two adjacent intervals separated by this threshold parameter.

[5] The willingness to pay function satisfies the "single crossing property" and therefore facilitates segmentation and guarantees the uniqueness, as well as the monotonicity ($0 \leq T^{***} \leq T^{**} \leq T^* \leq 1$) of the thresholds.

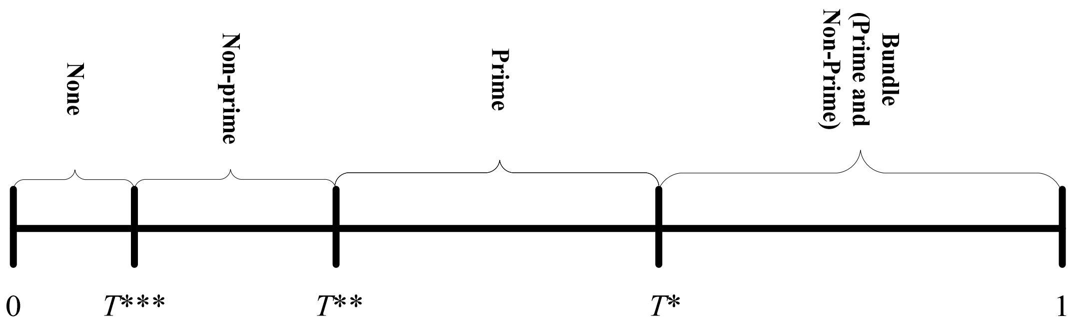

Figure 1. Market segmentation

To set up the model we define the selling prices for the bundle, prime, and non-prime products by p_B, p_P and p_N, respectively. The revenue optimization with mixed bundling model (*ROMB*), from the broadcaster's perspective, is:

$$\textbf{[ROMB]} \quad \max_{p_B, p_P, p_N \geq 0} \pi = p_B \int_{T^*}^{1} f(t)dt + p_P \int_{T^{**}}^{T^*} f(t)dt + p_N \int_{T^{***}}^{T^{**}} f(t)dt \tag{3.1}$$

subject to:

$$p_B \leq p_P + p_N, \tag{3.2}$$

$$\int_{T^*}^{1} f(t)dt + \int_{T^{**}}^{T^*} f(t)dt \leq q_P, \text{ and} \tag{3.3}$$

$$\int_{T^*}^{1} f(t)dt + \int_{T^{***}}^{T^{**}} f(t)dt \leq q_N. \tag{3.4}$$

The broadcaster's revenue from a market segment equals its size multiplied by the price of the product it corresponds to; the total revenue, π, in the objective function (3.1) is the sum of the revenues from each of the three segments that the broadcaster serves. Constraint (3.2), the bundle "survivability" constraint, prevents arbitrage opportunities for an advertiser to compose a bundle by separately buying a prime and a non-prime time products separately.[6] Constraints (3.3) and (3.4) model the limited prime and non-prime time available.

Advertisers self-select their purchases (or they may decide to not purchase any of the offered products) based on their willingness to pay and the product prices. (See Moorthy (1984), for an analysis of self-selection based market segmentation.) Consider the difference between an advertiser's willingness to pay and the price of the product he[7] purchases. This difference equals the *premium* the advertiser derives from the purchase. An advertiser will purchase a product only if his premium is nonnegative. Moreover, an advertiser will be indifferent, say, between buying only prime time and buying a bundle consisting of prime and non-prime time, if he extracts the same premium from either purchase. The following relationships between the purchasing premiums are invariant boundary conditions, regardless of the efficiency distribution $f(t)$.

$$\gamma T^{*} - p_B = \beta T^{*} - p_P \Leftrightarrow T^{*} = \frac{p_B - p_P}{\gamma - \beta}, \tag{3.5}$$

$$\beta T^{**} - p_P = \alpha T^{**} - p_N \Leftrightarrow T^{**} = \frac{p_P - p_N}{\beta - \alpha}, \text{ and} \tag{3.6}$$

$$\alpha T^{***} - p_N = 0 \Leftrightarrow T^{***} = \frac{p_N}{\alpha}. \tag{3.7}$$

Notice that the non-negativity of the thresholds implies

$$p_P \le p_B, \text{ and} \tag{3.8}$$

$$p_N \le p_P. \tag{3.9}$$

[6] Unless a systematic secondary market exists, an intermediary cannot purchase a bundle and then sell its components individually at a profit.

[7] Where necessary, we use masculine gender for the advertiser and feminine gender for the broadcaster.

Moreover, it is easy to see that p_N, as well as the premium for customers in each of the three categories, is nonnegative.

Before we analyze the situations that arise when at least one of the capacity constraints is binding, Proposition 3.1 considers the case when neither capacity constraint is binding.

Proposition 3.1. *If the prime and non-prime resource availability is sufficiently high, the optimal strategy for the broadcaster is pure bundling. The corresponding optimal threshold is the fixed point of the reciprocal of the hazard rate function of the distribution of advertisers, that is,* $T^* = \left(1 - F\left(T^*\right)\right) \Big/ f\left(T^*\right)$.

The following corollary uses Markov's inequality to establish an upper bound on the optimal revenue when the problem is not constrained by the inventory availability.

Corollary 3.2. *An upper bound on the broadcaster's total revenue* π *is* $\gamma \mathbf{E}\left[T\right]$, *where* $\mathbf{E}\left[T\right]$ *is the expected value of the efficiency, t. The actual revenue collected under the pure bundling strategy is* $\gamma\left(1 - F\left(T^*\right)\right)^2 \Big/ f\left(T^*\right)$.

Proof of Proposition 3.1 and Corollary 3.2: Consider the total revenue gained by the monopolist:

$$\pi = p_B\left[1 - F\left(T^*\right)\right] + p_P\left[F\left(T^*\right) - F\left(T^{**}\right)\right] + p_N\left[F\left(T^{**}\right) - F\left(T^{***}\right)\right].$$

We will derive first the conditions for the concavity of the revenue function. Let $K_1 = 2f(T^*)+T^*f'(T^*)$, $K_2 = 2f(T^{**})+T^{**}f'(T^{**})$, and $K_3 = 2f(T^{***})+T^*f'(T^{***})$, with $0 \leq T^*,T^{**},T^{***} \leq 1$. The Hessian matrix H associated with the revenue function is

$$H = \begin{bmatrix} -\frac{K_1}{\gamma-\beta} & \frac{K_1}{\gamma-\beta} & 0 \\ \frac{K_1}{\gamma-\beta} & -\frac{K_1}{\gamma-\beta}-\frac{K_2}{\beta-\alpha} & \frac{K_2}{\beta-\alpha} \\ 0 & \frac{K_2}{\beta-\alpha} & -\frac{K_2}{\beta-\alpha}-\frac{K_3}{\alpha} \end{bmatrix}.$$

Let $x = [x_1\ x_2\ x_3]$ be a three dimensional real-valued vector. The product $x^T Hx$ is equal to:

$$x^T Hx = -\frac{K_1}{\gamma-\beta}(x_1-x_2)^2 - \frac{K_2}{\beta-\alpha}(x_2-x_3)^2 - \frac{K_3}{\alpha}x_3^2.$$

Hence, the Hessian matrix is negative semi-definite, and therefore the revenue function is concave and admits a local maximum, as long as K_1, K_2 and K_3 are non-negative. The non-negativity assumption on K_1, K_2 and K_3 is satisfied by a large class of probability density functions bounded on the [0, 1] domain, including the Beta distribution, of which the uniform distribution is a special case (Johnson, Kotz, & Balakrishnan, 1994).

We will now use the first order KKT conditions to show that this local maximum is, in fact, unique, and therefore global. Noticing, for example, that

$$\begin{aligned} \frac{\partial \pi}{\partial p_B} &= \left[1-F\left(T^*\right)\right] + p_B\left(-\frac{dF\left(T^*\right)}{dp_B}\right) + p_P\left(-\frac{dF\left(T^*\right)}{dp_B}\right) \\ &= \left[1-F\left(T^*\right)\right] - \frac{p_B - p_P}{\gamma-\beta} f\left(T^*\right) \\ &= \left[1-F\left(T^*\right)\right] - T^* f\left(T^*\right), \end{aligned}$$

we can complete the rest of the first order conditions associated with problem *ROMB*:

$$\begin{aligned} 1-F\left(T^*\right)-T^* f\left(T^*\right) &= 0 \\ F\left(T^*\right)-F\left(T^{**}\right)+T^* f\left(T^*\right)-T^{**} f\left(T^{**}\right) &= 0 \\ F\left(T^{**}\right)-F\left(T^{***}\right)+T^{**} f\left(T^{**}\right)-T^{***} f\left(T^{***}\right) &= 0. \end{aligned}$$

Let the *hazard rate* function $h(t)$ be defined as $h(t) = f(t) / [1 - F(t)]$. From the first equation, we obtain the value of threshold T^* to be equal to the fixed point of the inverse of the hazard rate function, that is, $T^* = [1 - F(T^*)] / f(T^*)$. Substituting into the remaining equations, we obtain, similarly, that $T^{**} = [1 - F(T^{**})] / f(T^{**})$, and $T^{***} = [1 - F(T^{***})] / f(T^{***})$. Brouwer's fixed point theorem guarantees the existence of at least one such point; however due to the monotonicity of the hazard rate function, we can see that the solution must be unique, that is, $T^* = T^{**} = T^{***}$.

Since all thresholds are equal, it follows that the monopolist will only offer the bundle, cover the segment $1 - F\left(T^*\right)$, and collect total revenues $\pi = \gamma T^* \left[1 - F\left(T^*\right)\right]$. Using Markov's inequality, we can find an upper bound on the profit value as follows:

$$\Pr\left(t > T^*\right) \leq \frac{\mathbf{E}\left[T\right]}{T^*} \Leftrightarrow \gamma T^* \Pr\left(t > T^*\right) \leq \gamma T^* \frac{\mathbf{E}\left[T\right]}{T^*} \Leftrightarrow \pi \leq \gamma \mathbf{E}\left[T\right]. \square$$

The result in Proposition 3.1 seems to contradict previous bundling literature (McAfee, et al., 1989; Schmalensee, 1984) which demonstrates that the mixed bundling strategy weakly dominates both the pure bundling and pure components strategies. However, a critical difference between our model and previous work is that the advertisers have a common preferred ordering of the three products. In contrast, the previous research stream does not assume any such ordering of the products. Since the bundle is the most desirable option for every advertiser, the broadcaster offers only the bundle when the available prime and non-prime advertising time is unconstrained. This unconstrained case is unlikely to arise in reality, since all broadcasters are usually heavily constrained by the prime time resource availability.

In the next section we derive the analytical solution of the constrained optimization problem under the simplifying assumption that the efficiency parameter of advertisers is uniformly distributed. In Section 3.4, we extend the results numerically using a Beta distribution.

3.3 REVENUE MAXIMIZING STRATEGIES WHEN CAPACITY IS BINDING

Clearly, the capacity constraints in the *ROMB* model play a significant role in determining the broadcaster's optimal strategy. Particularly, the relative scarcity of the two resources, prime time and non-prime time, is the main driver of the analysis. In the television advertising market, the prime time resource availability constraint (3.3) is far more likely to be binding than the non-prime time resource availability constraint (3.4). Prime time on television is usually the slot from 8:00 pm until 11:00 pm Monday to Saturday, and 7:00 pm to 11:00 pm on Sunday. Hence, the ratio of prime to non-prime time availability is about 1:8 (or 1:6 on Sunday). In other media markets, the relative scarcity of the non-prime time constraint may also become an issue. For instance, in the billboard advertising market, the "non-prime time" resource is the limited availability of billboards on secondary roads (which are less traveled), whereas the "prime time" is the extensive availability of billboard advertisement space on major roads (which have more travelers). In such a market, it is the "non-prime time" capacity that is more likely to be binding. Finally, in internet advertising both types of capacity constraints may be binding. Each website has limited space for banner advertisements irrespective of whether it is the front page ("prime time") or a secondary page ("non-prime time"). In this section, we specify the distribution of the efficiency parameter of advertisers to be uniform and identify the impact of the two capacity constraints on the type of strategy followed by the broadcaster. We find that the following strategies can arise as the optimal solution of *ROMB*: no bundle is offered, that is, the pure components strategy, *PC*; only the bundle is offered, that is, the pure bundling strategy, *PB*; the bundle, as well as each separate time product is offered, that is, the full spectrum mixed bundling strategy, *MBPN*; the bundle and the prime time product are offered, that is, the partial spectrum mixed bundling strategy, *MBP*; the bundle and the non-prime time product are offered, that is, the partial spectrum mixed bundling strategy, *MBN*. Throughout the remainder of the paper we will refer to these abbreviations.

In our derivations, we will demonstrate that the optimal strategy critically depends upon the relative availability of q_P and q_N. We will show that, for instance, that the *MBN* strategy is optimal when q_P is scarce relative to q_N, and the *MBP* strategy arises in the opposite case. The *MBPN* strategy is the optimal strategy when the ratio of q_P to q_N is close to one, but they are both sufficiently large.

We will also show that the characterization of the solution when the partial spectrum mixed bundling strategies (*MBP* or *MBN*) are optimal is further contingent upon the overall availability of the more abundant resource. Specifically, even though the strategy itself, say *MBP*, remains the same, the solution characteristics (product prices and the shadow prices of the resources) depend on whether q_P is less than or greater than a half. Similarly, the characteristics of the solution corresponding to *MBN* depend on whether q_N is less than or greater than a half. To distinguish between these two cases, we designate by MBP^+ and MBP^- the partial spectrum mixed bundling strategies when q_P is greater than and when q_P is less than a half, respectively. We define the subcategories MBN^+ and MBN^- of *MBN* in a similar manner depending on the availability of q_N.

Figure 2 depicts the regions corresponding to the various strategies that we discussed above. For the uniform distribution, the unconstrained solution that we described in section 3.2 arises when q_P and q_N are both at least a half. In this case, at the optimal solution, the broadcaster never sells more than an aggregate quantity of one, split equally between the prime and non-prime advertising times. To depict the constrained solution, therefore, in Figure 2, we restrict attention only to the case when $q_P + q_N \leq 1$. The unconstrained solution in the figure is designated by the point, *PB*, where $q_P = q_N = ½$. The boundaries for the regions in Figure 2 will be explained in detail once the solution to model *ROMB* is derived.

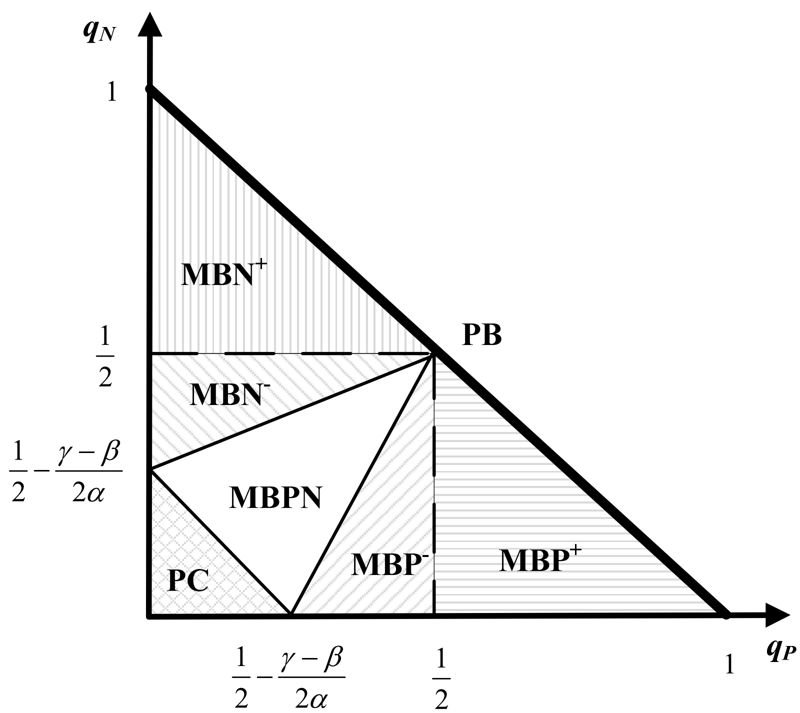

Figure 2. Representation of the optimal strategies

Replacing the general distribution by a uniform distribution in the model *ROMB* yields the following model, which we refer to as *ROMB_U.*

[ROMB_U] $\max \pi = p_B\left(1-\frac{p_B-p_P}{\gamma-\beta}\right)+p_P\left(\frac{p_B-p_P}{\gamma-\beta}-\frac{p_P-p_N}{\beta-\alpha}\right)+p_N\left(\frac{p_P-p_N}{\beta-\alpha}-\frac{p_N}{\alpha}\right)$ (3.10)

subject to:

$$p_B - (p_P + p_N) \le 0, \tag{3.11}$$

$$1 - \frac{p_P - p_N}{\beta - \alpha} \le q_P, \text{ and} \tag{3.12}$$

$$1 - \frac{p_B - p_P}{\gamma - \beta} + \frac{p_P - p_N}{\beta - \alpha} - \frac{p_N}{\alpha} \le q_N. \tag{3.13}$$

It is easy to see that the solution to the unconstrained case (when $q_P \ge ½$ and $q_N \ge ½$) is $p_B = \gamma/2$, $p_P = \beta/2$, and $p_N = \alpha/2$. This solution guarantees that only pure bundling arises since $T^* = T^{**} = T^{***} = ½$, and the broadcaster's revenues are $\gamma/4$. Note that this solution also guarantees that the arbitrage constraint $p_B \le p_P + p_N$ is satisfied since $\gamma \le \alpha + \beta$ by the concavity assumption.

3.3.1 Characterization of the different strategies

We now discuss the characterization of the constrained case. Proposition 3.3 describes the boundaries of the regions corresponding to the different strategies depicted in Figure 2, and Propositions 3.4 and 3.5 derive the optimal product and the shadow prices, respectively.

Proposition 3.3. *The optimal strategies as a function of the availability of q_P and q_N are as follows:*

(i) *The pure component strategy, PC, is optimal if*

$$0 < q_N + q_P < \frac{1}{2} - \frac{\gamma - \beta}{2\alpha}.$$

(ii) *The full spectrum mixed bundling strategy, MBPN, is optimal if*

$$\frac{\gamma - \beta}{\alpha} \le \frac{1 - 2q_N}{1 - 2q_P} \le \frac{\alpha}{\gamma - \beta}.$$

(iii) *The partial spectrum mixed bundling strategies, MBN^- and MBN^+, are optimal if 0< q_P < ½, and*

a. $\dfrac{1-2q_N}{1-2q_P} < \dfrac{\gamma-\beta}{\alpha}$ *and* $q_N < \dfrac{1}{2}$ *for* MBN^-,

b. $q_N \geq \dfrac{1}{2}$ *for* MBN^+.

(iv) *The partial spectrum mixed bundling strategies, MBP^- and MBP^+, are optimal if 0 < q_N < ½ and*

a. $\dfrac{1-2q_N}{1-2q_P} > \dfrac{\alpha}{\gamma-\beta}$ *and* $q_P < \dfrac{1}{2}$ *for* MBP^-,

b. $q_P \geq \dfrac{1}{2}$ *for* MBP^+.

(v) *The pure bundling strategy, PB, is optimal at a single point $q_P = q_N = ½$.*

Proof: In Proposition 3.1 we have established that the objective function of *ROMB* is concave. In particular, the *ROMB_U* model is a concave quadratic optimization program with linear constraints; therefore the first-order KKT conditions are both necessary and sufficient. The proof is straightforward once the first order conditions are expressed under the capacity scenario that both capacity constraints are binding and enforcing the increasing monotonicity of the thresholds. Optimizing *ROMB_U* under the assumption that both capacity constraints are binding, and solving for the optimal thresholds, we get:

$$
\begin{aligned}
T^* &= \frac{1}{2} + \frac{\alpha}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right) \\
T^{**} &= 1-q_P \\
T^{***} &= \frac{1}{2} + \frac{\gamma-\beta}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right).
\end{aligned}
$$

In order to maintain consistency, the thresholds must be ordered on the [0, 1] line segment, that is $0 \leq T^{***} \leq T^{**} \leq T^* \leq 1$ (this ordering is due to the single crossing property of the willingness to pay function). For example, the last inequality is equivalent to the following:

$$
\begin{aligned}
T^* \le 1 &\Leftrightarrow \frac{\alpha}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right) \le \frac{1}{2} \\
&\Leftrightarrow 1-q_P-q_N \le \frac{\gamma-\beta+\alpha}{2\alpha} \\
&\Leftrightarrow q_P+q_N \ge \frac{1}{2}-\frac{\gamma-\beta}{2\alpha}.
\end{aligned}
$$

The bundle is not offered when equality holds, therefore condition (i) from the proposition follows naturally. Similarly, we can examine the remaining inequalities:

$$
\begin{aligned}
T^{**} \le T^* \quad &\Leftrightarrow q_P + \frac{\alpha}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right) \ge \frac{1}{2} \\
&\Leftrightarrow \left(\gamma-\beta\right)q_P - \alpha q_N \ge \frac{\gamma-\beta-\alpha}{2}, \text{ and} \\
T^{***} \le T^{**} &\Leftrightarrow q_P + \frac{\gamma-\beta}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right) \le \frac{1}{2} \\
&\Leftrightarrow \alpha q_P + \left(\gamma-\beta\right)q_N \le \frac{\gamma-\beta-\alpha}{2}.
\end{aligned}
$$

Combining the two inequalities gives us condition (ii). Conditions (iii) and (iv) emerge from (ii) with the additional observation that a capacity constraint is non-binding if and only if the corresponding capacity is greater than ½ (Proposition 3.1 with $F(t) = t$ and $f(t) = 1$). Finally, using again Proposition 3.1 with $F(t) = t$ and $f(t) = 1$, we obtain $T^* = T^{**} = T^{***} = ½$ and the substitution into both capacity constraints yields condition (v). □

According to Proposition 3.3 when the available aggregate capacity is small (lower than $\frac{1}{2}-\frac{\gamma-\beta}{2\alpha}$), the broadcaster follows a pure component strategy where each advertiser can choose between advertising on prime time or on non-prime time but not both. Offering the bundle is suboptimal in this case given the extreme scarcity of the advertising time availability. When the aggregate capacity is larger than $\frac{1}{2}-\frac{\gamma-\beta}{2\alpha}$, and the discrepancy between the capacities available on prime time and non-prime time is relatively moderate (that is, $\frac{\gamma-\beta}{\alpha} \leq \frac{1-2q_N}{1-2q_P} \leq \frac{\alpha}{\gamma-\beta}$), it is optimal for the broadcaster to choose full segmentation of advertisers by offering all three different products. Notice that the size of the region expands as the relationship between the ratings parameters becomes more concave (that is, the fraction $(\gamma-\beta)/\alpha$ becomes smaller.) On the other hand, the *MBPN* region becomes smaller as $(\gamma-\beta)/\alpha$ becomes larger, that is, as $\gamma-\beta$ approaches α. In the extreme case, when $\gamma-\beta$ equals α (that is, the ratings are additive), the *MBPN* region becomes a line (and the *PC* region disappears). In this case, the *MBPN* strategy applies only when $q_P = q_N$. This is obvious since the ratings of the bundle exactly equal the sum of the prime and non-prime ratings.

When the availability of the non-prime time resource is much greater than that of the prime time resource ($\frac{1-2q_N}{1-2q_P} < \frac{\gamma-\beta}{\alpha}$), the broadcaster offers both the non-prime product and the bundle. Conversely, when the availability of the prime time resource is much bigger than that of the non-prime time ($\frac{\alpha}{\gamma-\beta} < \frac{1-2q_N}{1-2q_P}$), the broadcaster offers a choice between advertising just on prime time or buying a bundle. With significant abundance of one resource relative to the other, it pays to utilize the entire capacity of the more scarce resource as part of the bundle. Since advertisers have a higher willingness to pay for the bundle than for each component sold separately, the broadcaster uses the entire capacity of the scarcer resource in the form of the product that can command a higher price. Any remaining quantity of the more abundant resource, not sold as part of the bundle, is offered separately to the customers. According to part (v) of the Proposition, pure bundling arises only when the capacity of each category is large enough to obtain the solution of the unconstrained optimization (equal to ½). Notice, in fact, that the scarcity of the two resources, which determines the boundaries of the regions in Figure 2, is expressed in terms of $(1-2q_N)$ and $(1-2q_P)$. These expressions measure the extent to which the individual capacities fall short of the unconstrained optimal value of ½.

3.3.2 Optimal product prices and shadow prices

Having studied how and why the differing relative availabilities of the prime and non-prime time resources impact the regions where the different strategies apply, we now investigate the optimal pricing structure under the different strategies.

Proposition 3.4. *The optimal prices charged by the broadcaster in the different regions of Figure 2 are those listed in Table 1.*

Proof: Just like the proof of Proposition 3.3, we use the fact that the first order conditions are both necessary and sufficient. Additionally, the invariant boundary conditions (3.5) – (3.7) establish the relationships between thresholds and prices. For example, under the case of both capacity constraints binding, we derive the optimal price for the non-prime time product:

$$\left.\begin{array}{l} \alpha T^{***} - p_N = 0 \Leftrightarrow p_N = \alpha T^{***} \\ T^{***} = \dfrac{1}{2} + \dfrac{\gamma-\beta}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right) \end{array}\right\} \Rightarrow p_N = \frac{\alpha}{2} + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(1-q_P-q_N\right).$$

Similarly we obtain the remaining prices as

$$\begin{aligned} p_P &= \frac{\beta}{2} + \frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right) + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right) \\ p_B &= \frac{\gamma}{2} + \frac{\beta(\gamma-\beta)+\alpha(\gamma-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right) + \frac{2\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right). \end{aligned}$$

The other cases follow the same argument and are derived in an identical fashion. □

It is noteworthy that ignoring the arbitrage constraint (3.11) (that is, $p_B \le p_P + p_N$) and solving for the optimal prices yields a solution that automatically satisfies the constraint except possibly in the MBN^+ and the MBP^+ regimes. If $\alpha < 2(\gamma - \beta)$ (that is, when the concavity of the ratings parameters is moderate), the constraint might be violated when $q_P < \alpha / \left(2(\gamma - \beta)\right)$ under MBN^+ or $q_N < \alpha / \left(2(\gamma - \beta)\right)$ under MBP^+. Since the arbitrage constraint is binding in this case, incorporating it (that is, setting $p_P = p_B - p_N$ under MBN^+ or $p_N = p_B - p_P$ under MBP^+) still results in the desired outcome for the broadcaster. Specifically, no advertiser chooses to buy the prime time product under MBN^+ or the non-prime time product under MBP^+.

In Proposition 3.5, we solve for the Lagrange multipliers of the resource constraints. This analysis provides the foundation for our subsequent investigation into the relative marginal values of the two resources.

Table 1. Optimal prices for the *ROMB_U* model

STRATEGY	OPTIMAL PRICES	
PC	$p_B = \gamma - \beta q_P - \alpha q_N$ $p_P = \beta\left(1-q_P\right) - \alpha q_N$ $p_N = \alpha\left(1-q_P-q_N\right)$	(3.14)
MBPN	$p_B = \frac{\gamma}{2} + \frac{\beta(\gamma-\beta)+\alpha(\gamma-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right) + \frac{2\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right)$ $p_P = \frac{\beta}{2} + \frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right) + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right)$ $p_N = \frac{\alpha}{2} + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left[\left(\frac{1}{2}-q_P\right)+\left(\frac{1}{2}-q_N\right)\right]$	(3.15)
MBN$^-$	$p_B = \gamma\left(1-q_P\right) - \alpha\left(q_N-q_P\right)$ $p_P = \beta\left(1-q_P\right) - \alpha\left(q_N-q_P\right)$ $p_N = \alpha\left(1-q_N\right)$	(3.16)
MBN$^+$	$p_B = \frac{\gamma}{2} + (\gamma-\alpha)\left(\frac{1}{2}-q_P\right)$ $p_P = \max\left\{\frac{\beta}{2}+(\beta-\alpha)\left(\frac{1}{2}-q_P\right), \frac{\gamma-\alpha}{2}+(\gamma-\alpha)\left(\frac{1}{2}-q_P\right)\right\}$ $p_N = \frac{\alpha}{2}$	(3.17)
MBP$^-$	$p_B = \gamma\left(1-q_N\right) - \beta\left(q_P-q_N\right)$ $p_P = \beta\left(1-q_P\right)$ $p_N = \alpha\left(1-q_P\right)$	(3.18)
MBP$^+$	$p_B = \frac{\gamma}{2} + (\gamma-\beta)\left(\frac{1}{2}-q_N\right)$ $p_P = \frac{\beta}{2}$ $p_N = \max\left\{\frac{\alpha}{2}, (\gamma-\beta)\left(1-q_N\right)\right\}$	(3.19)

Proposition 3.5. *The shadow prices of the resources in the different regions are specified as follows:*

STRATEGY	**OPTIMAL SHADOW PRICES**	
PC	$\lambda_P = \beta(1-2q_P) - 2\alpha q_N$ $\lambda_N = \alpha(1-2q_P-2q_N)$	(3.20)
MBPN	$\lambda_P = \frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}(1-2q_P) + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}(1-2q_N)$ $\lambda_N = \frac{2\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}(1-q_P-q_N)$	(3.21)
MBN$^-$	$\lambda_P = (\gamma-\alpha)(1-2q_P)$ $\lambda_N = \alpha(1-2q_N)$	(3.22)
MBN$^+$	$\lambda_P = (\gamma-\alpha)(1-2q_P)$ $\lambda_N = 0$	(3.23)
MBP$^-$	$\lambda_P = \beta(1-2q_P)$ $\lambda_N = (\gamma-\beta)(1-2q_N)$	(3.24)
MBP$^+$	$\lambda_P = 0$ $\lambda_N = (\gamma-\beta)(1-2q_N)$	(3.25)

Proof: The proof relies on the first order conditions and solving for the Lagrange multipliers. □

To understand the relationship between the shadow prices that we report in Proposition 3.5, and the product prices in Proposition 3.4, note that the availability of one additional unit of a scarce resource results in both a direct effect of generating additional revenues from the sale of this unit (perhaps, partly separately and partly in the bundle), and an indirect effect of depressing the prices that the broadcaster can charge for the products. For instance, the availability of an additional unit of the prime time resource, when the strategy *PC* is employed, has the direct effect of generating extra revenues equal to p_P and an indirect effect of reducing the price of the prime time product at the rate of β and the price of the non-prime time resource at the rate of α. Hence, $\lambda_P = p_P - \beta q_P - \alpha q_N$. Substituting for p_P from Proposition 3.4 yields the expression for λ_P reported in (20). The explanation for the shadow price of the non-prime resource, λ_N, is similar. For the *MBPN* strategy, establishing the relationship between shadow prices and product prices is a bit more complicated since an additional unit of the scarce resource is partially allocated to the bundle and partially sold separately. Specifically, an additional unit of the prime time resource is allocated to the bundle at the rate of $\alpha/(\gamma - \beta + \alpha)$ and is sold separately at the rate of $(\gamma - \beta)/(\gamma - \beta + \alpha)$. Hence an additional unit of prime time resource generates direct extra revenues equal to $\alpha/(\gamma - \beta + \alpha)\, p_B + (\gamma - \beta)/(\gamma - \beta + \alpha)\, p_P$. The extra unit depresses prices according to Proposition 3.4 as follows:

- the price p_B at the rate of $\left((\gamma-\beta)(\alpha+\beta)+\alpha(\beta-\alpha)\right)/\left(\gamma-\beta+\alpha\right)$,
- the price p_P at the rate of $\left(\beta(\gamma-\beta)+\alpha(\beta-\alpha)\right)/(\gamma-\beta+\alpha)$, and
- the price p_N at the of $\alpha(\gamma-\beta)/(\gamma-\beta+\alpha)$.

Combining the direct and indirect effects yields the desired expression for λ_P in (3.21), and similarly for λ_N. We observe that an increase in either the prime or the non-prime time resource availability lowers the optimal prices of all three products even though the broadcaster offers a lower amount of the prime time product when q_N increases and of the non-prime time product when q_P increases. An argument that combines the direct and indirect effect similarly applies for the expressions (3.22) – (3.25).

3.3.3 Relative shadow prices

The shadow prices in Proposition 3.5 measure the extra cost the broadcaster might be willing to incur in order to obtain one additional unit of the scarce resource. In the case of television advertising, an increase in the available advertising time comes at the expense of programming time, and thus can potentially decrease the ratings and hence the advertiser's profits. The shadow prices in Proposition 3.5 provide an upper bound on the reduction in ratings that the broadcaster might be willing to tolerate in order to increase advertising time by one unit.

A related question that arises is how much more, or less, valuable to the broadcaster an additional unit of prime time is vis-à-vis an additional unit of non-prime time. In addition, how does this comparison change as we move from one regime to another? We summarize this comparison in Corollary 3.6.

Corollary 3.6. *The shadow price of a unit of the prime time resource is greater than a unit of non-prime time resource if the strategies PC, MBPN, and MBN (both MBN^- and MBN^+) are optimal. If strategy MBP is optimal, an extra unit of the non-prime time resource may become more valuable than an extra unit of the prime time resource. Specifically, the difference $\lambda_P - \lambda_N$ for the different regions is as follows:*

STRATEGY	DIFFERENCE IN THE OPTIMAL SHADOW PRICES	
PC	$(\beta-\alpha)(1-2q_P)$	(3.26)
MBPN	$(\beta-\alpha)(1-2q_P)$	(3.27)
MBN^-	$(\gamma-\alpha)(1-2q_P)-\alpha(1-2q_N)$	(3.28)
MBN^+	$(\gamma-\alpha)(1-2q_P)$	(3.29)
MBP^-	$\beta(1-2q_P)-(\gamma-\beta)(1-2q_N)$	(3.30)
MBP^+	$-(\gamma-\beta)(1-2q_N)$	(3.31)

Proof: Follows immediately from Proposition 3.5 by taking the difference between the Lagrange multipliers for the prime and non-prime time capacity constraints. □

To illustrate how the relative availability for the two resources affects the relative shadow prices, consider a value of $q_P < 1/2 - (\gamma - \beta)/(2\alpha)$. We select this choice of q_P because the available prime time inventory is typically relatively low, and as we gradually increase the value of q_N, the optimal strategy shifts, according to Figure 2, from *PC* to *MBPN* to *MBN*$^-$ and finally to *MBN*$^+$. We can do a similar analysis for the case $1/2 - (\gamma - \beta)/(2\alpha) < q_P < 1/2$. Selecting a value of q_P greater than or equal to a half does not result in a change in strategies as we increase the value of q_N, and so a similar analysis is not interesting in that case. Figure 3 depicts the relative shadow prices of the two resources when considering such an increase in q_N. (The solid dots in this and subsequent figures represent a shift in the strategy.)

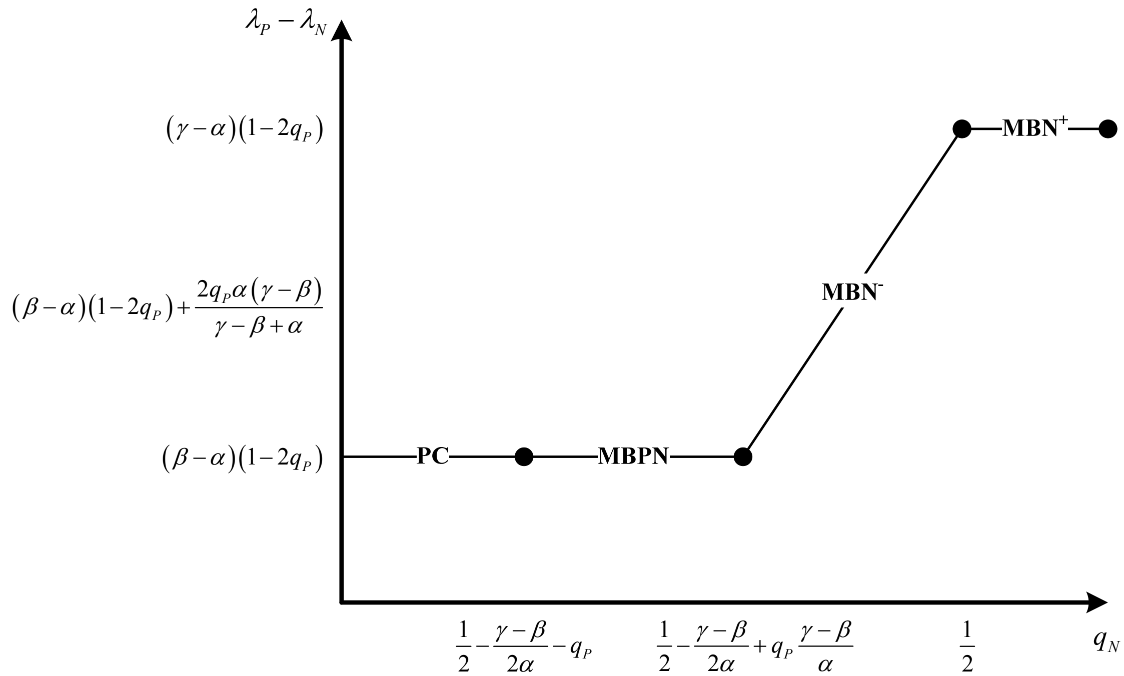

Figure 3. Relative shadow prices of the resources for any $q_P < \frac{1}{2} - \frac{\gamma-\beta}{2\alpha}$

The shadow price of the prime time resource is higher than that of the non-prime time resource since the broadcaster can charge higher prices from advertisers choosing to place advertisements on prime time. This higher price is proportional to the difference in the ratings of the prime and non-prime time products. Indeed, under the *PC* and *MBPN* regimes, the difference in the shadow prices is proportional to $(\beta - \alpha)$, which measures the difference in ratings between the pure components. Interestingly, the added segmentation of advertisers that is facilitated by bundling under *MBPN*, does not enhance the relative shadow price of the prime time resource. The reason for this result is that an additional unit of either resource is allocated in the same proportion towards the bundle, thus maintaining the relative desirability of the two resources to the broadcaster irrespective of whether full segmentation is feasible or not. Under MBN^+ regime, the difference in the shadow prices of the two resources is proportional to $(\gamma - \alpha)$, since this regime occurs under the extreme scarcity of the prime time resource, and each additional unit of the prime time resource is used only in the bundle, thus yielding the extra rating of γ rather than β.

A similar allocation of an extra prime time unit is optimal under MBN^- also. However, since there are no unused units of the non-prime time resource under this regime, each additional unit of the prime time that is sold requires directing a non-prime time unit from being sold as an independent component. As a result, the shadow price of the prime time resource under MBN^- is not as high as it is under MBN^+. Under MBN^-, the difference $\lambda_P - \lambda_N$ is an increasing function of q_N, or alternatively, since q_P is fixed for this analysis, an increasing schedule of the relative scarcity of the prime-time resource, until it reaches its maximum value when $q_N = ½$, and the MBN^+ region is reached. Note also that a bigger value of q_P reduces the difference $\lambda_P - \lambda_N$ for all regimes. Hence, as the prime time becomes less scarce, its importance relative to the non-prime time resource declines.

3.3.4 Incentives for improving the programming quality

We can now use the characterization of the optimal solution to assess the relative incentives of the broadcaster to increase the ratings of the time periods by improving the quality of the programming. We will evaluate those incentives by deriving the expression for the *Relative Incentive to Improve Ratings*, $RIIR = \frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha}$, where π^* corresponds to the broadcaster's optimal revenue.

In our analysis, we assume that $\frac{\partial \gamma}{\partial \alpha} = \frac{\partial \gamma}{\partial \beta}$. Specifically, improving the quality of programs on non-prime time has the same effect on the ratings of the bundle as an equivalent improvement in prime time programming. We refer to this as the *equal ratings-improvement effect* assumption. Given that we restrict attention to bundles with equal proportions of prime time and non-prime time resources, such an assumption seems reasonable. Substituting the optimal prices from Proposition 3.4 back into the objective function (3.10) yields the optimal equilibrium revenue, π^*.

Proposition 3.7 reports on the *RIIR* values, that is, the added incentive of increasing the ratings of prime time over non-prime time, for the different regions.

Proposition 3.7. *The RIIR values,* $\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha}$*, can be expressed as follows:*

STRATEGY	**RIIR VALUES** $\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha}$
PC	$(q_P - q_N)(1 - q_P - q_N) + 2q_P q_N$ (3.32)
MBPN	$\frac{\left[(\gamma - \beta)^2 + \alpha^2\right](1 - q_P - q_N)^2}{(\gamma - \beta + \alpha)^2} - 2\left(\frac{1}{2} - q_P\right)^2$ (3.33)
MBN$^-$	$(q_P - q_N)(1 - q_N - q_P)$ (3.34)
MBN$^+$	$-\left(\frac{1}{2} - q_P\right)^2$ (3.35)
MBP$^-$	$(q_P - q_N)(1 - q_N - q_P)$ (3.36)
MBP$^+$	$\left(\frac{1}{2} - q_N\right)^2$ (3.37)

Proof: Using Proposition 3.3 and substituting the optimal prices in the objective function of *ROMB_U* and taking the partial derivatives, we obtain, for example, for the PC case, the following:

$$\pi^* = \beta q_P (1 - q_P) + \alpha q_N (1 - q_N - 2q_P)$$

$$\left.\begin{aligned} \frac{\partial \pi^*}{\partial \alpha} &= q_N (1 - q_N - 2q_P) \\ \frac{\partial \pi^*}{\partial \beta} &= q_P (1 - q_P) \end{aligned}\right\} \Rightarrow RIIR = (q_P - q_N)(1 - q_P - q_N) + 2q_P q_N.$$

The other cases are derived in a similar fashion. □

Inspecting the expressions derived in Proposition 3.7, it immediately follows that the *RIIR* values depend on their relative scarcity. In general, irrespective of which bundling strategy is optimal, the broadcaster has greater incentives to improve the ratings of the resource that is more abundant. The value of *RIIR* is an increasing function of q_P and a decreasing function of q_N, implying that the broadcaster benefits more from upgrading the quality of the more plentiful resource.

The results of Proposition 3.7 allow us to also assess the implications of different bundling strategies on the incentives to improve the quality of the programming. To control for the relative size effect reported above, in conducting this assessment, we consider the symmetric case: $q_N = q_P = q$. Given this symmetry, we can only compare the *PC* regime with the *MBPN* regime since the partial spectrum mixed bundling strategies (*MBP* and *MBN*) arise when the availabilities of the two resources is asymmetric.

Corollary 3.8. *When $q_N = q_P = q$, the RIIR value, $\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha}$, can be derived as follows:*

i. *for the PC Strategy (that is , when* $q < \frac{1}{4} - \frac{\gamma - \beta}{4\alpha}$), $\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha} = 2q^2$

ii. *for the MBPN Strategy (that is , when* $\frac{1}{4} - \frac{\gamma - \beta}{4\alpha} < q < 1/2$),

$$\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha} = \frac{\left(\alpha + \beta - \gamma\right)^2 (1 - 2q)^2}{2\left(\gamma - \beta + \alpha\right)^2}$$

Proof: Follows from Proposition 3.7 by substituting $q = q_N = q_P$. □

Figure 4 graphically presents the results stated in Corollary 3.8.

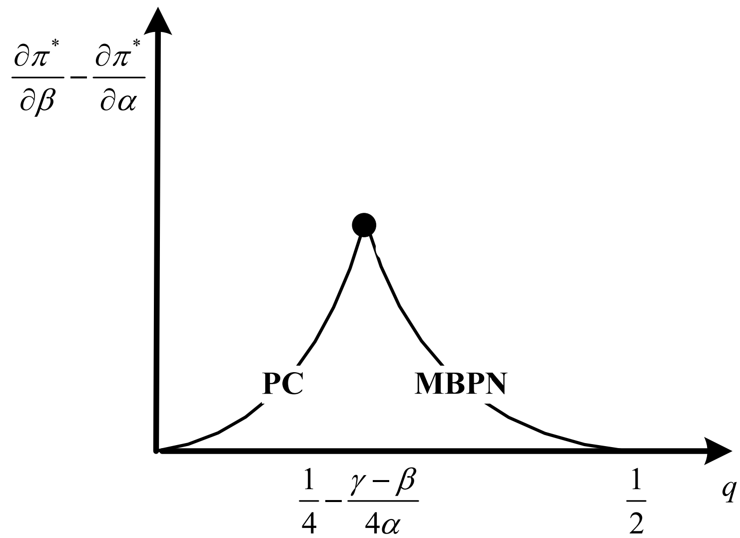

Figure 4. Relative incentive to improve ratings, *RIIR*

As illustrated in Figure 4, when the capacities of the prime time and non-prime time resources are comparable, it is always more advantageous to improve the ratings of the prime time product. However, whereas under the *PC* regime the relative benefit of enhancing the ratings of prime time over non-prime time programming is higher the larger the capacities are, the opposite is true for the *MBPN* region. The reason for the increase in the *RIIR* value in the PC region as q increases is the following. Given a value of q (recall that $q = q_N = q_P$ for this discussion), the price p_N is determined by the indifference relationship (3.7) of the customer with efficiency $t = 1 - 2q$, and equals $\alpha(1 - 2q)$. On the other hand, the price p_P is determined by the indifference relationship (3.6) of the customer with efficiency $t = 1 - q$, and equals $(\beta - \alpha)(1 - q) + p_N = \beta(1 - q) - \alpha q$. Improving the quality of the prime time programming by a unit results in an increase of $(1 - q)$ in p_P which is greater than the increase in p_N of $(1 - 2q)$ attributable to increasing the quality of non-prime programming. The difference between these two quantities, $(1 - q) - (1 - 2q) = q$, helps measure the difference in the change in optimal prime and non-prime product prices as the corresponding ratings increase, and is obviously an increasing function of q. This difference gets further pronounced because increasing the ratings of the non-prime product decreases p_P. Since the available quantities of both the resources are equal, the *RIIR* value, $\frac{\partial \pi^*}{\partial \beta} - \frac{\partial \pi^*}{\partial \alpha}$, increases as q increases.

For larger resource capacities and full segmentation, the relative advantage of improving the prime time programming diminishes. To gain insight into why this effect manifests, first consider the extreme case when q = ½ (where *MBPN* transitions to the *PB* strategy). Since the broadcaster offers only the bundle in this case, improving the ratings of the prime time product affects the *PB* revenue ($\gamma/4$) only through its impact on the ratings of the bundle. But by our "equal ratings-improvement effect" assumption, improving the ratings of the non-prime time product has an identical impact on the ratings of the bundle. Therefore, the *RIIR* value is zero. Another way to look at this is that since a bundle needs both a prime and non-prime product, their impact is equivalent as far as the bundle is concerned. Now consider the *MBPN* region. As q goes to ½, the broadcaster sells more of the bundle, and so (i) the amount of the prime time resource used in the bundle increases and (ii) the amount sold separately decreases. For reasons similar to those in the *PB* case above, the impact of improving prime time quality approximately equals the impact of improving non-prime time quality for the part used in the bundle, and the approximation becomes more exact as q approaches ½. So, the impact on *RIIR* due to the bundle decreases as q approaches ½. The impact on *RIIR* due to the individual component sale also reduces because the amount sold separately decreases. Therefore, the *RIIR* value decreases as q increases.

3.4 VALUE OF BUNDLING

We now analyze the economic benefit of bundling, first from the broadcaster's perspective and then from the advertisers' perspective. For the broadcaster, bundling only makes sense if her revenue when she does consider bundling as an option is at least equal to the revenue when she does not. On the other hand, we say that advertisers (as a group) derive positive value from bundling if their total premium when the broadcaster considers the bundling option exceeds the total premium if the broadcaster does not. Let *Bundle Included* (*BI*) refer to the situation when the broadcaster includes the bundle in the set of products considered for being offered to the advertisers, and *Bundle Excluded* (*BE*) refer to the situation when she does not. Note that *BI* is the default case that we studied in the previous sections. In Sections 3.4.1 and 3.4.2 we investigate the *value of bundling* from the *broadcaster's* viewpoint (VoB_B) and the *advertisers'* viewpoint (VoB_A) respectively. In Section 3.4.3 we determine the total (or, social) value of bundling $VoB = VoB_A + VoB_B$.

3.4.1 Broadcaster's Value of Bundling

We refer to the optimal revenue of the *BE* case as $\pi_{BE}{}^*$. Clearly, $VoB_B = \pi^* - \pi_{BE}{}^*$ is always nonnegative (since the broadcaster will simply not offer the bundle if it is not profitable to do so). For example, in the *PC* region (Figure 2), the optimal *BI* strategy is to not offer the bundle; hence $VoB_B = 0$. Next, whenever q_P and q_N both equal at least a half, the optimal *BI* strategy is to only offer the bundle with an optimal revenue of $\gamma/4$. In this case, the optimal *BE* strategy is to offer only the (more valuable) prime-time product to half the market and not cover the remaining market. The optimal revenue value $\pi_{BE}^{*} = \beta/4$, and $VoB_B = (\gamma - \beta)/4$.

To compute the VoB_B for the other BI regions (see Figure 2), we need to first determine the optimal BE strategies and the corresponding revenues. Doing so is straightforward, and Figure 2a presents the different BE strategies. Note that the optimal BE strategy offers, as mentioned above, only the prime-time product whenever $q_P \geq ½$ (region P^+) at a price $\beta/2$. For the remaining two regions, both the prime and non-prime products are offered, but optimal revenue expressions are different. In the region N^-P^- (where $q_P + q_N < ½$), the expressions for the revenue and the product prices are the same as they were for the PC strategy of situation BI. For region NP^- (where $q_P + q_N \geq ½$ and q_P is less than a half), the revenue is $\frac{\alpha}{4}+(\beta-\alpha)q_P(1-q_P)$, p_P is $\frac{\beta}{2}+(\beta-\alpha)(1/2-q_P)$ and p_N is $\alpha/2$. Note that both q_P and q_N are binding at the optimal solution for the N^-P^- region, only q_P is binding in the NP^- region, and neither resource is binding in region P^+ (except at the boundary $q_P = ½$).

Figure 5b represents the different regions for computing VoB_B. Since VoB_B depends on both π and π_{BE}, the expression for VoB_B would be different if the expression for either π or π_{BE} is different. Therefore, by taking the intersection between the regions in Figures 2 and 5a, we get the Figure 5b. Note that, as an example, the vertical hatched region in Figure 2b, $MBN^+ \wedge NP^-$, is the intersection of the regions MBN^+ of Figure 2 and NP^- of Figure 5a.

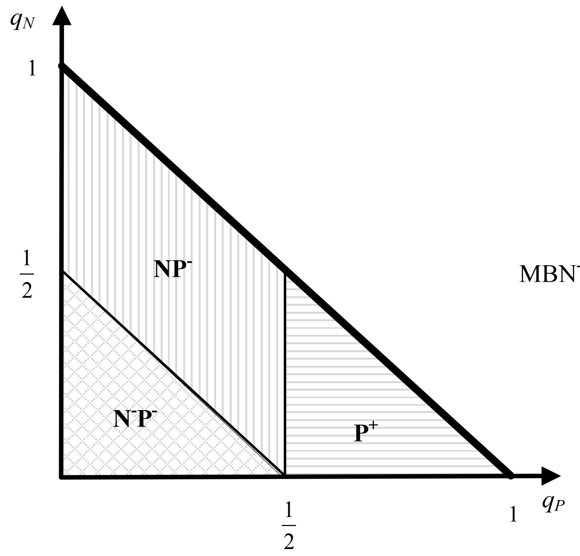

Figure 5a. Optimal strategies when bundling not considered

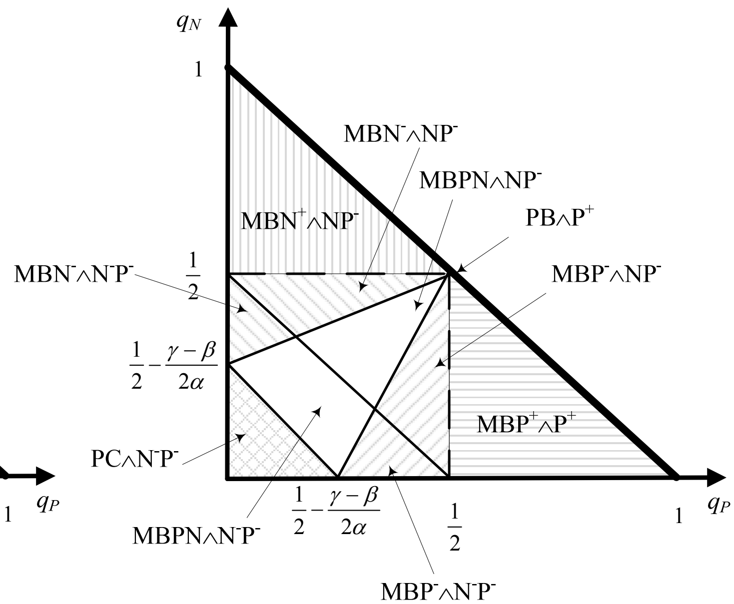

Figure 5b. Regions for investigating impact of bundling

Figure 5. Total *VoB* as a function of $q_N = q_P = q$

Proposition 3.9 lists the VoB_B expressions for the different regions of Figure 5b. The proof of the proposition comes directly from substituting the optimal prices as listed in Proposition 3 into the objective function of *ROMB_U*, which yields π^*. We then compute $\pi_{BE}{}^*$ in a similar fashion, by removing the bundle term from *ROMB_U*.

Proposition 3.9. *The optimal revenue values,* π^* and $\pi_{BE}{}^*$, and *the broadcaster's Value of bundling, VoB_B, are as given in Table 2:*

Proof: For the *BI* scenario, the results follow from substituting the optimal prices from Proposition 3.4 in the objective function (3.10). For the *BE* scenario, the similar construction is made by modifying *ROMB_U* so that there is no bundle considered from the beginning. □

Table 2. Broadcaster's VoB

STRATEGY	OPTIMAL REVENUE π^*	OPTIMAL REVENUE π_{BE}^*	VALUE OF BUNDLING, VoB_B
$PC \wedge N^- P^-$	$\beta q_P(1-q_P)$ $+\alpha q_N(1-q_N-2q_P)$	$\beta q_P(1-q_P)$ $+\alpha q_N(1-q_N-2q_P)$	0
$MBPN \wedge N^- P^-$	$\frac{\gamma}{4}-K_0\left(\frac{1}{2}-q_P\right)^2-K_1\left(\frac{1}{2}-q_N\right)^2$ $-2K_1\left(\frac{1}{2}-q_P\right)\left(\frac{1}{2}-q_N\right)$ $K_0=\frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}$ $K_1=\frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}$	$\beta q_P(1-q_P)$ $+\alpha q_N(1-q_N-2q_P)$	$\frac{\left[\alpha+\beta-\gamma-2\alpha(q_P+q_N)\right]^2}{4(\gamma-\beta+\alpha)}$
$MBPN \wedge NP^-$	$\frac{\gamma}{4}-K_0\left(\frac{1}{2}-q_P\right)^2-K_1\left(\frac{1}{2}-q_N\right)^2$ $-2K_1\left(\frac{1}{2}-q_P\right)\left(\frac{1}{2}-q_N\right)$ $K_0=\frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}$ $K_1=\frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}$	$\frac{\alpha}{4}+(\beta-\alpha)q_P(1-q_P)$	$(\gamma-\beta)\left[\frac{1}{4}-\frac{\alpha(1-q_P-q_N)^2}{\gamma-\beta+\alpha}\right]$
$MBN^- \wedge N^- P^-$	$(\gamma-\alpha)q_P(1-q_P)$ $+\alpha q_N(1-q_N)$	$\beta q_P(1-q_P)$ $+\alpha q_N(1-q_N-2q_P)$	$2\alpha q_P q_N$ $-(\alpha+\beta-\gamma)q_P(1-q_P)$
$MBN^- \wedge NP^-$	$(\gamma-\alpha)q_P(1-q_P)+\alpha q_N(1-q_N)$	$\frac{\alpha}{4}+(\beta-\alpha)q_P(1-q_P)$	$(\gamma-\beta)q_P(1-q_P)$ $+\alpha q_N(1-q_N)-\frac{\alpha}{4}$
$MBN^+ \wedge NP^-$	$\frac{\alpha}{4}+(\gamma-\alpha)q_P(1-q_P)$	$\frac{\alpha}{4}+(\beta-\alpha)q_P(1-q_P)$	$(\gamma-\beta)q_P(1-q_P)$
$MBP^- \wedge N^- P^-$	$(\gamma-\beta)q_N(1-q_N)$ $+\beta q_P(1-q_P)$	$\beta q_P(1-q_P)$ $+\alpha q_N(1-q_N-2q_P)$	$2\alpha q_P q_N$ $-(\alpha+\beta-\gamma)q_N(1-q_N)$
$MBP^- \wedge NP^-$	$(\gamma-\beta)q_N(1-q_N)+\beta q_P(1-q_P)$	$\frac{\alpha}{4}+(\beta-\alpha)q_P(1-q_P)$	$(\gamma-\beta)q_N(1-q_N)$ $+\alpha q_P(1-q_P)-\frac{\alpha}{4}$
$MBP^+ \wedge P^+$	$\frac{\beta}{4}+(\gamma-\beta)q_N(1-q_N)$	$\frac{\beta}{4}$	$(\gamma-\beta)q_N(1-q_N)$
$PB \wedge P^+$	$\frac{\gamma}{4}$	$\frac{\beta}{4}$	$(\gamma-\beta)/4$

Several noteworthy comments follow from the VoB_B expressions in Proposition 3.9. As expected, because the prime-time resource is abundant in the region $\text{MBP}^+ \wedge P^+$, VoB_B does not depend on q_P, but it increases as q_N approaches ½. Similarly, q_N is not binding in the region $MBN^+ \wedge NP^-$, and so VoB_B does not depend on q_N but increases as q_P approaches ½. Within both the MBPN regions ($MBPN \wedge N^-P^-$ and $\text{MBPN} \wedge NP^-$) the VoB is constant along the line $q_N + q_P = c$, for any c between $½ - (\gamma - \beta)/(2\alpha)$ and 1, but increases as c increases. The VoB_B is maximized when $q_N = q_P = ½$.

Now, let us fix the value of q_P such that $(\gamma - \beta)/(2(\alpha + \gamma - \beta)) < q_P < 1/2 - (\gamma - \beta)/(2\alpha)$ and assume that $\alpha > (\gamma - \beta)(1 + \sqrt{5})/2$. These conditions imply that as we increase q_N from zero, the resource availability coordinates will move from sequentially through the $PC \wedge N^-P^-$, $MBPN \wedge N^-P^-$, $MBPN \wedge NP^-$, $MBN^- \wedge NP^-$, and finally the $MBN^+ \wedge NP^-$ regions. The VoB_B will change as follows.

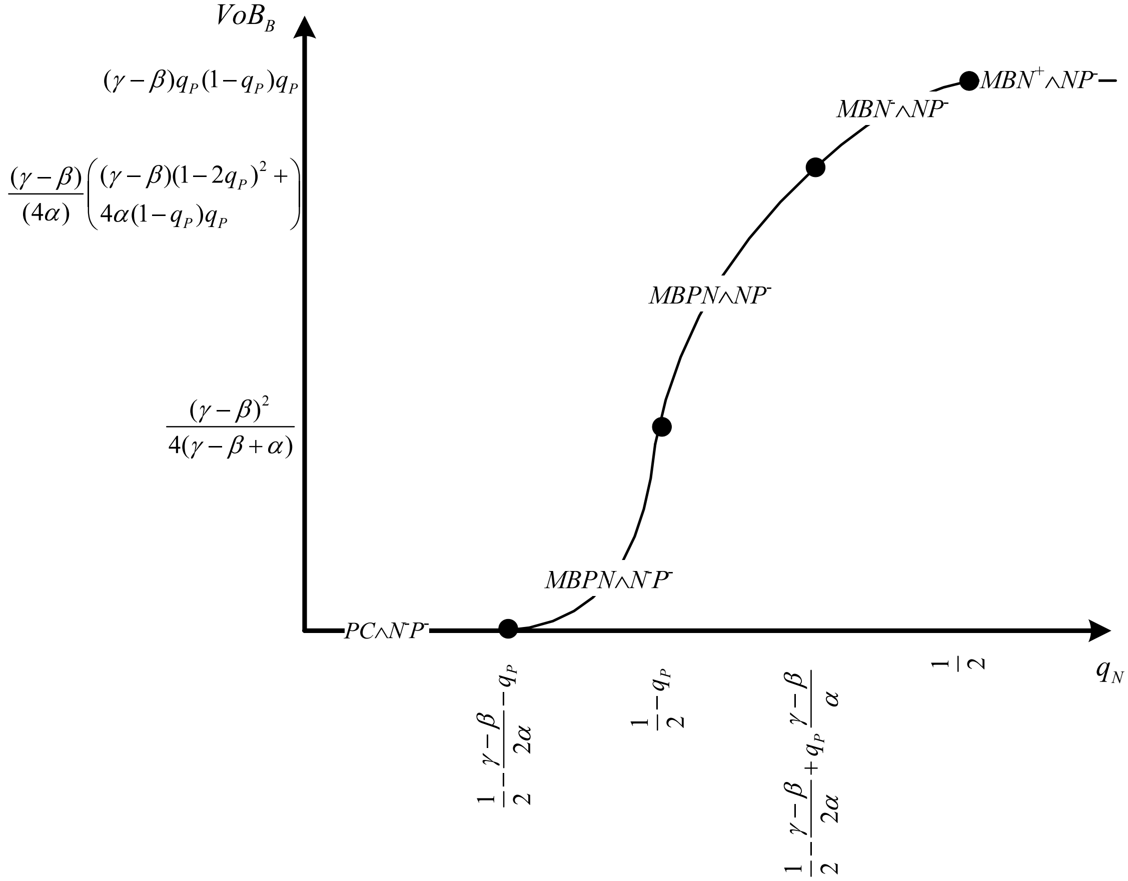

Figure 6. The broadcaster's VoB_B for q_P satisfying
$(\gamma-\beta)/(2(\alpha+\gamma-\beta)) < q_P < 1/2-(\gamma-\beta)/(2\alpha)$

We might have expected that VoB_B would be a concave function of q_N (keeping q_P fixed) rather than the quasi-linear that we observe in Figure 6. Indeed, the optimal revenue functions for both *BI* and *BE* cases for the $MBPN\wedge N^-P^-$ region are concave in q_N; however, when we take the difference of π and π_{BE} we get a convex function. (Another interesting observation is that $\partial^2\pi^* / \partial {q_N}^2 = -2\alpha(\gamma-\beta)/(\gamma-\beta+\alpha) > -2\alpha = \partial^2 {\pi_{BE}}^* / \partial {q_N}^2$.) The reason for this relationship is that the non-prime resource can be used as a part of the bundle in the *BI* case and therefore the rate of increase of the revenue function decreases at a lower rate. For the subsequent two regions

($MBPN \wedge NP^{-}$ and $MBN^{-} \wedge NP^{-}$), π^{*} is concave and $\pi_{BE}{}^{*}$ is constant in q_N, while for the $MBN^{+} \wedge NP^{-}$, both the revenue functions are constant in q_N.

3.4.2 Advertisers' Value of Bundling

The value of bundling for the broadcaster is thus always nonnegative and monotonically nondecreasing in q_N. However, the impact of bundling for the advertiser is less clear. As for the broadcaster, the value of bundling for the advertiser varies by region as in Figure 2b. But there are two issues to consider. First, since the bundle is not offered in the *BE* case, the products that the advertisers select will be different for the *BI* and *BE* cases except in the $PC \wedge N^{-}P^{-}$. Second, because different advertisers have different efficiencies, the same product might contribute different amounts to VoB_A.

Let $R_{BI}(t)$ and $R_{BE}(t)$ denote the premium that an advertiser with efficiency t draws under the *BI* and *BE* cases respectively. Both these premiums depend on the product purchased. $R_{BI}(t)$ is given below; the expressions for $R_{BE}(t)$ are similar.

$$R_{BI}(t) = \begin{cases} \gamma t - p_B & \text{if } T^{*} < t \le 1, \\ \beta t - p_P & \text{if } T^{**} < t \le T^{*}, \\ \alpha t - p_N & \text{if } T^{***} < t \le T^{**}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the total premium for all advertisers under the *BI* scenario, R_{BI}, is

$$\begin{aligned} R_{BI} &= \int_{T^{*}}^{1} (\gamma t - p_B) f(t)dt + \int_{T^{**}}^{T^{*}} (\beta t - p_P) f(t)dt + \int_{T^{***}}^{T^{**}} (\alpha t - p_N) f(t)dt \\ &= \frac{1}{2}\left[\begin{array}{l} (1-T^{*})\left(\gamma(1+T^{*}) - 2p_B\right) + (T^{*} - T^{**})\left(\beta(T^{*}+T^{**}) - 2p_P\right) + \\ (T^{**} - T^{***})\left(\alpha(T^{**}+T^{***}) - 2p_N\right) \end{array}\right]. \end{aligned}$$

Note that if a product is not offered, the thresholds defining the corresponding market segment are equal, and so this product does not contribute towards the consumer premium. Knowing the thresholds, which are easy to derive given the optimal prices in Proposition 3.3 and the boundary conditions (3.5) - (3.7), we can compute R_{BI}.

Since the thresholds will in general be different tor the *BE* scenario, we use the subscript *BE*. For consistency, we use T_{BE}^{**} to denote the lower end of the prime market segment, and T_{BE}^{***} to denote the lower end of the non-prime market segment. Since the bundle is not considered, T_{BE}^{*} is not relevant. Similarly, $p_{P(BE)}$ and $p_{N(BE)}$ denote the prices of the prime and the non-prime products under the *BE* scenario. The total premium for all advertisers under the *BE* scenario, R_{BE}, is then

$$R_{BE} = (1 - T_{BE}^{**})\left(\beta(1 + T_{BE}^{**}) - 2p_{P(BE)}\right)/2 + (T_{BE}^{**} - T_{BE}^{***})\left(\alpha(T_{BE}^{**} + T_{BE}^{***}) - 2p_{N(BE)}\right)/2.$$

Proposition 3.10 presents the values of VoB_A for the regions along the diagonal, that is, $PC \wedge N^- P^-$, $MBPN \wedge N^- P^-$, $MBPN \wedge NP^-$, and $PB \wedge P^+$.

Proposition 3.10. *The optimal consumer premium values, R_{BI} and R_{BE} , and the values of bundling for the advertisers, VoB_A, are as follows:*

STRATEGY	OPTIMAL CONSUMER PREMIUM		VALUE OF BUNDLING, VoB_A
	R_{BI}	R_{BE}	
PC∧N⁻P⁻	$\frac{\beta q_p{}^2 + \alpha q_n(q_n + 2q_p)}{2}$	$\frac{\left(\begin{array}{l}\beta q_p{}^2 + \\ \alpha q_N(q_N + 2q_p)\end{array}\right)}{2}$	0
MBPN∧N⁻P⁻	$\frac{\left(\begin{array}{l}\alpha^2(1-4q_P{}^2)+2\alpha((\gamma-\beta)(-1+2q_N{}^2) \\ +4(\gamma-\beta)q_N q_P + 2\beta q_P{}^2) \\ +(\gamma-\beta)(\gamma-\beta(1-4q_P{}^2))\end{array}\right)}{8(\gamma-\beta+\alpha)}$	$\frac{\left(\begin{array}{l}\beta q_p{}^2 + \\ \alpha q_N(q_N + 2q_p)\end{array}\right)}{2}$	$\frac{\left(\begin{array}{l}(\gamma-\beta-\alpha(1+2q_N+2q_P)) \\ (\gamma-\beta+\alpha(-1+2q_N+2q_P))\end{array}\right)}{8(\gamma-\beta+\alpha)}$
MBPN∧NP⁻	$\frac{\left(\begin{array}{l}\alpha^2(1-4q_P{}^2)+2\alpha((\gamma-\beta)(-1+2q_N{}^2) \\ +4(\gamma-\beta)q_N q_P + 2\beta q_P{}^2) \\ +(\gamma-\beta)(\gamma-\beta(1-4q_P{}^2))\end{array}\right)}{8(\gamma-\beta+\alpha)}$	$\frac{\alpha + 4(\beta-\alpha)q_P{}^2}{8}$	$\frac{\left(\begin{array}{l}(\gamma-\beta)(\gamma-\beta+ \\ \alpha(-3+4q_N{}^2+8q_N q_P+4q_P{}^2))\end{array}\right)}{8(\gamma-\beta+\alpha)}$
PB∧P⁺	$\gamma/8$	$\beta/8$	$(\gamma-\beta)/8$

To get some insight into how VoB_A changes, let us assume that $q_N = q_P = q$. We present VoB_A below, in Figure 7.

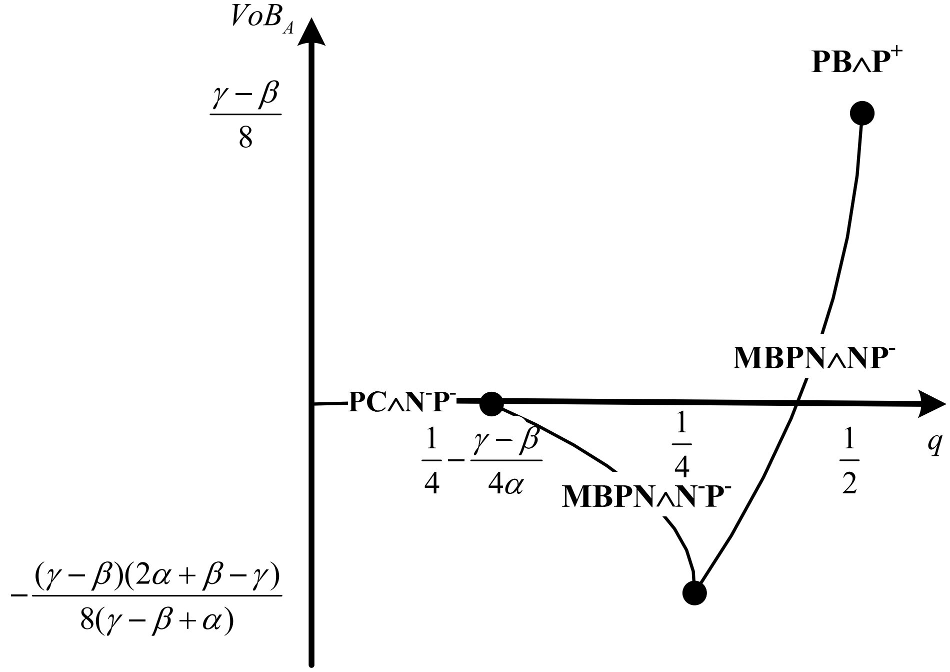

Figure 7. The advertisers' VoB_A as a function of $q_N = q_P = q$

The VoB_A is zero in the $PC\wedge N^-P^-$ region (because bundling is not used for either *BE* or *BI* cases). Continuing from zero in the $MBPN\wedge N^-P^-$, VoB_A decreases until q becomes ¼, and then increases until q equals ½, when it becomes $(\gamma - \beta)/8$. This is in contrast to the model in Schmalensee (1984), for which bundling always lowers the advertisers' value of bundling. Like the model in Salinger (1995), we find that the advertisers' value of bundling can be positive or negative, but unlike his analysis, we observe how the value of bundling changes with q. (He did not assume limited resource availability.) There are two ways in which the advertisers' value of bundling is affected. First, transitioning into the $MBPN\wedge N^-P^-$ from $PC\wedge N^-P^-$ region, there is a downward pressure on the price of the prime and non-prime products when bundling is introduced (see Corollary 3.11). This effect, which like Salinger, we call the *price* effect, tends to increase VoB_A. In addition to the price effect, the *bundle* effect is the loss in the consumer premium when a bundle is introduced. This effect is always nonpositive for the advertisers. The net effect is therefore always nonnegative in the $MBPN{\wedge}NP^-$ and the $PB{\wedge}P^+$ regions, but could be negative in the $MBPN{\wedge}N^-P^-$ region.

Corollary 3.11. *The price effects across the regions described in Figure 5b are as follows:*

STRATEGY	**OPTIMAL PRICES**		**DIFFERENCE ΔP**
	BI	**BE**	
$\boldsymbol{PC\wedge N^-P^-}$	$p_B = \gamma - \beta q_P - \alpha q_N$ $p_P = \beta(1-q_P) - \alpha q_N$ $p_N = \alpha(1-q_P-q_N)$	$p_P = \beta(1-q_P) - \alpha q_N$ $p_N = \alpha(1-q_P-q_N)$	*na* 0 0
$\boldsymbol{MBPN\wedge N^-P^-}$	$p_B = \frac{\gamma}{2} + \frac{\beta(\gamma-\beta)+\alpha(\gamma-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right)$ $+\frac{2\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right)$ $p_P = \frac{\beta}{2} + \frac{\beta(\gamma-\beta)+\alpha(\beta-\alpha)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_P\right)$ $+\frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left(\frac{1}{2}-q_N\right)$ $p_N = \frac{\alpha}{2} + \frac{\alpha(\gamma-\beta)}{\gamma-\beta+\alpha}\left[(1-q_P-q_N)\right]$	$p_P = \beta(1-q_P) - \alpha q_N$ $p_N = \alpha(1-q_P-q_N)$	*na* *positive / negative* Δp switches from negative to positive as q_N, q_P increase. *positive / negative*
$\boldsymbol{MBPN\wedge NP^-}$	$p_B = \gamma(1-q_P) - \alpha(q_N - q_P)$ $p_P = \beta(1-q_P) - \alpha(q_N - q_P)$ $p_N = \alpha(1-q_N)$	$p_P = \frac{\beta}{2} + (\beta-\alpha)(1/2 - q_P)$ $p_N = \frac{\alpha}{2}$	*na* *positive* *positive*
$\boldsymbol{PB\wedge P^+}$	$p_B = \gamma/2$ $p_P = \beta/2$ $p_N = \alpha/2$	$p_P = \beta/2$ $p_N = \alpha/2$	*na* 0 0

Proof: We derive the optimal prices for the *BI* scenario from Proposition 3.4. We get the optimal prices for the *BE* scenario by solving *ROMB_U* without the bundle decision variables and pricing constraints. □

3.4.3 Total (Social) Value of Bundling

The total value of bundling, $VoB = VoB_A + VoB_B$, measures the net economic impact of bundling. In region $PC \wedge N^-P^-$, the VoB is zero. In the $MBPN \wedge NP^-$ region, VoB can be positive or negative, depending on the values of q, α, β, and γ. Interestingly, when $q_N = q_P = q$, our discussion in the previous section indicates that total value of bundling is positive, and increases as q increases. Thus, at values of q close to ½, *both* the advertiser and the broadcaster are better off due to bundling. The broadcaster is better off because bundling allows her to better segment the market; the advertisers are better off because bundling gives some of them the opportunity to get the bundle which has the highest ratings, thus contributing the most to the consumer surplus. Finally, we note that VoB, VoB_A, and VoB_B are all maximized at $q_N = q_P = ½$. While q_N and q_P may be outside of the broadcaster's control, the result indicates a target to aim for so as to achieve a common optimal point from all three perspectives.

Proposition 3.12 presents the values of VoB for the regions along the diagonal, that is, $PC \wedge N^-P^-$, $MBPN \wedge N^-P^-$, $MBPN \wedge NP^-$, and $PB \wedge P^+$.

Proposition 3.12. *The optimal total value of bundling, VoB, is as follows:*

STRATEGY	TOTAL VALUE OF BUNDLING, *VoB*
PC∧N⁻P⁻	0
MBPN∧N⁻P⁻	$\frac{\left[\alpha+\beta-\gamma-2\alpha\left(q_P+q_N\right)\right]^2}{4\left(\gamma-\beta+\alpha\right)}+\frac{\begin{pmatrix}(\gamma-\beta-\alpha(1+2q_N+2q_P))\\(\gamma-\beta+\alpha(-1+2q_N+2q_P))\end{pmatrix}}{8(\gamma-\beta+\alpha)}$
MBPN∧NP⁻	$\left(\gamma-\beta\right)\left[\frac{1}{4}-\frac{\alpha\left(1-q_P-q_N\right)^2}{\gamma-\beta+\alpha}\right]+\frac{\begin{pmatrix}(\gamma-\beta)(\gamma-\beta+\\\alpha(-3+4q_N{}^2+8q_Nq_P+4q_P{}^2))\end{pmatrix}}{8(\gamma-\beta+\alpha)}$
PB∧P⁺	$3(\gamma-\beta)/8$

Proof: Follows from adding the broadcaster's value of bundling from Proposition 3.10 to the advertisers' value of bundling from Proposition 3.11. □

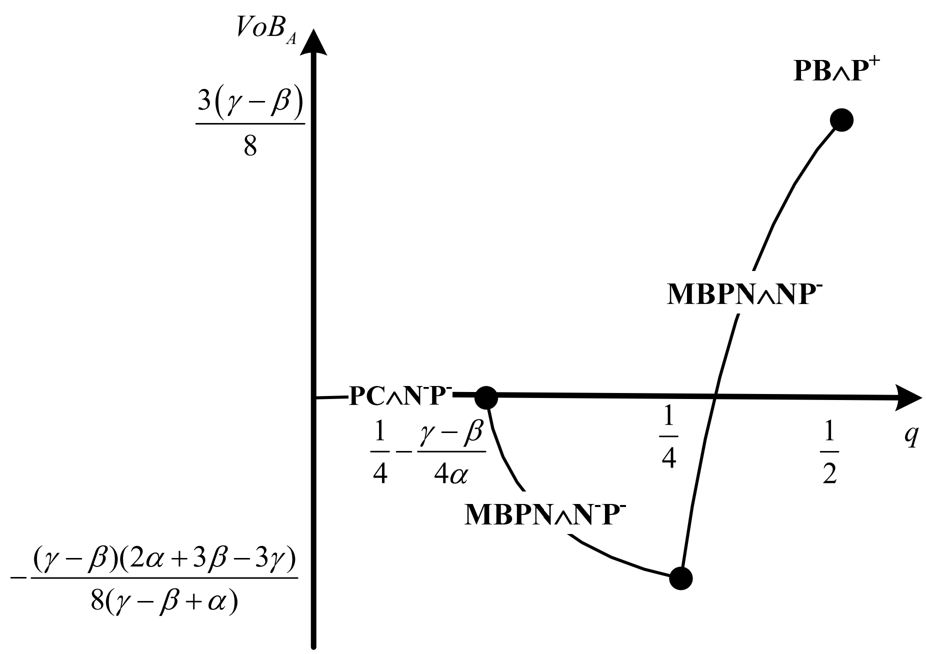

Figure 8. Total *VoB* as a function of $q_N = q_P = q$

Table 3 gives the *value of bundling* from the *broadcaster's* viewpoint (VoB_B) and the *advertisers'* viewpoint (VoB_A) respectively. The last column gives the total (or, social) value of bundling $VoB = VoB_A + VoB_B$. We derive these values numerically for the symmetric case, $q = q_N = q_P$, with $\alpha = 1$, $\beta = 2$ and $\gamma = 2.5$. The tabulated values are expressed as a function of q. Note that the pricing strategy changes, both with and without bundling, as q is increased.

Table 3. Value of Bundling
(Uniform Distribution; $\alpha = 1$, $\beta = 2$, $\gamma = 2.5$)

q_P	q_N	VoB_B	VoB_A	VoB
0.10	0.10	0.0000	0.0000	0.0000
0.20	0.10	0.0017	-0.0092	-0.0075
0.20	0.20	0.0150	-0.0325	-0.0175
0.30	0.30	0.0717	-0.0442	0.0275
0.40	0.40	0.1117	0.0025	0.1142
0.50	0.50	0.1250	0.0625	0.1875

As we can see from the table, VoB_B is zero initially, and then increases as q goes to ½. On the other hand, after remaining at zero initially, VoB_A becomes negative (due to the impact of full segmentation as a result of offering all three products) and then increases. As a result, the total value of bundling VoB is zero initially, becomes negative, and then increases. The reason is that when q is small, bundling is not used and so all three values of bundling are zero. Then VoB becomes negative because the increase in VoB_B does not fully compensate for the decrease in VoB_A, and the net impact is negative.

3.5 EXTENSIONS

In this section, we generalize our results by relaxing our model assumptions. First, rather than assuming that the advertiser efficiencies are uniformly distributed, we consider several other density functions, and investigate the robustness of our results. Next, we allow for bundles to be comprised of arbitrary number of units of the prime and the non-prime resources, and determine how the optimal composition of the bundle changes as the problem parameters change.

3.5.1 General density functions

The results that we have presented so far assume that the efficiency random variable has a uniform density function. A natural inquiry might be to check the sensitivity of our results to changes in the density function. For example, if the density distribution was left skewed or right skewed, how would the optimal strategies, product prices, and the total revenue change for the same level of resources? Or, if the density function was strictly concave and symmetric about an efficiency of one-half, how would the results compare with the uniform distribution?

We use the family of (standard) Beta distributions to model the density function of the efficiencies. The Beta distribution has two shape parameters, which we denote by *a* and *b*. We use the Beta distribution because it has the domain [0, 1] which equals our assumed efficiency range, and changing the parameter values generates the different shapes that are interesting from our perspective. Figure 9 gives the parametric settings and the four different shapes that we will investigate. Given the complexity of deriving the analytic solution for these more general density functions, we complement our analytical results with numerical computations. We assume that $\alpha = 1$, $\beta = 2$, and $\gamma = 2.5$. Since $\alpha = 1$ and $\beta = 2$, γ must lie in the open interval (2, 3) and so a value of 2.5 for γ denotes "medium" incentive to bundle, thus not favoring either a *PB* or a *PC* strategy.

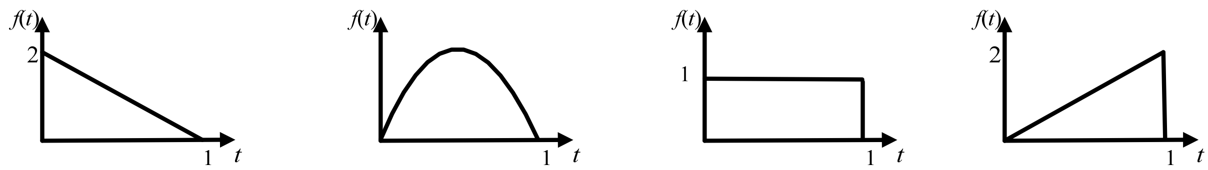

Figure 9 (a): a=1, b=2 **Figure 9 (b):** a=2, b=2 **Figure 9 (c)**: a=1, b=1 **Figure 9 (d):** a=2,b=1

Figure 9. Density functions

We refer to the advertisers having the efficiency distribution in Figure 9(a) as *parsimonious* advertisers because a large majority of them have a low willingness to pay. Similarly, we refer to advertisers in Figures 9(b), 9(c), and 9(d) as *centric* advertisers, *uniform* advertisers and *high-spenders* respectively (we have shown the uniform distribution in this figure for consistency with our later tables and figures).

Figure 10 presents the broadcaster's optimal strategies (determined numerically) for each of the four different types of advertisers as the availability of the two resources changes. Even though there are differences across the different distribution types that reflect the distributions' unique characteristics, the general structure of the optimal strategies is similar.

Comparing the parsimonious advertisers and the high-spenders cases (Figures 10(a) and 10(d)) we observe that the *PC* region is smaller for parsimonious advertisers. This difference is a consequence of parsimonious advertisers being concentrated near the low end of the efficiency scale. In order to extract greater revenue from them, the broadcaster offers full spectrum mixed bundling even when the availabilities of the two resources are low (and the relative availabilities are about the same). For the high-spenders case, the broadcaster uses the *PC* strategy for a greater range of resource availabilities because of the concavity assumption about the bundle ratings.

As we mentioned earlier for the uniform advertisers case, unconstrained optimization corresponds to both q_N and q_P values being at least a half. For the parsimonious advertisers case, we can use Proposition 3.1 to show that the unconstrained region begins at $q_N = q_P = 4/9$. Figure 10(a) reflects this observation. For the high-spenders case, again using Proposition 3.1, we can show that the unconstrained region begins at $q_N = q_P = 2/3$. Just like for the uniform advertisers case, these values of 4/9 and 2/3 do not seem to depend on the value of γ. Thus, the pure bundle is not offered for the high-spenders case when the sum of the resource availabilities is at most one, as we have assumed in this paper. Schmalensee (1984) has previously observed that mixed bundling reduces the heterogeneity in the customers, and therefore allows better price discrimination. A natural measure of heterogeneity is variance, and the distributions for both parsimonious advertisers and high-spenders have the same variance. Yet, for parsimonious advertisers, pure bundling is the optimal strategy for a larger region defined by q_N and q_P, and for high-spenders, mixed bundling is the optimal strategy for a larger region. This comparison of Figures 10(a) and 10(d) thus demonstrates that the *skewness* of the efficiency distribution, besides its heterogeneity (as measured by variance), seems to affect the benefits of mixed bundling.

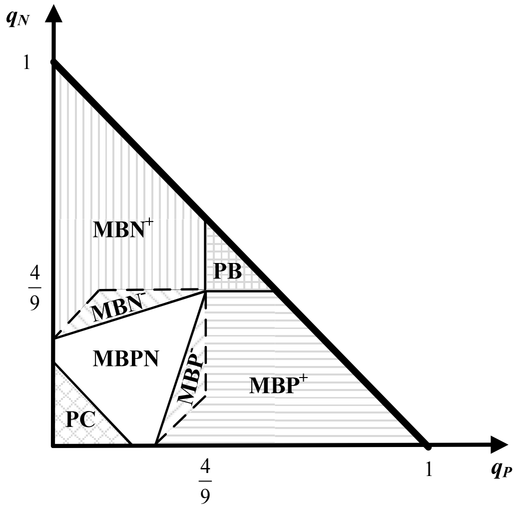

Figure 10 (a): Parsimonious advertisers

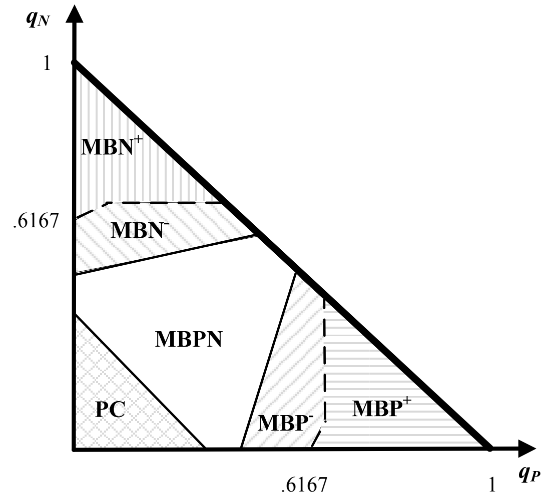

Figure 10 (b): Centric advertisers

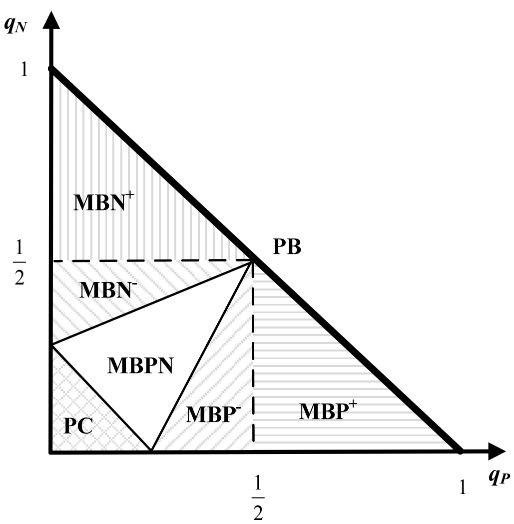

Figure 10 (c): Uniform advertisers

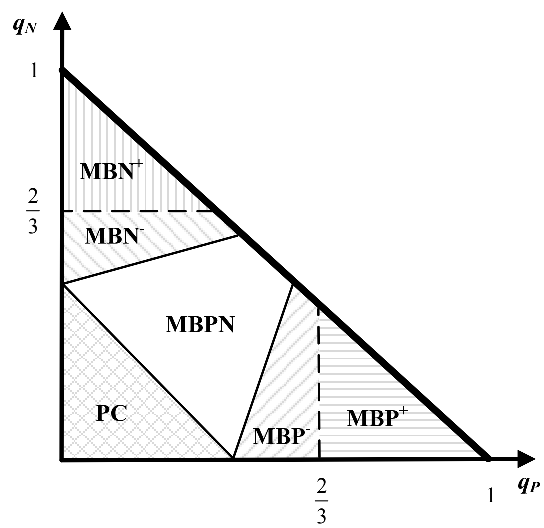

Figure 10 (d): High-spenders

Figure 10. Strategies for the different advertiser types

Tables 4, 5 and 6 show how the solution and the optimal strategy change as the distribution changes for the same values of resource availabilities. The T^* values are the lowest (keeping the resource availability constant) for parsimonious advertisers, and increase as we progressively go through centric and uniform advertisers; they are the highest for high-spenders. Thus, the broadcaster does not have to resort to bundling when the majority of the customers have a high willingness to pay. This is also borne out in Table 3, where the *MBPN* strategy appears more frequently for the parsimonious advertisers' case. As expected, the broadcaster charges the highest prices for the bundle in the high-spenders case.

Table 4. Strategies for Different Types of Advertisers ($\alpha = 1$, $\beta = 2$, $\gamma = 2.5$)

q_P	q_N	PARSIMONIOUS	CENTRIC	UNIFORM	HIGH-SPENDERS
0.10	0.10	MBPN	PC	PC	PC
0.10	0.20	MBPN	MBPN	MBPN	PC
0.10	0.30	MBN^-	MBPN	MBN^-	PC
0.10	0.40	MBN^-	MBN^-	MBN^-	MBPN
0.10	0.70	MBN^+	MBN^+	MBN^+	MBN^+
0.20	0.10	MBPN	MBPN	MBPN	PC
0.20	0.20	MBPN	MBPN	MBPN	PC
0.30	0.10	MBP^-	MBPN	MBP^-	PC
0.30	0.30	MBPN	MBPN	MBPN	MBPN
0.40	0.10	MBP^-	MBP^-	MBP^-	MBPN
0.40	0.40	MBPN	MBPN	MBPN	MBPN
0.50	0.50	PB	MBPN	PB	MBPN
0.70	0.10	MBP^+	MBP^+	MBP^+	MBP^+

Table 5. Threshold values for different Beta distributions ($\alpha = 1$, $\beta = 2$, $\gamma = 2.5$)

PARAMETER	q_P	q_N	TYPES OF ADVERTISERS			
			PARSIMONIOUS	CENTRIC	UNIFORM	HIGH-SPENDERS
T*	0.10	0.10	0.8352	0.9996	1.0000	1.0000
	0.10	0.20	0.7224	0.9231	0.9667	1.0000
	0.10	0.30	0.6838	0.8421	0.9000	1.0000
	0.10	0.40	0.6838	0.8042	0.9000	0.9659
	0.10	0.70	0.6838	0.8042	0.9000	0.9487
	0.20	0.10	0.7224	0.9231	0.9667	1.0000
	0.20	0.20	0.6344	0.8421	0.9000	1.0000
	0.30	0.10	0.6838	0.8421	0.9000	1.0000
	0.30	0.30	0.4939	0.7215	0.7667	0.9264
	0.40	0.10	0.6838	0.8042	0.9000	0.9659
	0.40	0.40	0.3791	0.6208	0.6333	0.8427
	0.50	0.50	0.3333	0.5276	0.5000	0.7517
	0.70	0.10	0.6838	0.8042	0.9000	0.9487
T**	0.10	0.10	0.6838	0.8042	0.9000	0.9487
	0.10	0.20	0.6838	0.8042	0.9000	0.9487
	0.10	0.30	0.6838	0.8042	0.9000	0.9487
	0.10	0.40	0.6838	0.8042	0.9000	0.9487
	0.10	0.70	0.6838	0.8042	0.9000	0.9487
	0.20	0.10	0.5528	0.7129	0.8000	0.8944
	0.20	0.20	0.5528	0.7129	0.8000	0.8944
	0.30	0.10	0.4523	0.6367	0.7000	0.8367
	0.30	0.30	0.4523	0.6367	0.7000	0.8367
	0.40	0.10	0.3675	0.5671	0.6000	0.7746
	0.40	0.40	0.3675	0.5671	0.6000	0.7746
	0.50	0.50	0.3333	0.5000	0.5000	0.7071
	0.70	0.10	0.3419	0.4215	0.5000	0.5773
T***	0.10	0.10	0.5843	0.7129	0.8000	0.8944
	0.10	0.20	0.5279	0.6490	0.7333	0.8367
	0.10	0.30	0.4523	0.6132	0.7000	0.7746
	0.10	0.40	0.3675	0.5671	0.6000	0.7530
	0.10	0.70	0.3419	0.4215	0.5000	0.5774
	0.20	0.10	0.5279	0.6490	0.7333	0.8367
	0.20	0.20	0.4839	0.6132	0.7000	0.7746
	0.30	0.10	0.4523	0.6132	0.7000	0.7746
	0.30	0.30	0.4136	0.5599	0.6333	0.7361
	0.40	0.10	0.3675	0.5671	0.6000	0.7530
	0.40	0.40	0.3562	0.5149	0.5667	0.6999
	0.50	0.50	0.3333	0.4724	0.5000	0.6595
	0.70	0.10	0.3419	0.4215	0.5000	0.5774

Table 6. Bundle prices for different Beta distributions ($\alpha = 1$, $\beta = 2$, $\gamma = 2.5$)

PARAMETER	q_P	q_N	TYPES OF ADVERTISERS			
			PARSIMONIOUS	CENTRIC	UNIFORM	HIGH-SPENDERS
p_B	0.10	0.10	1.6856	2.0169	2.2000	2.3431
	0.10	0.20	1.5728	1.9147	2.1167	2.2853
	0.10	0.30	1.4779	1.8385	2.0500	2.2233
	0.10	0.40	1.3932	1.7734	1.9500	2.1847
	0.10	0.70	1.3675	1.6278	1.8500	2.0004
	0.20	0.10	1.4418	1.8234	2.0167	2.2311
	0.20	0.20	1.3539	1.7471	1.9500	2.1690
	0.30	0.10	1.2464	1.6710	1.8500	2.1113
	0.30	0.30	1.1128	1.5574	1.7167	2.0359
	0.40	0.10	1.0770	1.5362	1.6500	2.0106
	0.40	0.40	0.9133	1.3924	1.4833	1.8958
	0.50	0.50	0.8333	1.2362	1.2500	1.7425
	0.70	0.10	1.0257	1.2452	1.4500	1.6291

Across the different resource availability combinations in Table 5, the T^{***} values are the lowest for parsimonious advertisers indicating that the broadcaster has to "dig deeper" into the market, when many of the advertisers have low willingness to pay. A further analysis of the T^{***} values shows that the location of the marginal advertiser, under all possible resource combinations, that the broadcaster chooses not to serve also varies. It is 0.5 for the uniform advertisers, and for centric customers, this value is 0.4215, which is achieved for both MBN^+ and MBP^+ cases. The reason for this change is that a lack of advertisers with high willingness to pay at the very top end in the centric advertisers case lowers T^* and the increase in the willingness to pay in the middle of the distribution lowers T^{***} (for the MBN^+ strategy) and T^{**} (for the MBP^+ strategy). The minimum T^{***} value (again, over all combinations of the resource availabilities) for parsimonious advertisers is 1/3, and 0.5773 (this data point does not appear in Table 5 as not all resource availabilities are included in the table) for high spenders. Despite the fact that the minimum T^{***} is the highest for high-spenders and the lowest for parsimonious advertisers, the maximum proportion of the market served (again, over all combinations of the resource availabilities) by the broadcaster is the highest (2/3) for high spenders and the lowest (4/9) for parsimonious advertisers. (It is 0.5 for uniform and 0.6167 for centric advertisers, respectively.) These values make sense because the broadcaster serves a greater proportion of the market if the majority of advertisers have a higher willingness to pay.

3.5.2 Bundling with unequal resource proportions

As we mentioned earlier, our model can be modified to allow for an unequal proportion of the two resources in the bundle. In this context, there are two different scenarios, differentiated by whether or not this proportion is fixed or can be chosen optimally. In each case, the solution approach differs slightly depending on whether or not we can redefine the non-prime product when we offer a bundle with unequal proportions of the prime and non-prime products.

First, consider the case when the proportion is fixed and a redefinition of the non-prime product is possible. In this case, we can simply recalibrate the units of measurement of the non-prime product. For instance, since the non-prime resource tends to be more plentiful than the prime time resource, one unit of the non-prime time resource can be calibrated to a supra-unitary multiple of the prime time unit. For example, if the non-prime time resource is ten times more plentiful, a unit of the non-prime time product can consist of ten minutes, while a unit of the prime time product can consist of only one minute. Hence, as long as the proportion of the two resources in the bundle has to remain fixed and the non-prime product can be redefined, the analysis we have done so far immediately carries over with one caveat: if the fixed proportion multiplied by α turns out to be greater than β, then we switch the names of the prime and non-prime products. We make this exchange in the terminology to satisfy our assumption that the prime time ratings exceed the non-prime time ratings.

When the proportion is fixed, but the units of the non-prime time cannot be recalibrated, the analysis is slightly different but the conclusions remain qualitatively similar to what we saw in Section 3.3.

The interesting case arises when the proportion is a decision variable. Here, we focus on the situation where we cannot redefine the non-prime product. When this proportion can be chosen optimally as a function of the parameters of the model (q_P, q_N, α, β, γ), the characterization of the regions depicted in Figure 2 is likely to be more difficult since each combination of capacity levels (q_P, q_N) leads to a different optimal proportion of the resources used in the bundle. Due to the analytical complexity of solving the problem with variable proportions, we illustrate the solution via numerical calculations.

Let the *bundle composition parameter*, θ, with $\theta > 0$, denote the number of units of the non-prime resource in the bundle with one unit of the prime resource. If the non-prime resource has high availability, we expect the optimal value of θ to be at least one. When we introduce the bundle composition parameter θ as a decision variable, we need to make three changes in the *ROMB_U* model. First, since the bundle needs θ units of the non-prime resource (with one unit of the prime resource), the non-prime resource constraint (3.13) changes to $\theta\left(1-\frac{p_B-p_P}{\gamma-\beta}\right)+\frac{p_P-p_N}{\beta-\alpha}-\frac{p_N}{\alpha}\leq q_N$. Second, to avoid the price arbitrage opportunity, so that bundle is "survivable" (Schmalensee, 1984), we need the constraint that the price of the bundle be no more than the sum of the prices of the components it comprises, that is, $p_B \leq p_P + \theta p_N$.

Finally, we have to assume a functional form for the ratings of the bundle, γ, as a function of α, β and θ. We use the specification $\gamma = \sqrt{\beta + \theta\alpha}$, and investigate two cases: $\alpha = 1$ and $\beta = 1.5$, and $\alpha = 1$ and $\beta = 2$. Such a specification for γ guarantees that the ratings of the bundle increase with the ratings of the two resources sold separately, and with θ, the number of units of the non-prime resource used in the bundle. Additionally, the functional form of γ maintains the concavity property we assumed in our original model with fixed proportions.

As expected, the optimal value of θ increases as q_N or q_P increases. Moreover, keeping β constant, θ seems to decrease as α increases (because the square root form of γ does not bring about a proportionate increase in the ratings of the bundle, and so we prefer to sell the nonprime resource by itself rather than in the bundle). The following proposition describes the regions for the different strategies. In this case, we end up using all the non-prime resource since $\theta > 1$, and so the non-prime resource is always binding. For simplicity, we do not distinguish between the MBP^+ and MBP^+ strategies and refer to the strategy where the prime product and the bundle is offered simply as *MBP*.

Proposition 3.13. *For α = 1, β = 1.5 and $\gamma = \sqrt{\beta + \theta\alpha}$, the optimal mixed bundling strategies for a given value of θ are as follows:*

(i) *The pure component strategy is optimal if*

$$0 < q_N + q_P < \frac{1}{2} - \frac{\gamma - \beta}{2\theta\alpha}.$$

(ii) *The full spectrum mixed bundling strategy, MBPN, is optimal if* $q_N + q_P \geq \frac{1}{2} - \frac{\gamma - \beta}{2\theta\alpha}$

and $\frac{\theta\alpha(\theta-1)+(\gamma-\beta)}{\theta\alpha} \leq \frac{1-2q_N}{1-2q_P} \leq \frac{\theta^2\alpha}{\gamma-\beta}$.

(iii) *The partial spectrum mixed bundling strategy, MBN, is optimal if* $q_P > 0$ *and*

$$\frac{1-2q_N}{1-2q_P} < \frac{\theta\alpha(\theta-1)+(\gamma-\beta)}{\theta\alpha}.$$

(iv) *The partial spectrum mixed bundling strategies, MBP is optimal if* $q_N > 0$ *and*

$$\frac{1-2q_N}{1-2q_P} > \frac{\theta^2\alpha}{\gamma-\beta}.$$

Proof: Exactly like Proposition 3.3. □

Figure 11 depicts these regions. Note that each of these regions specialize to the regions in Proposition 3.3 when $\theta = 1$. However, there seems to be one difference. When θ is greater than one, and with the α, β, γ as described above, the pure bundle is never offered.

As the value of θ increases, ceteris paribus, the *PC* region increases. The reason for this is again that the increase in the resources required by the bundle as θ increases is not commensurate with the improvement in its ratings. This expansion of the PC region cuts into the regions for the *MBN*, *MBPN*, and the *MBP* strategies. The line demarcating the *MBN* region from the *MBPN* regions shifts to the left; the reason is that the bundle requires more of the non-prime and so there is less of the non-prime available for sale by itself. The *MBPN* and *MBP* border also shifts to the left for a similar reason (because the bundle requires more of the non-prime resource and only a limited amount of it is available, the region where the prime needs to be sold by itself increases). As a consequence, increasing θ decreases the *MBN* region (because the requirement of the non-prime resource from the bundle increases).

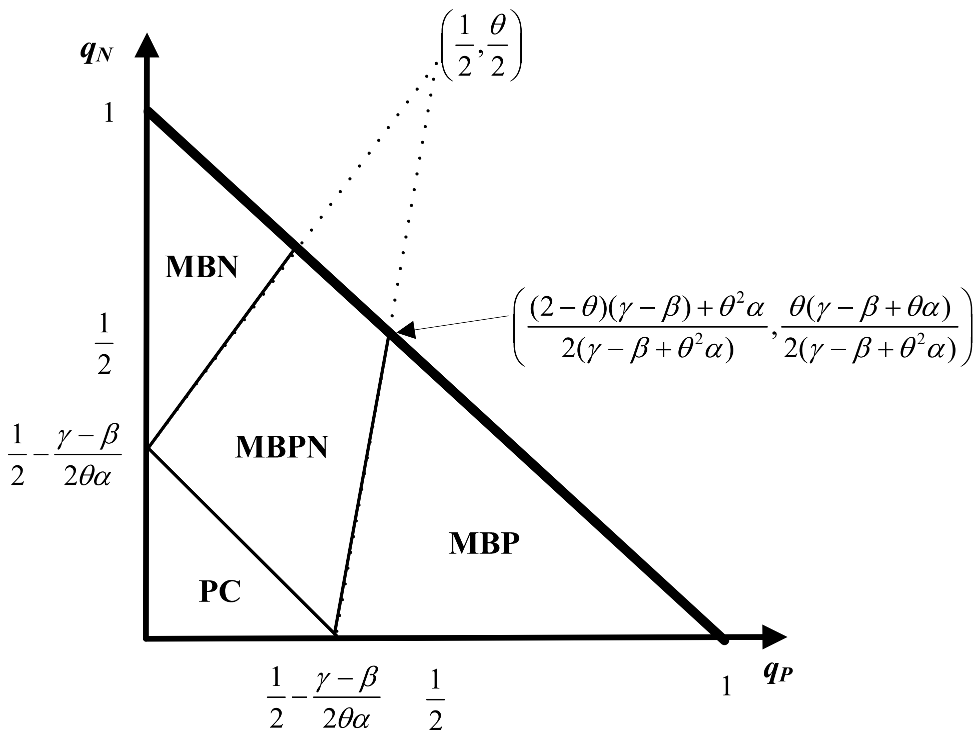

Figure 11. Optimal strategies as a function of the bundle composition parameter, θ

Figures 12(a) – 12(c) depict how θ changes with the problem parameters. In each case, the graphs do not include the *PC* region as the bundle is not offered in this region and so the value of θ is irrelevant. Figure 12(a) shows how the optimal θ changes with $q = q_N = q_P$. The smaller values of q in this graph correspond to the *MBPN* regime. As q increases, the optimal θ value increases because by consuming more of the non-prime resource as part of the bundle, the broadcaster can charge higher prices for the bundle while at the same time avoiding the downward pressure that selling the non-prime product by itself imposes on the price of the prime product. Indeed, as q increases, the optimal strategy shifts from *MBPN* to *MBP*$^{-}$. When the β/α

increases to two, θ jumps up because, relatively speaking, the prime product becomes more attractive and commands a higher price, and therefore the amount of the prime sold by itself increases. Consequently, the amount of the prime resource sold as part of the bundle, and hence the amount of bundle sold by itself decreases, and θ increases.

Figure 12(b) shows how θ changes as q_N increases keeping q_P constant at 0.1. Similarly, Figure 12(c) depicts how θ changes as q_P increases keeping q_N constant at 0.1. In both cases, increasing the resource availability tends to increase the value of θ, but in slightly different ways. As q_N increases, the strategy changes from *PC* to *MBPN* and then to MBN^-. However, as q_P increases, the strategy changes from *PC* to *MBPN* to MBP^- and then finally to MBP^+. Because θ is a decision variable, we always end up using all of the non-prime resource, but that is not so for the prime resource. Also, once we reach the MBP^- region, the value of θ no longer changes since the prime resource is not fully utilized in the bundle. Therefore, in Figure 12(c), we see that θ plateaus; the kink that just precedes the plateau is the point at which the strategy switches from *MBPN* to MBP^- region.

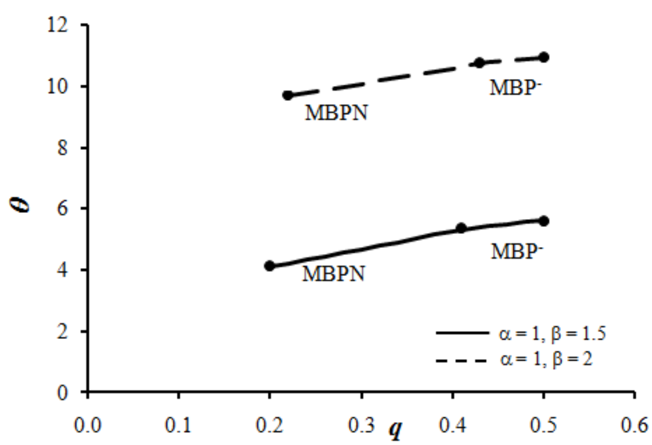

Figure 12 (a). θ as a function of $q = q_N = q_P$

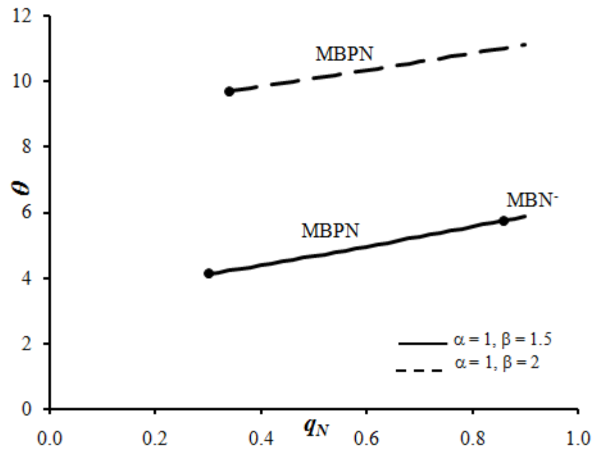

Figure 12 (b). θ as a function of q_N;q_P = 0.10

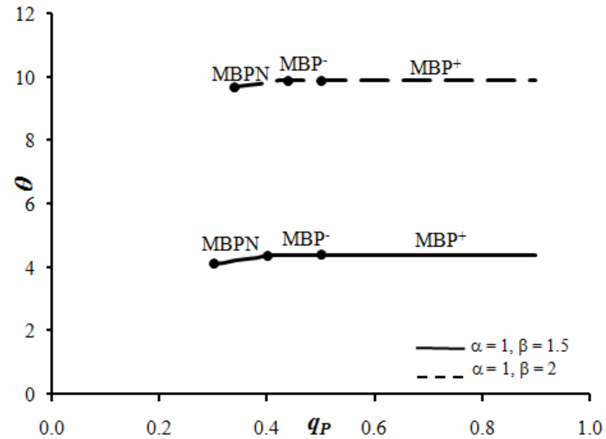

Figure 12 (c). θ as a function of q_P; q_N = 0.10

Figure 12. Change in θ with resource availability

3.6 CONCLUSIONS

In this chapter, we have examined bundling strategies when the bundle's components satisfy a universal preference ordering and have limited availability. While this research is motivated by television advertising, where a preference ordering of the products exists naturally, several other situations (e.g, billboard and internet advertising) also exhibit this characteristic. Our results show that the relative availabilities of the resources strongly influence the broadcaster's optimal strategy of implementing full spectrum mixed bundling (offering the bundle and each of the components), or partial spectrum mixed bundling (offering the bundle with one of the components), or not using bundling at all. Clearly, the resource availabilities also influence their marginal value to the broadcaster; we determine how much more valuable it is to increase the availability of one resource over the other. We also investigate the relative benefits of improving the quality of prime versus non-prime time programming. The robustness of the managerial guidance provided by this analytical work is substantiated by our numerical testing.

Our research points towards several promising research directions. First, we have assumed a monopolistic scenario with only one broadcaster. Introducing competition, where advertisers desiring to place commercials during, say, prime time have a choice of multiple networks, would both add complexity to and enhance realism of the model. Incorporating the advertisers' objectives such as recent work in targeted advertising (Chen & Iyer, 2002; Gal-Or & Gal-Or, 2005; Gal-Or, et al., 2006; Iyer, Soberman, & Villas-Boas, 2005), or combative advertising (Chen, Joshi, Raju, & Zhang, 2009) into this competitive bundling framework promises to be interesting. Second, we have assumed that the resource capacities are limited and that their marginal costs are zero (or, equivalently, that the resource availabilities are limited and the resource costs are sunk). It might be worth investigating how the results change if this marginal cost assumption does not hold. Third, it might be useful to investigate the optimal bundling strategies in the presence of multiple resource classes (for example, in internet advertising, the number of clicks needed from the home page to reach the advertisement). Fourth, our model is deterministic along the advertisers' willingness to pay; introducing stochastic elements with respect to this dimension (Ansari, Siddarth, & Weinberg, 1996; Venkatesh & Mahajan, 1993) might also be worthwhile. Fifth, we have also assumed that the marginal utility of the second unit of a product is zero. By removing this assumption, we can model situations where we can capture the effect of multiple views of an advertisement by the TV viewer. The current model allows for such an extension, by introducing a new product that is formed by two units of non-prime time. Depending on the positioning of this product on the line—either between the non-prime and the prime products, or between the prime and the bundle products—and by comparing the emerging corresponding bundling strategies to the original model, we can analyze the impact of multiple opportunities to see an advertisement.

Finally, we have assumed concavity of the rating function. This need not always be the case. Continuing with the television advertising situation, if there are multiple decision makers who have different viewing preferences, the advertiser may derive super-additive benefits from advertising during prime time and during prime time. Moreover, assuming that there is no secondary market that allows an intermediary to buy the components and assemble the bundle for sale and that the broadcaster can impose a restriction rationing each advertiser to buy at most one product, the price arbitrage constraint (3.2) may not be economically valid.

4.0 COMPETITIVE ENVIRONMENT

4.1 INTRODUCTION

In Chapter 3, we have addressed the problem of deriving optimal bundling strategies under a vertically differentiated market and monopolistic competition, with limited advertising time availability. In this chapter, we will extend the analysis to emphasize duopolistic competition. The intention here is to derive allocation policies and to examine the value of bundling in the television advertising market, under duopolistic competition, when the availability of the advertising time is limited. We choose to focus on this environment due to several reasons. First, the television advertising market captures a significant component of the US advertising market (about $64 billion is spent annually on TV advertising, out of the approximately $150 billion spent on advertising)[8]. Secondly, this market has an interesting structure, given by the variation in television watching habits. For obvious reasons, advertisers prefer placing advertisements during *prime time* (that garners largest audience sizes, but is limited to just the evening hours due to customer viewing habits), and if they cannot afford to, settle for the alternative, called *non-prime time*. Lancaster (1979) refers to this type of market as being *vertically differentiated*. (In a vertically differentiated market, everyone prefers having more of the attribute—in this case, audience—rather than less, but the valuation of the attribute is

[8] 2007 TNS media intelligence report (http://www.tns-mi.com/news/03252008.htm).

different. In contrast, a *horizontally* differentiated market is characterized by a large variety of products.) Thirdly, advertising time availability, especially during prime time, is becoming increasingly restrictive, either due to competitive pressures that force a cap on the amount of commercial time that can be broadcasters can use[9], or due to ongoing debate of regulating the non-programming broadcast time (Getz, 2006), as is the case currently in Europe. Considering the market size and the fact that broadcasters do not have much flexibility for increasing the amount of time they use for airing commercials, the limited resource availability situation in the television advertising market that we are considering is of significant economic consequence to broadcasters. Finally, the television advertising market can be thought of being representative of other vertically differentiated advertising markets, like online or billboard advertising.

In this setting, the broadcaster makes available for sale advertising time during both the prime and non-prime segments. The advertising products are available separately as well as a bundle, and advertisers select products based on their willingness to pay and the product price. Previous marketing literature on bundling has focused primarily on pricing (Ansari, et al., 1996; Hanson & Martin, 1990; Venkatesh & Mahajan, 1993; Wu, et al., 2008), or studying the optimality of bundling (Bakos & Brynjolfsson, 1999; Guiltinan, 1987; Venkatesh & Kamakura, 2003). Recently, Wu et al. (2008) have looked at the welfare function using a computational study. On the economics research front, the bulk of the investigations about welfare effects are in the context of bundling in a monopolistic setting (Dansby & Conrad, 1984; Salinger, 1995; Whinston, 1990). Our work provides an analysis of the value of bundling for both parties involved in the transaction, focused in an advertising environment where limited availability of

[9] 1999 ANA report on commercial clutter (http://www.kued.org/misc/pdfs/outreach/readyToLearn/tvclutter.pdf)

bundle components is a key issue. In the process, we also identify situations where bundling is a win-win proposition for both broadcasters and advertisers.

In the recent past, researchers from the economics and marketing domains have investigated bundling related issues in a competitive environment. Matutes and Regibeau (1992) analyzed the interactions between two players engaged in a duopolistic competition, and showed that the optimal strategy is for companies to provide compatible products (such that consumers could theoretically form their own bundle by purchasing each component from a different firm), but to offer a discount if all components are purchased from the same firm. If the components are "incompatible" (i.e., components from different competitors cannot form a bundle), then they argue that the optimal strategy is pure bundling. Our findings show that in a vertically differentiated market, the partial bundling spectrum can still be optimal, depending on the relative availabilities of the resources. If the market does not exhibit growth potential, it has been shown (Kopalle, et al., 1999) that the equilibrium strategy tends to be pure components. Armstrong and Vickers (2009) have shown that bundling can harm customer welfare, while Thanassoulis (2007) looks at customer welfare in the context of mixed bundling and finds that the customer welfare is either reduced or increased, depending on certain conditions. With respect to the optimality of bundling as a strategy, Chen (1997) shows that bundling is an equilibrium strategy in a duopoly where at least one good that could be part of the bundle is produced under perfect competition. Moreover, if both players in the duopoly commit to bundling, then they increase their profits, but the social welfare is reduced. This idea is confirmed by Gans and King (2006). Separately from the optimality of bundling question, Nalebuff (2004) shows that in a competitive model where a company has market power in two goods, it can use bundling as a strategy to create a barrier to entry in those markets.

Our analysis focuses on the value of bundling, defined as the sum of the broadcasters' and advertisers' respective values, under a duopolistic competition effect, and under the same model of vertical differentiation of the TV advertising industry. Just like in the previous essay, we will assume that one possible dimension of competition is exogenous (the ratings), and competition occurs only along the second dimension, namely price. Once we characterize the various equilibrium strategies, we can compute the value of bundling. Additionally, we assume that the advertisers can purchase time either separately, or a bundle, but only from one network or another. In other words, the advertisers are not allowed to form a bundle on their own, by purchasing one component from the first network, and the second from the other (to follow Matutes and Regibeau (1992) nomenclature, we say that the individual components are *incompatible*). We need to make this assumption in order to compute properly the broadcasters' value of bundling; if we allow compatibility, we can only capture the advertisers' value of bundling.

In the context of one dimensional competition in a vertically differentiated market, Shaked and Sutton (1982) and Moorthy (1988) provide the earliest results about equilibrium prices, though neither consider bundling or capacity issues. They show that the equilibrium strategy is maximum differentiation, with the higher quality firm choosing the higher price and vice-versa. These results are generalized to competition in a two dimensional universe by Vandenbosch and Weinberg (1995). Using a Hotelling spatial location model, they show that the equilibrium strategy has a maxmin structure, that is, the competitors will try to differentiate as much as possible along one dimension and as little as possible along the other. However, in these papers, unlike in our work, the goal is to derive optimal locations for each competitor (i.e. the decision variables are the coordinates of the firms).

4.2 DUOPOLY MODELS

The previous section derived the optimal value of bundling assuming a monopolistic environment. In particular, we identified the value of bundling for the broadcaster and the advertisers under various resource availability scenarios. In this section, we extend the results to a duopolistic environment, so that we can better capture any effects due to competition. Due to the exogenous nature of the ratings, a key operating assumption for this scenario is that the two networks set prices to maximize their revenues. However, the ratings still play an important part, because the relationship (the ordering) between the ratings of the two networks will still drive the model setup and the subsequent equilibrium analysis. With that in mind, let i = 1, 2 denote the index of the network, and let $J = \{N, P, B\}$ be the index set of the three products offered. Also, let $r^i = \{\alpha^i, \beta^i, \gamma^i\}$ be the generic ratings of network i. Let the prices for each product be $p^i_j, j \in J$, and let the indifference thresholds (the market segment delimiters) be $T^*_i, T^{**}_i, T^{***}_i$. These thresholds are derived from the boundary conditions implied by the self-selection model described in the monopolistic environment, adjusted for r^i and $p^i_j, j \in J$. Based on empirical data collected by two advertising trade organizations[10], we assume that each broadcaster has equal resource availability, q_P and q_N, and that both networks have the same availability of q_P and q_N.

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[10] American Association of Advertising Agencies and the Association of National Advertisers, "Television Commercial Monitoring Report", 2002.

There are multiple possible approaches for modeling competition in broadcast advertising market. First, we can assume that the two networks provide similar programming choices and so attract audiences with similar demographics. For example, two family channels such as NBC and ABC have similar viewer demographics; therefore they attract advertisers with similar requirements. On the other hand, channels might attract different audiences and therefore advertisements for different products. For example, ESPN might primarily attract companies such as Toyota and Nickelodeon might primarily attract companies such as Toys'R'Us. We model the first scenario as an extension of the one-dimensional model (because the advertisers are similar) that we developed in the previous chapter and the situation where the advertisers have different requirements using a two-dimensional model. Within the one-dimensional framework, there are two different situations. In the first case, one of the firms dominates the other one; for example, the Food Network dominates the newcomer TasteTV, and CNN dominates a smaller local news station, such as PCNS in Pittsburgh. We refer to the dominating firm as the strong firm and the dominated firm as the weak firm, and assume that the ratings of the non prime product of the strong firm are at least as high as the bundle ratings of the weak firm (see Figure 13 below). In the second one-dimensional case, the two competitors have similar strengths: the ratings of bundles from both firms are stronger than the ratings of the prime time products offered by both firms which are in turn stronger than the ratings of the non prime time products of both firms (see Figure 14). In this chapter, we analyze the strong/weak model. As we later show, this model generalizes the model in Moorthy (1988). In his analysis, Moorthy focuses on a duopolistic model where each competitor offers only one product, and there are no resource constraints. We capture this particular case in our analysis of the strong/weak duopolistic model, with limited resource availability.

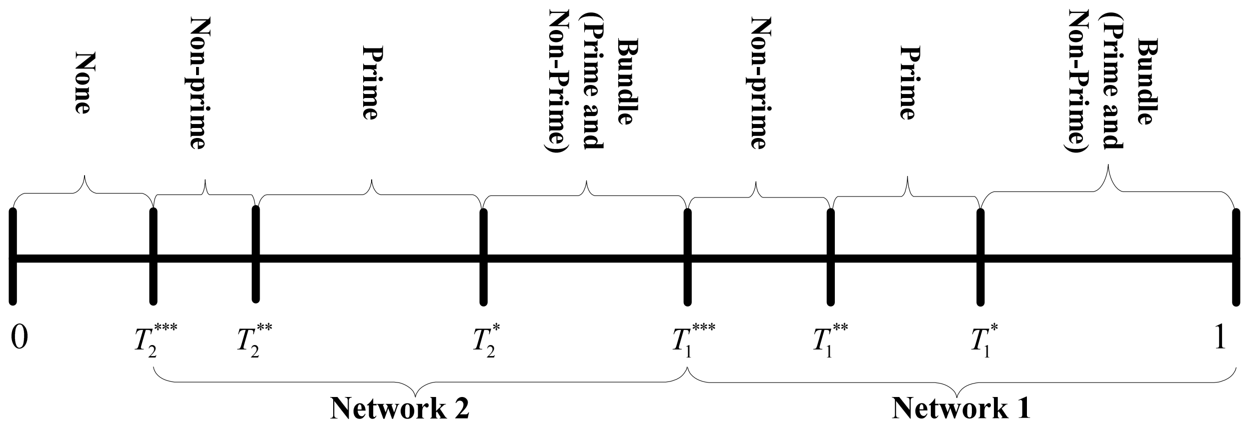

Figure 13. Competition with a dominant network

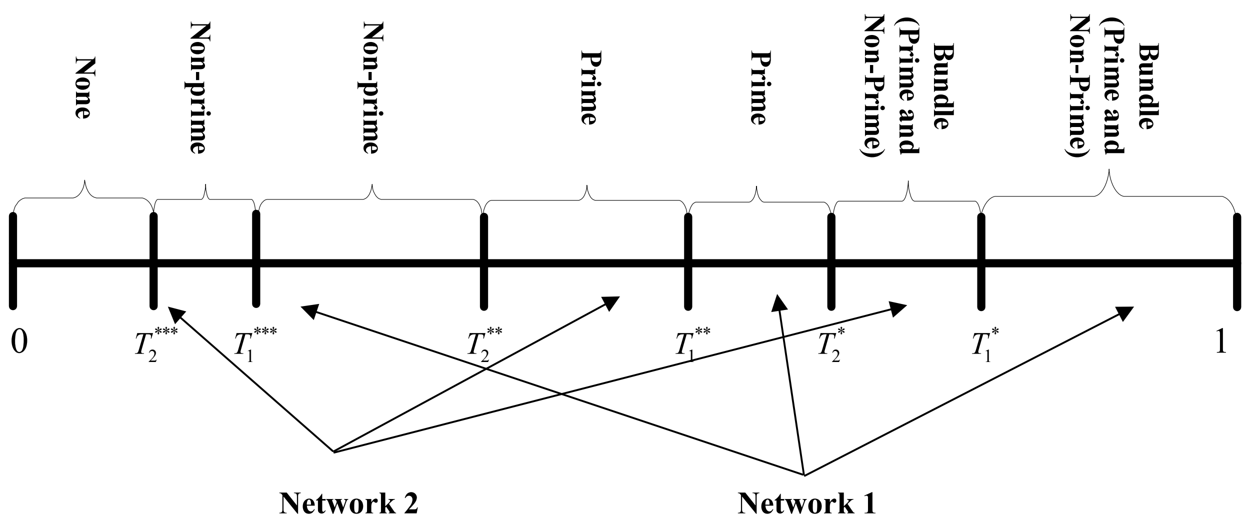

Figure 14. Competition between comparable networks

Returning to the analysis of the case presented in Figure 13 (strong vs. weak competition), assume that the worst rating of network 1 is better than the best rating of the second network (e.g. the ordering assumption is $0 \leq \alpha_2 \leq \beta_2 \leq \gamma_2 \leq \alpha_1 \leq \beta_1 \leq \gamma_1$). Furthermore, assume, without loss of generality, that $\alpha_2 = \alpha_1 / k_\alpha$, $\beta_2 = \alpha_1 / k_\beta$, and $\gamma_2 = \alpha_1 / k_\gamma$, with the multipliers $k_\alpha > k_\beta > k_\gamma \geq 1$. In order to preserve the concavity of the ratings function for the weak firm, we will further need to assume $k_\gamma \geq (k_\alpha k_\beta) / (k_\alpha + k_\beta)$. Since k_γ measures the closeness of the second network with respect to the first, we will refer to this parameter as the *relative weakness* of the second firm. In addition, we must also preserve the concavity assumption for the strong firm, as in the monopolistic model. Notice that we can thus capture either the situation when the best product (the bundle) of the weak network is not differentiated from the worst product (the non-prime time) of the strong network (when k_α, k_β, k_γ approach 1 from above), or a monopolistic model (when k_α, k_β, k_γ approach infinity).

Each network individually optimizes its revenues subject to its availability and pricing constraints. The equilibrium conditions are derived by simultaneously solving for the best response functions for each network since the objective functions are concave. For tractability reasons, unlike the monopolistic competition analysis, we will restrict ourselves only to the cases when both networks are unconstrained, or one or both networks are constrained with respect to prime time availability. (That is, we exclude the unlikely case where one or both networks are constrained regarding the non-prime resource.) We consider the unconstrained scenario to be a base case, and the situation where the prime availability is limited for both networks to be the more realistic scenario. The individual revenue maximization problem *ROMB_U*$_i$, as faced by each network i = 1, 2, is as follows:

[ROMB_U$_\mathbf{i}$] $$\max \pi^i = p_B^i\left(\tau_1^i - \frac{p_B^i - p_P^i}{\gamma_i - \beta_i}\right) + p_P\left(\frac{p_B^i - p_P^i}{\gamma_i - \beta_i} - \frac{p_P^i - p_N^i}{\beta_i - \alpha_i}\right) + p_N\left(\frac{p_P^i - p_N^i}{\beta_i - \alpha_i} - \tau_2^i\right) \quad (4.1)$$

subject to:

$$p_B^i - \left(p_P^i + p_N^i\right) \leq 0, \quad (4.2)$$

$$\tau_1^i - \frac{p_P^i - p_N^i}{\beta_i - \alpha_i} \leq q_P, \text{ and} \quad (4.3)$$

$$\left(\tau_1^i - \frac{p_B^i - p_P^i}{\gamma_i - \beta_i}\right) + \frac{p_P^i - p_N^i}{\beta_i - \alpha_i} - \tau_2^i \le q_N \tag{4.4}$$

$$\tau_1^1 \equiv 1; \tau_1^2 = \tau_2^1 \equiv \frac{p_N^1 - p_B^2}{\alpha_1 - \gamma_2}; \tau_2^2 \equiv \frac{p_N^2}{\alpha_2}. \tag{4.5}$$

Here, for convenience, we introduce the τ notations to capture the optimization model for each network *i* with one single set of equations.

The boundary conditions are as follows:

$$\gamma_1 T_1^* - p_B^1 = \beta_1 T_1^* - p_P^1 \Leftrightarrow T_1^* = \frac{p_B^1 - p_P^1}{\gamma_1 - \beta_1}, \tag{4.6}$$

$$\beta_1 T_1^{**} - p_P^1 = k_\gamma \gamma_2 T_1^{**} - p_N^1 \Leftrightarrow T_1^{**} = \frac{p_P^1 - p_N^1}{\beta_1 - \alpha_1}, \tag{4.7}$$

$$k_\gamma \gamma_2 T_1^{***} - p_N^1 = \gamma_2 T_1^{***} - p_B^2 \Leftrightarrow T_1^{***} = \frac{p_N^1 - p_B^2}{\left(k_\gamma - 1\right)\gamma_2}, \tag{4.8}$$

$$\gamma_2 T_2^* - p_B^2 = \beta_2 T_2^* - p_P^2 \Leftrightarrow T_2^* = \frac{p_B^2 - p_P^2}{\gamma_2 - \beta_2}, \tag{4.9}$$

$$\beta_2 T_2^{**} - p_P^2 = \alpha_2 T_2^{**} - p_N^2 \Leftrightarrow T_2^{**} = \frac{p_P^2 - p_N^2}{\beta_2 - \alpha_2}, \text{ and} \tag{4.10}$$

$$\alpha_2 T_2^{***} - p_N^2 = 0 \Leftrightarrow T_2^{***} = \frac{p_N^2}{\alpha_2}. \tag{4.11}$$

4.3 EQUILIBRIUM ANALYSIS

Just like in the monopolistic scenario, we will first characterize the possible equilibrium solutions in terms of the inventory availability. Since in this part of the work we are concerned only with the impact of the prime time resource, we will characterize the boundaries of the various regions only as a function of q_P and the relative weakness of the second network. For ease of exposition, we will denote an equilibrium strategy via a pair (x & y), with x being the strategy chosen by the strong network, and y the strategy chosen by the weak firm. Both x and y could theoretically take any value in the set $\{PC, PB, MBN, MBP, MBPN\}$. Observe that the strategy of the strong network depends on the relative weakness value (and is independent of the value of q_P. The following result establishes the boundaries, which we show graphically below, in Figure 15.

Proposition 4.1. *The equilibrium space for the strong/weak competition model is partitioned as follows:*

i) If $0 < q_P < \frac{k_\gamma - 1}{4k_\gamma - 1}$, MBN & MBN is a valid equilibrium if the second network is relatively weak, while PC & MBN is a valid equilibrium if the second network is relatively strong;

ii) If $\frac{k_\gamma - 1}{4k_\gamma - 1} < q_P < \frac{1}{2}$, MBN & PB is a valid equilibrium if the second network is relatively weak, while PC & PB is a valid equilibrium if the second network is relatively strong;

iii) If $q_P \geq \frac{1}{2}$, MBN & PB is a valid equilibrium if the second network is relatively weak, while PC & PB is a valid equilibrium if the second network is relatively strong.

Proof: For parts i) and ii), assume that in both $ROMB\_U_1$ and $ROMB\_U_2$ the prime time capacity constraint is binding. The Nash equilibrium is found by solving simultaneously the system of equations formed by the best response function of each firm. The best response functions are as follows:

$$p_B^1 = \frac{1}{2}\left(\gamma_1 - \beta_1 + 2p_P^1 + \lambda_{T^*}^1\right) \tag{4.12}$$

$$p_P^1 = \frac{1}{2\left(\gamma_1 - \alpha_1\right)}\left[\gamma_1\left(\lambda_P^1 + 2p_N^1\right) - \beta_1\left(\lambda_P^1 + \lambda_{T^*}^1 - 2p_B^1 + 2p_N^1\right) + \alpha_1\left(\lambda_{T^*}^1 - 2p_B^1\right)\right] \tag{4.13}$$

$$p_N^1 = \frac{1}{2\left(\beta_1 k_\gamma - \alpha_1\right)}\left[\beta_1 k_\gamma p_B^2 - \alpha_1\left(\left(k_\gamma - 1\right)\lambda_P^1 + k_\gamma p_B^2 + 2p_P^1 - 2k_\gamma p_P^1\right)\right] \tag{4.14}$$

$$p_B^2 = \frac{1}{2k_\gamma\left(k_\beta - 1\right)}\left[k_\beta\left(\lambda_P^2 + \left(k_\gamma - 1\right)\lambda_{T^*}^2 + p_N^1 + 2\left(k_\gamma - 1\right)p_P^2\right) - k_\gamma\left(\lambda_P^2 + p_N^1\right)\right] \tag{4.15}$$

$$p_P^2 = \frac{1}{2k_\beta\left(k_\alpha - k_\gamma\right)}\left[k_\beta k_\gamma\left(\lambda_{T^*}^2 - 2p_B^2\right) + k_\alpha\begin{pmatrix} k_\beta\left(\lambda_P^2 - \lambda_{T^*}^2 + 2p_N^2\right) - \\ k_\gamma\left(\lambda_P^2 - 2p_B^2 + 2p_N^2\right)\end{pmatrix}\right] \tag{4.16}$$

$$p_N^2 = \frac{k_\beta}{2k_\alpha}\left(\lambda_{T^*}^2 - \lambda_P^2 + 2p_P^2\right) \tag{4.17}$$

Solving simultaneously (4.12) - (4.17) we obtain the equilibrium prices. In order for all prices to be valid, the arbitrage pricing constraint must also be observed. Imposing the pricing constraints $p_B^i \le p_P^i + p_N^i, i = 1,2$ introduces two different strategies for network 1: *MBN* and *PC*, depending on the magnitude of the relative weakness parameter, whereas the magnitude of q_P drives two different strategies for network 2—*MBN* and *PB*. Part iii) is derived similarly, observing that all Lagrange multipliers are zero, since the operating assumption is that both networks are unconstrained. □

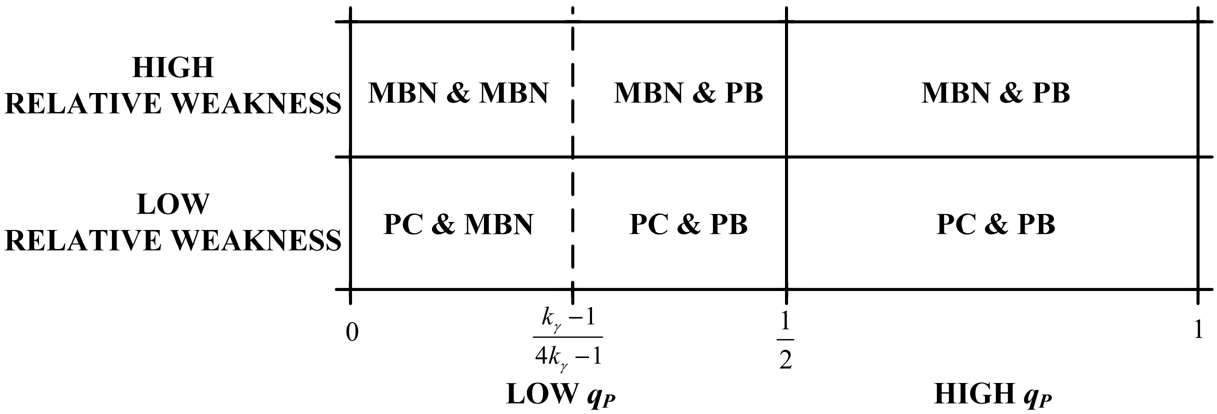

Figure 15. Valid equilibrium strategies

We summarize below, in the next two propositions, the equilibrium prices and corresponding thresholds under the assumption that both networks have no capacity constraints (the natural ordering of the thresholds coupled with the concavity assumptions automatically satisfy the pricing constraints).

Proposition 4.2. *When $q_P > 1/2$, the equilibrium strategy for the weak network is always pure bundling, PB. The strong network may choose either:*

i) *the partial spectrum mixed bundling MBN, if* $k_\gamma \geq 1 + \dfrac{3(\gamma_1 - \beta_1)}{4(\alpha_1 + \beta_1 - \gamma_1)}$.

ii) *pure components PC, otherwise.*

The equilibrium thresholds are as follow:

SCENARIO	THRESHOLD	FIRM 1	FIRM 2
MBN & PB	T_i^{*}	$\frac{1}{2}$	$\frac{k_\gamma - 1}{4k_\gamma - 1}$
	T_i^{**}	$\frac{1}{2}$	$\frac{k_\gamma - 1}{4k_\gamma - 1}$
	T_i^{***}	$\frac{2k_\gamma - 1}{4k_\gamma - 1}$	$\frac{k_\gamma - 1}{4k_\gamma - 1}$
PC & PB	T_i^{*}	1	$\frac{k_\gamma - 1}{4k_\gamma - 1}$
	T_i^{**}	$\frac{1}{2}$	$\frac{k_\gamma - 1}{4k_\gamma - 1}$
	T_i^{***}	$\frac{2k_\gamma - 1}{4k_\gamma - 1}$	$\frac{k_\gamma - 1}{4k_\gamma - 1}$

The equilibrium prices are as follow:

SCENARIO	PRICES	FIRM 1	FIRM 2
MBN & PB	p_B^i	$\frac{\gamma_1}{2} - \frac{3\alpha_1}{2\left(4k_\gamma - 1\right)}$	$\frac{\gamma_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$
	p_P^i	$\frac{\beta_1}{2} - \frac{3\alpha_1}{2\left(4k_\gamma - 1\right)}$	$\frac{\beta_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$
	p_N^i	$\frac{2\alpha_1\left(k_\gamma - 1\right)}{4k_\gamma - 1}$	$\frac{\alpha_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$
PC & PB	p_B^i	$\gamma_1 - \frac{\beta_1}{2} + \frac{3\alpha_1}{2\left(4k_\gamma - 1\right)}$	$\frac{\gamma_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$
	p_P^i	$\frac{\beta_1}{2} - \frac{3\alpha_1}{2\left(4k_\gamma - 1\right)}$	$\frac{\beta_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$
	p_N^i	$\frac{2\alpha_1\left(k_\gamma - 1\right)}{4k_\gamma - 1}$	$\frac{\alpha_2\left(k_\gamma - 1\right)}{4k_\gamma - 1}$

Proof: Solving simultaneously (4.12) - (4.17) and observing that all Lagrange multipliers are 0. The equilibrium prices are then substituted in (4.6) - (4.11) to derive the equilibrium thresholds. □

The proposition establishes that if the weak firm is "weak enough," the strong network does not have to offer the entire mixed bundling spectrum, nor does it have to offer pure components. Past results from the literature have focused, for instance, on the usage of bundling as a barrier to entry (Nalebuff, 2004), or on the equilibrium strategies on a generic market that is not vertically differentiated (Chen, 1997). In our case, surprisingly, the strong firm does not use the bundle as a deterrent, but rather the "worst" product in its portfolio (the non-prime time). As the dominated firm gets stronger and tries to rival the strong firm (k_γ approaches $1+.75\left[(\gamma_1-\beta_1)/(\alpha_1+\beta_1-\gamma_1)\right]$ from above), the strong network reacts by discounting both the non-prime and the bundle. When the prices go down enough such that the arbitrage pricing constraint becomes active (which will happen because the strong network decreases its non-prime price at a faster rate than either the price of the prime or of the bundle), the strong network switches to PC, and continues to discount just the non-prime price, effectively pushing the competitor out of the market. Therefore, it is this downward effect on prices (due to competition) that gradually forces the pricing constraint for the strong network to become binding. Once the pricing constraint is binding, the strong network is forced to switch to pure components, so that it can further depress the price of its non-prime time resource, without the need of affecting downwards its bundle prices, and thus the total revenue. If we denote by π_j^i the revenues of firm *i* under strategy *j*, then the total revenues collected by each firm are:

$$\pi_{MBN}^{1} = \frac{\gamma_1}{4} - \frac{\alpha_1\left(8k_\gamma + 1\right)}{4\left(4k_\gamma - 1\right)^2}, \tag{4.18}$$

$$\pi_{PC}^{1} = \frac{\beta_1}{4} - \frac{\alpha_1\left(8k_\gamma + 1\right)}{4\left(4k_\gamma - 1\right)^2}, \text{ and} \tag{4.19}$$

$$\pi_{PB}^{2} = \frac{\alpha_1\left(k_\gamma - 1\right)}{\left(4k_\gamma - 1\right)^2}. \tag{4.20}$$

We can see the competition effects on total revenue, by varying k_γ. When the weak firm threatens the competition by improving its programming quality, π^1 decreases due to network 1 depressing the prices of its non-prime and bundle, but too much of a threat results in firm 2 being pushed out of the market. In fact, the concavity of π^2 suggests the following corollary, which follows from optimizing π^2 with respect to k_γ:

Corollary 4.3. *The optimal response to the strong firm is for the weak network to choose $k_\gamma = 1.75$. At this value, regardless of the strong network's bundling strategy, network 2's equilibrium profit is maximized at $\alpha_1 / 48$, and the total market coverage is 87.5%, with the strong network capturing 58.33% of the market.*

Proof: Using Proposition 4.2 and substituting the equilibrium prices for network 2 into the objective function of *ROMB_U*$_2$, we obtain the optimal revenue for the weak network. Differentiating the total revenue with respect to the relative weakness parameter and solving the resulting equation gives us the optimal value of the parameter. Substituting back into the corresponding thresholds gives us the corresponding market shares. □

The consequence of the corollary is that there exists an equilibrium solution such that one can find an optimal separation between the two networks. At this level, depending on the concavity of the ratings, the strong firm's revenue will vary from $(12\beta_1 - 5\alpha_1)/48$ to $(12\gamma_1 - 5\alpha_1)/48$. A deviation from this strategy will translate into lost revenues for the second player, and any further threats would result in the eventual expulsion from the market, as the strong firm retaliates by giving non-prime time for free, in the limit. The important insight seems to be that the best approach for the weak firm is to not threaten the strong network when seeking to improve its ratings (as the corollary shows, the optimal value for k_γ is greater than 1). Figure 16 below presents the total revenues earned by the weak firm as a function of the relative weakness parameter.

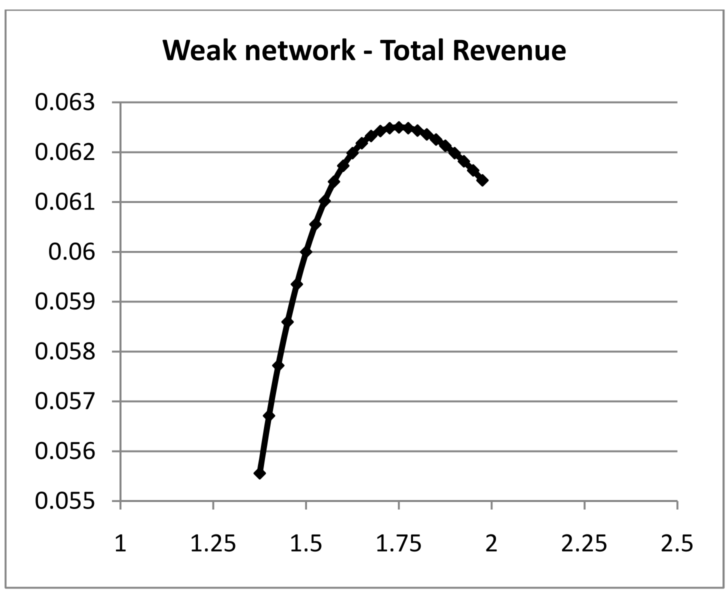

Figure 16. Total revenues of the weak network as a function of k_γ

In addition, as the competition becomes fiercer, the dominant firm depresses the price of the non-prime resource, until it gives it for free in the limit (under the pure components choice, in retaliation to a strong competitor). But this is not the only effect—the price of the bundle is reduced as well, by $\alpha_I/2$, so from a welfare perspective, even if the competition is damaging for the weak firm, the market segment that purchases the strong firm's bundle will be better off. Similarly, when the weak network has very low ratings, it will be forced to compensate by decreasing its prices until her products become essentially free goods. Consequently, the strong firm will behave like a monopolist.

An analysis of the market share covered by each network reveals that, as expected, stronger competition benefits the consumers, in terms of the size of the market that is collectively covered. When the relative weakness parameter is high, then, ceteris paribus, the strong network acts like a monopoly, and in the limit only half the market will be covered. On the other hand, if the relative weakness parameter is low, and thus the weak network threatens the strong network, the total market covered by both firms grows to 100% (with 2/3 of the market served by the strong network, and the remaining 1/3 covered by the weak network). Both market shares are decreasing in k_γ, which suggests that there is no interest in serving a higher fraction of the market as the weak network becomes weaker. Figure 17 below illustrates this phenomenon.

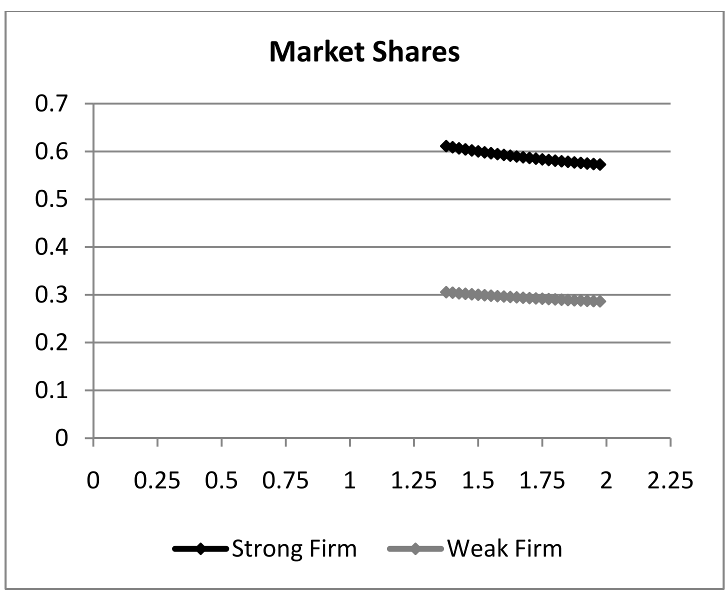

Figure 17. Market shares of both networks as a function of k_γ

The equilibrium prices derived in Proposition 4.2 also generalize Moorthy's (1988) results, since we consider a larger set of products. (In his paper, equations 4.5 and 4.6 are equivalent to our results for p_N^1 and p_B^2, if we define $k_\gamma = s_2/s_1$.) In addition, we provide here an analysis for a duopoly where both players offer a line consisting of three products (two individual components and a bundle), whereas his analysis is limited to competition with single product offerings.

We now shift our analysis to characterizing the equilibrium solution when at least one of the networks is constrained with respect to the prime time resource. First, we examine the situation where the strong network is constrained with respect to q_P, but the weak network is not. We summarize the results in Proposition 4.4.

Proposition 4.4. *When the strong network is constrained with respect to the prime time resource, but the weak network is unconstrained, the equilibrium strategy for the weak network is always pure bundling, PB. The strong network may choose either:*

i) the partial spectrum mixed bundling MBN, if

$$k_\gamma \geq 1 + \frac{3(\gamma_1 - \beta_1)(1 - q_P)}{2\left[\alpha_1 - (\gamma_1 - \beta_1)(1 - q_P)\right]}$$

ii) pure components PC, otherwise.

The equilibrium thresholds are as follow:

SCENARIO	THRESHOLD	FIRM 1	FIRM 2
MBN & PB	T_i^{*}	$1-q_P$	$\frac{k_\gamma-1}{4k_\gamma-1}$
	T_i^{**}	$1-q_P$	$\frac{k_\gamma-1}{4k_\gamma-1}$
	T_i^{***}	$\frac{2k_\gamma-1}{4k_\gamma-1}$	$\frac{k_\gamma-1}{4k_\gamma-1}$
PC & PB	T_i^{*}	1	$\frac{k_\gamma-1}{4k_\gamma-1}$
	T_i^{**}	$1-q_P$	$\frac{k_\gamma-1}{4k_\gamma-1}$
	T_i^{***}	$\frac{2k_\gamma-1}{4k_\gamma-1}$	$\frac{k_\gamma-1}{4k_\gamma-1}$

The equilibrium prices are as follow:

SCENARIO	PRICES	FIRM 1	FIRM 2
MBN & PB	p_B^i	$\gamma_1\left(1-q_P\right)+\alpha_1 q_P-\frac{\alpha_1\left(2k_\gamma+1\right)}{4k_\gamma-1}$	$\frac{\gamma_2\left(k_\gamma-1\right)}{4k_\gamma-1}$
	p_P^i	$\beta_1\left(1-q_P\right)+\alpha_1 q_P-\frac{\alpha_1\left(2k_\gamma+1\right)}{4k_\gamma-1}$	$\frac{\beta_2\left(k_\gamma-1\right)}{4k_\gamma-1}$
	p_N^i	$\frac{2\alpha_1\left(k_\gamma-1\right)}{4k_\gamma-1}$	$\frac{\alpha_2\left(k_\gamma-1\right)}{4k_\gamma-1}$
PC & PB	p_B^i	$\gamma_1-\left(\beta_1-\alpha_1\right)q_P-\frac{\alpha_1\left(2k_\gamma+1\right)}{4k_\gamma-1}$	$\frac{\gamma_2\left(k_\gamma-1\right)}{4k_\gamma-1}$
	p_P^i	$\beta_1\left(1-q_P\right)+\alpha_1 q_P-\frac{\alpha_1\left(2k_\gamma+1\right)}{4k_\gamma-1}$	$\frac{\beta_2\left(k_\gamma-1\right)}{4k_\gamma-1}$
	p_N^i	$\frac{2\alpha_1\left(k_\gamma-1\right)}{4k_\gamma-1}$	$\frac{\alpha_2\left(k_\gamma-1\right)}{4k_\gamma-1}$

Proof: From solving simultaneously (4.12) - (4.17). The equilibrium prices are then substituted in (4.6) - (4.11) to derive the equilibrium thresholds. □

The first observation is that the weak firm follows exactly the same strategy as in the above analysis, that is, they will go with *PB*, regardless of what the strong network chooses. Therefore, the same analysis we have done in the previous proposition will apply here as well. Secondly, due to the constraint on prime time, the tipping point from *MBN* to *PC* for the first network will change (and we note that in the limit, as q_P approaches ½, the switching point from Proposition 4.4 approaches the one from Proposition 4.2).

Proposition 4.5. *When the strong network is unconstrained, but the weak network is constrained with respect to prime time availability, the optimal strategy for the weak firm is always MBN. The strong firm may choose either:*

i) *partial spectrum mixed bundling MBN, if* $k_\beta \le 1 + \dfrac{k_\alpha - 1}{2k_\alpha^2 - 2k_\alpha + 1}$

ii) *pure components PC, otherwise.*

The equilibrium thresholds are as follows:

Scenario	Threshold	Firm 1	Firm 2
MBN & MBN	T_i^{*}	$\frac{1}{2}$	$\frac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]$
	T_i^{**}	$\frac{1}{2}$	$\frac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]$
	T_i^{***}	$q_P + \frac{(k_\gamma - 1)\left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma}$	$\frac{(k_\gamma - 1)}{k_\gamma \left[(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma\right]} \times \left[k_\gamma (k_\alpha - 1) - (k_\alpha - k_\gamma) q_P\right]$
PC & MBN	T_i^{*}	1	$\frac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]$
	T_i^{**}	$\frac{1}{2}$	$\frac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]$
	T_i^{***}	$q_P + \frac{(k_\gamma - 1)\left[2k_\alpha (1 - 2q_P) + (1 - q_P)\right]}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma}$	$\frac{(k_\gamma - 1)}{k_\gamma \left[(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma\right]} \times \left[k_\gamma (k_\alpha - 1) - (k_\alpha - k_\gamma) q_P\right]$

The equilibrium prices are as follows:

SCENARIO	**PRICES**	**FIRM 1**	**FIRM 2**
MBN & MBN	p_B^i	$\dfrac{(\gamma_1-\alpha_1)}{2}+p_N^1$	$\dfrac{\gamma_2(k_\gamma-1)}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}\times \left[(k_\alpha-1)-(k_\alpha-k_\gamma)(4q_P-1)\right]$
	p_P^i	$\dfrac{(\beta_1-\alpha_1)}{2}+p_N^1$	$\dfrac{\beta_2(k_\gamma-1)}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}\times \left[(k_\alpha-1)-(k_\alpha-k_\beta)(4q_P-1)-\left(\dfrac{k_\beta}{k_\gamma}-1\right)q_P\right]$
	p_N^i	$\dfrac{2\gamma_2(k_\gamma-1)}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}\times \left[k_\gamma(k_\alpha-1)-(k_\alpha-k_\gamma)q_P\right]$	$\dfrac{\alpha_2(k_\gamma-1)\left[(k_\alpha-1)-\left(\dfrac{k_\alpha}{k_\gamma}-1\right)q_P\right]}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}$
PC & MBN	p_B^i	$\gamma_1-\dfrac{(\beta_1+\alpha_1)}{2}+p_N^1$	$\dfrac{\gamma_2(k_\gamma-1)\left[(k_\alpha-1)-(k_\alpha-k_\gamma)(4q_P-1)\right]}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}$
	p_P^i	$\dfrac{(\beta_1-\alpha_1)}{2}+p_N^1$	$\dfrac{\beta_2(k_\gamma-1)}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}\times \left[(k_\alpha-1)-(k_\alpha-k_\beta)(4q_P-1)-\left(\dfrac{k_\beta}{k_\gamma}-1\right)q_P\right]$
	p_N^i	$\dfrac{2\gamma_2(k_\gamma-1)}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}\times \left[k_\gamma(k_\alpha-1)-(k_\alpha-k_\gamma)q_P\right]$	$\dfrac{\alpha_2(k_\gamma-1)\left[(k_\alpha-1)-\left(\dfrac{k_\alpha}{k_\gamma}-1\right)q_P\right]}{(2k_\alpha-1)(2k_\gamma-1)-k_\gamma}$

Proof: From solving simultaneously (4.12) - (4.17). The equilibrium prices are then substituted in (4.6) - (4.11) to derive the equilibrium thresholds. □

Proposition 4.6. *When both networks are constrained with respect to the prime time resource, the optimal strategy for the weak firm is always MBN. The strong firm may choose either:*

i) *partial spectrum mixed bundling MBN, if* $k_\beta \le 1 + \dfrac{k_\alpha - 1}{2k_\alpha^2 - 2k_\alpha + 1}$

ii) *pure components PC, otherwise.*

The equilibrium thresholds are as follows:

SCENARIO	THRESHOLD	FIRM 1	FIRM 2
MBN & MBN	T_i^*	$1 - q_P$	$\dfrac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha(1-2q_P) + (1-q_P)\right]$
	T_i^{**}	$1 - q_P$	$\dfrac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha(1-2q_P) + (1-q_P)\right]$
	T_i^{***}	$q_P + \dfrac{(k_\gamma - 1)\left[2k_\alpha(1-2q_P) + (1-q_P)\right]}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma}$	$\dfrac{(k_\gamma - 1)}{k_\gamma\left[(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma\right]} \times \left[k_\gamma(k_\alpha - 1) - (k_\alpha - k_\gamma)q_P\right]$
PC & MBN	T_i^*	1	$\dfrac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha(1-2q_P) + (1-q_P)\right]$
	T_i^{**}	$1 - q_P$	$\dfrac{(k_\gamma - 1)}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma} \times \left[2k_\alpha(1-2q_P) + (1-q_P)\right]$
	T_i^{***}	$q_P + \dfrac{(k_\gamma - 1)\left[2k_\alpha(1-2q_P) + (1-q_P)\right]}{(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma}$	$\dfrac{(k_\gamma - 1)}{k_\gamma\left[(2k_\alpha - 1)(2k_\gamma - 1) - k_\gamma\right]} \times \left[k_\gamma(k_\alpha - 1) - (k_\alpha - k_\gamma)q_P\right]$

The equilibrium prices are as follow:

SCENARIO	**PRICES**	**FIRM 1**	**FIRM 2**
	p_B^i	$\left(\gamma_1-\alpha_1\right)\left(1-q_P\right)+p_N^1$	$\dfrac{\gamma_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[\left(k_\alpha-1\right)-\left(k_\alpha-k_\gamma\right)\left(4q_P-1\right)\right]$
MBN & MBN	p_P^i	$\left(\beta_1-\alpha_1\right)\left(1-q_P\right)+p_N^1$	$\dfrac{\beta_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[\left(k_\alpha-1\right)-\left(k_\alpha-k_\beta\right)\left(4q_P-1\right)-\left(\dfrac{k_\beta}{k_\gamma}-1\right)q_P\right]$
	p_N^i	$\dfrac{2\gamma_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[k_\gamma\left(k_\alpha-1\right)-\left(k_\alpha-k_\gamma\right)q_P\right]$	$\dfrac{\alpha_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[\left(k_\alpha-1\right)-\left(\dfrac{k_\alpha}{k_\gamma}-1\right)q_P\right]$
	p_B^i	$\gamma_1-\alpha_1\left(1-q_P\right)-\beta_1 q_P+p_N^1$	$\dfrac{\gamma_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[\left(k_\alpha-1\right)-\left(k_\alpha-k_\gamma\right)\left(4q_P-1\right)\right]$
PC & MBN	p_P^i	$\left(\beta_1-\alpha_1\right)\left(1-q_P\right)+p_N^1$	$\dfrac{\beta_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[\left(k_\alpha-1\right)-\left(k_\alpha-k_\beta\right)\left(4q_P-1\right)-\left(\dfrac{k_\beta}{k_\gamma}-1\right)q_P\right]$
	p_N^i	$\dfrac{2\gamma_2\left(k_\gamma-1\right)}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}\times \left[k_\gamma\left(k_\alpha-1\right)-\left(k_\alpha-k_\gamma\right)q_P\right]$	$\dfrac{\alpha_2\left(k_\gamma-1\right)\left[\left(k_\alpha-1\right)-\left(\dfrac{k_\alpha}{k_\gamma}-1\right)q_P\right]}{\left(2k_\alpha-1\right)\left(2k_\gamma-1\right)-k_\gamma}$

Proof: From solving simultaneously (4.12) - (4.17). The equilibrium prices are then substituted in (4.6) - (4.11) to derive the equilibrium thresholds. □

It is interesting to notice that the weak network always follows a stable strategy: *PB* if it is unconstrained, and the next best, *MBN*, when it is constrained with respect to the prime-time availability. This suggests that if the weak network does not position itself on the line according to the maxmin principle (by optimizing k_γ), it is vulnerable due to its inflexibility that arises from its "the best or nothing" approach. On the other hand, the strong network does have added flexibility and can adapt both when it has information about the competitor's ratings, as well as when it does not.

4.4 VALUE OF BUNDLING WITH COMPETITION

We now analyze the broadcasters' value of bundling under competition. We present the results below, in Proposition 4.7.

Proposition 4.7. *The broadcaster's value of bundling,* $VOB_B^i, i = 1,2$*, as well as the aggregated broadcaster value of bundling,* VOB_B*, are as given in Table 7:*

Proof: Using Propositions Proposition 4.2 through Proposition 4.6, and substituting the equilibrium prices into the objective functions of $ROMB\_U_1$ and $ROMB\_U_2$, respectively, we obtain the optimal revenues when the bundle is considered by each network. Similarly, we derive the optimal revenues when the bundle is not considered by either network. The difference in revenues represents the broadcaster's value of bundling for each network. □

Table 7. Broadcaster's and aggregate value of bundling

VoB_B	FIRM 1	FIRM 2	AGGREGATE
MBN & PB	$\frac{\gamma_1-\beta_1}{4}$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$	$\frac{\gamma_1-\beta_1}{4}+\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$
MBN & PB	$(\gamma_1-\beta_1)q_P(1-q_P)$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$	$(\gamma_1-\beta_1)q_P(1-q_P)+\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$
PC & PB	0	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{k_\beta-1}{(4k_\beta-1)^2}\right]$
MBN & MBN	$\frac{\gamma_1-\beta_1}{4}$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{\gamma_1-\beta_1}{4}+\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$
MBN & MBN	$(\gamma_1-\beta_1)q_P(1-q_P)$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$(\gamma_1-\beta_1)q_P(1-q_P)+\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$
PC & MBN	0	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\alpha_1\left[\frac{k_\gamma-1}{(4k_\gamma-1)^2}-\frac{(k_\beta-1)(1-q_P)^2}{(2k_\beta-1)^2}\right]$

The first interesting observation is that when the second firm cannot differentiate between its prime and the bundle with respect to the strong network's non-prime time ratings, that is, both are just as poor compared to the strong network's non-prime ratings ($k_\gamma = k_\beta$), the weaker firm derives no value from bundling, while the strong network (and thus the aggregate measure) reverts to the monopolistic scenario analysis. This can be explained by the fact that any benefit derived from bundling by the second firm is negated by the cannibalization due to its prime product having the same rating. Therefore, customers will prefer buying the prime time product at a (presumably) cheaper price. In this situation it makes no difference if the firm is stronger (both multipliers approach 1) or weaker (both multipliers approach infinity), the effect is the same. The other interesting observation is that for firm 2, once k_γ has been optimized, maximization of VoB_B occurs when k_β is minimized subject to the constraint imposed by the concavity assumption. This behavior has also been observed in the two-dimensional competition models based on the Hotelling framework (Ansari, Economides, & Steckel, 1998), which have the same MaxMin equilibrium solution. Under the Hotelling framework, the competitors choose locations alongside each of the two dimensions (say, quality and service level), and the subsequent analysis shows that the equilibrium solution has maximum differentiation across one dimension—in our case, prime time—and minimum differentiation across the other—in our case, the bundle.

We now examine the advertiser's value of bundling. The results are summarized below, in Proposition 4.8.

Proposition 4.8. *The advertisers' value of bundling,* $VOB_A^i, i = 1,2$*, as well as the aggregated advertiser value of bundling, VOB_A, are as given in Table 8.*

We notice that VOB_A has the same behavior as VOB_B, that is, when the dominated firm does not differentiate, regardless of its status (weak or strong), we revert to the monopolistic scenario, and there is no bundling benefit for the second network.

Table 8. Advertisers' and aggregate value of bundling

VoB_A	FIRM 1	FIRM 2	AGGREGATE
MBN & PB	$\frac{\gamma_1-\beta_1}{8}$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$	$\frac{\gamma_1-\beta_1}{8}+$ $\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$
MBN & PB	$\frac{(\gamma_1-\beta_1)q_P^2}{2}$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$	$\frac{(\gamma_1-\beta_1)q_P^2}{2}+$ $\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$
PC & PB	0	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta}{(4k_\beta-1)^2}\right]$
MBN & MBN	$\frac{\gamma_1-\beta_1}{8}$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{\gamma_1-\beta_1}{8}+$ $\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$
MBN & MBN	$\frac{(\gamma_1-\beta_1)q_P^2}{2}$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{(\gamma_1-\beta_1)q_P^2}{2}+$ $\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$
PC & MBN	0	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{\alpha_1}{2}\left[\frac{k_\gamma}{(4k_\gamma-1)^2}-\frac{k_\beta(1-q_P)^2}{(2k_\beta-1)^2}\right]$

Proposition 4.9. *The total value of bundling, VOB, is as follows:*

VoB	Firm 1	Firm 2	Aggregate
MBN & PB	$\frac{3(\gamma_1-\beta_1)}{8}$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$	$\frac{3(\gamma_1-\beta_1)}{8}+\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$
MBN & PB	$(\gamma_1-\beta_1)q_P\left(1-\frac{q_P}{2}\right)$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$	$(\gamma_1-\beta_1)q_P\left(1-\frac{q_P}{2}\right)+\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$
PC & PB	0	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{3k_\beta-2}{(4k_\beta-1)^2}\right]$
MBN & MBN	$\frac{3(\gamma_1-\beta_1)}{8}$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{3(\gamma_1-\beta_1)}{8}+\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$
MBN & MBN	$(\gamma_1-\beta_1)q_P\left(1-\frac{q_P}{2}\right)$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$(\gamma_1-\beta_1)q_P\left(1-\frac{q_P}{2}\right)+\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$
PC & MBN	0	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$	$\frac{\alpha_1}{2}\left[\frac{3k_\gamma-2}{(4k_\gamma-1)^2}-\frac{(3k_\beta-2)(1-q_P)^2}{(2k_\beta-1)^2}\right]$

Proof: Using Proposition 4.7 and Proposition 4.8 and adding up the corresponding values across each scenario and for each network. □

4.5 CONCLUSIONS AND EXTENSIONS

In this chapter, we have extended the bundling study for a vertically differentiated monopolistic market (that we have studied in chapter 3), to include competition in the form of a duopoly. We have modeled two different types of duopolies: competition between a strong and a weak network, and competition between two comparable networks. For the strong-weak competition model, we derive the conditions for the equilibrium strategies as a function of the available capacities and the relative weakness of the second firm, as well as the properties of each equilibrium strategy. We then investigate the value of bundling in a duopoly.

In the equilibrium analysis, we find that the strong network chooses either *MBN* or *PC*, whereas the weak network chooses either *PB* or *MBN*. The main drivers of the equilibrium strategies are the relative scarcity of the prime time resource and the relative weakness of the second firm. Of notable importance is that in a strong/weak framework, unlike in a general duopoly, the strong firm can use the inferior product as a deterrent, rather than the bundle (in fact, the inferior product is used in such a fashion precisely so that the bundle is protected). In this market, the weak network survives only if the quality of its programming does not threaten the strong firm.

The value of bundling analysis suggests that, as expected, competition has a beneficial effect for the consumers (the advertisers) than in the monopolist case. Overall, bundling is a win-win proposition for both broadcasters and advertisers.

Possible future extensions would include finalizing the analysis for the comparable networks scenario and contrasting the results to the findings of the strong/weak analysis presented in this chapter, as well as extending the analysis to an arbitrary number of broadcasters. Separately, it would be interesting to consider a two-dimensional competition model between the two networks. In this two-dimensional framework (see Figure 18), the advertisers are distributed over a unit-square, with their locations measuring their efficiencies for each of the two networks. Each axis represents a network, and along each axis, according to the self-selection framework, there exist thresholds delimiting the different market segments. Unlike the one-dimensional case, we conjecture that the indifference regions would be defined by lines, with different slopes equal to the relative ratings of the competitors. A possible solution to the two-dimensional model would look like Figure 18.

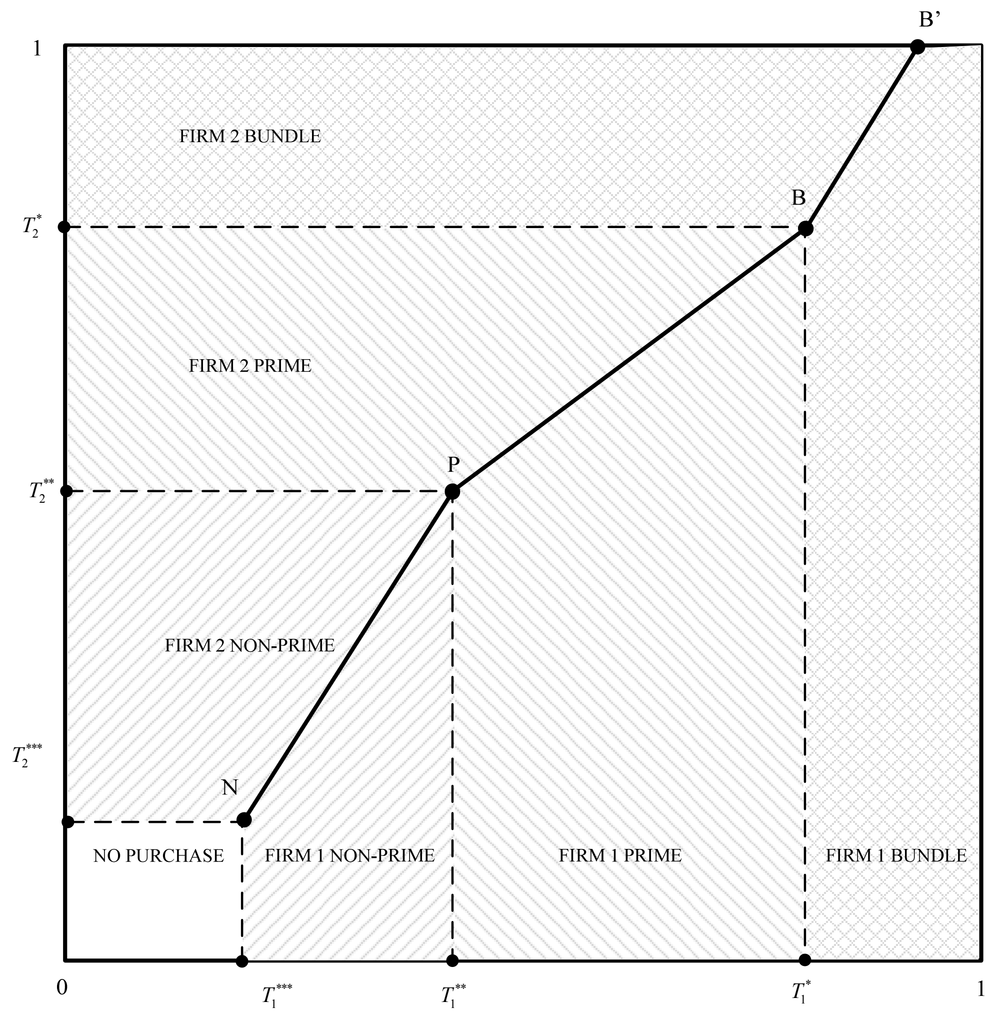

Figure 18. Two-dimensional competition model

Notice that each indifference line has a different slope, due to the fact that the relative ratings may very well be different across the entire unit square (i.e., $\frac{\alpha_1}{\alpha_2} \neq \frac{\beta_1}{\beta_2} \neq \frac{\gamma_1}{\gamma_2}$). In the particular case where the two competitors have equal ratings for each product type, then the three indifference lines would collapse into the 45 degree line.

5.0 MIXED BUNDLING WITH INDEPENDENTLY VALUED PRODUCTS

5.1 INTRODUCTION

In this chapter, we depart from the assumption of the vertically integrated market that existed in Chapters 3 and 4, and focus instead on the properties of the mathematical programming model that describes the general mixed bundling problem. This is an interesting problem because of its complexity, and also because of its potential real-world applications. For example, consider a travel website such as Expedia.com or Travelocity.com, which specialize in putting together travel packages (in effect bundles) of air fares, hotel rooms, car rentals, and/or tickets for local attractions. Assuming that a good demand forecasting system is in place (as discussed in the literature review section on forecasting), and given the high traffic that these type of websites typically experience, the problem of finding an optimal discount level for a travel package, as opposed to building the trip separately from components, in an online environment is very challenging. On one hand, there are many possibilities that can be combined together into bundles, if we consider the plethora of flight legs, hotel classification, car types, etc. On the other hand, the transaction occurs online, so a pricing solution has to be found fast, usually, while a web page is loading in between the submission of forms containing the pricing requests.

The general mixed bundling problem is characterized by rapid growth in both the number of constraints that have to be observed and in the number of variables. On one end, a pure bundling strategy has only one decision variable: at what price should the bundle be offered in order to maximize the revenue from the sale. Conversely, a pure components strategy has n decision variables – one for each component. A mixed-bundling strategy has 2^n -1 variables, reflecting all the possible combinations of products that can be formed from the base set, plus the components that can be sold separately. On the constraints part, the set of bundle prices have to satisfy a "no arbitrage" condition, which means that the seller should never offer a bundle at a price higher than the sum of its parts, since no one will buy it (assuming negligible building costs) and rather prefer to construct the bundle themselves. It is easy to see that, as the base set of components increases, the number of such constraints that have to be observed grows very, very fast. Moreover, in a realistic setting, the seller has limited resources, and so the sales decision will involve trade-offs between the products being offered. Additionally, other design parameters could be added to the complexity of the problem as well: the number of units of each component that goes into the bundle, the demand characteristic (stochastic vs. deterministic, as fitted by a forecasting model). For tractability reasons, we will limit ourselves in our analysis to a deterministic linear demand assumption, and we will base our analytical work on the two product/one bundle setting. We also assume that all products (that is, the individual components, as well as the bundle) are independently valued, and that, in essence, there exists a separate market for each product. These markets are linked via a common inventory and via the pricing constraints. While this assumption may seem limiting, we will see later on that it still provides a good basis for a rich discussion.

This chapter is organized as follows. Section 5.2 provides a small theoretical foundation that helps us understand why exactly the mixed bundling problem is difficult to solve, and what is its special structure. In this case, the underlying price function defined on the set of products is submodular, and the underlying structure of the unconstrained mixed bundling problem is that of a polymatroid. We discuss the definition and properties of polymatroids and explain under what conditions a greedy algorithm is optimal for this type of problem. Section 5.3 discusses our modeling assumptions and develops a nonlinear pricing model for a bundling situation when the resources have limited availability. The output from this model is a set of optimal product prices. In Section 5.4, we analyze the properties of the optimal prices and of the shadow prices. Section 5.5 starts the computational investigation by developing a heuristic method that computes fast sub-optimal bundle prices in the presence of inventory considerations. We also provide a theoretical discussion of the worst-case performance of the method, as well as a computational study that shows the practical performance of the method on a set of randomly generated instances. Section 5.6 addresses the need for a large scale optimization methodology for the mixed bundling problem through a column generation approach. We discuss the connection between the restricted master problem and its two sub-problems – the separation and the pricing sub-problems – and do a computational analysis of the performance of this method. We find that in all instances only a small fraction of constraints and variables are present in the final solution.

5.2 SUBMODULAR OPTIMIZATION

In order to have a better understanding of the complexities of the general mixed bundling problem, we need to define first several concepts that we will use throughout this chapter. For a better understanding of submodular optimization in the context of resource allocation problems, see Ibaraki and Katoh (1988), while for a good treatment of general submodular optimization topics, Topkis (1978) provides a good reference.

Let n be a positive integer and let $E = \{1, 2, ..., n\}$. Let 2^E denote the family of all subsets of E, and let $\mathcal{D} \subseteq 2^E$ be a given family of subsets of E. We will start with the following definitions.

Definition 5.1. If $X, Y \in \mathcal{D} \Rightarrow X \cup Y, X \cap Y \in \mathcal{D}$, we call $\mathcal{D}$ a *distributive lattice* with lattice operations union and intersection, and we denote it by $(\mathcal{D}, \cup, \cap)$.

Definition 5.2. A function $r : \mathcal{D} \to \mathbf{R}$ is *submodular* over the distributive lattice $\mathcal{D}$ if:

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y), \forall X, Y \in \mathcal{D}$$

Furthermore, a pair $(\mathcal{D}, r)$ where $\mathcal{D}$ is a distributive lattice and r is a submodular function over $\mathcal{D}$ is called a *submodular system*.

Definition 5.3. Define $\mathbf{R}^E = \left\{ x = (x_1, x_2, ..., x_n) \mid x_j \in \mathbf{R}, j \in E \right\}$. For a submodular system $(\mathcal{D}, r)$, $P(r) = \left\{ x \mid x \in \mathbf{R}^E, x(X) \leq r(X), \forall X \in \mathcal{D} \right\}$ is called the *submodular polyhedron* associated with $(\mathcal{D}, r)$. A subset of $P(r)$, $B(r) = \left\{ x \mid x \in P(r), x(E) = r(E) \right\}$ is called the *base polyhedron* of $(\mathcal{D}, r)$, and each $x \in B(r)$ is a *base* of $(\mathcal{D}, r)$.

Definition 5.4 (Edmonds, 1970). A system $(2^E, r)$ is a *polymatroid* if $r : 2^E \to \mathbf{R}$ satisfies the following axioms:

i) $r(\varnothing) = 0$

ii) monotonicity: $r(X) \leq r(Y), \forall X, Y \in 2^E; X \subseteq Y$

iii) submodularity of r

Notice that according to Definition 5.4 a pricing relationship for bundles satisfies the conditions of a polymatroid. Indeed, the price of nothing should be 0 (condition i). For any two bundles *X* and *Y*, such that *Y* contains at least as many items as *X*, the price of *X* should be lower than the price of *Y* (condition ii). Finally, for any two components (either individual components or bundles) *S* and *T*, the bundle $S \cup T$ should be priced lower than the sum of its components (the "no arbitrage" pricing constraint first introduced in model *ROMB_U* in Chapter 3; also condition iii). The following result provides us with the intuition for our own heuristic development later in the chapter.

Lemma 5.5. Consider the following optimization problem:

$$SMLP : \left\{ \max \sum_{j \in E} c_j x_j \mid x \in P(r),\, x \geq 0 \right\}$$

Problem *SMLP* defined over the polymatroid $\left(2^E, r\right)$ can be solved in polynomial time using a greedy algorithm.

Proof: See Ibaraki and Katoh (1988).

Notice that problem *SMLP* is a relaxation of the problem *ROMB_U* (there are no capacity constraints here).

5.3 THE GENERAL MIXED BUNDLING PROBLEM

In this section, we will examine the bundling problem formulated as a generalization of the well-studied resource allocation problem (Hochbaum, 1994; Ibaraki & Katoh, 1988). In this context, Federgruen and Groenvelt (1986) and later Zaporozhets (1997) have shown that a greedy algorithm based on allocating resources according to the marginal revenue is optimal if the underlying problem structure is a polymatroid. Unfortunately, the general bundling formulation does not have a polymatroidal structure (because of the capacity constraints), rendering a pure greedy approach ineffective. Related, Dyer and Frieze (1990) describe a polynomial algorithm for the particular allocation problem with a nested structure; as we will see later, the general model does not have this structure, since a component can be used in more than one bundle. A different approach is described by Hanson and Martin (1990), who look at an optimization model based on reservation prices and market segmentation.

Let $N = \{1, 2, ..., n\}$ denote the set of components which we can be used to form bundles. Let $B = \{1, 2, ..., b\}$ denote the set of all products (separate components and bundles) offered. Let 2^N denote the power set of *N*. For all $k \in N$, let $S(k) = \{I \subset B \mid k \in I\}$, and $T(J) = \{k \in N \mid k \in J\}$, that is, *S*(*k*) is the set of all products that use component *k*, and *T(J)* is the set of all components that are used in bundle *J*. Let $\mathbf{\Gamma} = \left(\gamma_J^k\right), k \in N, J \in B$ be the *bill of materials* matrix, that is, γ_J^k represents how many units of component *k* are used in packaging product *J* (for our purposes, we will assume that all γ_J^k are either 0 or 1, that is, there is only unitary consumption). Notice that under this provision, the set of possible bundles increases infinitely, because we could in theory create independently a given combination of any number of base products, mixed in any number of proportions.

Let $\mathbf{p} = (p_J), J \in B$ denote the price vector, and $\mathbf{d} = d_J(p_j), J \in B$ be the demand function of product *J*. Also, let $\mathbf{a} = a(p_J, \xi), J \in B, \xi \in \Xi$ denote the number of units of the J^{th}

product offered, under a random scenario ξ drawn from the set of all possible future realizations Ξ. Let $\mathbf{r} = r(p_J, \xi)$ be the revenue function, defined as $\mathbf{r} = \mathbf{p}^{\mathrm{T}}\mathbf{a}$. The general problem of stochastic revenue maximization with mixed bundling (*SMBRM*) in the presence of given inventory levels q_i of each component *i*, and subject to "no arbitrage" pricing constraints, can be formulated as follows:

[SMBRM] $$\pi_{SMBRM} = \max_{\mathbf{p},\mathbf{a}} \mathop{\mathbb{E}}_{j\in\Xi} \left\{ \sum_{i=1}^{n} \mathbf{r}(p_i, \xi_j) \right\}$$

subject to:

$$\sum_{I:\cup T(I)=T(J)} p_I \geq p_J, \forall J \in B \ s.t \sum_{I:\cup T(I)=T(J)} \gamma_I^k \geq \gamma_J^k, \forall k \in N \qquad (5.1)$$

$$\sum_{J\in S(i)} \gamma_J^j a(p, \xi_k) \leq q_i, \forall i \in N, k \in \Xi \qquad (5.2)$$

$$a(p_i, \xi_j) \leq d(p_i, \xi_j), \forall i \in 2^N, \forall j \in \Xi \qquad (5.3)$$

$$0 \leq p_i \leq u_i, i \in 2^N \qquad (5.4)$$

The first set of constraints imposes that the price of any bundle has to be i) at least the price of any of its component parts, and ii) the price of the bundle has to be at most the sum of the prices of its components. Thus, this set enforces a no "buyer arbitrage" condition – that is, we do not allow for the possibility of a particular consumer buying a bundle that contains more items that she needs initially, breaking it up, and discarding of the parts that are of no use to her; nor do we allow for someone to buy a bundle, break it up into an arbitrary number of sub-bundles and/or components, and resell everything for a profit. The second set of constraints is a regular inventory capacity constraint. Constraint set (5.3) imposes that the allocation should not exceed the demand, while the last constraint set says that the prices should not exceed a general upper bound *u*, for which the corresponding demand is 0.

In order to reduce the complexity of the model, throughout this section we will make the following simplifying assumptions:

a) the demand is deterministic;

b) the consumption rates are unitary, i.e. $\gamma_i^j = 1, \forall i, j$.

Proposition 5.6. *If demand is deterministic, then optimal allocations are made exactly at the observed demand levels.*

Proof: Optimality conditions imply this result. Let $\mathbf{a}^*$ be a solution in which the allocation is not made exactly at the observed demand level. If there is under-allocation, then we can improve the value of the objective function induced by $\mathbf{a}^*$ simply by increasing the allocation. No constraint will be violated. Hence, constraints (5.3) are always tight when demand is deterministic. □

The managerial insight of the above proposition is that when a manager knows exactly the demand for his products, she will allocate exactly up to the demand level, while observing the capacity constraints. As a result, the only decision variables in the revised model are the prices, since $a(p_i) = d(p_i), \forall i \in 2^N$.

The simplified model MBRM can be written as follows:

[MBRM] $\pi_{MBRM} = \max_{\mathbf{p}} \sum_{i=1}^{n} r(p_i)$

subject to:

$$\sum_{I:\cup T(I)=T(J)} p_I \geq p_J, \forall J \in B \tag{5.5}$$

$$\sum_{J \in S(k)} d_J(p_J) \leq q_k, k \in N \tag{5.6}$$

$$0 \le p_i \le u_i, i \in 2^N \tag{5.7}$$

In the formulation *MBRM*, the total number of variables is given by $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$. The first set of constraints can be counted using the definition of a *Bell number*. The Bell number of a set, denoted by B_n, counts the number of ways in which the set can be partitioned into non-empty subsets. It is given by the recurrence formula $B_{n+1} = \sum_{k=0}^{n} B_n \binom{n}{k}$. In our case, there are $\sum_{k=2}^{n} B_k \binom{n}{k}$ constraints of the type (5.5). Finally, the set of constraints (5.6) has exactly n constraints. Therefore, the problem has exponential growth in both the number of variables and in the number of constraints. The following table exhibits the combinatorial explosion of the problem, as the cardinality of set N increases:

Table 9. MBRM problem growth for selected values of *n*

n	ROWS	COLUMNS
10	677,556	1,023
20	4.75×10^{14}	1,048,575
30	1.03×10^{25}	1.07×10^{9}
40	2.35×10^{36}	1.09×10^{12}
50	3.26×10^{48}	1.12×10^{15}

5.4 BUNDLING WITH LINEAR DEMAND FUNCTIONS

Let $p_i = \alpha_i - \beta_i d_i$ be a general linear demand function. Since we are interested in finding out the optimal price values, we will make the transformation $A_i = \frac{\alpha_i}{\beta_i}, \eta_i = \frac{1}{\beta_i}$ such that the demand has the form $d_i = A_i - \eta_i p_i$ for each product (including the bundle). Let $\overline{A_i} = \sum_{j \in S(i)} A_j$ (i.e., $\overline{A_1} = A_1 + A_{12} + A_{13} + \ldots + A_{12\ldots n}$). Then, the first order KKT conditions applied to the deterministic version of the MBRM model imply the following optimal price vector

$$p_I^* = \frac{A_I + \eta_I \sum_{\{j \in I \| j|=1\}} \lambda_j}{2\eta_I}, \forall I \in B \tag{5.8}$$

where $\lambda_1, \ldots, \lambda_n$ are the Lagrange multipliers associated with the n inventory availability constraints (5.6). We can then show the following result.

Proposition 5.7. *Constraint set* (5.5) *is satisfied automatically if*

$$\sum_{I:\cup T(I)=T(J)} \frac{A_I}{\eta_I} \geq \frac{A_J}{\eta_J}, \forall J \in B$$

Proof: Consider a general inequality of type (5.5). According to (5.8), we can rewrite the term:

$$p_I + p_J - p_{I \cup J} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{A_I + \eta_I \sum_{\{k \in I \| k|=1\}} \lambda_k}{2\eta_I} + \frac{A_J + \eta_J \sum_{\{m \in J \| m|=1\}} \lambda_m}{2\eta_J} - \frac{A_{I \cup J} + \eta_{I \cup J}\left(\sum_{\{k \in I \| k|=1\}} \lambda_k + \sum_{\{m \in J \| m|=1\}} \lambda_m \right)}{2\eta_{I \cup J}} \geq 0$$

$$\Leftrightarrow \frac{A_I}{\eta_I} + \frac{A_J}{\eta_J} - \frac{A_{I \cup J}}{\eta_{I \cup J}} \geq 0$$

□

Corollary 5.8. *If there exists a relationship between the parameters of the demand functions as stated in Proposition 5.7, then constraint set* (5.5) *is redundant and can be dropped from the general model MBRM.*

Proof: From Proposition 5.7 the first set of inequalities is automatically satisfied if the demand parameters are suitably chosen. Hence, these inequalities can be dropped from the model. □

Proposition 5.7 allows us to formulate the conditions that imply whether or not the *i*-th capacity constraint is binding:

$$q_i + \sum_{j \in S(i)} p_j \geq \overline{A_i} \Leftrightarrow q_i \geq \overline{A_i} - \sum_{j \in S(i)} \frac{A_j}{2} \Leftrightarrow q_i \geq \frac{\overline{A_i}}{2}, i = 1, \ldots, n \tag{5.9}$$

Under the assumption of deterministic linear demand, the MBRM model is a quadratic maximization problem with linear constraints, so the first-order KKT conditions are both necessary and sufficient. Obviously, the solution depends on whether or not the quantities on hand make the inventory constraints binding, so we will have to analyze each possible scenario in detail. For the sake of simplicity, we will analyze a stylistic scenario with n = 2 possible components.

5.4.1 Unconstrained model

If the last two inequalities are not binding on the optimal solution, then analytically the two Lagrange multipliers associated with (5.6) are 0. Looking at (5.9), this implies

$$q_1 \geq \frac{A_1 + A_{12}}{2} \tag{5.10}$$

$$q_2 \geq \frac{A_2 + A_{12}}{2} \tag{5.11}$$

Then, the optimal solution becomes simply

$$p_1^* = \frac{A_1}{2\eta_1}, p_2^* = \frac{A_2}{2\eta_2}, p_{12}^* = \frac{A_{12}}{2\eta_{12}}, z^* = \frac{1}{4}\left(\frac{A_1^2}{\eta_1} + \frac{A_2^2}{\eta_2} + \frac{A_{12}^2}{\eta_{12}}\right)$$

5.4.2 Both capacity constraints binding

If the two quantities do not satisfy simultaneously (5.10) and (5.11), then complementary slackness dictates that $\lambda_4 > 0, \lambda_5 > 0$, where λ_4, λ_5 are the Lagrange multipliers associated with (5.6). The optimal solution is found by solving the first order conditions

$$q_1 - A_1 - A_{12} + \eta_1 p_1 + \eta_{12} p_{12} = 0$$
$$q_2 - A_2 - A_{12} + \eta_2 p_2 + \eta_{12} p_{12} = 0$$
$$p_1 = \frac{A_1 + \eta_1 \lambda_4}{2\eta_1}$$
$$p_2 = \frac{A_2 + \eta_2 \lambda_5}{2\eta_2}$$
$$p_{12} = \frac{A_{12} + \eta_{12}\left(\lambda_4 + \lambda_5\right)}{2\eta_{12}}$$

which has the unique solution

$$\lambda_4^* = \frac{(\eta_2+\eta_{12})(A_1+A_{12}-2q_1)-\eta_{12}(A_2+A_{12}-2q_2)}{\eta_1\eta_2+\eta_{12}(\eta_1+\eta_2)}$$

$$\lambda_5^* = \frac{-\eta_{12}(A_1+A_{12}-2q_1)+(\eta_1+\eta_{12})(A_2+A_{12}-2q_2)}{\eta_1\eta_2+\eta_{12}(\eta_1+\eta_2)}$$

$$p_1^* = \frac{\left[2\eta_1(\eta_2+\eta_{12})+\eta_2\eta_{12}\right]A_1-\eta_1\eta_{12}A_2+\eta_1\eta_2A_{12}-2\eta_1\left[(\eta_2+\eta_{12})q_1-\eta_{12}q_2\right]}{2\eta_1\left[\eta_1\eta_2+\eta_{12}(\eta_1+\eta_2)\right]}$$

$$p_2^* = \frac{-\eta_2\eta_{12}A_1+\left[2\eta_2(\eta_1+\eta_{12})+\eta_1\eta_{12}\right]A_2+\eta_1\eta_2A_{12}-2\eta_2\left[-\eta_{12}q_1+(\eta_1+\eta_{12})q_2\right]}{2\eta_2\left[\eta_1\eta_2+\eta_{12}(\eta_1+\eta_2)\right]}$$

$$p_{12}^* = \frac{\eta_2\eta_{12}A_1+\eta_1\eta_{12}A_2+\left[2\eta_{12}(\eta_1+\eta_2)+\eta_1\eta_2\right]A_{12}-2\eta_{12}(\eta_2q_1+\eta_1q_2)}{2\eta_{12}\left[\eta_1\eta_2+\eta_{12}(\eta_1+\eta_2)\right]}$$

In order to ensure the validity of the solution, we need to further impose the condition that both multipliers are nonnegative. This amounts to imposing the following boundary conditions:

$$(\eta_2+\eta_{12})q_1-\eta_{12}q_2 \le \frac{(\eta_2+\eta_{12})(A_1+A_{12})-\eta_{12}(A_2+A_{12})}{2} \tag{5.12}$$

$$(\eta_1+\eta_{12})q_2-\eta_{12}q_1 \le \frac{(\eta_1+\eta_{12})(A_2+A_{12})-\eta_{12}(A_1+A_{12})}{2} \tag{5.13}$$

$$q_1 < \frac{A_1+A_{12}}{2}, q_2 < \frac{A_2+A_{12}}{2} \tag{5.14}$$

5.4.3 One binding constraint

If only one of the capacity constraints is binding, then we have two possible sub-cases, depending on which one it is. Supposing that $q_2 < \frac{A_2+A_{12}}{2}, q_1 \ge \frac{A_1+A_{12}}{2}$, then the first order conditions resolve to:

$$q_2 - A_2 - A_{12} + \eta_2 p_2 + \eta_{12} p_{12} = 0$$

$$p_2 = \frac{A_2 + \eta_2 \lambda_5}{2\eta_2}$$

$$p_{12} = \frac{A_{12} + \eta_{12} \lambda_5}{2\eta_{12}}$$

$$\left(\eta_2 + \eta_{12}\right)\lambda_5 = A_2 + A_{12} - 2q_2$$

with the unique solution

$$\lambda_5^* = \frac{A_2 + A_{12} - 2q_2}{\left(\eta_2 + \eta_{12}\right)}$$

$$p_1^* = \frac{A_1}{2\eta_1}$$

$$p_2^* = \frac{\left(2\eta_2 + \eta_{12}\right)A_2 + \eta_2 A_{12} - 2\eta_2 q_2}{2\eta_2\left(\eta_2 + \eta_{12}\right)}$$

$$p_{12}^* = \frac{\eta_{12} A_2 + \left(\eta_2 + 2\eta_{12}\right)A_{12} - 2\eta_{12} q_2}{2\eta_{12}\left(\eta_2 + \eta_{12}\right)}$$

This solution holds under all circumstances, because to imply a negative value of the multiplier would mean that $A_2 + A_{12} - 2q_2 < 0 \Leftrightarrow q_2 > \frac{A_2 + A_{12}}{2}$, which contradicts the case setup. On the other hand, if $q_1 < \frac{A_1 + A_{12}}{2}, q_2 \geq \frac{A_2 + A_{12}}{2}$, the first order conditions are:

$$q_1 - A_1 - A_{12} + \eta_1 p_1 + \eta_{12} p_{12} = 0$$

$$p_1 = \frac{A_1 + \eta_1 \lambda_4}{2\eta_1}$$

$$p_{12} = \frac{A_{12} + \eta_{12} \lambda_4}{2\eta_{12}}$$

$$\left(\eta_1 + \eta_{12}\right)\lambda_4 = A_1 + A_{12} - 2q_1$$

with the unique solution

$$\lambda_4^* = \frac{A_1 + A_{12} - 2q_1}{(\eta_1 + \eta_{12})}$$

$$p_1^* = \frac{(2\eta_1 + \eta_{12})A_1 + \eta_1 A_{12} - 2\eta_1 q_1}{2\eta_1(\eta_1 + \eta_{12})}$$

$$p_2^* = \frac{A_2}{2\eta_2}$$

$$p_{12}^* = \frac{\eta_{12}A_1 + (\eta_1 + 2\eta_{12})A_{12} - 2\eta_{12}q_1}{2\eta_{12}(\eta_1 + \eta_{12})}$$

Putting together the boundary conditions from (5.10)-(5.14), we can represent graphically the regions that define the various bundling strategies. Figure 19 below summarizes the possible solutions.

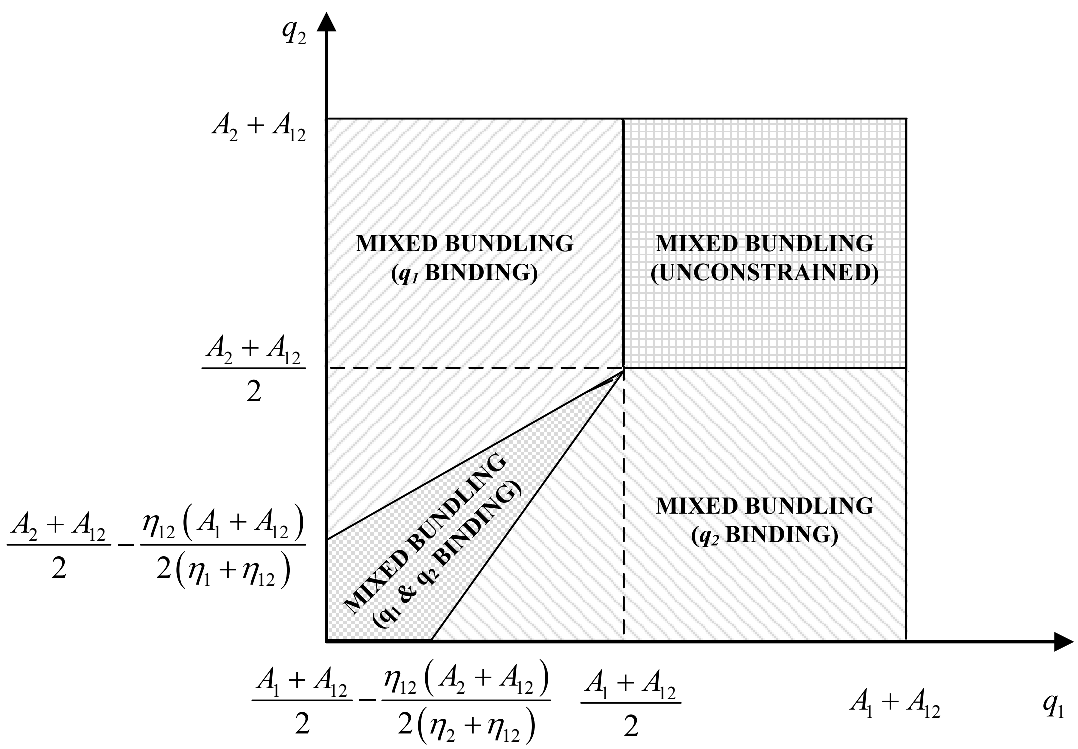

Figure 19. Mixed bundling strategies with independent linear demand functions

Notice how the variation in the two quantities affects the price levels, and accordingly the value of the optimal solution. When either of the quantities exceeds a certain threshold, then the optimal prices do not depend on it, and moreover, the optimal price of the base product based on that quantity is set independently of the other prices. What is also interesting is that the optimal prices when one (or both) of the quantities are below the given threshold, then the optimal prices of each bundle are inversely related to the quantities available of the parts that go in the bundle, and directly related to the quantities available from the products which are not included in the bundle. Finally, notice the connection between Figure 19 and Figure 2 from Chapter 3. In Chapter 3, we obtained the strategies using a self selection mechanism that did not explicitly model a linear demand function. However, the linear demand structure can be easily derived from the uniform distribution of advertisers. Using the notation from Chapter 3, for the bundle, the demand function is $q_B = 1-(p_B - p_P)/(\gamma-\beta)$, for the prime time product, it is $q_P = (p_B - p_P)/(\gamma-\beta)-(p_P - p_N)/(\beta-\alpha)$, and for the non-prime product it is $q_N = (p_P - p_N)/(\beta-\alpha) - p_N/\alpha$. In Chapter 3, the demand elasticities therefore depend on the ratings parameters, α, β, and γ, whereas in Chapter 5 they depend on the values of η_1, η_2, and η_{12}. Unlike Chapter 3, in this section the mixed bundling strategy emerges as the optimal strategy in each case, due to the independent valuation assumption. In this situation, it is never optimal to choose either pure bundling or pure components.

5.5 GREEDY HEURISTIC

Based on the observations presented so far, and on the properties of the demand function *d* (continuous, decreasing) and those of the revenue function (concave), we can derive an allocation algorithm, generalizing the previous work of Federgruen and Groenvelt (1986) and

Hochbaum (1994). In our situation, we need to account for the capacity constraints, which destroy the polymatroidal structure of the problem. We are grateful to Dr. Srinivas Bollapragada of GE Research for his ideas about this particular approach.

We will denote by $\Delta\mathbf{R} = \left(\Delta r_1, ..., \Delta r_B\right)$ the vector of marginal revenues (i.e. the changes in the objective function as a result of changing the corresponding demand by a small amount Δd_i. Let $\mathbf{Q} = \left(\overline{q}_i\right)$ be the current allocation vector, let $\mathbf{p} = \left(p_J\right)$ be the price vector, and let $g^{-1}(\cdot)$ denote the inverse of function *g*. The algorithm proceeds as follows:

Step 0: Initialization. Let $\mathbf{Q} = 0, p_i = u_i$ (set the initial quantities to zero and all prices at their upper bounds).

Step 1: Allocation. Let $i = \arg\max_{j \in B} \left\{\Delta r_j\right\}$. If $p_i \leftarrow p_i - g^{-1}(\Delta d)$ is not feasible, find the next best available p_i and decrement accordingly. Set $\overline{q_i} \leftarrow \overline{q_i} + \Delta d$ for those quantities that are affected by the price decrement.

Step 2: Stopping. If $\Delta\mathbf{R} \le 0$, stop. Report ***Q*** and ***p***. Otherwise, go to step 1.

Figure 20. Description of the greedy allocation algorithm

It is easy to see that Step 1 of the algorithm maintains primal feasibility at all iterations.

5.5.1 Worst-case heuristic performance

Since mixed bundling weakly dominates both pure components and pure bundling in this case, we can compute the optimal solutions for these situations and consider the maximum of those as our lower bound on the optimal solution. For ease of exposition throughout this subsection, assume a functional form for the demand function of type $d_J = a_J - b_J p_J, J \in B$.

For the pure components strategy, we need to solve the following problem:

$$\max_{d_1, d_2} \left\{ \pi_{PC} = p_1 d_1 + p_2 d_2 \mid d_1 \le q_1; d_2 \le q_2 \right\}$$

The optimal set of solutions is as follows:

$$d_1^* = \min\left\{q_1, \frac{a_1}{2}\right\}, d_2^* = \min\left\{q_2, \frac{a_2}{2}\right\}$$

$$\pi_{PC}^* = \begin{cases} \dfrac{q_1(a_1 - q_1)}{b_1} + \dfrac{q_2(a_2 - q_2)}{b_2}, \text{if } q_1 < \dfrac{a_1}{2} \text{ and } q_2 < \dfrac{a_2}{2} \\ \dfrac{q_1(a_1 - q_1)}{b_1} + \dfrac{a_2^2}{4b_2}, \text{if } q_1 < \dfrac{a_1}{2} \text{ and } q_2 \ge \dfrac{a_2}{2} \\ \dfrac{a_1^2}{4b_1} + \dfrac{q_2(a_2 - q_2)}{b_2}, \text{if } q_1 \ge \dfrac{a_1}{2} \text{ and } q_2 < \dfrac{a_2}{2} \\ \dfrac{a_1^2}{4b_1} + \dfrac{a_2^2}{4b_2}, \text{if } q_1 > \dfrac{a_1}{2} \text{ and } q_2 > \dfrac{a_2}{2} \end{cases}$$

Noticing that for any parameters $\frac{q_1(a_1 - q_1)}{b_1} + \frac{q_2(a_2 - q_2)}{b_2} \le \frac{q_1(a_1 - q_1)}{b_1} + \frac{a_2^2}{4b_2} \le \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2}$ and that $\frac{q_1(a_1 - q_1)}{b_1} + \frac{q_2(a_2 - q_2)}{b_2} \le \frac{q_2(a_2 - q_2)}{b_2} + \frac{a_1^2}{4b_1} \le \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2}$, we obtain that the maximum value of the objective function is $\pi_{PC}^* = \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2}$. Conversely, for the pure bundle strategy, we need to solve the following problem:

$$\max_{d_{12}} \left\{ \pi_{PB} = p_{12} d_{12} \mid d_{12} \le q_1; d_{12} \le q_2 \right\}$$

The optimal solution is as follows:

$$d_{12}^* = \min\left\{q_1, q_2, \frac{a_{12}}{2}\right\}$$

$$\pi_{PB}^* = \begin{cases} \frac{q_1\left(a_{12}-q_1\right)}{b_{12}}, \text{if } q_1 = \min\left\{q_1, q_2, \frac{a_{12}}{2}\right\} \\ \frac{q_2\left(a_{12}-q_2\right)}{b_{12}}, \text{if } q_2 = \min\left\{q_1, q_2, \frac{a_{12}}{2}\right\} \\ \frac{a_{12}^2}{4b_{12}}, \text{if } \frac{a_{12}}{2} = \min\left\{q_1, q_2, \frac{a_{12}}{2}\right\} \end{cases}$$

Once again, noticing that $\frac{q_1\left(a_{12}-q_1\right)}{b_{12}} \leq \frac{a_{12}^2}{4b_{12}}$ and $\frac{q_2\left(a_{12}-q_2\right)}{b_{12}} \leq \frac{a_{12}^2}{4b_{12}}$ we conclude that the maximum value of the objective function is $\pi_{PB}^* = \frac{a_{12}^2}{4b_{12}}$. Finally, for the mixed bundling strategy, we have to solve:

$$\max_{d_1, d_2, d_{12}} \left\{\pi_{MBRM} = p_1 d_1 + p_2 d_2 + p_{12} d_{12} \mid d_1 + d_{12} \leq q_1; d_2 + d_{12} \leq q_2\right\}$$

We can say from first principles that the more constrained the problem, the worse the objective value. Hence, the largest possible value for π_{MBRM} occurs when the problem is unconstrained, and the lowest occurs when all the constraints are binding on the optimal solution. Therefore, using the analytical results from the previous section, the optimal solution ranges between

$$\pi_{\min}^* = \frac{q_1\left[\left(b_2+b_{12}\right)\left(a_1-q_1\right)+b_2 b_{12}\left(\frac{a_{12}}{b_{12}}-\frac{a_2}{b_2}\right)\right]+q_2\left[\left(b_1+b_{12}\right)\left(a_2-q_2\right)+b_1 b_{12}\left(\frac{a_{12}}{b_{12}}-\frac{a_1}{b_1}\right)\right]+2b_{12}q_1q_2}{b_1 b_2 + b_2 b_{12} + b_1 b_{12}}$$

and $\pi_{\max}^* = \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{a_{12}^2}{4b_{12}}$. We will use $\pi_{LB} = \pi_{\min}^*$ as our lower bound.

In order to compute an upper bound, we will denote by *p'* and *d'* the values of price and demand after making a small allocation δ. We have $d' = d + \delta$, and $p' = p - \varepsilon$. It follows immediately that $\varepsilon = \delta / b$.

Let r_{12}, r_1, and r_2 respectively be the marginal changes in revenues after the allocation is made (i.e. $r_{12} = (p_{12}'d_{12}'-p_{12}d_{12}) / \delta$, etc.) Suppose, without loss of generality, that $r_{12} > r_1 > r_2$ and $r_{12} < r_1 + r_2$. Also, assume $q_1 << q_2$. By selection rule, we will allocate d to product 1 rather than the bundle, which is a suboptimal choice (it is suboptimal since allocating to the bundle is feasible, and by choice, the increase in revenue contributed by the bundle is higher than the increase in revenue contributed by product 1.) Hence, there will be a loss, which can be computed as:

$$\Delta = r_{12} - r_1 = \frac{\left(p_{12}^{'}d_{12}^{'} - p_{12}d_{12}\right)}{\delta} - \frac{\left(p_1^{'}d_1^{'} - p_1 d_1\right)}{\delta} = \frac{a_{12} - 2d_{12} - \delta}{b_{12}} - \frac{a_1 - 2d_1 - \delta}{b_1}$$

In the worst case, Δ is largest when $\delta \to 0$ and $d_1 = d_{12} = 0$ (first step of the greedy allocation algorithm). In this case, the loss is $\lim_{\delta \to 0} \Delta = \frac{a_{12}}{b_{12}} - \frac{a_1}{b_1}$. Let $r_1^{(k)}$ be the marginal revenue of product 1 at iteration k, and $p_1^{(k)}, d_1^{(k)}$ be the corresponding price and demand levels. Obviously, $p_1^{(0)} = \frac{a_1}{b_1}, d_1^{(0)} = 0$. We want to establish a recursion rule:

$$\begin{aligned}
r_1^{(1)} &= \frac{a_1 - \delta}{b_1} \\
r_1^{(2)} &= \frac{\left(p_1^{(1)} - \varepsilon\right)\left(d_1^{(1)} + \delta\right) - p_1^{(1)} d_1^{(1)}}{b_1} = \frac{\left(p_1^{(0)} - \varepsilon\right)\delta - \varepsilon\left(d_1^{(0)} + \delta\right) - \varepsilon\delta}{b_1} = \frac{a_1 - 3\delta}{b_1} = r_1^{(1)} - \frac{2\delta}{b_1} \\
r_1^{(3)} &= \frac{\left(p_1^{(2)} - \varepsilon\right)\left(d_1^{(2)} + \delta\right) - p_1^{(2)} d_1^{(2)}}{b_1} = \frac{\left(p_1^{(1)} - \varepsilon\right)\delta - \varepsilon\left(d_1^{(1)} + \delta\right) - \varepsilon\delta}{b_1} = r_1^{(2)} - \frac{2\delta}{b_1} \\
&\ldots \\
r_1^{(k)} &= r_1^{(k-1)} - \frac{2\delta}{b_1}
\end{aligned}$$

Hence, at each step, the loss is

$$
\begin{aligned}
\Delta_1 &= r_{12} - r_1^{(1)} \\
\Delta_2 &= r_{12} - r_1^{(2)} = r_{12} - \left(r_1^{(1)} - \frac{2\delta}{b_1} \right) \\
\Delta_3 &= r_{12} - r_1^{(3)} = r_{12} - \left(r_1^{(2)} - \frac{2\delta}{b_1} \right) = r_{12} - \left(r_1^{(1)} - \frac{4\delta}{b_1} \right) \\
&\dots \\
\Delta_k &= r_{12} - r_1^{(k)} = r_{12} - \left[r_1^{(1)} - \frac{2(k-1)\delta}{b_1} \right]
\end{aligned}
$$

The total loss is

$$
\Delta = \sum_{i=1}^{k} \Delta_i = k\left(r_{12} - r_1^{(1)}\right) - \frac{2\delta}{b_1} \sum_{i=1}^{k} (i-1) = k\left(\frac{a_{12}}{b_{12}} - \frac{a_1 - \delta}{b_1} \right) - \frac{2\delta}{b_1} \left[\frac{k(k-1)}{2} \right]
$$

The initial relationships $r_{12} > r_1 > r_2$ and $r_{12} < r_1 + r_2$ cannot always hold, since r_1 is decreasing at every step. To figure out what is the maximum number of iterations, we need to solve $r_{12}^{(0)} > r_1^{(k)} > r_2^{(0)}; r_{12}^{(0)} = r_1^{(k)} + r_2^{(0)}$. We have

$$
\begin{aligned}
& r_{12} - r_1^{(k)} - r_2 = 0 \Leftrightarrow \frac{a_{12}}{b_{12}} - \frac{a_1 - \delta}{b_1} + \frac{2(k-1)\delta}{b_1} - \frac{a_2}{b_2} = 0 \\
& \Rightarrow k = \frac{a_1 + \delta}{2} + \frac{b_1}{2} \left(\frac{a_2}{b_2} - \frac{a_{12}}{b_{12}} \right)
\end{aligned}
$$

It could also happen that $r_1^{(k)} = 0$, at some iteration k, that is

$$
\frac{a_1 - 2(k-1)\delta}{b_1} = 0 \Leftrightarrow k = 1 + \frac{a_1}{2\delta}
$$

Putting the two situations together, it follows that

$$
k = \max \left\{ \frac{a_1 + \delta}{2} + \frac{b_1}{2} \left(\frac{a_2}{b_2} - \frac{a_{12}}{b_{12}} \right), 1 + \frac{a_1}{2\delta} \right\}
$$

Substituting back into the total loss expression and taking the limit, we get:

$$\lim_{\delta\to 0}\Delta=\lim_{\delta\to 0}\sum_{i=1}^{k}\Delta_i=\lim_{\delta\to 0}\left[k\left(r_{12}-r_1^{(1)}\right)+\frac{\delta}{b_1}\sum_{i=1}^{k}2\left(i-1\right)\right]=k\left(\frac{a_{12}}{b_{12}}-\frac{a_1}{b_1}\right)$$
$$=\left(\frac{a_{12}}{b_{12}}-\frac{a_1}{b_1}\right)\max\left\{\frac{a_1+\delta}{2}+\frac{b_1}{2}\left(\frac{a_2}{b_2}-\frac{a_{12}}{b_{12}}\right),1+\frac{a_1}{2\delta}\right\}$$

Since we can make the same argument for product 2, then the worse that can happen is:

$$\Delta=\max\left\{\left(\frac{a_{12}}{b_{12}}-\frac{a_i}{b_i}\right)\left[\frac{a_i+\delta}{2}-\frac{b_i}{2}\left(\frac{a_{12}}{b_{12}}-\frac{a_i}{b_i}\right)\right],\left(\frac{a_{12}}{b_{12}}-\frac{a_i}{b_i}\right)\left[1+\frac{a_i}{2\delta}\right]\right\}, i=1,2$$

If we denote by π^* the optimal solution to the mixed bundling problem, and by π_H the solution given by the greedy heuristic, we have obtained that, in fact, $\pi^*-\pi_H\leq\Delta$. Putting together this information and our previously established lower bound, the performance of the greedy algorithm can be bounded by:

$$\frac{\pi^*}{\pi_H}\leq\frac{\pi_{LB}}{\pi_{LB}-\Delta}$$

Corollary 5.9. *The performance of the heuristic algorithm as a function of initial capacity is asymptotically optimal.*

Proof: Lemma 5.5 establishes that the greedy allocation is optimal if the capacity constraints are non-binding. As q_i approach infinity, the capacity constraints become non-binding, so the result in Lemma 5.5 applies. □

5.5.2 Computational results

In order to establish the practical performance of the heuristic, we have created a benchmark of test problems, varying the size of the set of basic components. Due to the size of the problem, we limited ourselves to small instances, so that we can compute the optimal solution with a non-linear solver and evaluate the optimality gap. We randomly picked the slopes and the intercepts of the linear demand functions from a uniform distribution with support [1, 10]. Similarly, we selected the initial quantities from a uniform distribution with support [0, 5t], $t = \left(\sum_{i=1}^{n} a_i / b_i \right) \Big/ n$.

For each problem size n, we benchmarked 30 different instances and aggregated the results. The summary is presented below, in Table 10.

Table 10. Greedy heuristic performance

COMPONENTS (n)	**AVG. GAP (%)**	**AVG. TIME (SECONDS)**
2	0.34%	0.063
3	0.82%	0.084
4	0.44%	0.209
5	0.90%	0.771
6	2.45%	1.984
7	2.71%	5.933
8	2.91%	17.922
9	2.96%	50.209

We notice a monotonic increase in the size of the optimality gap, as the problem size grows. This is not unexpected, as the possible ways in which the greedy selection mechanism can make a suboptimal choice increases with the number of available components. On the other hand, the gap is relatively small, with the highest around 3%, which could be very manageable for realistic pricing problems, especially since the average time needed to run the algorithm is amenable even for an online environment (like, say, pricing a set of products while the user is waiting for the response with a web browser). Unfortunately, the global solver runs out of memory after n = 9 base components, so we cannot judge the accuracy of the heuristic beyond this point, although we could test its running time. However, looking at the current data points, we would expect an exponential increase in the running time.

5.6 DECOMPOSITION FRAMEWORK

In this section we propose an efficient approach for solving the mixed bundling revenue-maximizing problem. We recognize the rapid growth in dimensionality of the generic problem, and we try to exploit the problem structure, based on the insight that not all possible bundles need to be considered explicitly. Therefore, the idea is to price bundles on the fly, and introduce them for consideration in the maximization problem only when, in fact, they are profitable. The tradeoff that we will have to judge is whether or not the computational gains that we theoretically make by not having to consider the entire product set are offset by the search for the next "best" bundle, as well as by validating that the pricing constraints are met at every iteration.

The rest of this section is organized as follows. We first present the idea behind the decomposition approach, and formulate a restricted master problem. We then augment the restricted master problem with a separation sub-problem and a pricing sub-problem. The separation sub-problem validates that, given a set of prices as inputs, either all pricing constraints are satisfied, or one pricing constraint is violated. If such a constraint is violated, it is added to the restricted master problem. The pricing sub-problem identifies a possible bundle (a new column) that can be added to the restricted master problem. Finally, we present results from a computational experiment that suggest this approach is promising for large-scale problems.

5.6.1 The restricted master problem

Recall the original formulation of the generalized deterministic MBRM problem:

[MBRM] $$\pi_{MBRM} = \max_{p_i} \sum_{i=1}^{n} r(p_i)$$

subject to:

$$\sum_{I:\cup T(I)=T(J)} p_I \geq p_J, \forall J \in B \tag{5.15}$$

$$\sum_{J \in S(k)} d_J(p_J) \leq q_k, k \in N \tag{5.16}$$

$$0 \leq p_i \leq u_i, i \in 2^N \tag{5.17}$$

The problem suffers from exponential growth in both variables and constraints; however the structure of the problem is such that there is an exponential increase in the number of constraints for each possible variable that is included in the model. Therefore, an interesting attack angle is that of decoupling the complicating variables from the problem, and adding them only on a needed basis. Initially, we would start optimizing with no bundles at all, only with the non-bundle products, so that there are no pricing constraints to observe. As we start adding potential revenue-increasing bundles, we start adding pricing constraints as needed, observing again that for a given bundle, it is possible that not all pricing constraints need to be added (as some pricing constraints might be either dominated by others).

We refer to the variant of *MBRM* without all bundles and pricing constraints explicitly included as the restricted master problem of *MBRM*, or *MBRM-RMP*, for short. Let $B' \subset B$ be the set of all products included in *MBRM-RMP*, and let *S'* be the set of all pricing constraints included in *MBRM-RMP*. The *MBRM-RMP* problem is then as follows:

[MBRM-RMP] $\pi_{MBRM-RMP} = \max_{p} \sum_{J \in B'} p_J d_J(p_J)$

subject to:

$$\sum_{\substack{I:\cup_I T(I)=T(J)\\ I:I \supset J}} p_I \geq p_J, \forall J \in S' \quad (5.18)$$

$$\sum_{J \in S(k)} d_J(p_J) \leq q_k, k \in N \quad (5.19)$$

$$0 \leq p_J \leq u_J, J \in B' \quad (5.20)$$

Initially, *S'* is initialized with the empty set, and *B'* is initialized with the subset of all base components. Using an iterative algorithm, we will connect the master problem with two sub-problems: a separation problem (*MBRM-SEP*) for identifying violated pricing constraints, and a pricing sub-problem (*MBRM-CG*) that generates new bundles that are added to the master problem. The pseudo-code for the decomposition algorithm is presented below.

Step 0: Let $B' = \{J \in B \mid |J| = 1\}$. Let $S' = \varnothing$

Step 1: Solve *MBRM-RMP*. Obtain the optimal solution vector ***p****.

Step 2: Using ***p**** as an input, solve the separation subproblem *MBRM-SEP* and see if there exists a pricing constraint for some bundle *J* that is violated. If it is, add the constraint to the master problem, let $S' = S' \cup \{J\}$ and go to step 1, otherwise go to step 3.

Step 3: Solve the column generation subproblem *MBRM-CG* and see if any new bundle prices favorably. If there exists such a bundle *J'*, then let $B' = B' \cup \{J'\}$ and go to step 1. Otherwise, stop. The vector ***p**** is an optimal solution to MBRM.

Figure 21. Description of the decomposition algorithm

The following sections describe in detail the two sub-problems that are invoked by the master problem.

5.6.2 The separation sub-problem

The separation sub-problem receives as input a vector of prices from the master problem and checks to see whether some pricing constraint is violated. If such a constraint exists, it is identified and added to the master problem. Resolving the new master problem should generate a new set of prices. At a basic level, the separation sub-problem is similar to a valid inequalities generator from a generic mixed integer programming solver. Just like the way a cut generator eliminates intermediate fractional solutions, the separation sub-problem eliminates prices that violate the pricing relationships.

Let $\bar{p}$ = $\boldsymbol{p^*}$ be the current solution to the master problem ($\bar{p}$ is a parameter). Let $w_J \in \{0,1\}$ be a binary variable indicating whether the price of product $J \in B'$ violates subadditivity. If $w_J = 1$, then a pricing constraint for bundle *J* has to be added to the master problem. Also, let $u_{JK} \in \{0,1\}$ be a binary decision variable equal to 1 if product *K* contains some components that can be used to assemble product *J*, and 0 otherwise. Finally, let the parameter $e_{iJ} = 1$ if $i \in T(J)$ and 0 otherwise.

Following a formulation originally presented in Hanson and Martin (1990), the separation problem *MBRM-SEP* is a binary integer program that can be written as follows:

[MBRM-SEP] $\pi_{MBRM-SEP} = \max \sum_{J \in B'} \left(\overline{p}_J w_J - \sum_{K \in B' \setminus J} \overline{p}_K u_{JK} \right)$

subject to:

$$e_{iJ} w_J \le \sum_{K \in B' \setminus J} e_{iK} u_{JK}, \forall J \in B', i \in N \quad (5.21)$$

$$\sum_{J \in B'} w_J \le 1 \quad (5.22)$$

$$w_J, u_{iJ} \in \{0,1\}, \forall i \in N, J \in B' \quad (5.23)$$

The idea behind MBRM-SEP is the identification of the maximally violated pricing constraint. If $\pi_{MBRM-SEP} = 0$ then all pricing constraints are satisfied for the current master problem. Otherwise, the violated pricing constraint $\sum_{\{K | u_{JK} = 1\}} p_K \ge p_J$ for which $w_J = 1$ and $u_{JK} = 1$ should be added to the master problem. Constraints (5.21) ensure that a valid set of pricing components are identified (otherwise the maximization problem can be solved trivially by setting $w_J = 1$ and $u_{JK} = 0$). Constraint (5.22) is a set covering constraint, ensuring that at most one bundle J with a violated pricing constraint is identified. Computationally, it can be experimented with dropping this constraint and adding several violated pricing constraints during the current iteration. On the other hand, the addition of several constraints at every single iteration might not be worthwhile if some of them are implied by other pricing relationships.

5.6.3 The pricing sub-problem

The pricing sub-problem is designed to add (or "price") new variables (that is, new bundles) into the master problem. Since *MBRM* is a convex optimization problem, we need to take several theoretical precautions to ensure that the column generation mechanism generates variables that ensure both convergence and optimality of the restricted master problem.

Let $\mathbf{A} \in \mathbf{R}_{+}^{|N| \times |B|}$ be the bill-of-materials for the products (the usage rates for each product), $\mathbf{p} \in \mathbf{R}_{+}^{|B|}, \mathbf{d} : \mathbf{R}_{+}^{|B|} \to \mathbf{R}_{+}^{|B|}$, and $\mathbf{q} \in \mathbf{R}_{+}^{|N|}$ be the price, demand functions, and capacity vectors. Also, let $\mathbf{C} \in \mathrm{E}^{|B| \times |B|}, \mathrm{E} = \{-1, 0, 1\}$ be a generic matrix corresponding to the coefficients of the pricing constraints (5.18). Then, problem *MBRM* can be re-written in matrix form as:

[MBRM] $\pi_{MBRM} = \max_{\mathbf{p} \geq 0} \mathbf{p}^{\mathrm{T}} \mathbf{d}$

subject to:

$$\mathbf{Cp} \leq 0 \tag{5.24}$$

$$\mathbf{Ad} \leq \mathbf{q} \tag{5.25}$$

Let $\boldsymbol{\sigma}, \boldsymbol{\omega}$ be the row vectors of Lagrange multipliers associated with constraints (5.24) and (5.25) , respectively. Consider the Wolfe dual of *MBRM*:

$$\pi_D = \min_{\mathbf{d}, \boldsymbol{\sigma}, \boldsymbol{\pi} \geq 0} \left\{ \mathbf{p}^{\mathrm{T}} \mathbf{d} + \boldsymbol{\sigma}^{\mathrm{T}} \left(\mathbf{Cp} \right) + \boldsymbol{\omega}^{\mathrm{T}} \left(\mathbf{Ad} - \mathbf{q} \right) \right\} \tag{5.26}$$

subject to:

$$\nabla_{\mathbf{d}} \left(\mathbf{p}^{\mathrm{T}} \mathbf{d} \right) + + \boldsymbol{\sigma}^{\mathrm{T}} \nabla_{\mathbf{d}} \left(\mathbf{Cp} \right) + \boldsymbol{\omega}^{\mathrm{T}} \nabla_{\mathbf{d}} \left(\mathbf{Ad} - \mathbf{q} \right) = 0 \tag{5.27}$$

Since *MBRM* is a convex problem, if the dual is also a convex problem, then (Bazaraa, Sherali, & Shetty, 2006) the duality gap is 0 (a condition that is satisfied for linear demand functions). The Lagrange multipliers will have the same interpretation as the shadow prices in a linear programming problem. In particular, $\boldsymbol{\omega}$, the Lagrange multiplier vector associated with constraint (5.25) can be interpreted as the fair price for a very small amount resource $\mathbf{q}$. Hence, the idea behind the column generation method is fundamentally a cost-benefit analysis: we will accept into the master problem a new bundle *J* if its contribution is greater than the "cost" incurred by consuming the resources that go into the bundle. Specifically, let $\mathbf{A_J}$ be the column of matrix **A** corresponding to bundle *J*, and define

$$\rho_J = \frac{\boldsymbol{\omega}^{\mathbf{T}} \mathbf{A_J} d_J}{p_J d_J} = \frac{\boldsymbol{\omega}^{\mathbf{T}} \mathbf{A_J}}{p_J} \tag{5.28}$$

to be the cost/benefit ratio associated with a bundle *J*

Then, the pricing problem involves finding the "best" such ρ_J, that is, we want to solve

$$\textbf{[MBRM-CG]} \quad J^* = \arg\min \left\{ \frac{\boldsymbol{\omega}^{\mathbf{T}} \mathbf{A_J}}{p_J} \,\middle|\, \frac{\boldsymbol{\omega}^{\mathbf{T}} \mathbf{A_J}}{p_J} \leq 1, \forall J \in B \setminus B' \right\} \tag{5.29}$$

The pricing problem *MBRM-CG* reduces thus to a sorting problem. The computational challenge is twofold: sorting a list of products that is theoretical exponential in size (all potential non-included bundles) and figuring out a priori what is the value of p_J in (5.29). The first problem can be mitigated (besides using an efficient sorting algorithm) by maintaining the sorting order and updating the list only if the Lagrange multipliers change. The problem of finding out the correct price can be solved by using as a proxy the price at which product *J* would have been offered if *J* were the only item in the product line, and ensuring that this price is higher than $\boldsymbol{\omega}^{\mathbf{T}} \mathbf{A_J}$. This approximation, coupled with the fact that the duality gap inherent in *MBRM-SEP* may not be 0 due to non-convexity of the Wolfe dual, establishes that the decomposition might not provide optimal solutions to the general *MBRM* model.

If problem *MBRM-CG* does not have a solution, then the current solution to the relaxed master problem is optimal. Indeed, if there exists a solution J^* to (5.29) such that $\rho_J^* \geq 1$, then the cost of adding bundle J^* to the solution is greater than its potential contribution, so the total revenue will decline. Since the revenue function is concave, it follows that the relaxed problem cannot be optimal.

5.6.4 Computational results

In this section we present several results that evaluate the effectiveness of the decomposition algorithm. Just like the evaluation of the heuristic introduced in section 5.5.2, our benchmark consists of randomly generated instances having the same parameters for the demand function as those presented previously. Unlike the heuristic case, we increased the size of the instances beyond the capabilities of the current non-linear solvers. The decomposition algorithm was implemented using the AMPL modeling language. We used CPLEX 8.1 with the built-in quadratic solver for the restricted master problem and the regular integer optimization solver for the separation sub-problem. Table 10 below presents the aggregate results. The "average rows" columns refers to the average number of constraints present in the master problem at the last iteration, while the "average columns" section refers to the total number of products included in the final solution. For a given size n, the results are averaged across 10 different instances.

Table 11. Aggregate results for the decomposition algorithm

COMPONENTS (n)	AVG. ROWS (CONSTRAINTS)	AVG. COLUMNS (BUNDLES)	AVG. TIME (SECONDS)
10	93	54	2.58
15	3193	1836	8.52
20	87353	57402	192.72
25	597233	354298	606.42

It is interesting to note that even for large size instances, the density of the problem (proportion of actual bundles present in the final solution as opposed to all possible bundles of size n) is really small, around 4.3%. This suggests that the decomposition approach can be an effective tool for these type of instances, since it does not need to consider explicitly every single bundle.

As expected, only a small fraction of the bundles turn out to be relevant. The same is true for the constraints, where the original exponential number of constraints is reduced to a more manageable size. On the other hand, the method has its limitation, as the algorithm runs out of memory around instances of size $n = 26$ (which is still better than the previous limit of $n = 9$ that exhausts the direct approach to solving *MBRM*).

5.7 CONCLUSIONS

In this chapter we have studied analytically and computationally the general mixed bundling problem when products have independent valuations. We have shown that the general deterministic problem is hard, due to both the exponential growth in both constraints and decision variables, as well as the non-polymatroid structure exhibited by the problem.

First, we examined a particular situation of the general mixed bundling problem, when there are only two components and the demand function is assumed linear, and found a noteworthy feature of the optimal solution. We notice a connection between the linear demand/independent valuation case and the self-selection model described in Chapter 3. Specifically, we find that in both cases the regions described by the capacity constraints are similar to one another. The similarity of the regions, even though the dominant strategy in each region is different between the settings, most likely comes from the fact that in Chapter 3 the consumers are assumed to be uniformly distributed, which implicitly induce a linear demand function of the type used in this chapter. The difference is that in Chapter 3 the pure components strategy is optimal when both resources are very scarce, and that pure bundling is optimal when both resources are plentiful whereas we do not find either the PC or the PB strategy to hold in the general linear demand case. The reason for this is twofold: on one hand, in this chapter there is no assumption about the vertical differentiation, and therefore there is no universal ranking of preferences, and on the other hand we in this case we have independent valuation, which results in both the PC and PB strategies being always dominated by some form of mixed bundling.

Next, we derive two different algorithms for solving the general case of the mixed bundling problem. The first method, a greedy heuristic, has the advantage of speed for smaller instances, but it can behave badly in terms of solution quality if the instance is characterized by relatively large scarcity (the performance becomes asymptotically optimal as the resource availability increases). The second algorithm, which is based on a decomposition framework, performs well on larger scale instances.

For the future, an interesting discussion would be to derive optimal strategies if the independent valuation assumption is removed, but the capacity constraints remain in place. In this case, the demand for the bundle would have a functional form that depends on the relationship between the demands for the components (and whether these are complements or substitutes). We would expect the boundaries of the optimal strategy to change, and it would be interesting to see if either pure components PC or pure bundling PB emerge as possible optimal strategies over the mixed bundling spectra. Additionally, we would like to modify the two algorithms to account for stochastic demands. In the decomposition case, such an extension would probably include an extra layer in the algorithm, in the form of an L-shaped approach. Similarly, the greedy heuristic would have to be modified in a fashion similar to the capacity control limits from revenue management, that is, we would expect to "protect" the more expensive product (the bundle) by pre-allocating a certain amount of capacity. It would also be interesting to relax the assumption of unitary resource consumption in the bundle, and see if there exists an efficient way to generate the additional pricing constraints that would be needed in such a model. Finally, it would be interesting to account for temporal effects and introduce a dynamic pricing formulation, to see whether the structure of the solution is influenced by the planning horizon, (e.g. if, for example, the mixed bundling strategy is optimal up to a certain time, after which some other strategy may become dominant), in addition to the scarcity of the resources.

6.0 CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

This dissertation addresses the general problem of mixed bundling in two separate environments: a vertically differentiated market where there exists a universal ordering of customer preferences, and a generic market where the products offered are independently valued. In both settings, a point of emphasis is the limited availability of the bundle components. Each of the three chapters contributes along a different research question, from characterization of optimal bundling strategies, to computational issues raised by the problem complexity. We present below the most interesting findings, alongside possible future investigative avenues.

In Chapter 3, we have examined bundling strategies when the bundle components satisfy a universal preference ordering and have limited availability, as in television advertising. The most important outcome of this work is to show that the relative availabilities of the resources strongly influence the broadcaster's optimal strategy of implementing full spectrum mixed bundling, or partial spectrum mixed bundling, or not using bundling at all. Clearly, the resource availabilities also influence their marginal value to the broadcaster; we determine how much more valuable it is to increase the availability of one resource over the other. We also investigate the relative benefits of improving the quality of prime versus non-prime time programming. The robustness of the managerial guidance provided by this analytical work is substantiated by our numerical testing.

Chapter 4 introduces competition in the form of a duopoly. We show several ways in which competition can be modeled, depending on the relative strength of each competitor, and derive the equilibrium strategies if one network dominates the other in terms of ratings. The weak firm always chooses pure or partial spectrum mixed bundling, while the strong firm chooses either partial spectrum mixed bundling or pure components. The interesting outcome here is that the strong firm can use the lower quality component as a barrier to entry or as a deterrent, if it feels threatened by the weaker firm.

In Chapter 5, we notice the connection between the linear demand/independent valuation case and the self-selection model described in Chapter 3. Specifically, we find that in both cases the graphical depiction of the strategies is similar in both cases. The similarity most likely comes from the fact that in Chapter 3 the consumers are assumed to be uniformly distributed, which implicitly induce a linear demand function of the type used in this chapter. The difference is that in Chapter 3 the pure components strategy is optimal when both resources are very scarce, and that pure bundling is optimal when both resources are plentiful whereas we do not find either the PC or the PB strategy to hold in the general linear demand case. This is due to the assumption of independent product valuations that appears in Chapter 5, which results in both the PC and PB strategies being always dominated by some form of mixed bundling. Separately, we also derive two different algorithms for solving the general case of the mixed bundling problem. The first method, a greedy heuristic, has the advantage of speed for smaller instances, but it can behave badly in terms of solution quality if the instance is characterized by relatively large scarcity (the performance becomes asymptotically optimal as the resource availability increases). The second algorithm, which is based on a decomposition framework, performs well on larger scale instances.

This rich research area has several potential interesting avenues for further research. For example, we have assumed that the resource capacities are limited and that their marginal costs are zero (or, equivalently, that the resource availabilities are limited and the resource costs are sunk). It might be worth investigating how the results change if this marginal cost assumption does not hold. It might also be useful to investigate the optimal bundling strategies in the presence of multiple resource classes (for example, in internet advertising, the number of clicks needed from the home page to reach the advertisement). In the analytical treatment in all chapters we have assumed that our model is deterministic along the advertisers' willingness to pay; introducing stochastic elements and exploring the stochastic formulation introduced at the beginning of Chapter 5 might also be worthwhile.

In Chapter 3, we have assumed concavity of the rating function. This need not always be the case. For example, in the broadcast of TV advertisings, if there are multiple decision makers who have different viewing preferences, the advertiser may derive super-additive benefits from advertising during prime time and during prime time. For example, Mattel might advertise during non-prime time to target children and during prime time to target the parent. To sell a big ticket item such as an automobile or a large kitchen appliance, both spouses (who may have different viewing habits) may need to be targeted, and so a company like Maytag may see advertisements during prime time and non-prime time as complementing each other. Due to joint decision making and complementarity of components, therefore, Mattel's or Maytag's willingness to pay for the bundle of advertisements may be greater than the sum of their willingness to pay for the components of the bundle. In this case, the price arbitrage constraint may be always binding. Moreover, assuming that there is no secondary market that allows an intermediary to buy the components and assemble the bundle for sale and that the broadcaster can impose a restriction rationing each advertiser to buy at most one product, the price arbitrage constraint (3.2) may not be economically valid.

In the competition framework, a natural extension would be to examine equilibrium strategies when both networks have relatively equal strength. It might also be of interest to study bundling-based competition when there exists a vertical as well as a horizontal dimension. From an optimization perspective, the problem of finding computationally the Bertrand-Nash equilibrium in the context of an arbitrary number of products and/or competitors is not solved so far.

In the context of Chapter 5, an interesting discussion would be to derive optimal strategies if the independent valuation assumption is removed, but the capacity constraints remain in place. In this case, the demand for the bundle would have a functional form that depends on the relationship between the demands for the components (and whether these are complements or substitutes (Venkatesh & Kamakura, 2003)). We would expect the boundaries of the optimal strategy to change, and it would be interesting to see if either pure components PC or pure bundling PB emerge as possible optimal strategies over the mixed bundling spectra. Additionally, we would like to modify the two algorithms to account for stochastic demands. In the decomposition case, such an extension would probably include an extra layer in the algorithm, in the form of an L-shaped approach. Similarly, the greedy heuristic would have to be modified in a fashion similar to the capacity control limits from revenue management, that is, we would expect to "protect" the more expensive product (the bundle) by pre-allocating a certain amount of capacity. It would also be interesting to relax the assumption of unitary resource consumption in the bundle, and see if there exists an efficient way to generate the additional pricing constraints that would be needed in such a model. Finally, it would be interesting to account for temporal effects and introduce a dynamic pricing formulation, to see whether the structure of the solution is influenced by the planning horizon, (e.g. if, for example, the mixed bundling strategy is optimal up to a certain time, after which some other strategy may become dominant), in addition to the scarcity of the resources.

After presenting all these potential research vectors, and after noting the challenges in modeling and analyzing bundling situations and its inter-disciplinary appeal, it is our belief that in all likelihood bundling will continue to be a fertile research area.

BIBLIOGRAPHY

Adams, W. J., & Yellen, J. L. (1976). Commodity Bundling and the Burden of Monopoly. *The Quarterly Journal of Economics, 90*(3), 475-498.

Altinkemer, K. (2001). Bundling E-Banking Services. *Communications of the ACM, 44*(6), 45-47.

Ansari, A., et al. (1998). The Max-Min-Min Principle of Product Differentiation. *Journal of Regional Science, 38*(2), 207-231.

Ansari, A., et al. (1996). Pricing a Bundle of Products or Services: The Case of Nonprofits. *Journal of Marketing Research, 33*(1), 86-93.

Araman, V., & Popescu, I. (2009). Media Revenue Management with Audience Uncertainty: Balancing Upfront and Spot Market Sales. *forthcoming in MSOM*.

Armstrong, M., & Vickers, J. (2009). Competitive Non-Linear Pricing and Bundling. *forthcoming in Review of Economic Studies*.

Bakos, Y., & Brynjolfsson, E. (1999). Bundling Information Goods: Pricing, Profits, and Efficiency. *Management Science, 45*(12), 1613-1630.

Bakos, Y., & Brynjolfsson, E. (2000). Bundling and Competition on the Internet. *Marketing Science, 19*(1), 63-82.

Bazaraa, M. S., et al. (2006). *Nonlinear Programming : Theory and Algorithms* (3rd ed.). New York: Wiley.

Beckmann, B. J., & Bobkowski, F. (1958). Airline Demand: An Analysis of Some Frequency Distributions. *Naval Research Logistics Quarterly, 5*, 43-51.

Belobaba, P. P. (1987). Air Travel Demand and Airline Seat Inventory Management *Ph.D. thesis, MIT*.

Belobaba, P. P. (1989). Application of a Probabilistic Decision Model to Airline Seat Inventory Control. *Operations Research, 37*(2), 183-197.

Bitran, G., & Ferrer, J.-C. (2007). On Pricing and Composition of Bundles. *Production and Operations Management, 16*(1), 93-108.

Bitran, G. R., & Mondschein, S. V. (1995). An Application of Yield Management to the Hotel Industry Considering Multiple Day Stays. *Operations Research, 43*(3), 427-443.

Bollapragada, S., et al. (2004). Scheduling Commercial Videotapes in Broadcast Television. *Operations Research, 52*(5), 679-689.

Bollapragada, S., et al. (2002). Nbc's Optimization Systems Increase Revenues and Productivity. *Interfaces, 32*(1), 47-60.

Bollapragada, S., & Mallik, S. (2008). Managing on-Air Ad Inventory in Broadcast Television. *IIE Transactions, 40*(12), 1107 - 1123.

Bollapragada, S., & Morton, T. E. (1999). Myopic Heuristics for the Random Yield Problem. *Operations Research, 47*(5), 713-722.

Boyd, A., & Bilegan, I. (2003). Revenue Management and E-Commerce. *Management Science, 49*(10), 1363-1386.

Chen, Y. (1997). Equilibrium Product Bundling. *The Journal of Business, 70*(1), 85-85.

Chen, Y., & Iyer, G. (2002). Research Note Consumer Addressability and Customized Pricing. *Marketing Science, 21*(2), 197-208.

Chen, Y., et al. (2009). A Theory of Combative Advertising. *Marketing Science, 28*(1), 1-19.

Chung, J., & Rao, V. R. (2003). A General Model for Bundles with Multiple-Category Products: Application to Market Segmentation and Optimal Pricing for Bundles. *Journal of Marketing Research, 40*(2), 115.

Cooper, W. L., et al. (2006). Models of the Spiral Down Effect in Revenue Management. *Operations Research, 54*(5), 968-987.

Cready, W. M. (1991). Premium Bundling. *Economic Inquiry, Huntington Beach, 29*(1), 173-181.

Dana Jr, J. D., & Petruzzi, N. C. (2001). Note: The Newsvendor Model with Endogenous Demand. *Management Science, 47*(11), 1488-1497.

Dansby, R. E., & Conrad, C. (1984). Commodity Bundling. *American Economic Review, 74*(2), 377-381.

Dyer, M. E., & Frieze, A. M. (1990). On an Optimization Problem with Nested Constraints. *Discrete Applied Mathematics 26*(2-3), 159-173.

Edmonds, J. (1970). *Submodular Functions, Matroids, and Certain Polyhedra.* Paper presented at the Proceedings of the Calgary International Conference on Combinatorial Structures and Their Applications, New York.

Elmaghraby, W., & Keskinocak, P. (2003). Dynamic Pricing in the Presence of Inventory Considerations: Research Overview, Current Practices, and Future Directions. *Management Science, 49*(10), 1287.

Federgruen, A., & Groenevelt, H. (1986). The Greedy Procedure for Resource Allocation Problems: Necessary and Sufficient Conditions for Optimality. *Operations Research, 34*(6), 909-918.

Fridgeirsdottir, K., & Roels, G. (2009). Dynamic Revenue Management for Online Display Advertising. *forthcoming in Journal of Revenue and Pricing Management*.

Fuerderer, R., et al. (1999). *Optimal Bundling: Marketing Strategies for Improving Economic Performance*. Berlin, Germany: Springer-Verlag.

Gal-Or, E., & Gal-Or, M. (2005). Customized Advertising Via a Common Media Distributor. *Marketing Science, 24*(2), 241-253.

Gal-Or, E., et al. (2006). Targeted Advertising Strategies on Television. *Management Science, 52*(5), 713-725.

Gallego, G., & Van Ryzin, G. J. (1994). Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons. *Management Science, 40*, 999-1020.

Gans, J. S., & King, S. P. (2006). Paying for Loyalty: Product Bundling in Oligopoly. *Journal of Industrial Economics, 54*(1), 43-62.

Geraghty, M. K., & Johnson, E. (1997). Revenue Management Saves National Car Rental. *Interfaces, 27*(1), 107-127.

Getz, M. (2006). Drowned in Advertising Chatter: The Case for Regulating Ad Time on Television. *Georgetown Law Journal, 94*(3), 1229-1264.

Guiltinan, J. P. (1987). The Price Bundling of Services: A Normative Framework. *Journal of Marketing, 51*(2), 74-85.

Hanson, W., & Martin, R. K. (1990). Optimal Bundle Pricing. *Management Science, 36*(2), 155-174.

Hitt, L. M., & Chen, P.-y. (2005). Bundling with Customer Self-Selection: A Simple Approach to Bundling Low Marginal Cost Goods. *Management Science, 51*(10), 1481-1493.

Hochbaum, D. S. (1994). Lower and Upper Bounds for the Allocation Problem and Other Nonlinear Optimization Problems. *Mathematics of Operations Research, 19*(2), 390-409.

Ibaraki, T., & Katoh, N. (1988). *Resource Allocation Problems : Algorithmic Approaches*. Cambridge, Mass.: MIT Press.

Iyer, G., et al. (2005). The Targeting of Advertising. *Marketing Science, 24*(3), 461-476.

Johnson, N. L., et al. (1994). *Continuous Univariate Distributions* (2nd ed.). New York: Wiley & Sons.

Kanetkar, V., et al. (1992). Price Sensitivity and Television Advertising Exposures: Some Empirical Findings. *Marketing Science, 11*(4), 359-371.

Kimes, S. E. (1999). Implementing Restaurant Revenue Management. *Cornell Hotel and Restaurant Administration Quarterly, 40*(3), 16-21.

Kimms, A., & Muller-Bungart, M. (2007). Revenue Management for Broadcasting Commercials: The Channel's Problem of Selecting and Scheduling the Advertisments to Be Aired. *International Journal of Revenue Management, 1*(1), 28-44.

Kopalle, P. K., et al. (1999). The Role of Market Expansion on Equilibrium Bundling Strategies. *Managerial and Decision Economics, 20*(7), 365-377.

Lan, Y., et al. (2008). Revenue Management with Limited Demand Information. *Management Science, 54*(9), 1594-1609.

Lancaster, K. (1979). *Variety, Equity, and Efficiency: Product Variety in an Industrial Society*. New York: Columbia University Press.

Lee, L. H., et al. (2007). A Heuristic to Solve a Sea Cargo Revenue Management Problem. *OR Spectrum, 29*(1), 123-136.

Lee, T. C., & Hersh, M. (1993). A Model for Dynamic Airline Seat Inventory Control with Multiple Seat Bookings. *Transportation Science, 27*(3), 252-265.

Lyle, C. (1970). A Statistical Analysis of the Variability in Aircraft Occupancy. *AGIFORS Symposium Proceedings, Terrigal, Australia, 10*.

Matutes, C., & Regibeau, P. (1992). Compatibility and Bundling of Complementary Goods in a Duopoly. *The Journal of Industrial Economics, 40*(1), 37-54.

McAfee, R. P., et al. (1989). Multiproduct Monopoly, Commodity Bundling, and Correlation of Values. *Quarterly Journal of Economics, 104*(2), 371.

McGill, J. I., & van Ryzin, G. J. (1999). Revenue Management: Research Overview and Prospects. *Transportation Science, 33*(2), 233-256.

Metters, R., et al. (2008). The "Killer Application" Of Revenue Management: Harrah's Cherokee Casino & Hotel. *Interfaces, 38*(3), 161-175.

Moorthy, K. S. (1984). Market Segmentation, Self-Selection, and Product Line Design. *Marketing Science, 3*(4), 288-307.

Moorthy, K. S. (1988). Product and Price Competition in a Duopoly. *Marketing Science, 7*(2), 141-168.

Nair, S. K., et al. (2001). An Application of Yield Management for Internet Service Providers. *Naval Research Logistics, 48*, 348-362.

Nalebuff, B. (2004). Bundling as an Entry Barrier. *Quarterly Journal of Economics, 119*(1), 159-187.

Pak, K., & Dekker, R. (2004). Cargo Revenue Management: Bid-Prices for a 0-1 Multi-Knapsack Problem. *Econometric Institute, 26*.

Petruzzi, N. C., & Dada, M. (1999). Pricing and the Newsvendor Problem: A Review with Extensions. *Operations Research, 47*(2), 183-194.

Phillips, R. (2005). *Pricing and Revenue Optimization*: Stanford University Press.

Polt, S. (1998). *Forecasting Is Difficult - Especially If It Refers to the Future*. Paper presented at the Reservations and Yield Management Study Group Annual Meeting.

Raz, G., & Porteus, E. L. (2006). A Fractiles Perspective to the Joint Price/Quantity Newsvendor Model. *Management Science, 52*(11), vi-vi.

Sa, J. (1987). *Reservations Forecasting in Airline Yield Management.* Cambridge, MA.

Salinger, M. A. (1995). A Graphical Analysis of Bundling. *Journal of Business, 68*(1), 85-98.

Savin, S. V., et al. (2005). Capacity Management in Rental Businesses with Two Customer Bases. *Operations Research, 53*(4), 617-631.

Schmalensee, R. (1982). Commodity Bundling by Single-Product Monopolies. *Journal of Law and Economics, 25*(1), 67-71.

Schmalensee, R. (1984). Gaussian Demand and Commodity Bundling. *The Journal of Business, 57*(1), S211-S230.

Shaked, A., & Sutton, J. (1982). Relaxing Price Competition through Product Differentiation. *Review of Economic Studies, 49*, 3-13.

Smith, B. C., et al. (1992). Yield Management at American Airlines. *Interfaces, 22*(1), 8-31.

Soman, D., & Gourville, J. (2001). Transaction Decoupling: How Price Bundling Affects the Decision to Consume. *Journal of Marketing Research, 38*(1), 30-44.

Stigler, G. (1963). United States V. Loew's Inc.: A Note on Block-Booking. *Supreme Court Review, 1963*, 152-157.

Stremersch, S., & Tellis, G. J. (2002). Strategic Bundles of Products and Prices: A New Synthesis for Marketing. *Journal of Marketing, 66*, 55-72.

Talluri, K. T., & Van Ryzin, G. J. (2004). *The Theory and Practice of Revenue Management*: Kluwer Academic Publishers.

Thanassoulis, J. (2007). Competitive Mixed Bundling and Consumer Surplus. *Journal of Economics & Management Strategy, 16*(2), 437-467.

Topkis, D. (1978). Minimizing a Submodular Function on a Lattice. *Operations Research, 26*, 305-321.

Vandenbosch, M. B., & Weinberg, C. B. (1995). Product and Price-Competition in a 2-Dimensional Vertical Differentiation Model. *Marketing Science, 14*(2), 224-249.

Venkatesh, R., & Kamakura, W. (2003). Optimal Bundling and Pricing under a Monopoly: Contrasting Complements and Substitutes from Independently Valued Products. *Journal of Business, 76*(2), 211-231.

Venkatesh, R., & Mahajan, V. (1993). A Probabilistic Approach to Pricing a Bundle of Products or Services. *Journal of Marketing Research, 30*(4), 494.

Weatherford, L. R. (1999). *Forecasting Issues in Revenue Management*. london, England.

Weatherford, L. R., & Bodily, S. E. (1992). A Taxonomy and Research Overview of Perishable-Asset Revenue Management: Yield Management, Overbooking, and Pricing. *Operations Research, 40*, 831-844.

Weatherford, L. R., et al. (1993). Modeling the Customer Arrival Process and Comparing Decision Rules in Perishable Asset Revenue Management Situations. *Transportation Science, 27*(3), 239-251.

Weatherford, L. R., et al. (2001). Forecasting for Hotel Revenue Management. *Cornell Hotel and Restaurant Administration Quarterly, 42*, 53-64.

Whinston, M. D. (1990). Tying, Foreclosure, and Exclusion. *American Economic Review, 80*(4), 837-859.

Wu, S.-y., et al. (2008). Customized Bundle Pricing for Information Goods: A Nonlinear Mixed-Integer Programming Approach. *Management Science, 54*(3), 608-622.

Yadav, M. S. (1994). How Buyers Evaluate Product Bundles: A Model of Anchoring and Adjustment. *Journal of Consumer Research, 21*(2), 342-353.

Yadav, M. S., & Monroe, K. B. (1993). How Buyers Perceive Savings in a Bundle Price: An Examination of a Bundle's Transaction Value. *Journal of Marketing Research, 30*(3), 350-358.

Zaporozhets, A. (1997). A Short Proof of Optimality of the Bottom up Algorithm for Discrete Resource Allocation Problems. *Operations Research Letters, 21*(2), 81-85.

Zhang, X. (2006). Mathematical Models for the Television Advertising Allocation Problem. *International Journal of Operational Research, 1*(3), 302-322.

Zhao, W., & Zheng, Y. S. (2000). Optimal Dynamic Pricing for Perishable Assets with Nonhomogeneous Demand. *Management Science, 46*, 375-388.