NILPOTENT CONJUGACY CLASSES OF REDUCTIVE P-ADIC LIE ALGEBRAS AND DEFINABILITY IN PAS'S LANGUAGE

by

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M.S., Indiana State University, 1994

Submitted to the Graduate Faculty of

the Department of Mathematics in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2006

UNIVERSITY OF PITTSBURGH

MATHEMATICS DEPARTMENT

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It was defended on

July 11th 2006

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ABSTRACT

NILPOTENT CONJUGACY CLASSES OF REDUCTIVE P-ADIC LIE ALGEBRAS AND DEFINABILITY IN PAS'S LANGUAGE

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We will study the following question: Are nilpotent conjugacy classes of reductive Lie algebras over p-adic fields definable by a formula in Pas's language. We answer in the affirmative in three cases: special orthogonal Lie algebras $\mathfrak{so}(n)$ for n odd, special linear Lie algebra $\mathfrak{sl}(3)$ and the exceptional Lie algebra $\mathfrak{g}_2$ over p-adic fields.

The nilpotent conjugacy classes in these three cases have been parameterized by Waldspurger ($\mathfrak{so}(n)$) and S. DeBacker ($\mathfrak{sl}(3)$, $\mathfrak{g}_2$). For $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$ we are required to extend Pas's language by a finite number of symbols.

ACKNOWLEDGMENTS

It gives me great pleasure to thank many people who have touched my life deeply and who have contributed to this thesis in some way.

It would be hard to overstate my gratitude to my advisor, Professor Hales, who taught me how to think clearly. His enthusiasm, teaching, insight and ability to explain things in simple words have influenced me immensely. I could not have imagined a better mentor for my Ph.D. and had it not been for his common sense, encouraging words, knowledge and cracking-of-the-whip, I would have never finished! I am also grateful to Professor Greg Constantine, Professor Jason Fulman and Professor Jeremy Avigad for their enthusiastic and insightful lectures which lead me to request them to be on my committee. Their comments and suggestions have made this thesis better and easier to read.

My parents, Vasant and Manda Mainkar, have been a constant source of strength and inspiration for me. They brought me up to treat education not just as a value but as a way of life. My debt to them is immeasurable.

Vaibhav, my husband, has shown a rare talent in diffusing the tension when I was stressed out. I thank him profusely for his patience with my early hours, spoiled weekends and tear sessions.

My sister-in-law, Veda, has gone through the same travails in Biochemistry and has shared her worst experiences when I was down and frustrated. I thank her for being a therapist when needed!

My closest friends, Dana Mihai, Jonathan Korman, Angela Reynolds and Anna Vainchtein have been a source of strength and have offered advice, coffee and food whenever I needed them the most.

I would like to thank my in-laws and my extended family for their love and for knowing when not to ask "when will you be done ?".

I take this opportunity to thank the faculty, students and staff in the department of mathematics for making the Ph.D. experience pleasant in their own way. I dedicate this thesis to my father. For without his encouragement, love (and nagging!) I would not have come this far.

In Memoriam: Aaji.

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1.0 INTRODUCTION

1.1 CONTRIBUTIONS

In this thesis I study the question of definability of nilpotent conjugacy classes in reductive p-adic Lie algebras. The idea of 'definability' is dependent on the language of logic in which the objects are being studied. For p-adic fields, Pas's language [see Section 2.2] is a good choice. For the purpose of studying nilpotent conjugacy classes and other representation theoretic objects; I have created a dictionary of formulae in this language for many p-adic objects. However, it is a first order language with no notion of sets. We are so accustomed to a higher order language that we have taken many ideas like taking quotients, letting quantifiers range over subsets of a structure for granted. I try to find mathematically correct alternative formulations of these ideas in a way that will allow us to define some of the commonly occurring objects in p-adic representation theory.

1.2 HISTORICAL BACKGROUND: MOTIVIC REPRESENTATION THEORY

In a lecture given at Orsay in 1995, M. Kontsevich introduced the concept of motivic integration. Since then it has become a tool of immense importance for mathematicians working in algebraic geometry, representation theory and harmonic analysis over p-adic fields. The theory of motivic integration has been developed and extended by Jan Denef and François Loeser [12] and presented as *arithmetic motivic integration.* Their work strengthens the belief that "All natural p-adic integrals are motivic."

A construction of Denef and Loeser [12] attaches a virtual Chow motive to every formula in Pas's language. These virtual Chow motives are thus independent of the p-adic field. We give a brief introduction to Pas's language in the next chapter.

Originally, this theory was designed for varieties defined over algebraically closed fields. The *arithmetic motivic integration* developed by Denef and Loeser [12] deals with varieties over various fields and may be viewed as geometrization of ordinary p-adic integration. The domain of integration here is restricted to *definable sets* as described below. In this context, *definability* means *field independence*. Our hope is that, if the objects appearing in this kind of integration are definable then we could use a computer and a suitable algorithm to do calculations regardless of the specific value of p.

This thesis is a small part of an effort initiated by T. C. Hales [18] to relate various objects arising in representation theory of p-adic groups to geometry. It is conjectured that many basic objects in representation theory should be motivic in nature. If the conjecture is true, it will enable us to do computations without relying on the specific value of 'p' [17, 18]. In his paper, Hales [18] achieves the goal for p-adic orbital integrals by showing that under general conditions p-adic orbital integrals of definable functions are represented by virtual Chow motives. In her thesis, J. Gordon [14] proves that character values of a class of depth-zero representations of symplectic groups ($Sp(2n)$) and special orthogonal groups ($SO(2n+1)$) over p-adic fields can be represented as virtual Chow motives. Showing that the concepts of representation theory of p-adic groups are definable is the first step towards that goal.

'Definable' means describable by a formula in a formal language $\mathcal{L}$. Formulae in language $\mathcal{L}$ are strings of symbols and variables of $\mathcal{L}$, logical connectives and quantifiers. More precisely, the class of formulae in $\mathcal{L}$ contains the following:

- all atomic formulae of the language
- for every formula ϕ, it should also contain $\neg\phi$
- if ψ is in the language and y is a variable symbol, then $\forall y\, \psi$ and $\exists y \psi$

- if ψ and ϕ are two formulas in the collection, then it also contains $\phi \wedge \psi$ and $\phi \vee \psi$

We use a *first order language* in this thesis. In first-order languages, the formulae have to be finite in length and only individual elements of the language can be quantified. For example, in the first-order language of groups, we cannot have quantifiers ranging over normal subgroups of a group G, since a normal subgroup is not an *element* of G. While definability is a restrictive condition, it supplies rich and interesting classes of sets and functions. Hence, definability is highly desirable.

We choose a language of logic due to Pas [27] as will be seen in Section 2.2. Our choice of language stems from the desire to express objects of p-adic representation theory in a field independent way. We call a mathematical object *definable* if it can be described (defined) by a formula in Pas's language. As we describe in the next chapter, this language makes no reference to the specific value of 'p' [27]. The objects found in our proofs will be formulae in this language of (somewhat new entities called) *virtual sets*. While the language is somewhat limited, it is small enough to admit a quantifier elimination process [27].

For a finite dimensional semisimple Lie algebra $\mathfrak{g}$, its adjoint group G [defined in Section 3.2] acts on it via conjugation giving us adjoint orbits. In order to understand the structure of reductive Lie groups and algebras, it is often necessary to study the conjugacy classes in the group and adjoint orbits in the Lie algebra.

Over an algebraically closed field ($\mathbb{K}$), we have the Jordan Decomposition theorem which allows us to separate adjoint orbits into two extremes: semisimple and nilpotent. Moreover, the number of nilpotent conjugacy classes is finite. The fact that nilpotent orbits are finite in number means that the study of nilpotent orbits is in some sense the discrete part of the study of adjoint orbits. In characteristic zero, these orbits have been classified by Dynkin and Kostant [8]. The work of Bala and Carter gives a unified treatment of nilpotent orbits in characteristic zero or in large characteristic, and the Bala-Carter theorem has made a significant contribution to representation theory [**?**].

Nilpotent orbital integrals are important for the representation theory and harmonic analysis of the group $G(\mathbb{F})$, where $\mathbb{F}$ is not necessarily closed. An irreducible, smooth representation of $G(\mathbb{F})$ on a complex vector space determines an invariant distribution on $\mathfrak{g}(\mathbb{F})$. In characteristic zero, Harish-Chandra's local character expansion says that this distribution has an asymptotic expansion in a neighborhood of 0 in $\mathfrak{g}(\mathbb{K})$ which coincides with a linear combination of Fourier transforms of nilpotent orbital integrals. The coefficients in this expansion are a subject of considerable interest in the representation theory of $G(\mathbb{F})$. In the study of p-adic groups, they appear prominently in the Shalika Germ expansion [32, 35]. Along with these orbits, if other components of the expansion are shown to be definable, then it would imply that Shalika germs exist independent of primes.

Over an arbitrary p-adic field ($\mathbb{F}$) the Bala-Carter parameterization of nilpotent orbits is not sufficient to give all the orbits. The orbits of classical Lie groups have been extensively studied by Waldspurger. He gives a satisfactory parameterization in those cases [39]. Furthermore, a result of Moy and Prasad establishes a strong connection between the nilpotent conjugacy classes of $G(\mathbb{F})$ and those of some reductive groups over the corresponding (finite) residue field $\mathfrak{f}$. DeBacker [9, 11] uses their result and a geometric object called an *affine building* [32] to count the number of classes. An *affine building* is a geometric structure that carries important information about $G(\mathbb{F})$. To various parts of this object are attached other reductive groups over the corresponding finite residue field. Moy and Prasad's result and DeBacker's use of affine buildings enable us to count the number of orbits over $\mathbb{F}$. This number depends on the characteristic of the residue field.

In this thesis, we show that nilpotent conjugacy classes of $\mathfrak{so}(n)$ (n odd), $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$ are indeed definable in (an extension of) Pas's language. Our treatment of $\mathfrak{so}(n)$ relies on Waldspurger's parameterization [39]. He gives combinatorial data as parameters for the nilpotent conjugacy classes of $\mathfrak{so}(n)$ for odd n but excludes the case where n is even.

Roadmap In Section 2.1 we give a brief introduction to p-adic fields. Although brief, it is sufficient to justify the choice of Pas's language. Section 2.2 is devoted to a detailed description of this language. Sections 3.1 and 3.2 cover some background material provide a context for the objects discussed in this thesis. Section 3.2 is a summary of various results on nilpotent conjugacy classes in semisimple Lie algebras over algebraically closed fields.

Section 3.5.2 contains almost all the formulae (in Pas's language) developed in course of this thesis and required for the results. In chapter 4 we prove definability of nilpotent conjugacy classes in special orthogonal Lie algebras over a p-adic field. In Chapter 5 we present 'affine apartments' which exploit the affine structure of reductive groups. We also present some well-known results which have motivated this research. Chapter 5 will hopefully be helpful in understanding Chapters 6 and 7.

With the knowledge of parahorics discussed in chapter 5 we show in chapters 6 and 7 respectively that the nilpotent conjugacy classes of $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$ are definable in an extension of Pas's language called $\mathcal{L}_{ext}$ [see Section 2.2.3].

Notation: By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we mean the integers, rational numbers and real numbers respectively. Unless stated otherwise $\mathbb{K}$ will denote an algebraically closed field. More importantly, $\mathbb{F}$ is always a p-adic field. The (finite) residue field of $\mathbb{F}$ is denoted by $\mathfrak{f}$. Finally, $\mathfrak{g}$ is a finite dimensional Lie algebra (most of the time semisimple) and G is its adjoint group.

2.0 PRELIMINARIES

2.1 *P*-ADIC FIELDS

The underlying field of all the objects appearing in this thesis is a p-adic field. Such fields were introduced by number theorists to facilitate calculations involving congruences modulo p^n. Just as the reals are obtained by completing rational numbers with respect to an Archimedean metric, the p-adic numbers are obtained by completing the rationals with respect to a *non-Archimedean* metric. More precisely:

Definition 1. Let $\mathbb{Q}$ denote the field of rational numbers and p a prime integer. Then the **p-adic norm** $|\ |_p$ is defined as follows:
Given $x \in \mathbb{Q}^{\times}$, $\exists$ unique $r, m, n \in \mathbb{Z}$ such that $(m, n) = 1$ and $p \not| m,\ p \not| n$ and $x = p^r \frac{m}{n}$. Then $|x|_p = p^{-r}$. Set $|0|_p = 0$.

Definition 2. The completion of $\mathbb{Q}$ with respect to the p-adic norm $|\ |_p$ is denoted by $\mathbb{Q}_p$. $\mathbb{Q}_p$ is called a p-adic field.

Note. Any finite extension of $\mathbb{Q}_p$ is also called a p-adic field.

Example 3. In $\mathbb{Q}_5$ we have the 5-adic norm. Consider the following expansions
$37 = 2 \times 5^0 \ + \ +2 \times 5^1 \ + \ 1 \times 5^2$
$1945 = 0 \times 5^0 + 4 \times 5^1 + 2 \times 5^2 + 0 \times 5^3 + 3 \times 5^4$

Observe in the examples that the coefficients in the power series expansion were 0, 1, 2, 3 or 4. As seen in the next paragraph, this is not a coincidence. By virtue of the norm, we get a valuation on p-adic numbers. The *valuation* v of a p-adic number is the exponent r appearing in Definition 1. We say $v(x) = r$. Furthermore, the norm (or the valuation) allows us to define the following structures:

1. $\mathfrak{o} = \{a \in \mathbb{Q}_p \,|\, |a|_p \leq 1\}$ is called the ring of integers or the *valuation ring*.
2. $\mathfrak{p} = \{a \in \mathbb{Q}_p \,|\, |a|_p < 1\}$ is a maximal ideal in $\mathfrak{o}$ called the *valuation ideal*.
3. The residue class field of $\mathbb{Q}_p$ is the field $\mathfrak{o}/\mathfrak{p} = \mathbb{F}_p$. The residue field $\mathfrak{f}$ of any p-adic field is a finite extension of $\mathbb{F}_p$.
4. The *valuation group* is the set of all exponents of p appearing in definition 1.

$\mathbb{Q}_p$ is a non-archimedian, topologically non-discrete, totally disconnected, locally compact field with a finite residue class field. These fields are also called *local fields* of characteristic zero. They have a rich structure. Of particular interest is the fact that we can define

$$\mathfrak{p}^n = \{a \in \mathbb{Q}_p \,|\, |a|_p \leq p^{-n}\}$$

and use it to get an exhaustive filtration of $\mathbb{Q}_p$ by compact open subgroups

$$\{0\} \subsetneq \ldots \subsetneq \mathfrak{p}^3 \subsetneq \mathfrak{p}^2 \subsetneq \mathfrak{p} \subsetneq \mathfrak{p}^0 = \mathfrak{o} \subsetneq \mathfrak{p}^{-1} \subsetneq \ldots \subsetneq \mathbb{Q}_p.$$

When we talk about p-adic fields, we have 3 structures in mind: the p-adic numbers, the finite residue field and the valuation group. This dictates our choice of language, in the sense of logic.
We now define *valuation* v' on an extension of $\mathbb{Q}_p$.

Definition 4. Let $\mathbb{F}$ be a p-adic field, i.e an extension of $\mathbb{Q}_p$. Let $a' \in \mathbb{F}$. Then the normalized discrete valuation of a', called v' is given by

$$v'(a') = \tfrac{1}{f} v(N_{\mathbb{F}/\mathbb{Q}_p}(a')).$$

Here f denotes the degree of extension of the residue field of $\mathbb{F}$ over $\mathbb{F}_p$ and 'N' denotes the norm of the extension $\mathbb{F}/\mathbb{Q}_p$.

Definition 5. Let $\mathbb{F}$ be a p-adic field. Then, ϖ is called a *uniformiser* of the valuation on $\mathbb{F}$ if the valuation of ϖ is 1.

2.2 PAS'S LANGUAGE

Since we desire a field-independent description, we find it convenient to use Pas's language which allows us to exploit the structure of a p-adic field without referring to its individual features such as uniformiser of the valuation [27].

Pas's language is a *first order language* with three sorts of variables; variables for the elements of the valued field ($\mathbb{F}$), variables for the elements of the residue field ($\mathfrak{f}$) and variables for elements of the value group (Γ). It contains symbols for standard field operations in the valued field and in the residue field (i.e. addition and multiplication) along with symbols for the usual operation (only addition) in the value group (Γ). In addition, both field sorts have a symbol for equality ($=$). The value sort has symbols $\leq$, $\geq$ and $\equiv_n$ for congruence modulo each non-zero $n \in \mathbb{N}$. With $\mathbb{Z}$ as a structure for Γ, these symbols have the usual meaning.

Let $\mathbb{L}_{\mathbb{F}}$ be the language of fields for the field sort ($\mathbb{F}$-sort) and $\mathbb{L}_{\mathfrak{f}}$ the language of fields for the residue field sort ($\mathfrak{f}$-sort). For the value group sort, let $\mathbb{L}_{\Gamma}$ be the language of ordered Abelian groups with an element ∞ on top given by

$$\mathbb{L}_{\Gamma\infty} = \{+,\, 0,\, 1,\, \infty,\, \leq\}$$

Then the following is Pas's language $\mathcal{L}$:

$$\mathcal{L} \,=\, (\mathbb{L}_{\mathbb{F}},\, \mathbb{L}_{\mathfrak{f}},\, \mathbb{L}_{\Gamma},\, val,\, \overline{ac})$$

Note. With $\mathbb{Z}$ as a structure for the value group, $\mathbb{Q}_p$ is a structure for the language $\mathcal{L}$. [See Section 2.1]

Moreover, in the valued field sort, there are symbols 0 and 1 respectively for the additive and multiplicative identities. Using these, we formally add symbols denoting other integers to this language.

Example 6. Let $P(t)$ be a formula in Pas's language, with t as a free variable. $P(-1)$ is the abbreviation for

$$\exists x\ P(x) \wedge (x + 1 = 0)$$

The language contains symbols for existential ($\exists$) and universal ($\forall$) quantifiers for each sort. In particular, we have six symbols;

$$\forall_{\mathbb{F}} \quad \forall_{\mathfrak{f}} \quad \forall_{\Gamma} \quad \exists_{\mathbb{F}} \quad \exists_{\mathfrak{f}} \quad \exists_{\Gamma}$$

Whether the quantifiers range over the field sort, the residue field sort or the value group sort will generally be clear from the context. If there is a possibility of confusion, we will attach the respective sort symbol to the quantifier as shown above. Once the *sort* of variable symbols used is clear, we will use them in a way that indicates that meaning.

Pas's language also has standard symbols for logical disjunction ($\vee$), conjunction ($\wedge$)and negation ($\neg$). In addition, we use the following standard logical abbreviations for implication ($\Rightarrow$), bi-conditional ($\Leftrightarrow$), and exclusive or ($\veebar$) respectively:

- $\phi \Rightarrow \psi$ for $\neg\phi \vee \psi$
- $\phi \Leftrightarrow \psi$ for $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$
- $\phi \veebar \psi$ for $\neg(\phi \Leftrightarrow \psi)$

Pas's language includes a function symbol '*val*' for the valuation map from the valued field to the value group and another function symbol for an angular component '$\overline{\text{ac}}$' from the valued field to the residue field. We will explain the role of these symbols in the next section after we introduce structures for this language.

2.2.1 Pas's Structures

We make a distinction between the variable symbols used and their interpretation. Here we discuss structures (in the model theoretic sense) for Pas's language $\mathcal{L}$ [27]. We will state explicitly the conditions on these structures.

2.2.1.1 Conditions on Pas's Structures

Definition 7. An *SPL* is a structure **R** for Pas's language that consists of the following:

1. A structure for the field sort $(\mathbb{F}, +_{\mathbb{F}}, -_{\mathbb{F}}, \cdot_{\mathbb{F}}, 0_{\mathbb{F}}, 1_{\mathbb{F}})$, where $\mathbb{F}$ is the domain for the field sort. $\mathbb{F}$ is assumed to be a valued field of characteristic 0.
2. A structure for the residue field sort $(\mathfrak{f}, +_{\mathfrak{f}}, -_{\mathfrak{f}}, \cdot_{\mathfrak{f}}, 0_{\mathfrak{f}}, 1_{\mathfrak{f}})$. $\mathfrak{f}$ is assumed to be a finite field.
3. For the value group sort: $(\mathbb{Z}, +, 0, 1, \infty, \leq)$
4. *val*, valuation function on $\mathbb{F}$. [See Section 2.2.1.2]

5. An angular component map $\overline{ac}$ on $\mathbb{F}$. [See section 2.2.1.2]

An example of an SPL is a p-adic field.

Remark 8. We mention in passing that in his paper [27] Pas places an additional condition that $\mathbb{F}$ be Henselian. It is required for the quantifier elimination proved in that paper. This condition is not used in this paper.

Let $\mathbf{R}$ be the domain of the structure. A structure with domain $\mathbf{R}$ attaches a set $A(\mathbf{R})$ to every virtual set A [defined in Section 2.2.2] and an interpretation $\theta^{\mathbf{R}}$ to every formula θ.

Since the three sorts of this language are fields, finite fields and Abelian groups respectively; the language may be equipped with field and group axioms. Thus we have the theories of fields and Abelian groups at our disposal. In the following sections we prove some theorems where we will need to make use of the theory of fields. We use the notation (even though R is a structure and not a model)

$$R \models \phi$$

to indicate that a formula, ϕ, in Pas's language holds in SPL R.

2.2.1.2 Function Symbols: $\overline{ac}$ and val

Here we explain the role played by the function symbols $\overline{ac}$ and val. Let $\mathbb{F}$ be a valued field with valuation

$$\mathrm{val} : \mathbb{F} \to \mathbb{Z} \cup \{\infty\}$$

We write

$$\mathfrak{o} = \{x \in \mathbb{F} \,|\, \mathrm{val}(x) \geq 0\} \text{ and } \mathfrak{p} = \{x \in \mathbb{F} \,|\, \mathrm{val(x)} > 0\}$$

for the valuation ring and valuation (maximal) ideal respectively. Denote the residue field $\mathfrak{o}/\mathfrak{p}$ by $\bar{\mathbb{F}}$. The set of units of $\mathfrak{o}$ is denoted by $\mathfrak{u}$, i.e.

$$\mathfrak{u} = \{x \in \mathfrak{o} \,|\mathrm{val(x)} = 0\}$$

The valuation map allows passage from the valued field sort to the valuation group sort.

Definition 9. An *angular component* map modulo $\mathfrak{p}$ on $\mathbb{F}$ is a map

$$\overline{ac} : \mathbb{F} \;\rightarrow\; \mathfrak{f} \qquad x \;\mapsto\; \overline{ac}(x)$$

such that

1. $\overline{ac}(0) \;=\; 0$
2. the restriction of $\overline{ac}$ to $\mathbb{F}^*$ is a multiplicative morphism from $\mathbb{F}^*$ to $\mathfrak{f}^*$
3. the restriction of $\overline{ac}$ to $\mathfrak{u}$ coincides with the canonical projection from $\mathfrak{o}$ to $\mathfrak{f}$.

The angular component map allows passage from the valued field sort to the residue field sort. To illustrate how the functions *val* and $\overline{ac}$ work, consider the following example:

Example 10. Let $\mathbb{F}$ be the field $\mathbb{Q}_5$. Every non-zero element in $\mathbb{Q}_5$ can be written in the form

$$\textstyle\sum_{i=N}^{\infty} a_i 5^i$$

where N is an integer, $a_i \,\in\, \{0,\, 1,\, 2,\, 3,\, 4\}$ and $a_N \,\neq\, 0$.

Then $\mathrm{val}(\sum_{i=N}^{\infty} a_i 5^i) \,=\, N$ and $\overline{ac}(\sum_{i=N}^{\infty} a_i 5^i) \,=\, a_N$. So, from Example 6 we have;

$$\mathrm{val}(37) = 0 \;\text{ and }\; \overline{ac}(37) = 2$$

This language is highly restrictive with no notion of sets. More specifically, the set membership predicate $\in$ is absent. We introduce *virtual sets* into the language by means of various logical formulae. The notion of virtual sets is similar to what Quine [28] refers to as '*virtual classes*'.[1]

[1] Quine states,"...classes are freed of any deceptive hint of tangibility, there is no reason to distinguish them from *properties*. It matters little whether we read $x \in y$ as 'x is a member of the class y' or 'x has the property y'."[28, pg. 120, 2nd paragraph]

2.2.2 Virtual Sets

A *virtual set* is a construct of the form

$$\{\, x \;:\; \phi(x) \,\}$$

where ϕ is a formula in Pas's language with free variables x_1, x_2, $\ldots$, x_n and x is a multi-variable symbol

$$x = (x_1, x_2, \ldots, x_n)$$

In this case, we say that the variable symbol x has *length* n.
We write

$$y \in \{\, x \,:\, \phi(x) \,\} \;\text{ for }\; \phi(y) \tag{2.1}$$

Thus a serviceable '∈' of (ostensible) class membership can be introduced as a purely notational adjunct [29]. The whole combination $y \in \{x : \phi(x)\}$ reduces to $\phi(y)$ so there remains no trace of the existence of a class $\{x : \phi(x)\}$. We could rephrase $y \in \{x : \phi(x)\}$ by $(\exists\, x)\,\big((x = y) \wedge \phi(x)\big)$ but we prefer to view $\in$ and class abstraction as fragments of the entire combination of (1).

When we write $x \in V$, we mean $V(x)$. (This is an extension of the notation $\phi(x)$.) It is also to be understood that the length of x is the same as the number of free variables used in the formula defining V.

The 'virtual set theory' shares some aspects of set theory. We note that the usual set operations union, intersection and a notion of subset are present. If A and B are virtual sets defined by formulae $\phi(x)$ and $\psi(x)$ respectively, then:

• $A \cup B$ is a virtual set defined by $\{x : \phi(x) \vee \psi(x)\}$

• $A \cap B$ is a virtual set defined by $\{x : \phi(x) \wedge \psi(x)\}$

• We say that A is a subset of B and denote it by $A \subset B$, where $A \subset B$ is an abbreviation of the formula $\forall\, x(\phi(x) \Rightarrow \psi(x))$. Since x is a multi-variable symbol, $\forall\, x$ is a quantified n-tuple.

• $x \notin A$ means $\neg\, \phi(x)$.

Note. Although a set can be a member of another set, a *virtual set* cannot be a member of another *virtual set*. Thus $A \in B$ is not permissible.

Here are two examples of virtual sets:

- The ring of integers $\mathfrak{o}$ of any valued field is a virtual set defined thus:

$$\{x : \mathrm{val}(x) \geq 0\}$$

- The maximal ideal $\mathfrak{p}$ in o is a virtual set defined thus:

$$\{x : \mathrm{val}(x) > 0\}$$

We conclude this section with one more definition.

Definition 11. Let $\Psi(x, y)$ be a formula with free variables $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$. We define a *virtual set with parameters y* by

$$u \in \{x : \Psi(x, y)\} \quad \text{for} \quad \Psi(u, y)$$

where $u = (u_1, u_2, \ldots, u_n)$. For an example, see Formula 51 in Section 3.5.2.

One should note that the quantifiers are *not* allowed to range over virtual sets. Hence, there is no such expression as $\forall V$ where V is a virtual set.

Remark 12. In Section 3.5.2 we prove some facts in linear algebra using virtual sets defined by formulae in Pas's language. Many of the proofs are classical and at times, instead of giving the entire proof, we say,“ ... the rest of the proof is classical.” However, caution must be exercised in making such statements. It may not always be possible to ‘lift’ proofs from classical mathematics and fit them into the Pas's language. Virtual set theory is more restrictive than set theory. Concepts and objects of set theory may not always have virtual set analogues.

Remark 13. In the most recent version of motivic integration, Cluckers and Loeser avoid some of the aforementioned difficulties by using a category theoretic construct called *definable subassignments* [7]. Their setting admits a good dimension theory and makes a general integration version possible.

2.2.3 Extension of Pas's Language

In chapter 6 we will see that the parameterization of nilpotent orbits is closely linked with the number of cubic residues in the residue field $\mathfrak{f}$. This number is clearly dependent on p, the characteristic of $\mathfrak{f}$, in particular on its congruence class modulo 3. We get around this difficulty by extending Pas's language to include a finite number of variable symbols of the residue field sort. More precisely, we consider

$$\mathcal{L}_{\text{ext}} = \mathcal{L} \cup \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

where λ_i are constants of the residue field sort.

Let $\mathbf{R}$ be a structure for $\mathcal{L}_{\text{ext}}$, then the constants are to be interpreted so that they satisfy the following conditions:

1. $\nexists_{\mathfrak{f}}\, \gamma$, $\gamma \neq 0$ such that $\lambda_i = \gamma^3 \lambda_j$. This ensures that the λ_i represent distinct cubic classes.
2. $\forall\, \delta\ \exists_{\mathfrak{f}}\, \gamma$ such that $(\delta = \gamma^3\lambda_1) \vee (\delta = \gamma^3\lambda_2) \vee \ldots \vee (\delta = \gamma^3\lambda_n)$. This ensures that we consider *all* cubic classes.

Thus we have two different extensions $\mathcal{L}_{\text{ext}}$, where n is depends on whether $q \equiv 1 (\operatorname{mod} 3)$ or $q \equiv 2 (\operatorname{mod} 3)$. A structure for $\mathcal{L}$ is a p-adic field with $q \equiv 2 (\operatorname{mod} 3)$. A structure for $\mathcal{L}_{\text{ext}} = \mathcal{L} \cup \{\lambda_1, \lambda_2, \lambda_3\}$ is a p-adic field with $q \equiv 1 (\operatorname{mod} 3)$, where q is the cardinality of the residue field.

3.0 LIE ALGEBRAS AND NILPOTENT CONJUGACY CLASSES

This chapter is a brief review of the theory of Lie Algebras. Section 3.2 provides a discussion and some results about parameterization of nilpotent conjugacy classes.

3.1 LIE ALGEBRAS

The purpose of this section is to merely provide basic definitions and theorems that will be used in the subsequent chapters. For a detailed treatment of Lie algebras, see [22]. Let $\mathbb{K}$ be a field.

Definition 14. A Lie Algebra is a vector space L over $\mathbb{K}$, together with an operation $B : L \times L \to L$ that satisfies the following axioms:

1. B is bilinear.
2. $B(x, x) = 0$ for all $x \in L$
3. $B(x, (y, z)) + B(y, (z, x)) + B(z, (x, y)) = 0$

The binary operation 'B' is called a bracket operation.

Remark 15. In section 3.5.2 we write a formula for a finite dimensional Lie algebra in Pas's language.

We will assume that L is finite dimensional throughout this thesis.

Example 16. The Euclidean space $(\mathbb{R}^3, \times)$ with the standard cross product of vectors as a bracket operation is a Lie Algebra over $\mathbb{R}$.

Our interest is only in matrix algebras and we give three examples of Lie algebras that are of importance for this thesis.

Example 17. The set of all $n \times n$ matrices with entries in field $\mathbb{K}$, denoted by $\mathfrak{gl}(n, \mathbb{K})$ is a Lie algebra where the bracket operation is given by the commutator, i.e., $B(X, Y) = [X, Y] = XY - YX$ where juxtaposition denotes the usual matrix multiplication. This Lie algebra is called the general linear algebra.

Example 18. The set of all $n \times n$ matrices with entries in field $\mathbb{K}$ with trace zero is a Lie algebra with respect to the same bracket operation as above. This Lie algebra is denoted by $\mathfrak{sl}(n, \mathbb{K})$ and is called the special linear algebra.

Example 19. Let J be a non-zero symmetric matrix. Then the set

$$\{X \in \mathfrak{gl}(n, \mathbb{K}) \mid JX + {}^{t}XJ = 0\}$$

is a Lie algebra. It is denoted by $\mathfrak{so}(n, \mathbb{K})$ and is called the *special orthogonal* algebra.

A subset of a Lie algebra that is also a Lie algebra is called a *subalgebra*. Of particular interest are the following subalgebras.

1. The *center* $Z(L) = \{X \in L \,|\, B(X, Y) = B(Y, X)\}$ (assume that char($\mathbb{K} \neq 2$).
2. The *centralizer* of an element X in L is denoted by L^X and is given by
$$\{Y \in L \,|\, B(X, Y) = 0\}.$$
3. The *maximal Abelian subalgebra* H is a maximal subalgebra satisfying the condition
$$\forall X, Y \in \; H \;\; B(X, Y) = 0.$$
4. An *ideal* I of L is a subspace of L satisfying the property
$$X \in I, Y \in L \Rightarrow B(X, Y) \in I.$$

Ideals arise naturally as kernels of homomorphisms of Lie algebras and they help us to analyze the structure of Lie algebras. Their role is summarized in the following definitions.

1. A Lie algebra is called *simple* if its only ideals are 0 and itself.

2. The *derived series* for L is given by

$$L^{(0)} = L \;\; L^{(1)} = B(L, L) \;\; \ldots \;\; L^{(i)} = B(L^{(i-1)}, L^{(i-1)})$$

   L is *solvable* if $L^{(n)} = 0$ for some n.

3. L is *semisimple* if 0 is its only maximal solvable ideal. This is one of the most important classes of Lie algebras. All the Lie algebras discussed in this thesis are semisimple. Note, however, that $\mathfrak{gl}(n, \mathbb{K})$ is not semisimple.

A *representation* of a Lie algebra is a vector space homomorphism (linear transformation) from $L \rightarrow \mathrm{End}(V)$ that preserves the bracket operation, where V is a finite dimensional vector space. A representation of immense importance is:

Definition 20. The *adjoint* representation of a Lie algebra L is a map $ad : L \longrightarrow \mathrm{End}(L)$ taking $X \mapsto \mathrm{ad_X}$ given by $\mathrm{ad_X(Y) = [X, Y]}$.

Let V be a vector space over $\mathbb{K}$ and X an element of End(V). Recall, from basic linear algebra that we call X *semisimple* if every X-invariant subspace has an X-invariant complement. We say that X is *nilpotent* if $\exists n$ such that $X^n = X \circ X \circ \ldots \circ X = 0$. The *adjoint representation* allows us to lift this terminology to Lie algebras.

Definition 21. We say that $X \in L$ is semisimple if ad_X is semisimple as an endomorphism. We say that $X \in L$ is nilpotent if ad_X is nilpotent as an endomorphism.

The reason for considering these two extreme cases is that over an algebraically closed field, we have the following theorem.

Theorem 22 (Jordan Decomposition). *: Let X be an endomorphism of a finite dimensional vector space V over $\mathbb{K}$. There exist unique operators $X_s, X_n \in End(V)$ satisfying the conditions*

$$X = X_s + X_n, \qquad X_s \textit{ semisimple}, \; X_n \textit{ nilpotent}, \qquad X_s, X_n \textit{ commute}$$

From now on we will assume that $\mathbb{K}$ is algebraically closed.
We close this section with one more definition.

Definition 23. Let L be a Lie algebra. We say that H is a *Cartan subalgebra* if it is a maximal Abelian subalgebra consisting of only semisimple elements.

For example, in $\mathfrak{sl}(3)$, a Cartan subalgebra is the set of all matrices of the form

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -(a_1+a_2) \end{pmatrix}.$$

3.2 NILPOTENT CONJUGACY CLASSES

The material in this section is drawn heavily from [8] and [20]. The concept of conjugacy classes is fundamental to the study of groups. Recall that in finite groups, the class equation states that a group G is the disjoint union of its center and conjugacy classes that consist of more than one element. For any group, class functions are constant on conjugacy classes. There is an analogous notion of 'conjugacy classes' in Lie algebras. By conjugation we mean the adjoint action of the adjoint group of a Lie algebra. Elements of this adjoint group act on the algebra as automorphisms. Recall that the Lie algebra of $GL(n,\mathbb{R})$ is $\mathfrak{gl}(n,\mathbb{K})$ and it acts on $\mathfrak{gl}(n,\mathbb{K})$ by conjugation, giving the familiar similarity classes of matrices. Now we will define the 'adjoint group of a Lie algebra'.

Definition 24. Let $\mathfrak{g}$ be a semisimple Lie algebra. We define the following two groups

1. The automorphism group is given by:
   $\mathrm{Aut}(\mathfrak{g}) = \{\phi \in GL(\mathfrak{g}) \mid B(\phi(x),\phi(y)) = \phi(B(x,y)), \forall x,y \in \mathfrak{g}\}$. $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group.
2. $G_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g})^\circ$ where $\mathrm{Aut}(\mathfrak{g})^\circ$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$. G_{ad} is called the *adjoint group.*

Definition 25. Let $\mathfrak{g}$ be a Lie algebra and G its adjoint group. Let X be any element of $\mathfrak{g}$. Then, the conjugacy class of X is denoted by $\mathcal{O}_X$ and is given by

$$\{Y \in \mathfrak{g} \mid \exists h \in G \ \ (\mathrm{ad}(h)Y = X)\}$$

Note. When we say *orbit* we mean *adjoint orbit*. In the context of Lie algebras, we use the words *orbit* and *conjugacy class* interchangeably.

Finally, an adjoint orbit $\mathcal{O}_X$ is called *nilpotent* (resp. *semisimple*), if X is nilpotent (resp. semisimple). That the orbits should fall naturally into these two categories can be seen by an example.

Example 26. Suppose X is in $\mathfrak{sl}(2)$ and is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, over $\mathbb{K}$, its characteristic polynomial is $(t - r_1)(t - r_2)$. Corresponding to the cases $r_1 = r_2$ or $r_1 \neq r_2$, we conclude that any X is similar to either

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

for some a. In the former case, X is nilpotent and in the latter, it is semisimple. Clearly, there are infinitely many semisimple classes.

Thus the semisimple orbits in $\mathfrak{sl}(2)$ can be parameterized by the set $\mathbb{K}/(a \sim -a)$. Recall that the Weyl group $\mathcal{W} = \{1, s_\alpha\}$ acts on a Cartan subalgebra $\mathfrak{h}$ by reflecting the origin. Thus we may identify $\mathbb{K}/(a \sim -a)$ with $\mathfrak{h}/\mathcal{W}$. This result extends to any semisimple algebra.

Theorem 27 (Classification). *: Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathcal{W}$ the associated Weyl group. Then the set of semisimple orbits is in bijective correspondence with $\mathfrak{h}/\mathcal{W}$. In particular, there are infinitely many semisimple classes. [8, pgs. 25-26]*

Although the set of all semisimple orbits is infinite, it has a finite subset of a certain semisimple orbit called *distinguished*. The so-called weighted Dynkin diagrams parameterize them. We will define them later for nilpotent elements. The work of Jacobson-Morozov and Dynkin-Kostant proves that there is a bijective correspondence between the set of nilpotent orbits and the set of distinguished semisimple orbits in $\mathfrak{g}$. In particular, the nilpotent orbits in $\mathfrak{g}$ are finite. Richardson [31] proves the result generally.

Theorem 28 (Richardson's Finiteness Theorem). *: Let H be a closed subgroup of $G = GL(n, \mathbb{K})$, and $\mathfrak{h}$ its Lie algebra. Let $\mathfrak{h}$ satisfy the following condition:*

(*) There exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$, stable under Ad(H), for which $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
Then, any G-orbit in $\mathfrak{g}$ intersects $\mathfrak{h}$ in only finitely many H-orbits. In particular, $\mathfrak{h}$ has only finitely many nilpotent orbits.

Note that condition (*) is satisfied by all reductive (hence semisimple) Lie algebras. The following theorem in positive characteristic is due to Lusztig [25]

Theorem 29. *If $\mathfrak{g}$ is simple and char $\mathbb{K}$ is positive, then $\mathfrak{g}$ has only finitely many nilpotent orbits.*

In classical cases, the weighted Dynkin diagrams can be replaced by partitions. Now we quote some parameterization results with examples.

Theorem 30. *In $\mathfrak{sl}(n)$, there is a one-to-one correspondence between the set of nilpotent orbits and the set of partitions of n. The correspondence sends a nilpotent element X to the partition determined by the block sizes in its Jordan normal form. [8]*

Here is the explicit correspondence:
Let $[d_1, d_2, \ldots, d_k]$ be a partition of n satisfying the conditions $d_1 + d_2 + \ldots + d_k = n$ and $d_1 \leq d_2 \leq \ldots \leq d_k > 0$.

Definition 31. Let J_i be the i by i matrix given by $\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$. J_i is called a Jordan block of type i.

Then

$$X_{[d_1, d_2, \ldots, d_k]} = J_{d_1} \oplus J_{d_2} \oplus \ldots \oplus J_{d_k}$$

is a nilpotent element of $\mathfrak{sl}(n, \mathbb{K})$. Henceforth, we will refer to $X_{[d_1, d_2, \ldots, d_k]}$ as the nilpotent element associated with the partition $[d_1, d_2, \ldots, d_k]$. Two distinct partitions give disjoint nilpotent classes.

Example 32. In $\mathfrak{sl}(3)$, $n = 3$. The partitions of 3 are: [3], [1 2], [1 1 1]. The corresponding nilpotent elements (representatives) are given by,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively.

Theorem 33. *Nilpotent orbits in $\mathfrak{so}(n)$, n odd are in one-to-one correspondence with the set of partitions of n in which even parts occur with even multiplicity [8].*

Example 34. In $\mathfrak{so}(5)$, $n = 5$. The partitions that give nilpotent orbits are

$$[1\ 1\ 1\ 1\ 1],\ [1\ 1\ 3],\ [1\ 2\ 2],\ [5].$$

As will be seen in Section 3.5.2 and the subsequent chapters, our treatment of many objects appearing in this thesis is 'linear algebraic'. Orbits (nilpotent or semisimple) have a rich topological and algebro-geometric structure but they are not vector spaces. In chapter 5 we will need to refer to the *dimension* of a nilpotent orbit. This is the dimension of an orbit as a variety. We do not have a corresponding notion in Pas's language. We overcome this difficulty by using the following result:

Lemma 35. *Let $\mathcal{O}_X$ be the orbit containing the element X. Then $\mathfrak{g}^X$ is a subalgebra of $\mathfrak{g}$ and $dim(\mathcal{O}_X) = dim(\mathfrak{g}) - dim(\mathfrak{g}^X)$*

3.3 THE EXCEPTIONAL LIE ALGEBRA $\mathfrak{g}_2$

In Section 3.1 we introduced Lie algebras as matrix algebras. However, they have also been classified according to their rank using abstract root systems. We give a brief review of the classification. We give definitions of concepts that will be invoked in later chapters and leave out those that are not directly relevant to our work. Consider E, a finite dimensional vector space over $\mathbb{R}$. Any non-zero vector $\alpha \in E$ determines a reflection σ_α with reflecting plane

$$P_\alpha = \{\beta \in E \mid \langle \beta, \alpha \rangle = 0\}$$

where $\langle , \rangle$ denotes a non-degenerate inner product on E. Then,

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

Definition 36. A subset Φ of E is called a (reduced) root system in E if the following axioms hold

1. Φ is finite, spans E and 0 is not in Φ.
2. $\alpha \in \Phi \Rightarrow$ the only multiples of α in Φ are $\alpha, -\alpha$.
3. $\alpha \in \Phi \Rightarrow \sigma_\alpha(\Phi) = \Phi$
4. $\alpha, \beta \in \Phi \Rightarrow \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$

Definition 37. The *rank* of Φ is given by the dimension of E.

The *classification theorem* states that corresponding to the rank ($= l$) we have four families of Lie algebras called *classical Lie algebras* and 5 others called *exceptional Lie algebras*. The subscript in the labels for exceptional Lie algebras denotes the rank of the algebra.

- $A_l \, (l \geq 1)$ with dimension $l(l+2)$
- $B_l \, (l \geq 2)$ with dimension $l(2l+1)$
- $C_l \, (l \geq 3)$with dimension $l(2l+1)$
- $D_l \, (l \geq 4)$with dimension $l(2l-1)$
- E_6 with dimension 78
- E_7 with dimension 133
- E_8 with dimension 248
- F_4 with dimension 52
- G_2 with dimension 14

Example 38. The Lie algebra $\mathfrak{sl}(3)$ has rank 2 and it corresponds to the root system given by $\Phi = \{\alpha, -\alpha, \beta, -\beta, \, \alpha+\beta, -(\alpha+\beta)\}$.

Example 39. The Lie algebra $\mathfrak{g}_2$ has rank 2. In chapter 7, we will realize $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(7)$. It is the Lie algebra corresponding to the root system given by

$\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha+\beta, -(\alpha+\beta), 2\alpha+\beta, -(2\alpha+\beta), 3\alpha+\beta, -(3\alpha+\beta), 3\alpha+2\beta, -(3\alpha+2\beta)\}$.

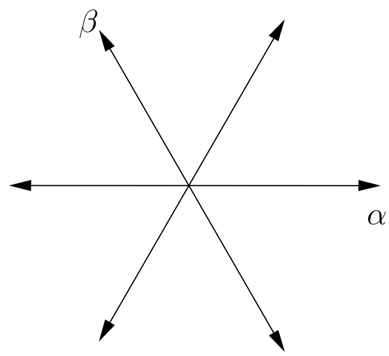

Figure 3.1: Root system for $\mathfrak{sl}(3)$

This correspondence can be made precise in terms of a Cartan subalgebra. Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{h}$ be its Cartan subalgebra. Since $\mathfrak{h}$ is Abelian, $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ gives a commuting family of semisimple endomorphisms of $\mathfrak{g}$. Thus $\mathfrak{h}$ has a root space decomposition $\mathfrak{h} = c_{\mathfrak{g}}(\mathfrak{h}) \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ where:

- for $\alpha \in \mathfrak{h}^*$, the dual of $\mathfrak{h}$, we define: $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, B(h,x) = \alpha(h)x \;\forall\, h \in \mathfrak{h}\}$
- $\Phi = \{\alpha \in \mathfrak{h}^* \,|\, \alpha \neq 0 \wedge \mathfrak{g}_\alpha \neq 0\}$.

The root system Φ determines $\mathfrak{g}$ up to isomorphism. Every Φ has an associated semisimple Lie algebra.

For future reference, we introduce, the notion of a *co-root*.

Definition 40. Let $\alpha \in \Phi$. The corresponding co-root $\alpha^\vee \in E^*$ is given by

$$\langle \lambda, \alpha^\vee \rangle = \tfrac{2\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle}$$

3.4 NILPOTENT CONJUGACY CLASSES OVER P-ADIC FIELDS

Here we consider nilpotent conjugacy classes of Lie algebras over p-adic fields. Let $\mathbb{F}$ be a p-adic field. Recall example 26. Over $\mathbb{F}$ more conjugacy classes are to be expected. It turns out that the characteristic of the residue field plays a major role in determining the the total

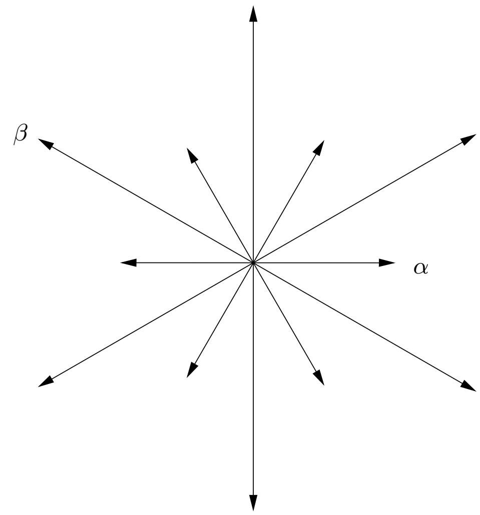

Figure 3.2: Root system for $\mathfrak{g}_2$

number of conjugacy classes. Indeed, in $\mathfrak{sl}(3)$ they are found to be related to the number of cubic classes in the residue field. In $\mathfrak{g}_2$ the number of orbits is linked with the isomorphism classes of separable cubic algebras over $\mathbb{F}$.

3.5 A LIST OF FORMULAE IN PAS'S LANGUAGE

3.5.1 Introduction

We wish to speak about linear algebra in this language, so we will start with vectors. By a *vector* x we mean an n-tuple $(x_1, x_2, \ldots, x_n)$ where the x_i are variable symbols of either the valued field sort or the residue field sort. Hence, when we say

$$\forall x \ \ (x \in V)$$

we really mean

$$\forall x_1, \forall x_2, \ \ldots, \ \forall x_n \ \ \big((x_1, x_2, \ldots, x_n) \in V\big)$$

Example 41. If x is an n-tuple $(x_1, x_2, \ldots, x_n)$ of variables of the field sort and y is an n-tuple $(y_1, y_2, \ldots, y_n)$ of variables of the field sort as well, then $x + y$, too, is an n-tuple $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Example 42. If the context is an $n \times n$ matrix, we will use variable symbols x_{ij}, y_{ij}, and so forth rather than labeling the n^2 entries in a sequence $x_1, x_2, \ldots, x_{n^2}$

Example 43. If X is an $n \times n$ matrix (x_{ij}) of variable symbols of the valued field sort; then $\exists_{\mathbb{F}} X$ is an abbreviation of $\exists_{\mathbb{F}} x_{11}, \exists_{\mathbb{F}} x_{12}, \ldots, \exists_{\mathbb{F}} x_{nn}$

And finally, we define an operation on matrices:

Definition 44. If A is an $n \times n$ matrix (a_{ij}) of variable symbols of the valued field sort and B is an $m \times m$ matrix (b_{ij}) of variable symbols of the valued field sort then $A \bigoplus B$ is an $(n+m) \times (n+m)$ matrix $(a)_{ij}$ where

$$(a \oplus b)_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i, j \leq n; \\ b_{i-n, j-n} & \text{if } n+1 \leq i, j \leq n+m; \\ 0 & \text{otherwise.} \end{cases}$$

The following section gives a long list of formulae. While the list seems tedious, it contains formulae for all the objects needed in the proof of our main result. We hope that this will allow us to present a short and clean proof.

3.5.2 Formulae

1. If V is a non-empty virtual set, let $\mathrm{Lin}(V)$ be the formula:
   $\underline{0} \in V \wedge \forall \lambda_1, \forall \lambda_2 \, \forall x_1, \forall x_2 \, (x_1, x_2 \in V \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in V)$
   Here λ_1 and λ_2 are variable-symbols of the valued field sort (or residue field sort) and x_1 and x_2 are vectors of variable-symbols of the valued field sort (or residue field sort). We use $\mathrm{Lin}(V)$ to define a *virtual vector space* over the valued field (or the residue field, respectively).

   Definition 45. Let T_R be the theory consisting of sentences that are true for all SPL R. If $T_R \models \mathrm{Lin}(V)$, then we say that V is a virtual vector space.

(In the first order language of rings, our structure would be a ring. In that case, Lin(V) would assert that V is a module.)

2. Let Lin-ind($e_1, \ldots, e_n, V$) be the formula:

   $\forall\, \lambda_1, \lambda_2, \ldots, \lambda_n \; ((\sum_{i=1}^{n} \lambda_i e_i \;=\; 0) \;\Rightarrow\; (\lambda_1 \;=\; \ldots \;=\; \lambda_n \;=\; 0) \wedge V(e_1) \;\wedge\; V(e_2) \;\wedge\; \ldots \;\wedge\; V(e_n)$

   This formula asserts the linear independence of vectors $e_1, e_2, \ldots, e_n$ in V where V is a virtual set with M free variables and e_i are vectors of length M each consisting of terms.

3. Let Lin-comb($e_1, e_2, \ldots e_m, u$) be the formula:

   $\exists\, \lambda_1, \ldots, \lambda_m \; (u \;=\; \sum_{i=1}^{m} \lambda_i e_i)$

   This formula states that u is a linear combination of $e_1, e_2, \ldots, e_m$.

4. Let Span($e_1, e_2, \ldots, e_m, V$) be the formula:

   $\forall\, v \; \big(V(v) \;\Leftrightarrow\; \text{Lin-comb}(e_1, e_2, \ldots e_m, v)\big)$

   This states that V is the span of vectors $(e_1, e_2, \ldots, e_m)$.

5. Let Basis($e_1, e_2, \ldots, e_m, V$) be the formula:

   Lin-ind($e_1, e_2, \ldots, e_m, V$) $\wedge$ Span($e_1, e_2, \ldots, e_m, V$)

   This formula states that $(e_1, e_2, \ldots, e_m)$ is a basis for V.

6. For m, a fixed natural number; let Dim(m, V) be the formula:

   $\exists\, e_1,\, e_2,\, \ldots,\, e_m\;$ Basis($e_1, e_2, \ldots, e_m, V$)

   We wish to point out that, here, m is not a variable in Pas's language.

7. At times we will need to say that a vector space has odd (resp. even) dimension. We will be dealing with only finite dimensional vector spaces so, *a priori*, there will be an upper bound n on the dimension.

   - Let Odd-Dim(n, V) be the formula:

   $$\text{Dim}(1, V) \;\vee\; \text{Dim}(3, V) \;\vee\; \ldots \;\vee\; \text{Dim}(2k-1, V) \;\text{ where } n-1 \;\leq 2k-1 \;\leq\; n$$

   The formula asserts that the virtual set V is a vector space of odd dimension that is less than or equal to n.

   - Let Even-Dim(n, V) be the formula:

   $$\text{Dim}(0, V) \;\vee\; \text{Dim}(2, V) \;\vee\; \ldots \;\vee\; \text{Dim}(2k, V) \;\text{ where } n-1 \;\leq 2k \;\leq\; n$$

8. Let Int-comb$(e_1, e_2, \ldots e_m, u)$ be the formula:

   $\exists\, \lambda_1 \ldots \lambda_m\ (\mathrm{val}(\lambda_\mathrm{i}) \geq 0) \wedge u = \sum_{i=1}^{m} \lambda_i e_i$

   This formula asserts that u is an integral combination of vectors $(e_1, e_2, \ldots, e_m)$.

9. Let Int-basis$(e_1, \ldots, e_n, L)$ be the formula:

   $$\text{Lin-ind}(e_1, \ldots, e_n) \wedge (\forall w \in L \ \text{Int-comb}(e_1, \ldots, e_n, w))$$

10. Let $V = U \oplus W$ be the formula:

    $(W \subset V) \wedge (U \subset V) \wedge (U \cap W = \{\underline{0}\}) \wedge \big(\forall\, v \in V\, (\exists w \in W, u \in U \ v = u + w)\big)$

    The formula states that V is the direct sum of U and W. (The lack of conditions on U, W and V is intentional. This decomposition allows us to talk about direct sums of lattices, vector spaces or modules.)

11. Let Q-space$(U, V/W)$ be the formula:

    $$\mathrm{Lin}(V) \wedge \mathrm{Lin}(W) \wedge \mathrm{Lin}(U) \wedge V = U \oplus W$$

    Observe that the quotient of a vector space by a subspace is identified with its complement in the decomposition.

    Remark 46. Henceforth, objects defined on quotient spaces will be identified with objects on complements.

12. Let Q-Basis$(e_1, \ldots, e_n, U, V/W)$ be the formula:

    $$\text{Q-space}(U, V/W) \wedge \text{Basis}(e_1, e_2, \ldots, e_n, U)$$

    This says that the vectors $e_1, \ldots, e_n$ form a basis for the quotient space $V/W = U$.

13. A <u>lattice</u> in a linear space V is an integral-span of a basis of V. Let Lattice(L, V) be the formula:

    $\mathrm{Lin}(V) \wedge (L \subset V) \wedge$

    $\exists\, e_1, \ldots, e_n \big(\text{Basis}(e_1, \ldots, e_n, V) \wedge \forall w\, (w \in L \iff \text{Int-comb}(e_1, \ldots, e_n, w))\big)$

    This asserts that the virtual set L is a lattice in V.

14. Let lattice$(e_1, e_2, \ldots, e_m)$ be the virtual set:

$$\{u \,|\, \exists_{\mathbb{F}} \alpha_1, \ldots, \alpha_m \;\; \mathrm{val}(\alpha_i) \geq 0 \, (u = \sum_{i=1}^{m} \alpha_i e_i)\}$$

Remark 47. What is the difference between this formula and the earlier one? In the previous formula we assert that L, a 'known' virtual set, is a lattice; whereas, in this formula we construct a lattice. It seems as though we are splitting hair here. We are not! This allows us to use (say) L as an abbreviation for a virtual set. The formula

$$L = \mathrm{lattice}(e_1, e_2, \ldots, e_m)$$

will thus mean "Label this particular virtual set as L."

15. Similarly, let vectorspace$(e_1, e_2, \ldots, e_m)$ be the virtual set:

$$\{u \,|\, \exists_{\mathbb{F}} \alpha_1, \ldots, \alpha_m \, (u = \sum_{i=1}^{m} \alpha_i e_i)\}$$

16. Let L, $\tilde{L}$ and V be virtual sets. Let J be an M by M matrix of terms, where M is the number of free variables in V.

The formula Dual-lattice$(L, \tilde{L}, J, V)$ is given by:

$$\mathrm{Lattice}(L, V) \;\wedge\; (\tilde{L} \subset V) \;\wedge$$

$$\forall\, w \in V \left(w \in \tilde{L} \iff \left(\forall\, v \; (v \in L \Rightarrow \; \mathrm{val}({}^t v J w) \geq 0) \right) \right)$$

This asserts that $\tilde{L}$ is the dual of lattice L with respect to matrix J.

17. Let sym-bil-nd(J, V, n) denote the formula

$$\exists\, e_1, \ldots, e_n \left(\mathrm{Lin}(V) \;\wedge\; \mathrm{Basis}(e_1, \ldots, e_n, V) \wedge \det(A) \neq 0 \;\wedge\; (A_{ij} = A_{ji}) \right)$$

where $A_{ij} = {}^t e_i J e_j$.

Here, e_i are vectors of variable symbols of length M and J is an M by M matrix of terms, where M is the number of free variables in V.

Lemma 48. *Under these definitions, a 'dual-lattice' is a lattice. More precisely, if R is an SPL, then:*

$R \models \text{sym-bil-nd}(J, V) \Longrightarrow$

$\big(\text{Dual-lattice}(L, \tilde{L}, J, V) \;\; \Rightarrow \;\; \text{Lattice}(\tilde{L}, V))\big)$

J is an $M \times M$ matrix of terms, V is a virtual set with M free variables, L is a virtual set with M free variables, and $\tilde{L}$ is a virtual set with M free variables.

Proof. Let R be an SPL. Then,

$R \models \text{sym-bil-nd}(J, V) \Longrightarrow$

$$\exists\, e_1,\, \ldots,\, e_n \quad \text{Lin}(V) \,\wedge\, \det(A) \neq 0 \quad \wedge \quad A_{ij} \,=\, A_{ji} \quad 1 \,\leq\, i \,\leq\, n,\;\; 1 \,\leq\, j \,\leq\, n$$

where $A_{ij} =^t e_i J e_j$.

The proof is constructive in the sense that using the basis $\{e_1,\, \ldots,\, e_n\}$ for lattice L, we will produce a basis $\{e'_1,\, \ldots,\, e'_n\}$ such that $\tilde{L}$ is a lattice with respect to this basis. In other words, we will show that

$R \;\models\; \exists\, e'_1,\, \ldots,\, e'_n$ such that

$\text{Basis}(e'_1,\, \ldots,\, e'_n,\, V) \,\wedge\, (\forall\, w(w \,\in\, \tilde{L} \Leftrightarrow \text{Int-comb}(e'_1,\, \ldots,\, e'_n, w))$

Refer to formula (13).

Define e'_i as follows:

$$e'_i \;=\; \sum_{j=1}^{M} \alpha_{ij} e_j \;\; \text{such that} \;\; {}^t e'_i J e_j \;=\; \delta_{ij} \tag{3.1}$$

We need to show that

$\text{Basis}(e'_1, e'_2, \ldots, e'_n, V) \;\wedge\; \Big(\forall\, w \,\in\, V\, \big(w \in \tilde{L} \;\Leftrightarrow\; \text{Int-comb}(e'_1,\, \ldots,\, e'_n, w)\big)\Big)$

To say that these e'_i's exist and are unique is equivalent to saying that the α_{ij}'s exist and are unique.

For each i, equation (3.1) gives a system of n linear equations in n variables. Since $A_{ij} \,=\, {}^t e_i J e_j$ is a square non-degenerate matrix (i.e $\det(A_{ij}) \,\neq\, 0$); the α_{ij}'s exist and are unique. Thus the e'_i's are uniquely defined and form a basis of V.

The rest of the proof is classical.

∎

18. A lattice L is said to be *Almost Self Dual* if the following hold:

$$\mathfrak{p}\tilde{L} \subset L \subset \tilde{L}$$

While $A \subset B$ is a formula in Pas's language, a comment is needed on the meaning of $\mathfrak{p}\tilde{L}$. It is the following virtual set:

$$\{v \in V : \exists \alpha \in \mathfrak{p}, \exists w \in \tilde{L}(v = \alpha w)\}$$

Let $\mathrm{ASD}(L, J, V)$ be the following formula:

$\mathrm{Lin}(V) \wedge \mathrm{Lattice}(L, V) \wedge \text{Dual-lattice}(\tilde{L}, L, V, J) \wedge (L \subset \tilde{L}) \wedge (p\tilde{L} \subset L)$

19. We need a formula for lattices of quotient spaces.

Recall that we will identify quotients of vector spaces with orthogonal complements. Let $\text{Q-Lattice}(L, U, V/W)$ denote the formula:

$$\text{Q-space}(U, V/W) \wedge \mathrm{Lattice}(L, U)$$

20. Let $\mathrm{Gram}_{ij}(e_1, e_2, \ldots, e_m, J)$ be the entry ${}^te_iJe_j$. Here J is an M by M matrix of terms.
21. Let $\text{Gram-det}(e_1, e_2, \ldots, e_m, J)$ be the determinant of matrix $({}^te_iJe_j)$.
22. Let $\Theta(\mathrm{sq}, J, V)$ be the formula:

$\forall e_1, \ldots, e_n \Big(\mathrm{Basis}(e_1, \ldots, e_n, V) \Rightarrow$
$(\exists \xi \in \mathfrak{f}\ \xi \neq 0 \wedge \xi^2 = \overline{\mathrm{ac}}(\text{Gram-det}(e_1, \ldots, e_n, J)))\Big)$

This states that the Gram-determinant of the quadratic form on V, given by matrix J is a square class in the residue field.
23. Let $\Theta(\mathrm{nsq}, J, V)$ be the formula:

$\forall e_1, \ldots, e_n \Big(\mathrm{Basis}(e_1, \ldots, e_n, V) \Rightarrow$
$(\nexists \xi \in \mathfrak{f}\ \xi \neq 0 \wedge \xi^2 = \overline{\mathrm{ac}}(\text{Gram-det}(e_1, \ldots, e_n, J)))\Big)$

This states that the Gram-determinant of the quadratic form on V, given by matrix J is a non-square class in the residue field.

24. Let Q-dim$(L_1,\ L_2,\ V,k)$ be the formula:

$(L_1 \subset L_2)\ \wedge\ \text{Lin}(V)\ \wedge\ \text{Lattice}(L_1,V)\ \wedge\ \text{Lattice}(L_2,V) \wedge$

$\exists e_1,\ldots,e_n \in V$

$\big(\exists\,\alpha\ (\text{val}(\alpha)=1)\ \wedge\ (\text{Int-basis}(e_1,\ldots,e_n,L_2))\ \wedge\ \text{Int-basis}(\alpha e_1,\ldots,\alpha e_k,e_{k+1},\ldots,e_l,L_1)\big)$

This formula asserts that the dimension of the vector space L_2/L_1 (over the residue field) is k.

25. As in 7, we will write formulae stating that the dimension of the aforesaid quotient is odd (resp. even).

• Let Odd-Qdim$(n,L_1,\ L_2,\ V)$ be the formula:

Q-dim$(L_1,\ L_2,\ V,1)\ \vee\ $ Q-dim$(L_1,\ L_2,\ V,3)\ \vee$

$\ldots\ \vee$ Q-dim$(L_1,\ L_2,\ V,2k-1)$ where $n-1\ \le\ 2k-1\ \le\ n$

• Let Even-Qdim$(n,L_1,\ L_2,\ V)$

Q-dim$(L_1,\ L_2,\ V,0)\ \vee\ $ Q-dim$(L_1,\ L_2,\ V,2)\ \vee$

$\ldots\ \vee$ Q-dim$(L_1,\ L_2,\ V,2k)$ where $n-1\ \le\ 2k\ \le\ n$

26. Let Anisotropic$(e_1,e_2,\ \ldots,\ e_m,J,V)$ be the formula:

Lin$(V)\ \wedge$ Lin-ind$(e_1,e_2,\ \ldots,\ e_m,V)\ \wedge$

$\forall\,\lambda_1,\ \ldots,\ \lambda_m\ ({}^t(\sum_{i=1}^m \lambda_i e_i)J(\sum_{i=1}^m \lambda_i e_i)\ =\ 0) \Rightarrow (\lambda_1\ =\ \ldots\ =\ \lambda_m\ =\ 0)$

Recall that in Lin-ind$(e_1,e_2,\ \ldots,\ e_m,V)$ the e_i are vectors of terms, V is a virtual set with M free variables and J is an M by M matrix of terms.

This formula states that if V is a vector space and if J is the matrix of a quadratic form on V then the linearly independent vectors $\{e_1,\ \ldots,\ e_m\}$ span a subspace of the anisotropic kernel of V.

27. Let Dim-aniso(m,J,V) be the formula:

$\exists\,e_1,\ e_2,\ \ldots,\ e_m$ Anisotropic$(e_1,e_2,\ldots,\ e_m,J,V)\ \wedge$

$\nexists\,e_1,\ e_2,\ \ldots,\ e_{m+1}$ Anisotropic$(e_1,e_2,\ldots,\ e_{m+1},J,V)$.

This asserts that m is the dimension of the anisotropic kernel of V.

28. Let Iso-aniso(V,J_V,W,J_W) be the formula:

$\exists e_1,\ \ldots,e_m,e'_1,\ \ldots,e'_m$

$\big($ Anisotropic$(e_1,\ \ldots,\ e_m,J_V,V)\ \wedge$ Anisotropic$(e'_1,\ \ldots,\ e'_m,J_W,W)\ \wedge$

$\left({}^{t}e_i J_V e_i \;=\; {}^{t}e'_i J_W e'_i \;\forall\, i\; 1 \;\leq\; i \;\leq\; m\right) \wedge$

Dim-aniso$(m, J_V, V) \wedge$ Dim-aniso(m, J_W, W)

This formula asserts that the vector spaces have isomorphic anisotropic kernels under their respective quadratic forms.

29. Now we would like to be able to talk about direct sums of vector spaces formed by annexing two arbitrary vector spaces. Let e be a vector of terms of length n. Let f be a vector of terms of length m. We construct a vector of terms of length $n + m$ by concatenating e with f. We denote this by $e \oplus f$. Thus, if e is given by $(e_1, e_2, \ldots, e_n)$ and f by $(f_1, f_2, \ldots, f_m)$, then $e \oplus f$ is given by $(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_m)$, where the free variables e_i are distinct from the free variables f_j.

Moreover, if e and h have length n and f and k have length m; then define

$$(e \oplus f) + (h \oplus k) := (e + h) \oplus (f + k)$$

Let Dir-sum(V, W, U) denote the formula:

Lin(V) $\wedge$ Lin(W) $\wedge$

$\Big(\forall\, f\; (f \in U) \iff \big(\exists\, f_v \in V \;\; \exists\, f_w \in W\; (f = f_v \oplus f_w)\big)\Big)$

Lemma 49. *The direct sum of two vector spaces is a vector space. More precisely, let R be an SPL. Then,*

$R \models$ Dir-sum$(V, W, U) \Rightarrow$ Lin(U)

Proof. Now the symbol $\lambda(e \bigoplus f)$ will denote a vector of terms of length $n + m$ where the first n terms are that of the vector λe (scalar multiplication by the field constant λ) and the remaining n terms are those of the vector λf.

$R \models$

$\forall\, f\, \forall\, e\;\; \big(\, f \in U \;\wedge\; e \in U \wedge$ Dir-sum$(V, W, U)\,\big)$

$\Rightarrow \big((\exists\, f_v\, \exists\, f_w\;\; f_v \in V,\; f_w \in W\; (f = f_v \oplus f_w)) \;\wedge$

$\quad (\exists\, e_v\, \exists\, e_w\;\; e_v \in V,\; e_w \in W\; (e = e_v \oplus e_w))\big)$

$\Rightarrow \forall\, \lambda_1\, \forall\, \lambda_2\; \Big(\lambda_1 f \;+\; \lambda_2 e = \lambda_1(f_v \oplus f_w) \;+\; \lambda_2(e_v \oplus e_w)$

$$= \lambda_1 f_v \bigoplus \lambda_1 f_w + \lambda_2 e_v \oplus \lambda_2 e_w$$
$$= (\lambda_1 f_v + \lambda_2 e_v) \oplus (\lambda_1 f_w + \lambda_2 e_w)\Big)$$

$\mathrm{Lin}(V) \Rightarrow \lambda_1 f_v + \lambda_2 e_v \in V$

$\mathrm{Lin}(W) \Rightarrow \lambda_1 f_w + \lambda_2 e_w \in W$

$\Rightarrow \lambda_1 f + \lambda_2 e \in U \Rightarrow \mathrm{Lin}(U)$ $\square$

30. Now we will define a *Virtual function* as a relation. Let V and W be non empty virtual sets. Denote by $f(V,W)$ the virtual set satisfying the following 3 conditions:
   a. $f(V,W) \subset \{(v,w) \,|\, v \in V,\ w \in W\}$
   b. $\big((v_1,w_1) \in f(V,W) \wedge (v_1,w_2) \in f(V,W)\big) \Rightarrow w_1 = w_2$
   c. $\forall v \in V\ \ \exists w \in W((v,w) \in f(V,W))$

   Then we say that f is a virtual function from V to W and that V is the domain of f. Furthermore, the virtual set $f(V)$ given by
   $$\{w \in W \,|\, \exists\, v((v,w) \in f(V,W))\}$$
   is the range of f. Naturally, we say, $f(v) = w$ if $((v,w) \in f(V,W))$.
31. Linear Transformation: We say that α is a linear transformation from V to W if the following hold:
   a. α is a virtual function from V to W.
   b. $\mathrm{Lin}(V) \wedge \mathrm{Lin}(W)$
   c. $\forall v_1, v_2 \in V\ \alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$
   d. $\forall\, a\ \forall v \in V\ \alpha(av) = a\alpha(v)$
32. Linear Functional: We say that α is a linear functional from V to W if α is a linear transformation and $\mathrm{Dim}(1,W)$ holds.
33. Virtual Binary Operation: Let V be a non empty virtual set. We say that the virtual set $B(V \times V, V)$ defines a binary operation on V if $B(V \times V, V)$ is a virtual function. Furthermore, if $((v_1,v_2),v_3) \in B(V \times V, V)$, then we write $v_1 B v_2 = v_3$.
34. Algebra: Let V be a virtual set. Let $\mathrm{Alg}(V,B)$ denote the formula

   $\mathrm{Lin}(V) \wedge B(V \times V, V)\ \wedge\ \big(\forall u,v,w\ \in\ V(\exists\, 1_V \in V\ 1_V B u = u B 1_V = u)\ \wedge$

   $(\forall \lambda_1, \lambda_2\ (uB(\lambda_1 w + \lambda_2 v) = \lambda_1(uBw) + \lambda_2(uBv)) \wedge ((\lambda_1 w + \lambda_2 v)Bu = \lambda_1(wBu) + \lambda_2(vBu)\big)$

We use $\mathrm{Alg}(V,*)$ to define a virtual algebra over the valued field (or the residue field, respectively).

Definition 50. Let T_R be the theory consisting of sentences that are true for all SPL **R**. If $T_R \models \mathrm{Alg}(V,*)$, then we say that V is a virtual algebra.

35. Lie Algebra: Let V be a nonempty virtual set. Let Lie-Alg(V,B) denote the formula

$$\mathrm{Alg}(V,B) \wedge (\forall x \in\ V\, xBx = 0) \wedge \big(\forall x,y,z\ \in\ V\, xB(yBz)) + yB(xBz)) + zB(xBy)) = 0\big)$$

Definition 51. Let $\mathfrak{g}$ be a virtual Lie algebra. Let $X = (X_{ij}) \in \mathfrak{g}$ be a parameter. We say that $\mathfrak{g}^X$ is a virtual set with parameters $X = (X_{ij})$ given by $\{Y \in \mathfrak{g} | XY - YX = 0\}$. The virtual set $\mathfrak{g}^X$ is called the centralizer of X.

Theorem 52. $\mathbf{R} \models \forall X \in \mathfrak{g}\ \ (\mathrm{Lin}(\mathfrak{g}^X))$

Proof. $\forall X\ (X \in g^X)$
$\Rightarrow \forall\lambda_1\forall\lambda_2\,\forall Y_1, \forall Y_2\ (XY_1 - Y_1X = 0) \wedge (XY_2 - Y_2X = 0))$
$\Rightarrow \lambda_1(XY_1 - Y_1X) = 0 \wedge \lambda_2(XY_2 - Y_2X) = 0$
$\Rightarrow X(\lambda_1Y_1) - (\lambda_1Y_1)X = 0 \wedge X(\lambda_2Y_2) - (\lambda_2Y_2)X = 0$
$\Rightarrow X(\lambda_1Y_1) - (\lambda_1Y_1)X + X(\lambda_2Y_2) - (\lambda_2Y_2)X = 0$
$\Rightarrow X(\lambda_1Y_1 + \lambda_2Y_2) - (\lambda_1Y_1 + \lambda_2Y_2)X = 0$
$\Rightarrow \lambda_1Y_1 + \lambda_2Y_2 \in\ \mathfrak{g}^X$
This proves that $\mathfrak{g}^X$ is a vector space.

□

4.0 SPECIAL ORTHOGONAL ALGEBRA $\mathfrak{so}(r)$, r ODD

We follow Waldspurger's treatment of nilpotent orbits in the classical p-adic Lie algebras [39]. As $\mathbb{F}$ changes, so do the characteristic and the cardinality of its residue field. Hence it is natural that some invariants associated with the residue field should play a role in determining these parameters. Waldspurger uses the theory of quadratic forms over finite fields. (In chapter 5 we will see a different parameterization of nilpotent orbits of semisimple Lie algebras, where a special type of orbits in Lie groups over the residue field is used.) In the next section we give an introduction to quadratic forms and their invariants. All vector spaces appearing in this chapter are *finite dimensional*.

4.1 FUNDAMENTALS OF QUADRATIC FORMS

Definition 53. Let V be a vector space (resp. module) over a field (resp. commutative ring) $\mathbf{F}$. A function $Q_V : V \to \mathbf{F}$ is called a quadratic form on V if

1. $Q_V(\alpha v) = \alpha^2 Q_V(v)$ for all $\alpha \in \mathbf{F}$ and $v \in V$.
2. The function $B : V \times V \to \mathbf{F}$ given by

$$B(v, w) = Q_V(v + w) - Q_V(v) - Q_V(w)$$

is bilinear.

Let $\mathbf{F}$ be such that its characteristic is not 2. For all $v, w \in V$ define

$$q_V(v, w) = \tfrac{1}{2}(Q_V(v + w) - Q_V(v) - Q_V(w))$$

This is a symmetric bilinear form. The factor $\frac{1}{2}$ gives $q_V(v, v) = Q_V(v)$. The pair (V, q_V) is called a *quadratic space*. Quadratic forms can be represented by matrices.

Definition 54. Matrix of a quadratic form: Let $\{e_1, \ldots, e_n\}$ be a basis of V. The *matrix of* q_V with respect to this basis is the matrix

$$A = (a_{ij}) \text{ where } a_{ij} = q_V(e_i, e_j)$$

In this chapter we are concerned with the invariants of quadratic spaces. The dimension of V is an obvious candidate and is our first invariant. We denote this by $d(V)$. For the second invariant, observe that in Definition 54, if we change the basis from $\{e_i\}$ to $\{e'_i\}$, we get a new matrix. Let C be the change of basis matrix. Then

$$A' = CA^tC \Rightarrow \det(A') = \det(C)\det(A)\det({}^tC).$$

Since $\det(C) = \det({}^tC)$, we get $\det(A') = \det(A)(\det(\mathrm{C}))^2$. Thus $\det(A)$ is invariant up to multiplication by a non-zero square element of $\mathbf{F}$. In other words, the square class of the determinant is another invariant. We will discuss this in Section 4.1.1 in the context of finite fields.

Vectors of a quadratic space can be classified based on the quadratic form.

Definition 55. 1. A vector v in V is called *isotropic* if $q_V(v, v) = 0$. Otherwise, it is called *anisotropic.*

2. A subspace W of V is called *isotropic* if $q_V(w, w) = 0$ for a non-zero $w \in W$.

3. If (V, q_V) contains no non-zero isotropic vectors, it is called an *anisotropic space.*

4. A subspace W of V is called *totally isotropic* if $q_V(w_1, w_2) = 0$ for all $w_1, w_2 \in W$.

Theorem 56. *All maximal totally isotropic subspaces of* (V, q_V) *have the same dimension. This dimension is called the* Witt Index.

Proof. See [33, pg.17]. ∎

A quadratic space (V, q_V) is called *nonsingular* or *nondegenerate* if for $x \in V$ there is a $y \in V$ such that $q_V(x, y) \neq 0$. We are interested in decomposing (V, q_V) in terms of isotropy. Before stating the decomposition theorem, we define an important map between quadratic spaces.

Definition 57. Let (V, q_V) and (W, q_W) be two quadratic spaces. A one-to-one linear transformation T from V to W is called an *isometry* if

$$\text{for all } v_1, v_2 \in V \text{ we have } q_W(Tv_1, Tv_2) = q_V(v_1, v_2)$$

Theorem 58 (Witt Decomposition). *: Let (V, q_V) have Witt Index m. Then*

$$V = H_1 \perp H_2 \perp \ldots \perp H_m \perp V_a,$$

where V_a is anisotropic, is uniquely determined up to isometry and is called the anisotropic kernel *of (V, q_V) (or if V is fixed, of q_V). The H_i are 2-dimensional nondegenerate isotropic quadratic spaces. By $H_i \perp H_j$ we mean $H_i \oplus H_j$ with the condition that $q_V(H_i, H_j) = 0$.*

Proof. See [33, pg. 17-18]. □

4.1.1 Quadratic Forms Over Finite Fields

Now suppose that $\mathbf{F}$ is the finite field $\mathbb{F}_q$. As discussed earlier, one of the invariants of the quadratic form is the dimension of the vectorspace V, denoted by $d(V)$. The other invariant is the image of $(-1)^{[\frac{d(V)}{2}]}\det(q_V)$ in $\mathbb{F}_q^*/\mathbb{F}_q^{*2}$, where $[.]$ denotes the integer part of the quantity inside. We denote this image by $\eta(q_V)$. For finite fields, these 2 invariants determine the quadratic form uniquely. [34, IV.1.7]

We are now ready to discuss the case $\mathfrak{so}(r)$. The next section is based on the discussion in [39, I.5, I.6, I.7].

4.2 PARAMETERIZATION OF NILPOTENT ORBITS

Let us fix some notation first.

- $\mathbb{F}$ is a p-adic field, ϖ is any uniformizer of $\mathbb{F}$.
- $\mathfrak{o}$ is the valuation ideal of $\mathbb{F}$.
- $\mathfrak{f}$ is the residue field of $\mathbb{F}$.
- $\mathfrak{g}$ is the Lie Algebra $\mathfrak{so}(r)$ over $\mathbb{F}$, with r odd. (In this chapter, r will always be odd.)

- $X \in \mathfrak{g}$ is a nilpotent element.
- (V, q_V) is the underlying vector space of $\mathfrak{g}$ with q_V as the quadratic form in the definition of $\mathfrak{g}$.
- $d(V)$ is the dimension of V.
- $P(r)$ is the set of partitions of r
- If $\Lambda = (\lambda_j)$ is a partition of r, then $c_i(\Lambda)$ denotes the number of λ_j that equal i.
- $\tilde{P}(r)$ is the subset of $P(r)$ consisting of partitions Λ of r with the property that for any even $i \geq 2$, $c_i(\Lambda)$ is even.

  Note. For this chapter, a partition Λ will always belong to $\tilde{P}(r)$.

Recall Theorem 33 from Chapter 3 and the subsequent example. Over an algebraically closed field, the set $\tilde{P}(r)$ completely parameterizes the orbits of $\mathfrak{so}(r)$. For each partition Λ in $\tilde{P}(r)$, we get a representative X such that its matrix representation contains $c_i(\Lambda)$ Jordan blocks of length i. This correspondence is bijective. Two distinct partitions give two distinct orbits. Obviously, this does not change when we consider $\mathfrak{g}$ over $\mathbb{F}$. However, Λ alone is no more sufficient to determine the orbits uniquely and we need more parameters.

4.2.1 Parameters For Nilpotent Conjugacy Classes In $\mathfrak{so}(r, \mathbb{F})$

Let $X, \mathfrak{g}, \Lambda, c_i(\Lambda)$ be as introduced earlier. The element X acts on V as a linear operator. Since X is nilpotent, there exists some n such that $X^n = 0$ and the nullspace of X^n is the entire space V. We now move to the residue field in two steps. First, we define a new quadratic space over $\mathbb{F}$ that depends on X.

Definition 59. 1. For all $i \geq 1$, i odd, let

$$V_i = \ker(X^i)/[\ker(X^{i-1}) + X \ker(X^{i+1})] \tag{4.1}$$

2. Let $\tilde{q}_i$ be the quadratic form on $\ker(X^i)$ given by

$$\tilde{q}_i(v, v') = (-1)^{(\frac{i-1}{2})} q_V(X^{i-1}(v), v') \tag{4.2}$$

3. Let q_i be the quadratic form on V_i given by taking the quotient. This is nondegenerate.

For each Λ we now have a family $(V_i,\, q_i)$ of quadratic spaces.

Theorem 60. *The set $\{\Lambda, (q_i)\}$ subject to the condition [to be discussed in Section 4.3]:*

$$\bigoplus_{i\,odd} q_i \sim_a q_V$$

parameterizes the nilpotent conjugacy classes of $\mathfrak{so}(r)$ completely. The relation $\sim_a$ indicates that the two forms have the same anisotropic kernel.

A quadratic form is uniquely determined by its invariants. In the orthogonal case, d_i (the dimension of V_i) is one of them. For the remaining invariants we make a transition to the residue field $\mathfrak{f}$ as follows.

For each i, let L_i be a lattice (i.e. an $\mathfrak{o}$-module) in V_i that generates V_i over $\mathbb{F}$. Its *dual* is given by

$$\tilde{L}_i = \{v \in V_i : \forall w \in L_i,\, q_i(v,w) \in \mathfrak{o}\} \tag{4.3}$$

We choose L_i so that it satisfies the property

$$\tilde{L} \supset L \supset \mathfrak{p}\tilde{L} \tag{4.4}$$

Such a lattice called an *almost self-dual* lattice. Now define the following quotients:

$$l_i' = L_i/\mathfrak{p}\tilde{L}_i,\;\; l_i'' = \tilde{L}_i/L_i \tag{4.5}$$

These quotients are, in fact, vector spaces over $\mathfrak{f}$. They inherit their quadratic forms from q_i and are both of the orthogonal type. More precisely,

$$q_{l_i'}(\overline{v},\overline{w}) = \overline{q_i(v,w)} \quad \text{for}\, v,\; w\; \in\; L, \tag{4.6}$$

$$q_{l_i''}(\overline{v},\overline{w}) = \overline{\varpi q_i(v,w)} \quad \text{for}\, v,\; w\; \in\; \tilde{L}, \tag{4.7}$$

where ϖ is any uniformizer of the valuation on $\mathbb{F}$.

The invariants of (V_i, q_i) are now given by those of $(l_i', q_{l_i'})$ and $(l_i'', q_{l_i''})$. Recall our discussion in Section 4.1.1. Let $\eta_i' = \eta(q_{l_i'})$, $\eta_i'' = \eta(q_{l_i''})$. The invariants of $(l_i', q_{l_i'})$ and $(l_i'', q_{l_i''})$ are $(d(l_i'),\, \eta_i')$ and $(d(l_i''),\, \eta_i'')$ respectively.

We want invariants of the 'isometry class' of (V_i, q_{V_i}), so they should also encode information about the anisotropic kernel. We note that, in the orthogonal case, the anisotropic kernels of $q_{l'}$ and $q_{l''}$ do not depend on L. These kernels together with dimension d_i (the dimension of V_i) determine the isomorphism class of (V_i, q_{V_i}). For the anisotropic kernel, we need to worry about only the reduction of the dimensions of these vector spaces mod $2\mathbb{Z}$, since over a finite field of characteristic other than 2, every quadratic space is isotropic [33, pg. 39]. Let d'_i (respectively d''_i) be the reduction of $d(l')$ (resp., $d(l'')$) in $\mathbb{Z}/2\mathbb{Z}$. They satisfy the condition:

$$d'_i + d''_i \equiv d_i \bmod 2\mathbb{Z}.$$

We now state a theorem for the orthogonal case. This has been rephrased for the purpose of this thesis.

Theorem 61 (J. L. Waldspurger). *: [39, I.3 & I.6] Let $\mathbb{F}$ be a finite extension of the field $\mathbb{Q}_p$ with $\mathfrak{f}$ as its residue field. Let V be a vector space over $\mathbb{F}$ with dim $V = d$, where d is odd and*

$$p \geq 3d + 1$$

Let $J = (J_{ij})$ where

$$J_{ij} = \begin{cases} 1 & \text{if } i+j=d+1; \\ 0 & \text{if otherwise.} \end{cases}$$

Let $\mathfrak{g}$ = Lie algebra (V, J). Let $\Sigma = \big(\Lambda,\, (d'_i, d''_i, \eta'_i, \eta''_i)\big)$. Then the set of nilpotent conjugacy classes are in bijection with the set $\{\Sigma\}$ and are denoted by N_Σ, where:

- *$\Lambda \in P(d)$ is a partition of d satisfying the condition:*

$$\forall i \in 2\mathbb{Z} \;\; c_i(\Lambda) \in 2\mathbb{Z}.$$

- *$\forall i \notin 2\mathbb{Z}$, if $c_i(\Lambda) \neq 0$ we have*

$$d'_i \in \mathbb{Z}/2\mathbb{Z},\; d''_i \in \mathbb{Z}/2\mathbb{Z} \text{ and } d'_i + d''_i \equiv c_i(\Lambda) \bmod (2\mathbb{Z})$$

- *$\eta'_i \in \{s, ns\}$, $\eta''_i \in \{s, ns\}$ where s and ns denote square classes and non-square classes in the field $\mathfrak{f}$, respectively.*

Furthermore, one imposes the condition:

$$\bigoplus_{i\,odd} q_i \sim_a q_V \qquad\qquad (*, \Sigma)$$

The relation $\sim_a$ indicates that the two forms have the same anisotropic kernel. For now, we will refer to this condition as 'condition $(*, \Sigma)$'.

Proof. See Waldspurger [39]. □

4.3 THE RELATION $\bigoplus_{i\,odd} q_i \sim_a q_v$

This is based on J.L. Waldspurger's personal notes [40]. Let V be a finite dimensional vector space on a field F equipped with a non-degenerate quadratic form. Then there exists an orthogonal decomposition due to Witt [see Theorem 58].

$$V = H_1 \perp H_2 \perp \ldots \perp H_m \perp V_a,$$

where V_a is anisotropic and is uniquely determined upto isometry. Also, q_{V_a} is the restriction of q_V on V_a

Definition 62. We say that $(V, q_V) \sim_a (V', q_{V'})$ if $(V_a, q_{V_a}) \cong (V'_a, q_{V'_a})$

Let (V, q_V) be a finite dimensional quadratic space over $\mathbb{F}$, where q_V is non-degenerate. Let X be a nilpotent element of the orthogonal Lie Algebra of (V, q_V). Then X acts a linear operator on V.

Recall that the family $(\Lambda, (q_i))$ parameterizes the conjugacy class of X. Due to the condition on Λ, if i is even the corresponding V_i is even dimensional. It has a trivial anisotropic kernel. Whereas, if i is odd, the form q_i turns out to be equivalent to the $\sum_{j\in J} a_j x_j^2$, where $|J| = c_i(\Lambda)$ and the quadratic form on the anisotropic kernel is of the form ax^2 where a is a non-zero element of the field $\mathbb{F}$. Thus the anisotropic kernel of V_i is non-trivial only when i is odd.

Hence, q_V is $\sim_a$ to $\bigoplus_{i\,odd} q_i$.

4.4 DEFINABILITY OF NILPOTENT CONJUGACY CLASSES IN $\mathfrak{so}(r)$, r ODD: THE MAIN THEOREM

We will now show that the conjugacy classes parameterized by the set $\{\Sigma\}$ and the condition $(*, \Sigma)$ are definable. Recall that $\Sigma = (\Lambda,\, (d'_i,\, d''_i,\, \eta'_i,\, \eta''_i))$.

4.4.1 A Brief Outline

What are we trying to do here?

In the odd orthogonal case the nilpotent conjugacy classes are uniquely parameterized by the family $(\Lambda, (V_i, q_i))$ [refer to equation (4.1) and (4.2) and the subsequent comment in Section 4.2]. Each (V_i, q_i) is uniquely determined by the 4-tuple $(d'_i,\, d''_i,\, \eta'_i,\, \eta''_i)$ where:

1. $d'_i = \overline{0}$ (resp. $\overline{1}$) means that the dimension of the vector space $l'_i = L/\mathfrak{p}\tilde{L}$ (over the residue field) is even (resp. odd). In our case, L is any *almost self dual* on the quotient space V_i given by equation (4.1) in Section 4.2.
2. $d''_i = \overline{0}$ (resp. $\overline{1}$) means that the dimension of the vector space $l''_i = \tilde{L}/L$ (over the residue field) is even (resp. odd).
3. $\eta'_i = \mathrm{sq}$ (resp. nsq) means that the Gram-determinant of the quadratic form on l'_i given by equation (4.6) in Section 4.2 is a square (resp. non-square)in the residue field.
4. $\eta''_i = \mathrm{sq}$ (resp. nsq) means that the Gram-determinant of the quadratic form on l''_i given by equation (4.7) in Section 4.2 is a square (resp. non-square)in the residue field.

In the proof, we fix $n = \mathrm{Dim}(V, \mathbb{F})$ and select a partition of n satisfying the condition $\forall i \in 2\mathbb{Z}$ $c_i(\Lambda) \in 2\mathbb{Z}$. For each i such that $c_i(\Lambda) \neq 0$, select a 4-tuple for the parameters $(d'_i,\, d''_i,\, \eta'_i,\, \eta''_i)$ from the set $\{\overline{0}, \overline{1}\} \times \{\overline{0}, \overline{1}\} \times \{\mathrm{sq}, \mathrm{nsq}\} \times \{\mathrm{sq}, \mathrm{nsq}\}$. We claim that there is a formula in Pas's language for each of the aforementioned four statements and for the condition $(*, \Sigma)$. (This condition is satisfied by the quadratic forms and quotient spaces (V_i, q_i) and (V, q) considered here.)

Finally, the main claim is that the virtual set cut out by these formulae is either empty or a nilpotent conjugacy class. The definition of the parameters indicates that there are $2^4 = 16$ possible choices for the 4-tuple $(d_i',\, d_i'',\, \eta_i',\, \eta_i'')$. Some of these will be ruled out by the condition $d_i' + d_i'' \equiv c_i(\Lambda) \pmod{2\mathbb{Z}}$ but many options remain. It will be extremely cumbersome to write out all these options together. Hence, we will state as clearly as possible how they are to be pieced together instead of presenting a long formula containing concatenated conjunctions and disjunctions.

Here is the statement of the theorem and its proof:

4.4.2 The Statement

Theorem 63. *1. For $\Sigma = (\Lambda,\, (d_i',\, d_i'',\, \eta_i',\, \eta_i''))$, $S_d = \{\Sigma\}$ is a finite, field-independent set, and there exists a formula ϕ_Σ in Pas's language for each $\Sigma \in S_d$.*

2. The condition (,Σ) can be expressed by a formula $\phi_{*,\Sigma}$ in Pas's language.*

3. Let $\mathbb{F}$ be a p-adic field [see Section 2.1] and let its finite residue field be $\mathfrak{f}$.
Let V be a virtual set such that $\mathrm{Lin}(V) \wedge \mathrm{Dim}(d, V)$ holds.
Let J be a matrix of terms satisfying the condition $J_{ij} = J_{ji}$. Let $\mathfrak{g}$ be the virtual set $\{Y : {}^tYJ + JY = 0\}$. (Here Y is a matrix of terms of the valued field sort.)
Then

$$\{X \in \mathfrak{g} : \phi_\Sigma(X) \wedge \phi_{*,\Sigma}(X)\} \tag{4.8}$$

is either empty or a nilpotent conjugacy class in $\mathfrak{g}$.

4. For each $\mathbb{F}$, every nilpotent class appears exactly once in this set.

Proof. :

1. Any integer d has a finite number of partitions and thus, only a finite number of them appear as Λ in the set $\{\Sigma\}$. The partitions depend only on d and not on the field.
2. Let $\Lambda \in \tilde{P}(d)$, there is a unique J_Λ - the Jordan block matrix - associated with the partition Λ. Let $\mathrm{J}_\Lambda(X)$ denote the formula:
   $\exists (g_{ij})_{1 \le i,j \le d}\ (g_{ij})X = J_\Lambda(g_{ij}) \ \wedge\ det(g_{ij}) \neq 0$
   This states that X is conjugate to J_Λ.

Λ is now fixed for the rest of the proof.

3. For each $i \notin 2\mathbb{Z}$ such that $c_i(\Lambda) \neq 0$ the following are virtual sets with a parameter X ranging over $n \times n$ matrices.
   - $K_i := K_i(X) := \mathrm{Ker}(X^i) := \{v \in V | \, X^i(v) = 0\}$ for all $i \in 2\mathbb{Z}$ with $c_i(\Lambda) \neq 0$
   - $W_i := W_i(X) := \{y \in V \,|\, \Phi(y, X)\}$

   where $\Phi(y, X)$ is the formula
   $\exists\, y_1, \, y_2, u_2, \, \big(y = y_1 + y_2 \ \wedge \ X^{i-1}(y_1) = 0 \ \wedge \ X(u_2) \ = \ y_2 \wedge X^{i+1}(u_2) = 0\big)$

   The virtual set W_i replaces the space $[\ker(X^{i-1}) + X \ker(X^{i+1})]$ in Section 4.2.

   We now fix i till the last step. Thus $c_i(\Lambda)$ is fixed, call it c_i.

4. We need a formula for the set of elements in S_d that correspond to $(d_i', \, d_i'', \, \eta_i', \, \eta_i'')$. This lengthy construction is divided into five steps. To keep us on track, we will give appropriate parallel references to Waldspurger's treatment from Section 4.2. In the final formula, all the quantities will be bound by appropriate quantifiers.

   Step 1 First, we need to cut out a formula that gives an 'almost self-dual' lattice in $V_i = K_i/W_i$. Note that we will use the labels V_i, K_i and W_i in the sense of Formula 14 in Section 3.5.2.
   We have

   $$(\text{Q-space}(V_i, K_i/W_i) \ \ \wedge \ \ \text{Basis}(e_{i_1}, \, \ldots, \, e_{i_{c_i}}, V_i) \ \ \wedge \ \text{ASD}(L_i, {}^tX^{i-1}J, V_i))$$

   where:
   $K_i = K_i(X)$, $V_i = \text{vector space}(e_{i_1}, \, \ldots, \, e_{i_{c_i}})$ and $L_i = \text{lattice}(e_{m_1}, \, \ldots, \, e_{m_{c_i}})$
   Call this formula $\phi_i^{(1)}(X, e_{m_1}, \, \ldots, \, e_{m_{c_i}})$.

   Note. The i refers to the fixed i and the superscript (1) refers to Step 1.

Step 2 Now we need to cut out a formula that states that the dimension of the quotient space l_i' is even (odd) respectively. Recall that this number is bounded above by c_i. This would be:

Even-Qdim$(c_i, \mathfrak{p}\tilde{L}_i, L_i, V_i)$ (resp. Odd-Qdim$(c_i, \mathfrak{p}\tilde{L}_i, L_i, V_i)$)

where:

$\tilde{L}_i = \{u \mid \mathrm{val}({}^t e_{m_j}\, {}^t X^{i-1} J u) \geq 0 \text{ for } j = 1, \ldots, c_i\}$ and L_i, V_i are as in Step 1.

Call these formulae $\phi^{(2)}_{i,\epsilon}(X, e_{m_1}, \ldots, e_{m_{c_i}})$. Here ϵ refers to 'odd' or 'even'.

Step 3 Now suppose that the value for the parameter η_i' is sq (resp. nsq). This is given be the following formula:

$\Theta(\mathrm{sq}, {}^t X^{i-1} J, V_i)$ (resp. $\Theta(\mathrm{nsq}, {}^t X^{i-1} J, V_i)$)

Piecing together one formula each from steps 2 and 3 gives the pair (d_i', η_i'). Call these formulae $\phi^{(3)}_{i,\epsilon}(X, e_{m_1}, \ldots, e_{m_{c_i}})$. Here ϵ refers to 'square' or 'non-square'.

Now we need to construct formulae for the pair (d_i'', η_i''). Recall, d_i'' is the dimension of the vector space $l'' = \tilde{L}/L$ modulo $2\mathbb{Z}$ [See Section 4.2(7)].

Step 4 The formula for $d_i'' = \overline{0}$ (resp. $\overline{1}$) is given by:

Even-Qdim$(c_i, L_i, \tilde{L}_i, V_i)$ (resp. Odd-Qdim$(c_i, L_i, \tilde{L}_i, V_i)$).

where V_i, L_i and $\tilde{L}_i$ are as above.

Call these formuale $\phi^{(4)}_{i,\epsilon}(X)$. Here ϵ refers to 'odd' or 'even'.

Step 5 The formula for $\eta_i'' = \mathrm{sq}$ (resp. nsq) is given by:

$\forall e_1', \ldots, e_{c_i}' \Big(\mathrm{Basis}(e_1', \ldots, e_{c_i}', V_i) \Longrightarrow$

$\exists \eta \in \mathfrak{o} \wedge \exists \xi \in \mathfrak{f}^*$ such that

$\mathrm{val}(\eta) = c_i + \mathrm{val}(\text{Gram-det}(e_1', \ldots, e_{c_i}', {}^t X^{i-1} J) \wedge \xi^2 = \mathrm{ac}(\eta) \wedge$

$\mathrm{ac}(\eta) = \mathrm{ac}(\text{Gram-det}(e_1', \ldots, e_{c_i}', {}^t X^{i-1} J))$

where V_i is as above.

The formula for 'nsq' follows similarly.

Call these formulae $\phi^{(4)}_{i,\epsilon}(X)$. Here ϵ refers to 'square' or 'non-square'. Piecing together one formula each from steps 4 and 5 gives the pair (d_i'', η_i'').

5. Finally, we show that the condition $(*, \Sigma)$ is definable. Note that if $(*, \Sigma)$ is not satisfied by the parameters, then the parameters give an empty conjugacy class. Now recall that

$$\bigoplus_{i\, odd} q_i \sim_a q_V \qquad (*, \Sigma)$$

is a concise notation for

$((V_1 \bigoplus V_3 \bigoplus \ldots \bigoplus V_j)_a, (q_1 \bigoplus q_3 \bigoplus \ldots \bigoplus q_j)_a) \cong (V_a, q_a)$

where j is the largest odd integer less than or equal to d for which $c_j(\Lambda) \neq 0$ and the subscript a refers to the anisotropic part. This given by the formula [refer to formula 28]:

Iso-aniso$(V_1 \bigoplus V_3 \bigoplus \ldots \bigoplus V_j,\ J \bigoplus {}^tX^2 J \bigoplus \ldots \bigoplus {}^tX^{j-1}J,\ V,\ J)$

where:

$V_i =$ vectorspace$(e_{m_1}, \ldots, e_{m_{c_i}})$

6. How do we piece all this together to present a virtual set in the form given by equation (4.8)?

Recall $\Sigma = (\Lambda, (d_i', d_i'', \eta_i', \eta_i''))$. Now, for each $\Lambda \in \tilde{P}(d)$, consider

$$\{X \in \mathfrak{g} \,|\, \exists_{\mathbb{F}}\, e_{m_1}, \ldots, e_{m_{c_i}} \ \ \phi_\Sigma(X, e_{m_1}, \ldots, e_{m_{c_i}}) \wedge \phi_{*,\Sigma}(X, e_{m_1}, \ldots, e_{m_{c_i}})\} \qquad (4.9)$$

where $1 \leq i \leq n$, and i ranges over all odd numbers appearing in the partition Λ. (For brevity, we use the notation $i \in \Lambda$ to indicate this condition on i.)

- $\phi_\Sigma(X, e_{m_1}, \ldots, e_{m_{c_i}})$ is the conjunction

$$J_\Lambda(X) \wedge (\textstyle\bigwedge_{i \in \Lambda} \phi_i(X))$$

where, $\phi_i(X, e_{m_1}, \dots, e_{m_{c_i}})$ stands for:
$\phi_i^{(1)}(X, e_{m_1}, \dots, e_{m_{c_i}}) \wedge \phi_{i,\epsilon}^{(2)}(X, e_{m_1}, \dots, e_{m_{c_i}}) \wedge \phi_{i,\epsilon}^{(3)}(X, e_{m_1}, \dots, e_{m_{c_i}}) \wedge \phi_{i,\epsilon}^{(4)}(X) \wedge \phi_{i,\epsilon}^{(5)}(X)$
combining the formulae from step 1 and one each (for the choice of ϵ) from steps 2 to 5.

- $\phi_{*,\Sigma}(X)$ is the formula
Iso-aniso($V_1 \bigoplus V_3 \bigoplus \dots \bigoplus V_j,\ J \bigoplus {}^tX^2J \bigoplus \dots \bigoplus {}^tX^{j-1}J,\ V,\ J$)

In conclusion, the virtual set given by equation (4.9) is either empty or a nilpotent conjugacy class in $\mathfrak{g}$.

This gives definability in the orthogonal case. $\square$

5.0 AFFINE APARTMENTS

In Chapter 4 we saw that the orbits of classical semisimple Lie algebras can be parameterized in terms of invariants of quadratic forms over finite fields. What about exceptional Lie algebras? They are not defined in terms of a quadratic form and hence, a more general approach is needed to study their orbits. The technique discussed in this chapter relies on the structure theory of reductive p-adic Lie algebras.

Buildings were first introduced by J.Tits in 1950s to carry out Felix Klein's *Erlangen Program* for exceptional groups. Tits gave concrete geometric interpretations of exceptional groups [36]. These have been generalized in various ways. Our interest lies in an interpretation called *Bruhat-Tits Building* which is a Euclidean building [5, 6].

Lie (Chevalley) groups have a rich structure that comes from two sources; firstly they are constructed from the Chevalley data (root system, simple roots). This exists independent of the underlying field. The geometrization of this data is the *spherical building*. Secondly, much of the structure exists due to the fact that p-adic fields possess a non-trivial discrete valuation. This valuation equips the field $\mathbb{F}$ with a filtration by open compact groups

$$\{0\} \subsetneq \ldots \subsetneq \mathfrak{p}^3 \subsetneq \mathfrak{p}^2 \subsetneq \mathfrak{p} \subsetneq \mathfrak{p}^0 = \mathfrak{o} \subsetneq \mathfrak{p}^{-1} \subsetneq \ldots \subsetneq \mathbb{F}.$$

Using this, we can define many interesting open, compact subgroups of Chevalley groups. The geometrization of this data is called *Affine Buildings*. *Apartments* are a family of subsets of buildings satisfying certain conditions. For this thesis we are interested in apartments rather than buildings and hence, we will define them more carefully for each building in the subsequent sections.

Locally, an affine apartment looks like a spherical apartment, as will be seen in figures 5.2 and 5.3. First we recall a few facts about the spherical apartment. We will assume some familiarity with the structure theory of Lie algebras. As will be seen, both apartments are Euclidean spaces. A spherical apartment has action by reflections, whereas an affine apartment has action by reflections and translations. This chapter is based on DeBacker's notes [10] and on [9],[30].

5.1 SPHERICAL APARTMENTS

Tits introduced spherical buildings in a series of papers [36, 37, 38]. We fix some notation for the rest of the chapter.

Notation 64.
- Let G be a Chevalley (semisimple Lie) group and $\mathfrak{g}$ its Lie algebra.
- Let T be a maximal torus in G defined over $\mathbb{Z}$.
- $X^*(T)$ is the set of characters of T.
- $X_*(T)$ is the set of co-characters of T.
- Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$.
- E is a Euclidean space (over $\mathbb{R}$).
- E^* is its dual, i.e. the space of linear functionals on E.
- Φ is the (reduced) root system of G [see Section 3.3].
- Let $\langle\, ,\, \rangle$ be the map $X_*(T) \times X^*(T) \to \mathbb{Z}$.

Definition 65. The spherical apartment of G is the Euclidean space $\mathcal{A}_s = E^*$ i.e. $\mathcal{A}_s$ is the Euclidean space containing the (abstract) co-root system of G.

Definition 66. The root system Φ endows $\mathcal{A}_s$ with a *hyperplane structure* given by hyperplanes $H_\alpha = \{\lambda \in F \mid \langle \alpha, \lambda \rangle = 0\}$, where $\alpha \in \Phi$.

Geometrically σ_α [see Section 3.3, Definition 36] is the reflection about the hyperplane H_α. These hyperplanes divide E^* into regions called *Weyl chambers.* The *Weyl group* $\mathcal{W} = N_G(T)/T$ acts transitively on these chambers.

The spherical apartment of $\mathfrak{sl}(3)$ is as follows:

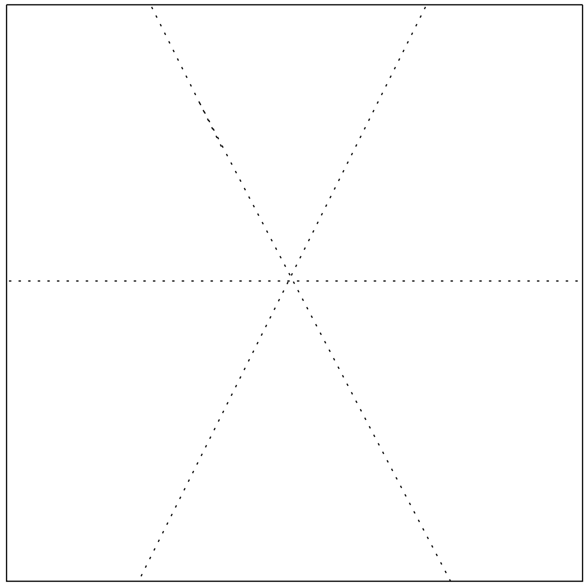

Figure 5.1: Spherical apartment for $\mathfrak{sl}(3)$

5.2 AFFINE APARTMENT

Iwahori and Matsumoto introduced a new type of BN pair for a Chevalley group over p-adic fields, where B and N are its subgroups satisfying some conditions [16]. This leads to a triangulation of Euclidean spaces. Later Bruhat and Tits [6] constructed a Euclidean building for reductive linear algebraic groups.

In the subsequent sections and chapters, by $G(\mathfrak{o})$ (resp. $\mathfrak{g}(\mathfrak{o})$), we mean $\mathfrak{o}$ points of G (resp. $\mathfrak{g}$).

With the notation developed above, we give a simple definition of an affine apartment.

Definition 67. $\mathcal{A} = X_*(T) \bigotimes \mathbb{R}$ is called the *affine apartment* of G attached to T.

Just as the spherical apartment has a hyperplane structure, so does the affine apartment. Let $\Psi = \{\gamma + n : \gamma \in \Phi, n \in \mathbb{Z}\}$. Each $\psi = \gamma + n \in \Psi$ defines an affine function on $\mathcal{A}$ by

$$(\gamma + n)(\lambda \otimes r) := r\langle \lambda, \gamma \rangle + n.$$

Thus for each $\psi = \gamma + n \in \Psi$, we can define a hyperplane

$$H_\psi = \{x \in \mathcal{A} \,|\, (\gamma + n)(x) = 0\}$$

These hyperplanes give us a polysimplicial decomposition of $\mathcal{A}$. A polysimplex occurring in this decomposition is called a *facet*. Let $x \in \mathcal{A}$ and Φ_x be the set of roots α for which $\alpha(x)$ is an integer. Now define

$$\mathcal{H}_n = \{x \in \mathcal{A} \,|\, |\Phi_x| = n\}.$$

Then an *n-facet* is a connected component of $\mathcal{H}_n$. The closure of a 0-facet is called an *alcove*. We will refer to any n-facet as a 'facet' in this thesis.

We have an extended Weyl group analogous to the Weyl group for spherical chambers.

Definition 68. The group $\tilde{W} = N_G(T)/T(\mathfrak{o})$ is called the *extended* or *affine Weyl group* and it acts transitively on alcoves.

We close this section with a visualization of the affine apartments of $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$.

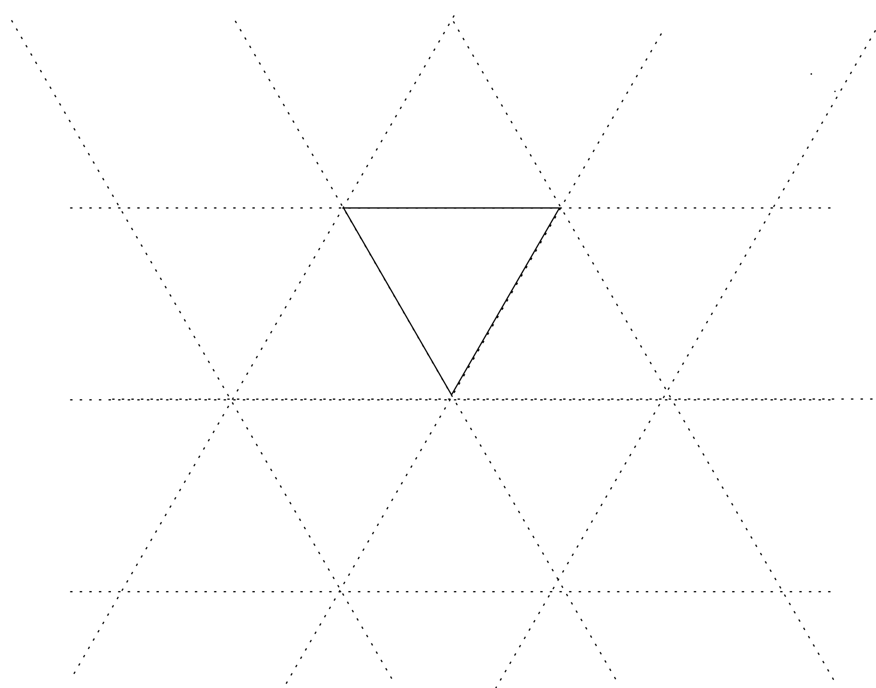

Figure 5.2: Affine apartment of $\mathfrak{sl}(3)$

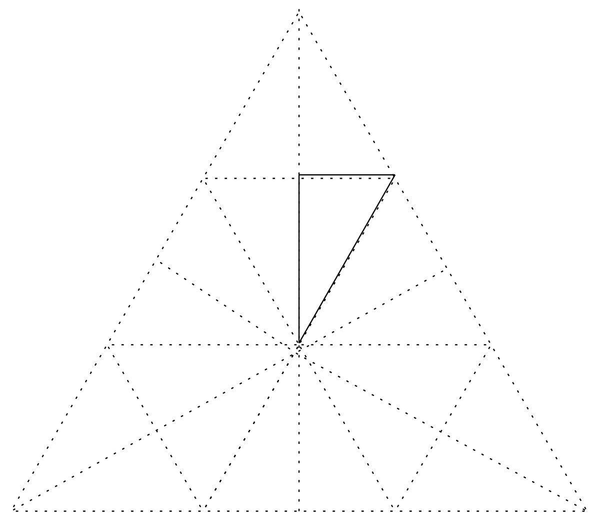

Figure 5.3: Affine apartment of $\mathfrak{g}_2$

5.3 OBJECTS ATTACHED TO AN AFFINE APARTMENT

For each facet F, we will define three types of objects. The first two are over $\mathbb{F}$ and are called parahoric subgroup and pro-unipotent radical. Parahoric subgroups are the affine analogues of parabolic subgroups. We will define them in this section.

For each $\gamma \in \Phi$, we have a root group U_γ with generator u_γ and a root space $\mathfrak{g}_\gamma$. Let $\mathfrak{g}_\gamma$ be spanned by X_γ. Recall that the discrete valuation on $\mathbb{F}$ endows it with a filtration. We use this to introduce a filtration on G (resp. $\mathfrak{g}$). More precisely:

Definition 69. Use the set $\{\gamma + n \,|\, n \in \mathbb{Z}\}$ to index the following filtration on G (resp. $\mathfrak{g}$).

$$U_{\gamma+n} \subsetneq U_{\gamma+n+1} \text{ where } U_{\gamma+0} := G(\mathfrak{o}) \cap U_\gamma$$

$$\mathfrak{g}_{\gamma+n} \subsetneq \mathfrak{g}_{\gamma+n+1} \text{ where } \mathfrak{g}_{\gamma+0} := \mathfrak{g}(\mathfrak{o}) \cap \mathfrak{g}_\gamma \text{ and } \mathfrak{g}_{\gamma+n} := \mathfrak{g}(\mathfrak{o}^n) \cap \mathfrak{g}_\gamma$$

Notation 70. For all n in $\mathbb{Z}$, let $\mathfrak{p}^n X_\gamma = \{aX_\gamma \,|\, a \in \mathfrak{p}^n\}$

Definition 71. Let F be a facet in $\mathcal{A}$. Let $\Psi = \{\gamma + n \,|\, \gamma \in \Phi,\, n \in \mathbb{Z}\}$. For $\psi = \gamma + n$, define the following:

1. Parahoric group: Let $G_F := G_x := \langle T(\mathfrak{o}), \mathfrak{p}^{-\lfloor \gamma(x) \rfloor} U_\gamma \,:\, \forall \gamma \in \Phi \rangle$. Then, G_F is called the

parahoric group attached to F.

2. Pro-unipotent radical: Let $G_F^+ := G_x^+ := \langle T(\mathfrak{p}+1), \mathfrak{p}^{1-\lceil \gamma(x) \rceil} U_\gamma \ : \ \forall \gamma \in \Phi \rangle$. Then, G_F^+ is called the *pro-unipotent radical* group associated with F. It is a normal subgroup of G_F.
3. The lattice corresponding to G_F is given by $\mathfrak{g}_F := \langle \mathfrak{h}(\mathfrak{o}), \mathfrak{p}^{-\lfloor \phi(x) \rfloor} X_\gamma \ : \ \forall \gamma \in \Phi \rangle$
4. The lattice corresponding to G_F^+ is given by $\mathfrak{g}_F^+ := \langle \mathfrak{h}(\mathfrak{p}), \mathfrak{p}^{1-\lceil \phi(x) \rceil} X_\gamma \ : \ \forall \gamma \in \Phi \rangle$.
5. Object over the residue field $\mathfrak{f}$: Denote by $L_F(\mathfrak{f})$ the quotient $\mathfrak{g}_F / \mathfrak{g}_F^+$.

These definitions will be clearer once we see the example in Section 5.5.

Note. The terms *parahoric, pro-unipotent* are used for groups. The corresponding Lie algebras (lattices) do not have separate or special names. For the purpose of this thesis, when we say *parahoric* (resp. *pro-unipotent*), we mean the *Lie algebra* corresponding to the *parahoric* (resp. *pro-unipotent*) group.

5.4 PARAMETERIZATION OF NILPOTENT CONJUGACY CLASSES

In this section we relate the nilpotent conjugacy classes of $\mathfrak{g}$ to a special type of orbit in finite groups of Lie type using the following theorem. These special type of orbits are called *distinguished orbits*. An orbit is said to be *distinguished* if it does not intersect any proper Levi subalgebra of $\mathfrak{g}$.

Theorem 72 (Barbash and Moy [4]). *If F is a facet and $\bar{\mathcal{O}}$ is a distinguished nilpotent orbit in $L_F(\mathfrak{f}) = \mathfrak{g}_F / \mathfrak{g}_F^+$, then there exists a unique nilpotent orbit in $\mathfrak{g}$ of minimal dimension which intersects the pre-image of $\bar{\mathcal{O}}$ in $\mathfrak{g}_F$ non-trivially.*

The theorem does not say that the correspondence between facets and distinguished orbits in $L_F(\mathfrak{f})$ is bijective. In deed, it is possible that two different facets may give the same orbit. DeBacker [9] refines this correspondence by defining an equivalence relation on the set of facets in two stages. At first, he focusses only on the facets.

Definition 73. Let F be a facet in $\mathcal{A}$. Then $A(F)$ is the smallest affine space containing F.

Definition 74. Suppose F_1 and F_2 are two facets in $\mathcal{A}$. If there is $w \in \tilde{W}$ such that $A(F_1) = A(wF_2)$, then we write $F_1 \sim F_2$.

Remark 75. Two alcoves are always equivalent by definition. Hence, we consider an alcove to be a 'building block' of the affine apartment.

Now let I_d be the set consisting of pairs $(F, \bar{\mathcal{O}})$ where $\bar{\mathcal{O}}$ is a distinguished $G_F(\mathfrak{f})$-orbit in $L_F(\mathfrak{f})$. Define a relation:

Definition 76. Suppose $(F_1, \bar{\mathcal{O}}_1)$ and $(F_2, \bar{\mathcal{O}}_2)$ are two elements of I^d. We say that

$$(F_1, \bar{\mathcal{O}}_1) \sim (F_2, \bar{\mathcal{O}}_2)$$

provided that there exists $n \in N_G(T)$ such that

1. $A(F_1) = A(nF_2)$
2. $\imath(\bar{\mathcal{O}}_1) = {}^n\bar{\mathcal{O}}_2$, where $\imath$ is the canonical isomorphism $L_{F_1}(\mathfrak{f}) \simeq L_{nF_2}(\mathfrak{f})$.

We now state a theorem due to DeBacker.

Theorem 77 (S. Debacker [9]). *: Suppose char $\mathfrak{f} = p$ is sufficiently large. Let F be a facet in $\mathcal{A}$ and $\bar{\mathcal{O}}$ a distinguished $G_F(\mathfrak{f})$-orbit in $L_F(\mathfrak{f})$. The map that sends $(F, \bar{\mathcal{O}}) \in I^d$ to the unique nilpotent G-orbit of minimal dimension which intersects the pre-image of $\bar{\mathcal{O}}$ non-trivially, induces a bijective correspondence*

$$I^d/\sim \leftrightarrow \mathcal{O}(0),$$

where $\mathcal{O}(0)$ is the set of all nilpotent orbits in the $\mathfrak{g}$.

5.5 A SIMPLE EXAMPLE

Here we will calculate parahorics and pro-unipotent radicals for $\mathfrak{sl}(2, \mathbb{F})$ in detail. Let char $\mathfrak{f} \neq 2$. Note that $\mathfrak{sl}(2)$ is rank 1, hence the affine apartment is a one-dimensional affine space. Its root system Φ consists of only two roots, α and $-\alpha$. Let us fix an alcove. This alcove contains three facets; F_1, F_2 and F_3. We will show how to compute the corresponding lattices for facet F_1.

Calculations for Facet F_1

We fix a basis for the root spaces:

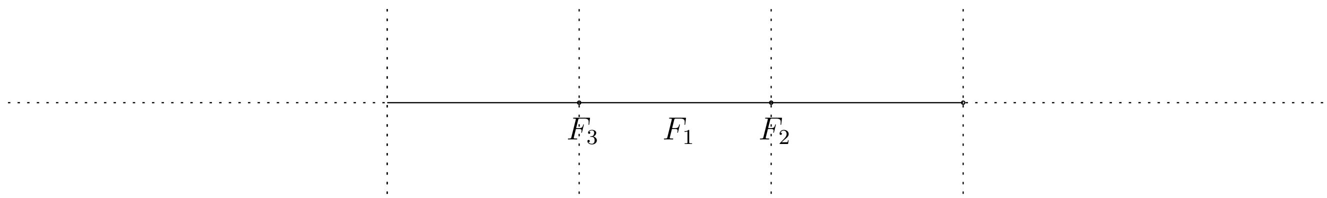

Figure 5.4: Affine apartment of $\mathfrak{sl}(2)$

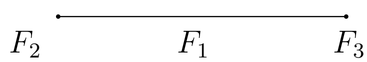

Figure 5.5: Alcove for $\mathfrak{sl}(2)$

$$X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

To compute the parahoric and pro-unipotent radical lattices, we observe that x lies between $H_{\alpha+0}$ and $H_{\alpha-1}$.

The plane $H_{\alpha+0}$ is the same as $H_{-\alpha-0}$. For $y \in H_{\alpha+0}$, we have $(\alpha+0)(y) = 0$, hence $\alpha(y) = 0$ and $-\alpha(y) = 0$. Similarly, for $z \in H_{\alpha-1}$, we have $(\alpha-1)(z) = 0$, hence $\alpha(z) = 1$ and $-\alpha(z) = -1$.

Now, x lies between $H_{\alpha+0}$ and $H_{\alpha-1}$ means that $0 < \alpha(x) < 1$. Thus $\lfloor \alpha(x) \rfloor = 0$ and $\lceil \alpha(x) \rceil = 1$. This yields the following:

$\mathfrak{p}^{-\lfloor \alpha(x) \rfloor} = \mathfrak{p}^{-0} = \mathfrak{o}$ $\qquad$ $\mathfrak{p}^{1-\lceil \alpha(x) \rceil} = \mathfrak{p}^{1-1} = \mathfrak{o}$

$\mathfrak{p}^{-\lfloor -\alpha(x) \rfloor} = \mathfrak{p}^{-(-1)} = \mathfrak{p}$ $\qquad$ $\mathfrak{p}^{1-\lceil -\alpha(x) \rceil} = \mathfrak{p}^{1-0} = \mathfrak{p}$

Parahoric $\mathfrak{g}_F$	Pro-Unipotent $\mathfrak{g}_F$	$L_F(\mathfrak{f})$
$\mathfrak{g}_{F_1} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}$	$\mathfrak{g}_{F_1}^+ = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$	$L_{F_1}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & 0 \\ 0 & \mathfrak{f} \end{pmatrix}$
$\mathfrak{g}_{F_2} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}$	$\mathfrak{g}_{F_2}^+ = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} \end{pmatrix}$	$L_{F_2}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} \end{pmatrix}$
$\mathfrak{g}_{F_3} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix}$	$\mathfrak{g}_{F_3}^+ = \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$	$L_{F_3}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} \end{pmatrix}$

How does the theorem work? Assume that char($\mathfrak{f}$) is not 2. Here is a list of distinguished orbits in the respective Lie algebras over $\mathfrak{f}$.

- In $\mathfrak{gl}(1, \mathfrak{f})$, there is only one distinguished orbit and it is 0.
- In $\mathfrak{sl}(2, \mathfrak{f})$ there are 2: $\begin{pmatrix} 0 & \text{sq} \\ 0 & 0 \end{pmatrix}$, where sq is a square in $\mathfrak{f}$ and $\begin{pmatrix} 0 & \text{nsq} \\ 0 & 0 \end{pmatrix}$, where nsq is a non-square in $\mathfrak{f}$.

  The pre-image of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{g}_{F_1}$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

  The pre-images of $\begin{pmatrix} 0 & \text{sq} \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{g}_{F_2}$ and $\mathfrak{g}_{F_3}$ are, respectively, $\begin{pmatrix} 0 & \varpi^{-1} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

  The pre-images of $\begin{pmatrix} 0 & \text{nsq} \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{g}_{F_2}$ and $\mathfrak{g}_{F_3}$ are given by $\begin{pmatrix} 0 & \varpi^{-1}\epsilon \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$ respectively. Here ϵ is a non-zero element of $\mathfrak{o} \backslash \mathfrak{o}^2$.

This gives five representatives for distinct orbits in $\mathfrak{sl}(2, \mathbb{F})$ as expected.

5.6 TWO LEMMAS

We saw earlier that affine apartments are Euclidean spaces that capture the p-adic structure of the underlying field. While affine apartments are not definable in Pas's language; the parahorics and pro-unipotent radicals are. This has been verified for each example considered in this thesis. However, we would like to point out that this is true for any reductive group and hence we state a lemma to that effect.

Lemma 78. *Let $\mathbb{F}$ be a p-adic field, let $\mathfrak{g}$ be a definable reductive Lie algebra with root system Φ. Let F be a facet in the affine building of $\mathfrak{g}$. Let $\mathfrak{h}$ be a definable Cartan subalgebra of $\mathfrak{g}$. Then its parahoric $\mathfrak{g}_F$ and pro-unipotent radical $\mathfrak{g}_F^+$ (Lie algebras) are definable.*

Proof. Let Φ be the root system for $\mathfrak{g}$. We will assume that $\mathfrak{g}$ can be realised as a matrix algebra and use matrix notation for elements of the algebra. Furtheremore, suppose $\mathfrak{g}$ has $n \times n$ free variables. Let F be a facet. We will write $\mathfrak{g}_F$ and $\mathfrak{g}_F^+$ as direct sums [See Formula 10 in Section 3.5.2]. Let F be a facet.

- Let $\mathfrak{h}(\mathfrak{o})$ be the virtual subset of $\mathfrak{h}$, given by

$$\{h \in \mathfrak{h} \mid \mathrm{val}(h_{ij}) \geq 0\}.$$

- Let $\mathfrak{h}(\mathfrak{p})$ be the virtual subset of $\mathfrak{h}$, given by

$$\{h \in \mathfrak{h} \mid \mathrm{val}(h_{ij}) \geq 1\}.$$

- Let X_ϕ be an $n \times n$ matrix of terms such that $\mathrm{Basis}(X_\phi, \mathfrak{g}_\phi)$ holds, where $\mathfrak{g}_\phi$ is the root space corresponding to root ϕ.
- For any integer m, let $X_\phi \mathfrak{p}^m$ be the virtual set get given by

$$\{v \in \mathfrak{g} \mid \exists a\ (v = aX_\phi)\ \wedge\ (\mathrm{val}(a) = m)\}$$

Then we write:

$$\mathfrak{g}_F = \mathfrak{h}(\mathfrak{o}) \bigoplus\nolimits_{\phi(x)\in\mathbb{Z}} X_\phi \mathfrak{p}^{-\phi(x)} \bigoplus\nolimits_{\phi(x)\notin\mathbb{Z}} X_\phi \mathfrak{p}^{-\lfloor \phi(x) \rfloor}$$

$$\mathfrak{g}_F^+ = \mathfrak{h}(\mathfrak{p}) \bigoplus\nolimits_{\phi(x)\in\mathbb{Z}} X_\phi \mathfrak{p}^{-\phi(x)+1} \bigoplus\nolimits_{\phi(x)\notin\mathbb{Z}} X_\phi \mathfrak{p}^{1-\lceil \phi(x) \rceil}.$$

$\square$

Remark 79. Observe that if $\phi(x)$ is not an integer, then there exist integers n_1 and n_2 such that

$$n_1 < \phi(x) < n_2$$

holds.

$\Rightarrow$ $\lfloor \phi(x) \rfloor = n_1$ and $\lceil \phi(x) \rceil = n_2 \quad \Rightarrow \quad 1 - \lceil \phi(x) \rceil = 1 - n_2 = n_1$. Thus, if $\phi(x) \notin \mathbb{Z}$, then $X_\phi \mathfrak{p}^{-\lfloor \phi(x) \rfloor} = X_\phi \mathfrak{p}^{1-\lceil \phi(x) \rceil}$

Hence the quotient $\mathfrak{g}_F/\mathfrak{g}_F^+$ is nothing but

$$\mathfrak{h}(\mathfrak{o})/\mathfrak{h}(\mathfrak{p}) \bigoplus_{\phi(x)\in\mathbb{Z}} (X_\phi \mathfrak{p}^{-\phi(x)}/X_\phi \mathfrak{p}^{1-\lceil \phi(x) \rceil})$$

In the introduction to this chapter, we said that locally an affine apartment looks like a spherical apartment. By that we mean that the hyperplane structure of the spherical apartment of $\mathfrak{g}_x/\mathfrak{g}_x^+$ is the same as the local hyperplane structure in the affine apartment $\mathcal{A}$ at x. This can be seen in both our examples, see Figures 5.2 and 5.3. This is true in general as the preceeding remark suggests. We have a lemma:

Lemma 80. *Let $\mathfrak{g}$ be a reductive Lie algebra with affine apartment $\mathcal{A}$. Let $x \in \mathcal{A}$. Then $L_x(\mathfrak{f})$ is ismorphic to the generalized Levi subalgebra of $\mathfrak{g}(\mathfrak{f})$ corresponding to the root system Φ_x. Under this isomorphism, we have the following correspondence between the respective generators*

$$\mathfrak{p}^{-\alpha(x)} X_\alpha \;\mapsto\; \overline{X}_\alpha$$

$$\mathfrak{h}(\mathfrak{o}) \;\mapsto\; \mathfrak{h}(\mathfrak{f})$$

Proof. See Theorem 3.17 in [30]. □

Consider Figure 5.2. At each vertex, we see a hyperplane structure of the spherical apartment of $\mathfrak{sl}(3)$. In fact, as we see in Chapter 6, we do get $\mathfrak{sl}(3)$ at each vertex. We will see many instances of this lemma in the next two chapters.

6.0 NILPOTENT CONJUGACY CLASSES IN $\mathfrak{sl}(3, \mathbb{F})$.

We present this case as a prelude to $\mathfrak{g}_2$. Another reason for choosing this case is that the orbits of $\mathfrak{sl}(3, \mathbb{F})$ appear in the parameterization of the orbits of $\mathfrak{g}_2(\mathbb{F})$. We recall two main ideas from Chapter 5.

Theorem 81 (D.Barbash and A.Moy [4]). *If F is a facet and $\bar{\mathcal{O}} \subset L_F(\mathfrak{f}) = \mathfrak{g}_F/\mathfrak{g}_F^+$ is a distinguished nilpotent orbit, then there exists a unique nilpotent orbit in $\mathfrak{g}$ of minimal dimension which intersects the preimage of $\bar{\mathcal{O}}$ nontrivially.*

Theorem 82 (DeBacker [9, 11]). *Suppose p is sufficiently large. The map that sends $(F, \overline{\mathcal{O}}) \in I^d$ to the unique nilpotent G orbit of minimal dimension which intersects the pre-image of $\overline{\mathcal{O}}$ non-trivially induces a bijective correspondence $I^d/\sim \rightarrow \mathcal{O}(0)$, where $\mathcal{O}(0)$ is the set of nilpotent G-orbits in $\mathfrak{g}$.*

Fix an alcove in the affine building of $\mathfrak{sl}(3)$. We use DeBacker's equivalence relation in Theorem 77 to label the corresponding non-equivalent facets. Thus, we have:

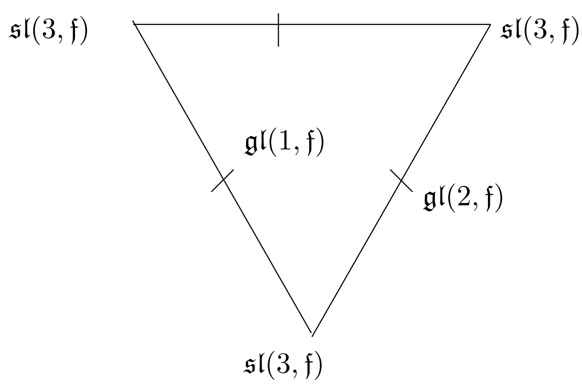

Figure 6.1: An alcove for $\mathfrak{sl}(3)$

F_1 : attached object $= \mathfrak{gl}_1^2(\mathfrak{f})$ $\qquad$ F_2 : attached object $= \mathfrak{gl}_2(\mathfrak{f})$,

F_3 : attached object $= \mathfrak{sl}_3(\mathfrak{f})$, $\qquad$ F_4 : attached object $= \mathfrak{sl}_3(\mathfrak{f})$,

F_5 : attached object $= \mathfrak{sl}_3(\mathfrak{f})$.

Since the parahorics and pro-unipotent radicals associated with each facet are purely p-adic, it makes sense to speak of 'definability' of these objects. We have already shown in Section 5.6 that they are definable in Pas's Language. However, here we show explicitly for this example that they are definable.

Consider the following basis elements for the root spaces. We will assign them our label and also include the familiar label.

$$Y_1 = Y_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_2 = Y_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_3 = Y_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Y_4 = Y_{-\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_5 = Y_{(\alpha+\beta)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_6 = Y_{-(\alpha+\beta)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

6.1 PARAHORICS AND PRO-UNIPOTENT RADICAL

Consider the following alcove of $\mathfrak{sl}(3)$. This diagram will guide us throughout these calculations.

Let $\mathfrak{h}$ be the Cartan sub-algebra of $\mathfrak{sl}(3, \mathbb{F})$ containing matrices of the type $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -(a_1+a_2) \end{pmatrix}$, where $a_1, a_2 \in \mathbb{F}$. We will now calculate the parahorics and pro-unipotent radicals for all facets.

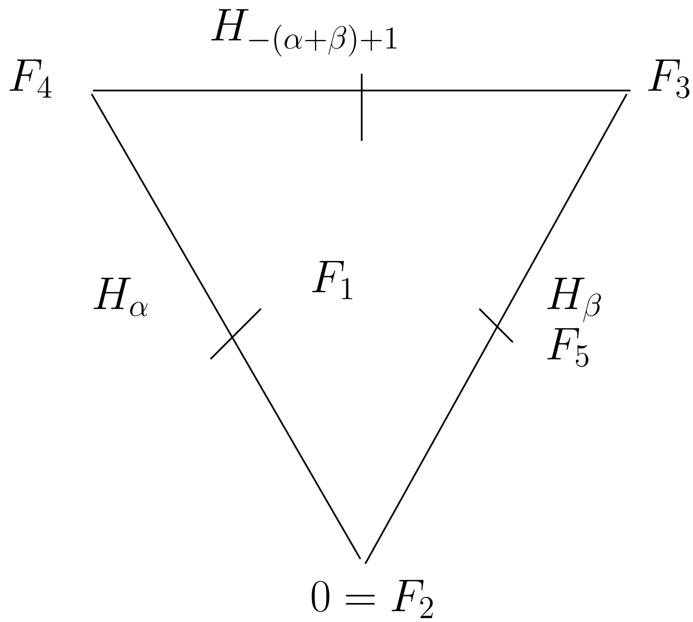

Figure 6.2: An alcove for $\mathfrak{sl}(3)$ and its hyperplanes

1. Facet F_1

   $x \in F_1$ means that x lies between the hyperplanes $H_{-(\alpha+\beta)+1}$ and $H_{\alpha+\beta}$.

$$\mathfrak{g}_{F_1} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \qquad \mathfrak{g}^+_{F_1} = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \end{pmatrix} \qquad L_{F_1}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & 0 & 0 \\ 0 & \mathfrak{f} & 0 \\ 0 & 0 & \mathfrak{f} \end{pmatrix}$$

2. Facet F_2

   $x \in F_2$ means that x lies on the hyperplanes H_α, H_β and $H_{\alpha+\beta}$.

$$\mathfrak{g}_{F_2} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \qquad \mathfrak{g}^+_{F_2} = \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \qquad L_{F_2}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

3. Facet F_3

   $x \in F_3$ means that x lies on the hyperplanes $H_{-(\alpha+\beta)+1}$, H_β and $H_{\alpha+1}$.

$$\mathfrak{g}_{F_3} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \qquad \mathfrak{g}^+_{F_3} = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \qquad L_{F_3}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

4. Facet F_4

$$\mathfrak{g}_{F_4} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}^{-1} & \mathfrak{o} \end{pmatrix} \qquad \mathfrak{g}^+_{F_4} = \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{p} \end{pmatrix} \qquad L_{F_4}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

5. Facet F_5

$$\mathfrak{g}_{F_5} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \qquad \mathfrak{g}^+_{F_5} = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \qquad L_{F_5}(\mathfrak{f}) = \begin{pmatrix} \mathfrak{f} & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} \\ 0 & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

Theorem 83. *The Lie algebra $\mathfrak{sl}(3,\mathbb{F})$, the parahorics and pro-unipotent radicals associated with each facet are definable in Pas's language.*

Proof. They are virtual sets given as follows:

- $\mathfrak{sl}(3,\mathbb{F})$ is the following virtual set

$$\{Y | \sum_{i=1}^{3} Y_{ii} = 0\}$$

  where Y is a 3×3 matrix of terms of the valued field sort.

- $\mathfrak{g}_{F_1} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V \; (v = h + u) \; \mathrm{val}(h_{ii}) \geq 0, u = \sum_{i=1}^{6} b_i x_i$ $(\mathrm{val}(b_1), \mathrm{val}(b_4), \mathrm{val}(b_5) \geq 0, \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_6) \geq 1)\}$

- $\mathfrak{g}^+_{F_1} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V \; (v = h + u) \; \mathrm{val}(h_{ii}) > 0, u = \sum_{i=1}^{6} b_i x_i$ $(\mathrm{val}(b_1), \mathrm{val}(b_4), \mathrm{val}(b_5) \geq 0, \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_6) \geq 1)\}$

- $\mathfrak{g}_{F_2} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) \geq 0\; u = \sum_{i=1}^{6} b_i x_i$
  $\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_4), \mathrm{val}(b_5), \mathrm{val}(b_6) \geq 0\}$

- $\mathfrak{g}^{+}_{F_2} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) > 0,\; u = \sum_{i=1}^{6} b_i x_i$
  $\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_4), \mathrm{val}(b_5), \mathrm{val}(b_6) \geq 1\}$

- $\mathfrak{g}_{F_3} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) \geq 0, u = \sum_{i=1}^{6} b_i x_i$
  $\mathrm{val}(b_3), \mathrm{val}(b_4) \geq 0 \wedge \mathrm{val}(b_2), \mathrm{val}(b_6) \geq 1 \wedge \mathrm{val}(b_1), \mathrm{val}(b_5) \geq -1\}$

- $\mathfrak{g}^{+}_{F_3} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) > 0, u = \sum_{i=1}^{6} b_i x_i$
  $(\mathrm{val}(b_1), \mathrm{val}(b_5) \geq 0 \wedge \mathrm{val}(b_3), \mathrm{val}(b_4) \geq 1\, \mathrm{val}(b_2), \mathrm{val}(b_6) \geq 2)\}$

- $\mathfrak{g}_{F_4} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) \geq 0, u = \sum_{i=1}^{6} b_i x_i$
  $(\mathrm{val}(b_1), \mathrm{val}(b_2) \geq 0 \wedge \mathrm{val}(b_3), \mathrm{val}(b_6) \geq 1 \wedge \mathrm{val}(b_4), \mathrm{val}(b_5) \geq -1)\}$

- $\mathfrak{g}^{+}_{F_4} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) > 0, u = \sum_{i=1}^{6} b_i x_i$
  $(\mathrm{val}(b_1), \mathrm{val}(b_2) \geq 1 \wedge \mathrm{val}(b_3), \mathrm{val}(b_6) \geq 2 \wedge \mathrm{val}(b_4), \mathrm{val}(b_5) \geq 0)\}$

- $\mathfrak{g}_{F_5} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) \geq 0, u = \sum_{i=1}^{6} b_i x_i$
  $(\mathrm{val}(b_1), \mathrm{val}(b_3), \mathrm{val}(b_4), \mathrm{val}(b_5) \geq 0 \wedge \mathrm{val}(b_2), \mathrm{val}(b_6) \geq 1)\}$

- $\mathfrak{g}^{+}_{F_5} = \{v \in V \,|\, \exists h \in \mathfrak{h}, u \in V\; (v = h + u)\; \mathrm{val}(h_{ii}) > 0, u = \sum_{i=1}^{6} b_i x_i$
  $(\mathrm{val}(b_1), \mathrm{val}(b_5) \geq 0\, \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_4), \mathrm{val}(b_6) \geq 1)\}.$

$\square$

6.2 DEFINABILITY OF NILPOTENT CONJUGACY CLASSES IN $\mathfrak{sl}(3)$

6.2.1 A Brief Outline of the strategy

Theorem 81 gives a convenient way of locating the required orbits. Fix a facet F. We start with a nilpotent distinguished element in the Lie algebra over the residue field. Then we consider its pre-image in $\mathfrak{g}_F$. The orbit we desire intersects this pre-image non-trivially but there may be more than one orbit intersecting this pre-image. We want the orbit with minimal dimension, (by dimension, we mean dimension of the orbit as an algebraic variety). There are many difficulties with this approach as it stands. Firstly, there are no quotients in Pas's language. Recall Formula 11 from Section 3.5.2. Secondly, we have no notion of an algebraic variety or its dimension as yet. Lastly, the number of conjugacy classes is dependent on the number of cubic classes in the residue field $\mathfrak{f}$. We overcome these difficulties as follows:

- We think of a pre-image of the distinguished nilpotent element in $L_F(\mathfrak{f})$ as the sum of two elements, one summand lies in the pro-unipotent $\mathfrak{g}_F^+$ and another is an element whose angular component is this distinguished nilpotent element.
- For the dimension argument we use the result $\dim \mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{g}^X$ [see Lemma 35 in Section 3.2] and argue that the elements in the orbit of minimal dimension have centralizers of maximal dimension. Centralizers are vector spaces over the valued field and we have a formula for the dimension of a vector space. The dimension of the orbit (hence that of the centralizer) does not change from $\mathbb{K}$ to $\mathbb{F}$.
- We extend Pas's language to include a finite number of variable symbols of the residue field type, see Section 2.2.3.

Theorem 84. *Let $\mathbb{F}$ be a p-adic field and $\mathfrak{f}$ its residue field such that char $\mathfrak{f} \neq 2, 3$. Let $\mathfrak{g}$ be the virtual set given by*

$$\{X \mid \sum_{i=1}^{3} X_{ii} = 0\}$$

where X is a 3×3 matrix of terms of the valued field sort. Let (F_i, e_i) be an equivalence class on the set of facets in the affine apartment of $\mathfrak{g}$ given by Definition 74, where F_i is a facet in a fixed alcove of the affine apartment of $\mathfrak{sl}(3)$ and e_i is a representative of a distinguished *nilpotent orbit in the object $L_{F_i}(\mathfrak{f})$. Then,*

$$\mathcal{O}(F_i, e_i) \qquad i = 1, \ldots, 5$$

is a nilpotent conjugacy class of $\mathfrak{g}$, where $\mathcal{O}(F_i, e_i)$ is definable in the extension of Pas's language $\mathcal{L}_{ext} = \mathcal{L} \bigcup \{\lambda_1, \ldots, \lambda_n\}$, where $\mathcal{L}_{ext}$ coincides with Pas's langugae $\mathcal{L}$ if $|\mathfrak{f}| \equiv 2 (\bmod 3)$. If $|\mathfrak{f}| \equiv 1 (\bmod 3)$, then $\mathcal{L}_{ext} = \mathcal{L} \bigcup \{\lambda_1, \lambda_2, \lambda_3\}$ [See Section 2.2.3].

Proof. $\forall\, x, y \in \mathfrak{g}$ let $x \sim y$ be the abbreviation of the formula

$$\exists Z\, (Z = Z_{ij}\ 1 \leq i, j \leq 3)\ (\det(Z) \neq 0)\ \wedge\ (ZxZ^{-1} = y)$$

1. Facet F_1

   Let $e_1 = \{0\}$ and let $\mathcal{O}(F_1, e_1)$ be the virtual set $0_{3\times 3}$.

   - For facets F_2 to F_5, we write a formula that will allow us to lift e_i to its pre-image in $\mathfrak{g}_{F_i}$. Let $\phi(e_i, v)$ be the formula

   $$\exists\, u, \tilde{e}_i \in \mathfrak{g}_{F_i},\ y \in \mathfrak{g}^+_{F_i}\ (u = y + \tilde{e}_i)\ \wedge\ (v \sim u)\ \wedge\ \overline{\mathrm{ac}}(\tilde{e}_i) = e_i$$

   - For facets F_2 to F_4, we write the following formula: $\phi(z_1, z_2, z_3, e_i, v)$ is the formula

   $$\phi(e_i, v)\ \wedge\ z_1 z_3 \neq 0.$$

2. Facet F_2

   Let e_2 be a matrix of terms of the residue field sort given by $\begin{pmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}$. For each $k = 1, \ldots, n$, let $\mathcal{O}_k(F_2, e_2)$ with $k = 1, \ldots, n$ be the virtual set given by

   $$\{v\ \in\ \mathfrak{g}\,|\ (\dim(\mathfrak{g}^v, 2))\ \wedge\ (\exists\, z_1, z_2, z_3\ \phi(z_1, z_2, z_3, e_2, v)\ \wedge\ (z_1(z_3)^2 = \lambda_k))\}$$

3. Facet F_3

   Let e_3 be the same as e_2. For each k, let $\mathcal{O}_k(F_3, e_3)$ be the virtual sets given by

   $$\{v\ \in\ \mathfrak{g}\,|\ \dim(\mathfrak{g}^v, 2)\ \wedge\ (v\ \notin \mathcal{O}(F_2, e_2)) \wedge (\exists\, z_1, z_2, z_3\, \phi(z_1, z_2, z_3, e_3, v)\ \wedge\ z_1(z_3)^2 = \lambda_k)\}$$

4. Facet F_4

   Let e_4 be the same as e_2. For each k, let $\mathcal{O}_k(F_4, e_4)$ be the virtual sets given by:

   $\{v\ \in\ \mathfrak{g}\,|\, v\ \notin\ (\mathcal{O}(F_2, e_2) \cup \mathcal{O}(F_3, e_3))\ \wedge\ \dim(\mathfrak{g}^v, 2)\ \wedge\ (\exists\, z_1, z_2, z_3\, (\phi(z_1, z_2, z_3, e_4, v)\ \wedge\ (z_1(z_3)^2 = \lambda_k))\}$

5. Facet F_5

Let e_5 be a matrix of terms of the residue field sort given by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Let $\mathcal{O}(F_5, e_5)$ be the virtual set given by

$$\{v \in \mathfrak{g} \,|\, (v \notin (\mathcal{O}(F_2, e_2) \cup \mathcal{O}(F_3, e_3) \cup \mathcal{O}(F_4, e_4)) \wedge \phi(e_5, v)\}$$

$\square$

7.0 NILPOTENT CONJUGACY CLASSES IN $\mathfrak{g}_2$

We now focus our attention on the exceptional Lie algebra $\mathfrak{g}_2$. It has rank 2. For its affine apartment, recall Figure 5.3 from Chapter 5. Once again, we consider the group action by its adjoint group G_2. The p-adic fields considered here have residue characteristic $\neq 2, 3$. Furthermore, we assume that $\mathbb{F}$ is complete and that $\mathfrak{f}$ is perfect.

7.1 G_2 AND $\mathfrak{g}_2$

Before we can talk about the definability of nilpotent conjugacy classes, we have to prove that G_2 and $\mathfrak{g}_2$ are definable.

7.1.1 The Lie Algebra $\mathfrak{g}_2$

Theorem 85. *The exceptional Lie algebra $\mathfrak{g}_2$ is definable.*

Proof. We realize $\mathfrak{g}_2$ as a sub-algebra of $\mathfrak{so}(7)$, see [22].

Let J be the matrix of terms given by

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & I_3 & 0 \end{pmatrix}$$

where I_3 is the 3 by 3 matrix with 1's on the diagonal and 0's elsewhere. Then, $\mathfrak{so}(7)$ is the virtual set given by

$$V = \{(a_{ij}) \in \mathfrak{gl}(7) \mid J(a_{ij}) + {}^t(a_{ij})J = 0\}$$

Definition 86. We define some notation that will be used throughout this chapter.

1. Let e_{ij} be a 7 by 7 matrix of terms with 1 as the entry in the ith row and jth column and 0 elsewhere.
2. Let $\mathrm{diag}(b_1,\, b_2,\, \ldots,\, b_n)$ be the notation for an n by n diagonal matrix whose diagonal entries are $b_1,\, b_2,\, \ldots,\, b_n$ respectively.
3. Define $d_n = e_{n+1,n+1} - e_{n+4,n+4}, 1 \leq n \leq 3$.

   Let $\mathfrak{h}$ be the virtual set given by

   $$\{(h_{ij} \in V \mid \exists_{\mathbb{F}} a_1, a_2(h_{ij} = a_1 d_1 + a_2 d_2 - (a_1 + a_2) d_3)\}$$

   This is a Cartan subalgebra. Every element in $\mathfrak{h}$ is diagonal and of the form

   $$\mathrm{diag}(0, a_1, a_2, -(a_1 + a_2), -a_1, -a_2,\, a_1 + a_2),$$

   where a_1, a_2 are variables of the valued field sort.
4. $X_1 = X_\beta = e_{3,2} - e_{5,6}$ $\quad X_2 = X_{-\beta} = e_{2,3} - e_{6,5}$ $\quad X_9 = e_{2,4} - e_{7,5}$

   $X_{10} = e_{4,2} - e_{5,7}$ $\quad X_{11} = e_{3,4} - e_{7,6}$ $\quad X_{12} = e_{4,3} - e_{6,7}$
5. Let λ be such that $\lambda^2 = 2$. The note below makes this more precise.

   $X_3 = X_\alpha = \lambda(e_{1,5} - e_{2,1}) - (e_{6,4} - e_{7,3})$ $\quad X_4 = X_{-\alpha} = \lambda(e_{1,2} - e_{5,1}) - (e_{3,7} - e_{4,6})$

   $X_5 = \lambda(e_{1,6} - e_{3,1}) + (e_{5,4} - e_{7,2})$ $\quad X_6 = \lambda(e_{1,3} - e_{6,1}) + (e_{2,7} - e_{4,5})$

   $X_7 = \lambda(e_{1,4} - e_{7,1}) - (e_{2,6} - e_{3,5})$ $\quad X_8 = \lambda(e_{1,7} - e_{4,1}) - (e_{5,3} - e_{6,2})$

Note. By $\lambda^2 = 2$ we mean that $\lambda = \sqrt{2}$, however 2 may not have a square root in $\mathbb{F}$. In other words, we are considering a quadratic extension as a vector space over $\mathbb{F}$. More precisely, let $v_1 = 1$ and $v_2 = \sqrt{2}$. Then we get $\sqrt{2}v_1 = v_2$ and $\sqrt{2}v_2 = 2v_1$.
The matrix representing this linear transformation is $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Hence $\lambda = \sqrt{2}$ is to be treated as an abbreviation of $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. In fact, each X_i is a 14 by 14 matrix. With this understanding, we leave X_i as they are. This affects neither the calculations nor the definability.

We define $\mathfrak{g}_2$ to be the virtual set given by

$$\{v \in V \mid \exists_{\mathbb{F}} h \in \mathfrak{h}, w \in V\ \big(\mathrm{span}(X_1, X_2, \ldots, X_{12}, w) \wedge (v = h + w)\big) \tag{7.1}$$

□

We write the basis elements explicitly since they will be used in the next section frequently. Recall that the roots of $\mathfrak{g}_2$ are $\{\alpha, -\alpha, \beta, -\beta, \alpha+\beta - (\alpha+\beta), 2\alpha+\beta, -(2\alpha+\beta), 3\alpha+\beta, -(3\alpha+\beta), 3\alpha+2\beta, -(3\alpha+2\beta)\}$ and $X_1, X_2, \ldots, X_{12}$ are the basis elements of the root spaces corresponding to these roots respectively as can be seen by the commutator relations at the end. The indices will respect this order the rest of the chapter. We will show what the first four X_i are. The rest follow from Item 4 in Definition 7.1.

$$X_1 = X_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_2 = X_{-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = X_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_4 = X_{-\alpha} = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To give a complete list, we have:

$X_1 = X_\alpha,\ X_2 = X_{-\alpha},\ X_3 = X_\beta,\ X_4 = X_{-\beta}$

$X_5 = X_{\alpha+\beta},\ X_6 = X_{-(\alpha+\beta)},\ X_7 = X_{2\alpha+\beta},\ X_8 = X_{-(2\alpha+\beta)}$

$X_9 = X_{3\alpha+\beta},\ X_{10} = X_{-(3\alpha+\beta)},\ X_{11} = X_{3\alpha+2\beta},\ X_{12} = X_{-(3\alpha+2\beta)}$

7.1.2 The group G_2

First, we write additional formulae that will enable us to talk about groups in Pas's Language.

7.1.2.1 List of Additional Formulae

1. Let G be a non-empty virtual set. Let Bin-Op$(G, *)$ be the abbreviation for '* is a binary operation on G'. In particular, * is a virtual function [Formula 30] from $G \times G \to G$ and we have

$$\forall g, h \in G \big(g * h \in G\big).$$

   Note: By definition, G is closed under the operation $*$.

2. We say that a binary operation * on G is associative if the following holds

$$\forall g_1, g_2, g_3 \in G \ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$$

   and denote it by Assoc-Bin$(G, *)$.

3. Let G be a non-empty virtual set. We say that G is a group with * as a binary operation if

$$\text{Bin-op}(G, *) \wedge \text{Assoc-Bin}(G, *) \wedge (\exists e \in G\, (\forall g \in G \ g * e = g = e * g))\wedge$$
$$(\forall g \in G, \ \exists g' \in G \ g * g' = g' * g = e)$$

   We denote this formula by Grp$(G, *)$.

4. Let H, G be non-empty virtual sets. SubGrp$(H, G, *)$ denotes the formula

$$(H \subset G) \wedge \text{Grp}(G, *) \wedge \text{Grp}(H, *)$$

7.1.2.2 Constructing G_2 We construct G_2 using a basis for the root spaces of $\mathfrak{g}_2$ and the following theorem. We quote this theorem without any background, our interest lies in choosing generators for G_2 that will allow us to construct the group in Pas's language.

Let G be a reductive group, T its maximal torus. We fix a Borel subgroup B of G containing T and set $N = N_G(T)$. Let $\mathcal{W}$ be the Weyl group of G given by $N_G(T)/T$.

Theorem 87 (Bruhat Decmposition). *: $G = \coprod_{\omega \in \mathcal{W}} B\omega B$ with $B\omega_1 B = B\omega_2 B$ iff $\omega_1 = \omega_2$ in $\mathcal{W}$ [21].*

Strategy for constructing G_2

- The generators for the unipotent radical of Borel group can be calculated by exponentiating $X_i(t)$. Since $X_i(t)$ are nilpotent, we do not have to worry about convergence. We denote them by $x_i(t)$. The $x_i(t)$ generate the root-groups U_i.
- We now find generators for the Weyl group. The Weyl group is D_{12}, the Dihedral group of order 12. It is a Coxeter group with generators σ_α and σ_β (representing simple reflections). These generators satisfy the relation

$$(\sigma_\alpha \sigma_\alpha)^6 = 1$$

We denote their matrix represenations by n_α and n_β respectively. They are calculated as follows [21, Section 27 and 28]:

For $\phi \in \Phi$, let $\varepsilon_\phi : \mathbf{G}_a \to U_\phi$ be the natural isomorphism, where $\mathbf{G}_a$ is the additive group $(\mathbb{F}, +)$. Then the matrix representation of $\varepsilon_\phi(t)$ is $x_\phi(t)$.

$$n_\alpha = \varepsilon_\alpha(1)\varepsilon_{-\alpha}(-1)\varepsilon_\alpha(1) \qquad n_\beta = \varepsilon_\beta(1)\varepsilon_{-\beta}(-1)\varepsilon_\beta(1) \tag{7.2}$$

In our notation, $n_\alpha = n_3 = x_3(1)x_4(-1)x_3(1)$ and $n_\beta = n_1 = x_1(1)x_2(-1)x_1(1)$

The generators for the root groups are obtained by exponentiating basis elements X_i. We write

$$x_i(t) = \exp(tX_i).$$

Since X_i are nilpotent, we do not have to worry about convergence. In fact, we get $X_i^3 = 0$. So

$$x_i = \exp(tX_i) = I + tX_i + \frac{t^2 X_i^2}{2}. \tag{7.3}$$

Moreover, if $X_i^2 \neq 0$, then the term $\frac{t^2 X_i^2}{2}$ contains a λ^2. Since λ^2 is 2, there are no denominators in x_i. For example, $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ are as follows:

$$x_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 & 1 \end{pmatrix}$$

$$x_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & 1 \end{pmatrix} \qquad x_4(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the basis elements X_i are nilpotent, the generators $x_i(t)$ are unipotent.

Definition 88.
- Let x_i, i odd be the generators $x_i(t)$ corresponding to the positive roots.
- Let $T(a, b)$ be given by $\mathrm{diag}(1, a, b, a^{-1}b^{-1}, a^{-1}, b^{-1}, ab)$.
- Let l_1 and l_2 take integer values from 1 to 6.

Then, the group G_2 is a virtual set given by:

$\{g \in \mathfrak{gl}(7) \mid \bigvee_{l_1,l_2} (\exists\ a_1, b_1, a_2, b_2, t_1, t_3, \ldots, t_{11},$
$(g = T(a_1, b_1)x_1(t_1)x_3(t_3)x_5(t_5)x_7(t_7)x_9(t_9)x_{11}(t_{11})n_1^{l_1}n_2^{l_2}T(a_2, b_2)))\}$

7.2 PARAHORICS AND PRO-UNIPOTENT RADICALS

This section is devoted to computing lattices corresponding to the parahorics and unipotent radicals associated with each facet. We follow the steps outlined in the $\mathfrak{sl}(3)$ case [see 6.1]. Our interest lies in their definability. Since their actual matrix form plays no role in the proof of the result, we will not show them for all facets. We will describe them as virtual sets in all cases and give their matrix form, as an example, in one case. Recall the definitions of parahoric and its pro-unipotent radical from Definition 71.

Parahoric: $\mathfrak{g}_F = \langle \mathfrak{h}(\mathfrak{o}), \mathfrak{p}^{-\lfloor \phi(x) \rfloor} X_\gamma \ : \forall \gamma \in \Phi \rangle$

Pro-unipotent: $\mathfrak{g}_F^+ := \langle \mathfrak{h}(\mathfrak{p}), \mathfrak{p}^{1-\lceil \phi(x) \rceil} X_\gamma \ : \forall \gamma \in \Phi \rangle$.

Object over the residue field: $L_F(\mathfrak{f}) = \mathfrak{g}_F / \mathfrak{g}_F^+$.

Fix an alcove and label the facets as follows.

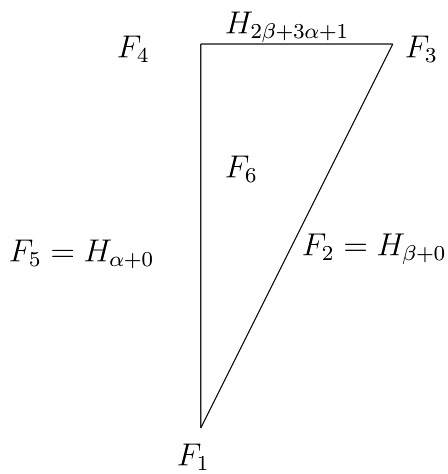

Figure 7.1: An alcove for $\mathfrak{g}_2$

1. <u>Facet F_1</u>

   $x \in F_1 \Rightarrow (x \in H_\alpha,\ x \in H_\beta,\ x \in H_{2\beta+3\alpha},\ x \in H_{\alpha+\beta},\ x \in H_{3\alpha+\beta},\ x \in H_{2\alpha+\beta})$.

   Facet F_1 is a vertex and all hyperplanes H_ϕ (where $\phi \in \Phi$) pass through it. Hence, all the matrix entries change when we go from $\mathfrak{g}_{F_1}$ to $\mathfrak{g}_{F_1}^+$ (except the seven places with zeroes, of course). We get $L_F(\mathfrak{f}) = \mathfrak{g}_2(\mathfrak{f})$

2. <u>Facet F_2</u>

Observe that $x \in F_2 \Rightarrow\ x \in H_{\beta+0} = H_{-\beta+0} \Rightarrow\ \pm\beta(x) = 0$

$$\mathfrak{g}_{F_2} = \begin{pmatrix} 0 & \lambda\mathfrak{p} & \lambda\mathfrak{p} & \lambda\mathfrak{o} & \lambda\mathfrak{o} & \lambda\mathfrak{o} & \lambda\mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & 0 & \mathfrak{o} & \mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & 0 & \mathfrak{p} \\ \lambda\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & 0 \\ \lambda\mathfrak{p} & 0 & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \lambda\mathfrak{p} & \mathfrak{p} & 0 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{o} & \mathfrak{o} & 0 & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \end{pmatrix} \qquad \mathfrak{g}_{F_2}^{+} = \begin{pmatrix} 0 & \lambda\mathfrak{p} & \lambda\mathfrak{p} & \lambda\mathfrak{o} & \lambda\mathfrak{o} & \lambda\mathfrak{o} & \lambda\mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 0 & \mathfrak{o} & \mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & 0 & \mathfrak{p} \\ \lambda\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 0 \\ \lambda\mathfrak{p} & 0 & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \lambda\mathfrak{p} & \mathfrak{p} & 0 & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \lambda\mathfrak{o} & \mathfrak{o} & \mathfrak{o} & 0 & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \end{pmatrix}$$

$$L_{F_2}(\mathfrak{f}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{f} & \mathfrak{f} & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{f} & \mathfrak{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathfrak{f} \end{pmatrix}$$

$\mathfrak{g}_{F_2} = \big\{v \in \mathfrak{g}_2 | \ \exists h \in h\,(v = h + \sum_{i=1}^{12} b_i x_i)\,\mathrm{val}(h_{ii}) \geq 0$

$\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_5), \mathrm{val}(b_7), \mathrm{val}(b_9), \mathrm{val}(b_{11}) \geq 0$

$\mathrm{val}(b_4), \mathrm{val}(b_6), \mathrm{val}(b_8), \mathrm{val}(b_{10}), \mathrm{val}(b_{12})\ \geq\ 1\big\}$

$\mathfrak{g}_{F_2}^{+} = \{\{v \in \mathfrak{g}_2 \mid \exists h \in h(v = h + \sum_{i=1}^{12} b_i x_i)\,\mathrm{val}(h_{ii})\ \geq\ 1$

$\mathrm{val}(b_3), \mathrm{val}(b_5), \mathrm{val}(b_7), \mathrm{val}(b_9), \mathrm{val}(b_{11}) \geq 0$

$\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_4), \mathrm{val}(b_6), \mathrm{val}(b_8), \mathrm{val}(b_{10}), \mathrm{val}(b_{12}) \geq 1\}$

3. <u>Facet F_3</u>.

Observe that $x \in F_3 \Rightarrow x \in H_\beta = H_{-\beta} \Rightarrow \pm\beta(x) = 0 \;\; x \in H_{2\beta+3\alpha-1},\, x \in H_{(3\alpha+\beta)-1}$

$$L_{F_3}(\mathfrak{f}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 \\ 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ 0 & 0 & 0 & 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \\ 0 & 0 & 0 & 0 & \mathfrak{f} & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

$\mathfrak{g}_{F_3} = \big\{ v \in \mathfrak{g}_2 \mid \exists h \in h(v = h + \sum_{i=1}^{1} 2b_i x_i)\, \mathrm{val}(h_{ii}) \geq 0$

$\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_3), \mathrm{val}(b_5), \mathrm{val}(b_7) \geq 0 \;\; \mathrm{val}(b_9), \mathrm{val}(b_{11}) \geq -1$

$\mathrm{val}(b_4), \mathrm{val}(b_6), \mathrm{val}(b_8), \mathrm{val}(b_{10}), \mathrm{val}(b_{12}) > 0 \big\}$

$\mathfrak{g}^+_{F_3} = \{ v \in \mathfrak{g}_2 \mid \exists h \in h(v = h + \sum_{i=1}^{1} 2b_i x_i)\, \mathrm{val}(h_{ii}) \geq 1$

$\mathrm{val}(b_3), \mathrm{val}(b_5), \mathrm{val}(b_7), \mathrm{val}(b_9), \mathrm{val}(b_{11}) \geq 0$

$\mathrm{val}(b_1), \mathrm{val}(b_2), \mathrm{val}(b_4), \mathrm{val}(b_6), \mathrm{val}(b_8) \geq 1 \, \mathrm{val}(b_{10}), \mathrm{val}(b_{12}) \geq 2$

4. Facet F_4

$x \in F_4 \Rightarrow x \in H_{-(2\beta+3\alpha)+1} = H_{2\beta+3\alpha-1}, x \in H_{\alpha+0} = H_{-\alpha+0}$

$$L_{F_4}(\mathfrak{f}) = \begin{pmatrix} 0 & \mathfrak{f} & 0 & 0 & \mathfrak{f} & 0 & 0 \\ \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{f} & \mathfrak{f} & 0 & 0 & \mathfrak{f} \\ 0 & 0 & \mathfrak{f} & \mathfrak{f} & 0 & \mathfrak{f} & 0 \\ \mathfrak{f} & 0 & 0 & 0 & \mathfrak{f} & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{f} & 0 & \mathfrak{f} & \mathfrak{f} \\ 0 & 0 & \mathfrak{f} & 0 & 0 & \mathfrak{f} & \mathfrak{f} \end{pmatrix}$$

Note that $\mathfrak{g}_{F_4}$ and $\mathfrak{g}^+_{F_4}$ are calculated as above.

5. Facet F_5

$x \in F_5 \Rightarrow x \in H_{\alpha+0}$

Hence, $L_{F_5}(\mathfrak{f}) = \begin{pmatrix} 0 & \mathfrak{f} & 0 & 0 & \mathfrak{f} & 0 & 0 \\ \mathfrak{f} & \mathfrak{f} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{f} & 0 & 0 & 0 & \mathfrak{f} \\ 0 & 0 & 0 & \mathfrak{f} & 0 & \mathfrak{f} & 0 \\ \mathfrak{f} & 0 & 0 & 0 & \mathfrak{f} & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{f} & 0 & \mathfrak{f} & 0 \\ 0 & 0 & \mathfrak{f} & 0 & 0 & 0 & \mathfrak{f} \end{pmatrix}$

Again $\mathfrak{g}_{F_5}$ and $\mathfrak{g}^+_{F_5}$ are calculated the same way.

6. Facet F_6

$x \in F_6 \Rightarrow (\phi{+}n)(x) \neq 0$ for any ϕ in the root system. Hence, $L_{F_6}(\mathfrak{f}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{f} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{f} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathfrak{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathfrak{f} \end{pmatrix}$

7.3 DISTINGUISHED NILPOTENT ORBITS IN OBJECTS $L_F(\mathfrak{f})$

Here we give a list of representatives of distinguished nilpotent orbits in each Lie algebra $L_F(\mathfrak{f})$. We also show how to get their pre-images in the parahoric sub-algebras of $\mathfrak{g}_2(\mathbb{F})$ using Lemma 80 from Chapter 5. We begin the discussion with $\mathfrak{g}_2(\mathfrak{f})$ [20].

1. $\mathfrak{g}_2(\mathfrak{f})$ class representatives

   Let $\overline{X}_1 = \overline{\mathrm{ac}}(X_1)$.

G_2-Class	Representative
1	0
A_1	$\overline{X}_1$
$\tilde{A}_1$	$\overline{X}_3$
$G_2(a_1)$ (subregular)	$\overline{X}_1 + \overline{X}_7$
	$\overline{X}_1 + \eta\overline{X}_7$
	$\overline{X}_1 + \mu\overline{X}_9$ or $\overline{X}_1 + \overline{X}_7 + \zeta\overline{X}_9$
G_2 (regular)	$\overline{X}_3 + \overline{X}_1$

where η is a fixed non-square in $\mathfrak{f}$

μ is a fixed non-cube in $\mathfrak{f}$ and ζ is such that $x^3 - 3x - \zeta$ is irreducible.

Note. For the subregular case, the two representatives correspond to the 2 cases where $q \equiv 1 (\bmod 3)$ or $q \equiv 2 (\bmod 3)$

Only the regular and subregular orbits are distinguished. Once again, we only give the matrix representation of e_1, a representative of the regular orbit.

Regular: Let e_1 be given by $\overline{X}_1 + \overline{X}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & \overline{\lambda} & 0 & 0 \\ -\overline{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Here $\overline{\lambda}$ is to be understood the same way λ was in Definition 86.

Notation 89.

- $e_1 = \overline{X}_1 + \overline{X}_3$
- $e_1^1 = \overline{X}_1 + \overline{X}_7$
- $e_1^2(\eta) = \overline{X}_1 + \eta\overline{X}_7$
- $e_1^3(\mu) = \overline{X}_1 + \mu\overline{X}_9$
- $e_1^4(\zeta) = \overline{X}_1 + \overline{X}_7 + \zeta\overline{X}_9$

Remark 90. Notice that the last subregular orbit has two representatives depending on whether $q \equiv 1 (\bmod 3)$ or $q \equiv 2 (\bmod 3)$. What if, for a particular p-adic field with $q \equiv 1 (\bmod 3)$, the second representative $e_1^4(\zeta)$ yields another subregular orbit different from the *intended* orbit, that is an orbit which does not intersect the pre-image of $e_1^3(\mu)$? This problem is avoided as follows. By quadratic reciprocity [see Lemma 95 in the Appendix], when $q \equiv 2 (\bmod 3)$, all elements of $\mathfrak{f}$ are cubes. Suppose we were looking at $\mathbb{F}$ such that $q \equiv 2 (\bmod 3)$. Then, the pre-image of $e_1^3(\mu)$ is empty since no non-cubes exist, whereas the pre-image of $e_1^4(\zeta)$ would give the appropriate orbit. Now suppose, we look at $\mathbb{F}$ such that $q \equiv 1 (\bmod 3)$, then the pre-image of $e_1^3(\mu)$ gives the required orbit, and the pre-image of $e_1^4(\zeta)$.

2. $\mathfrak{sl}(3)$

Recall, from Chapter 6, that the only distinguished nilpotent orbit in $\mathfrak{sl}(3)$ is its regular orbit. Consider $\begin{pmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}$ where $z_1, z_2,$ and z_3 are elements of the residue field with $z_1 z_3 \neq 0$. This is a representative for the regular orbit. We now need to lift it to $\mathfrak{g}_2(\mathbb{F})$. We use Lemma 80 to show how $\mathfrak{sl}(3)$ is embedded in $\mathfrak{g}_2$. We conclude from Figure 7.1 that $\mathfrak{sl}(3)$ corresponds to the simple root system $\{\beta, 2\beta + 3\alpha\}$. However, F_3 lies on $H_{2\beta+3\alpha-1}$ and not on $H_{2\beta+3\alpha}$. This is reflected in the map $\mathfrak{sl}(3) \rightarrow \mathfrak{g}_2$. Recall the basis for $\mathfrak{sl}(3)$ from Chapter 6. Thus we have, $\mathfrak{sl}(3) \rightarrow \mathfrak{g}_{F_3} \subset \mathfrak{g}_2$ by

$$Y_\beta \mathfrak{f} \rightarrow X_\beta \mathfrak{o} = X_1 \mathfrak{o}$$

$$Y_\alpha \mathfrak{f} \rightarrow X_{2\beta+3\alpha} \mathfrak{p}^{-1} = X_{11} \mathfrak{p}^{-1}.$$

Observe that X_1 and X_{11} are two of the six long roots in $\mathfrak{g}_2$. The remaining four Y_i get mapped to the remaining four long roots so that they preserve the respective commutator relations.

We call this map ρ_3.

- For each k, let

$$e_3^k(z_1, z_2, z_3) := e_3^k := \rho_3(z_1 Y_\alpha + z_2 Y_\beta + z_3 Y_{\alpha+\beta}). \tag{7.4}$$

3. $\mathfrak{gl}(2,\mathfrak{f})$

A representative of its distinguished nilpotent orbit is given by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We use Lemma 80 to show how $\mathfrak{gl}(2)$ is embedded in $\mathfrak{g}_2$. We see in Figure 7.1 that we get $\mathfrak{gl}(2)$ on two different facets F_2 and F_5; the respective local hyperplane structures correspond to two different roots. Observe that F_2 is contained in H_β and F_5 is contained in H_α.

Let ρ_2 be the map $\mathfrak{gl}(2,\mathfrak{f}) \to \mathfrak{g}_{F_2} \subset \mathfrak{g}_2$ given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_1 \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_2$$

Let ρ_5 be the map $\mathfrak{gl}(2,\mathfrak{f}) \to \mathfrak{g}_{F_5} \subset \mathfrak{g}_2$ given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_3 \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_4$$

4. $\mathfrak{so}(4,\mathfrak{f}) = A_1 \times \tilde{A}_1$

This vertex is at the intersection of hyperplanes H_α and $H_{(2\beta+3\alpha)-1}$ and $\mathfrak{so}(4,\mathfrak{f})$ is the Lie algebra corresponding to the root system $\{\alpha, -\alpha, 2\beta+3\alpha, -(2\beta+3\alpha)\}$. Thus $\mathfrak{so}(4,\mathfrak{f})$ is made of two copies of $\mathfrak{sl}(2,\mathfrak{f})$; one generated by $\overline{X}_\alpha$ and another by $\overline{X}_{2\beta+3\alpha}$. The fact that vertex F_4 is on the hyperplane $H_{(2\beta+3\alpha)-1}$ and not on $H_{(2\beta+3\alpha)}$ plays a role in the way $\mathfrak{so}(4,\mathfrak{f})$ embeds into $\mathfrak{g}_2(\mathbb{F})$. More precisely; we have:

$$\begin{aligned} \overline{X}_\alpha \mathfrak{f} &\mapsto X_\alpha \mathfrak{o} \\ \overline{X}_{2\beta+3\alpha}\mathfrak{f} &\mapsto X_{2\beta+3\alpha}\mathfrak{p}^{-1} \end{aligned}$$

Thus, let $\rho_4 : \ \mathfrak{so}(4,\mathfrak{f}) \longrightarrow \mathfrak{g}_2(\mathbb{F})$ be the map above.

We realize $\mathfrak{so}(4,\mathfrak{f})$ as the set of 4 by 4 matrices of the form $\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$, where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ B and C are skew-symmetric and are given by $B = \begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & c_2 \\ -c_2 & 0 \end{pmatrix}$. In other words, $\mathfrak{so}(4,\mathfrak{f})$ is the Lie algebra preserving the quadratic form given by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where I is the two by two identity matrix. To see that $\mathfrak{so}(4,\mathfrak{f})$ contains two copies of $\mathfrak{sl}(2,\mathfrak{f})$, we define the following Lie algebra isomorphism. For the first copy, let s_1 be the map

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \overline{X}_\alpha.$$

For the second copy, let s_2 be the map

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \overline{X}_{2\beta+3\alpha}.$$

Recall, in $A_1 = \mathfrak{sl}(2,\mathfrak{f})$ the distinguished orbit has as its representative: $\begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$ where ϵ is a square class in the residue field $\mathfrak{f}$.

Let $n_{sq} \in \mathfrak{so}(4)$ be $s_1\left(\begin{pmatrix} 0 & \mathrm{sq} \\ 0 & 0 \end{pmatrix}\right) \oplus s_2\left(\begin{pmatrix} 0 & \mathrm{sq} \\ 0 & 0 \end{pmatrix}\right)$, where sq is a square in $\mathfrak{f}$.

Let $n_{nsq} \in \mathfrak{so}(4)$ be $s_1\left(\begin{pmatrix} 0 & \mathrm{nsq} \\ 0 & 0 \end{pmatrix}\right) \oplus s_2\left(\begin{pmatrix} 0 & \mathrm{nsq} \\ 0 & 0 \end{pmatrix}\right)$, where nsq is a non-square in $\mathfrak{f}$.

Finally, let

$$e_4^1 = \rho_4(n) \quad \text{and} \quad e_4^2 = \rho_4(n) \tag{7.5}$$

7.4 DEFINABILITY OF NILPOTENT CONJUGACY CLASSES IN $\mathfrak{g}_2$

We are now ready to present the main result of this chapter. Consider the following alcove. Recall from Section 6.2.1 that, first for each facet F_i, we start with a distinguished nilpotent element in the Lie algebra $L_{F_i}(\mathfrak{f})$. We lift it to its pre-image in the corresponding parahoric and obtain all orbits intersecting this pre-image. We will now write a formula (rather, a template) capturing this idea. This formula works for all facets.

Let $i = 1, 2, \ldots, 6$ be the index running over all facets.

$$\text{Let } j = \begin{cases} 0,1,2,3,4 & \text{if } i = 1; \\ 1,2 & \text{if } i = 4; \\ 1,2,\ldots,n & \text{if } i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

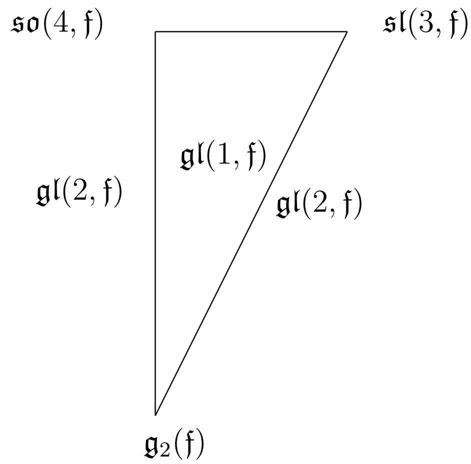

Figure 7.2: An alcove for $\mathfrak{g}_2$

Notation 91. 1. Let $x \sim y$ be the abbreviation of $\exists H \in G_2 \ (HxH^{-1} = y)$.

2. When we write $\overline{ac}(w) = b$, we mean $\overline{ac}(w_{l,m}) = b_l$, where $l, m = 1, \ldots, 7$.

3. By e_i^j, we mean a representative of a distinguished nilpotent orbit in $L_{F_i}(\mathfrak{f})$ as defined in Section 7.3.

4. Let $\phi(e_i^j, v)$ be the formula

$$\exists\, u, \tilde{e}_i^j \in \mathfrak{g}_{F_i}, \exists\, y \in \mathfrak{g}_{F_i}^+ \ (u = y + \tilde{e}_i^j) \wedge (\overline{ac}(u) = e_i^j) \wedge (\mathrm{val}(\tilde{e}_i^j) = 0)\,(v \sim u)$$

5. Let non-sq(γ) be the formula

$$\gamma \neq 0 \wedge (\nexists \xi)(\xi^2 = \gamma).$$

6. Let non-cb(γ) be the formula

$$\gamma \neq 0 \wedge (\nexists \xi)(\xi^3 = \gamma).$$

7. Let poly(γ) be the formula

$$\gamma \neq 0 \wedge (\nexists \xi)(\xi^3 - 3\xi - \gamma = 0).$$

Theorem 92. *Let $\mathbb{F}$ be a p-adic field and $\mathfrak{f}$ its residue field. Let char $\mathfrak{f}$ be sufficiently large, set $p > 16$. Let $\mathfrak{g}_2$ be the virtual Lie algebra given by Equation 7.1. Let (F_i, e_i) be an equivalence class on the set of facets in the affine apartment of $\mathfrak{g}_2$ given by Definition 74, where F_i is a facet in a fixed alcove of the affine apartment of $\mathfrak{g}_2$ and e_i is a representative of a distinguished nilpotent orbit in the object $L_{F_i}(\mathfrak{f})$. Let i and j be as in Notation 91. Then,*

$$\mathcal{O}(F_i, e_i^j)$$

is either empty or a nilpotent conjugacy class of $\mathfrak{g}$, where $\mathcal{O}(F_i, e_i^j)$ is definable in the extension of Pas's language $\mathcal{L}_{ext} = \mathcal{L} \bigcup \{\lambda_1, \ldots, \lambda_n\}$, where $\mathcal{L}_{ext}$ coincides with Pas's langugae $\mathcal{L}$ if $|\mathfrak{f}| \equiv 2 (\bmod 3)$. If $|\mathfrak{f}| \equiv 1 (\bmod 3)$, then $\mathcal{L}_{ext} = \mathcal{L} \bigcup \{\lambda_1, \lambda_2, \lambda_3\}$ [See Section 2.2.3].

Proof. Recall that an object is definable if it is of the form

$$\{x \mid \phi(x)\}$$

where ϕ is a formula in Pas's language. Also, recall that from Section 7 that $\mathfrak{g}_2$ and $\mathfrak{g}_{F_1}$, $\mathfrak{g}^+_{F_1}$, $\mathfrak{g}_{F_2}$, $\mathfrak{g}^+_{F_2}$, $\mathfrak{g}_{F_3}$, $\mathfrak{g}^+_{F_3}$, $\mathfrak{g}_{F_4}$, $\mathfrak{g}^+_{F_4}$, $\mathfrak{g}_{F_5}$, $\mathfrak{g}^+_{F_5}$, $\mathfrak{g}_{F_6}$, $\mathfrak{g}^+_{F_6}$ are definable. As before the number of orbits may be more than the number of facets.

1. Facet F_1

   The Lie algebra associated with this facet is $\mathfrak{g}_2(\mathfrak{f})$. Recall that it has 4 distinguished orbits: 1 regular and 3 subregular. Let e_1^0 be a matrix of terms of the residue field sort as given by Notation 89 from Section 7.3.

   Then $\mathcal{O}(F_1, e_1^0)$ is the virtual set given by

   $$\left\{v \in \mathfrak{g}_2 \,|\, \phi(e_1^0, v)\right\}.$$

   This is the regular orbit in $\mathfrak{g}_2$. Let $e_1^1, e_1^2, e_1^3, e_1^4$ be matrices of terms of the residue field sort respectively, given by Notation 89 from Section 7.3.

   Let $\mathcal{O}(F_1, e_1^1)$ be the virtual set given by

   $$\{v \in \mathfrak{g}_2 \,|\, \phi(e_1^1, v) \wedge (\dim(\mathfrak{g}^v, 4))\}$$

   Let $\mathcal{O}(F_1, e_1^2)$ be the virtual set given by

   $$\left\{v \in \mathfrak{g}_2 | \, \exists\, \eta(\text{non-sq}(\eta)) \; \phi(e_1^2(\eta), v) \; \wedge \dim(\mathfrak{g}^v, 4) \wedge (\neg\phi(e_1^1, v))\right\}$$

For the next virtual set, we refer to Remark 90. Let $\mathcal{O}(F_1, e_1^3, e_1^4)$ be the virtual set given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \Big(\big(\exists \, \mu \text{ non-cb}(\mu) \wedge \phi(e_1^3(\mu), v) \wedge (\nexists \, t. \, t^2 = -3)\big) \vee$$
$$\big(\nexists \mu. \text{ non-cb}(\mu) \, \exists \, \zeta \text{ poly}(\zeta) \, \phi(e_1^4(\zeta), v)\big)\Big) \wedge \dim(\mathfrak{g}^v, 4) \wedge (\neg\phi(e_1^1, v)) \wedge (\neg\phi(e_1^2, v))\Big\}$$

2. <u>Facet F_2</u>

   Recall that the object associated with this facet is $\mathfrak{gl}(2, \mathfrak{f})$. Let e_2^0 be a matrix of terms of the residue field sort as given in Section 7.3.

   Then, $\mathcal{O}(F_2, e_2^0)$ is the virtual set given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \phi(e_2^0, v) \wedge \dim(\mathfrak{g}^v, 8) \wedge (\textstyle\bigwedge_{j'=0,\ldots,4} \neg\phi(e_1^{j'}, v))\Big\}$$

3. <u>Facet F_3</u>

   Recall that the object associated with this facet is $\mathfrak{sl}(3, \mathfrak{f})$. Let e_3^k be a matrix of terms of the residue field sort as given in Equation (7.4) in Section 7.3.

   For each k, let $\mathcal{O}(F_3, e_3^k)$, where $k = 1, \ldots, n$ be the virtual sets given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \exists z_1, z_2, z_3 \ \dim(\mathfrak{g}^v, 4) \wedge \phi(e_3^k(z_1, z_2, z_3), v) \wedge \overline{\text{ac}}(z_1 z_3^2) = \lambda_k)\Big\}$$

4. <u>Facet F_4</u>

   Recall that the object associated with this facet is $\mathfrak{so}(4, \mathfrak{f})$. Let e_4^1, e_4^2 be matrices of terms of the valued field sort given by Equation (7.5).

   Let $\mathcal{O}(F_4, e_4^1)$ be the virtual set given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \phi(e_4^1, v) \wedge (\dim(\mathfrak{g}^v, 4)) \wedge (\neg\phi(e_2^0, v)) \wedge (\textstyle\bigwedge_{j'=0,\ldots,4} \neg\phi(e_1^{j'}, v))\Big\}$$

   Let $\mathcal{O}(F_4, e_4^2)$ be the virtual set given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \phi(e_4^2, v) \wedge (\dim(\mathfrak{g}^v, 4)) \wedge (\neg\phi(e_2^0, v)) \wedge (\textstyle\bigwedge_{j'=0,\ldots,4} \neg\phi(e_1^{j'}, v))\Big\}$$

5. <u>Facet F_5</u>

   Recall that the object associated with this facet is $\mathfrak{gl}(2, \mathfrak{f})$. Let e_5^0 be a matrix of terms of the residue field sort as given in Section 7.3.

   Let $\mathcal{O}(F_5, e_5^0)$ be the virtual set given by

$$\Big\{v \in \mathfrak{g}_2 \,|\, \phi(e_5^0, v) \wedge (\neg\phi(e_4^1, v) \wedge (\neg\phi(e_4^2, v) \wedge (\neg\phi(e_2^0, v)) \wedge (\textstyle\bigwedge_{j'=0,\ldots,4} \neg\phi(e_1^{j'}, v))\Big\}.$$

6. <u>Facet F_6</u>

   Recall, that Lie algebra associated with this facet is $\mathfrak{gl}(1, \mathfrak{f})$. The only distinguished orbit in $\mathfrak{gl}(1)$ is the 0-orbit. The orbit $\mathcal{O}(F_6, 0)$ is the virtual containing the zero-element, i.e., 0. Let

$$\mathcal{O}(F_6, e_6^0) = 0$$

Each $\mathcal{O}(F_i, e_i^j)$ gives a nilpotent conjugacy classes of $\mathfrak{g}_2$. Thus nilpotent conjugacy classes of $\mathfrak{g}_2$ are definable in $\mathcal{L}_{ext}$. $\square$

8.0 CONCLUSION

In this thesis we focussed on $\mathfrak{so}(n)$, $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$ because, while they differ in their treatment, they have all been already parameterized. What made the definability of their nilpotent conjugacy classes possible?

The parameterization of the orbits in $\mathfrak{so}(n)$ was based on the concepts of quadratic forms. We do not have a theory of quadratic forms in Pas's language but this difficulty could be overcome using an appropriate matrix representation of the quadratic form on $\mathfrak{so}(n)$. Similar parameterizations exist for $\mathfrak{sp}(2n)$ and $\mathfrak{u}(n)$. It is not hard to show that the orbits of $\mathfrak{sp}(2n)$ are definable, the proof would be similar to the $\mathfrak{so}(n)$ case. However, we face a problem with $\mathfrak{u}(n)$. Unitary Lie algebras are, in general, not definable in Pas's language. We would prefer not to restrict ourselves to only those p-adic fields that have $\sqrt{-1}$. For a more universal approach, we could "define" $\sqrt{-1}$ the way we defined $\sqrt{2}$ in Note 86. With this in mind we hazard a conjecture:

Conjecture 93. *With appropriate restrictions on p, the nilpotent conjugacy classes of the Unitary Lie algebra $\mathfrak{u}(n)$ are definable in Pas's language.*

Before we discuss the use of affine apartments in the second half of the thesis; we recall that we realized $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(7)$. Since a parameterization and result for $\mathfrak{so}(7)$ already exist, we could consider studying how the orbits of $\mathfrak{so}(7)$ intersect those of $\mathfrak{g}_2$ and use the result. (This was a suggestion by Kay Magaard.)

The cases $\mathfrak{sl}(3)$ and $\mathfrak{g}_2$ are treated differently. Their orbits are parameterized using affine apartments, these are not purely p-adic objects but the parahoric and pro-unipotent radical Lie algebras are p-adic objects. The examples of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ show that the number of orbits depends on the number of square and cubic classes in the residue field $\mathfrak{f}$ respectively.

This leads to a conjecture

Conjecture 94. *Let $\mathbb{F}$ be a p-adic field with a sufficiently large p. Then, for every n, there exists an extension of Pas's language $\mathcal{L}_{ext} = \mathcal{L} \cup \{\gamma_1, \ldots, \gamma_m\}$ such that the nilpotent conjugacy classes of $\mathfrak{sl}(n)$ are definable. Here γ_i are constant-symbols of the residue field sort.*

The result due to Barbasch and Moy [see Theorem 81] allows us to lift the distinguished orbits over the residue field to an orbit over the p-adic field. One of the important facts assisting us in definability of orbits in $\mathfrak{g}_2$ was that the dimensions of the centralizers of all orbits were known. From $\mathbb{K}$ to $\mathbb{F}$, an orbit may split into more orbits but the dimension of the centralizer remains the same. We used this knowledge thus by-passing the problem of constructing a formula that says that the dimension of a vector space is minimal/maximal.

Can this result be extended to other semisimple Lie algebras of higher ranks? The answer is 'yes' for classical Lie algebras. With exceptional Lie algebras F_4, E_6, E_7, E_8, over an algebraically closed field, the orbits have been parameterized in terms of the conjugacy classes of Levi algebras [3]. However, we are faced with the issue of definability of the algebras. One way of constructing F_4 requires using Octonions and their isometries; another requires Octonions and its projective plane [1]. In order to define projective planes in Pas's language, we need the notion of quotient spaces. We have tackled this issue using orthogonal complements [see formula 11 in Section 3.5.2]. It appears that we could answer in the affirmative for F_4.

The use of Pas's language to reformulate p-adic representation theory gives rise to many important directions for research. Is this language powerful enough to express other representation theoretic objects? We take many such objects for granted: the notion of roots is entrenched in the study of Lie algebras. Roots however are p-adic and real number objects.

APPENDIX

A LEMMA ABOUT QUADRATIC RESIDUES

Lemma 95. *Let p be an odd prime not equal to 3 and let q be a power of p. Then -3 is a non-square in $\mathfrak{f} = \mathbb{F}_q$ iff $q \equiv 2 \,(\mathrm{mod}\, 3)$.*

Proof. First note that there are two possibilities for the congruence class of p modulo 3, $p \equiv 1\,(\mathrm{mod} 3)$ or $p \equiv 2\,(\mathrm{mod}\, 3)$.

If $p \equiv 1\,(\mathrm{mod}\, 3)$, then $q = p^n \;\Rightarrow\; q \equiv 1^n = 1\,(\mathrm{mod} 3)$. Whereas, if $p \equiv 2\,(\mathrm{mod} 3)$, then the congruence class of q modulo 3 is determined by the exponent of p. If q is an odd power of p, we get $q = p^{2n+1} = (p^2)^n p \equiv (2^2)^n 2 \equiv 1^n 2 \equiv 2(\mathrm{mod}\, 3)$. If q is an even power of p, then $q = p^{2n} = (p^2)^n \equiv (2^2)^n \equiv 1^n \equiv 1(\mathrm{mod}\, 3)$.

Thus, $q \equiv 2\,(\mathrm{mod}\, 3) \;\Rightarrow\; p \equiv 2\,(\mathrm{mod}\, 3)$.

<u>Case 1</u> $q \equiv 2\,(\mathrm{mod} 3)$.

Let $\left(\frac{a}{b}\right)$ be the Legendre symbol, where b is any prime and a any integer.

$$\begin{aligned}
\left(\frac{-3}{p}\right) &= \left(\frac{3}{p}\right)\left(\frac{-1}{p}\right) \\
&= \left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}} \\
&= \left(\frac{p}{3}\right)(-1)^{p-1} \\
&= \left(\frac{2}{3}\right)(-1)^{p-1} \\
&= \left(\frac{2}{3}\right) = -1
\end{aligned}$$

This says that -3 is not a square in $\mathbb{F}_p$. Since $\mathbb{F}_q$ is an odd-power extension of $\mathbb{F}_p$, we can conclude that -3 is not a square in $\mathbb{F}_q$.

Case 2 $q \equiv 1 \,(\text{mod} 3)$. This can happen in two ways $p \equiv 2 \,(\text{mod} 3)$ (with q an even power of p). Even though, -3 is not a square in $\mathbb{F}_p$, it is a square in $\mathbb{F}_q$, since $q = p^{\text{even}}$.

Another possibility is $p \equiv 1 \,(\text{mod} 3)$. Then using our earlier computation, we get

$$\begin{aligned} \left(\frac{-3}{p}\right) &= \left(\frac{p}{3}\right)(-1)^{p-1} \\ &= \left(\frac{1}{3}\right) = 1 \end{aligned}$$

Thus we have -3 is a non-square in $\mathfrak{f} = \mathbb{F}_q$ iff $q \equiv 2 \,(\text{mod} 3)$. $\square$

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