EQUILIBRIUM AND EXPLANATION IN 18TH CENTURY MECHANICS

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The received view of the Scientific Revolution is that it was completed with the publication of Isaac Newton's (1642-1727) *Philosophiae Naturalis Principia Mathematica* in 1687. Work on mechanics in the century or more following was thought to be merely filling in the mathematical details of Newton's program, in particular of translating his mechanics from its synthetic expression into analytic form. I show that the mechanics of Leonhard Euler (1707–1782) and Joseph-Louis Lagrange (1736–1813) did not begin with Newton's Three Laws. They provided their own beginning principles and interpretations of the relation between mathematical description and nature. Functional relations among the quantified properties of bodies were interpreted as basic mechanical connections between those bodies. Equilibrium played an important role in explaining the behavior of physical systems understood mechanically. Some behavior was revealed to be an equilibrium condition; other behavior was understood as a variation from equilibrium. Implications for scientific explanation are then drawn from these historical considerations, specifically an alternative account of mechanical explanation and unification. Trying to cast mechanical explanations (of the kind considered here) as Kitcher-style argument schema fails to distinguish legitimate from spurious explanations. Consideration of the mechanical analogies lying behind the schema are required.

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1.0 INTRODUCTION

The superposition of equilibria in mechanics is a principle as fecund as the superposition of figures in geometry. —J. L. Lagrange[1]

We now regard Newton's laws as the synthesis of all that had gone before, but the 18th century mathematicians seldom looked at it that way. —Thomas Hankins[2]

This thesis describes changes in mechanics that occurred during the 18th century, changes which accompanied the shift from geometry to analytic calculus as the dominant formalism for expressing and investigating truths about nature. The Scientific Revolution was begun, in as much as we can say such things have beginnings and endings, with the rejection of scholasticism and its world view based in Aristotelian cosmology. The scholastics explained the motion of natural bodies in terms of the ordering of everything in the cosmos. The Earth was at the center and the sun and the heavens above the moon; the four sub-lunar elements all moved in straight lines toward their properly ordered locations, the quintessences moved in perfect circles around the center above the moon, and all moved according to their natures.

The Aristotelian cosmos was abandoned for many reasons, but Copernicanism was easily the most significant, removing, as it did, the center and foundation of the Aristotelian explanatory framework. The project of mechanics became one of a simultaneous search for new descriptions and new explanations of the nature and motion of bodies. The search was a complex interplay of empirical observation, metaphysical reasoning, mathematical innovation and social and cultural contingencies. Its chief result however has been characterized as the mathematization of nature. In

[1]Lagrange (1788, 13).
[2]Hankins (1970, 152).

order to avail themselves of a mathematical mechanics the opponents of scholasticism needed some accompanying theory of the relation between mathematical description, explanation and nature. The new mechanics was not to be merely mathematical but a new empirical description of the world.

One view of the mathematics-nature relation, held for example by Isaac Newton (1642-1727), is that mathematics, in particular geometry, can describe nature because it can graphically represent naturally generated trajectories. The most crucial and innovative techniques of Newton's *Philosophiae Naturalis Principia Mathematica* (Newton, 1687) are those for dealing with continually varying geometric curves and their generation, treating the curves as the results of motions generated by forces. Newton's new mathematical methods were constructed in order to deal with the infinitesimally short motions he understood as making up the generation of curvilinear trajectories. Those methods, coupled with axioms that related the infinitesimal motions to forces—his Laws of motion—enabled Newton to derive curves from particular force laws. The primary object of mathematical investigation was therefore the curve, understood as a trajectory generated by motion. It was only in the not-so-straightforward way just described that Newton's mathematical language described forces.

On the other hand, developments in the methods of analysis during the 18th century by the likes of Leonhard Euler (1707–1783) and Joseph-Louis Lagrange (1736–1813) came to emphasize instead the importance of functions as the primary object of mathematical investigation. Functions were seen to lie behind both geometric and algebraic representations. The representation of nature by a function achieves something more than does the mere pictorial capabilities of geometry. First, when an *equation* is invoked as a representation of a physical process, a new kind of explanation is introduced as well as an abstract representation of quantities. The equation is explanatory in one sense because it gives an analytic form of the trajectory and so reveals, as Euler put it, the nature of the curve. But Euler also means something more by this. An analytic equation and geometric curve both represent the same underlying object, namely a function. If a physical system shares a mathematical description with some other system such as the balance then they also share the underlying functional description. This relation between systems through analogy of mathematical description is explanatory insofar as there is a mechanical understanding of one case which is taken to be self-evident. The mechanical workings of that self-evident case are shown to be an instantiation of the function, now understood as describing an inferential structure among physical quantities. In particular, cases to which analogies are formed are taken to have the self-evidently explanatory force of either symmetry (though not so named), conservation of state or the minimization of quantities. Functions which describe those systems either equate to zero or can be shown, through the new calculus of variations, to have stationary values. This kind of reasoning thus introduced into 18th century mechanics concepts seminal for classical and modern physics.[3]

The beauty and elegance of analytic mechanics is due greatly to notation, its most conspicuous feature, and it may be notational conspicuousness that led to an underestimation of the rôle this approach to physics had in shaping our modern scientific view. Today, the designations 'classical mechanics' and 'Newtonian physics' are used interchangeably. As far as mechanics is concerned, historians of the 18th century seem to take the product of the Scientific Revolution to be all but identical with Newtonian physics, making allowance only for notational advances offered by the new differential calculus. At best, analytic mechanics is seen as discovering mathematical implications of Newtonian laws—a kind of completion of his physics to be sure, but not in any significant way changing it.

This view of theoretical advances in the 18th century as mere notational re-expression of the *Principia* takes it for granted that any change in formalism is merely notational and that notational change can be had without conceptual consequences. This may be plausible from a modern view point, post relativity, gauge symmetry, the equivalence of matrix and wave mechanics, and post a century of formal axiomatic systems. Any modern definition of a formal system begins with laying down the formal grammar as a matter of conventional choice. But formal systems are one thing and "science on the hoof" is another. The conceptual apparatus employed when grappling with understanding phenomena surely has an influence on at least the initial form that understanding will take.

A comparison of the foundations which Euler provided for the science of mechanics in his *Mechanica* with those of Newton's *Principia* reveals that the relation between the two was far more

[3]The mechanics of the 18th century has been referred to as rational mechanics and analytic mechanics. Newton also refers to his system of natural philosophy as rational mechanics in his first preface to *Principia*. I prefer, therefore, not to use the term rational as I intend to emphasize differences with Newton's mechanics, and also for the fact that the "rational" moniker tends to obscure the empirical content of 18th century mechanics. Analytic is the word of choice for those in the 18th century who see themselves as casting mechanics in a new form. Leonhard Euler, for example, stresses that his approach in *Mechanica* is analytic, as opposed to the synthetic methods of the ancients used by Newton and his generation.

complex than a mere translation. Euler did not begin with Newton's laws and derive analytic results from those. Rather, he re-conceived the entire science from top to bottom, taking the occupation of space to be the essential feature of bodies from which the rest of mechanics could be derived necessarily. Mechanics, before him, had been too groundlessly asserted, in his estimation. Newton's laws can be found in *Mechanica*, but they have been deconstructed and their content spread across various theorems and corollaries. Their fundamental role, which for Newton connected his mathematics with natural, dynamic phenomena, had been obviated by Euler's founding his mechanics on a functional conception of the underlying mathematics. Functions provided the necessary explanatory underpinning that allowed mathematics to describe and explain nature. Euler was, in fact, transitional in this regard as he split forces into those he treated functionally (*potentiae*) and those he still maintained a "metaphysical" commitment to, in particular *vis inertia*. Lagrange, on the other hand, found all the mechanical explanation he needed in the functions alone.

So it is implausible that how mechanics was expressed, at a time when the content of the theory itself was up for grabs, did not influence how it was conceived of and vice versa. Debates at the time about the foundations of the *Principia*, and about mechanics as a whole, are well known and tended to fall along the same lines as the geometry versus calculus debates.[4] As for the 18th century being a period of chasing down mathematical implications, we will see this is a description completely inadequate to the complexity of the works and their foundations. It was more often the case that Newton's laws were seen as following *from* the more fundamental discovered principles and not the reverse. What made these principles more fundamental, such as the Principle of Virtual Velocities or the Principle of the Lever for Lagrange, was their direct explanatory rôle in cases whose real behavior was taken to be self-evident. The balance, for example, is a paragon of equilibrium. If the weights and their distances from the fulcrum are equal then the balance must be at equilibrium. If a more complicated case can be reduced, through mathematical means or otherwise, to a balance, then that case will thereby be explained *as a kind of equilibrium*. Lagrange's block-and-tackle model of a mechanical system is a perfect example of this method (see Chapter 4.) A physical system of bodies and forces acting on one another is re-conceived as a system of weights and pulleys. The mathematical description of the latter—the *equation* derived to describe its behavior—then reveals the equivalence of the original system to a lever. Further examples from

[4] See, especially, Part Two of (Guicciardini, 1999).

Lagrange show how he extended this notion to cases of dynamic equilibrium, where balance-like equations express equilibrium conditions but nonetheless include motions. Note that this method of grounding complicated cases in self-evident ones is at once both an argument from experience with simple mechanisms and a logical deduction, a feature obscured by the label "rational" mechanics.

There were also debates at the same time over the appropriateness of the new analytic calculus, as opposed to pure geometry, for physics. Both sides of the debate seemed to agree that the geometer himself plays a greater role, in some sense, in the reasoning when arriving at synthetic demonstrations. Where they differed was in evaluating the merits of that greater role. I hope to cast light on this difference by pointing out differences in the formalisms and their capabilities for modeling nature. As alluded to with Newton, there is an emphasis on trajectory in geometrical modeling, while the analytic approach modeled physical systems through functions governing their behavior. This difference made the analytic solutions more easily transported to other problems. The equation/function model represents more of the mechanical structure of the physical system in that it contains more place-holders for features relevant to the solution of the problem, these place-holders being the variables. All of the constraints among the properties required for a complete description of the mechanical system came to be represented in the equations. There was no need for the brilliant but complicated sorts of geometric auxiliary structures I describe Newton as employing. (See Chapter 5.)

This representational or descriptive difference was connected with a difference in the explanation of mechanical phenomena. As already described, an emphasis on equation (principle or function) as opposed to trajectory (path or curve), gave rise to the increased importance of equilibrium. This led to an emphasis on entities rather than activities in explanations with functions describing connections among states of physical systems. The argumentative and explanatory force of the equilibrium picture stems from the system sharing a mathematical characteristic (i.e. obeying the same principle or equilibrium equation) with paradigmatic cases of systems at equilibrium like the balance or lever. This is an historical curiosity since symmetry notions are treated today as an inheritance of the geometrical tradition. For instance, (Hon and Goldstein, 2005) relates the history of the geometrical and mathematical notion of symmetry, especially its changes around the 18th century. But, they point out, this new geometric notion of symmetry lacks the transformational aspect that our modern notion has, where symmetries lie behind dynamical changes in a system. I will suggest how a kind of symmetry takes on just this dynamical aspect in mechanics. It should not be surprising that the notion of dynamic symmetry has roots not just in geometry but also in mechanical notions of conservation and equilibrium.

Finally, this historical raw material is used as a case study in scientific explanation. The distinction between explanation and description is well-known in the philosophy of scientific explanations. Positions on explanation can be categorized according to what they offer as distinctive of explanations as opposed to "mere" descriptions, whether it be Laws (D-N model), statistical relevance, support for counter-factuals. The following will, in part, take a case-study approach to the issue. Newton and Euler-Lagrange differ in the description and related explanation of mechanical phenomena (interacting bodies in motion), providing an opportunity for an examination of how description influences the kind of explanation. The geometric case corresponds to explanations through law-governed processes. The functional case corresponds to explanations through analogy of formal structure to simple, understood cases, such as the balance. The interesting question in the latter case is how Euler and Lagrange understand the formal similarity.

From the case study I then draw some philosophical morals for scientific explanation, particularly a criticism of the unification programme of Philip Kitcher. I argue that this approach is mistaken in its top-down approach which tries to eliminatively reduce explanation to unification. By considering the methodology of mechanical explanation, as exemplified historically by Euler and Lagrange, I argue that unification still requires an account of explanation. What I argue is that, although unification is an important aim of science, it is not merely unification but explanatory unification that is sought.

Despite the received view of so called "rational mechanics" as merely deriving the mathematical implications of Newton's program for mechanics, there was actually a substantial critique of the state of mechanics put forward in the 18th century. Unlike most Englightenment *philosophes*, the physicists of that period closely read Sir Isaac Newton's *Principia* and, as a consequence, they held a critical attitude towards the mechanics it embodied. Moreover, that criticism was directed to the state of mechanics generally, and was not framed as if Newton was the science's main architect. Their aims for a new science of motion were more encompassing of Archimedean, Galilean and early modern influences then fits comfortably with the Newtonian label attributed to them.

Criticism was given on two related fronts: formalism and explanation. On the one hand, 18th century figures, largely on the Continent, were pushing for the wholesale adoption of the new calculus on the grounds that the old synthetic geometric methods obscured a deep understanding of the phenomena. In particular, synthetic methods made it difficult to apply solutions from one problem to another, even very similar problem. On the other, there was dissatisfaction with the explanations being offered and the foundational principles were, to use Euler's words, "too thoughtlessly asserted." (Euler, 1736, preface)

The areas of the two criticisms were connected in that explanations were being re-interpreted in light of the new formalism (with its new quantities and operations, such as virtual velocities or differentials), while the formalism (the calculus) was being interpreted through mechanical explanations. It was thought the proper understanding and use of the calculus, as a new description of nature, would facilitate a clearer understanding of the operation of nature. This clearer understanding would allow one to see how the mechanics behind one problem would apply to other problems. Thus, people like Euler and Lagrange were deeply engaged with the question of the *representation of nature by mathematics* and not merely mathematics. They wanted not only a unified description but a unified explanation to go with the new mathematical description.

Newtonian mechanics is a key example for Kitcher. The Newtonian program (which Kitcher dubs "dynamic corpuscularianism") sought to

> complete the unification of science by finding further force laws analogous to the law of universal gravitation. Dynamic corpuscularianism remained popular so long as there was promise of significant unification. Its appeal began to fade only when repeated attempts to specify force laws were found to invoke so many different (apparently incompatible) attractive forces that the goal of unification appeared unlikely. (Kitcher, 1981, 513)

But the Newtonian programme did not appeal to many of the most important figures in 18th century mechanics from the very outset, and for reasons very different than the unpromising prospects for unification. A theme of the criticisms of Euler and Lagrange was that while a mechanics founded on uninterpreted laws may be descriptively accurate of a wide range of phenomena it provides unsatisfactory explanations. That Euler and Lagrange found 17th century mechanics explanatorily inadequate, despite their recognition of the unification achieved by Newton's programme, is an historical detail that ought to spell trouble for Kitcher and his account. At best we can say that one of his favorite case studies is historically mistaken. At worst, the rational mechanics programme of the 18th century suggests that Kitcher's account conflates unification with explanation.

There may be many important innovations in this period, but my focus here is the use of equilibrium and its connections with least-action, dynamics, conservation and symmetry—all important concepts in even the most advanced modern physics—as well as scientific explanation. If anything could be called a paradigm of mechanical philosophy in the 18th century it would be the use of equilibrium in solving mechanical problems and arriving at explanations in terms of mechanisms. All of these considerations provide a correction to the historical view that the addition of calculus was merely a notational change, adding nothing *conceptually* to Newtonian mechanics.

1.1 AN HISTORIOGRAPHICAL NOTE

In what follows I do not intend to claim that the invention of the calculus *caused* there to arise a certain view of nature. The calculus might have resulted in, given different historical circumstances, an entirely different view of mathematics and nature than the one I describe here. Moreover, not all views of the relation between mathematics and nature were the same, even among the practitioners of mechanical calculus, so that one can hardly ascribe necessity to any one of them.

Nevertheless, a number of important commonalities can be pointed out between, on the one hand, mathematical representation and, on the other, the conception of nature and change around the time of the Scientific Revolution—commonalities which are much stronger, it turns out, for calculus than for geometry. This will be the real point, that the *shift* from geometry to the calculus amplified the effect of the new formalism—the practitioners were working with and evaluating both formalisms. Feeling there were limitations to geometry, and seeing a freedom in the calculus (to be spelled out) increased the resonance of the new formalism with the belief of 18th century mechanicians that they were seeing nature in a new and deeper way. Not all who used the calculus denounced explicitly, as Lagrange did, any continuing utility for geometric methods, but all were enamored with its promise of increased mathematical and explanatory power. This new way of seeing nature had been enabled by Newton of course, but Newton's mechanics were geometrical mechanics. The 18th century gave rise to a new kind of mechanical explanation, based on functions, analysis and equilibrium, which we would do well to understand today.

2.0 HISTORIOGRAPHY OF THE SCIENTIFIC REVOLUTION

Leonhard Euler (1707-1783) was born in Basel, Switzerland, into a religious family. His father was a pastor of the Reformed Church; his mother was the daughter of a pastor. He matriculated by age 13, had his Masters at 16 and his PhD by 19. According to the index compiled in 1910 and 1913 by the Swedish Mathematician Gustav Eneström, Euler published 866 distinct works. Euler was producing work at such a rate that nearly one third of these (273) were not published until after his death. It took the 60 years from 1784–1762 to catch up to his prolific writings. Perhaps even more remarkable is that, according to the Eneström index, 366 of these works were written in the last 12 years of his life, during which time he was blind in both eyes.

Yet today, when we consider who are the main architects of the Scientific Revolution, Euler is eclipsed by figures like Galileo, Descartes, Leibniz and Newton. Euler is remembered primarily as a mathematician, despite the fact that over half his work is on subjects we would today call either physics, engineering or astronomy. His are no mean contributions either. Moreover, a cursory look at the journals and books of the 18th century reveals that not only Euler but most of his contemporaries, such as Lagrange, were seriously engaged with not just questions about mathematical methods but their physical interpretation and application.

In the Introduction I described a 'received view' of the 18th century. I also characterized that view as dismissing 18th century mechanics as a mere translation of Newtonian science into calculus. This chapter surveys the major historiographies of the post-Newtonian period, with the goal of identifying features of this received view. I will identify three narratives of the scientific revolution, narratives which lead naturally to the 18th century being under-emphasized and, more specifically, to the *Principia* being a convenient bookend to the period. These narratives are characterized by what they take the real conceptual shift of the Scientific Revolution. Their themes are constructed from the viewpoint that whatever the important conceptual shift was, it culminated in the *Prin-*

cipia. I call these narratives, after their themes, the Astronomical, Rational Reconstructionist, and Newtonian Physics.

In *What was Mechanical about Mechanics: the concept of Force between Metaphysics and Mechanics from Newton to Lagrange* (Boudri, 2002), J. Christiaan Boudri also argues for the thesis that the contribution of the 18th century to our modern scientific outlook has been dramatically underestimated. Boudri, however, focuses on force and metaphysics and the vis viva controversy, implicitly accepting the dichotomy that to talk about mathematics is not to talk about metaphysics. That is, his position is that the received view fails to understand the 18th century because it looks at only the mathematics.

The thesis of this work is therefore different in that I intend to pay attention to how the mathematical changes themselves resonated with changes in our understanding of physics. The conceptual change to be considered here has to do with the understanding of physical phenomena as evidenced by the kinds of explanations used by physicists in the 18th century. This change is explicitly related to the change in mathematical methods.

A related theme, not addressed here, would be to consider the question What kind of theory change does a change in formalism represent? In particular, the change wrought by the invention and adoption of the calculus, as I will describe it, seems to be beyond merely normal science versus problem solving as it involves a change in world view through a change in its models. On the other hand, analytic mechanics does not seem to constitute a wholesale paradigm change either. It seems neither description is adequate, and that this case falls between the cracks. Thus, scientific "revolutions" are not so dichotomous as Kuhn suggests (or as he has been read as suggesting.) Analytic mechanics had important influences on our world view—the rise of symmetry and equilibrium, less emphasis on activities in mechanisms.

A similar point has been made recently in Maglo (2003) but in regard to Netwon's gravitational theory. An analogy is drawn with the "eclipsing" of Darwinian theory after its advent, followed by the eventual acceptance of the theory. The point of Maglo's analysis is to describe what goes on during this eclipsing period and how it fits (or does not fit) into an overall Kuhnian picture. Interesting results lie down this road, but I bracket them off here.

2.1 THE RECEIVED VIEW OF 18TH CENTURY MECHANICS AND THE SCIENTIFIC REVOLUTION

The literature on the Scientific Revolution is vast and even a short list must include Cohen (1980, 1985, 2002); de Gandt (2001); Dijksterhuis (1986); Dobbs (2000); Downing (1997); Dugas (1955); de Gandt (1999); Garber (1999); Greenberg (1986); Guerlac (1981); Hall (1962, 1975); Heimann (1977); Jacob (2000); Koyré (1965); Lakatos (1978); Meli (1993); Truesdell (1968); Westfall (2000); Osler (2000). [1]

In an attempt to make this literature tractable, and view it through a lens amenable to the task at hand, I have identified three broad narratives of the historiography of the Scientific Revolution. Each of these narratives sees the *Principia* as a kind of high point of that revolution and so casts what comes after as a wrapping up. Even so, the consolidation of the Newtonian revolution, so understood, deserves careful historical consideration as the consolidation of a revolution is every bit as important to our understanding of its legacy.

It should be noted too though, that the beginnings of the Scientific Revolution have also been debated. Even its demarcation at all has been questioned. “There was no such thing as a the Scientific Revolution, and this is a book about it.” (Shapin, 1996, 1) The narratives of the Scientific Revolution identified here will have at least one assumption in common with my own treatment of the importance of the 18th century: the Scientific Revolution begins with, and is primarily constituted by, a rejection of Aristotelian / Scholastic natural philosophy.

2.1.1 The Astronomical Narrative

For some, the Scientific Revolution was about astronomy. It ended when Newton answered all those questions, save a detail or two, with which Copernicus had begun it.

The scientific revolution, then, had its beginnings with Copernicus, and culminated in Newton’s theory of universal gravitation. Once the question of how the planets moved around the Sun had been settled, there was nothing much left to discuss. All that remained was the discovery and perfection of new mathematical techniques which made tractable the harder problems yet to be

[1] Another excellent resource is the website maintained by Robert A. Hatch at http://www.clas.ufl.edu/users/rhatch/pages/03-Sci-Rev/SCI-REV-Home/.

solved.

> [After the *Principia*] the astronomical and cosmological issues that had so troubled the world since Copernicus' time were regarded as **settled for good**; it only remained for mathematicians to arrange the details of the Newtonian universe in somewhat more exact order. (Hall 1962)

The emphasis of this view is on Newton's theory of universal gravitation. Gravitation allowed for the unification of celestial and terrestrial phenomena. The theory also exemplified Newton's methodology of explaining phenomena through matter and forces. With these three conceptual features in place it was considered that the *physics* of the revolution were complete. Only their description and use was to be developed.

However, it should be pointed out that the mathematical details which were to "be arranged" included the shape of the Earth, the Principle of Least Action, the dynamics of fluids, optics and the nature of light and the correct understanding of the collision of bodies. My critique of this narrative, then, is that the unification of phenomena achieved by Newton was incomplete. The problems addressed in the 18th century had as much to do with the physical interpretation and application of the new mathematics as it did with merely developing the techniques. What was missing from Newton's achievement, and what was the conceptual burden of 18th century mechanical investigation, was a theory of explanation of phenomena that went beyond the mathematical description enabled by Newton. This went hand-in-hand with what amounted to a rejection of the synthetic approach to mathematics which Newton actually used in his descriptions of phenomena.

2.1.2 The Rationalist Reconstructionist

This view of the Scientific Revolution can be characterized as follows. Newton gave us his three laws. Everything else was merely mathematical demonstrations of theorems from those axioms. There was no need for further physical or metaphysical speculation about causes, nature of forces, or the explanation of gravity. The great success of 18th century mechanics was avoiding those questions.

> Since [Newton's] time no essentially new principle has been stated. All that has been accomplished in mechanics since his day, has been a deductive, formal, and mathematical development of mechanics on the basis of Newton's laws. (Mach 1960/1883)

This view can be substantially critiqued by showing, as I do in Chapters 3, 4 and 5, that Euler and Lagrange did not simply base their mechanics on Newton's laws. New principles and different laws are rallied. The principles they choose are based on the explanatory backdrop available for them. It is true that part of this backdrop has to do with the problems solvable by choosing other principles. The important point though is that this very solvability is taken as indicating deeper *physical* understanding.

To be charitable to this narrative though, it should be recognized that the view is reconstructionist. It denies that any "essentially new principle" results from the mechanics of the 18th century. The Principle of Least Action, for instance, or what we today call the Euler-Lagrange equations, can all be shown to be reducible to Newton's three laws. These are not, of course, precisely the three laws as Newton laid them down, and this is part of the point. But I will leave off criticizing this view as simply a-historical.

2.1.3 Newtonian Physics

A more open, and much more recent, view of the Scientific Revolution recognizes something like a continued philosophical and scientific investigation, but views all of these developments (at least all successful ones) as part of a Newtonian tradition or Newtonian science.

Cohen, in the Introduction to *The Cambridge Companion to Newton*, says:

> As the spectacular success of [the science coming out of Newton's *Principia*] became increasingly evident during the course of the eighteenth century, the problem took on the added dimension of explaining how such knowledge is possible. (Cohen 2002)

What Cohen is suggesting is a Newtonian science and epistemology. This leaves unspecified though, what the essential character of this science and of this epistemology is.

On Newtonian science, Lisa Downing (echoing Voltaire) claims:

> ... a Newtonian is someone who advocates some significant portion of the physical and cosmological theories of Newton and who is an attractionist. (Downing 1997)

where the "attractionist" sees no need to settle the question of gravity's operation. Euler and Lagrange in particular, however, were not attractionists. It is true that, in solving many problems, gravity appeared as an unexplained force. But Euler himself was a plenist and, in "On the force of percussion and its true measure" (Euler, 1746), argued explicitly that all change in the motion of bodies was due to contact action.

It seems too, that, an essential character of Newtonianism, especially for its epistemological dimension, would have to be the method of synthesis and analysis. This methodology is explicitly set out by Colin Maclaurin in *An Account of Sir Isaac Newton's Philosophical Discoveries* (Maclaurin, 1748/1968).

> In order to proceed with perfect security, and to put an end for ever to disputes, he proposed that, in our enquiries into nature, the methods of *analysis* and *synthesis* should be both employed in a proper order; that we should begin with the phaenomena, or effects, and from them investigate the powers or causes that operate in nature; that, from particular causes, we should proceed to the more general ones, till the argument end in the most general: this is the method of *analysis*. Being once possest of these causes, we should then descend in a contrary order; and from them, as established principles, explain all the phaenomena that are their consequences, and prove our explications: and this is the *synthesis*. It is evident that, as in mathematics, so in natural philosophy, the investigation of difficult things by the method of *analysis* ought ever to precede the method of composition, or the *synthesis*. For in any other way, we can never be sure that we assume the principles which really obtain in nature; and that our system, after we have composed it with great labour, is not mere dream and illusion. (Maclaurin, 1748/1968, 8–9, emphasis in original)

Not only are both methods important, their order is crucial. Synthesis demonstrates the phenomena, but prior analysis is required to ensure that the descriptions obtained are not only accurate but true. True principles are those which really obtain in nature.

Maclaurin saw in this two-stage methodology a kind of immunity in Newton's philosophy.

> By distinguishing these [methods] so carefully from each other, he has done the greatest service to this part of learning, and has secured his philosophy against any hazard of being disproved or weakened by future discoveries. (Maclaurin, 1748/1968, 9)

Thus, on Maclaurin's view, it is not necessarily the theory of gravity, or Newton's discoveries on the properties of light, that characterized his real contribution. It was the methodology of analysis and synthesis which "has opened matter for the enquiries of future ages, which may confirm and enlarge his doctrines, but can never refute them." (Maclaurin, 1748/1968, 10)

As I will show in Chapters 3 and 6, however, the distinction between analysis and synthesis, and which method was to be preferred, was debated by Euler. The importance of synthesis to Newton, I argue, was not just to demonstrate phenomena, but it was also to put the phenomena back together, as it were. *Mathematical* analysis, as distinct from the analysis to causes, reduced motions to a finite misrepresentation, on Newton's view. An important aspect to his proofs was in obtaining the infinitesimal limit of the geometrical representation. The synthetic character of Newton's approach, and his preference for geometry as opposed to the Leibnizian functional approach employed by Euler and Lagrange, had just as much to do with the correct understanding of real motions in nature. Only through synthesis did one obtain the correct fluxional understanding of change as not just composed of a series of disconnected, static moments. On this point, Euler and Lagrange disagreed with Newton. The methods of analysis revealed the functional descriptions of motions and it was functions, interpreted as describing mechanical connections, that provided the proper understanding of physical phenomena (see Chapter 6.)

2.2 A PARALLEL COMPLEXITY IN POST-NEWTONIAN MATHEMATICS

The idea that Euler and Lagrange saw the function as the important new mathematical object of investigation is a central feature of recent histories of post-Newtonian mathematics. These histories reveal the rich complexity of the mathematical relations between Newton and the continent. The period following Newton and the adoption of calculus on the continent is not merely a matter of Leibnizian notation applied to Newton's calculus. Of particular importance to Euler and Lagrange is the emergence of the concept of the function as the object which unites synthesis and analysis. Literature relevant to the adoption of calculus is Boyer (1959); Bos (1993); Fellman (1988); Greenberg (1986); Guicciardini (1999, 1989). Relevant to the notion of function are Ferraro (2000); Fraser (1997, 1989); Youskevitch (1976). A succinct statement of the transition in mathematics is provided by A.P. Youskevitch: "…the calculus about variable quantities and their differentials (or fluxions) became a calculus about functions and their derivatives." (Youskevitch, 1976, 314)

By far, the most authoritative source on post-Newtonian mathematics is Niccolo Guicciardini's *Reading the Principia* (Guicciardini, 1999). The thesis of that book was that not only was the transition from the synthetic methods of Newton's *Principia* to the calculus a complex one, it was one in which even British Newtonians, and not just Continental Leibnizians, were actively engaged.

In this book I will prove that the programme of translation of the *Principia* into calculus language

> was not an exclusively Continental affair. Newton and a restricted group among his disciples (David Gregory, De Moivre, Cotes, Keill, Fatio de Duillier) were able to apply the analytical method of fluxions to some problems concerning force, motion and acceleration. (Guicciardini, 1999, 5)

Guicciardini exhaustively explores the various geometric, fluxional, and analytic methods employed by the readers of the *Principia* as they wrestled with its implications.

It is again important to stress, however, that it was the analytical method of *fluxions* that was important to the Newtonians. Guicciardini highlights the importance to Newton and his followers of the continuity of their mathematical methods with Ancient geometry. Newton reinforces this continuity, as Guicciardini quotes, by noting the important connection between his fluxional approach and motion.

> The geometry of the ancients had, of course, primarily to do with magnitudes, but propositions on magnitudes were from time to time demonstrated by means of local motion: as, for instance, when the equality of triangles in Proposition 4 of Book 1 of Euclid's *Elements* were demonstrated by transporting either one of the triangles into the other's place. Also the genesis of magnitudes through continuous motion was received in geometry: when for instance, a straight line were drawn into a straight line to generate an area, and an area were drawn into a straight line to generate a solid. . . . If times, forces, motions and speeds of motion be expressed by means of lines, areas, solids or angles, then these quantities too can be treated in geometry. Quantities increasing by continuous flow we call fluents, the speeds of flowing we call fluxions and the momentary increments we call moments. (Newton, 1967-1981, 455, Vol. **8)**

The momentary quantities which analysis treated were merely properties of the continuously flowing quantities. Understanding this relation to the true, flowing quantities was crucial to relating the mathematical methods to nature. With Euler and Lagrange, functional relations among instantaneous, differential quantities could be given a mechanical interpretation. These functions expressed the true physical relations that lay behind the motions. Analysis was not only pragmatically to be preferred, it was conceptually deeper.

An important insight of Guicciardini's, and supportive of my own view, is that

> [t]he original Newtonian and Leibnizian calculus had dealt mainly with the study of geometrical objects (typically, curves) . . . . The eighteenth-century calculus did not have an immediate geometrical interpretation: mathematicians began to think mainly about equations (e.g. differential or partial differential equations.) (Guicciardini, 1994, 314)

As I argue throughout this thesis, it is a new *mechanical* interpretation of those equations, especially their relation to equilibrium and Least Action Principles, that makes 18th century a significantly original period in the development of physics.

2.3 SUMMARY

The historical view of the Scientific Revolution and of 18th century mechanics I want to espouse here is most closely aligned with that of Eduard Dijksterhuis, as put forward in *The Mechanization of the World Picture* (Dijksterhuis, 1986).

> With the appearance of Newton's *Principia* the era of transition from ancient and medieval to classical science was concluded; the mechanization of the world-picture had in principle been accomplished; natural scientists had been furnished with an aim which they were to pursue for two centuries as the only conceivable one and which was to inspire them to great achievements. The moment has now come to raise the questions that have already been touched upon from time to time: what is the significance of the change that took place? What do we understand by the mechanization of the picture that scientists form of the physical world? In what does the mechanistic character consist that is henceforth to be typical? What is the meaning of the word 'mechanical', which from now on was to be linked so liberally with a great many scientific terms—problem, model, fact, law, phenomenon, conception? (Dijksterhuis, 1986, 495)

These are indeed the important questions to ask. But I disagree that they can be answered fully while at the same time holding the view that the transition from medieval to classical science was concluded with the appearance of the *Principia*. The mechanical picture, the interpretation of the relation between mathematics and nature in terms of mechanisms, which is characteristic of the development of classical science, and especially modern physics, is formulated to a great extent after, and as a critical response to, the *Principia*. We shall begin by asking Was Euler Newtonian?

3.0 WAS EULER NEWTONIAN?

3.1 INTRODUCTION

Was Euler Newtonian? *Yes and no*, depending on what one takes to be essential for Newtonianism and hence what counts as a significant difference. The focus here is on mechanics. If we compare Newton's *Principia* with Euler's *Mechanica* we see the two texts share many characteristics: both, for instance, have a Euclidean style of presentation, are written in Latin, and contain a mix of mathematical symbols, diagrams and expositive text. But in Euler's work, for the most part, the text is less prolix, the diagrams more spare, and differential equations predominate.

A more informative comparison could be made by examining, side by side, the treatment of some same proposition by both authors. But an inconvenient difference between these works emerges in attempting such a comparison. None of the propositions given in the first two books of the *Principia* appear identically in *Mechanica*, although most do occur implicitly in corollaries or as special cases of more general propositions.[1] Among those which do appear, some appear in an entirely different context (many of Newton's propositions which deal with bodies moving on conic sections freely, as Euler would call it, appear in the 2nd book of *Mechanica* as corollaries to non-free motion problems.) Where and how propositions of the *Principia* occur in *Mechanica*, especially Newton's three laws, is a complicated affair which I discuss in some detail in this chapter.

Even the fact that Euler is using and developing a new mathematical style, namely analysis, can be seen as similar to Newton, despite the fact that Euler's style of presentation itself is different.

[1] Most of the ones that do not appear are specific to the geometric methods that Newton applies. Newton's third book, on the other hand, is a special case. It deals with the real motions of the planets, comets and the moon and is the place where the third law plays its greatest role. Some of these results appear in Scholia in *Mechanica* but there is nothing like a "System of the World" in Euler's text.

Newton sees the *Principia* as also providing new mathematical methods in order to deal with motion and the generation of curves. However, he chose to emphasize an intellectual connection with the past and so presents his improvements in the *Principia* as synthetic, *geometrical* methods rather than algebraic analysis.[2] This is partly for reasons historians of science have spelled out elsewhere, but also because Newton saw geometry as a subdiscipline of the science of motion in general.

For Newton, geometry had in fact always been about motions (he attributes this view to the "ancients"). Lines are generated by the motion of a point, surfaces by the motion of a line, and solids by the motion of a surface. Plane geometry restricted those motions by requiring, through its axioms, that the result of the motions have certain properties—they were circles of a given radius, lines of a given length, or angles subtending chords or arcs of a certain length. Newton's new mathematical principles of natural philosophy extended mathematics to the universal treatment of the motion of natural bodies by providing a different set of axioms and definitions which dictated how motions are generated in time by forces. That is, rather than take the motions only as completed and dictate the properties of the result as a beginning, universal mechanics would treat of the actual process of moving bodies. The new axioms were essentially his Laws of Motion, though the Definitions provided part of the foundation as well. § 3.3 argues this on the basis of Newton's Preface to the 1st edition of the *Principia*.

Euler has a different view of the connection between geometry and the new mathematics he employs. For Euler, what analysis reveals is that the central object of investigation in mechanics is the function.[3] Geometric figures and algebraic equations (as well as infinite series and the transcendental functions) are merely different ways of representing the same mathematical object. The mathematics of mechanics is then descriptive of natural phenomena insofar as the functions derived are instantiated in nature. Euler was not necessarily concerned with absolute values of those functions though, as functions always relate quantities and those relations have a formal character which is the same whatever baseline magnitude is chosen for the quantities. The foundation Euler was then required to provide for his mechanics would allow him to treat mechanics functionally.

[2]The analysis / synthesis distinction, at the time, was most often construed simply as employing algebraic derivations as opposed to geometric constructions in a diagram. Maclaurin has more to say on this distinction though, in reference to Newton's philosophical system, using the terms to characterize its two stages. (See Chapter 2.)

[3]See references in Chapter 2 above, especially Bos (1993) and Youskevitch (1976).

The chief relation he employs is to space, therefore his first definitions and propositions define how bodies occupy space and the relations of speed and time to the places a body occupies (or the relations of those quantities between bodies.) § 3.6 spells this out on the basis of the early parts of *Mechanica*.

The upshot of these different foundational approaches can be seen in the following two statements (emphasis is mine.)

- Newton:

> RULE IV: In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as *accurately or very nearly true*, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions. (Newton, 1962/1686, 400)

- Euler:[4]

> In the second Chapter therefore, I pursue what kind of effect an arbitrary force [potentia] must exercise [exercere] in a free point either resting or moved. From this are constructed the true principles of Mechanics, by means of which whatever pertains to the alteration of motion must be explained. Since these laws have until now been too groundlessly asserted, I therefore demonstrate them in such a way so that they are understood as *not only certain but also necessarily true*. (Euler, 1736, 10, emphasis mine.)

For Newton, the phenomena from which the forces of nature are induced are motions. But he maintains two reservations with respect to those forces. First, he famously "feigns no hypotheses" with respect to the cause of the forces themselves. Second, he allows that the force laws induced are subject to revision, given new phenomena, and that other force laws may be compatible with those same motions (although the simplest should be preferred by his Rule I: *We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearance.* Newton (1962/1686)) This is because the force laws themselves are not the actual phenomena but are induced from the phenomena, which are motions.[5] In this sense, the forces are "outside" the

[4]The Latin for the Euler quotation is the following.

> In secundo igitur Capite persequor, cuiusmodi effectum quaeque potentia in punctum liberum sive quiescens sive motum exercere debeat. Hinc conficiuntur vera Mechanicae principia, ex quibus, quicquid ad motus alterationem pertinet, explicari debet; quae, cum adhuc nimis leviter essent confirmata, ita demonstravi, ut non solum certa, sed etiam necessario vera esse intelligantur.

[5]Newton does not treat impact problems at length in the *Principia*. A short, mostly qualitative discussion occurs in the Scholion to the laws, and there Newton does not apply the new mathematical methods but only considers collision

phenomena. As I will argue below, this resonated with his chosen mathematical approach. As Newton puts it, the constructions of the circles and lines were required from outside the domain of geometry. The same thing then applies to forces with respect to the *mathematics* of mechanics.

For Euler, the situation is different in two respects. First, phenomena, for Euler, include imagined situations about which we can reason, especially by the principle of sufficient reason. From the counterfactual case of a sole body in infinite and empty space (which he calls a phenomenon), Euler "derives" the equivalent of Newton's First Law and from that that bodies are endowed with a vis inertia. Secondly, phenomena in general are to be understood in terms of the quantities present and the functions which they instantiate. In particular, Euler argues in the Preface to *Mechanica* for a distinction between Statics, which treats of the comparison and equilibrium of forces, and Mechanics, the science of motion. The distinction is important, Euler argues, because of the "very different role of forces in the latter." Because of the motion of bodies, forces will have a different functional character in Mechanics than in Statics. Namely, both the strength and direction of their operation on the body will change with respect to where the body is as it moves. Euler's mechanics will deal directly with forces through their functional relation on space.

Furthermore, as already pointed out, functions relate quantities to one another and though the the actual values of those quantities might differ, depending on the chosen description of the situation, their functional relation remains the same. For example:

> 82. And furthermore, we will not be overly concerned with absolute motion, as . . . such relative [motion] is constrained [contineatur] by the same laws. And therefore this relative motion we will often transform [mutabimur] into others of the same sort, in such a way nevertheless, that the laws related are observed: if, that is, we consider [the motion] in relation to another body which is made to progress also uniformly in a straight line. In this way it will not cease to move equally in a right line, and this can be done in innumerable ways, out of which that which is the most convenient will be selected. (Euler, 1736, 33)

These innumerable transformations will change the values of the motions and positions of the bodies and the directions of the forces. What is retained is the functional form, defined by which quantities the function depends on and the algebraic operations on those quantities.[6]

in connection with his laws. This supports the point that it is the *laws* that are about dynamics. For the rest of the *Principia*, the actions of forces are always as a pressure or what would later be called vis mortua. The kind of force phenomena I have in mind are the percussion experiments Galileo described. I take the vis viva controversy, which carries on for some 60 years after the publication of the *Principia* as further evidence for the claim that it left open the question of the representation of dynamical phenomena.

[6] The equivalence of the laws for a uniformly moving body was also observed by Newton of course, but the dif-

The necessity Euler attributes to his mechanics seems to derive from his relating everything to the location of bodies in space. The property of occupying space he believed derived necessarily from the nature of bodies and through his definitions of space and motion. For Euler, the mathematical representation he employed—analysis based on functions—allowed him to deal with forces and motions through their relations to bodies and space, thus his talk of deeper analysis, genuine solutions and certainty.

Ironically, the lack of direct representation of forces in the mathematics of Newton's mechanics resonated with his attaching a greater importance to forces. A real commitment to forces and their dynamic role as generators of motion was required for the mathematics to be about the motions of real bodies. This commitment took the form of assuming the relation between motions and forces in the axioms. Euler, on the other hand, by basing mechanics on a functional foundation, made position in space the foundational mechanical attribute. This required a real commitment to the nature of bodies such that they occupied space and so had real positions. Motions were changes in position expressible as functions of position; forces were changes in changes in position, also expressible as functions of position. Euler's treatment of forces thus came in the "deductive" stage of his mechanics and the relations between forces and change of position were theorems for Euler (see below on Newton's First and Second Laws.)

This illuminates a distinction Euler is to make between potentia and vis. Potentia are forces which are responsible for the change in state, either resting or moving, of a body and it is these forces that are treated functionally. Vis, on the other hand, is a more general notion of force or power which includes potentia but also includes vis inertia, the force responsible for a body's occupation of space. Considering this, we see that Euler was really a transitional figure with respect to Newton. He maintained a real commitment to an underlying force in the form of the vis inertia. In later papers Euler will reduce the vis inertia to the force of impenetrability and there discuss the role of impenetrability in changing the state of other bodies through contact. But the role of vis inertia in *Mechanica* goes no further than to establish the occupation of space

ference is the terms in which it was understood. In general, when Newton was considering the motion of one body with respect to another, it was an occasion to invoke the third law, considering the mutual interaction between the two bodies. The third law guaranteed that interaction would cancel and thus the center of gravity of the two bodies would continue to move according to the first law. For Euler, uniformly moving reference frames require only a coordinate transformation for which the functions remain the same (details are given below.) Thus, the third law does not appear as proposition or corollary in *Mechanica*.

by a body. Also, just as with potentia, the nature of this force was deduced, not induced, from the phenomena. Thus Euler was non-Newtonian in the following respect: he saw a functional understanding as providing Mechanics with a necessity that was lacking in prior presentations of Mechanics, including Newton's; he brought forces under direct mathematical treatment in a way I will attempt to show Newton did not. This led to the identification and then abandonment of metaphysical forces, such as vis inertia, which were not mathematically represented.

In the following section I look at Euler's Preface to *Mechanica*: the Statics-Mechanics distinction, the role of analysis, and the connection to Newton. I then elaborate in §§ 3.3 and 3.4 Newton's views on the mathematics of the *Principia* on the basis of his Preface to the first edition and through consideration of Newton's mathematical lemmæ. In § 3.5 I discuss generally the nature of Euler's foundational project, in connection with Descartes and Newton, and then, at § 3.6, present Euler's mechanical foundations and their connection to functions. Following this is further discussion of Euler's treatment of Newton's first two laws of motion. The absence of the third law is then considered in § 3.8.3. Finally, I provide examples of Euler's mathematical methods in *Mechanica*, along with tables summarizing the complex relation between Newton's three laws of motion and Euler's mechanics.

3.2 THE PREFACE TO *MECHANICA*

Those who hold the view that Euler was following in the great Newton's footsteps will find surprising the following passage from the Preface to *Mechanica*.

> For this reason I do not know if, besides Hermann's *Phoronomia*, there has publicly appeared a work in which the science of motion is treated on its own and enriched with distinct discoveries. (Euler, 1736, 12)

The omission of Newton's *Principia* here is remarkable. Euler does say of Newton that his *Principiae*, "by means of which the science of motion was greatly advanced, are composed in a not dissimilar fashion." (Ibid.) Not dissimilar, that is, to the fashion of Hermann's *Phoronomia*, that fashion being the "perplexing" [distineo] style of synthetic demonstrations that "hides the analysis, by means of which a complete knowledge of the subject is attained."[7] It we take Euler at his word then, in his estimation Newton greatly advanced the science of motion but did not treat it on its own or enrich it with distinct discoveries. Euler's analytic approach is no mere translation *into* analytic form of Newton, or even of Hermann, to whom Euler really gives top billing. Euler's mechanics is rather an attempt at complete knowledge of the subject *through* analysis—to offer "genuine solutions". Granted, prefatorial rhetoric may not be entirely persuasive, but there is substance to Euler's distancing himself from Newton.

If, at some level, *Mechanica* represents the casting of Newtonian mechanics in analytic form it is not without significant changes. For instance, Euler derives the area law, Newton's fundamental Proposition 1, toward the middle of the text as a trivial corollary to Prop 74; the statics-mechanics distinction is highlighted in the preface, a distinction Newton does not make;[8] a distinction is introduced among forces (potentia-vis); velocities are quantified through a height-of-fall; the first law is presented as two theorems; the second law is given in conjunction with the definition of a technical term, *potentia*, which is distinct from the general term vis; Newton's third law is given neither as an axiom or theorem by Euler. In later papers the equivalent of the third law is cashed out in collision cases in terms of the constancy of the action of inertia. Inertia will either preserve the state of motion of a body or give up some of that action to another body. The amount of motion (proportional to quantity of matter and speed) given to that other body is equal to the amount lost in the body in which the inertia inheres.

Despite these differences, Euler did intend to incorporate results from the *Principia*. He tells us he has "endeavored to gather together all the problems treated by Newton and others and solved up till now and to offer genuine solutions using an analytical method." At a handful of places Euler respectfully refers to Summus Neutonus. So Newton is not without influence. It is this incorporation of Newtonian propositions along with "professional" deference which I argue gives *Mechanica* the appearance of a translation.

[7]The choice of the word *distineo* is perhaps telling here, given its connotation of distancing or dividing the reader from the subject. It is precisely this kind of division I am arguing is present between the mathematical representation of the phenomena in Newton's approach, a kinematical representation, and one's dynamical understanding of that phenomena. This leads to Euler's and Newton's differing assessments of the necessity of the laws of mechanics. Synthesis gives you forces through the indirect representation by motions, analysis gives them to you directly through functions.

[8]Newton's Scholion to the laws argues for a unification of the established science of machines and his mechanics of motion. It is safe to assume this includes statics. See below.

Leaving the Preface for a moment, Scholion 1, which follows Proposition 8 and 9 of Euler's text, reveals at least one of the editions of the *Principia* from which he was gathering results. Euler writes

> 68. Authors [Auctores] have combined these laws [leges] of absolute rest and motion together into one. Newton offers it as follows in the Principia Phil.: Every body perseveres in its state of resting or moving uniformly in a straight line, except to the extent that it is compelled to change its state by impressed forces [viribus].

In Euler's original Latin the quotation from Newton reads

> Omne corpus persevare in statu suo quiscendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

This is verbatim the 1713 edition. The first edition (1686) had ended with *illud mutare* rather than *illum mutare* as in the 2nd, while for the third edition (1726) illum had been changed back to illud and suum was added: "nisi quatenus illud a viribus impressis cogitur statum suum mutare." It is clear though, from remarks in various Scholion, that he had also access to the third edition (he refers to the ultimus editione) and to Motte's English edition of 1729. (Euler, 1736, 274)[9] Euler prefers the second edition phrasing of the First Law, which emphasizes that the forces change the state without the body intermediating. This fits with Euler's commitment, which I describe below, that bodies can never act to change their own state.[10] This passage exemplifies more the differences than the similarities between the texts of Euler and Newton, such as Euler seeing Newton's first law as three theorems with different demonstrations (one for the conservation of rest and one for each of the direction and speed.)

In the passage from the Preface above, Euler alluded to there not having appeared "a work in which the science of motion is treated on its own and enriched with distinct discoveries." Part of the

[9] So although Euler never published in English, presumably he could read it.

[10] Here is the Latin for the three versions of the final clause:

1686 …nisi quatenus a viribus impressis cogitur statum illud mutare.

1713 …nisi quatenus a viribus impressis cogitur statum illum mutare.

1726 …nisi quatenus illud a viribus impressis cogitur statum suum mutare.

Illud is a neuter pronoun, and is either nomitive or accusative case. Illum is a masculine pronoun and can only be accusative case. Statum is masculine and accusative; corpus is neuter. Suum indicates possession. So illud must refer to 'that body', by gender agreement, but can be either the object or subject of the clause. Illum refers to 'that state' where state is the object of the transitive verb mutare. So the first strictly reads by the impressed forces that body is compelled to change the state; the second, by the impressed forces that state is compelled to change; third, by the impressed forces that body is compelled to change its state.

reason for that, he tells us, is that until that point Statics had been conflated with Mechanics. Statics is "that science which deals with the equilibrium and comparison of forces" while Mechanics is the distinct science of motion.

> Although forces [potentiae] are also considered in the latter subject [Mechanics], since motion is both generated and diminished by them, the reason in the latter for treating them is quite different than in the former [Statics]. (Ibid.)

Euler tells us more about the different role of forces in Mechanics after Definition 11. Definition 11 is part of the complex of definitions and propositions that together carry the same content as Newton's second Law. It defines the direction or determination [directio] of the action of *potentiae*, Euler's technical term for motion-changing forces as distinct from the general notion of vis.

> 104. In Statics, where all [bodies] are assumed to remain at rest, all potentia are set to perpetually [perpetuo] keep their same direction. But in Mechanics, where bodies arrive perpetually in another position [locum], the direction of the potentia acting on it will be continuously [continuo] changed. At the different locations of the body, either the directions of the potentiae will be parallel with one another or convergent to a fixed point or some other law [legem] will hold, from which will arise the treatment in Mechanics of such changeable potentiarum.(Euler, 1736, 41)

In statics, the bodies and the forces do not change locations or orientation. In mechanics, the forces (potentia) still do not change orientation and their directions follow some law. The positions of the bodies, however, do change and so the direction of action of those potentia on the bodies changes with them. But even though a distinction is being made between mechanics and statistics, the two actually remain quite close. The changing "action" of the potentia is characterized as a function of the location of the body. Some law will hold dictating the direction of the potentia everywhere the body is and if the body were not moving then the potentia's action on the body would be perpetually the same.[11] Thus Euler's Statics is implicitly a special case of his Mechanics. Nonetheless, Euler tells us, "To avoid therefore all ambiguity it will help to call that science which deals [agit] with the equilibrium and comparison of forces Statics, leaving the name Mechanics only for the other science of motion, in which sense these words are now everywhere commonly used."(Euler, 1736, 7)[12]

[11] Euler does not consider forces that depend on time in *Mechanica*. This is part of the overall focus on functional relations to place that I am arguing for.

[12] Ad vitandam igitur omnem ambiguitatem iuvabit illam scientiam, quae de potentiarum aequilibrio et comparatione agit, Staticam appellasse, alteri vero motus scientiae soli Mechanicae nomen reliquisse, quo quidem sensu hae voces iam passim sunt usurpari solitae.

Another key distinction for the science of Mechanics is that how bodies react to the impressing of motions depends on how they are constituted by points. These first two books on mechanics, the ones I consider here, deal solely with the application of forces to points, though the long term project was to build upon those foundations.

> I based the divisions of the work not only on the differences of bodies which are moved, but also on their state, either free or non-free. The intrinsic character of bodies supplied me with this division, so that I first treated the motion of bodies that are infinitely small and are like points, then proceeded to bodies of finite size: those that are rigid, then those that are flexible, then those that consist of particles entirely distinct from each other. (Euler, 1736, 8-9)

As Ed Sandifer has pointed out, this constitutes Euler's "lifelong project for mechanics."[13] The treatments of finite bodies, rigid, flexible and the rest, come in later works, especially *Theory of the motion of rigid bodies* Euler (1765) and his three-part treatment of the dynamics of fluids (the bodies which "consist of particles entirely distinct from each other) Euler (1757b,c,a).

Euler goes on to spell out his version, as Newton did, of the mechanics-geometry analogy, but also to make a point about the importance of the fact that it is bodies in motion that are being considered.

> For in like fashion in geometry, in which the dimension of bodies is treated, the treatment customarily begins from points, and indeed the motion of bodies of finite size cannot be explained unless first the motion of points, out of which bodies must be conceived as being composed, is first carefully investigated. For the motion of a body having finite magnitude cannot be otherwise investigated and determined unless it be determined what sort of motion is had by the particles or points of it. In this way the treatment of the motion of points is the foundational and principle part of all Mechanics, on which the remaining parts all rest. I have devoted the first two books to an inquiry into the motion of points, in the first of which I have considered free points, in the second of which points that are not free. By far the majority of what I have discussed in these books extends beyond single points and the motion of many finite bodies can be determined from them, but not of all, in particular not of those in which the individual parts move among themselves. For from the fact that a point projected in a vacuum describes a parabola it is seen that any finite body, when projected, ought to move in a parabola. But the law of motion of the individual parts is not thereby determined, and this investigation belongs to the following books, in which the motion of finite bodies is determined. In a similar fashion Newton rightly carried over those things which he had proved about the motion of bodies attracted by centripetal forces [viribus], which holds of points as such. (Euler, 1736, 9)

Newton had also considered motion the important factor in the difference between geometry and mechanics. However, for him that difference was that geometry (before his synthetic method

[13] Online preprint, available at http://people.wcsu.edu/sandifere/History/Preprints/Talks/RWU Euler 202003.doc.

of fluxions) had relied on motion but did not treat motion. For Euler the point is more subtle: geometry treats the dimension of bodies, that dimension being built up out of points; Mechanics treats the motion of bodies, that motion being built up out of the motion of points. Mechanics thus has an extra order of complexity. It must treat the geometric make up of bodies as well as their mechanical make up. The points of the bodies are not just spatially related but also dynamically.

I take it the last sentence of the long passage quoted, that Newton "rightly carried over those things which he had proved about motion", refers to Section XII, Book I of the *Principia*, in which Newton treats the attractive forces of spherical bodies, and to Section VII of Book 2, which is the treatment, widely recognized as unsuccessful, of fluids. Both pale by comparison to the depth with which Euler would treat the various kinds of finite bodies. The increased power provided by analytic mechanics has a basis in not only the formalism but the underlying physical conception, according to which the points of a body are related functionally. Euler incorporates forces into his mathematics by representing them through their relation to space. This reveals equilibrium conditions of the points that make up a body which render more complicated problems tractable.

Another passage from the Preface reinforces this claim that Euler is seeking a deeper understanding of mechanics, if not a new mechanics altogether.

> But in all writings which are written without analysis it happens most in Mechanics that the reader, although convinced of the truth of those things which are put forward, nevertheless does not achieve clear and distinct [claram et distinctam] knowledge of them, and so can barely solve the same questions by his own devices when they are altered even a little, unless he engages in analysis and explicates the same propositions using an analytical method. This often happened to me when I began to read through Newton's *Principia* and Hermann's *Phoronomia*: although I seemed to myself to have understood the solutions to many problems, still I could not solve other problems that differed even a little. (Euler, 1736, 13)

I treat, in the next chapter, the question of how it is that analysis, understood functionally as it is by Euler and later Lagrange, provides clear and distinct knowledge while synthesis does not—what the representational difference amounts to and the argumentative strategies it enables. What I argue there is that Euler's claim of the frequent lack of clear and distinct knowledge in Mechanics synthetically set out is more than just an enumerative claim. More, that is, than the claim that it just so happens that it is most often in *mechanical* texts that the reader is left helpless in the face of new problems without analytic solutions. There is a reason for it. Namely, the role that forces play in algebraically represented mechanics as opposed to synthetic or geometric approaches. Functional representations provide place holders in the solution to a problem for the forces required to solve the next problem. In Newton's mechanics, a geometric representation of the forces must be found through the motions, which is what the mathematics really represents in the formalism. When new motions are considered, new representations are likewise required. Part of this claim is spelled out further at §3.3 where I argue that Newton's methods only indirectly represent forces and so the actual mathematics is not strictly dynamical. In a later chapter I take up the more general claim about the representation of forces in functions.

Before leaving the Preface, I offer one final, and hopefully provocative, quotation from the preface which indicates where Euler thinks the beginnings of mechanics do lie. "[T]he fundamentals of Mechanics were first laid down by Galileo when he investigated the fall of heavy objects." (Ibid.) The fundamentals were Galilean, only Hermann had enriched them with distinct discoveries. Newton's *Principia* greatly advanced the science but his synthetic approach rendered those advances, in all practicality, useless beyond what was there established. One can "barely solve the same questions by his own devices when they are altered even a little . . . ." I now turn to Newton's mathematical methods in the *Principia*.

3.3 NEWTON'S PREFACE TO THE FIRST EDITION

This section argues that Newton's mathematical methods, especially those of the *Principia*, were essentially kinematic and that the role of forces in the *mathematics* of the *Principia* was limited to the axioms of motion. The picture put succinctly is this: the mathematics allows one to treat geometric curves as the result of infinitesimal motions; the Laws of motion allow one to then take those infinitesimal motions as indirectly representing forces through their effects. Newton's mathematics applied to natural phenomena represent motions directly but require the the axiomatic introduction of forces. His mathematical principles of natural philosophy therefore require a strong commitment to the need for underlying forces as responsible for motion and change, given that natural philosophy is about motions and forces. The nature of those forces must be induced from the phenomena of motion and its mathematical description, and this applies as much to the axiomatic Laws of motion as it does to the $1/R^2$ form of the law of gravitational attraction. This is revealed by considering the connection he makes between the science of mechanics and geometry.

Newton's view, as it had developed through 1686, on this connection can be gleaned from the preface to the first edition of the *Principia*. For Newton, "rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated."(Newton, 1962/1686, xvii) 'Accurate proposal and demonstration' is the key feature. The aim of the *Principia* was to expand the scope of demonstrations possible in mathematical mechanics through the proposing of new axioms, chiefly the Laws of Motion. The method is meant to be analogous to geometry as Newton construes it. He points out that "what is perfectly accurate, is called geometrical; what is less so is called mechanical." But geometry is only perfectly accurate because the lines and circles upon which it relies are taken to be perfectly drawn.

> Geometry does not teach us to draw these lines, but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters upon geometry, then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. (Newton, 1962/1686, xvii)

The properties of the constructions are postulated as the axioms of geometry. Newton aims to provide analogous axioms which will perfect mechanics. Just as geometry does not base its demonstrations on actually constructed circles or triangles but on perfect ones, mechanics ought to begin with perfect forms of motions produced by forces. The label rational, which Newton applies to his mechanics, refers to this idealization of the subject.

The connection between geometry and mechanics is even stronger than analogy though, for according to Newton,

> geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring. (Ibid.)

Geometry is that part of mechanics achieved by fixing the domain of application through defining certain constructions. With these constructions fixed, the domain of application is limited to the measurement of fixed quantities (distances and angles.)

> But since the manual arts are chiefly employed in the moving of bodies, it happens that geometry is commonly referred to their magnitude and mechanics to their motion. (Ibid.)

The construction of a line or circle is thus, from the point of view of universal mechanics, a motion of a body. But the motion is taken as completed, as static, in the limited practice of plane geometry. The properties of the figure, the magnitudes to which geometry commonly refers, are stipulated when restricting universal mechanics to geometry. "The solution of these problems is required from mechanics, and by geometry the use of them, when so solved, is shown; and it is the glory of geometry that from those few principles, brought from without, it is able to produce so many things." (Ibid.) The principles of mechanics too will be brought from without. We should therefore consider forces as strictly outside the mathematical methods of Newton's mechanics. He gives geometric representations of forces to be sure, but those representations are through the motions and enabled by the axioms. In the same way, the motions that produce circles and lines are outside geometry.

Universal mechanics then will require both a different set of axioms and new methods to make use of those axioms. The axioms of universal mechanics—the axioms that define properties of motion in terms of the character of their generation—dictate how motions are generated by forces.[14] The mathematics must provide a way also of treating motions. "I have, in this treatise, cultivated mathematics as far as it relates to philosophy." (Ibid.) But "the whole burden of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of nature, and then from these forces demonstrate the other phenomena." The mathematics to be cultivated allows for the derivation of curves from motions, those motions being taken as representative of forces. But the reverse "investigation" of forces from the motions is an inductive process. The mathematics allows one to show only that, *given a certain understanding of the relation between forces and motion*, particular motions can be derived.

The role of forces is encompassed entirely in the axioms (the three Laws of Motion), which define how motions are generated by forces, and those definitions which define measures for forces in terms of motions (Definitions VI–VIII.) The later propositions, such as Proposition VI, which introduce further geometric representatives of forces all rely on the axioms and earlier definitions. Motions are then the key to the new *mathematics* just as they had always been, in Newton's estimation, for geometry. Newton has both an algebraic and a geometric way of treating motion. The

[14] We have only to look at the axioms of the *Principia* to see that this is what Newton does. But there is also explicit evidence that Newton sees this as what he does. E.g., from the mathematical papers: "In mechanics it is lawful to postulate: To cut a given body by a given knife-edge carried straight through; To cut a given body by a given knife-edge rotating round a given axis; and To move a given body by a given force in a given direction."(Newton, 1967-1981, vol. 8, 177)

geometric way, which is the method employed in the *Principia*, will be discussed a little further on when I look at Newton's method of first and last ratios from the Lemmæ. However, it might be claimed that Newton, despite his geometric presentation, was thinking in terms of an underlying analysis. So I first argue that even the analysis Newton employs has the same character as his synthesis. Namely, the method itself emphasizes motions rather than forces. It is the axioms which allow the mathematics to be about motions generated by forces, but this places the forces outside the mathematics. In the same way, motions were outside geometry for Newton.

Newton's algebraic way of treating motion is through his method of fluxions. The fluxional calculus is conceived as centrally about motions—more generally about changes, but for bodies in motion it is change in position with time. Fluxions are the speeds with which fluent quantities change. Newton gives his view on fluxions explicitly in a paper published in 1711 but written around 1689/90.

> I consider here Mathematical Quantities not as composed out of smallest parts but as described by continuous motion. Lines are described and I describe their production not by comparison of parts but through continuous motion of points, surfaces through the motion of lines, solids through the motion of surfaces, angles through the rotation of laterum, and time through continuous flowing, and thus for all the rest. These generations [geneses] truly take place in natural things and are discerned daily. And in this manner the Ancients should be considered as showing the generation of a rectangle by a line moving along a non-moving line.
>
> I will consider therefore such quantities which will be increased by the increase of equal times and, according to either a greater or lesser velocity [velocitate] by which they increase and are generated, will come out [evadunt] either greater or lesser; . . . this velocity of motion or increase is called *Fluxion* and the generated quantity is called *Fluent*. . . .

Newton's algebraic method for treating motions is fundamentally kinematic, being first and foremost concerned with motions and time. There is no mention of forces here either. Notice that the talk of generation is not talk of the *generation of motion*. Rather, the motion *is* the generation. The motion of a point generates the line, the motion of the line generates the figure, etc. Newton's mathematical methods are about motion and the generation of paths or curves but not motion and the generation of motion. It is the axioms that are about the generation of motion.

One might consider it possible to bring forces into a hierarchy of fluxions by construing motion more generally as change. That is, one might say that velocity is the fluxion of position, acceleration is the fluxion of velocity, and then take acceleration as the representative of force. But this is only to bring accelerations into the hierarchy, providing force again with an indirect representation.

Furthermore, it is well known that the form of the 2nd law that Newton uses in the *Principia* does not take forces as proportional to accelerations. Rather it takes impulse, the change in quantity of motion, as the measure of the force. The units are important.Acceleration as a fluxion would be a ratio of the change of motion over time. An impulse is the combined proportion of the change in speed with time and mass.

So Newton's method of fluxions itself is not strictly but only indirectly dynamical. When applied to mechanics, it requires the action of forces to be imposed from without, as geometry required the construction of its figures from without. What matters is that this lack of forces in the mathematics results in a commitment to underlying forces behind the phenomena mathematically represented. Newton has to build the connection between forces and motion into his axioms. This is true moreover, given that when mechanics is applied to the real world, the phenomena are motions from which the forces of nature are induced. The main device of the method of fluxions is to treat the ratios among motions, just as it will be in his geometrical approach. In mechanics proper, the axioms will allow the limits of certain of these ratios to be taken as proportional to forces but the forces are then represented indirectly through those motions.

The claim is not that this feature of the mathematics is the cause of Newton's commitment to underlying forces. Rather Newton constructs his mathematics to reflect his philosophical commitment to the need for forces as underwriters of the phenomenon of change.[15] The success of his mathematics merely reinforces this pre-existing belief. When 18th Century mechanicians such as Euler or Lagrange reconceptualize mechanics in analytic form the result is a mathematics in which forces are directly represented. In the long run this does cause, I argue, the eventual obviating of the need for an extra-theoretic commitment to forces.

What is new about Newton's fluxional form of analysis then, is what is new about geometry as a whole in his hands. It allows one to treat of motion and the generation of curves. Reasoning in the geometrical style though, is still his preferred way of carrying out mathematical demonstrations, even in this new domain of moving bodies. Next I discuss in greater detail Newton's geometrical or synthetic approach in the *Principia* to the mathematics of motion and change.

[15] McGuire (1968).

3.4 NEWTON'S MATHEMATICAL METHODS IN THE LEMMAS

Just as Newton's fluxional/algebraic method was kinematic, the geometrical methods he actually uses in the *Principia*, as set out in the first eleven lemmæ, are also kinematic. The first is most interesting insofar as it sets the tone for all of Newton's proof methods.

> Lemma I
> *Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.*

The statement of the lemma clearly makes appeal to kinematic intuitions. The quantities and ratios *converge continually in time*, *approach* ever nearer, and ultimately *become* equal. Granted, this same sort of language is used today in discussing mathematical series or convergence and in proofs in general. For Newton though, this is more than a way of talking—it is the way the relation between geometry and mechanics is understood. Just as plane curves can be understood as generated by compass and ruler constructions, the paths of real bodies in space can be understood as curves generated by motions, with forces only represented indirectly through the motions, but being *understood* as responsible for them.

The generation of curves takes place in time. And so, importantly, the first Proposition of the *Principia* is one which gives control of time in the diagrams by representing it as an area.

> Proposition I. Theorem I
> *The areas which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described.* (Newton, 1962/1686, 40)

There are, though, really two roles which time plays in Newton's mathematics, and it is this proposition which introduces the second role. The distinction and connection between them must be properly understood. The first is the role played in the lemmæ and further demonstrations, whereby we are asked to imagine that quantities are vanishing, points are approaching one another, distances are evanescing. Again, this is more than a way of talking.

> [B]y the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives: that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in a like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish.(Newton, 1962/1686, 39)

Similarly for nascent quantities. This is nothing like the modern understanding of a limit in which terms are ordered in a series. In the modern understanding, the terms have some index. To say later terms are ever closer or ever approach to the limit really is just a temporal *facon de parler*, where later merely means those terms having an indice in the ordering greater than a certain number. So one might think that Newton's use of time is just a stand-in for the index by which the series is ordered. But one cannot make sense of his ultimate quantity in this way. In order for the ultimate quantity to be a real quantity this vanishing must be real. Newton really means in time. This accords with his understanding of paths as a whole. He understands paths of bodies as being generated in time, therefore the elements that make up the path must also be temporal. It is tiny, just-disappeared, uniform motions in time that make up the overall large motion in time.

The second role of time occurs at a level above the first and is the more familiar. We are thinking in one time when we imagine the limits being approached, but when we are considering the theorems as applying to real motions, we are thinking in real time. We can call it internal and external time respectively. This is best illustrated by considering Newton's first proposition (see Figure 1.) We begin with the finite figure of motions and deflections. This is occurring in external time. The body moves, for example, from A to B where it is understood to be 'hit' by the force BV and deflects along the path BC, where it is again deflected, etc. Now we freeze that process of moving, freezing the entire rectilinear path, and introduce another process of moving to the limit in internal time. We "augment" the number of triangles while diminishing their breadth *in infinitum*, eventually giving us (we pretend) a curvilinear path. We now understand the curvilinear path as a process of infinitely-many infinitely-short processes of movement and deflection. We have to understand the path of a body in this way in order for the proofs to be intelligible as they rely on the time aspect for achieving their ultimate values. So it is not just that Newton's approach emphasizes time—it emphasizes time on two levels: an infinity of temporal processes within the overall process in time. The internal time of the limits is reduced to the infinity of instants that make up the overall external time of the curvilinear motion.

The importance of time is part of the kinematic nature of Newton's methods. The indirect representation of forces through motions also needs to be illustrated. The proof of the parallelogram rule in a corollary to the Laws, presumably a fundamentally dynamic result about forces, nevertheless relies on a robust sense of the time and of motions. The parallelogram law relies on the two

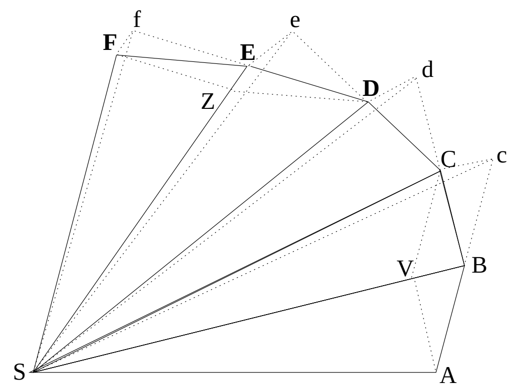

Figure 1: Newton's First Proposition, *Principia*

motions happening in the same amount of time and on the two forces acting at the *beginning* of the same instant. (See Fig. 2.)

> Corollary I
>
> *A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.*
>
> If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B, and by the force N impressed apart in the same place, should be carried from A to C, let the parallelogram ABCD be completed, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second Law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. (Newton, 1962/1686, 14)

In general time serves as the common measure of choice between motions of bodies and actions of forces for Newton (for Euler, place in absolute space is the common measure of choice.) Notice too that the seemingly dynamic parallelogram rule is being mediated by velocities. The proof strictly shows that the parallelogram rule holds for directional motions, but the rule is then inferred to hold for forces as well by taking the motions to be produced by forces at the beginning of the instants over which they motions occur. It is the second Law which justifies this move.

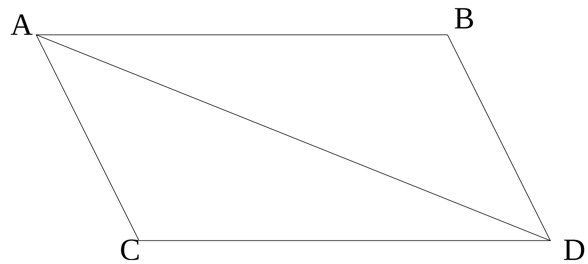

Figure 2: The parallelogram rule of motions

Newton clearly enables the move to consider forces directly, and we see this in the parallelogram rule. One could imagine the sides of the parallelogram as representing the magnitudes and directions of the forces and the rule as simply a geometric representation of how vectors add. This is not how Newton understands it. Newton *enables* this move but does not himself make it. In demonstrating the proposition he argues using the nature of the motions. Because the motion along AC is parallel to the direction BD the point arrives at line BD in the same time it would have arrived there had it been moving only in the direction AB.

Something further needs to be pointed out for the parallelogram rule. Compare its diagram with another from the scholium to the Laws. (See Fig. 3.) The parabolic trajectory E for this parallelogram results from the motion along AB being uniform while the motion along AC is accelerated. The motion AC is the "motion arising from [the body's] gravity."(Newton, 1962/1686, 21). That is, there is a force acting along the direction AC but not along AB.[16] "When a body is falling, the uniform force of its gravity acting equally, impresses, in equal intervals of time, equal forces upon that body, and therefore generates equal velocities."(Newton, 1962/1686, 21) The original parallelogram rule, which results in the body travelling directly along the diagonal, only holds therefore if we either assume both forces are acting continuously along AB and AC or only act at the beginning instant at A. If they are acting continuously then the distances, AB and AC, which occur in the same time, are proportional to the time squared and the ratio between them is linear. Only if one motion is accelerated and the other not will their ratio be non-linear and

[16]At least no accelerative force is acting along AB, though the force of inertia is. Newton requires an underwriting force for all change and he considers uniform motion a kind of change (Cite Ted again). This is another key difference between he and Euler which I discuss below.

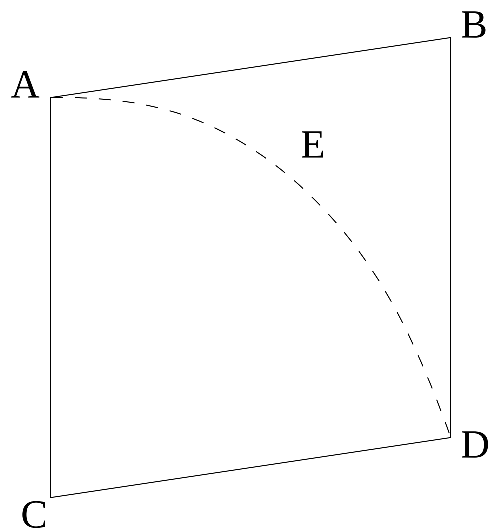

Figure 3: The parabolic trajectory of a projectile

hence the path across the parallelogram not directly along the diagonal. Moreover, for the rule to apply in later propositions, e.g. Proposition I, Newton must take the forces to be only acting at the beginning of the motions. For this reason, the demonstration of the parallelogram rule states that the body is carried "with a uniform motion … from A to B."(Newton, 1962/1686, 14)

I now turn to the remaining lemmæ. Lemmæ II, III and IV would be familiar to anyone who has taken integration in an introductory calculus class. Newton shows that a curvilinear path can, in the limit, be captured by a rectilinear construction—for example, a series of parallelograms lying beneath the curve or a series above it.

Lemma V simply states "*All homologous sides of similar figures, whether curvilinear or rectilinear, are proportional; and the areas are as the squares of the homologous sides*"(Newton, 1962/1686, 32) and no demonstration is given. This result is important at Lemma IX.

Lemmæ VI, VII and VIII establish the equivalence, in the limit, of an arc, its chord and tangent: Lemma VII, Corollary III states "And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other."(Newton, 1962/1686, 33)

Lemma IX, X and XI deserve closer attention. Referring to Figure 4, for Lemma IX we are to take curve ABD and line AE as given in position and the angle between them is thus likewise

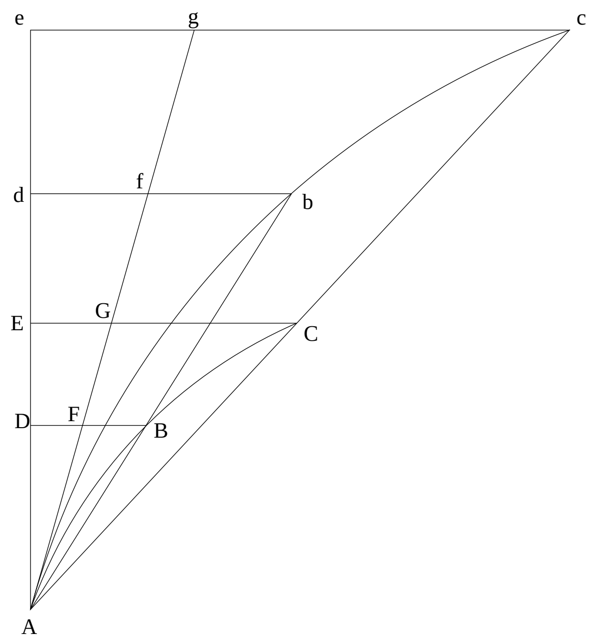

Figure 4: Diagram for Lemma IX, *Principia*. "*The area of the triangles* ABD, ACE, *will ultimately be to each other as the squares of homologous sides.*"

given.[17] The points B and C are to "come together" in A. The Lemma then asserts "The area of the triangles *ABD, ACE,* will ultimately be to each another as the squares of homologous sides" (This is where Lemma V comes in.) The homologous sides of import are AD, AE. At Lemma X we take the sides AD, AE to represent times and the lines DB, EC etc. to represent "velocities generated in those times". That is, we take the given curve to be a velocity-time graph. The (curvilinear) areas ABD, ACE then represent distances. In, other words, these two lemmæ show that the time-squared law holds at the very beginning (and very end) of arbitrary motions.

Lemma XI now establishes a different square law.

Lemma XI

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately as the square of the subtense of the conterminous arc.

[17] The diagram given in the *Principia*, and as reflected here, is constructed with an acute angle between the curve ABD and line AE. It is not obvious that the proof should also hold for an obtuse angle. In the following Lemma the line and curve are taken to be a velocity-time diagram and so it is also not obvious that that result should hold for arbitrary velocity curves. (I think I can show the proof holds though.)

BD and bd are subtenses of the angle of contact (See Fig. 5). AD, Ad are subtenses of the conterminous arcs. The theorem then claims that BD:bd :: AD^2:Ad^2. Corollary III to this Lemma, which

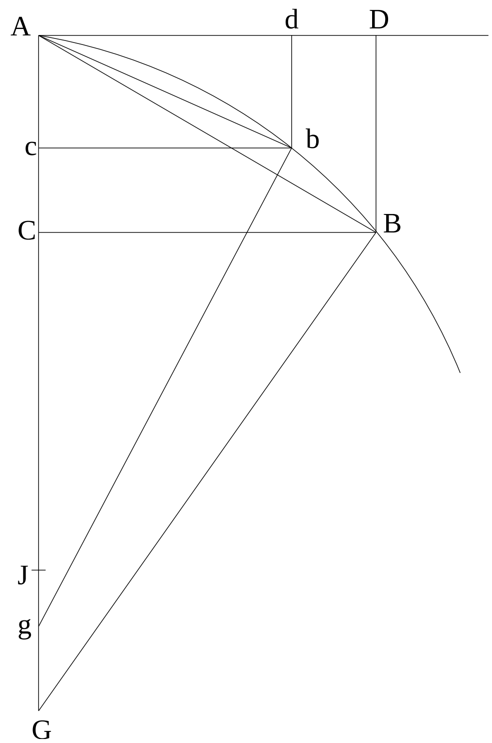

Figure 5: Diagram for Lemma XI, *Principia*.

establish the proportionality of the versed sine to time squared, plays a crucial role in Proposition VI of Book I. Corollary III states "And therefore the versed sine is as the square of the time in which a body will describe the arc with a given velocity." This claim is based on merely the geometrical result that the versed sines of arcs are as the squares of the arc lengths, which is Corollary II. For a uniform motion along the arc, the arc lengths are as the times. And we assume, for infinitesimal motions along the arc, that the motion is uniform since the force only acts at the beginning of the instant of that motion.

Prop VI relies on this time squared corollary. The conclusion of Prop VI is that *the centripetal force in the middle of the arc will be directly as the versed sine and inversely as the square of the time.*(Newton, 1962/1686, 48). It also relies on the fourth corollary to Prop I, which establishes the proportionality between the versed sines and centripetal force. Referring to Fig. 1, the versed sines are halves of the diagonals of the parallelograms inscribed at, e.g. ABCV. These diagonals represent the motions that are *caused by* the impulsive forces acting at each point of deflection, these points becoming continuous in the limit. Note, the conclusion of Prop VI can be restated as the versed sine is proportional to the force compounded with the square of the time. The conclusion then, is a combination of the two Corollaries just mentioned: one establishing the proportionality of the versed sine to time squared, the other establishing its proportionality to the force. Rearranging for force makes it proportional to the versed sine and inversely to the time squared. Note that the force here is represented by the versed sine. The versed sine is half of the deflection which is a motion. The force is therefore represented by a motion, that representation relying on the second Law. The force is not present in the Lemmæ on which Prop I and Prop VI rely.

These lemmæ demonstrate the importance of internal time to the proofs, relying as they do on evanescent ratios. But they also illustrate the absence of forces. The axioms of motion which precede the first book introduce forces as generators of the motions to which the lemmæ apply. Euler's differing treatment of the laws and the role of forces in them will be discussed following the next section. We must first consider Euler's foundations for mechanics as enabling a functional representation, both of motions and of forces.

3.5 IF NOT NEWTONIAN, THEN CARTESIAN?

In 1982 Stephen Gaukroger suggested that Euler was, in a particular sense, Cartesian. "To simplify somewhat for Euler as for d'Alembert it was a question of squeezing Newtonian mechanics into a Cartesian shape." (Gaukroger, 1982, 134) The project was Cartesian, he says further on, "…in the sense that there was a clear attempt to derive basic concepts of mechanics from the essence of body" (Gaukroger, 1982, 139).

At a number of places Euler claims that impenetrability is the source of all forces and the essential feature of bodies. There is no such explicit claim in *Mechanica*, but they can be found in other papers (Euler, 1753b, 1752, 1750), as well as in his later installment in the science of mechanics, *The Theory of Motion of Solid or Rigid Bodies* (Euler, 1765). In "On the Force of Percussion and its True Measure" (Euler, 1746), he says rather that all forces are excited by inertia, while at other places Euler takes impenetrability to be a precondition for inertia. All of this is surely worthy of independent cataloging. What matters here, however, is that in all cases what Euler is attempting to do is to distinguish bodies from void space. For example, in "Reserches sur l'origine de Forces", we have

> ...impenetrability is that property of bodies in virtue of which a body being in a place, in so far as it occupies that space, will not allow another body to occupy that same space at the same time. The condition which I infer in this definition, [which answers all objections], is *insofar as it occupies that space.*(Euler, 1753b, 425–426)

The objections mentioned have to do with putative examples from nature where two bodies would seem to occupy the same space, e.g. a sponge absorbing water. This is why occupancy is the salient point for Euler with regard to these challenges. In the sponge case, it is "not the particles of the sponge [the water] penetrates, but the pores," (Ibid.) driving out the air that was there. The same reasoning applies to light traveling through a transparent body or to ether "freely traversing" any body. "From this it is clear how much it belongs to the essence of bodies to be impenetrable, since without this property they would not be capable of occupying any space and ... there would be no difference between them and void space."(Euler, 1753b, 427)

If it is non-Newtonian to look beyond the principles of mechanics for some further and, as Euler would call it, metaphysical foundation then this alone would make Euler non-Newtonian. One can construe this search for foundations as an *interpretation* of mechanics, answering as it does, the question of what the world would have to be like for mechanics to be true. What this section and the next aim to show is that Euler does provide, through the foundations of mechanics given in his *Mechanica*, a picture of what the world must be like in order for functions to be intelligible descriptions of phenomena. This picture is non-Newtonian, as claimed, in that it sees mechanical descriptions as necessary while moving away from the need for an induction of the activity of forces as explanatory.

At the same time, Euler's mechanics cannot be construed as entirely Cartesian either. Even granting he is Cartesian-like for having a foundational approach which considers the essence of bodies, this similarity does not make Euler's project uniquely identifiable as Cartesian (as opposed to Leibnizian, say), let alone make it as radically rationalist as Descartes' might be taken to be. In fact, Newton could be seen as providing just such a foundation in his second law, taking mass to be an essential feature of body. Nor did Euler endorse the Cartesian identity of extension with body since he deems it necessary for the intelligibility of mechanics that a body and the space it occupies be essentially distinct. As for Euler being a Cartesian rationalist, he does not treat the laws of mechanics which govern the motion of bodies as if they could be derived from first principles alone. Some appeal to the world is required, but the connection is subtle, as I will illustrate next. For Euler, actually carrying out experiments oneself does not seem to be required for the assurance of the indubitable truth of mechanical principles. Though he left us little evidence to tell us which experiments he did or did not perform, or even witness, this paucity itself suggests that justifying his own mathematics with experimentation was no great concern. He does provide us though, with a succinct example of his position on the real world status of mechanical principles in "Reflections on Space and Time": "These two truths being so indubitably constant, it is absolutely necessary that they be founded in the nature of bodies."(Euler, 1750, 324) It is this example I discuss more next.

The two truths referred to in this last quotation are the two parts of the conservation of state (which Newton calls the 1st law): a body at rest remains at rest and a body in motion continues to move with the same speed and direction unless, in either case, it is acted on by some external force. For Newton and most others this was one Law. In fact, it was one axiom. In 1750, at the time of writing "Reflections on Space and Time", Euler continued to consider them two distinct truths, but in *Mechanica* Gaul was yet divided in three—a theorem and demonstration were needed to establish the preservation of rest, of speed and of direction. In *Mechanica*, the principle of sufficient reason is used for rest and direction, another argument for the constancy of speed. The argument for the constancy of speed though, depends on a corollary to the theorem of rest.

The theorem for rest is:

> Proposition 7
> Theorem
> 56. A body resting absolutely must remain at rest forever, unless it is drawn into motion by an external cause.

The demonstration begins by imagining a body "to exist in infinite and empty space." "It is clear," Euler argues, "there is no reason whereby, to this or that region, it should rather be moved. As a result of this lack of a sufficient reason why it should be moved, it must perpetually rest." The lack of sufficient reason being posited has to do with the preferability of one region of space over another [quare potius] to which the body might move. It is not the lack of any force or principle in the body to cause the motion. It cannot be any such lack, on pain of circularity, for the presence of a principle inherent in the body and responsible for its resting is going to be *required* for the conclusion of the demonstration.

This occurs when Euler wants to extrapolate from the imagined case to the world. “Neither, in fact, does this reasoning cease in the world, although it could be objected that in the world there is sufficient reason whereby the body might preferably go to some region [in hanc potius quam illam plagam cedat].” This is because the principle of sufficient reason cannot ever provide, as Euler puts it, a “true and essential cause”.

> Although one should not believe that the lack, in infinite and empty space, of a sufficient reason for a particular motion is the cause of its persistence in rest, there is no doubt that the cause of that phenomenon is present in the nature of the body. That is to say, the lack of a sufficient reason cannot be regarded as the true and essential cause of that event, but it demonstrates the existence of the true cause, and does so rigorously. Indeed, it shows that there is in the nature of the thing a hidden [occultam], true, and essential cause, whose influence does not cease even as the lack of a sufficient reason itself disappears.(Euler, 1736, 27)

Euler takes the imagined case of a body in a vacuum, and what we know must be true of that case by reasoning, to be a real phenomenon. The imagined phenomenon is real enough, that is, that we can draw conclusions about the nature of bodies from it.

> [S]ince it is true that in empty space a body at rest must remain at rest, some reason for that fact is also posited in the nature of body, on account of which a body in the actual world that is at one time at rest will, unless impelled by a distinct cause, be forced to remain at rest.(Euler, 1736, 27)

This is thought experiment with a vengeance. The principle of sufficient reason tells us what the phenomenon must be but does not provide us with the true and essential cause of that phenomenon. That cause must be posited in the nature of the body. Euler takes this positing to be rigorous. Corollary 1 to Proposition 7 states

> 57. It is therefore a law founded in the very nature of things [lex in ipsa rerum natura fundata], that all resting bodies must remain at rest unless solicited to motion by some other external cause.

‘Other external cause’ refers to other than that internal cause in the body responsible for preserving its rest.

Euler even attributes the same sort of reasoning to Archimedes with regard to the balance.[18]

[18] As Bernard Goldstein has pointed out to me, and I have verified, you will find no such thing in Archimedes. However, Goldstein thinks it likely that Euler has read this in Leibniz.

> Thus the demonstration of Archimedes of the equilibrium of a balance, both sides similar to themselves [utrinque sibi similis], evinces the truth of the matter not only in a vacuum, but also in the actual world. But another genuine reason for that equilibrium is given, which also has a place in the actual world. (Euler, 1736, 27)

What is interesting about this claim is its suggestion that something like symmetry might also be the basis for imagined phenomenon. However, it is more likely the case that Euler rather saw the similarity of both sides of the balance as providing another occasion for application of the principle of sufficient reason.

Euler goes on to also argue from the principle of sufficient reason to establish that a body moving in infinite and empty space would continue to move in the same direction—there is no reason to prefer one direction over another. A different kind of reasoning was needed however, to establish that that same body could not change its speed. From the theorem for rest he derives that a body at rest and not acted on by any force must always have been at rest. Therefore, no body moving and not acted on by any force can ever come to rest. The demonstration of the conservation of motion therefore relies crucially on the theorem for rest and so relies also, albeit indirectly, on the principle of sufficient reason. (See below, where I compare Newton and Euler on the first Law.)

Thus, the conservation of state was three truths for Euler in 1736. But they all rigorously demonstrated the same true and essential cause, given at Definition 9.

> Definition 9
>
> *Vis inertia is that faculty, inherent in all bodies, of remaining at rest or of continuing direct uniform motion.*

The foundation of this principle is thus a matter of Metaphysics according to Euler, as it is Metaphysics that inquires into the nature of bodies (Euler, 1750). This is one way in which Euler moves away from Newton—or at least away from what Newtonians claim to be doing (See discussion of Maclaurin in Chapter 2.) He employs rationalist (a priori) style premises and arguments. He has argued from imagined situations (a body in infinite and empty space) to laws necessarily founded in the nature of bodies. This is in addition to empirical phenomena, such as he musters in (Euler, 1750). There it is the indubitably constant agreement of the principles with nature which serves as the basis for his metaphysical assertions about the nature of bodies.[19] Interestingly, even his

[19] I have in mind the quotation, already given above: "These two truths being so indubitably constant, it is absolutely necessary that they be founded in the nature of bodies."(Euler, 1750, 324)

appeal to the principle of sufficient reason is a kind of empirical claim since he takes the imagined example to be a real phenomenon. Thus we see emerging the necessity of the connection between phenomena and the principles of mechanics—the same necessity that was seen lacking in those principles as "too groundlessly asserted."

Euler's project starts in the middle then, as it were, and proceeds in two directions. As the middle starting point, the basic, constant principles of mechanics are *deduced* from the phenomena of moving and resting bodies, both imagined cases and their indubitable constancy in nature. (Maclaurin called this the analysis phase of Newtonian philosophy). One can then proceed in a downward, "metaphysical", direction. The ability of the principles of mechanics to always describe correctly the motion of bodies, in heaven and on Earth (Euler, 1750), demands, according to Euler, that those principles be founded in the nature of bodies. Euler takes both an argument from the principle of sufficient reason and the indubitable constancy of the principles of mechanics to guarantee the existence of certain features of the nature of bodies (and other features of space and time as we will see.)

Mechanics can also proceed in an upward direction from the principles (Maclaurin: Newtonian synthesis phase), deriving further principles or particular results that apply to bodies, or interacting systems of bodies, in the world. This procedure corresponds precisely to the plan of Euler's *Mechanica*.[20] Euler bases the divisions of that book on the way in which points (are conceived to) compose a body [quibus corpora composita concipienda] as I described above. He also introduces a free / non-free distinction among the states bodies can be in. "...I based the divisions of the work on both the differences among the bodies which move and on their state [statu], either free or non-free."(Euler, 1736, 9) The *Mechanica* therefore also has a major division into volumes based on this free or non-free distinction. The distinction is not, as one might expect, that between bodies being acted on by forces and those that are not. Rather, it is free in the sense of free fall for Galileo. Free bodies are those that can move as they ought *given their intrinsic motion and the forces acting on them.* Non-free cases are those such as the pendulum or bodies that moves on a surface or given line—what today would be called constrained motions.

Non-free motions are treated in the second volume of *Mechanica* which consists of four chap-

[20] See e.g. Ed Sandifer's "Euler's lifelong project for Mechanics" for an outline of how this plan proceeds throughout Euler's career. Reference in note 13, this chapter.

ters: non-free motion in general, motion on a given line in a vacuum, motion on a given line in a resistive medium and motion on a surface. Euler does not treat these constraints explicitly as force constraints. Rather, those components of the motion that would otherwise take the body from the constraining path are simply "absorbed" [absorbere]. He does not say those motions are canceled or counteracted or resisted. Those are the sorts of words you would use if you were thinking Newton's Third Law was in the background. He talks of these constraint cases as cases of pressure—but it is the body doing the pressing, that pressure being absorbed. For Euler, stationary obstacles do not provide an occasion for the explicit application of Newton's third law (see 3.8.3.)

We might think, of course, that some force must be acting to counteract the pressure of the constrained body. But in not making this explicit, Euler is indicating the direction in which mechanics is moving. It is moving away from the reliance on a commitment to the action of forces where a simple functional constraint will do.

This second volume then represents two things. On the one hand, Euler, as he promised, is incorporating (and offering "genuine solutions" to) a number of problems that are standard fare at the time. Namely, problems involving motions along paths having certain defining characteristics. Examples are

- Arcus isochronos (Euler cites previous treatments of Bernoulli and Saurin)

Proposition 14

Problema

106. If there is constructed an infinity of similar curves AM, AM etc. (Fig. 6) beginning from a fixed point A, find the curve CMM which intersects all those curves AM, AM etc where a body descending on the curves having run for equal times will be found, driven by a uniform and everywhere downwardly directed potentia.

- *Linea aequabilis pressionis* (Line of uniform pressure)

Proposition 25

Problem

224. If a body is drawn downward perpetually by some force, find the curve AM on which a body descends pressing equally everywhere.

- Linea aequabilis descensus (Line uniformly descended)

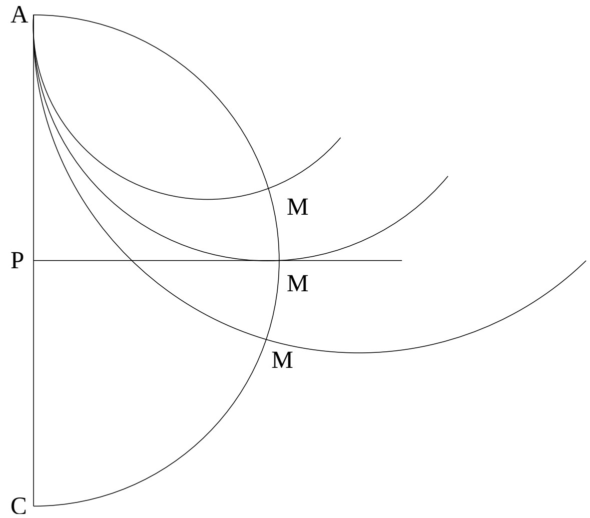

Figure 6: Arc of isochrony for given paths under a downward *potentia*

Proposition 28

Problem

If a body tending downward is solicited by some potentia, find the curve AM, above which the descending body is carried downward by a uniform motion, or recedes uniformly for horizontal AB.

- Brachistochrone (Proposition 40, a variational calculus problem)

These proofs serve to demonstrate the power of treating problems functionally. For each problem set out, the demonstration begins by assuming the solution, as in the traditional analytic approach. The solution will have some characteristic equation constructed from the features specified in the setting out of the problem. However, this equation will not have the functional dependencies required for the solution. The given equation is manipulated following the rules of infinitesimal analysis to obtain an equation defining the curve sought and depending on the quantities asked for in the problem. The problem is then considered solved.

The key feature of the three examples is a recognition of the functional dependence of the equations on the variables. For instance, consider the first example (Arc of isochrony), where similar curves are concerned. The parameter of the curves is taken as a variable. The function which defines the time of descent along each curve is then taken to depend on this parameter. At the same time, that function must be a constant quantity for each curve. This is given in the problem as the nature of the solution—we seek the arc for which the time along each curve to its intersection with that arc is the same. Therefore, the differentials of the function with respect to the parameter must be zero. This condition is sufficient to determine the unique equation of the arcus isochronis. Steps in these proofs refer to other publications of Euler's in which he has laid the groundwork for what will become the method of variations. In general this method relies on recognition of the zero first derivative of the equation defining the curve—which is, in essence, an equilibrium condition.

So the second book showcases the analytic method, but it also represents a crucial step in Euler's plan for Mechanics as he has laid it out. Points, when they come together to form bodies, will be treated as non-freely moving points. Rigid bodies, for instance, are comprised of points having fixed relational properties. Those points cannot obey freely the forces acting on them. The methods for dealing with constraints are the methods for dealing with finite bodies. Finite bodies are therefore constrained systems of points. The next section will begin to describe how the foundations for mechanics which Euler provides are constructed to enable this functional representation.

3.6 EULER ON MOTION AND ABSOLUTE SPACE—DEFINITIONAL FOUNDATIONS

Euler's *Mechanica* is, like Newton's *Principia*, a Euclidean-style presentation, proceeding from definitions to propositions which are either theorems or problems to be solved. Rather than identify anything as an axiom Euler takes the definitions to be self-evident—in fact, necessary. What is not a definition is either a theorem or a problem to be solved. The first chapter, "On Motion in General", begins appropriately with a general definition of motion.

Definition 1

1. *Motion is the translation [translatio] of a body out the place [loco] which it had occupied [occupabat] to another. Rest however, is the remaining [permansio] of a body in the same place [loco].*(Euler, 1736, 13)

In two corollaries to this definition Euler argues that therefore only bodies can be said to move or rest, as only they can occupy space; and all bodies, since they occupy some space, must either continue to occupy the same space or not and so must either move or rest. Succinctly, all bodies move or rest; everything that moves or rests is a body.

Euler then defines place.

> Definition 2
> 4. *Place [locus] is part of immeasurable or infinite space [spatii immensi seu infiniti], in which the whole World [universus mundus] stands. Place taken in this sense, is customarily called absolute, as distinguished from relative place [loco], of which mention will be made next.*

The whole World is at rest with respect to the immeasurable or infinite space that contains it. Parts of that world take up parts of its space. Euler uses 'absolute' to distinguish motion, rest and place with respect to the space of the whole World, according to what he calls customary usage, but nowhere in *Mechanica* does he describe the space itself as absolute—only immeasurable or infinite space. The definition of motion is now restated with this notion of place unpacked: "Therefore, when a body successively occupies one and then another part of this immeasurable space, it moves; but if it perseveres continually in the same spot [sede], it rests."

This is a difference from both Newton and Descartes. Newton begins with defining the quantity of matter, saying "it is this quantity that I mean hereafter everywhere under the name of body or mass" (Newton, 1962/1686, 1). And although body and space are intimately related for Euler, they are crucially distinct in that it is bodies *which occupy* space. Descartes identifies body with extension, a body with its space. For Euler, the relation between body and space is the two-place relation of occupation. Although the definition of motion is given first, when it comes to bodies it is their occupying space that is conceptually prior to their possessing motion, as is clear from the first corollary. "Therefore, the ideas of motion and rest cannot fall to anything but those which occupy space. Wherefore, as it is proper to bodies to occupy space, only of bodies can it be said that they are moved or rest."[21] [22]

Newton gives his definition of place in the Scholion to the Definitions: "Place is a part of space which a body occupies [occupat], and is according to the space, either absolute or relative." His

[21] [Motus igitur et quietis ideae in alias res cadere non possunt, nisi quae locum occupant. Quare cum hoc sit corporum proprium, locum occupare] (Euler, 1736, 13).

[22] As Euler argues elsewhere, only by understanding bodies and their positions in this way will the principles of mechanics be intelligible. The point is made in several places. For example, Euler (1746, 1750, 1753a).

and Euler's definitions share the Latin "Locus est pars spatii ..." and the notion that the body occupies a part of space. However, Euler defines place [locus] with respect to the space in which the whole universe *stands*, so place cannot be either absolute or relative but is always absolute.

Moreover, Newton goes on to say that situation, *situs*, "properly has no quantity", and is not place but is rather a property of place. Euler introduces *situs* (position) in a second corollary to the definition of place.

> COROLLARY 2
>
> 6. Fixed *termini* of that space, to which bodies are referred, are customarily formed [concipi] by the mind. And such a relation [relatio] is called position [situs]. Bodies therefore which preserve the same position with respect to these *terminorum* are said to rest. On the other hand, those whose positions are changing, are said to move.

For now, nothing is being assumed about the imagined fixed termini—only that they are imposed by an act of conceiving. Fixed termini will provide for absolute rest and motion. Far from being unquantifiable, as situs is for Newton, these positions [situs] or relations to imposed termini are the very way we judge place. Judging place will be a mathematical notion. More on this follows.

Two Scholia follow that fill out the idea more clearly and will take us to the definition of relative motion.

> SCHOLION 1
>
> 7. The expressions being accepted according to this signification set out [*significatione expositae*], they are customarily called absolute motion and absolute rest. And these are the true and genuine definitions of those expressions; namely, they are appropriate to the laws of motion [leges motus].

Euler explains below why these expressions are appropriate to the laws of motion. The 'signification set out' refers to Definition 2 of (absolute) rest and motion—that is, rest and motion understood in terms of the place occupied the body occupies in the space of rest of the whole universe. Euler now tells us more about the imagined, imposed [termini] of Corollary 2. Scholion 1 continues:

> Since, however, we can form no certain [certam] idea of immeasurable [immensi] space and of its [terminorum], mention of which is made in the given definitions, instead of this immeasurable space and its [terminorum] we customarily consider a finite space and corporeal limits [finitum spatium corporeos limitesque], by which we judge [iudicamus] the motion and rest of bodies.

True and genuine notions of motion and rest are understood with respect to absolute space. But since we can form no certain idea of that space, we must introduce another space, customarily a finite one and having corporeal limits, by which we can form certain ideas of motion and rest. It seems choosing corporeal limits must be more than customary though. Euler will speak of the motion or resting of relative space but he has already told us that only of bodies can it be said that they rest or move. So in speaking of the motion of space we must understand the motion of the body by which the space is defined. We choose, therefore, convenient *corporeal* limits of a finite space, such as, e.g., the Earth or the fixed stars. Relative motion and rest are the change or lack of change in the relative *situm* (position) of a body with respect to some other body (or bodies.)

What Euler means by a 'certain idea' we have to gather from the rest of the first chapter. It will become clear that a function is sufficient for a clear idea. We are left wondering in what sense are the relative notions *in*appropriate to the laws of mechanics? They are perfectly appropriate for the practice of mechanics, it turns out, but the laws are not properly understood merely in their terms. The practical appropriateness of relative motion and rest can be gleaned from Scholion 2, which comes next, just before the explicit definition of relative motion and rest.

> SCHOLION 2
>
> 8. What has been said here of of immeasurable [immenso] and infinite space and of its *terminis*, ought to be considered as conceived purely of mathematics [ut conceptus pure mathematici]. Which, although to metaphysical speculations they appear contrary, nevertheless they are rightly put to our purpose. For in fact we do not assert to be given in this way an infinite space which has fixed and immobile limits.[23] Whether it exists or not we are not required to care as such, since if absolute motion and absolute rest are to be contemplated by someone, he must represent to himself such a space and the state of the body, either rest or motion, is judged according to it. (Euler, 1736, 14)

Metaphysical speculations tell us that the space by which absolute motion and rest ought to be judged, the space of the *universus mundus*, must be infinite and boundless. But in order to have certain ideas of motion and rest we must judge them relative to a space with limits. And so we conceive of a purely mathematical space having such limits. Euler is bracketing these metaphysical questions, saying that the mathematization of the space does not "give" us such a space, that is, give it to exist.

> For computation is in this fashion set up [instituetur] most commodiously, so that abstracting mentally from the world [mundo] we imagine an infinite space and vacuum and conceive bodies arranged [collocata] in it, which if they retain their position [situm] in this space are to judged to be absolutely at rest, and to be absolutely in motion if in contrast they move from one part of this space to another.(Euler, 1736, 14)

[23] Namque non asserimus dari huiusmodi spatium infinitum, quod habeat limites fixos et immobiles.

Euler has defined, in effect, the notion of a physical reference frame. The role of the *termini* is to give us limits or reference points by which to judge positions. Although he first mentions customarily considering a finite space with corporeal limits, here he is allowing an infinite space. Abstracting from the world means the removal of the world, leaving the "container" space, which is infinite and empty. We then imagine bodies arranged in that space which, as an abstraction, we are free to impose limits on. Such a space is not "given" to exist in this way, but is invented to facilitate computation. The summary point of Scholion 1, Euler later tells us, is that "every idea we have of motion is relative (§ 7) . . . ." Presumably he means every certain idea we have, for he has just given us a metaphysical idea of absolute motion. So by certain then, he must mean an idea which lends itself to computation. The certain ideas of motion Euler will employ are functions specifying those motions in relative space. Times and forces will be likewise given.

Relative motion is now properly defined, as change of situation with respect to whatever space we find convenient, not just those we take as limits fixed in absolute space.

DEFINITION 3

> 9. *Relative motion is change of position [situs] with respect to whatever space we adopt for convenience [pro lubito assumpti]. And relative rest is perseverance in the same position with respect to that space.*

(Euler gives the standard ship-earth analogy as examples of relative motion.) In cases where the limits of the space chosen for convenience "are in fact" at rest with respect to absolute space (i.e., with respect to the limits we conceive absolute space to have), relative motion and absolute motion will be the same (Corollary 1.)

There are therefore two components to Euler's view on the relation between space and the science of mechanics. On the one hand, the true and absolute properties of motion and rest must be understood in relation to the space of the World. And locations are also absolute in this sense. But for calculation, limits are required in order to judge position. Positions and locations will concur absolutely and relatively when the limits chosen are at rest absolutely. However, Euler will place restrictions on our knowledge of absolute rest.

To this point, Euler has provided us with three explicitly identified definitions (Definitiones 1–3): one for motion simpliciter; one for absolute place (locus), which coupled with Definition 1 defines absolute motion / rest; and a definition for relative motion. But the latter depends on the notion of *situ* introduced in Scholion 1 and *situ* in turn is defined with respect to imagined corporeal limits. Euler also relies on the implicit definition of body as anything that has a place.

Newton, on the other hand, begins with two definitions (I and II) which define measures of the quantity of matter and of the quantity of motion respectfully; three definitions (III–V) which characterize innate force, impressed force and centripetal force; and three definitions (VI–VIII) providing different measures of centripetal force (its absolute quantity, accelerative quantity and motive quantity.)

As for the other notions that Euler takes care to define, Newton says

> I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.(Newton, 1962/1686, 6)

Euler turns these "well known to all" notions into definitional foundations for his mechanics and then derives the principle of inertia and the effects of forces from those. Whereas Newton will take the force laws as axioms and argue for their truth on the basis of evidentiary support derived from the agreement of the motions they predict with phenomena.

Euler now begins to derive propositions, though first advising us, in a Scholion to the definition of relative motion:

> Scholion
>
> 12. It is evident that the relative state of motion or rest of bodies can differ in innumerable ways: for as we take one and then another space as that with respect to which motion and rest is determined, they will produce different relative motions and relative rests. Thus the fixed stars move with respect to the earth, but each of them is at rest with respect to the others. And the planets move both with respect to the earth and with respect to the fixed stars. In the following, however, I want to understand motion and rest as absolute, unless I expressly warn that what I say concerns the relative notions.

He has already told us that it is absolute rest and motion that are the true and genuine notions appropriate to the laws of motion. The relation between absolute and relative motions, their respective spaces and the laws of mechanics will be explicated in what follows.

> PROPOSITION 1
>
> THEOREM
>
> 13. *Every body which is carried to another place [locum] either by absolute or relative motion moves through all intervening places [loca] and does not move from the first to the last all at once.*

The proof of this is given as two cases, one for absolute motion and one for relative.

The argument for the absolute case asserts that if the body did not pass through the intervening spaces it would require annihilation of that body in one place and its production de novo in the last. This cannot happen, by the laws of nature, unless a miracle occurs. However, the picture Euler provides of what does happen is not very satisfactory. The body "thus proceeds from the first into the very next" [ex primo in proximum quendam]. It is not clear how proximum quendam solves the problem though, and why this does not require the same annihilation and production of the body. But let's return to that in a moment.

The relative case turns on the fact of the real motion of the space which the motion of the body is relative to. If that space is at rest then the relative and absolute motions of the body are the same and the absolute case applies. If the space is moving absolutely however, then *its* motion will be covered by the absolute case so that the space must go over every intervening point. "For this reason," Euler asserts, "the relative motion [of the body] will be successively made through each intervening place."

In both cases I think Euler probably takes the burden of proof to be on those who would deny the proposition. For now this argument is enough as below he will treat moving spaces functionally and argue that the form of the laws is the same whenever we consider a resting or uniformly moving space.

The are three important upshots of this Proposition, given in two corollaries and a Scholion. In the Scholion, Euler reinforces the point he made in the Preface about the importance of considering the motion of not just the body but all the points that comprise it. Rotating bodies for instance may have an axis that is absolutely at rest but its parts (except for those on the axis) will each have an absolute motion.

> In a similar fashion it is necessary to consider all bodies so that the location and motion of all their particular parts are investigated, not just the location and motion of the body considered as a whole. (Euler, 1736, 16)

The first corollary states that all motion must take some time; the second asserts the existence of the path [via] of the body or the space it runs through [spatium percursum], every point of which the body must attain. (*Spatium percursum* is the technical phrase Euler will use throughout *Mechanica*.) We will see how these two corollaries underwrite the use of functions through the rest of the book. Functions are defined for each point of their domain. In particular, speeds and times will be given as functions of the body's location.

Definition 4 and 5 follow, giving the standard characterizations of uniform motion (equal spaces in equal times) and non-uniform motion (either equal spaces in unequal times or vice versa), and then defining the measure of speed [celeritatem] or velocity [velocitatem] in terms of uniform motions. Uniform motion also gives a measure of time.

> [I]f a uniform motion is given, we have an accurate measure of time, which cannot be known except from motion. For in measuring the spaces which a body in uniform motion traverses, the ratio of the times in which it traverses them is known. (Euler, 1736, 17)

The measure of time can only be known through motions and hence through the ratio of spaces. In later papers (Euler, 1750), Euler will assert that whatever metaphysical notion of time, and of space, one might have, it must agree with those mathematical notions employed in mechanics. The same distinction, that is between metaphysical and mathematical notions, is being made explicitly in *Mechanica* with respect to space.

The next Proposition (2) and its corollaries establish the various ratios among distance, time and speed and how they may be used as measures for one another. Euler tells us, at Scholion 2 to Proposition 2, he will employ (Rhenic) feet for distance and seconds for time, and then continues

> Below we will encounter an easier method of determining speeds, which we will thereafter use, but it arises from this method and is easily recast in its terms. (Euler, 1736, 20)

The "method" is taking the ratio of feet traversed to the time of the motion. And so the speed of a body which traverses, e.g., 60 feet in 20 seconds is 3 (and not 3 feet per second.) The easier method will be to give speed simply as feet, the number of feet being the distance which a body would have to travel under a given force in order to achieve that speed. The ease derives from thereby not having to consider units (See example and discussion below.)

3.7 AN IMPORTANT DIFFERENCE: UNDERSTANDING THE GENERATION OF MOTION

Given next is yet another theorem important to the functional treatment of mechanics. In the proof to this theorem Euler also draws an analogy between the analytic treatment of motion and geometry.

PROPOSITION 3
Theorem

33. *In any non-uniform motion the smallest elements of space can be conceived as traversed by means of uniform motion.*

DEMONSTRATION

For just as the elements of curved lines are treated as straight lines in geometry, non-uniform motion in mechanics is resolved into infinite uniform motions. For either the elements are in fact traversed by means of uniform motions, or the change of speed through the elements of this sort is so small that increase or decrease can be neglected without error. In either case the truth of the proposition is apparent. Q.E.D.

The concept of truth being employed is 'description without error'. What the demonstration is meant to establish is that, given the existence of small enough elements of the motion, the mathematics can be applied without error to real motions. This highlights the importance of Proposition 1. If a body occupies every place in a path then presumably the elements are small enough for the application of infinitesimal mathematics. The elements of the motion are not points but themselves motions. The "analysis" of non-uniform motion is into uniform ones. The issue of the size is sidestepped by functionally relating each elemental motion to a point. Euler does not need to hold that a uniform motion occurs in a point but rather *at* a point. Thus, his idea is different than Newton's. For Newton, the straight lines which, in the limit, make up the curve are the infinitesimal motions which make up the overall non-uniform motion.

This proposition introduces infinitely small quantities and so introduces some important considerations for Euler. First, "34. ... all change of speed in non-uniform motion is to be considered as happening at the beginnings of single elements, since we suppose that whole elements are traversed with uniform motion." This allows, in the "notation of the analysis of the infinitely small", representation of the speeds in successive elements as c, $c+dc$, $c+2dc+ddc$, etc. This treatments appears to be the same as the Newtonian picture, such as that employed in Proposition 1, of a series of impacts by forces, impressing changes in velocity. But Euler's view is more subtle, aided by a functional understanding, as we will see next.

As the second important consideration raised by Proposition 3, Euler recognizes a potential problem with the infinitesimal representation when considering the beginnings of motion from rest.

SCHOLION

36. The force of the given proofs depends on the fact that the change of speed that can occur while an infinitely small element is traversed must be infinitely small and vanish in comparison with the speed that the body already has; for if it were comparable, a finite motion would be generated in an instant, which is absurd. However, this proposition does not seem admissible if the motion and speed are infinitely small, in which case a momentary increase or decrease of speed can have a finite ratio to the former. But we will see more concerning this below, when the generation of motion is considered.

The question of how to understand the very beginning of motion from rest is also addressed by Galileo, asserting that body achieves every speed, as slow as you like in the finite time it takes to achieve its final speed (get ref.) Newton partly addressed the issue at Lemma IX and X, as we saw above, deriving Galileo's time squared law for the initial spaces at the beginning of motion. His proof in Lemma IX however, made no restrictions on the nature of the given curve which represented speed versus time. Euler addresses the issue squarely, at Proposition 4 and corollaries, through functions.

PROPOSITION 4
PROBLEM

37. *Suppose a body to be moved with an arbitrary non-uniform motion through the line AM, and the speed of the body to be given at every point; it is necessary to determine the time in which the arc AM is completed.*

The language here is compatible with functions—the speed at every point must be given—and he will avail himself of the elemental treatment of motions he has just established. The proof proceeds as follows. Euler's first move is to substitute analytic notation for geometric. Thus, distance along the curve AM becomes s, element of the curve Mm is denoted ds, and the speed, c, "will be a certain function of that same s." Now the time in which element Mm is run through is given by $\frac{ds}{c}$. As argued above, we can consider bodies to move uniformly through elements of space with the speed they have at the beginning of the element, in this case c. The total time for path AM is then found by treating all the elements alike and adding their times up, i.e. integrating $= \int \frac{ds}{c}$.

The crucial feature in what follows is the constant of integration. "Of course, some constant should be added to the integral that would make the whole time $= 0$ if it is supposed that $s = 0$, according to the familiar rules of integration." Now comes the kicker, enabled by the generalized treatment through a functional representation in the example that follows.

> 38. Let the speed at M be as an arbitrary power of the already described space AM, that is $c = s^n$; then $\int \frac{ds}{c} = \frac{s^{1-n}}{1-n}$. To which it is not necessary to add a constant, if $n < 1$ or has a negative value. ...But if $1 - n$ is a negative number, we will have
>
> $$\int \frac{ds}{c} = \frac{-1}{(n-1)s^{n-1}}\,.$$
>
> To which the constant $\frac{1}{(n-1)0^{n-1}}$, i.e. an infinite quantity should be added ....

The conclusion Euler draws is that an infinite amount of time is therefore required to move through s, "wherefore [the body] will remain forever in A and never leave that point." The corollary following this then makes the substantial claim that "[i]n the World no other causes subsist [alii casus subsistere] except those by which the speeds of motions are at least initially as the powers of the space run through [spatiorum percursorum] taken to exponents less than unity." (Euler, 1736, 22)

This is the more general form of what Galileo pointed out in the *Dialago*, where he discussed the impossibility of a motion for which speed was proportional to distance. However, Galileo drew the opposite conclusion, saying that such a motion would be instantaneous, not infinitely long. The two arguments are not incompatible though because what Galileo rightly derived was that all distances would be covered in the *same* time. He did not show that that time had to be zero. The issue is whether the ratios he employed applied to the beginning of motion, i.e. zero distance and speed. If so, then the ratio to which all speed / distance ratios were proportional would have to be $0/0$.

Newton considered the beginning of motion from rest from the point of view of velocities versus time and so missed this point. Also, he considered forces as proportional to the quantity of motion generated and so a force that might not generate a motion was never a consideration for him.

However, the real flaw of Newton's program, with respect at least to this issue, is that its perspective fails at precisely the point it is needed. In understanding the operation of forces throughout the *Principia* we are forced to employ two competing intuitions. One the one hand, we imagine the force acting over a short time and producing in the body a velocity. A stronger force will produce a greater velocity in that time; same force but greater mass results in a lesser impressed velocity. The competing intuition is then to imagine that process of impressing as *not* occurring over the given interval of time but occurring rather as one great impact at the beginning of the element.

This is a failing of Newton's perspective in that, I have tried to argue, the program is fundamentally about processes and changes; the continual flowing of quantities. But when it comes to forces, that picture breaks down, and we are given an instantaneous, impact kind of picture. This picture glosses over important details when the beginning of motion is considered because it relates an impact with a resultant motion. The mathematics analyzes paths as a process of generation by motions, but the generation of motion by forces remains an *unanalyzed* process. This reinforces the point that forces are alien to the real formalism for Newton and he never really brings them under his mathematics of change.

The functional picture, on the other hand (and, incidentally, the way Galileo thought about it), relates a velocity to each point in space (Euler's preference) or to time. The function reconstructs the actual process. Euler is thus beginning from a different perspective, focusing on motion and its relation to space and not on forces and accelerations. In particular, he reduces motions to speeds in each element and the functional relation between those speeds and location and considers their causes in the same terms. Also worth noting is that Euler states the conclusion he draws about what there is in nature in terms of causes and not forces. He has not yet, however, given us his technical notion of forces (potentia), nor shown how to treat them mathematically.

Definition 6 and 7 further illustrate the privileging of space that Euler is using and the functional understanding behind it. These define a scale [scala] of speeds and a scale of times.

DEFINITION 6

> 48. *A scale of speeds is a curve whose ordinates represent the speeds which a moving body has in the corresponding places of the space that it traverses.*

Similarly, at definition 7, for times. The relation between the two Euler treats by analysis. "[I]f we label the space AM $= s$; the speed at M, i.e. MN, $= c$; and the time in which AM is run through, i.e. MT, $= t$, then $t = \int \frac{ds}{c}$. The quadratures being granted, the curve AT can thus be constructed from a given curve AM." A Problem is then given, requiring the construction of the scale of speeds from a given scale of times. The construction is geometric. This provides a good example of Euler's accepting either geometrical or analytic representations of what is really the important mathematical object—the functional relation between time, space and speed.

An important Scholion follows this proposition which will take us to Euler's introduction of Newton's first law, which I consider in the next section.

> SCHOLION
>
> 55. It is to be noted here that what has so far been given of the scales of speeds and times is not only to be observed in absolute motion but also pertains to relative [motion]. For natural motion [motus natura] itself is not yet considered, nor anything is assumed which is not proper to absolute motion. Now though, we shall put forward certain propositions which are peculiar to absolute motion and by means of which an internal distinction between absolute and relative motions can in a certain way be perceived.

Euler will now describe for us why absolute rest and motions are the ones "appropriate to the laws of motion."

3.8 NEWTON'S THREE LAWS, EULER'S SEVERAL THEOREMS, OF MOTION

The point of this section is to show the complexity of the relations between Euler's mechanics and Newton's three laws of motion. The analysis is kept to a minimum here. A more general discussion of what is new about Euler's mechanics is given at other places in the thesis. Here, the goal is only to demonstrate the inadequacy of the view that Euler's *Mechanica* merely derives further consequences from Newtonian laws through the aid of analytic calculus.

3.8.1 The First Law

Beginning at Proposition 7, Euler introduces what amounts to Newton's First Law, but in three stages. It is not given as an axiom but is divided into a set of theorems and the set is derived from the definitions thus far introduced. One demonstration is required for the principle as it applies to bodies at rest while another two apply to the case of motion. I have already discussed this proposition at length in § 3.5 above (See p. 43) with regard to Euler's foundations for Mechanics. Here I discuss it in respect to Newton, and it will help to restate the Theorem.

Proposition 7
Theorem

56. *A body resting absolutely must remain at rest forever, unless it is drawn into motion by an external cause.*

The content of Newton's First Law has two components: first, rest and uniform motion in a right line are treated equivalently as states; secondly, there is a change in state if and only if an external force is impressed on the body. Euler, as described above, breaks these components up into three theorems (See Figure 3.10). There is a demonstration for the case of rest, for the case of motion with constant speed, and a demonstration that absolute motion proceeds in a right line.

The second half of the principle is given at prop 8.

Proposition 8
Theorem

63. A body having some absolute motion will move uniformly forever, and has moved at every point of time before now with the same speed, unless an external cause acts on it or has acted it.

Demonstration

For if a moving body does not forever retain the same speed, then its speed must either increase or decrease. In the latter case it inclines toward rest, which cannot happen, as it can never obtain rest (§62). In the former case it would have to be regarded as having moved forward from rest, which would be equally absurd. In addition, if we imagine this body as placed in infinite and empty space and consider the path which it is taking and has taken, there is no reason why it should have a greater or smaller speed in one place than it does in another place, wherefore it will always have to move with the same speed. Q.E.D.

This proposition is understood to apply to an undisturbed, naturally moving body. In corollaries to the previous proposition (the principle of rest), Euler has argued that a never-disturbed body that is at rest must have always been at rest. Therefore, a never-disturbed body *not* at rest can never come to rest. But this does not rule out that a naturally moving body might lose half its speed on some finite interval—such a body would never achieve rest, at least in a finite time. Euler considers exactly such a motion at Prop 4, Corollary 4 (Euler, 1736, § 43).

The real argument then seems to be the one from the principle of sufficient reason—though this too merely begs the question. But we wish to understand the structure of Euler's method, not the the success of his argument. With regard to the principle of sufficient reason, it too is only part of the story.

> 75. Although we have derived the perseverance in rest and the uniform continuity of motion in a straight line from the principle of sufficient reason, we have also observed that the latter is not the efficient cause of the the phenomenon, for that is located in the nature of body. This cause of the conservation of state that depends on the nature of bodies is what is called the power of inertia.

A body in a vacuum, by the principle of sufficient reason, we know would not change its state. But even given such true knowledge, a physical principle is still required to underwrite that knowledge. Our knowing does not make it so. But this is still a 'rational' move of sorts. A property of bodies is revealed by our thinking about their behaviour in a certain abstract situation (namely, in a void.)

Proposition 9

Theorem

> 65. A body provided with absolute motion will proceed in a right line, or the space which it describes will be a right line.

Demonstration

> There is in fact no reason, if this body is imagined placed in infinite and empty space, that it would deviate to any other direction than a right line. From which is concluded, that the progression of the motion in a right line depends on the very nature of the body. On account of which, in the actual world, where in fact this principle of sufficient reason does not hold, one must nevertheless conclude that every moving body must move in a straight line, unless, of course, it is hindered from doing so. Q.E.D.

This propositon and the previous one are combined to give the motion part of Newton's first law. "66. From these two propositions we obtain the universal law: every body endowed with motion proceeds uniformly in a straight line." Euler argues here again from the principle of sufficient reason. The conclusion here is only that the direction will be a right line. Another argument was given above for the uniformity of speed. We can only speculate as to why he did not choose to use the same sort of argument for the conservation of speed as well. But there is something different about the two cases. On the one hand we imagine an infinite and empty space and then argue that there is nothing about any region of that *space* that would provide a reason for deviating towards it rather than some other. In the speed case we would imagine an empty and infinite space and then have to argue that there is nothing about any other speed that would provide sufficient reason for change. The connection to empty and infinite space is either spurious or would require some extra steps in the argument. So fashioning the same pattern of argument for the conservation of speed is awkward at best.

Speculation of this sort is unnecessary, however, for the overall argument of this thesis. It is enough to show that Euler does not simply adopt Newton's laws as axioms. In fact, in this case Euler's position on inertial motion is very much like Newton's. There is a faculty inherent in bodies which is responsible for conserving their state of motion; rest and motion are both states which inertia acts to preserve. Euler felt the need for further argument to establish these principles and the existence of the faculty. But Euler is also moving away from Newton by introducing his potentia-vis distinction. It is potentiae which are responsible for the change in states. Euler is introducing a difference among explanations required. The main role that vis inertia seems to play for Euler is in justifying the extrapolation of the case of a body in infinite and empty space to the real world. This distinction is the first step to abandoning the need for vis inertia and for imposed explanations in general. By the time we have arrived at Lagrange, there will be a complete reliance on the mathematics to indicate which things provide explanations and how.

3.8.2 Second Law

We saw with the first law that Euler points out to the reader where he has established it. In the case of the second law no such convenient sign posts have been left. Therefore, we need to unpack Newton's second Law for its content so we know what to look for in Euler. These two definitions and the law are needed.

DEFINITION II

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.

DEFINITION VIII

The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.

LAW II

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

These must always be understood as statements about ratios and proportionalities. For instance, one might think that because mass does not change we can reduce the idea of a change in the quantity of motion to a change in velocity. This is not so though, because the changes only have a measure through comparing one with another. Only when changes are in the same body will the ratio of the changes in motion be proportional only to the velocities. In comparing forces therefore, their quantity always depends on not just the velocity produced but also the masses in which those velocities are produced.

There are dangers lurking in the 'given time' notion and we saw them above. In understanding the operation of forces throughout the *Principia* we are forced to employ two intuitions. One the one hand, we imagine the force acting over a short time and producing in the body a velocity. A stronger force will produce a greater velocity in that time; same force but greater mass results in a lesser impressed velocity. The competing intuition is to imagine that process of impressing as *not* occuring over the given interval of time but occurring rather as one great impact at the beginning of the element.

Euler's treatment of the basic action of forces on bodies is very carefully spelled out in an entire chapter of *Mechanica*—the second chapter, entitled "On the effect of *potentia* acting on a free point". The details of that chapter relevant to his understanding of the second law are given here. Key questions are: does Euler give anything like an $F = ma$ version of the law, does he give an impulse version in the way that Newton does, or something else? How does his understanding of the 2nd law relate to functions?

One aspect of Euler's careful treatment is that he does not give one overarching form of the second Law The content of Newton's version of the law can only be found in Euler spread over several theorems, definitions, and scholia which characterizes the relation between potentiae (Euler's technical term for forces) and motions. For Euler, the most important feature of mechanics is that it is the science of motion. The bodies considered can be in motion and this effects the operation of a potentia on those bodies. He thus considers individually the different ways in which the motion of a body may be related to the direction of the potentia. At the same time, the basic measure for the comparison of potentia is founded in equilibrium and is taken from Statics.

Euler begins his careful treatment by introducing a technical term, *potentia*, a species of the more general *vis*, defining it as whatever is capable of changing the state of a body.

DEFINITION 10

99. Potentia *is a* vis *either inducing a body of out of rest into motion or altering the motion of that body.*

Potentia is distinct from, e.g. vis inertia, which does not change the state of the body in which it inheres. The direction of the potentia is also introduced as a technical term.

DEFINITION 11

103. The direction of a potentia *is a right line, following which the body endeavours to move.*

Following this definition is the passage discussed above in pointing out the connection between Statics and Mechanics for Euler. Euler treats the direction of a potentia, at least here in *Mechanica*, as something fixed in time. The direction may follow some rule dictating that it has a different direction at different places in space, but the rule does not change with time. As a body moves the direction of the potentia *relative to the body* can change in time though.

The measure and comparison of potentia is taken from Statics. The fundamental measure has to do with the ratios of potentiae. The ratio between potentia a and b is $m{:}n$ when it is the case that if a is applied m times to a point A and b is applied n times to the same point but in the opposite direction, then the point will remain in equilibrium. Euler gives the formulae $na = mb$ and $a : b = m : n$. Thus the basic notion of the measure of force is intuitively an equilibrium notion.

Another important distinction, from Definitions 12 and 13, is between absolute and relative potentia. A potentia is absolute when it has the same effect on a body whether it be moving or resting. A relative potentia, such as friction, has a different effect depending on the motion of the body on which it acts. The measure for potentia taken from statics, which applies to bodies in equilibrium and so at rest, can now be extended to absolute potentia and their effect on moving bodies.

This is done at Proposition 14, Euler's equivalent of the parallelogram rule. Prop 14 is a problem which asks for the effect of an absolute potentia on a body having some motion, given that the effect on the same body at rest is known. (See Figure 7.) The given motion is with speed c along direction AB. The known effect of the potentia is characterized as: If the body were resting at A, the potentia would move the body through $AC = dz$ in time dt Let $AB = cdt$. With the potentia acting on the body it will be found, after time dt not at B but at D and the measure of the

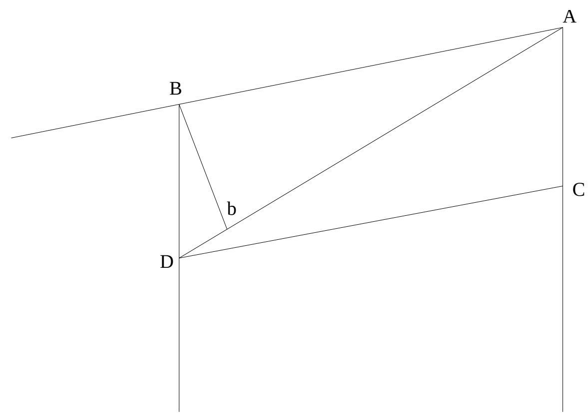

Figure 7: Effect of potentia on a moving body

effect of the potentia will be given by BD. That effect must be equal to the effect on the resting body, that is AC. So the length $BD = AC$. Moreover, since the time dt is infinitely small the direction of the potentia will not change and so BD is also parallel to AC. And so the body will run through space AD in time dt. The change in speed produced, dc, is given by $\frac{Db}{dt}$.

The effect of time is next considered and it is shown that the increment of speed is proportional to the time. The direction of the potentia is kept fixed.

PROPOSITION 15
Problem

130. Given the increment of speed, which a certain potentia *induces in a point A in a small time [tempusculo] dt, to find the increment of speed which that same potentia will produce in the same point in time $d\tau$.*

The answer is given in the solution.

SOLUTION

...Consequently the increments of speed are proportional to the times in which they are produced.

The derivation of the solution is interesting in that it treats time through elements of space. Point A has speed c in direction AB (Fig. 8.) The effect of the potentia is taken to be space ao which it would draw the body through, if it were resting, in time dt.

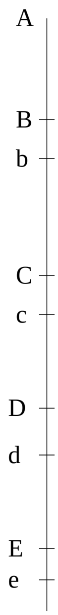

Figure 8: A falling body

AB is the space the body would travel by speed c in time dt, Ab is the space it travels with the addition of the potentia. Therefore $Bb = ao$. This space is infinitely small so we treat it as travelled with uniform motion. Then in the next instant, if the potentia did not act for the next dt, we could take $bC = Ab$. The effect of the potentia is added again, $Cc = ao$. And so on. Putting this together, Euler obtains

$$Ab = AB{+}ao;\ bc = AB{+}2ao;\ cd = AB{+}3ao;\ dc = AB{+}4ao\,. \tag{3.1}$$

> Therefore, $\frac{ao}{dt}$ will be the increment of speed [celeritatis] produced by the potentia in time dt; $\frac{2ao}{dt}$ will be the increment of speed [celeritatis] acquired in $2dt$; similarity $\frac{3ao}{dt}$ the increment in time $3dt$; and generally time ndt increases the speed of the point by element $\frac{nao}{dt}$. (Euler, 1736, 48)

That is, we derive the increase of distance and divide those by the element of time to arrive at the increase of speed. Now let $d\tau$ equal some ndt, i.e. $n = \frac{d\tau}{dt}$. The increase of speed over those n elements of dt is the increase of speed in time $d\tau$ and is equal to $\frac{nao}{dt} = \frac{d\tau ao}{dt^2}$. Considering this value in ratio to the increment of speed for dt, which is $\frac{ao}{dt}$, "produces this proportionality [analogia]: the increment of speed in time dt is to the increment of speed in time $d\tau$ as dt to $d\tau$. [24]

This proposition is another nice illustration of the transitional nature of Euler's mathematical approach. The initial solution is reasoned in terms of ratios and proportionalities and sounds very Newtonian or classical. But the proportional reasoning is accomplished with the aid of n, which is a constant of proportionality. We can see this also in the corollaries to the proposition. For instance, corollary 2 and 3 consider a body beginning at rest, and in them Euler derives Galileo's law of free fall for uniform acceleration. If a body begins at rest, then in the equation for increments of speed, Eqn. 3.1, AB, which is due to the initial speed over the given time, drops out and the increases in speed are just the $ao, 2ao, 3ao$, etc.. At corollary 3 then we take c to be the speed acquired, from

[24] Note the proof assumes $d\tau > dt$ by some integer multiple, but the result is easily generalized to any rational number by introducing some intermediary $d\tau'$. Say, for instance, that $d\tau = \frac{n}{m}dt$. Take $d\tau' = ndt$. The proportionality then obtains for $d\tau'$ and dt. Now take $d\tau' = md\tau$ and the proportionality holds for them as well. Proportionality among ratios is transitive and reflexive so it obtains between $d\tau$ and dt. I'm not sure what to say about "incommensurable" times here.

rest, after time t and while covering space s. The proportionality between time and speed is cashed out in functional form as $t = nc$. Employing an earlier result (§ 37), $t = \int \frac{ds}{c}$.

> This produces, therefore, $nc = \int \frac{ds}{c}$ or $ncdc = ds$ and this [produces] $s = \frac{nc^2}{2} = \frac{t^2}{2n}$. Therefore, the first spaces at the beginning of motion are described in duplicate ratio of the time or speed through the space acquired. (Euler, 1736, 49)

Euler freely moves back and forth between proportions and equations which employ constants of proportionality.

In yet another theorem Euler introduces the proportionality of potentia to the quantity of matter.

PROPOSITON 16

Theorem

136. Potentia *q will have the same effect on point b as potentia p on point a if q:p = b:a*. This theorem also takes assumes that some known measure of the effect of a potentia is given. We take $q = np$ for some n. Then $b = na$. What the latter means is that we can imagine body b divided in n equal parts [in n partes aequales divisum], the parts being equal in the sense that the effect of some given potentia on each of them is the same. And each part is acted on by an n-part of the potentia np. Assuming all the parts remain conjoined, then we can "find no discrepancy" between the two cases: either a body $= na$ acted on by a potentia $= np$ (conceived of as one body, one potentia) or n similar parts each being acted on by p. That is, the effect of np on na is the same as p on a.

"Vis intertiae," Euler goes on in the theorem of Prop 17, "of any body is proportional to the quantity of material of which it consists." The quantity of material is measured in the counting manner of the previous proposition. He has earlier (§ 74) defined *vis inertia* as that faculty inherent in a body by which it remains at rest or continues in its state of motion. The proposition just completed has shown us that the effect of a potentia on a body is inversely proportional to the quantity of matter. In other words, the ability of a body to resist a change in its state is proportional to its quantity of matter. And so vis inertia, the ability to resist change in state, is proportional to the quantity of matter.

No mention is made of *mass* in Proposition 17 or in its corollaries, but in a scholion to the proposition before we have

SCHOLION 1

> 189. This last proposition provides the foundation for measuring vim inertiae, for that always depends on this reasoning, by which the material of a body or mass must be considered in Mechanics. It is necessary to attend to the number of points out of which the moving body is composed, and to which the mass of the body is put proportional. The points ought to be reckoned [censeri] equal with each other, not those which are equally small, but on which the same potentia will exert equal effects. If therefore, all matter in this way in equal parts or elements is conceived divided, it is necessary to assess [aestimari] the quantity of matter of any body by the number of points out of which it is composed. We will show in the following proposition vim inertiae is proportional to this number of points or quantity of matter [materiae].
> 189. Propositio ista fundamentum complectitur ad vim inertiae metiendam, hac enim nititur omnis ratio, quare corporum materia seu massa in Mechanicis considerari debeat. Attendi enim oportet ad punctorum numerum, ex quibus corpus movendum est conflatum, eique massa corporis proportionalis est ponenda. Puncta vero ea inter se aequalia censeri debent, non quae aeque sunt parva, sed in quae eadem potentia aequales exerit effectus. Si igitur universam materiam in huiusmodi aequalia puncta seu elementa concipiamus divisam, quantitatem materiae cuiusque corporis ex numero punctorum, ex quibus est compositum, aestimari necesse est. Vim autem inertiae proportionalem esse huic punctorum numero seu quantitati materiae in sequenti propositione demonstrabimus.

Here too, Euler is weakening the notion of vis inertia required for his mechanics. In practice, it is treated through a more convenient measure, a heuristic device for obtaining the relative quantity of matter between bodies. Notice the measure itself relies on the basic measure Euler has introduced from statics. The parts which we are to count are those on which the same potentia will have the same effect. But at Prop 13 we saw that the same effect is judged by equilibrium.

3.8.3 Third Law

Law III is, for Newton, crucial to his treatment of systems of bodies (e.g. *systemate corporum plurium* (Newton, 1713, 17)) as it allows for treating the motion of several bodies through the motion of their center of gravity. But this is more than a convenient problem-solving device. It is the device by which the motions of points that make up a finite body are unified, mathematically and physically. It is their mutual gravitation—their reciprocal and equal action on one another—which unifies them physically. This action underwrites their mathematical treatment through their center of gravity. Thus Newton's third law plays its greatest part in the third book.

Moreover, the third law props up the solar system for Newton. The post-Copernicus universe has been set adrift as it were. It no longer has its stationary center. The sun does not provide a new center either because it is not at the center and the planetary motions are no longer circles. The gravitation of the sun pulls on the planets and keeps them in orbit about it, but how can the sun pull if it itself is not stationary?

Newton also identifies a role for the third law in collision and mechanics, and through that role a justification for the law.

> The power and use of machines consist only in this, that by diminishing the velocity we may augment the force, and the contrary; from whence, in all sorts of proper machines, we have the solution of this problem: *To move a given weight with a given power*, or with a given force to overcome any other given resistance.(Newton, 1962/1686, 27)

This he relates to the third law as follows.

> But to treat of mechanics is not my present business. I was aiming only to show by those examples the great extent and certainty of the third Law of Motion. For if we estimate the action of the agent from the product of the velocities of the several parts, and the forces of resistance arising from the friction, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate action will be always contrary to the reaction.(Newton, 1962/1686, 28)

This broadens the notions of action and reaction. The second law tells us that the effect of a force is to change the quantity of motion and there is no reason to think that the action of the force should be something different than its effect. In the above though, action is to be estimated from the product of the velocities and the resistances. The third law then applies under these suitably extended notions of action and reaction.

For Euler the understanding of bodies and their composition is more complex. It is not simply a matter of the mutual gravitation among parts. Mechanics is a project divided up according to the different ways in which bodies can be comprised of their points, whether rigidly, elastically, fluidly, as a gas, etc. Bodies have a center of gravity but this is a technical term and it is defined in terms of the motion of points of a body. It is really the centre of mass. More importantly, it is not gravity which is responsible for points moving with their center of gravity.

At Definition 14 toward the end of Chapter II, Euler introduces *vis restituens*, an "imaginary and infite *vis*, which brings back together the parts of bodies separated by *momento* and returns them to their original state." The theorem at Prop 22 demonstrates that the *vis restituens* will, in fact, cause the points to be brought back together at the center of gravity after it has moved in accordance with the resultant force. There is no identification of this restoring force as necessarily gravitational though. It is an imagined force and potentially infinite. With Newton's gravitational force, coupled with the third law, the points remain in contact with the center of gravity because of the opposite and equal action among them. For Euler the point is to allow treatment of force in the more general context as being responsible for the nature of bodies in whatever way they are comprised by their points.

3.9 CONCLUSION OF THE CHAPTER

In this chapter I have endeavored to show something of the complexity of the relation between Euler's own science of mechanics and that of Newton. It is not simply a matter of Euler beginning from Newton's Laws of motion and using analytic methods to re-derive the motions of bodies in various situations. The project of mechanics itself is reconceived in a way suggested by the mathematics. Functional descriptions are employed by Euler to relate motions and forces to positions in space. An abstract, mathematical space is introduced in place of real absolute space. This space has limits imposed on it which provide reference points for determining positions. The nature of bodies as occupiers of space and as distinct from space itself is crucial. The properties belonging to those bodies, their states of motion and the forces acting on their parts, can then be expressed as functions on space.

Euler's distinction between potentia and vis is likewise related to this representation. Potentia have differing relations to bodies as they move through space, their direction of action being determined in part by their own functional relation to space and the location of the body in that space. How they operate on the body is then given by what the function specifies is their direction for each point of space which the body occupies. The picture is less temporal then it is spatial.

3.10 TABULAR SUMMARY OF EULER ON NEWTON'S THREE LAWS

In the following tables, the translations are mine. The Latin for Newton is from the 2nd edition (1713) of the *Principia Mathematica*, the edition I have argued Euler is most likely working from. For Euler, the translations are made from *Mechanica*.

Table 1: Law I: Comparing Newton (left) with Euler

LAW I *Every body continues in its state of resting, or moving uniformly in a right line, except insofar as that state is compelled to change by impressed forces.* *Corpus omne persevare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*	PROPOSITION 7 Theorem *56. A body resting absolutely must remain perpetually at rest, unless it is incited to motion by an external cause.* *56. Corpus absolute quiescens perpetuo in quite perseverare debet, nisi a causa externa ad motum sollicitetur.* PROPOSITION 8 Theorem *63. A body having an absolute uniform motion will be moved perpetually, and it will move with the same speed at all times, unless an external cause drives or bears on it.* *63. Corpus absolutum habens motum aequabiliter perpetuo movebitur, et eadem celeritate iam antea quovis tempore fuit motum, nisi causa externa in id agat aut egerit.* PROPOSITION 9 Theorem *65. A body provided with absolute motion will proceed in a right line, or the space which it will describe will be a right line.* *65. Corpus absoluto motu praeditum progredietur in linea recta, seu spatium quod describit, erit linea recta.*

Table 2: Law II: Comparison of Newton (left) and Euler.

DEFINITION II *The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.* *Quantitas Motus est mensura ejusdem orta ex Velocitate et Quantitate Materiae conjunctum.* DEFINITION VIII *The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.* *Vis centripetae Quantitas Motrix est ipsus mensura proportionalis Motui, quem dato tempore generat.* LAW II *The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.* *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*	DEFINITION 10 *99.* Potentia *is a* vis *either inducing a body of out of rest into motion or altering the motion of that body.* *99. Potentia est vis corpus vel ex quiete in motum perducens vel motum eius alterans.* DEFINITION 11 *103. The direction of a* potentia *is a right line, following which the body endeavours to move.* *103. Directio potentiae est linea recta, secundum quam ea corpus movere conatur.* PROPOSITION 15 Problem *130. Give the increment of speed, which a certain* potentia *induces in a point A in a small time, find the increment of speed which that same potential will produce in the same point in time $d\tau$.* *130. Dato celeritatis incremento, quod quaedam potentia in puncto A tempusculo dt producit, invenire incrementum celeritatis quod eadem potentia in eodem puncto tempusculo $d\tau$ producit.* SOLUTION: ...Consequently the increments of speed are proportional to the times in which they are produced. PROPOSITON 16 Theorem *136.* Potentia *q will have the same effect on point b as potentia p on point a if q:p = b:a.*

Table 3: Law III: Comparison of Newton (left) and Euler.

LAW III	No role in *Mechanica*
To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.*	

4.0 LAGRANGE'S USE OF EQUILIBRIUM

Joseph-Louis Lagrange was born Giuseppe Lodovico Lagrangia in Turin in 1736, the year the first book of Euler's *Mechanica* was published. Euler would become a mentor to Lagrange. It was in a letter to Euler that 19 year old Lagrange communicated his solution to the half-century old isoperimetric problem, in which he used what we now call the calculus of variations. Lagrange succeeded Euler as the director of mathematics at the Berlin Academy in 1766, on the elder's recommendation, and would remain in Prussia for 20 years.

This chapter has the following straightforward structure. I'll first describe the use that Lagrange makes of equilibrium in the Statics section of his *Mecanique Analytique*. Second will be a description of the use he makes of equilibrium in the Dynamics section. The final section will be a comparison of both uses.

4.1 STATICS

Lagrange's *Mecanique* is divided into to two sections, Statics and Dynamics, each section beginning with a history of the principles employed in the domain up to that point. Statics and Dynamics are distinguished according to whether the body or bodies are in motion or not.

The basic formula of Statics for Lagrange, is the Principle of Virtual Velocities.

> 1. The general law of equilibrium in machines is that the forces or powers [puissances] are between them reciprocally as the speeds of the points where they are applied, measured [estimées] along the direction of these powers.
>
> In this law consists that which one commonly calls the *Principe des vitesses virtuelles*, the Principle recognized for a long time as the fundamental Principle of equilibrium . . . and that one can consequently regard as a kind of axiom of Mechanics. (Lagrange, 1788, 12)

Lagrange then goes about "reducing this Principle to a formula." The reduction amounts to a derivation or argument which relies crucially on two things. First, on rearranging mathematical terms so that they all appear on the left-hand side of an equation and are equated to zero. Secondly, the argument relies on the notion of functions.

He begins by introducing the symbols to be used, providing a discussion and explanation of their meaning along the way. The powers [puissances] are P, Q, R, etc.. From the points at which these forces are applied we imagine drawing lines p, q, r, etc. along the direction of the powers and we "designate in general, by dp, dq, dr, etc., the variations or differences [différences] of these lines, insofar as they result from any change infinitely small in the position of the different bodies or points of the system." (Lagrange, 1788, 13)

We are to understand these variations as the "spaces run through in the same instant by the powers" and so they express "the speeds [vitesses] of these powers estimated along their directions". The variations are thus differential velocities, though Lagrange has not yet defined a virtual velocity.

To proceed, Lagrange first imagines the case of three acting powers, P, Q, R, and the system at equilibrium. Then take one of the points to be fixed, say the point at which R acts, and consider the equilibrium condition of the remaining two powers. For the remaining two points to be at equilibrium they must be "disposed in a manner" such that they are constrained to move in contrary directions to one another and so the values of dp and dq will have opposite signs. Lagrange assumes P, Q, R etc are always positive. Therefore, since the principle states the powers must be reciprocally as the speeds of the points to which they are applied, and as dp and dq we can treat as speeds, then according to the principle applied to these two forces we have

$$\frac{P}{Q} = -\frac{dq}{dp}\,.$$

This can be improved upon, Lagrange says, as

$$Pdp + Qdq = 0\,. \tag{4.1}$$

Similarly, considering the other combinations of powers on their own, we have

$$Pdp + Rdr = 0 \qquad \text{and} \qquad Qdq + Rdr = 0 \tag{4.2}$$

and the trick now is to argue for their combination. This cannot be straightforwardly done as for the first equation we assumed dr was zero, for the second that dq was zero and the third assumed dp is zero. Combining them would then seem to demand that all three simultaneously be zero.

This is where the function notion comes to bear. Lagrange points out that, in order for there to be equilibrium among the powers they cannot move independently of one another. There is, therefore, a given relation among dp, dq, dr and consequently among the finite quantities p, q, r.

> Therefore, in virtue of this relation, whatever it may be, the variable p can be regarded as a function of the two other variables q and r; and its differential dp can be, consequently, expressed in general as $dp = mdq + ndr$.

Notice that m and n are what we would now call the partial differentials of p with respect to q and with respect r. A similar, general formula will hold for each of the differentials. These formulae are general in the sense that none of the points are being held fixed. The only constraint being maintained is the non-independence of the variables. These formulae allow us to make substitutions for the differentials which appear in each of the three equations. These differentials will then be generally true and so allow one to treat the three equations as a system of equations.

> ...the term Pdp which is found in the first two equations can be represented by $Pmdq$ in the first of these equations, and by $Pndr$ in the second; so that the sum of these two terms will be $P(mdq+ndr) = Pdp$. One can prove in the same way, by regarding q as a function of p and r, that the sum of the two terms Qdq which enter in the first and in the third equation will reduce simply to Qdq ...equally the two terms Rdr which are found in the last two equations, will reduce to Rdr, (p and q being variables at the same time in dr). So that the sum of the three particular equations found above, will be found, in regarding p, q, r as variables at the same time $Pdp+Qdq+Rdr = 0$; the formula of equilibrium of three arbitrary powers P,Q,R. (Lagrange, 1788, 14-15)

Consider the three equations again:

$$Pdp + Qdq = 0 \tag{4.3}$$

$$Pdp + Rdr = 0 \tag{4.4}$$

$$Qdq + Rdr = 0\,. \tag{4.5}$$

In the first equation we are taking $dr = 0$. In that case we can replace $dp = mdq + ndr = mdq$. In the second equation $dq = 0$ and so there $dp = ndr$. So we actually have to combine the Pdp of the first equation with the Pdp of the second to obtain the correct value of Pdp. Similarly combining the first and third will give us the correct Qdq; combining second and third, the correct Rdr. We can therefore sum all three, the argument goes, and regard p, q, r as variables at the same time, none of them being held fixed. The result is what Lagrange is here calling the Principle of Virtual Velocities.

The important distinction is between the "particular equations", where one of the variables was fixed, and the general equations which incorporate the functional relation among the terms. The representation dp is not the same quantity in each particular equation. Understanding the functional dependencies of dp provides the proper way to combine the three equations. To say there is a function among the variables is the same as saying the variables are not independent. This will be an important distinction for understanding the difference between unconstrained differentials of quantities as opposed to their constrained variations, which we will see below. What is important to see here is that the functional relation or constraint among these variables is a condition of their being at equilibrium. Equally important is the physical, or indeed, *mechanical* reasoning going on behind the formalism. Only mechanically do you understand that dp is not the same in each equation. Only on the justification of further mechanical reasoning can you then combine the equations.

The principle of virtual velocities, then, in principle and then equation forms are the following.

Principle of Virtual Velocities (PVV) If an arbitrary system of any number of bodies or mass points, each acted upon by arbitrary forces, is in equilibrium and if an infinitesimal displacement is given to this system, in which each mass point traverses an infinitesimal distance which expresses its virtual velocity, then the sum of the forces, each multiplied by the distance that the individual mass point traverses in the direction of this force, will always be equal to zero (Lagrange, 1788, 23).

$$Pdp + Qdq + Rdr + \text{ etc.} = 0 \tag{4.6}$$

Notice the rhetorical difference between the two. It is as if Lagrange is sweeping away the 17th century altogether by first presenting the turgid, long-winded first expression, which he magically transforms into the elegant equation form. Especially powerful is the "etc.". And all these terms, on to infinity if needed, come to zero.

Although there is no notational difference to go along with it, Lagrange is careful to distinguish

dp, dq, dr from ordinary differentials. These are virtual velocities.

virtual velocity The velocity which a body in equilibrium would take if the state of equilibrium ceased to exist, that is, the velocity that the body would have in the first instant of motion.

These definitions come from his prefatory survey to the statics of the "Various Principles of Statics".

As stated though, the PVV is really a principle of work. At equilibrium the system does no *net* work. This is clear from the formulaic representation of the principle, Eqn. 4.6, where P, Q, R are forces, and dp, dq, dr are differentials of arbitrary lines in directions of the applied forces. The equation expresses a sum of work differentials, that is infinitesimal forces times distances.

Notice that even at the Static level there is already some dynamics creeping in in the form of allowable motions that preserve equilibrium. In fact, this characterization of equilibrium is defined in terms of motions, albeit infinitesimal or first motions. We've moved from the simple (naïve) version of equilibrium where "nothing happens" to a kind of recognition of invariance. In other words, on this form of equilibrium, things can happen to a system—even motion—and yet it remains at equilibrium. According to this principle a balance can be rotated through any angle, for instance, and remain at equilibrium. What will matter, it turns out, and especially in the dynamic case, is that there is a balanced exchange of certain quantities.

Returning to Equation 4.6, although this equation will be used most often as the starting point in the following definitions, suggesting virtual work is the proper foundation of the *Mecanique*, Lagrange derives this formula from another. This alternate form also represents the short prose version of the principle.

$$\frac{P}{Q} = -\frac{dq}{dp} \tag{4.7}$$

If we can assume that dq, dp are virtual velocities, this is the mathematical expression of the principle stated in its ratio form. In general, the distinction between a virtual displacement (i.e., a virtual velocity) and a true differential like dp is crucial. In fact, it is in pointing out this difference to Euler that Lagrange makes his first great impression on him. Virtual displacements are displacements compatible with the constraints of the system and hence, not entirely independent of one another in the way that differentials are supposed to be. What makes the conflation of the two okay here is that this equation and the "work" equation (Equation 4.6) have the constraints built in—they define the virtual velocities as those displacements or differentials compatible with the imposed condition on equilibrium. In fact, the only constraint operating at this general a level, since we don't know anything about the nature of the system, is that it is at equilibrium.

This allows us to recognize—in the equation—a new characterization of equilibrium. We begin with the short prose version, which we see as characteristic of machines (more on this in a minute). Then, by algebraic manipulations that preserve both the mathematical equality as well as the physical one, we arrive at the longer form. From this we can read the condition on the total of all force $\times$ distance.

What makes these displacements virtual velocities is that all the differential displacements occur in the same time due to the mechanical constraint among the parts. This is how the Principle of equation 4.7 is recognized, first by Galileo. Because the lever is a machine and so its parts are connected together, the motions of those parts occur in equal times.

Also notice about the virtual form that it is only partly spatial. The differentials are of lines in the direction of the forces but there's no need for a geometric representation of the forces. Neither should this equation be read as providing a geometric interpretation. These differentials aren't like the vanishing deflections of Newton in the first proposition of the *Principia*, for example. In the Newtonian case the force is responsible for deflection of the path from rectilinear motion to the curve and taking that deflection to represent that force is a natural move—the length of the deflection is proportional to the strength of the force. What we have in the Lagrangian case is purely an algebraic representation of the forces. One cannot say here that force is proportional to the differentials since the equation expresses an *inverse* ratio between forces and their displacements. Moreover, the differentials are never used to eliminate forces from the equations—both are used together in the virtual work equation.

The most fascinating part of the statics is the physical model Lagrange gives for the principle of virtual velocities. More precisely, he describes the model as a demonstration of the principle of the pulley, though the difference seems to be so subtle as to be no difference at all, as we'll see.

The pulley model begins with a mechanical system of interconnected mass points to be modeled. Acting on each mass is assumed a force, represented by the letters $P, Q, R, \ldots$. Now imagine instead that each force is represented by a block and pulley assembly with a weight attached corresponding to the mass point. Since all points are part of the same system we take all the pulley sub-systems to share the same rope, using whatever manner of idler pulleys are required to route the rope from block to block. To the tag end of this rope is attached a unit weight while the other end is fixed.

The number of pulley loops in each block and pulley is determined by the force acting on the mass attached to that block. If the force is 2 then the block and pulley has two loops and hence a mechanical advantage of 2.[1]

What we have now, in effect, is two coupled mechanical systems: the system of block and pulleys that represents the mass points and forces of the system being modeled as well as the one weight system of the unit weight attached to the end of the rope. Because of the coupling, the equilibrium condition of one system will dictate that the second is also at equilibrium. This is the key. The equilibrium of the lone weight is easy to recognize: *it is not moving*.

How does this condition translate into equilibrium for the modeled system? Notice that the forces $P, Q, R, \ldots$ also represent the number of loops in each block-and-pulley. If we let dp, dq, dr represent the respective movements of the mass points then $Pdp + Qdq + Rdr \ldots$ represents the total change in length of the rope in all blocks. Therefore, the condition that the weight attached to the tag-end of the rope not move entails that sum of $Pdp, Qdq, Rdr, \ldots$ equal zero, which is precisely Equation 4.6.

Now we might worry about the pulley model itself being a mechanical system since using it as the basis of a model for another mechanical system would seem to involve a vicious circularity. If I can't understand a mechanical system unless I understand a system of pulleys and if understanding a system of pulleys requires understanding it as a mechanical system than this approach to modeling is either unnecessary or impossible. In either case it would be useless.

But it is not useless and why it is not illustrates the usefulness of mechanical equality. The pulley mechanical system is supposed to embody intuitive conditions for equilibrium, conditions which can be expressed in equation form. The real work to be done is in demonstrating the applicability of that equation to other systems, with the demonstration in turn relying on supposed intuitions about things that "make no difference". E.g., a force pulling on a body is the same force whether the pulling is by a rope or gravity or a nailed on two-by-four. These aspects make no dif-

[1] In principle a force of any rational number could be represented in this way, not just the integers, by choosing the appropriate "unit" weight. E.g. if the forces to be modeled were 22/7 and 1.4, choosing 70 for the unit weight would dictate 220 loops for the first force and 98 for the second.

ference and this last step, of porting an equation from one system to another, therefore amounts to further modeling through what is really idealization or abstraction. In other words, there is already very sophisticated physical modeling going on at the outset of analytic / algebraic mechanics in the 18th Century. If one system can be equated to another for which we can intuitively recognize its equilibrium conditions then we only have to map those conditions back on to the original system to find *its* equilibrium conditions.

The example just given should illustrate how this occurs. Another example would be Archimedes' proofs of lever laws by analyzing a balance with two different weights into a balance with identical weights distributed at different places. Then the conditions for equilibrium of the original system is just the conjunction of the equilibrium for each subsystem.

As yet another example consider James Bernoulli's analysis of Hüygens' compound pendulum as a coupled system of simple pendula. The complication here, and why it proved so difficult to solve, is that the coupling involves feedback and hence non-linearities. Even though it is still the case that the equilibrium conditions of the compound system are a conjunction of the simple conditions, this conjunction does not slide over into a simple mathematical combination. Another wrinkle of course is that we're now talking about dynamic equilibrium and that case is more complicated so we'll return to it below.

Before turning to Dynamics though, there is one further aspect of Lagrange's view of equilibrium that I'd like to point out. Lagrange notes that one might object to his pulley-block model that requiring no motion at all is a stronger condition than is required for equilibrium of the system. Since the only possible spontaneous movement of the unit weight is downward movement (it won't move up on its own) then even displacements that allow upward movement of the weight are allowable equilibrium conditions. In other words, the system remains in equilibrium if

$$Pdp + Qdq + Rdr + \ldots \geq 0 \tag{4.8}$$

since a positive total displacement *within* the pulley-block system means more rope is taken up (the length of rope wrapped around the pulleys is greater) and hence the tag end is shorter. Only a total displacement less than zero would allow a lowering of the weight.

Lagrange again demonstrates the sophistication of his view of the relation between mathematics and physics in his first response to this objection. He first points out that, since the equation governing the equilibrium is a differential equation,

> the relations between these quantities are linear and one or several among them will necessarily be indeterminate and could be taken as positive or negative. Therefore, the values of all those quantities are such that they could change sign.(Lagrange, 1788, 25)

Lagrange also makes a second and equally interesting *converse* objection. Assuming that the equality does hold, there's something unique about the condition when equated to zero. Namely, when it obtains for a set of values of displacements that set will contain a negative value for every positive one. I.e. we can change all the signs of the displacements and still have an equilibrium condition. This provides a reason (a principle of sufficient reason, in fact) for why that condition should be an equilibrium condition since for any displacement of the system allowed there is a completely symmetrical but opposite displacement also allowed.

4.2 DYNAMICS

When it comes to the dynamical cases, that is cases where motions (both velocities and accelerations) are involved, there are two ways to think of equilibrium. One is internal: the equilibrium of the interactions between the parts of a machine as it moves and changes. The other is external: the equilibrium between the net non-zero forces and the (accelerative) motions produced. As we'll see below, for Lagrange accelerative motions are not the same as accelerations. We can understand external equilibrium as obtaining between causes and effects but the effects of forces, for Lagrange, are not accelerations but accumulating velocities. What this means and its importance will become clear below.

The first kinds of equilibria are going to be simple extrapolations of static equilibria, most often understood by Lagrange and others as lever interactions. This is the case, e.g., for James Bernoulli's analysis of the compound pendula. Consider the pendula first as a coupling of two simple pendula, one short and one long. The coupling is going to influence the way the two simple pendula behave: the short pendulum will be slowed down from its natural frequency by the longer pendulum while the longer pendulum is going to be sped up from its natural frequency by the shorter pendulum. This was Bernoulli's insight as was the further insight that these two interactions could be viewed as two lever interactions and hence their equilibria found. Lagrange describes it this way.

> The velocity of the first weight is thus transfered to the second and as this transfer is made by means of a mobile lever about a fixed point, it must follow the law of equilibrium of forces applied to this lever. Thus it must occur in such a manner, that the ratio of the loss of velocity of the first weight to the gain of velocity of the second is reciprocal to the length of the arm of the lever.(Lagrange, 1788, 177)

But Lagrange also goes on to point out that Bernoulli errs in considering only total velocities transferred, e.g. average speed over a swing, rather than the differential transfer. As he continues

> the idea of referring to the lever the forces resulting from the velocities gained or lost by the weights is very astute and provides the key to the true theory. But James Bernoulli was mistaken when he considered the velocities acquired during an arbitrary finite time, instead he should have only considered the elementary velocities acquired during an instant of time and compared them with those that gravity would impress during the same instant. (Lagrange, 1788, 177)

This instantaneous aspect facilitates the shift from the static case to the dynamic one. The common currency of both cases is the virtual motion or the counterfactual motion—the motion that would just occur in the first instant if allowed. The difference is that in the dynamic case the net virtual motion that would result is non-zero.

Come now to the second kind of equilibrium: equilibrium between forces as the causes and the effected motion. It will be easiest simply to present the equation and then break it down.

$$S\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right)m + S\left(P\delta p + Q\delta q + R\delta r + \ldots\right)m = 0 \tag{4.9}$$

This is clearly an equilibrium equation. It is obtained in fact, by transposing an equation in which the second summation appears on the right side of the equation. The equation is derived, in other words, by setting the first summation equal to the second. We'll see that Lagrange justifies setting these quantities equal as a representation of an underlying physical equilibrium.

The S operator in this expression is a summation operator over the masses involved in the system. Each mass point may have a different m and will have its own coordinates, x, y, z, forces $P, Q, R, \ldots$ and force centers $p, q, r, \ldots$. The force centers are arbitrary points along the line of action. Since the equation only depends on variations of these lengths the actual lengths is arbitrary. Also, since the forces, by definition, act *toward* their centers they act to shorten the lines p, q, r and so the differentials dp, dq, dr will be negative.

The instantaneous velocity of each mass point is given by dx/dt (and so on for each of the three coordinates). "And these velocities, if the body was then left to itself, would remain constant in the following instants, according to the fundamental principles of the theory of motion." (Lagrange, 1788, 185) To understand this fundamental principle of motion will require a digression into Lagrange's understanding of force and its effect.

A distinction must be marked between accelerations and accumulation of velocities. Lagrange does not assume that the effect of forces is acceleration. Rather, forces "produce velocities that increase with time." The difference is subtle but important, for it turns on what can and cannot be explained.

> In mechanics, the effect of the force acting alone must be assumed known and the art of this science consists of deducing uniquely the composed effects which must result from the combined and modified action of the same forces.

The point is that by assuming that the accelerations are the result of the accumulation of impressed *velocities*—an accumulation that can be calculated and delivers the curve—the only mysterious, or at least assumed, relation left is the force-velocity cause-effect relationship. Lagrange is interpreting the force-acceleration relation as an accumulation of impressed velocities which are the real effect of forces. Accelerations are understood as the accumulation of the impressed velocities. These impressed velocities are differential when the instants in which they occur are infinitesimal.

If the action of forces were understood merely as the impressing of infinitesimal increments of velocity then there would be no point in arguing that the effect of forces were anything other than accelerations. However, the difference is important for Lagrange since he views the acceleration of bodies by forces as merely an extension of the composition of forces. Just as two forces in different directions but impressed at the same time can be composed so too can two forces at different times be composed within one body. Furthermore, because the impression only takes place in an instant the impression must be one of a straight line motion. Impressing a curved motion would require a finite amount of time. Thus the only motion a body can receive is a composition of straight line motions.[2] The picture is really one of impetus, rather than inertial movement.

This is why bodies, when no further force acts on them, will continue to move with uniform rectilinear motion. This is the only motion that can remain impressed in them in the absence of any further force. All curved paths require the continual impression of deflecting velocities from the straight line. Motion is a state for Lagrange, just as it was for Newton or Euler or Descartes, but no force of inertia is required to preserve the continued motion. If any motion did act in the direction of inertial motion the result would actually be acceleration through the composition of the accumulating differential velocities.

Having said that, there are forces that enter into the equation above that Lagrange calls accelerative forces. Properly speaking, what is entered in the equation is the *measure* of those forces, $ddx/dt, ddy/dt, ddz/dt$. "These increments can be treated as new velocities impressed on each body and by dividing them by dt, one will have immediately the measure of the forces of acceleration used to produce them ...." (Lagrange, 1788, 185) These velocity increments multiplied by the mass, m, give a measure of the accelerative forces. They are called accelerative forces only to distinguish them from retarding forces, i.e. forces that are negative with respect to the virtual velocities of the bodies. Their effect and their measure are still velocities, albeit differential increments of velocity, and their total effect is

$$S\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right)m.$$

The masses here serve as part of the measure of the force too. The same virtual velocity in two bodies of different masses actually indicates a stronger force acting on the more massive body. Thus the effect of the forces acting on all bodies in the system must weighted according to the mass of each of those bodies. The next step is determining what this sum should be equated to in order to solve it for the resulting motions.

Now, whatever the resulting motions are, we can imagine forces that, if they were to act on the system of bodies could generate further motions that counteracted the already existing resulting motions and hence resulted in the system being at equilibrium.

[2] To understand what an alternative view might be like imagine that the effect of a force on a body is to impose a sinusoidal motion of a given frequency. Then the motion of a body at any given instant is determined by the combination of all the imposed frequencies up to that point.

> . . . if it is imagined that upon each body the motion that it must follow is impressed in the opposite direction, it is clear that the system would be at rest. Consequently, these motions should cancel those that the bodies would have and that they would have followed without their mutual interaction. Thus there must be equilibrium between all these motions or between the forces which can produce them. (Lagrange, 1788, 180)

The sum of the effects of those forces with the effects of the accelerative forces just considered would then be zero. In the dynamical equation given those forces are $P, Q, R \ldots$. The equation is to be understood, then as expressing the equilibrium between the forces acting on the system, as the causes or inputs, and the resulting motions, which are measures of the effects or outputs.

The input-output language is not entirely anachronistic. Lagrange's view is very much that the forces acting on the system are modified by the system itself. Evidence for this is the way he talks about the compound pendulum case that we have already looked at. "[I]nstead [Bernoulli] should have only considered the elementary velocities acquired during an instant of time and compared them with those that gravity would impress during the same instant." (Lagrange, 1788, 177) Gravity acts on the system and would tend to impress velocities on the bodies of the system. However, the mechanical connections between the parts cause those impressed velocities to be both shared among the parts and composed in mathematically capturable ways. For instance, if gravity is one of the forces $P, Q, R, \ldots$, its effect, in terms of the resulting virtual displacements, is determined by the entire system of other forces, the masses and their displacements. The reason that gravity doesn't have its full or normal effect is due to an implicit equilibrium condition holding among the internal parts of the machine.

> If a motion is impressed upon several bodies so that they are forced to move consistent with their mutual interaction, it is clear that these motions can be viewed as composed of those which the bodies would actually follow and of other motions which are negated, from which it follows that these latter motions must be such that the bodies following only these motions are in equilibrium.(Lagrange, 1788, 179)

In other words, some of the displacements impressed by gravity on the system are counteracted by the displacements impressed on other parts of the systems. These parts of the system are then in equilibrium with one another. The system in a sense, acts a displacement sink—velocities go in, but they don't come out.

4.3 EQUILIBRIUM FROM STATICS TO DYNAMICS

As Lagrange points out

> it is obvious that this formula only differs from the general formula of statics [Eqn. 4.9] by the terms resulting from the forces $md^2x/dt^2, md^2y/dt^2, md^2z/dt^2$ which produce the acceleration of the body ....(Lagrange, 1788, 186)

This makes the move from statics to dynamics explicit. When the forces applied to the system, symbolized by P, Q, R, etc, are such that their effect is zero then the second sum in the dynamical equation is zero. This leaves the sum of *accelerative* forces also equal to zero, which is another kind of dynamic equilibrium. It does not necessarily entail that all the motions, $dx/dt, dy/dy, dz/dy$, are zero, only that their total effect is zero. This latter condition can be rearranged to show that it does entail that the center of gravity of the system does not undergo any accelerations. In other words, uniform rectilinear motion of the center of gravity of the system is compatible with equilibrium of the system.

If we reinterpret the pulley-block example in light of this generalization we see that the condition on possible virtual displacements can be read as a kind of symmetry condition. The terms being added in the dynamical and the static equations are what Lagrange calls moments but which are, in modern terms, virtual work terms. They are forces times distances. Equilibrium of the system is then the condition that net work done by the system is zero. This doesn't rule out all motion since some motions, e.g. some movements of the weights in the pulley-block system, could result in no net work being done. The equation therefore identifies an entire family of system configurations that may be reachable from a given equilibrium configuration without changing the energy of the system.

Of course, energy is not a concept yet available to Lagrange. But in the dynamic cases considered, where the static forces do not sum to zero, the equation dictates that the quantity of moments different than zero is equivalent to the effect of the motions produced in the system as a whole—that is, the motion of the center of gravity of the system. What will later be identified as the kinetic energy of the system. It is a short move, but not one made by Lagrange, from the idea of equilibrium between two quantities, expressed mathematically, to reification of that quantity. This will occur with Hamilton, for example, along with a greater emphasis on the forces themselves as the basic explanatory elements as opposed to the configurations and virtual displacements of the bodies of the system.

4.4 CONCLUSION

What has been describe so far involved no geometry. This is the most salient and oft pointed out character of Lagrange's *Mecanique Analytique*. The only occurrence of something like geometrical reasoning occurs early in the text in the prefatory chapter to the statics section and there the geometrical aspect of it is neither very convincing or very deep.

The example occurs in demonstrating that "the weight of any body depends only on its total mass, and not on its shape" (Lagrange, 1788, 12). We're asked to

> imagine a triangular plane loaded by two equal weights at the two corners of its base and by one of double weight at its top. This plane, which is supported on a straight line or fixed axis passing through the midpoints of the two sides of the triangle, will obviously be in equilibrium since each of the sides can be visualized as a lever loaded at each end by an equal weight and with its fulcrum on the axis passing through its midpoint.(Lagrange, 1788, 13, see Fig. 9.)

In the figure the weights at A and B are 1 unit and the weight at C is 2 units. Considering the side AC as a lever then A supports half of the weight at C and the other half is supported by BC considered as a second lever.

Equivalently, Lagrange points out, we can consider the two shaded lines as levers. Lever AB is obviously in equilibrium given the midpoint as its fulcrum. But this fulcrum is one end of the transverse lever loaded with the double weight at the other end C. We thus have one balance supporting another. But no geometric argument is given or even deemed necessary in this case. "*It is obvious* that the base lever is in equilibrium with respect to the transverse lever which it carries at its midpoint." (Lagrange, 1788, 13, my emphasis.) There is an interesting move here. What's claimed is not just the each lever is in equilibrium but that each is in equilibrium *with respect to each other.*

We've seen, I think, in the examples given, how powerful a notion this combination of systems in equilibrium idea is. Lagrange himself recognizes that "the superposition of equilibria in mechanics is a principle as fecund as the superposition of figures in geometry." (Lagrange, 1788, 13)

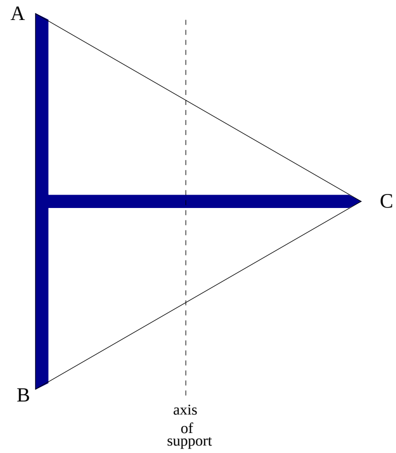

Figure 9: The weight of a body depends only on its mass and not its shape

My claim is that this is understatement. The superposition of equilibria in mechanics is *far more fecund* than any similar idea in geometry.

5.0 THE MOTION OF A BODY IN RESISTIVE MEDIA: A COMPARISON OF DEMONSTRATIONS

The following gives a detailed description of the solutions to one of the few propositions which can be found in common in Newton, Euler and Lagrange—the motion of a projectile through a resistive medium and attracted downwardly by gravity.

Close attention is paid to representation, whether a mathematical feature corresponds to some real-world aspect or is a geometric one (or both), and to how and why. With regard to the why, we will find intuitive the fact that particular real-world aspects of the problem are being represented.

We have expectations that certain factors are going to be relevant to the solution of the problem and so finding their representations set out in the course of the solution, particularly at the beginning, makes sense.

There are various reasons for our expectations. Some of the factors we expect to be relevant derive simply from the stating of the problem. The statement of the problem tells you what you are given and indicates the things you are supposed to solve the problem from. Problems are in fact worded this way, following a schematic such as Problem: Given x,y,z it is required to find (or construct, or define or determine) w, where 'w' can be a curve, a speed, a proportionality. . . which of these exactly though, will be an important feature in what follows.

Other factors are expected to be relevant for only implicit reasons. They are not explicit in that they are not listed as givens in the setting out of the problem. They are nonetheless expected given that the context is mechanics. A mechanical solution can be expected to be given in terms of things like forces, masses, other velocities, etc. Such quantities are all implicitly expected in mechanical contexts as they are the sorts of quantities that might enter into mechanical, causal explanations of phenomena or they are quantities that appear in known laws or principles of mechanics. For example, although it is not likely that the quantity time-squared has ever been described as the cause of the distance travelled by a uniformly accelerated body, it is nevertheless a perfectly reasonable mechanical quantity given the well known time-squared law. And so a representation provided in a diagram for the square of time would be mechanically expected. What we will see in the examples however, is that it is not simply the case that mechanics is mechanics is mechanics. What might be implicitly expected in a solution from the *Principia* may not be top of the list in an Eulerian or Lagrangian proof. The representation of time is an example.

Moreover, implicilty expected factors are also determined by the kind of mathematics being used. This is where another important difference arises between geometric and algebraic mechanics: in the geometric case there are representations introduced which may be implicitly expected given the geometric context but that would not otherwise be expected from the mechanical context alone. This is almost necessary in geometric contexts given the need to build connections among structures in the diagram. The harder the problem the more elaborate are the geometric structures required to obtain a solution, and so it is more probable that unexpected combinations of real represented quantities will arise. This distinction in expectations will be referred to as geometrically expected versus mechanically expected and examples are given below.

Finally, there will be some geometrical features introduced into diagrams which are not even geometrically expected. Rather their utility is demonstrated in earlier lemmae or propositions. I refer to these as *auxilliary structures* and an example from Newton is given in this chapter, where he employs a hyperbola to represent the relation among speed, time and distance for motion in a medium having a resistance proportional to velocity. Such representations almost never occur within the functional or algebraic approach. What is introduced into the mathematical representation when using functions is (almost always) implicitly expected for mechanical reasons (except, perhaps, in instances such as a substitution of variables to facilitate an integral.)

As we will see below, two of the roles that auxiliary structures and geometrically expected entities play in diagrammatic proofs are 1. to geometrically represent certain complex mathematical relations among quantities; 2. to allow the "moving around" of quantities in the diagram by introducing new structures to represent quantities already represented at other places (e.g. the sides of similar triangles provide two different representations of the same ratio.) In analytic mechanics, functions fulfill the first role, while moving quantities around, as per the second role, is understood rather as moving the same representation around rather than as the introduction of a new representation somewhere else. The variable is moved to the other side of the equation, for example. This latter feature has the side-effect that quantities so represented appear individuated, and as part of the ontology of the problem. This is an interesting effect when the quantity being represented and manipulated is one that might have otherwise, in a geometric context say, been considered a kind of activity, such as a motion or an attraction.

5.1 THE MODERN TREATMENT

The following is a detailed analysis of the way a solution to the problem of motion by a projectile through a resistive medium would be obtained today, using college-level physics. The standard procedure, in what is called in the textbooks 'Newtonian Mechanics', is to begin with the second-order force equations as given in the problem and integrate to obtain the equations of motion. The problem is standardly considered solved when the equations of motion are given. The position of the body in space at each point in time is given. Since one is integrating a second-order equation, two constants of the integration are required, usually initial velocity and position. These constants allow for agreement between the numerical values of the model and real world values, such as where the projectile is with respect to our chosen frame of reference at the time we have chosen to label zero.[1]

For a body moving through a resistive medium, where that resistance is proportional to the velocity, the equation describing the operation of the forces is

$$F = -kv\,, \tag{5.1}$$

where k is a constant of proportionality introduced to provide the correct kind of units (so that we have units of force on both sides of the equation) and the correct scale between units. Our choice of unit for velocity and for force is arbitrary, depending on what we take to be the standard against which we measure. Once the choice of units is made nature dictates what the pairs will be. That is,

[1] Other constants than initial velocity and position could be used. For example, the angular momentum may be given or total energy. This would require using an arbitrary constant of integration and manipulating the solutions to the integrals to obtain expressions for the known quantities. E.g., if the force equation is integrated to give some $v(t) + c$ equation then the total energy can be used to determine the constant of integration c from Kinetic Energy $= \frac{1}{2}mv^2$. But in the end it is only the constants of the integration that are arbitrary.

nature dictates the function between values. k allows us to match our representation of the function to the real one.

This equation can be expressed in second-order form by using "Newton's second Law" $F = ma = m\frac{d^2x}{dt^2}$, so that we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\frac{dx}{dt} \tag{5.2}$$

In this case however, it is more convenient, recognizing that we have v on the right hand side, to integrate this equation instead:

$$\frac{dv}{dt} = -\frac{k}{m}v \tag{5.3}$$

obtaining $\ln v = -k/m\,t + c$ or

$$v(t) = v_0 e^{-\frac{k}{m}t}\,, \tag{5.4}$$

($v_0 = e^c = v(0)$.) The functional notation used here is often read as 'the function v operates on the argument t'.[2] But this is not the way that real velocity is understood—velocity is not an operator.[3] It is the value the function outputs for that time t which describes the real quantity. In fact, $v(t)$ is better understood as second-order notation meant to represent something about the right hand side of the equation. Namely, it is a reminder that the right hand side operates on t as the independent variable. When we read $v(t)$ as describing a physical value, it means we have chosen to take t as the independent variable and specify velocities in relation to it.

From this functional representation of $v(t)$ we can obtain the equation of motion by another integration. Velocity represents instantaneous change of position and we can represent instantaneous changes in calculus as derivatives. So $\frac{dx}{dt} = v_0 e^{-\frac{k}{m}t}$ or

$$x(t) = x_f - \frac{mv_0}{k}e^{-kt/m}\,. \tag{5.5}$$

The modern solution privileges time and considers the values of other physical quantities as dependent on them. In your college physics class you learn to recognize certain, simple features of equations such as this, giving them physical interpretations. The second term, for instance, is a decay term, exponentially going to zero as time goes to infinity. As the second term goes to zero, $x(t)$ approaches ever closer to the final value x_f, though never quite reaching it in finite time.

[2]Euler is the first to use functional notation in various papers written in 1734. See below.

[3]A remark about QM is in order, where the connection between observables and operators is much tighter.

When considering a projectile also affected by gravity the same procedure can be applied in the y-direction. (Newton, it turns out, only solves the problem for resistance acting in the horizontal direction.)

5.2 NEWTON

Newton treats the motion of bodies in resistive media in Book II of the *Principia*. In our 'modern' solution, the second step was to employ what is today taken to be Newton's second law. This gave us the differential equation which was then integrated. The form of the law used was $F = m\frac{d^2x}{dt^2}$. This is not, of course, Newton's form of the second law and so there is a difference in how he proceeds.

First we must recognize that Newton's demonstrations, working in the synthetic fashion, employ ratios and proportionalities. Also, his second law actually states that the quantity of motive force is proportional to the change in motion (not force = mass $\times$ acceleration.) Using more modern notation for the moment, we could represent the resistive media relation as

$$\frac{v_1}{v_2} \propto \frac{\Delta v_1}{\Delta v_2}\,. \tag{5.6}$$

v_1 and v_2 are two velocities and their ratio is set proportional to the changes in those velocities. Time does not appear here. In fact, for Newton, these velocities would really be lengths, these being used as measures for the velocity on the assumption that they are distances travelled in the same time. The Δv quantities represent the changes in quantity of motion at the place in the motion where the respective velocities obtain. We might imagine these speeds over the motion as being discretized and then take the subscripts to indicate successive velocities, $v_1, v_2, v_3 \ldots$. The Δ factors would then be given by $v_2 - v_1, v_3 - v_2, \ldots$.

This relation is convenient for taking pairs and comparing them, but a more "continuous" representation can be had by expressing the resistive media relation as

$$\frac{v_1}{\Delta v_1} \propto \frac{v_2}{\Delta v_2} \propto \frac{v_3}{\Delta v_3} \ldots. \tag{5.7}$$

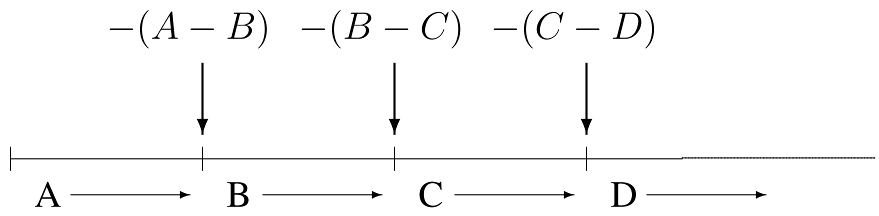

Figure 10: Speeds and resistive impulses at successive equal time increments.

This is, in fact, how Newton proceeds. His begins, at Lemma I to Proposition 1, Theorem 1 of Book II, with representing successive quantities (not necessarily speeds) as A, B, C, D etc.. If these quantities are proportional to their differences then those quantities are in continuous proportion: “let A be to A-B as B is to B-C and C is to C-D, etc. and then by division it will be that A is to B as B is to C and C to D, etc. *Q.E.D.*”(Newton, 1713, 211). This rule and the rule that quantities in continued proportion follow a geometric progression would be well known to any geometer of the time. (The definition of a geometric progression today is that the ratio of consecutive terms is a constant.)

This Lemma is then applied to the speeds in a resitive medium.

PROPOSITION II
THEOREM II

> *If a body is resisted in ratio to its speeds [velocitas], and if only by its* vi insita *is it moved through a* Medium similare*, and supposing equal times: the velocities in the beginning of each time are in Geometric progression, and the spaces described in each time are as the velocities.*

The time is divided in equal parts and the *vis resistens* is assumed to act with a single impulse at the beginning of each temporal part. Consider the figure 10. The speeds through each instant are given beneath the line; the differences between each element are given above, shown acting at the beginnings of those elements. The impulse between element A and B, e.g. is represented by their difference, $A - B$, with a negative sign for resistance. This impulse is proportional to the speed A, so the Lemma attaches and the consecutive speeds are then in continuous proportion.

This result is applied for a projectile moving through a resistive medium and being drawn downward by gravity. To obtain a geometric diagram of the path the geometric progression of speeds will need a diagrammatic representation.

PROPOSITION IV
PROBLEM II

If there be posited a vim gravitas in a Medium everywhere uniform and similar, and which tends perpendicularly to the plane of the Horizon; to find the motion of a projectile in that Medium subjected to a resistance proportional to the velocity [velocitatem].

Newton breaks the solution into the actual construction of the motion, which ends with *Q.E.I.*, followed by an explanation and demonstration of why the construction meets the requirements of the solution asked for. The latter demonstration ends with *Q.E.D*. The problem asks for the motion of the projectile. That is, construct not just the path of the projectile but also specify its velocity at each point on that path.

Newton begins by constructing line DP (See Fig. 11) which represents both the direction and the magnitude of the velocity. The direction is given by the angle between the line DP and the line representing the horizontal. Angles are interesting quantities because they are defined in terms of direction and so have no scale. They do, however, have an orientation. We could begin by assuming a given line as the direction of the horizontal and then take angles relative to that. The problem as set out though, appears to take the initial direction of the projectile as given and the horizontal as relative to that. The diagram though uses the horizontal on the page as the conventional reference point.

The length of the line DP will serve as the baseline magnitude for the velocities. Other representations of velocities in the diagrams can be made by lengths relative to the line DP. This representation requires two things. We have to understand these line lengths as scale representations of real lengths, with the same scale employed throughout the diagram. To interpret these lengths as relative velocities, we must assume both distances would be travelled in some same amount of time T by the bodies having those respective velocities. This second implicitly understood temporal factor makes some distances represented in the diagram actually representations of velocities. The real initial velocity is a given of the problem, therefore other real velocities will be given because of the diagrammatic relation of their representations to the representation of the initial velocity. Note also, that we can properly call these velocities given that DP fixes an orientation for directions.

Moreover, in the case of both the angle and the length of the line, it is unimportant to the

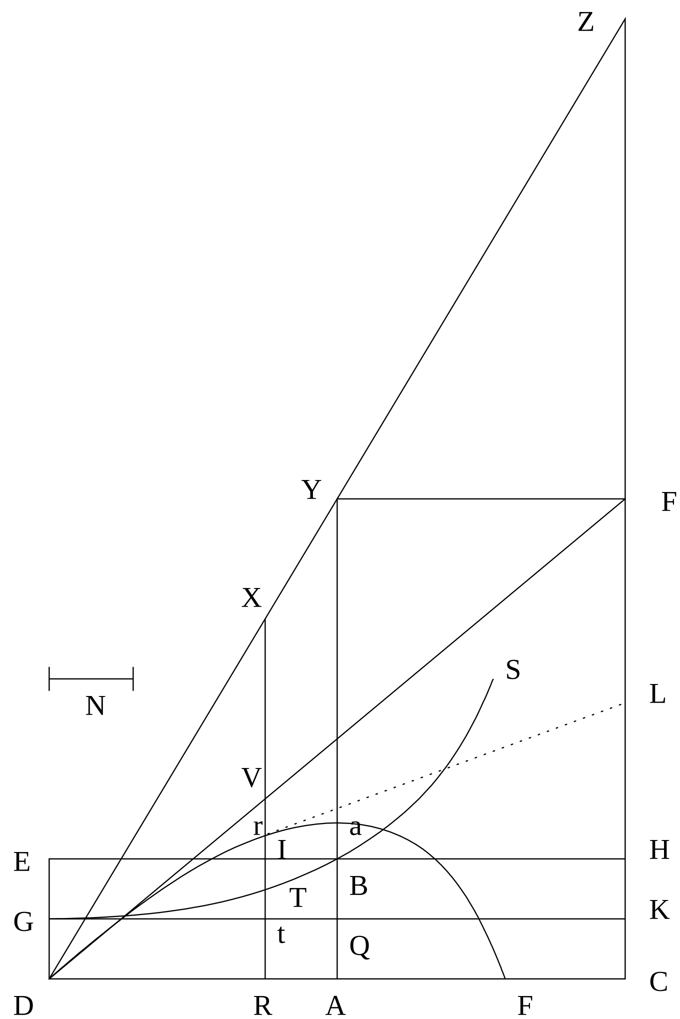

Figure 11: Path of a projectile moving in a resisting medium.

solution provided that they accurately represent any particular real quantities. The point of the problem is to demonstrate how, if certain things were given, other things could be derived. If the solution is to be actually applied, careful attention would have to be paid to the relation between the representation and the real quantities. Attention would have to be paid not only to the scale of the lengths but also to the implicitly assumed unit of time.

Continuing with the demonstration, a perpendicular to the horizontal is then constructed below P, cutting the horizontal line at C and making the line segment DC. Point A is introduced between D and C so that DA and AC have a specific ratio. The ratio of DA to AC is to be proportional to ratio between the initial force of resistance due to motion in the upward direction and vim gravitas. Or, "which is the same," take instead the ratio between the "rectangle under DA and DP to the rectangle under AC and CP" (Newton, 1713, 216) and this ratio will be proportional to that between the initial resistance due to the *whole motion* (rather than just the upward component) and the vim gravitas. (The ratio DP/CP is the inverse of the ratio between the whole initial motion and the vertical part of that motion.)

The ratio of the mentioned rectangles has the mechanical interpretation provided by the ratio of forces to which they are proportional but the rectangles themselves do not pictorially represent anything (understood literally as rectangles they are not even features of the diagram.) Neither is it clear what is given by multiplying DA by DP or AC by CP. These quantities in this guise are not mechanically expected. In this regard, thinking of DA times DP as a rectangle (and similarly AC times CP) is misleading. It is closer to their real meaning to think of the ratio $DA : AC$ compounded with the ratio $DP : CP$. That is, to think of the construction as the compounding of the ratio between the vertical component of resistance and gravity with the ratio of the whole resistance to the vertical component. Taking the ratio of the sides DP and CP is, I would argue, geometrically expected, as that ratio could be used with similar triangles. It is also mechanically expected in that it represents the slope of the initial velocity (an angle of elevation in artillery terms.) Note that CP also represents the height that the projectile would have obtained after time T had there been no gravity and no resistance. Time T is the implicit feature of the proof mentioned above and determined by the representation of the initial velocity as the distance DP. But as regards the original rectangles, it is not clear at all why either quantity—initial resistance times initial velocity, or the force of gravity times the height the projectile would have obtained—should be relevant to the solution. These quantities certainly do not appear in any laws or well known principles of motion nor do they lend themselves to causal explanations of the resulting projectile motion. (How could the 'the force of gravity times the height the projectile would have obtained' act on anything?) So interestingly, these rectangles are neither mechanically nor geometrically expected, while the identical compound ratio is expected in both ways.

Even if we could provide an intepretation of these rectangles, we will see below the introduction of further rectangles having the same ratio. These would seem to be redundant to the solution of the problem given the representation we already have of the forces to which they are proportional. This, however, is a generic feature of geometric demonstrations in that quantities need to be "moved around" the diagram by showing how they can be represented in different structures throughout the figure. This was pointed out above and it will be seen again below when we turn to the demonstration after the construction.

First, we see next in the construction an example of what I am calling an auxiliary structure. A hyperbola $GTBS$ is described having the given asymptotes DC and CP. In the preceding few propositions Newton has demonstrated that this auxiliary structure can be used to represent the relation between speeds, distances and times that hold when a body moves through a resistive medium. At Prop II, he has demonstrated that for a body travelling in a medium having resistance proportional to speed, the velocities at the beginning of equally distributed instants will be in a geometric progression while the spaces are as the velocities. In a corollary to Proposition II he introduces the hyperbola as a structure which captures these relations. In the Figure 5.2 AC is the initial speed and DC is the speed remaining as time progresses. The time will be given by the area $ABGD$. "Now if that area [$ABGD$] is augmented uniformly in time by the motion of point D, line DC will be decreased in Geometric ratio to the velocity and the parts of line AC described in equal times will decrease in the same ratio." This is a known relation of hyperbolae presumably (found in Archimedes?) Newton understands the property in terms of motions. In fact, he most likely was thinking of fluxions in the background. The area is the fluent that represents time. The length of the line DC is also a fluent. The motion of the point D represents the change in the velocity (given by DC) while also representing the distance travelled (AC). This representation is specific, however, to a medium that resists in proportion to the speed. A different auxiliary structure needs to be constructed if the resistance is proportional to, say, the square of the speed.

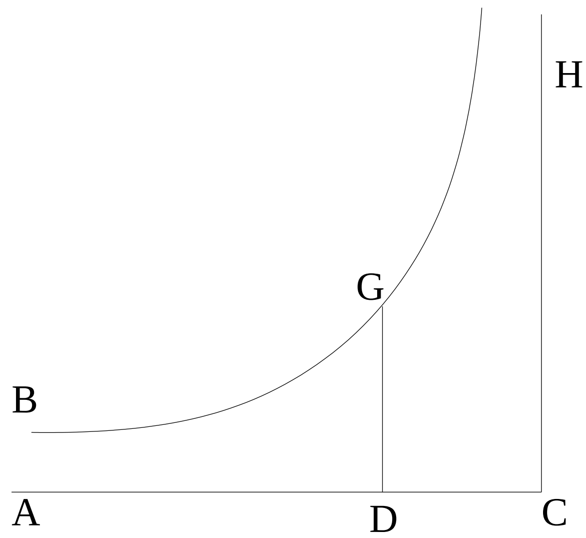

Figure 12: The hyperbola auxiliary structure

The introduction of the hyperbola now suggests further features be entered in the diagram, such as line segments DG, AB. And with DG entered it is a simple matter to construct the parallelogram $DGKC$. The side, GK, of this parallelogram cuts AB at a point which is labelled Q. These steps in the proof are all geometrically expected but they are mechanically anything but. Examining the auxiliary structure we can see that the ordinates (such as GD or AB) must be proportional to what would be $1/$ the acceleration in modern units, i.e. distance over time-squared. In a Newtonian context it is even less clear what such units would represent in mechanical terms. This can be seen by unit analysis. We are given that the abscissa is speed and the area is time. Now, since the area is the integral under the curve, its units (time) must be equivalent to the units of the abscissa (distance / time) multiplied by the units of the ordinate. I.e., rearranging, the units of the ordinate must be time-squared over distance. So an interpretation can be provided for the diagrammatic features DG, AB but I maintain that the quantity $1/$ acceleration is mechanically unexpected. In particular, because this quantity is associated with an auxiliary structure, it most likely will not play a role in a similar but altered version of this problem. For in the altered version a different auxiliary structure will be used and the expected geometric features will then most likely have different interpretations in terms of mechanical quantities (algebraic combinations of speeds, accelerations, times etc.)

As a specific example of this, I point out that Newton will consider the case of a resistance proportional to the square of the velocity (Book 2, Section II.) In that case an auxiliary hyperbola is also used to represent the speed-distance-time relations. However, there the abscissa represents time (not speed), the ordinates are velocites (not inverse of accelerations) and the areas are distances (not times.)

Returning to the construction, the next steps are

- to construct a line segment N having a length whose ratio to QB is in proportion to the ratio of DC to CP. This element serves interesting but strictly mathematical purposes, to be described in a moment, but does not represent any feature of the physical phenomenon. It will be eliminated before the final result is obtained.
- erect, at some point R, the perpendicular RT and label its intersections I,t and V.
- line EH is to be entered, although we are not told so explicitly. This is a geometrically expected element. Its details do not even need to be given, yet we can surmise what they must be from examining the diagram. We expect, that is, that EH will be parallel to DC and pass through the point B. These two "constraints" are apparent from the diagram and they determine the line segment. This is the sort of basic background knowledge that provides part of the geometric context in which such elements are expected.

Next comes the most important element which is point r. r is a point on the trajectory we are seeking to construct and is specified as follows. On the perpendicular take [cape] the distance Vr equal to $\frac{tGT}{N}$. "Or, which is the same, take Rr equal to $\frac{GTIE}{N}$." The construction of point r appears to require a calculation—the area tGT is divided by the length N and Vr is set equal to that length. Vr is not specified through a ratio and proportionality. However, in the calculation presented in the demonstration we will see that there is an implied ratio to unity. How this comes to the same thing as taking Rr equal to $\frac{GTIE}{N}$ is also found in the explanation after the construction. It follows from simple algebraic manipulations (this is taken from the demonstration proper which follows after *Q.E.I.*):

> N is to QB as DC is to CP or DR to RV [by similar triangles]. And so RV is equal to $\frac{DR \times QB}{N}$, and Rr (that is $RV - Vr$ or $\frac{DR \times QB - tGT}{N}$) equals $\frac{DR \times AB - RDGT}{N}$.

Area $GTIE$ is equal to these last two equal areas, $DR \times QB - tGT$ and $DR \times AB - RDGT$, and so Rr is equal to $\frac{GTIE}{N}$. But from this it is not yet clear why we should enter Vr first and not simply Rr. Rr represents the height of the projectile at horizontal location R (it gives the y-coordinate for a given x-coordinate, as we might put it today.) Again, our mechanical expectations are confounded by the geometrical procedure. We press on and hope that the need for Vr will become clear.

Returning to the construction, where we have just taken Rr equal to $\frac{GTIE}{N}$, it is now noted that (as per the auxiliary construction) area $DRTG$ will represent the time the projectile takes to arrive at R from D and hence the time after which "the Projectile ... arrives at r, describing the curved line $DraF$, which point r always touches."(Newton, 1713, 216) Recall that in the auxiliary structure RC represents the speed of the particle relative to the medium but DR represents this distance. This is a convenient feature of the case of resistance proportional to speed, whereby the distances are as the speeds. But it is unique to that case.

The curve reaches a maximum height of AB, after which it approaches the asymptote PLC. "And the velocity [of the projectile] in any point r is as the Tangent to the Curve rL. ***Q.E.I.***" (Ibid.) None of these last three features pointed out are obvious. The projectile must reach some maximum but why above A? (Recall that A was chosen so that DA to AC is as the initial resistance to the force of gravity.) And considering the forces present we can see that the lateral motion will become less and less while the downward motion will be perpetually accelerated, so that the motion will approach some vertical line, but why CP? Finally, the velocity of the projectile at each point r will have a direction tangential to the curve of course, but it is not at all obvious that distance rL should represent the magnitude of that velocity. We now turn to the demonstration and corollaries and see what further explanations are provided there.

After showing that Rr is equal to $\frac{GTIE}{N}$, as discussed above, the a distinction is made between the sideways motion [latus] and the upwards motion [ascensus]. In the auxiliary construction no attention is paid to the direction of the motion. The speeds, distances and times are all taken along whatever path the body travels, and under the assumption that the resistive medium is everywhere uniform.

Now we must distinguish at least the motion in the downward direction, along which direction gravity also acts. Newton takes DR to represent the sideways distance travelled and Rr to represent the altitude. The respective motions are described by these lines in the sense that they give the position of the body. But also, using the representation of the time provided by the auxiliary hyperbola, we can understand these as positions relative to times, thus they specify in fact motions, not just distances.

Notice however, that by taking DR to be the sideways distance this makes the path to which the resistance is applied only the sideways path. The proof is already complicated enough. More to the point, the proof as given does not provide us with any real guidance as to how one might accomodate a varying force in the vertical direction. It is this sort of deadend I believe Euler is referring to with regard to the failings of the geometric approach to mechanics.

Continuing on, the demonstration notes that at the beginning of the motion the area $RDGT$ will equal rectangle $DR \times AQ$. And so at the beginning the line Rr is to DR as $AB - AQ$ (which follows from the fact that $Rr = \frac{DR \times AB - DR \times AQ}{N}$.) Since $AB - AQ = QB$, Rr is to DR as QB to N. But N was constructed so that the ratio QB to N is as CP to DC. Now the point of N emerges, for CP to DC is "precisely as the upward motion to the lateral motion at the beginning". Therefore QB to N is precisely as the upward motion to the lateral motion at the beginning, as so Rr to DR is as those motions. The proof now finishes with a kind of mathematical induction.

> Therefore, as Rr will always be as the altitude and DR always as the longitude, and because Rr to DR is initially as the altitude to the longitude, it is necessary that Rr will always be to DR as the altitude to the longitude. And for this reason the body will move in line $DraF$, which point r perpetually touches [tangit]. ***Q.E.D.*** (Ibid.)

We might think that the conclusion (the ratio Rr to DR is always as the altitude to longitude) follows immediately from the fact that Rr is always as the atltitude and DR is always as the longitude. But with ratios we need to be more careful. Consider some other point R' along DC and the point r' above it on the curve. $R'r'$ is also always as the altitude. What links Rr and DR together is firstly the point R which they share, making Rr the altitude at the longitude DR; secondly that their motions are in proportion to the longitudinal and vertical motions; and thirdly, that they are initially in proportion to the altitude and longitude.[4]

[4]We can also see the point this way. Let h be the real altitude, l the longitude. Take $y = ah + c$, c being some constant; and $x = bl$. Now y is always as h; x is always as l. But $\frac{y}{x} = \frac{ah+c}{bl} = \frac{ah}{bl} + cbl$. Given the condition that y/x is initially as h/l however, the constant c would have to be zero.

5.3 EULER

In *Mechanica*, beginning with Definition 15 onward, Euler represents speeds through a height of fall (*altitudo debitam*). It will help to discuss this representation first, which is employed in solving the problem being considered, before turning to the solution to that problem. And to understand that representation we will have to begin with the two Propositions which precede it.

The representation is introduced in the third chapter of the first book, the chapter which treats "The rectilinear motion of a free point acted on [sollicitati] by a potentia." The distinguishing feature of the material treated in this chapter is that the potentia acts in the same direction as the motion. The result of this is stated in a theorem.

PROPOSITION 24
Theorem

189. *When the directions of the potentia and the motion are directed in to same position [sitae], the motion will be rectilinear.*

Potentiae can have two effects on bodies, changing either their direction or their speed (or both). Two change the direction their action must be along a line different than the motion the body already has.

PROPOSITION 25
PROBLEM

193. *A point resting in A is pulled [protrahatur] in line AP by a uniform potentia or at every point is sollicited the same by a force [vi]; to determine the speed of the point in every* loco P.

The mass or vis intertia is designated A (which we must distinguish in context from the point A); the potentia is designated g, which "will be constant or the same quantity everywhere." Space [spatium] $AP = x$ and the speed in P which we seek $= c$. The element of spatii Pp will be dx; the increment of speed which the body recieves from gravity while element Pp is [absolvitur], is dc.

DEFINITION 15

200. *Hereafter, for speed, we will call required altitude [altitudinem debitam] that particular altitude , by which a heavy thing descending above the earth will acquire that speed.*

200. *Altitudinem celeritati cuidam debitam vocabimus posthac eam altitudinem, ex qua grave in superficie terrae descendens eandem illam acquirit celeritatem.*

It is assumed that gravity is uniform over the fall. ‘Required altitude’ (*altitudo debita*) is used more or less as a technical term by Euler as set out in this definition, and it will be designated v. Technical phrase is probably more apt, as the use will usually be like “let the speed at (some) point M be due to altitude [debeatur altitudini] v.” The speed it designates is denoted c. In the first corollary to this definition, Euler points out that this definition of v dictates that “altitudo debita is as the square of the speed to which it refers.”(Euler, 1736, 69) In effect (and anachronistically), Euler’s chosen measure is kinetic energy rather than speed. One of the fundamental equations he will then use to measure the effect of an arbitrary potential will be (also speaking anachronistically) a work equation.

An example of the usage of this measure will be seen in the proof following. But first, if Euler is not using the concepts of energy or work, how does he understand the *altitudo debita* measure of speed? In part it is merely a convenient way of working out the units. But more importantly it is the direct representation of the effect of potentia in terms of space.

As we saw above, especially where I considered the Preface to *Mechanica*, the most significant distinguishing feature of Mechanics is that it is the science of bodies in motion. Mechanics is distinct from Statics because the way forces act on bodies in motion is different than when they are stationary. The question of what form of the “second law” should be used must be answered, for Euler, by asking how the motion of a body effects the way the potentia acts on the body. What he discovers at Propositions 19 and 20 is that

$$cdc = \frac{npds}{A} \tag{5.8}$$

where c is the speed, p is potentia, A the “matter or quantity of points”, s is distance and n is a constant which gives the ratio between p and vis gravitas.

Let’s consider Proposition 19 in detail.

PROPOSITION 19
THEOREM

150. *If a point is moved in direction AM . . . and is urged [sollicitetur], as it runs through [percurrit] space Mm, by potentia p drawing along the same direction; the increase of speed which the point acquires will be as the urging potential demands [ducta] in the time during which Mm is run through.*

DEMONSTRATION

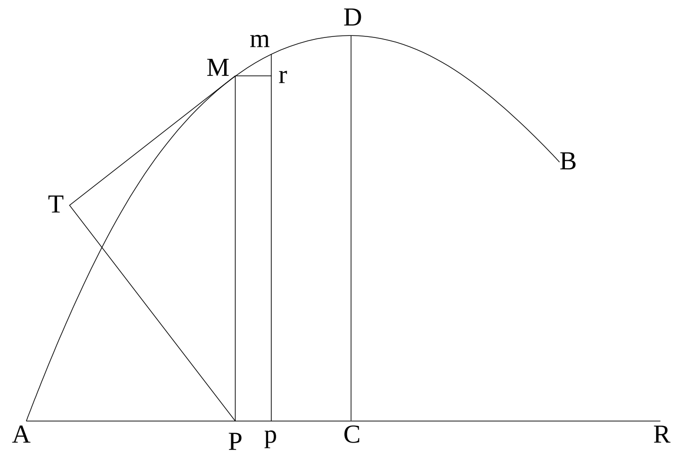

Figure 13: Euler's diagram for the path of a projectile moving in a resisting medium.

Let the element of time [tempusculum] be dt, and [allow that] the point complete space Mm in that time if it is not disturbed by the potentia but progresses uniformly with the speed it has in M. Then the effect of the potentia consists in that, as the point is dragged further [protrahatur] through some other μm, which space is equal to that through which the same potentia in the same time dt would drag the point if it were resting, assuming the potentia is absolute (§ 111). The increment of speed is proportional to this space for a given time. But since the potentia is the same, the increment of speed is as the element of time dt (§ 130). Wherefore, since the space $m\mu$ or increase of speed for a given time must be as potentia p, the increase of speed for whatever time and potentia will be as pdt, i.e. as the potentia demands in an element of time [tempusculum].

COROLLARY 1

151. If the point has speed c at M and space $Mm = ds$, then $dt = \frac{ds}{c}$, by which time the determined element Mm described by uniform motion is put. Since dc is as pdt, then also dc is as $\frac{pds}{c}$ or cdc is as pds. Therefore, the square of the speed is as potentia ducta in the element of space run through. (Euler, 1736, 55)

We see now the point of the squared relation. On the one hand the speed is as the space in the simple measure. That is, for a given time, the greater the speed the greater the space. But the amount of speed that can be impressed by a potentia is also greater in a greater space. So speed is doubly as the space. The point of this proposition is to eliminate time and arrive at the relation between speed and space when a body is both moving and acted on by a potentia.

PROPOSITION 106
PROBLEM

870. *If a vis sollicitans tends everywhere normal to the line AP given in position and a body is moved in some resistive media; to determine the curve AMB in which the body will be moved and the motion of that body.*

SOLUTION

Let the *vis*, which disturbs [sollicitat] the body at M, $= P$, the direction of which will be MP. Let the speed of the body in M be due to altitude v and the vis resistens there be $= R$. If element Mm is taken and mp drawn, then $AP = x$, $PM = y$, and $Mm = ds$. $Pp = Mr = dx$ and $mr = dy$. Draw to tangent MT from P perpendicular PT. By this [perpendicular] resolve vis P into the normal

$$\frac{P \cdot PT}{PM} = \frac{Pdx}{ds}$$

and the tangential

$$\frac{P \cdot MT}{MP} = \frac{Pdy}{ds}.$$

All the geometrical elements are translated into the notation of differential calculus, as well as the force (vis) acting on the body which is set $= P$.[5] The last two equalities follow from the similarity of triangles PTM and Mrm and the definitions of sine and cosine.

The solution concludes:

Because this tangential vis retards the motion of the body its negative is taken. Positing therefore, that the osculating radius in $M = r$, then

$$\frac{Pdx}{ds} = \frac{2v}{r} \text{ and } dv = -Pdy - Rds$$

(§ 866). From these equations not only the curve but the motion of the body can be found. Q.E.I.(Euler, 1736, 313)

The second equation is not describing the change in speed but rather the change in the 'altitude required for the speed'.

[5] P is also used in the diagram to label a point. We might be tempted to think the use of the same letter in two different ways indicates an implicit recognition by Euler of the two different formalisms, geometric and analytic, that he is employing. However, on the diagram for the arc of isochrony, Fig. 6, one label M is used for different points. That is, he duplicates labels even within the same formalism. It is more likely that Euler simply expects us to be able to keep the two apart from context.

6.0 MECHANICAL EXPLANATION, UNIFICATION AND THE 18TH CENTURY

6.1 INTRODUCTION

The preceding chapters have shown that the conceptual foundations of mechanics for Euler and Lagrange were importantly different from that of Newton. These differences went along with a re-thinking of the representational relation between mathematics and nature. For Newton, curves constructed in the limit of infinitesimal line segments represented the trajectories of bodies under the influence of forces. But at the same time, the analysis of those trajectories into rectilinear segments was, for him, an artifice. The limit had to be imagined by the geometer, and only in the limit was the real trajectory captured—not before the limit was achieved, not after the limit was achieved, but in the *very moment of arriving* at the limit. Unsurprisingly, synthetic geometry was the most important mathematical method for Newton because only in synthesizing the geometric elements, especially through the last conceptual step of imagining the limit, did one reconstruct the proper picture of the trajectory as a developing, flowing, whole.

The picture which emerged with Euler and Lagrange was more of a static one. Functional relations among variables represented mechanical constraints on the properties of bodies. Some of those properties were instantaneous differential values. Instead of the Newtonian, flowing, developing trajectory—the fluxional conception—the motion of a body was represented in analytic calculus as a series of instantaneous, frozen slices. At each point the state of a body was completely determined by a set of instantaneous quantities. In Euler's early work in *Mechanica*, for example, each body had a determinate state of motion at each point in space as it travelled. In fact, it was only near the end of the century (thanks in great part to Lagrange) that the idea of an equation of motion, giving position as a function of time, and so relating the trajectory directly to time, became the standard solution in physics. For Euler and Lagrange, the trajectory of a body was no more than the sum of the instantaneous states, and so analysis offered the truer picture of motion.

Along with these foundational differences went a difference in explanatory standards. The descriptive differences of the analytic calculus were connected with a different understanding of mechanics. There was the increased importance of equilibrium to explanations, either as a condition which explained phenomena directly or as a backdrop against which a phenomenon was explained as a difference from equilibrium. This went hand in hand, straightforwardly, with the use of equations as descriptions of phenomena. In static cases, equations expressed the equilibrium conditions directly. In dynamic cases, equations rewritten with zero on one side expressed an instantaneous, dynamic equilibrium between all the relevant, explanatory quantities—even though some of these quantities may have been non-zero velocities, and so not strictly speaking part of a static or equilibrium condition in the traditional sense.

There was also a difference in the use of mechanical models as ways of interpreting the functional relations among quantities that the various principles of mechanics expressed. Mechanical models were used to interpret why two quantities would be multiplied together, for instance, or why a series of terms would be added together. All of this, it will be shown, can be tied together by the fact that equilibrium conditions are a constitutive part of any mechanism and play an important role in mechanical explanation.

In this chapter, then, I draw some morals for the philosophy of scientific explanation from this important period in the development of modern physics. The plan of the chapter is to first point out limitations of attempts to eliminate scientific explanation in favor of unification. Next is described a schema of mechanical explanation which uses entities, characterized by their properties, and activities, characterized either by changes in the values or qualities of properties or by coupling among those properties. These couplings represent mechanical paradigms, particularly paradigmatic mechanical operations, and it is our understanding and acceptance of the operation of the underlying mechanisms that licenses the inferences we make regarding the derivation of phenomena. The derivations are then explanatory insofar as the inferences are based in these mechanical models—the derivations alone are not explanations.

This account of scientific explanation is therefore both similar to and distinct from standard unification accounts and from causal-mechanical accounts. It is similar in that argument and the deduction of phenomena still play a central role. On the present account, however, explanation is not reducible to unification. The mechanical explanation scheme presented here is unifying because of its descriptive capabilities, but it is explanatory because of its connection with mechanisms. Nevertheless, recognizing the importance of unification as a scientific aim, I will describe how mechanical descriptions provide for unifications.

Also, understanding based in mechanical models does not require analysis of the concept of causation as is required for the causal-mechanical account. My account will substitute an unanalyzed acceptance of mechanical operations for an analysis of causation. This does not undercut the satisfaction of what I take to be the overriding desiderata of any account of scientific explanation. That is, to

- provide an historically accurate description of at least one important period, or even better, a tradition, in science (as no account will capture explanation in all contexts and all periods in history;)

- present scientific explanation in a way which satisfies our intuitions about a large number of particular cases;

- present scientific explanation in a way which emphasizes the *subjective* character of explanation, particularly the issue of making phenomena *intelligible through* description and not merely describing them.

In line with these criteria, it is not my intention to provide necessary and sufficient conditions for scientific explanations. An account of scientific explanation should be every bit as dependent on context as explanations themselves are. Mechanical explanation is an important kind of scientific explanation, but I am not here defending the claim that it is the only kind of scientific explanation. This chapter is programmatic and outlines novel but important features of mechanical explanation, particularly the historical role of equilibrium and its legacy as a precursor to symmetry and conservation in modern physics.

6.2 KITCHER'S UNIFICATION PROGRAMME

The best known treatment of scientific explanation as unification is due to Philip Kitcher (Kitcher, 1981, 1989).[1] On his account, explanations are explanations because they fit an argument pattern or schema which is part of an overall unifying system.

At the core of Kitcher's account, like the deductive-nomological accounts of explanation before, is a deductive relation between the explananda and explanans. A statement of the phenomena to be explained is deducible from the set of sentences provided as the explanation. Not all deductions are explanatory, however. On the D-N model it is required that the premises of the explananda also crucially contain (i.e. without which the argument is not valid) a law of nature. Kitcher, instead, requires that the deduction fit an argument pattern which is part of a unifying system. Kitcher motivates his account with the intuition that explanations are connected with unifications. I think this intuition sound, but there is good reason to balk at the idea that explanation can be eliminatively reduced to unification as Kitcher holds. We might expect good explanations to unify. We may even therefore expect that unifying power is the mark of a good explanation. But if unification is going to eliminate explanation than unification must not only be sufficient, it must be necessary to explanation. That is, explanations are always and only unifying. This begins to strain, I think, our intuitions.

The main idea of Kitcher's model of scientific explanation is that each instance of an explanation fits an *argument pattern*, that the particular explanation can be constructed by filling in variables (dummy letters, in Kitcher's terminology (Kitcher, 1981, 516)) of the pattern. Arguments are a series of sentences. Argument patterns are a series of *schematic* sentences which have some of their non-logical vocabulary replaced by variables. Schematic sentences of the argument patterns, when filled in in the appropriate way, must be members of the set of statements accepted as true by the majority of the scientific community. Kitcher denotes the set of sentences accepted by scientists as K. Some of these sentences will be premises in the filled in argument patterns; some will be conclusions. A set of argument patterns is a unifying system or a *systematization*. The unifying power of each set is evaluated, in part, in terms of the number of sentences in K derivable from the patterns in the set, using sentences also in K. The explanatory store $E(K)$ is

[1] See also Friedman (1971).

then the systematization which best unifies K.

A somewhat awkward feature of Kitcher's account has been noted. His is a top-down account of explanation where instances of explanation are explanatory only because they fit a pattern which belongs to $E(K)$, the best systematization of K. When a new systematization comes along, many former explanations will not fit with the new unifying system. What one cannot say is that these old explanations are less explanatory, or explanatory in a qualified sense. They are simply no longer explanatory at all if they do not fit a pattern in the new unifying schema (Woodward, Summer 2003). The so-called "winner takes all" objection has been raised by, for instance, Sober (1999). I would argue that this is a symptom of a larger problem of one-sided top-down approaches: while they rightly emphasize the patterns which lie behind explanations, they pay too little attention to what is required for a particular instance to fit a pattern in the appropriate, explanatory way.

Just as not all derivations are explanations, neither are all unifications good explanations. One can imagine trivial unifications that simply list all the sentences in K, or systematizations whose only argument pattern is any sentence from K as premise and conclusion. Any measure on the success of unification by a systematization must therefore take into account other factors. Kitcher suggests a characteristic he calls the *stringency* of the systematizing argument patterns. Stringency has something to do with how easily argument patterns can be filled in—both of the trivial argument patterns just suggested are obviously the opposite of stringent. We are not given, however, an exact analysis of stringency and this is the weak point in the account. What the notion of stringency masks are explanatory considerations which are an important part of the way scientists have historically constructed and evaluated explanations.

Kitcher is aware, in part, of this lacuna.

> Thus, without trying to provide an exact analysis of the notion of stringency, we may suppose that the stringency of a pattern is determined by two different constraints: (1) the conditions on the substitution of expressions for dummy letters, jointly imposed by the presence of nonlogical expressions in the pattern and by the filling instructions; and, (2) the conditions on the logical structure, imposed by the classification. (Kitcher, 1981, 518)

The classification describes "the inferential characteristics of the schematic argument: its function is to tell us which terms in the sequence are to be regarded as premises, which are to be inferred from which, what rules of inference are to be used, and so forth." (Ibid.) Thus, in a nutshell, the stringency of a unifying argument pattern depends on what features of a real physical system can be identified with the dummy letters and what inferences we can form on the basis of the relations among those identified features. What I show in the next few sections is that these are precisely the features that mechanical explanations lay out. So what stringency amounts to, when evaluated with respect to argument patterns from mechanics, is how well the patterns explain. An account of explanation cannot eliminate explanations while relying on stringency because stringency cannot be evaluated without considering explanation.

A similar conclusion has been reached by Franz-Peter Griesmaier (Griesmaier, 2005). As just mentioned, argument patterns have an attached set of filling instructions. According to Kitcher, patterns must come with filling instructions for the dummy letters—filling instructions such as "occurrences of 'α' are to be replaced by an expression referring to the body under investigation; occurrences of 'β' are to be replaced by an algebraic expression referring to a function of the variable coordinates and of time ...." (Kitcher, 1981, 517) The filling instructions, in effect, say what sorts of things can play the required roles. Griesmaier argues that any adequate set of filling instructions will depend on a notion of explanatory adequacy. The relevant point here is that his criticism turns on a feature of explanatory adequacy having to do with the way the variables in an argument hang together. Through a pair of examples he shows that one cannot simply fill variables while considering those variables only in isolation. The objects and properties which are used in the explanation have relations to one another that are not entirely captured by the schematic argument.

> What I want to suggest is that this particular instantiation of the above argument pattern is unacceptable because the filling instructions are all wrong. They are wrong in virtue of not answering to the roles played by the dummy letters in the argument pattern. Those roles put restrictions on possible combinations of replacements. These restrictions in turn are related to what we intuitively judge as explanatorily relevant. (Griesmaier, 2005, 4X)

This therefore makes Kitcher's account of explanation depend on a notion of explanatory adequacy. The unification account cannot therefore be an eliminative account of explanation.

When we consider mechanical explanation below this will become more clear. We will see that the logic of the argument—what licenses the inferences being made—often depends on the mechanisms which lie behind the connections of the variables. I will not argue here that this is true for all scientific explanation, but only that unification does not capture everything of *mechanical* explanation. While unification and explanation are closely connected, an independent account of explanation is needed. It is the structure of the argument pattern around the dummy letters which determines, in part, what sort of things can fill in the dummy letters. Again, the schematic sentences express relationships among the dummy letters and whatever objects or properties are chosen to fill them must also be able to fulfill their roles in those relationships.

Let me say, lastly, that I believe Kitcher's program offers the best hope of an account of scientific unification, especially one which can move past old reduction/anti-reduction debates. This is because of its use of argument schema for unification rather than an appeal to reducing levels. But what I want to offer in this chapter is an amendment to the unifying pattern program by considering how the patterns are arrived at. The development of mechanical explanation is explicitly a search for the proper parts and relations among those parts for describing the behavior of physical systems. While unification is certainly an important aim of science, that aim arises from the desire to unify phenomena under a common *explanatory* system. What makes the system, or its patterns, explanatory requires explication independently of how systems unify. To put it another way, Kitcher requires that putative unifying sets of argument patterns be assessed for the number of conclusions that can be drawn from the set as well as the number and the stringency of the argument patterns. What ought to be added to this list is the explanatory value of the patterns—suggesting, of course, that the unification account of explanation has gone cart-before-horse.

In the remainder of the chapter I outlay my own account of scientific explanation, emphasizing the distinction between mechanical description and mechanical explanation. When we want to explain the behavior of some physical system, one way to proceed is by describing that system in terms of its parts and the roles they play in generating that behavior. Descriptions in those terms facilitate explanation by making plain an analogy between the system and mechanical paradigms having parts playing analogous roles. The chief mechanical paradigm is a balance or, more generally, a system at equilibrium. Equilibrium conditions provide background or normal conditions for the system and a properly specified mechanism will have clear equilibrium conditions. The behavior to be explained might then be revealed to be an equilibrium state, or it may be understood as a deviation from equilibrium.

6.3 MECHANICAL DESCRIPTION

At least since Galileo and the rejection of the Aristotelian system of natural philosophy, Mechanics, the science of motion, has been an attempt at explaining anew the movement of bodies through its proper description. Description and explanation have been used together, one helping to refine the other. Descriptions must include, point out, or isolate the relevant explanatory factors. What those factors are and which descriptions are tried is guided by what is taken to be explanatory. Sometimes we build descriptions to fit with what we take to be explanatory, other times we read explanations off of accurate descriptions.[2] This interplay between what the parts of a system are and what they do is a defining character of the development of mechanical explanations. In this section I focus on mechanical description in terms of parts, where the parts are individuated by the properties they possess as well as the role they play in the mechanical system.

With the mathematization of mechanics, more than ever one could put *descriptions* at the forefront, with views about what was explanatory then being read off of, or influenced by the nature of, those descriptions. This is, in part, the way many historians have characterized the Newtonian revolution: as a rejection of metaphysical explanatory frameworks in favor of bare, mathematically expressed, empirically adequate descriptions. As I have argued, however, this is not the way to understand Newton. His mathematical descriptions were constructed to describe the process which he took to explain motion, namely, the generation of motion by the action of forces. Euler and Lagrange, on the other hand, are both excellent examples of the mathematics-first approach. For them, the mathematics, specifically analytic calculus, was the starting point. Explanatory interpretations were constructed to fit with what was mathematically described. And although metaphysics was still to play an important role, namely as the description of the fundamental properties of bodies, what those properties were was determined by the role they played in the mathematical descriptions.

The basic descriptive and mathematical element of the 18th century was the function. To explain the functional relation among quantities, Euler and Lagrange appealed to mechanical analo-

[2] Sadi Carnot's (1796–1832) derivation of the formula and description for the heat engine on the basis of the explanatory assumption that heat was a fluid would be an example of the former (Carnot, 1960); attempts at providing a hidden-variables interpretation for the equations of non-relativistic quantum mechanics could be construed as the latter.

gies. Mechanisms were understood as constructed out of parts whose properties and activities were constrained by the connections of the mechanism. Parts were individuated and represented by the set of quantified properties they possessed. Functions expressed constraints among these quantified properties. The functions were thus a description of the mechanical behavior of the system of parts. This can be seen in Lagrange's treatment of the Principle of Virtual Velocities as described above (Chapter 4.) The principle expressed a function among the displacements and the masses of the block and pulley system and that function represented the mechanical constraint due to the coupling among the positions of the masses provided by the rope. Mechanical systems in general are only *systems* because of the connection among the parts and so are to be described, mathematically, by functions which describe the constraints caused by those connections.

Sets of properties are what we now call, generally, the state of the system. One can assign a variable for each of the properties in the set and an axis for each of the variables. Exact values of the properties pick out a point in state space which we call *the* state of the system. State equations assign values to the properties, usually expressing the variables of the state as functions of time. In classical mechanics the state involves the position and the velocity of the body. Equations which give the state of the system (the positions of its parts) as functions of time are called the equations of motion.

This approach can be generalized to systems which require more than just the position of its parts for a full description of the state of the system. The state of a neuron, for instance, can be described by perhaps the concentration of sodium and potassium ions, the permeability of the membrane and the current flow. In specifying the mechanism behind the operation of a neuron we likewise include constraints among those properties. Some of the constraints may be expressed as functions, as between the concentrations and voltage across the membrane; some may be qualitative as the relation between 'depolarized' and the relative concentrations and locations of the ions.

Often, the constraints also represent parts of the mechanism, as with the coupling of positions due to the rope in the Lagrangian example. But along with these constraints will always go an activity. Here I am endorsing the dualist view of mechanisms, as composed of entities and activities, put forward by Machamer et al. (2000). The constraints or couplings allow some parts of the mechanism to act on other parts. Focusing too much on the state part of physical descriptions tends to obscure the important role that activities play in mechanisms. In the next section I will turn to explanation and the role that activities play. The connection between mechanical couplings and the use of equilibrium in explanation will also be treated.

6.4 MECHANICAL EXPLANATION

The basic picture of explanation I have in mind goes as follows. To begin simply, explanations are requested when events occur and it is not already understood how those events come about. Explanations provide a story about the "coming about", about the objects and processes which result in the event of interest. The kind of events focused on here are those which are most often dealt with in science. You have some physical system, some real world system, and you want to understand its behavior, where behavior is characterized as observable changes in a system's properties. Given this focus, then, explanations are requested when a system's properties change and we want to understand how these changes are a response to other, often earlier, properties of the system.

I am avoiding characterizations of explanation as answers to questions such as "what *caused* the behavior to come about?", or even straightforward "why" questions. As mentioned before, this account will be absent any particular analysis of causation. As for "why" questions, when explanations are given against background equilibrium conditions, the question answered is rather "why did the system *not* behave in a way which I would have expected given my present understanding?" That is, the kind of explanations I will focus on have a negative aspect to them. They provide reasons for the system not to be in its normal or equilibrium condition.

This is the main point. Explanations are here characterized as a development in our understanding beginning with a broadly satisfactory background which includes some expectations as to how a system ought to behave. These expectations will be based in mechanisms we have reason to think are already operating in the system because of either earlier success with those explanations or, for a system we have no experience with, because we assume it to be like certain other systems we do have experience with. At the same time, the behavior to be explained somehow does not fit with that background. The job of the explanation is to solve this tension and it does so either through a change in the description of the system or a change in the background. A beginning attempt at the details of how these are achieved through explanation is the goal of this section.

Let us begin with the background or context of explanation. The context of an explanation includes background assumptions about what can count as an *explainer* and what sorts of behavior, phenomena, and other properties do not require explanation. Mechanical explanation, just as any other explanation, requires a background or context, specifically, a normal set of 'property values'. Examples are the resting state of the neuron; stable values in the population of mammals; water at room temperature; inertial motion of a body. Some of the background can be provided by equilibrium conditions of the system.

Given the importance of equilibrium to the mechanical explanations of Euler and Lagrange, we should consider what the notion of equilibrium might provide to an account of explanation, as one part of explanatory contexts and as part of the characterization of mechanisms. An important method for arriving at quantities which are explanatory and belong in mechanical descriptions is the use of equilibrium. We can again take our cue from the way Euler and Lagrange approached the solution of mechanical problems. What is advantageous about equilibrium is that it provides a starting point for arriving at a mechanical explanation with at least a rough description of the behavior. This gives some guidance as to what the parts and activities of a mechanism might be because we know that, whatever they are, they must be arranged in such a way as to cancel out in their overall effects on the system such that equilibrium, under whatever description we have given it, is achieved.

There are two distinct cases of equilibrium: the kind which needs explaining in a given context and the kind which is part of the context itself and so does not require explanation.

- Equilibrium requiring explanation aids in discovering explanatory factors. It provides a starting point and a context for explanation.
- Equilibrium, once explained, then serves as a context for further explanation as deviation from equilibrium. Alternatively, explanations can also be given as a return or drive towards equilibrium.

Equilibrium of the latter kind will need generalizing to something like dynamic equilibrium. Take, for example, inertial motion. For Newton, even inertial motion requires an explanation, namely the *vis insita*, a force. Euler, on the other hand, attributes inertial motion to some inherent property of bodies, though not a force. This general feature of nature he calls the conservation of state.[3] However, because he argues to this conclusion from the Principle of Sufficient reason, the explanation of inertial motion is actually a negative explanation. There is no reason for the body, in the absence of any external forces, to change its state. Thus inertial motion needs no explanation and can be thought of as a kind of unexplained but explaining equilibrium. What *does* require explanation is any change in the state of motion or of resting of a body.

Mechanical explanations turn heavily on equilibrium and a properly described mechanism will have easily recognizable equilibrium conditions. Equilibrium conditions have two requirements: equilibrium behavior must be readily understood, as must the kinds of conditions which can bring about that equilibrium. In most cases in classical mechanics, equilibrium behavior is the lack of motion. With the balance, for example, equilibrium occurs when the balance arm is static. One condition which brings about that behavior is that there be equal weights at equal distances from the fulcrum. Notice that the equilibrium condition is straightforward and from it we can draw conclusions about the arrangement of the parts of the system. The description of the balance exhibits a symmetry in this case between the parts. The symmetry between the weights, their magnitude and distance from the equilibrium, removes any reason for the system to be in any state other than a static one.

Symmetry can then be generalized to proportionalities which also bring about equilibrium. Part of the explanatory and descriptive project is to discover what the proper proportionalities are which must obtain for equilibrium. The natural extension, and the earliest achieved, was to recognize that an inverse proportionality between the weights and distances would also result in an equilibrium between the actions of the two weights within the machine. (It was this picture which was further generalized to the Principle of Virtual Velocities by Lagrange for his block and tackle model, a more general kind of balance arrangement with an arbitrary number of interacting masses.)

From here, we can go on to surmise the conditions which would result in the balance not being at equilibrium, namely removing the symmetry or proportionality. This picture can be further

[3]He also refers to the inherent property of the body as inertia, pointedly dropping the *vis* after first discussing how inertia, as a property which preserves the state of motion rather than altering it, is not a proper force. "…[I]t would be absurd to attribute to the same body an effort to conserve its state and at the same time to change it." (Euler, 1746, 21–22).

generalized, moving beyond symmetry to broader cases where normal behavior is disturbed to bring about some other behavior. Disturbances can be described as a change in some property. The explanation will be provided by the fact that the change to be explained is of a property coupled to that property whose change is offered as the explanation.

The phenomenon to be explained is described in terms of properties, different than the norm, which are coupled to other properties which also differ from norm. The differences in these other properties, as explainers, are accepted as unexplained for the purposes of the explanation, and so as part of the background. The coupling may be thought of as causal, nomic, probabilistic or deterministic. Part of what makes them explanatory is they are accepted as requiring no explanation themselves because they are *mechanical* couplings, and analogous to mechanistic paradigms—ropes, rods, channels of air or fluid flow and the like.

These kinds of couplings among properties are accepted as explanatory, moreover, because of the inferences they allow. The mechanical connection between bodies allows for the propagation of changes. Change in the quantification or qualification of the properties of one body or entity entails a change in the properties of another body determined by the nature of the mechanical connection. Mechanical connections are physical things but they licence entailments. Bodies connected by a rigid rod, for example, have their motions coupled due to the fact that the rod imposes a constraint on the distance between them (one can infer something about the motion of one body from the motion of another.) As another example, fluid or charge reservoirs connected by a channel of flow (such as a vesicle or an axon) can have coupled volumes or charges.

Notice that these inferences are grounded in the mechanical actions that lie behind the couplings or constraints. These actions, and the parts of the mechanism that produce them, are often only implicit in the functions used to describe them. It is in this way that one can drive a wedge between description and explanation. The functional constraints between state properties are adequate descriptions, but what explains the constraints, by which we infer the changes to be explained, are the actions of the connecting parts of the mechanism.

The mathematical representation of the couplings of course plays an important role in the derivation of the descriptions. Especially important are the treatment of these couplings as mathematical constraints on equations when solving problems. Euler's treatment of bodies moving on surfaces is illustrative just for this feature. Rather than treat the surfaces explicitly as providing forces to influence the motion of the body, Euler uses the equations of the surface as constraints in solving the system of equations. But what is understood is the mechanical arrangement operating behind the use of those constraints. It is this understanding that makes mechanics explanatory and not merely a mathematical exercise.

The goal of mechanical explanation, then, is to use these two broad categories—properties and their coupling—to formulate descriptions of the behavior of systems. It begins with properties coupled in an equilibrium arrangement. This is understood as an equilibrium among the activities of those parts to which the properties belong, as well as the parts of the mechanism whose activities constrain the coupled parts. The equilibrium could be characterized as the lack of change in any of the properties, as when no parts of a system are moving, or as the constancy in the value of some function of those properties, such as the value of the kinetic energy. Such conditions will go a long way to describing the behavior of many systems.

These conditions also provide a starting point for further explanations of the behavior of a system as differing from equilibrium. Given the equilibrium conditions we can also know which properties are outside equilibrium values. Some of these properties can be used to explain why other properties are also outside their equilibrium ranges. Having a functional description of the coupling among properties allows one to give values to the differences. There are therefore two kinds of advances made with respect to an explanatory background. Some behavior can be accounted for as a difference from equilibrium. In other cases, that difference can only be accommodated by changing the description of the equilibrium condition itself, and thus the background, through the introduction of new properties or new activities. It is through advances of both kinds that unifications are achieved.

6.5 MECHANICAL UNIFICATION

It remains to briefly say something about the role of mechanical explanation in unification. One of the things emphasized in this account of explanation is the connection between descriptions and explanations and how this goes along with the use of both entities and activities in the development of mechanical explanations. These developments can also be connected with unification. New entities and new activities can allow us to describe the behavior of two systems where, before, each had its own description and set of properties. But also, describing a system as having different parts (entities) playing new roles (activities) may allow us to explain behavior through analogy to systems we already feel we understand. In this latter case the parts are the same only insofar as they have analogous properties and behave in analogous ways. The systems could nevertheless be physically very different. Not only do both of these kinds of unification need to be considered in order to understand the work of Euler and Lagrange in the 18th century, this also illustrates again the importance of activities to explanation.

New descriptions are often justified because they allow for unification. Finding descriptions which allow for the explanation of the behavior of two seemingly very different physical systems, like a magnet and an electrical wire, is no easy task and so, when achieved, the tendency is to treat those descriptions as revealing some more fundamental structure. Recognizing this tendency, and evaluating it from a Humean perspective, is, it seems, a chief motivation of top-down accounts of unification such as Kitcher's. Such accounts seek to treat our predelictions to interpret unification instead in a pragmatic light. Rather than justify the tendency as accurate, and allowing that unifications do, in fact, reveal deeper structure, they focus on unification as enough and deny that anything more need be read into it.

But as already said, unification is not explanation. Justifying explanations through unification is not the same as identifying explanations with unifications. In particular, we have seen that the activities which license the inferences made according to an argument schema of a unifying system are as important as the schema itself and require independent consideration. I too want to avoid arguing here that unifications do, in fact, reveal deeper structure, or that unifications necessarily lead to reductions. An account of unification can be given, however, as it occurs through the development of mechanical explanations in terms of entities, their properties, and activities. That is, unifications in science are not merely unifications—they are explanatory.

I want to emphasize two aspects of unification therefore. The one is unification through descriptions, where advances in the entities and activities we use in argument schema result in unifications as characterized by Kitcher. This kind of unification has already been suggested as part of the development of mechanical explanations. Schematic arguments are series of sentences expressing relations among dummy letters (variables). The dummy letters represent objects and their properties; the relations expressed are constraints or couplings among those properties. Through adopting new objects, properties and constraints, new schematic arguments can be constructed.

But another aspect of the unification of domains occurs in science when the description of phenomena in one domain reveals an analogy with phenomena in another domain. With the right kind of analogy, the phenomena in both are then unified under the same *explanatory* schema. The basic structure of the analogy is of two systems having similar parts behaving in similar ways. The phenomena first needs to be described as a system of entities and activities and this was the essential character of the explanatory schema of mechanisms. The important difference is that, with analogy unification, the actual physical objects of two unified systems can be quite different.

As an example, consider the wide variety of systems and phenomena that can be described as a simple harmonic oscillator: from pendula and springs to electrical currents, biological feedback systems or population dynamics. The analogy among these systems is facillitated by a common mathematical description. Any system with a stable equilibrium point can be described, mathematically and to an order of approximation, as a harmonic oscillator. In this way we can understand the behavior of very complex systems—systems for which we do not have a full accounting of all their parts or of all the ways which they directly act on one another—through an analogy to simple mechanical systems we do understand.

Taking on board this sort of analogical unification allows us to think about unification in science in a new way. It allows for a *methodogical* unification where the point is not just to unify particular systems or domains under one descriptive or explanatory schema. Mechanical explanation is unifying as an approach to understanding systems. It does not require reducing systems to the level of forces and masses. Systems can be understood in terms of objects and properties coupled to one another in functional relations analogous to those describing simple machines—what could be called models of intelligibility. The same mechanism can be instantiated in a vast variety of physical systems and at many different levels of description.

6.6 SUMMARY

At the outset of this chapter I had pointed out that a crucial difference for Euler and Lagrange was their re-thinking the representational relation between mathematics and nature. This highlights an important feature of the historical development of mechanical description and explanation. By putting descriptions first and then addressing the explanatory question afterwards, Euler and Lagrange were seeking a new way of thinking about how mathematics represented nature. The question of the representation of nature by mathematics is really two questions which ought to be kept distinct. The first has to do with how mathematical quantities and the variables that stand in for them relate to the world; the second with what a functional relation between such quantities represents in nature.

It seems to be taken for granted that an answer to the first question is easy, and that having an answer to it is the same as having an answer to the second. Measurement allows us both to attach numbers to real world properties and, through coordinated measurements of various quantities, to discover the functional relation that holds *in nature* among those quantities. In practice, the process is not so straightforward though. The assumptions behind this characterization of the process have been fairly well problematized in philosophical literature in the form of the theory ladenness of observation. Choosing what to measure will depend on assumptions about the processes involved in the phenomena, what objects are present and what properties they have available for measurement. This is the important point for the present discussion—that there is an interplay in arriving at our descriptions of phenomena between our choice of objects and their behavior.

Note that this can occur in one way in terms of the level at which we make our description. We can choose different levels of scale, for instance, and the kinds of behaviors we describe will be likewise affected. The kinds of behavior an organism exhibits at the level of their cells is different than at the level of their organs; the activities and properties we can use to describe a gas or liquid are different than those which apply to its individual particles. In other words, there are descriptive emergent properties in many systems. And there are likewise mechanisms at many different levels.

Euler and Lagrange engaged in this interplay between the properties we use to describe systems and the kinds of functional connections between those properties. Not only were they investigating what the fundamental properties of bodies were, they were allowing that their mathematical investigations into mechanics might reveal further properties and functional relations beyond simply force, mass and velocity. The quantity of action and the Principle of Least Action is one such case. These arose in the investigation of more complex systems like strings and other oscillating bodies. The descriptions and functions which made these problems mathematically tractable also introduced new quantities and principles into the vocabulary of mechanics. At the same time, they sought to interpret the resulting functions through well known mechanical models: the balance, levers, pulleys and ropes.

Descriptive fine-tuning occurs with the discovery of mechanisms and mechanical explanations. To put it roughly, one starts by attempting to capture gross behaviors of a system and ascribes to the system a small number of parts behaving in simple ways. It is usual to begin with the most common background conditions one finds the system in and to attempt to account for what is the corresponding most common kind of behavior of the system. To accommodate other behaviors in other situations will then require tweaking of the components to allow for a system which is more responsive to its environment, connecting differences in the background state with differences in the resulting behavior. Responsiveness can be built into a machine, however, by either adding more parts or by making the parts themselves capable of more complicated behavior by attributing to them more properties and capacities.

Accommodating richer behavior requires a description having more dimensions. A richer description may take into account a greater number of features in the environment and it will represent more properties of the system. Describing the motion of the planets, for instance, one begins with two bodies, then three, then more, etc. (I am talking here about the description and not the solution of the problem.) This is an increase in the dimension of the description of the environment. More dimensions requires more variables. Variables, roughly, represent properties and can be either quantitative or qualitative. A set of variables taken together represents the state of the system or of the environment. We can therefore increase the dimension of our state descriptions by either increasing the number of parts or by increasing the number of variables that attach to each part or to the environment.

Another way to enrich our descriptions without changing the dimension of the state itself is by allowing for more complicated relations among the properties we already have. In doing this we ascribe not just new variables to the state but also new functional relations. Functions represent mechanical coupling, describing how states evolve or expressing constraints on the state. The two are not independent. The couplings you take to be present and operating will depend on how you carve the world up into objects. Two objects may instead be thought of as one object having a greater number of properties. This can be done by almagamating several parts into one object. Neurons can be treated at the level of their ions and membranes or they can be treated as one object whose properties and activities are the combined result of the behavior of internal details. This sort of modularity is an important advantage of mechanical explanation.

The point is, what is required of a state description will be influenced by the search for an adequate explanation, and vice versa. A mechanical explanation will rely on constraints among the properties included in the state. For the state to be adequate, we must be able to set up constraints that properly connect the quantity or quality of the properties as the state evolves so that the resulting behavior of interest can be accurately derived. The parts of the system, identified in its description, will be the ones which possess the required properties.

Newton invented the mathematical methods of *Principia* in order to avail himself of descriptions which fit with the explanation of motion by forces. The great achievement of the *Principia* was Newton's ability to derive incredibly accurate descriptions of motion from the seemingly meager explanatory resources of forces and masses. There were many in the 18th century, and many historians after, who took the essence of Newtonianism to be the reduction of all phenomena to a description which could be explained in terms of only forces acting on bodies (Kitcher is an example.) Others in the 18th century, such as Euler and Lagrange, were more catholic in the quantities they allowed as explanatory, requiring only that the quantities be present in some functional relation that could be mechanically interpreted. Thus their free use of quantities like action and virtual velocities. This is not done by rejecting the methodology of mechanical explanation though, but rather by extending it by recognizing analogies between the functional relations among the new quantities and functional relations that describe already accepted mechanisms.

7.0 CONCLUSIONS

It is a view of convenience that sees the culmination of the Scientific Revolution in a single magnum opus of one great man. It reduces the equally great minds of the century following to the task of merely trying to comprehend what has happened. We have seen the complexities this view glosses over. Historians of mathematics have revealed the complexities that lay beneath the historical narrative of "Newton invented the calculus but it is Leibniz' notation the continent (and we) adopt."

I have tried to show that a similar level of complexity can be found behind the narrative of "Newton's *Principia* was translated into the analytic calculus in the 18th century by the Bernoullis, Euler, d'Alembert, Lagrange et al." In particular, a detailed comparison was given of the foundations for the science of Mechanics employed by Newton as opposed to Euler. The most significant thing revealed there was that the idea that Euler's mechanics analytically derives consequences from Newton's three laws is hopelessly naive: the first law is itself derived, the second law is replaced, and the third law is absent.

The change in mathematics allowed Euler and Lagrange—and, in their estimation, in fact *required* them—to re-invent the mathematics-nature relation. The space of mathematics and the algebraic form of functions, expressed in terms of the variables defined on that space, were recognized as abstract representations of relations among quantities in nature. The function concept provided an extra layer of abstraction between nature and the description. This was to be recognized in the way that either geometric diagrams or analytic equations could both represent the same mathematical object. Different physical systems could instantiate the same function. The physical quantities which characterized the system stood in mechanical relations which could be described functionally (such as the square of speed was proportional to force $\times$ displacement.) The functions, in turn, could be represented algebraically or geometrically. In particular, for Lagrange, the equation that characterized all mechanical systems (the Principle of Virtual Velocities) was an extension of the equation which described the balance.

With the changes in descriptions came changes in explanations. Functions, states and equilbrium were all concepts employed in explanations by 18th century figures. These concepts changed the way explanations worked. The two-step, synthesis-analysis scientific method of Newton, as characterized by Maclaurin, was replaced with a rational mechanical philosophy, according to which, functional relations among quantities could be discerned in nature. The most important relations were equilibrium equations and what needed to be empirically discovered were the quantities among which the equilibrium held. These sets of quantities would become characterized as states. Once the states and their functional (equilibrium) relations were known, the analytic calculus (or algebra) provided the methods for investigating and deriving the consequences of these relations, understood as expressing mechanical connections. Changes in the values of some properties of the state were coupled to other changes by the functions relating the properties. These couplings could be used to derive descriptions of the behavior of physical systems and the resulting descriptions were explanatory because the couplings were interpreted as describing mechanical connections.

Of greatest importance was that these mechanical interpretations of the functional relations provided explanations which were felt missing from the mathematical unification of Newton's *Principia* alone. This fact was used to drive a wedge between description and explanation, and to critique attempts at reducing explanation to unification. Exploring this aspect of the 18th century programme in mechanics thus informs current philosophical debates on scientific explanation and unification. I hope to have shown that explanation via mechanisms provides an insight into physics as it stands today. It is further hoped that the continuation of this research will provide insights into the interpretation of quantum mechanics, as it relates to its classical background, and to pedagogical approaches to science education.

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